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BRIEF NOTES

A few properties of the resonant frequencies of a piezoelectric body

J. S. YANG (PRINCETON)

THIS PAPER PRESENTS a constraint variational formulation for the resonant frequencies of a piezoelectric body. The formulation is in a nonnegative form which is then used to prove a few properties of the lowest resonant frequency.

1. Introduction

THE RAYLEIGH QUOTIENT for the eigenvalue problem for the resonance of a finite piezoelectric body is an indefinite form which has stationary values with saddle point behaviour [2]. This has hindered many theoretical approaches. In this paper, it is proved that on a properly chosen subspace of the admissible functions, the Rayleigh quotient assumes a nonnegative form which immediately leads to a few useful conclusions on the properties of the lowest resonant frequency of the piezoelectric body. These properties can be considered as the generalization of the corresponding properties in classical elasticity.

2. Governing equations

Let the finite space region occupied by the piezoelectric body be Ω, the boundary surface of Ω be S, the unit outward normal of S be n_i, and S can be partitioned in the following way

$$S_u \cup S_\sigma = S_\phi \cup S_D = S,$$
$$S_u \cap S_\sigma = S_\phi \cap S_D = 0.$$

For the free vibration of a piezoelectric body, the governing equations and boundary conditions are [1]

$$-c_{ijkl}u_{k,lj} - e_{kji}\phi_{,kj} = \rho\omega^2 u_i \quad \text{in } \Omega, \tag{2.1}$$

$$-e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki} = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$u_i = 0 \quad \text{on } S_u, \tag{2.3}$$

$$\sigma_{ji}(u_i,\phi)n_j = (c_{jikl}u_{k,l} + e_{kji}\phi_{,k})n_j = 0 \quad \text{on } S_\sigma, \tag{2.4}$$

$$\phi = 0 \quad \text{on } S_\phi, \tag{2.5}$$

$$D_i(u_i,\phi)n_i = (e_{ikl}u_{k,l} - \epsilon_{ik}\phi_{,k})n_i = 0 \quad \text{on } S_D, \tag{2.6}$$

where ρ is the mass density, $c_{ijkl}, e_{ijk}, \epsilon_{ij}$ are material constants, u_i is the displacement, and ϕ is the electric potential. The material constants satisfy

$$c_{jikl} = c_{ijkl} = c_{jilk} = c_{klji},$$
$$e_{kji} = e_{kij},$$
$$\epsilon_{ij} = \epsilon_{ji},$$
$$c_{ijkl}u_{i,j}u_{k,l} \geq 0, \quad \epsilon_{ij}\phi_{,i}\phi_{,j} \geq 0.$$

Values of ω (resonant frequencies) are sought corresponding to which nontrivial solutions of u_i and ϕ exist.

3. The Rayleigh quotient

The Rayleigh quotient for the variational formulation of the above eigenvalue problem is known [2]. Let

$$H(\Omega) = \{u_i, \phi | u_i = 0 \quad \text{on } S_u, \phi = 0 \quad \text{on } S_\phi\},$$

where u_i and ϕ are assumed to be smooth enough for all the differentiation on them. Then, the expression for the Rayleigh quotient is [2]

$$R(u_i, \phi) = \frac{\int_\Omega (c_{ijkl}u_{i,j}u_{k,l} - \epsilon_{ij}\phi_{,i}\phi_{,j} + 2e_{ikl}u_{k,l}\phi_{,i})\, d\Omega}{\int_\Omega \rho u_i u_i \, d\Omega}. \tag{3.1}$$

When $u_i, \phi \in H(\Omega)$, the stationary values of $R(u_i, \phi)$ are the eigenvalues ω^2 in Eqs. (2.1)–(2.6), and the stationary values are assumed when u_i and ϕ are the corresponding eigenfunctions. The stationary values of the $R(u_i, \phi)$ in Eq. (3.1) are generally not simple minima but are of saddle point nature [2].

It has been shown [2, 3] that for the stationary solutions, with Eqs. (2.2), (2.6) and integration by parts, the value of the Rayleigh quotient is

$$\omega^2 = \frac{\int_\Omega (c_{ijkl}u_{i,j}u_{k,l} + \epsilon_{ij}\phi_{,i}\phi_{,j})\, d\Omega}{\int_\Omega \rho u_i u_i \, d\Omega}. \tag{3.2}$$

4. A constraint variational formulation

The derivation from Eqs. (3.1) to (3.2) suggests a constraint variational formulation of the eigenvalue problem (2.1)–(2.6). First Eqs. (2.2) and (2.6) can be put directly on the admissible functions for the Rayleigh quotient (3.1). To be exact, let

$$H'(\Omega) = \{u_i, \phi | u_i = 0 \text{ on } S_u, \ \phi = 0 \text{ on } S_\phi, -e_{ikl}u_{k,li} + \epsilon_{ik}\phi_{,ki} = 0 \text{ in } \Omega,$$
$$e_{ikl}u_{k,l} - \epsilon_{ik}\phi_{,k})n_i = 0 \quad \text{on } S_D\}.$$

Then, on $H'(\Omega)$, after integration by parts, the Rayleigh quotient assumes the following form

$$R'(u_i, \phi) = \frac{\int_\Omega (c_{ijkl}u_{i,j}u_{k,l} + \epsilon_{ij}\phi_{,i}\phi_{,j})\, d\Omega}{\int_\Omega \rho u_i u_i \, d\Omega}.$$

Here $R'(u_i, \phi)$ is nonnegative, hence it has minima on $H'(\Omega)$. We therefore have the following constraint variational formulation

$$\omega^2 = \min_{u_i, \phi \in H'(\Omega)} R'(u_i, \phi).$$

The advantage of the above constraint formulation is that it is a minimum principle which can be used to prove the properties of its minima, for example, the lowest resonant frequency.

5. The effect of S_u

S_u is the part of the boundary of the piezoelectric body on which u_i is prescribed. All the admissible functions for $R'(u_i, \phi)$ must vanish on S_u. Consider a new eigenvalue problem which differs from Eqs. (2.1)–(2.6) only in that S_u shrinks a little to S_u^0 such that $S_u^0 \subset S_u$. Denote the eigenvalues of this new problem by $(\omega^0)^2$. Since $u = 0$ on S_u implies $u_i = 0$ on S_u^0, we have

$$H'(\Omega) \subset H^0(\Omega),$$

where

$$H^0(\Omega) = \{u_i, \phi | u_i = 0 \quad \text{on } S_u^0,\ \phi = 0 \quad \text{on } S_\phi,\ -e_{ikl}u_{k,li} + \epsilon_{ik}\phi_{,ki} = 0 \quad \text{in } \Omega \\ (e_{ikl}u_{k,l} - \epsilon_{ik}\phi_{,k})n_i = 0 \quad \text{on } S_D\}.$$

Therefore

$$\omega^2 = \min_{u_i, \phi \in H'(\Omega)} R'(u_i, \phi) \geq \min_{u_i, \phi \in H^0(\Omega)} R'(u_i, \phi) = (\omega^0)^2.$$

6. The effect of c_{ijkl} and ρ

If two materials differ in their elastic constants and densities in the following way

$$c_{ijkl} \leq \overline{c}_{ijkl},$$
$$\rho \geq \overline{\rho},$$

and everything else remain the same, then, on $H'(\Omega)$

$$R'(u_i, \phi) = \frac{\int_\Omega (c_{ijkl}u_{i,j}u_{k,l} + \epsilon_{ij}\phi_{,i}\phi_{,j})\, d\Omega}{\int_\Omega \rho u_i u_i\, d\Omega} \\ \leq \frac{\int_\Omega (\overline{c}_{ijkl}u_{i,j}u_{k,l} + \epsilon_{ij}\phi_{,i}\phi_{,j})\, d\Omega}{\int_\Omega \overline{\rho} u_i u_i\, d\Omega} = \overline{R}(u_i, \phi).$$

Equation (6.1) immediately implies the following

$$\omega^2 = \min_{u_i, \phi \in H'(\Omega)} R'(u_i, \phi) \leq \min_{u_i, \phi \in H'(\Omega)} \overline{R}(u_i, \phi) = (\overline{\omega})^2.$$

References

1. H. F. Tiersten, *Linear piezoelectric plate vibrations*, Plane Press, 1969.
2. E. P. EerNisse, *Variational method for electroelastic vibration analysis*, IEEE Trans. Sonics and Ultrasonics, SU-14, 59–67, 1983.
3. J. A. Lewis, *The effect of driving electrode shape on the electrical properties of piezoelectric crystals*, Bell. Sys. Tech. J., **40**, 1259–1280, 1961.

DEPARTMENT OF CIVIL ENGINEERING AND OPERATIONS RESEARCH
PRINCETON UNIVERSITY, PRINCETON NJ., USA.

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