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An extension of the static shakedown theorem to a certain class of inelastic materials with damage

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IN THIS PAPER, an extension of the static shakedown theorem (Melan's Theorem) to elastic-plastic damaging material behaviour is presented. Damage is taken into account by using the Ju-model, and the generalized standard material model with plastic hardening.

1. Introduction

IF STRUCTURES are subjected to variable loads, the determination of the field quantities describing the state of structure during the process of deformation is very often less important than the answer to the question, whether the failure with respect to certain functional requirements occurs or not.

A particular kind of failure is the unlimited accumulation of plastic strains during the deformation process, characterized by the fact that there exists no instant beyond which no additional plastic deformations occur. If, on the other hand, plastic strains developing in the initial phase of the process generate such distributions of residual stresses that, starting from a certain time instant, the considerd body reacts purely elastically to the applied loads, we say that the body "shakes down".

The foundations of the theoretical achievements in this field of research were given by MELAN [10] and KOITER [5] who proved, under the assumptions of geometrical linearity, elastic-perfectly plastic or linear and unlimited hardening material bahaviour, and the validity of an associated flow law, the classical statical and kinematical theorems of shakedown, respectively.

Extensions of these theorems to broader classes of problems including the change of temperature, hardening, influence of geometrical changes and damage have attracted much interest in the last years. Material hardening effects have been investigated, among others, by MAIER [9] who presented a discrete structural model which accounts for linearized ("second order") geometrical effects. KÖNIG [6] and KÖNIG and SIEMASZKO [7] developed a method of stability analysis of shakedown process taking into account nonlinear strain-hardening rule, ZARKA *et al.* [21] presented an approach to inelastic analysis of structures used in conjunction with the classical tools of engineers such as any elastic finite elements method based code, and MANDEL [11] extended Melan's theorem to hardening material behaviour using the "Generalized Standard Material Model" which has been developed by HALPHEN and NGUYEN QUOC SON [2]. They all consider the case of linear and unlimited hardening. Weichert and GROSS-WEEGE [19] studied the case of linear and limited kinematical hardening using a simplified two-surface yield condition. For similar problems, STEIN, ZHANG and KÖNIG [16] propose a micromechanical overlay model with limited nonlinear hardening. The geometrically nonlinear problem including hardening has been studied by WEICHERT [18, 20], GROSS-WEEGE [1] and SACZUK and STUMPF [13] who used the "Generalized Standard Material Model".

More recently, SIEMASZKO [14] presented a step-by-step method of non-shakedown analysis for elastic-plastic discrete structures. This method accounts for nonlinear geometrical effects, nonlinear hardening and progressive damage of the material. The damage is taken into account by a porosity parameter which is interpreted as a void volume fraction. The evolution of this parameter describes the changes of internal imperfections caused by nucleation, growth and diffusion, using the material softening function developed by PERZYNA [12] combined with isotropic hardening.

In the presend paper, an extension of shakedown theory to damaging material behaviour is proposed using the energy-based isotropic elasto-plastic damage models given by J.W. JU [3]. Here, damage is taken into account by an internal scalar-valued parameter D, describing the effects caused by growth and coalescence of microcracks or microvoids, leading to the degradation of the material properties.

2. Basic assumptions and constitutive relations

In this paper, different physical phenomena on the level of local material behaviour are considered, namely: Linear elastic behaviour, plastic behaviour characterized by the existence of a convex yield surface in the space of generalized stresses, the validity of normality rule, linear limited hardening (see e.g. [2, 19]), and material damage following the theory of JU [3] and CHABOCHE-LEMAITRE [8]. For simplicity, all considerations are limited to the geometrically linear theory; extensions in the sense of e.g. MAIER [9] or others [1, 18–20] are possible.

2.1. Adopted model of material damage

Continuous damage mechanics has been introduced and employed extensively to describe the progressive degradation experienced by the mechanical properties of materials prior to the intiation of macrocracks. KACHANOV [4] was the first to introduce the concept of damage in the framework of continuum mechanics. For the case of isotropic damage and using the concept of effective stress, the scalar damage variable D is defined

$$D = \frac{S - \widetilde{S}}{S}, \tag{2.1}$$

where $\widetilde{S}$ is the effective (net) resisting area corresponding to the damaged area S.

Using the hypothesis of strain equivalence [8], the effective stress $\widetilde{\sigma}$ can be obtained from Eq. (2.1) by equating the force acting on the damage area S with the force acting on the hypothetical undamaged area $\widetilde{S}$ (see Fig. 1)

$$\sigma S = \widetilde{\sigma}\widetilde{S}, \tag{2.2}$$

where σ is the Cauchy stress acting on the damaged area S. From Eqs. (2.1) and (2.2), one obtains

$$\widetilde{\sigma} = \frac{\sigma}{1 - D}. \tag{2.3}$$

The value $D = 0$ corresponds to the undamaged (virgin) state, $D = D_c$ defines the complete local rupture ($D_c \in [0, 1]$), and $D \in (0, D_c)$ corresponds to a partly damaged state. It should be noted that the effective stress $\widetilde{\sigma}$ can be considered as a fictitious stress acting on an undamaged equivalent (fictitious) area $\widetilde{S}$ (net resisting area).

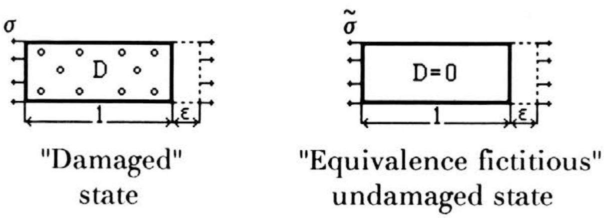

FIG. 1. Hypothesis of strain equivalence.

Linear kinematical hardening is taken into account by using internal parameters according to the concept of "generalized standard material" developed by HALPHEN and NGUYEN [2]. For this, generalized elastic strains $\mathbf{e}^e$, generalized plastic strains $\mathbf{e}^p$ and effective generalized stresses $\tilde{\mathbf{s}}$ are introduced, defined by the sets

$$\mathbf{e}^e = [\boldsymbol{\varepsilon}^e, \boldsymbol{\omega}], \qquad \mathbf{e}^p = [\boldsymbol{\varepsilon}^p, \boldsymbol{\kappa}], \qquad \tilde{\mathbf{s}} = [\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\pi}}]. \tag{2.4}$$

Here, $\boldsymbol{\varepsilon}^e$ and $\boldsymbol{\varepsilon}^p$ are, respectively, the observed "elastic-damage" and "plastic-damage" parts of the total strain equivalence tensor $\boldsymbol{\varepsilon}$, defined by their components referred to a Cartesian system of coordinates $[0, X_1, X_2, X_3]$

$$\begin{aligned} \varepsilon_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}), \\ \varepsilon_{ij} &= \varepsilon^e_{ij} + \varepsilon^p_{ij}, \end{aligned} \tag{2.5}$$

with u_i being components of the displacements vector $\mathbf{u}$ of a characteristic particle of the body. Latin indices run from 1 to 3 if not stated otherwise, summation convention over repeated indices is adopted, a comma denotes the partial derivative of the considered quantity with respect the coordinate following the comma. The effective stresses are represented by the effective stress tensor $\tilde{\boldsymbol{\sigma}}$, and the quatities $\boldsymbol{\omega}$, $\boldsymbol{\kappa}$ and $\tilde{\boldsymbol{\pi}}$ are the r-dimensional vectors of internal elastic and plastic parameters and "effective back-stresses", respectively. The dimension r depends upon the particular choice of the hardening model.

2.2. Thermodynamic basis

Following [3], the existence of a locally averaged free energy function is assumed; i.e.

$$W(\boldsymbol{\varepsilon}^e, D, \boldsymbol{\kappa}) = (1 - D)W^0(\boldsymbol{\varepsilon}^e, \boldsymbol{\kappa}) = W_{ed}(\boldsymbol{\varepsilon}^e, D) + W_{pd}(\boldsymbol{\kappa}, D), \tag{2.6}$$

where W_{ed} and W_{pd} are the uncoupled elastic and plastic potential energy functions of the damaged material, respectively.

In the isothermal case, W_{ed} and W_{pd} are given by

$$\begin{gathered} W_{ed}(\boldsymbol{\varepsilon}^e, D) = (1 - D)W_e(\boldsymbol{\varepsilon}^e), \qquad W_{pd}(\boldsymbol{\kappa}, D) = (1 - D)W_p(\boldsymbol{\kappa}); \\ W_{ed} = \frac{1}{2}(1 - D)L^{-1}_{ijkl}\varepsilon^e_{ij}\varepsilon^e_{kl}, \qquad W_{pd} = \frac{1}{2}(1 - D)Z^{-1}_{mn}\kappa_m\kappa_n, \\ n, m = 1, 2, \ldots, r. \end{gathered} \tag{2.7}$$

$\mathbf{L}$ and $\mathbf{Z}$ are constant, positive definite and time-independent tensors of elastic moduli and internal elastic moduli, respectively, with the symmetries $L_{ijkl} = L_{klij} = L_{jikl} = L_{ijlk}$

and $Z_{mn} = Z_{nm}$ defined by

$$L^{-1}_{ijkl} = \frac{\partial^2 W^0}{\partial \varepsilon^e_{ij} \partial \varepsilon^e_{kl}}, \qquad Z^{-1}_{mn} = \frac{\partial^2 W^0}{\partial \kappa_m \partial \kappa_n},$$

where in Eq. (2.7)$_1$ W_e denotes the initial elastic stored energy function of the undamaged (virgin) material. As a consequence of the 2$^{\text{nd}}$ thermodynamical law, the Clausius–Duhem (reduced dissipation) inequality takes the form

$$-\dot{W} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \geq 0 \tag{2.8}$$

for any admissible process.

Assuming that the unloading processes are purely elastic, we obtain

$$\boldsymbol{\sigma} = \frac{\partial W_{ed}}{\partial \boldsymbol{\varepsilon}^e} = (1 - D)\mathbf{L}^{-1} : \boldsymbol{\varepsilon}^e, \tag{2.9}$$

or, using (2.3),

$$\widetilde{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}}{1 - D} = \mathbf{L}^{-1} : \boldsymbol{\varepsilon}^e, \tag{2.10}$$

where the symbol (:) indicates tensorial product contracted in two indices.

The associated damage variable is the scalar $-Y$, defined by

$$-Y = -\frac{\partial W}{\partial D} = W^0. \tag{2.11}$$

Hence, the undamaged energy function $W^0(\boldsymbol{\varepsilon}^e, \boldsymbol{\kappa})$ is the thermodynamic force $(-Y)$ conjugate to the damage variable D (see also [3]). This is at variance with LEMAITRE and CHABOCHE [8] and SIMO and JU [15], who considered only the elastic part of the damage energy $W_e(\boldsymbol{\varepsilon}^e)$. Assumption of the elastic damage energy alone is the non-optimal choice, thus in a sense contradicting the experimental evidence that plastic variables also contribute to the initiation and growth of microcracks.

Taking time derivative of Eq. (2.6) and substituting into Eq. (2.8), we obtain the dissipative inequality

$$\sigma_{ij}\dot{\varepsilon}^p_{ij} - Y\dot{D} - \pi_n \dot{\kappa}_n \geq 0, \qquad n = 1, 2, \ldots, r. \tag{2.12}$$

If one assumes that the phenomenon of plastic flow can occur without damage, in the same way the phenomenon of damage may be assumed to occur without appreciable macroscopic plastic flow [3, 8, 15, 17]; then we must have separately

$$-Y\dot{D} \geq 0 \quad \text{and} \quad \sigma_{ij}\dot{\varepsilon}^p_{ij} - \pi_n \dot{\kappa}_n \geq 0. \tag{2.13}$$

2.3. Elastic part of the material law

It is assumed that for the elastic part of the material damage law the linear relationship

$$e^e = \mathcal{L} : \widetilde{s} \tag{2.14}$$

is valid. In the indicial notation we get

$$\varepsilon^e_{ij} = L_{ijkl}\widetilde{\sigma}_{kl}, \qquad \omega_m = Z_{mn}\widetilde{\pi}_n, \qquad n = 1, 2, \ldots, r. \tag{2.15}$$

2.4. Plastic part of the material law

For the plastic part of the material damage behaviour we assume the existence of a convex and fixed yield surface $\mathcal{F}$ in the space of effective generalized stresses $\widetilde{\mathbf{s}}$ with

$$\mathcal{F}(\widetilde{\mathbf{s}}, k) \le 0\,, \tag{2.16}$$

for all physically admissible states of effective stress, where k denotes a time-independent scalar. Convexity and normality rule can then be expressed by the condition

$$(\mathbf{s} - \widehat{\mathbf{s}}) : \dot{\mathbf{e}}^p \ge 0\,. \tag{2.17}$$

In indicial notation we get

$$(\sigma_{ij} - \widehat{\sigma}_{ij})\dot{\varepsilon}^p_{ij} + (\pi_n - \widehat{\pi}_n)\dot{\kappa}_n \ge 0\,, \tag{2.18}$$

where a dot denotes the rate of the considered quantity, and a superposed symbol ($\widehat{\ }$) characterizes arbitrary admissible field fulfilling inequality (2.16). In the case of kinematical hardening following Prager's hardening rule, the evolution of the internal plastic parameters κ is linked to the plastic strain rates by [2, 19]

$$\dot{\kappa}_n = -\dot{\varepsilon}^p_{ij}, \qquad \text{for} \quad i = j, \quad n = \frac{1}{2}(i + j), \qquad \text{for } i \ne j, \qquad n = i + j + 1\,. \tag{2.19}$$

The evolution of internal elastic parameters ω is in general given by

$$\dot{\omega}_n = -\dot{\kappa}_n, \qquad n = 1, 2, \ldots, r \tag{2.20}$$

so that for initially virgin material we have

$$\omega_n = -\kappa_n, \qquad n = 1, 2, \ldots, r \tag{2.21}$$

for all times [19].

3. The extended shakedown theorems

Using the material description given in Sec. 2, Melan's theorem can be extended for damaging kinematically hardening material. We consider a body $\mathcal{B}$ of volume V with a sufficiently smooth surface S consisting of the disjoint parts S_F and S_u, where statical and kinematical boundary conditions are prescribed, respectively. Volume forces are denoted by $\mathbf{f}^*$, surface tractions by $\mathbf{p}^*$, and the given displacements on S_u by $\mathbf{u}^*$ varying locally within fixed bounds.

In the sequel, the notion of a "purely elastic reference problem" is introduced, differing from the original problem only by the fact that the material reacts in this case purely elastically with the same elastic moduli as for the elastic part of the material law in the original problem [5]. All quantities related to this reference problem are indicated by superscript c. We assume that the solution of this reference problem is given, i.e. that the following system of equations is fulfilled:

$$\begin{aligned} \sigma^c_{ij,j} &= -f_i{}^* && \text{in } V\,, \\ n_j\sigma^c_{ij} &= p^*_i && \text{on } S_F\,, \\ u^c_i &= u^*_i && \text{on } S_u\,, \\ \varepsilon^c_{ij} &= \frac{1}{2}(u^c_{i,j} + u^c_{j,i}) && \text{in } V\,, \\ \varepsilon^c_{ij} &= L_{ijkl}\widetilde{\sigma}^c_{kl} && \text{in } V\,, \end{aligned} \tag{3.1}$$

where $\mathbf{n}$ is the outward normal vector to S_F.

The body $\mathcal{B}$ shakes down with respect to the given loading history, if there exists a time-independent field of effective generalized stresses $\overset{\circ}{\widetilde{\mathbf{s}}}^{\rho}(x) = [\overset{\circ}{\widetilde{\boldsymbol{\rho}}}(\mathbf{x}), \overset{\circ}{\widetilde{\boldsymbol{\pi}}}(\mathbf{x})]$ such that for all times $t > 0$ the following conditions hold:

$$
\begin{aligned}
&\overset{\circ}{\boldsymbol{\rho}}_{ij,j} = 0 && \text{in } V\,,\\
&n_j \overset{\circ}{\boldsymbol{\rho}}_{ij} = 0 && \text{on } S_F\,,\\
&\mathcal{F}(\widetilde{\mathbf{s}}^c(\mathbf{x}, t) + \overset{\circ}{\widetilde{\mathbf{s}}}^{\rho}(\mathbf{x}), k) < 0 && \text{in } V\,.
\end{aligned}
\tag{3.2}
$$

For the proof of the theorem the quadratic form W is introduced by the formula

$$
W = \frac{1}{2} \int\limits_{(V)} \left[(\rho_{ij} - \overset{\circ}{\rho}_{ij}) \frac{L_{ijkl}}{1 - D} (\rho_{kl} - \overset{\circ}{\rho}_{kl}) + (\pi_m - \overset{\circ}{\pi}_m) \frac{Z_{mn}}{1 - D} (\pi_n - \overset{\circ}{\pi}_n) \right] dV\,. \tag{3.3}
$$

In the following, it is assumed that damage is limited throughout the entire process so that $(1 - D)$ is, at any time, a positive-valued scalar. Then W is non-negative due to the fact that the tensors $\mathbf{L}$ and $\mathbf{Z}$ are positive definite. In Eq. (3.3), $\boldsymbol{\rho}$ denotes the time-dependent difference between the unknown, true state of stress in the body $\mathcal{B}$ and the given, time-dependent state of stress in the purely elastic comparative body $\mathcal{B}^c$, so that

$$
\overset{\circ}{\rho}_{ij} = \sigma^+_{ij} - \sigma^c_{ij}, \qquad \rho_{ij} = \sigma_{ij} - \sigma^c_{ij}, \qquad \pi^+_m = \overset{\circ}{\pi}_m\,, \tag{3.4}
$$

and

$$
\rho_{ij} = (1 - D)\widetilde{\rho}_{ij}, \qquad \dot{\rho}_{ij} = (1 - D)\dot{\widetilde{\rho}}_{ij} - \dot{D}\widetilde{\rho}_{ij}\,. \tag{3.4$_2$}
$$

Here, $\mathbf{s} = [\boldsymbol{\sigma}, \boldsymbol{\pi}]$ is the actual state of stress and $\mathbf{s}^+ = [\boldsymbol{\sigma}^+, \boldsymbol{\pi}^+]$ is a "safe" state of stress defined by the fact that inequality (2.16) is satisfied in the strict sense.

The time-derivative of Eq. (3.3) then yields, with Eqs. (3.4),

$$
\begin{aligned}
\dot{W} = & \int\limits_{(V)} [(\sigma_{ij} - \sigma^+_{ij}) L_{ijkl} \dot{\widetilde{\rho}}_{kl} + (\pi_m - \pi^+_m) Z_{mn} \dot{\widetilde{\pi}}_n]\, dV \\
& - \frac{1}{2} \int\limits_{(V)} [(\sigma_{ij} - \sigma^+_{ij}) L_{ijkl} (\sigma_{kl} - \sigma^+_{kl}) + (\pi_m - \pi^+_m) Z_{mn} (\pi_n - \pi^+_n)] \frac{\dot{D}}{(1 - D)^2}\, dV\,,
\end{aligned}
\tag{3.5}
$$

where $\dot{\widetilde{\boldsymbol{\rho}}}$, $\dot{\widetilde{\boldsymbol{\pi}}}$ and $\dot{D}$ are time-derivative of the effective residual stress, effective back-stresses and of the damage variable, respectively.

With

$$
\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij}, \qquad \kappa_m + \omega_m = 0\,, \tag{3.6$_1$}
$$

$$
\varepsilon_{ij} = \varepsilon^c_{ij} + \varepsilon^p_{ij} + \triangle\varepsilon_{ij}, \qquad \triangle\varepsilon_{ij} = L_{ijkl}\widetilde{\rho}_{kl}, \qquad \omega_m = Z_{mn}\widetilde{\pi}_n \tag{3.6$_2$}
$$

and

$$
-\triangle Y = \frac{1}{2}[(\sigma_{ij} - \sigma^+_{ij}) L_{ijkl} (\sigma_{kl} - \sigma^+_{kl}) + (\pi_m - \pi^+_m) Z_{mn} (\pi_n - \pi^+_n)]\,, \tag{3.6$_3$}
$$

we get from Eqs. (3.6)

$$
\dot{W} = \int\limits_{(V)} [(\sigma_{ij} - \sigma^+_{ij})(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^c_{ij} - \dot{\varepsilon}^p_{ij}) - (\pi_m - \pi^+_m)\dot{\kappa}_m]\, dV \tag{3.7}
$$

$$+ \int\limits_{(V)} \triangle Y \frac{\dot{D}}{(1-D)^2}\, dV . \tag{3.7 [cont.]}$$

From the Gauss theorem it follows that

$$\int\limits_{(V)} (\sigma_{ij} - \sigma^+_{ij})(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^c_{ij})\, dV = \int\limits_{(V)} (\sigma_{ij} - \sigma^+_{ij})(\dot{u}_{i,j} - \dot{u}^c_{i,j})\, dV$$
$$= - \int\limits_{(V)} (\sigma_{ij,j} - \sigma^+_{ij,j})(\dot{u}_i - \dot{u}^c_i)\, dV + \int\limits_{(S)} n_j(\sigma_{ij} - \sigma^+_{ij})(\dot{u}_i - \dot{u}^c_i)\, dS = 0 . \tag{3.8}$$

Finally

$$\dot{W} = - \int\limits_{(V)} [(\sigma_{ij} - \sigma^+_{ij})\dot{\varepsilon}^p_{ij} + (\pi_m - \pi^+_m)\dot{\kappa}_m]\, dV + \int\limits_{(V)} \triangle Y \cdot \frac{\dot{D}}{(1-D)^2} dV$$
$$= - \int\limits_{(V)} (\mathbf{s} - \mathbf{s}^+)\dot{e}^p\, dV + \int\limits_{(V)} \triangle Y \cdot \frac{\dot{D}}{(1-D)^2}\, dV . \tag{3.9}$$

From (2.13) and (2.17) it follows that $\dot{W} \leq 0$. For the strong inequality $(3.2)_3$, $\dot{W}$ is equal to zero only for $\dot{\mathbf{e}}^p = \mathbf{0}$ and $\dot{D} = 0$, and otherwise it is negative. So Melan's argument holds: As W is non-negative by definition, plastic dissipation and damage dissipation are limited and so plastic flow and damage evolution cease beyond a certain time instant if a field $\overset{\circ}{\tilde{\mathbf{s}}}{}^{\rho}$ exists, fulfilling the relations (3.2). We say that $\mathcal{B}$ shakes down in this case.

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