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Surface stress waves in a transversely isotropic nonhomogeneous elastic semispace
Part I. Equations of motion and equations of a Rayleigh-type surface wave

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PLANE STATE of strain of a transversely isotropic medium, nonhomogeneous in the plane of isotropy, is considered; equations of motion expressed in terms of stresses are used (Sec. 3) to formulate the problem of propagation of Rayleigh-type surface stress waves. It is proved that in the case of a harmonic wave, the problem may be reduced to a solution of a fourth order, ordinary differential equation with variable coefficients (3.30), (3.31) for the "stress function" $\beta(z)$, i.e. the normal stress σ_{zz} amplitude distribution in the semispace along the vertical direction; the remaining amplitudes, Eqs. (3.17), are expressed in terms of $\beta(z)$. The form of Eq. (3.31) makes it possible to formulate and discuss in Sec. 4 several particular cases of the media under consideration. Equations (4.19) or (4.21), (4.22) are proved to be identical with those derived by J. IGNACZAK in [1]. The methods of solution of the equations of motion are presented in Sec. 5. Conclusions are drawn in the final Sec. 6.

1. Introduction

IN THE LINEAR elastodynamics of anisotropic and nonhomogeneous bodies the general equations of motion and the corresponding initial and boundary conditions are written in terms of two different sets of unknowns representing either the displacement vector or the stress tensor. The remarkable achievements of the modern technology are based mainly on the observation and investigation of the wave processes, also those involving the surface waves. The experimental and phenomenological investigations are far ahead of the corresponding theory, first of all as far as the qualitative analysis of the wave processes is concerned. An important difficulty follows from the complex form of the phenomenon, the material coefficients being functions of position and direction in the space and leading to a coupled form of the equations, difficult to separate.

The stress waves occurring in materials of complex structure have been considered to be important for the engineering practice since the beginning of the present century for at least the following two reasons. First, the phenomenon may be utilized in diagnostic equipment aimed at measuring the elastic properties, material faults detection and transmission of information. Such applications are used mainly in the cases of sinusoidal stress or ultrasonic pulses. Another important reason results from the damages or total fracture of the structures produced by instantaneous stress pulses due to the impacts or earthquakes.

The impact loading is well known to be characterized by two important properties: the propagation speed of the stress wave and a certain rate of strain changes, i.e. the data which are usually disregarded in the static analysis, when the process is assumed to occur simultaneously in the entire structure.

The stress wave propagation velocity depends upon the direction of propagation and if the stress-strain proportionality condition is violated, also upon the stress level. The stress wave velocities in a non-simple (complex) medium are determined by averaging the motions over the local regions of the body containing various elements or particles.

It is obvious that during the impact loading the energy is concentrated within small volumes and, thus, the impact loading of a short duration accumulating small amounts of energy in a relatively small volume may lead to various types of damages, fracture or crack propagation, the phenomenon frequently encountered in high efficiency and precision machines (e.g. turbine generators) or, generally, in seismic zones.

In view of the finite duration of stress wave propagation and the strain rate-dependent material properties characterizing the impact processes occurring in engineering structures, anisotropy and inhomogeneity of the materials has to be accounted for at the early stage of engineering design of structures subject to dynamic loading.

The simplet model of an inhomogeneous body is a layered medium consisting of homogeneous elements of known (theoretically or experimentally determined) wave transmission properties. Such a medium was, for instance, used by the seismologists to refine the measurement techniques and to interpret the earthquake recordings (Ewing, Jardetzky, Press 1957), thus contributing to the knowledge of the earth crust structure; several works deal with problems of that type. Knowledge of some solutions in this field increased, however, the demand for further, more refined investigations (mainly theoretical) of media exhibiting more complicated internal structure.

The present paper is aimed at demonstrating the possibility of analysis of wave processes occurring in elastic media of a complex internal structure; the mathematical apparatus used should make it possible to analyze the processes at least in the qualitative sense. Thus we shall start with the general method of derivation of the field equations written in terms of displacements and stresses (Sec. 2); in Sec. 3 the general stress equations will be reduced to the case of transversely isotropic inhomogeneous medium filling a halfspace, Sec. 3.1. In such a halfspace the Rayleigh-type surface wave problem will be formulated as the eigenfunction problem of an ordinary fourth order differential equation with variable coefficients, Sec. 3.2. The equation obtained ((3.30) or (3.31)) may be reduced by a limiting procedure to the dynamic equation known from the literature [2]. If the semispace considered is isotropic and nonhomogeneous, the result obtained assumes the form given by J. Ignaczak in 1963, [1, 2].

The qualitative analysis of Eq. (3.31) presented in Sec. 4 demonstrates five particular types of wave phenomena; the first three of them (1–3) represent approximate solutions, and the remaining two (4, 5) — the accurate ones, depending on the material properties of the medium.

1. Transversely isotropic body of a "small nonhomogeneity", Sec. 4.1;
2. "Weakly anisotropic" nonhomogeneous body, Sec. 4.2;
3. "Weakly anisotropic" body with a "small nonhomogeneity", Sec. 4.3;
4. Transversely isotropic homogeneous body, Sec. 4.4;
5. Isotropic nonhomogeneous body, Sec. 4.5.

Section 5 is devoted to the method of solution of the equations constructed for an elastic, transversely isotropic nonhomogeneous medium, the ordinary, fourth order differential equation with variable coefficients being reduced to a set of two ordinary, second order differential equations with variable coefficients, satisfying certain definite boundary conditions. Several conclusions are presented in Sec. 6.

The equations written in terms of stresses for a transversely isotropic body may be transformed to the corresponding relations expressed in terms of displacements. Further limiting procedures applied to both types of equations lead to the static Beltrami–Michell stress equations (representing the compatibility conditions) and the Navier–Lekhnitskiĭ displacement equations, cf. [3]. Some static problems concerning the transversely isotropic bodies for a semispace were solved, on the basis of Navier's solution, by W. NOWACKI (1954), Z. MOSSAKOWSKA (1955), J. MOSSAKOWSKI (1956), papers [4–7]; in the dynamic case cf. the work by S. KALISKI [8], based on the Lamé equations.

The present paper may be referred directly to the problem of propagation of seismic waves and can be applied to the analysis of the effects of anisotropy and nonhomogeneity of the surface layer subject to dynamic loadings. The proposed model of a continuous complex elastic semispace, together with the equations of motion and the Rayleigh-type stress wave formulations represent, from the theoretical point of view, a certain extension of the methods of analysis of the surface layers known up-to-date.

The solutions obtained in the paper are original except the Eq. (4.21) which was derived earlier in [1]; they may be viewed as filling an important part of the gap in the description of an elastic, anisotropic and nonhomogeneous medium, allowing for an additional qualitative (sometimes even quantitative) insight into the behaviour of such a medium; this is due to the Eq. (4.12) which accounts for all the anisotropy and nonhomogeneity parameters introduced here.

2. Linear equations of elastodynamics for an anisotropic nonhomogeneous body written in terms of displacements and stresses

The considerations are based on the following relations:

a) linear kinematic relations

$$\varepsilon_{ij}(x,t) = \frac{1}{2}(u_{i,j}(x,t) + u_{j,i}(x,t)) = u_{(i,j)}(x,t), \tag{2.1}$$

b) equations of dynamic equilibrium

$$\sigma_{ij,j}(x,t) + F_i(x,t) = \rho(x)\ddot{u}_i(x,t), \qquad \sigma_{ij} = \sigma_{ji}, \tag{2.2}$$

c) constitutive equations

$$\sigma_{ij}(x,t) = C_{ijkl}(x)\varepsilon_{kl}(x,t) \tag{2.3}$$

or

$$\varepsilon_{ij}(x,t) = \varkappa_{ijkl}(x)\sigma_{kl}(x,t),$$

d) compatibility conditions

$$e_{pkl}e_{qmn}\varepsilon_{km,ln} = 0. \tag{2.4}$$

Substitution of Eqs. $(2.3)_1$ and (2.1) into (2.2) yields the displacement equations of motion of the medium (under isothermal conditions)

$$[C_{ijkl}(x)u_{k,l}(x,t)]_{,j} + \rho(x)(f_i(x,t) - \ddot{u}_i(x,t)) = 0. \tag{2.5}$$

On eliminating u_i and ε_{ij} from Eqs. (2.1)–(2.3) we obtain the equation of motion written in terms of stresses (under isothermal conditions),

$$\begin{aligned} 2\varkappa_{ijkl}(x)\ddot{\sigma}_{kl}(x,t) = {} & [\rho^{-1}(x)\sigma_{ik,k}(x,t)]_{,j} + [\rho^{-1}(x)\sigma_{jk,k}(x,t)]_{,i} \\ & +[\rho^{-1}(x)F_i(x,t)]_{,j} + [\rho^{-1}(x)F_j(x,t)]_{,i}. \end{aligned} \tag{2.6}$$

The following notations have been introduced in the above equations:

u_i components of the displacement vector,
ε_{ij} components of the strain tensor,

σ_{ij} components of the stress tensor,

x spatial coordinate (x_1, x_2, x_3) or (x, y, z) in a rectangular coordinate system,

t time,

e_{pkl} permutation symbol,

$C_{ijkl}(x)$ 4th order tensor of elasticity moduli as a function of a spatial coordinate,

$\varkappa_{ijkl}(x)$ 4th order tensor of compliance moduli as a function of a spatial coordinate,

$\rho(x)$ density of the medium,

$F_i(x,t) = \rho(x) f_i(x,t)$ body force.

Symmetry of the matrices $[\sigma_{ij}]$ and $[\varepsilon_{ij}]$ implies the relations

$$C_{ijkl} = C_{jikl} = C_{jilk}\,, \tag{2.7}$$

what reduces the number of independent elastic constants from 81 to 36. The next relation

$$C_{ijkl} = C_{klij}$$

produces the additional reduction to 21 independent constants.

The transversely isotropic body is characterized by 5 different elastic parameters. Moreover, the strong elipticity condition

$$C_{ijkl} a_i b_k a_j b_l > 0$$

should be satisfied by any pair of vectors a and b. Other notations: a dot over a symbol denotes differentiation with respect to time, indices following the comma denote partial differentiation with respect to the corresponding spatial coordinates.

If $\sigma_{ij}(x,t)$ is a solution of Eq. (2.6) under homogeneous initial conditions, then the displacement vector will be expressed by the equation (J. Ignaczak 1963, [1])

$$u_k(x,t) = \rho^{-1}(x) \int\limits_0^t (t-\tau)[\sigma_{ij,j}(x,\tau) + F_i(x,\tau)]\, d\tau\,. \tag{2.8}$$

Relation (2.8) was later generalized by M. E. Gurtin in 1964 to include the case of nonhomogeneous initial conditions in the form [9, 19]

$$u_i(x,t) = \rho^{-1}(x) \int\limits_0^t (t-\tau)[\sigma_{ij,j}(x,\tau) + F_i(x,\tau)]\, d\tau + u_i|_{t=0} + t\dot{u}_i|_{t=0}\,. \tag{2.9}$$

3. Transversely isotropic elastic, nonhomogeneous body (plane strain)

3.1. Equations written in terms of stresses

In the theory of elasticity use is made of the fact that when the anisotropic body structure exhibits a symmetry of any kind, the same symmetry occurs in the elastic properties of the body (see Eq. $(2.3)_{1,2}$ and $(2.7)_{7,8}$. We may thus search for a model of an elastic anisotropic, nonhomogeneous body of a simplified structure.

Let us assume the body to have the property that each point of that body lies on a plane within which all directions are equivalent from the point of view of the elastic properties; all these planes are parallel to each other. Thus, there exists a plane of isotropy or, in other words, at all points of the body there exists a single "distinguished" direction and an infinite number of principal directions in the plane passing through the given point and perpendicular to the "distinguished" principal direction. Another definition: each point of the body lies on an axis of symmetry of infinite order (sixth order would be sufficient), i.e. on an axis of rotational symmetry. Such a body is called the transversely isotropic body.

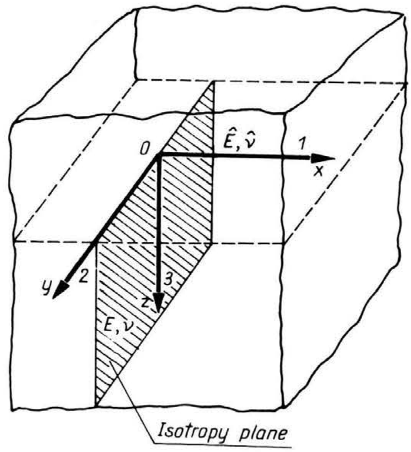

FIG. 1. Transversely isotropic nonhomogeneous medium.

The rectangular coordinate system (x, y, z) is assumed, the z-axis being directed vertically, and the x-axis representing the axis of rotational symmetry. The plane of isotropy is determined by the axes (y, z), as shown in Fig. 1. Using the "technical" constants instead of the compliances introduced in Sec. 2, the generalized Hook's law for a body with a plane of isotropy is written in the form (cf. [3, 4])

$$
\begin{aligned}
\varepsilon_{11} &= -\frac{\widehat{\nu}}{\widehat{E}}(\sigma_{zz} + \sigma_{yy}) + \frac{1}{\widehat{E}}\sigma_{xx}\,,\\
\varepsilon_{22} &= \frac{1}{E}(\sigma_{yy} - \nu\sigma_{zz}) - \frac{\widehat{\nu}}{\widehat{E}}\sigma_{xx}\,,\\
\varepsilon_{33} &= \frac{1}{E}(\sigma_{zz} - \nu\sigma_{yy}) - \frac{\widehat{\nu}}{\widehat{E}}\sigma_{xx}\,,\\
\varepsilon_{21} &= \frac{1}{2\widehat{G}}\sigma_{yx}\,,\\
\varepsilon_{31} &= \frac{1}{2\widehat{G}}\sigma_{zx}\,,\\
\varepsilon_{32} &= \frac{1}{2G}\sigma_{zy} = \frac{1+\nu}{E}\sigma_{zy}\,,
\end{aligned}
\tag{3.1}
$$

with the following notations:

- E Young's modulus in tension and compression in the directions lying within the plane of isotropy, $E = 2\mu(1 + \nu)$,
- $\widehat{E}$ Young's modulus in tension and compression in the directions lying in the plane perpendicular to the plane of isotropy,

ν Poisson's ratio determining the transversal contraction in the plane of isotropy due to tension in that plane,

$\widehat{\nu}$ Poisson's ratio determining the transversal contraction at tension occurring in the direction normal to the plane of isotropy,

$G = \mu, \widehat{G} = \widehat{\mu}$ shear moduli in the plane of isotropy and in the planes normal to that plane, respectively.

If we assume that the plane of isotropy is nonhomogeneous, the elastic constants will be point-dependent. Assume the sole dependence on the coordinate z, so that

$$E = E(z), \qquad \widehat{E} = \widehat{E}(z),$$
$$G = G(z) = \frac{E(z)}{2(1+\nu(z))}, \qquad \widehat{G} = \widehat{G}(z) = \frac{\widehat{E}(z)}{2(1+\widehat{\nu}(z))}. \tag{3.2}$$

Relations (3.2) should now be substituted into Eqs. (3.1).

Assume now the state of strain in the plane (x, y) to be two-dimensional, i.e. only the strains $\varepsilon_{11}, \varepsilon_{33}, \varepsilon_{13} = \varepsilon_{31}$ are different from zero, and $\varepsilon_{22} \equiv \varepsilon_{21} \equiv \varepsilon_{23} \equiv 0$. According to Eq. (3.1) and to the relation $\varepsilon_{22} = 0$ we obtain

$$\frac{1}{E}(\sigma_{yy} - \nu\sigma_{zz}) = \frac{\widehat{\nu}}{\widehat{E}}\sigma_{xx},$$

whence

$$\sigma_{yy} = lE\sigma_{xx} + \nu\sigma_{zz}, \qquad l = \frac{\widehat{\nu}}{\widehat{E}}. \tag{3.3}$$

Hence the generalized Hooke's law for a transversely isotropic body with the nonhomogeneity described by variable shear modulus being a function of $z \geq 0$ in the state of plane strain takes the form

$$\begin{aligned}
\varepsilon_{11}(x, z; t) &= -f(z)\sigma_{zz}(x, z; t) - B_1(z)\sigma_{xx}(x, z; t),\\
\varepsilon_{33}(x, z; t) &= B(z)f(z)g(z)\sigma_{zz}(x, z; t) - f(z)\sigma_{xx}(x, z; t),\\
\varepsilon_{13}(x, z; t) &= B_2(z)\sigma_{xz}(x, z; t),
\end{aligned} \tag{3.4}$$

where

$$f(z) = l(z)(1+\nu(z)), \qquad g(z) = l^{-1}(z)(1-\nu(z)), \qquad l(z) = \frac{\widehat{\nu}(z)}{\widehat{E}(z)}, \tag{3.5}$$

$$B(z) = \frac{1}{E(z)}, \qquad B_1(z) = l^2(z)E(z) - \frac{1}{\widehat{E}(z)}, \qquad B_2(z) = l(z) + \frac{1}{\widehat{E}(z)}. \tag{3.6}$$

The linearized relations between the displacement vector and the strain tensor in the two-dimensional system $(\alpha, \beta) = (1, 3)$ are expressed by the formula

$$\varepsilon_{\alpha\beta}(x, z; t) = \frac{1}{2}(u_{\alpha,\beta}(x, z; t) + u_{\beta,\alpha}(x, z; t)). \tag{3.7}$$

It should be noted that $u(x, z; t) \in C^3$.

Differentiation with respect to time yields

$$\ddot{\varepsilon}_{\alpha\beta}(x, z; t) = \frac{1}{2}(\ddot{u}_{\alpha,\beta}(x, z; t) + \ddot{u}_{\beta,\alpha}(x, z; t)). \tag{3.8}$$

The equation of dynamic equilibrium (with no body forces) has the form

$$\sigma_{\alpha\beta,\beta} = \rho\ddot{u}_\alpha\,, \tag{3.9}$$

whence it follows, under the assumption of ρ = const, that

$$\begin{aligned}\sigma_{\alpha\gamma,\gamma\beta} &= \rho\ddot{u}_{\alpha,\beta},\\ \sigma_{\beta\gamma,\gamma\alpha} &= \rho\ddot{u}_{\beta,\alpha}\end{aligned} \tag{3.10}$$

and

$$\rho(\ddot{u}_{\alpha,\beta} + \ddot{u}_{\beta,\alpha}) = \sigma_{\alpha\gamma,\gamma\beta} + \sigma_{\beta\gamma,\gamma\alpha}\,. \tag{3.11}$$

On comparing the Eqs. (3.8) and (3.11) we obtain the conditions of equilibrium which combine the acceleration of the strain tensor with the second spatial derivatives of the stress tensor.

$$\ddot{\varepsilon}_{\alpha\beta}(x,z;t) = (2\rho)^{-1}(\sigma_{\alpha\gamma,\gamma\beta}(x,z;t) + \sigma_{\beta\gamma,\gamma\alpha}(x,z;t))\,. \tag{3.12}$$

Substitution of the constitutive relations (3.4) in (3.12) enables us to write down the equation of motion in terms of stresses

$$\begin{aligned}-f(z)\ddot{\sigma}_{zz}(x,z;t) - B_1(z)\ddot{\sigma}_{xx}(x,z;t) &= \rho^{-1}(\sigma_{xx,xx}(x,z;t) + \sigma_{xz,zx}(x,z;t))\,.\\ B(z)f(z)g(z)\ddot{\sigma}_{zz}(x,z;t) - f(z)\ddot{\sigma}_{xx}(x,z;t) &\\ &= \rho^{-1}(\sigma_{zx,xz}(x,z;t) + \sigma_{zz,zz}(x,z;t))\,,\\ 2B_2(z)\ddot{\sigma}_{xz}(x,z;t) = \rho^{-1}(\sigma_{xx,xz}(x,z;t) &+ \sigma_{xz,zz}(x,z;t)\\ &+ \sigma_{zx,xx}(x,z;t) + \sigma_{zz,zx}(x,z;t))\,.\end{aligned} \tag{3.13}$$

3.2. Equations of surface waves

If the surface waves are sought for in the halfspace $z \geq 0$, particularly the Rayleigh-type waves, the set of equations of motion (3.13) must be completed by the following boundary conditions:

$$\begin{aligned}\sigma_{zz}(x,0;t) &= \sigma_{xz}(x,0;t) = 0\,,\\ \sigma_{zz}(x,\infty;t) &= \sigma_{xz}(x,\infty;t) = 0\,.\end{aligned} \tag{3.14}$$

The equations of motion in the form (3.13) have not been derived and analyzed in the literature thus far (according to my knowledge). J. IGNACZAK derived the equations of motion in terms of stresses for an arbitraty medium in the form (cf. [1] and Eqs. (2.6))

$$\begin{aligned}2\varkappa_{ijkl}(x)\ddot{\sigma}_{kl}(x,t) = [\rho^{-1}(x)\sigma_{ik,k}(x,t)]_{,j} &+ [\rho^{-1}(x)\sigma_{jk,k}(x,t)]_{,i}\\ &+[\rho^{-1}(x)F_i(x,t)]_{,j} + [\rho^{-1}(x)F_j(x,t)]_{,i}\,.\end{aligned} \tag{3.15}$$

The equations were then used to analyze the problem of Rayleigh waves in a nonhomogeneous, isotropic elastic halfspace under the assumption that the state of strain is two-dimensional ($u_3 = 0$); the equations have the form (cf. [1])

$$\frac{1}{\mu(x)}[\ddot{\sigma}_{\alpha\beta}(x,\tau) - \frac{\lambda(x)\delta_{\alpha\beta}}{2\lambda(x)+2\mu(x)}\ddot{\sigma}_{\gamma\gamma}(x,\tau)] = \sigma_{\alpha\gamma,\gamma\beta}(x,\tau) + \sigma_{\beta\gamma,\gamma\alpha}(x,\tau) \tag{3.16}$$

and should be satisfied in the region $|x_1| < \infty$, $0 < x_2 < \infty$ under the conditions

$$\sigma_{22}(x_1,0;\tau) = \sigma_{12}(x_1,0;\tau) = 0\,,$$

$$\sigma_{22}(x_1,\infty;\tau) = \sigma_{12}(x_1,\infty;\tau) = 0, \qquad x_1 < \infty\,,$$

where $\sigma_{\alpha\beta}, \alpha, \beta = 1,2$ denotes the dimensionless stress tensor in the plane state of strain $\sigma_{\alpha\beta} = \widehat{\sigma}_{\alpha\beta}/\mu_0$, and the remaining symbols have the meanings as follows:

$$\mu = \widehat{\mu}/\mu_0, \qquad x_\alpha = \widehat{x}_\alpha/x_0, \qquad \tau = t\sqrt{\mu_0}/x_0\sqrt{\rho_0}, \qquad \lambda = \widehat{\lambda}/\mu_0,$$

$\widehat{\lambda} = \widehat{\lambda}(\widehat{x}_\alpha)$, $\widehat{\mu} = \widehat{\mu}(\widehat{x}_\alpha)$ — Lamé coefficients,
$\widehat{x}_\alpha$ Cartesian coordinates,
ρ_0 constant density of the medium,
$\mu_0 = \widehat{\mu}_0(\widehat{x}^0_\alpha)$,
$\widehat{x}^0_\alpha$ a fixed point of the plane (x_1, x_2),
x_0 a characteristic length,
τ dimensionless time.

After the digression referring to the origins of the statement of the problem under consideration, let us now return to the analysis of Eq. (3.13); its solution is assumed in the form of harmonic functions

$$\begin{aligned} \sigma_{xx}(x,z;t) &= \alpha(z)\exp[i(sx-pt)]\,,\\ \sigma_{zz}(x,z;t) &= \beta(z)\exp[i(sx-pt)]\,,\\ \sigma_{xz}(x,z;t) &= \gamma(z)\exp[i(sx-pt)]\,, \end{aligned} \tag{3.17}$$

where α, β, γ are functions of the only variable z; they decrease, not necessarily exponentially, with $z \to \infty$, and $2\pi/p$, $2\pi/s$, $C_R = p/s$ denote the period, length and propagation speed (Rayleigh velocity) of the wave, respectively. Substitution of Eqs. (3.17) in Eqs. (3.13) and (3.14) yields

$$\begin{aligned} p^2[B_1(z)\alpha(z) + f(z)\beta(z)] &= \rho^{-1}[-s^2\alpha(z) + is\gamma'(z)]\,,\\ p^2[f(z)\alpha(z) - B(z)f(z)g(z)\beta(z)] &= \rho^{-1}[is\gamma'(z) + \beta''(z)],\\ 2B_2(z)p^2\gamma(z) &= \rho^{-1}[is\alpha'(z) + \gamma''(z) - s^2\gamma(z) + is\beta'(z)] \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \beta(0) = \gamma(0) = 0\,,\\ \beta(\infty) = \gamma(\infty) = 0\,, \end{aligned} \tag{3.18$'$}$$

primes denoting differentiation with respect to z.

Equation (3.18) may be replaced with the system of Eqs. (3.19)

$$\begin{aligned} &p^2[(B_1(z) + f(z))\alpha(z) + f(z)(1 - B(z)g(z))\beta(z)]\\ &\qquad\qquad = \rho^{-1}[-s^2\alpha(z) + \beta''(z) + 2is\gamma'(z)]\,,\\ &p^2[(B_1(z) - f(z))\alpha(z) + f(z)(1 + B(z)g(z))\beta(z)] = \rho^{-1}[-s^2\alpha(z) - \beta''(z)]\,,\\ &-p^2\gamma(z) = (2B_2(z)\rho)^{-1}[is\alpha'(z) + is\beta'(z) - s^2\gamma(z) + \gamma''(z)]\,. \end{aligned} \tag{3.19}$$

Equation $(3.19)_2$ is used to determine

$$\alpha(z) = -\mathcal{H}^{-1}\left[f(z)(1 + B(z)g(z)) + \frac{1}{\rho p^2}D^2\right]\beta(z)\,, \tag{3.20}$$

where

$$\mathcal{H} = \left[B_1(z) - f(z) + \rho^{-1}\left(\frac{s}{p}\right)^2\right], \qquad D = \frac{d}{dz}. \tag{3.21}$$

Equation $(3.19)_3$ is now used to determine

$$2is\gamma(z) = \mathcal{K}^{-1}D[2isD\gamma(z) - 2s^2(\alpha(z) + \beta(z))], \tag{3.22}$$

while Eq. $(3.19)_1$ makes it possible to determine the expression $(2isD\gamma)$ occurring in the formula (3.22),

$$2isD\gamma(z) = [\rho p^2(B_1(z) + f(z)) + s^2]\alpha(z) - D^2\beta(z) + \rho p^2 f(z)(1 - B(z)g(z))\beta(z).$$

Hence, this result and Eq. (3.22) yield

$$2is\gamma(z) = \mathcal{K}^{-1}(z)D\{\mathcal{W}\}\beta(z), \tag{3.23}$$

where

$$\begin{aligned} \mathcal{K}(z) &= s^2 - 2B_2(z)\rho p^2 = s^2\left[1 - 2B_2(z)\rho(\frac{p}{s})^2\right], \\ \{\mathcal{W}\}\beta(z) &= \mathcal{L}\alpha(z) - D^2\beta(z) + \mathcal{M}\beta(z). \end{aligned} \tag{3.24}$$

In the last formula we have introduced the notation

$$\begin{aligned} \mathcal{L} &= \rho p^2(B(z) + f(z)) - s^2, \\ \mathcal{M} &= \rho p^2 f(z)(1 - B(z)g(z)) - 2s^2. \end{aligned} \tag{3.25}$$

Inserting Eq. (3.20) into $(3.24)_2$, making use of the relation

$$\mathcal{M} = \mathcal{L} - [\rho p^2(B(z)f(z)g(z) + B_1(z)) + s^2],$$

regrouping and rearranging the terms we arrive at the operator $\mathcal{W}$ written in the form

$$\mathcal{W} = -\mathcal{A}(z)[D^2 - s^2(1 - \mathcal{R}(z))] + 4s^2\mathcal{E}(z), \tag{3.26}$$

where

$$\begin{aligned} \mathcal{A}(z) &= 2\mathcal{H}^{-1}(z)B_1(z), \qquad \mathcal{R}(z) = \rho\left(\frac{p}{s}\right)^2[(B_1(z))^{-1}f^2(z) + B(z)f(z)g(z)], \\ \mathcal{E}(z) &= \mathcal{H}^{-1}(z)\left[f(z) - B_1(z) - \frac{1}{2}\rho^{-1}\left(\frac{s}{p}\right)^2\right]. \end{aligned} \tag{3.27}$$

It should be noted that operator $\mathcal{W}$ in the form (3.26) acting on the function $\beta(z)$ plays an important role since it generates the fundamental differential equation governing the problem under consideration.

Thus the Eq. (3.23) determining the shear stress amplitude $\gamma(z)$ will be expressed by the formula

$$2is\gamma(z) = \mathcal{K}^{-1}(z)D\{(-\mathcal{A}(z))[D^2 - s^2(1 - \mathcal{R}(z))] + 4s^2\mathcal{E}(z)\}\beta(z). \tag{3.28}$$

Equation $(3.18)_2$ may be used to derive the differential relations to be satisfied by β. The values of $\alpha(z)$ and $\gamma(z)$ given by Eqs. (3.20), (3.28) are substituted in

$$B(z)f(z)g(z)\beta(z) + \frac{1}{\rho p^2}D^2\beta(z) - f(z)\alpha(z) + \frac{1}{\rho p^2}isD\gamma(z) = 0, \tag{3.29}$$

Rearranging the corresponding terms we obtain

$$
\begin{aligned}
(3.30)\qquad [B(z)f(z)g(z)-f^2(z)\mathcal{H}^1(z)(1+B(z)g(z))]\beta(z)+\frac{1}{\rho p^2}[1+f(z)\mathcal{H}^{-1}(z)]D^2\beta(z)\\
+\frac{1}{\rho p^2}\frac{1}{2}D\mathcal{H}^{-1}(z)D(-\mathcal{A}(z)[D^2-s^2(1-\mathcal{R}(z))]\beta(z)\\
+\frac{1}{\rho p^2}\frac{1}{2}D\mathcal{K}^{-1}(z)D4s^2\mathcal{E}(z)\beta(z)=0.
\end{aligned}
$$

This equation may also be written in the form

$$
\begin{aligned}
(3.31)\qquad (D\mathcal{K}^{-1}(z)D-1)(-\mathcal{A}(z)[D^1-s^2(1-\mathcal{R}(z))]\beta(z)\\
+(2[1+\mathcal{H}^{-1}(z)(f(z)-B_1(z))]D^2+4s^2D\mathcal{K}^{-1}(z)D\mathcal{E}(z))\beta(z)\\
+2\rho p^2\Bigg\{B(z)f(z)g(z)+\mathcal{H}^{-1}(z)\Bigg(f(z)B(z)f(z)g(z)\\
+B_1(z)\left[\rho^{-1}\left(\frac{s}{p}\right)^2-B(z)f(z)g(z)\right]\Bigg)\Bigg\}\beta(z)=0.
\end{aligned}
$$

It is seen that function $\beta(z)$ should satisfy the ordinary, linear differential equation of fourth order with variable coefficients, which are determined by means of the material characteristics of the medium, i.e. the elastic constants of anisotropy and nonhomogeneity. Of course, the inverse operations $\mathcal{H}^{-1}(z)$, $\mathcal{K}^{-1}(z)$ are assumed to exist.

The problem of the Rayleigh surface waves is then reduced to the analysis of Eq. (3.31) with the following boundary conditions:

$$
(3.32)\qquad
\begin{aligned}
&\beta(0)=\beta(\infty)=0,\\
&D\{\mathcal{H}^{-1}(z)B_1(z)[D^2-s^2(1-\mathcal{R}(z))]-2s^2\mathcal{E}(z)\}\beta(z)\Big|_{\substack{z=0\\ z=\infty}}=0.
\end{aligned}
$$

The solution consists in determining the non-vanishing function $\beta=\beta(z)$ satisfying Eqs. (3.31), (3.32), and the parameter $C_R=C_R(s)>0$, which is the Rayleigh wave propagation speed depending, in general, on the wave number s, the anisotropy and the nonhomogeneity characteristics. Once the function $\beta(z)$ is determined, the stress amplitude $\alpha(z)$, $\gamma(z)$ appearing in Eqs. (3.17) may be found according to the formulae (3.20), (3.23). This is the reason to call $\beta(z)$ a stress function.

4. Discussion of the equation for the stress function in a transversely isotropic nonhomogeneous semispace

Equation (3.31) for the stress function in a transversely isotropic, nonhomogeneous semispace (in accordance with the assumptions made earlier) is written in a form which may be divided into three parts, each of them contributing an essential information concerning the physical properties of the medium. The first two terms have the form

$$
(4.1)\qquad (D\mathcal{K}^{-1}(z)D-1)(-\mathcal{A}(z))[D^2-s^2(1-\mathcal{R}(z))]\beta(z),
$$

$$
(4.2)\qquad (2[1+\mathcal{H}^{-1}(z)(f(z)-B_1(z))]D^2+4s^2D\mathcal{K}^{-1}(z)D\mathcal{E}(z))\beta(z).
$$

The third term is represented by the remaining expressions at the left-hand side of Eq. (3.31). The terms will be discussed in the order exposed in the Introduction.

4.1. Transversely isotropic body of a "small nonhomogeneity"

The second term of Eq. (3.31) given by (4.2) (cf. Eq. (3.31)) may be proved to vanish identically when the medium becomes homogeneous. In the case when

$$T(z)\beta(z) = (T_1(z) + T_2(z))\beta(z) \ll 1\,, \tag{4.3}$$

where

$$T_1(z) = 2[1 + \mathcal{H}^{-1}(f(z) - B_1(z))]D^2\,,$$
$$T_2(z) = 4s^2 D\mathcal{K}^{-1}(z)D\mathcal{E}(z)$$

term (4.2) may be considered to be small as compared with the leading term (4.1), and Eq. (3.31) may be replaced by the equation

$$\begin{aligned}(D\mathcal{K}^{-1}(z)D - 1)(-\mathcal{A}(z))[D^2 - s^2(1 - \mathcal{R}(z))]\beta(z) \\ +2\rho p^2\Big\{ B(z)f(z)g(z) + \mathcal{H}^{-1}(z)\Big(f(z)B(z)f(z)g(z) \\ +B_1(z)\Big[\rho^{-1}\Big(\frac{s}{p}\Big)^2 - B(z)f(z)g(z)\Big]\Big)\Big\}\beta(z) = 0\,.\end{aligned} \tag{4.4}$$

This equation may find an application to the approximate description of a surface wave propagating in a transversely isotropic medium characterized by "small nonhomogeneity". The accuracy of the solution depends on the estimated value of the term (4.3); let us consider the function $T(z)$.

Substitution of the magnitude defined by Eqs. (3.5), (3.6), (3.21), $(3.24)_1$, $(3.27)_3$ into Eqs. $(4.3)_{2,3}$ yields the results

$$\begin{aligned}T_1(z) = 2 + \frac{2}{1+\widehat{\nu}(z)}\Big[\widehat{\nu}(z)(1+\nu(z)) + \Big(1 + \frac{E(z)}{\widehat{E}(z)}\widehat{\nu}^2(z)\Big)\Big] \\ \times\Big[\frac{\widehat{\nu}(z)}{1+\widehat{\nu}(z)}\Big(\frac{E(z)}{\widehat{E}(z)}\widehat{\nu}(z) - \nu(z)\Big) + \frac{2-\widehat{\Omega}(z)}{\widehat{\Omega}}\Big]^{-1} D^2\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}T_2(z) = -4D\frac{1}{1-\widehat{\Omega}(z)}D\Big[\frac{\widehat{\nu}(z)}{1+\widehat{\nu}(z)}\Big(\frac{E(z)}{\widehat{E}(z)}\widehat{\nu}(z) - \nu(z)\Big) + \frac{1-\widehat{\Omega}(z)}{\widehat{\Omega}(z)}\Big] \\ \times\Big[\frac{\widehat{\nu}(z)}{1+\widehat{\nu}(z)}\Big(\frac{E(z)}{\widehat{E}(z)}\widehat{\nu}(z) - \nu(z)\Big) + \frac{2-\widehat{\Omega}(z)}{\widehat{\Omega}(z)}\Big]^{-1},\end{aligned} \tag{4.6}$$

where

$$\widehat{\Omega}(z) = \frac{C_R^2}{\widehat{c}_2^2(z)}\,.$$

Consequently, assuming the expressions occurring in Eq. (4.5), (4.6) to be independent of z (what means that we are passing to the case of an anisotropic homogeneous medium), and introducing the relations

$$h = \frac{E}{\widehat{E}}, \qquad k = \frac{\widehat{\nu}}{\nu}\,, \tag{4.7}$$

we obtain

$$(4.8)\quad T = T_1 + T_2 = 2\left\{1 + \frac{1}{(1-\widehat{\Omega})(1+k\nu)}\left(3(k\nu + k\nu^2 - hk^2\nu^2 + 1) - \widehat{\Omega}[k\nu(1+\nu) + (1 - hk^2\nu^2)] - 2\frac{1}{\widehat{\Omega}}(1+k\nu)\right) \times \left[\frac{k\nu^2}{1+k\nu}(hk-1) + \frac{2-\widehat{\Omega}}{\widehat{\Omega}}\right]^{-1}\right\} D^2 .$$

Passing with h and k to 1,

$$(4.9)\quad \lim_{h,k\to 1} T = \left[1 + \frac{1}{1-\widehat{\Omega}}\left\{3 - \widehat{\Omega} - 2\frac{1}{\widehat{\Omega}}\right\}\frac{\widehat{\Omega}}{2-\widehat{\Omega}}\right] D^2 = 0 .$$

This means that under $\widehat{E} \to E$ and $\widehat{\nu} \to \nu$ and for a homogeneous medium, term Eq. (4.2) vanishes.

4.2. "Weakly anisotropic" nonhomogeneous body

It may be shown that if the medium is isotropic and nonhomogeneous, the third term of Eq. (3.31) will be identically zero. Thus, assume now that the term

$$(4.10)\quad 2\rho p^2\left\{(B(z)f(z)g(z)) + \mathcal{H}^{-1}(z)\left(f(z)(B(z)f(z)g(z)) + B_1(z)\left[\rho^{-1}\left(\frac{s}{p}\right)^2 - (B(z)f(z)g(z))\right]\right)\right\}\beta(z)$$

is small as compared with the other terms of Eq. (3.31); the equation assumes then the simplified form

$$(4.11)\quad (D\mathcal{K}^{-1}(z)D - 1)(-\mathcal{A}(z))[D^2 - s^2(1 - \mathcal{R}(z))]\beta(z) + (2[1 + \mathcal{H}^{-1}(z)(f(z) - B_1(z))]D^2 + 4s^2 D\mathcal{K}^{-1}(z)D\mathcal{E}(z))\beta(z) = 0 ,$$

describing the stress function in a "weakly anisotropic", nonhomogeneous medium.

4.3. "Weakly anisotropic" body with a "small nonhomogeneity"

The two particular cases considered in Secs. 4.1, 4.2 imply the approximate form of the equation for the stress function in a "weakly anisotropic" medium with "small anisotropy",

$$(4.12)\quad (D\mathcal{K}^{-1}(z)D - 1)\mathcal{A}(z)[D^2 - s^2(1 - \mathcal{R}(z))]\beta(z) = 0 .$$

This equation may be applied to determining the approximate solution of the surface wave problem formulated earlier in Sec. 3.2, cf. Eqs. (3.31), (3.32).

4.4. Transversely isotropic homogeneous body

Let us now present the accurate form of the stress function $\beta(z)$ in a transversely isotropic, homogeneous semispace. In such a case the coefficients of the equation are independent of z and Eq. (3.31) assumes the form

$$(4.13)\qquad \overline{\mathcal{A}}(\overline{\mathcal{K}}^{-1}D^2-1)[D^2-s^2(1-\overline{\mathcal{R}})]\beta(z) -(2[1+\overline{\mathcal{H}}^{-1}(\overline{f}-\overline{B}_1)]+4s^2\overline{\mathcal{K}}^{-1}\overline{\mathcal{E}})D^2\beta(z) -2\rho p^2\left\{\overline{B}\,\overline{f}\overline{g}+\overline{\mathcal{H}}^{-1}\left(\overline{f}(\overline{B}\,\overline{f}\overline{g})+\overline{B}_1\left[\rho^{-1}\left(\frac{s}{p}\right)^2-\overline{B}\,\overline{f}\overline{g}\right]\right)\right\}\beta(z)=0,$$

where

$$\overline{\mathcal{A}}=2\overline{B}_1\overline{\mathcal{H}}^{-1}=2\frac{1}{1+k\nu}(hk^2\nu^2-1)\left[k\nu^2\frac{hk-1}{1+k\nu}+\frac{2-\widehat{\Omega}}{\widehat{\Omega}}\right]^{-1},$$

$$\overline{\mathcal{K}}=s^2(1-\widehat{\Omega}),$$

$$\overline{\mathcal{R}}=\frac{1}{2}\widehat{\Omega}\left[2-\frac{\nu+1}{hk^2-1}\right],$$

$$(4.14)\qquad \overline{\mathcal{H}}=\left[\overline{B}_1-\overline{f}+\rho^{-1}\left(\frac{s}{p}\right)^2\right]=\frac{1}{2\widehat{\mu}}\left[k\nu^2\frac{hk-1}{1+k\nu}+\frac{2-\widehat{\Omega}}{\widehat{\Omega}}\right],$$

$$\overline{f}-\overline{B}_1=\frac{1}{2\widehat{\mu}(1+k\nu)}[k\nu(1+\nu)-(hk^2\nu^2-1)],$$

$$\overline{\mathcal{E}}=\overline{\mathcal{H}}^{-1}\left[\overline{f}-\overline{B}_1-\frac{1}{2}\rho^{-1}\left(\frac{s}{p}\right)^2\right]=\left[k\nu\frac{1+\nu}{1+k\nu}-\frac{hk^2\nu^2-1}{1+k\nu}-\frac{1}{\widehat{\Omega}}\right]\times\left[k\nu^2\frac{hk-1}{1+k\nu}+\frac{2-\widehat{\Omega}}{\widehat{\Omega}}\right]^{-1};$$

$$\overline{B}\,\overline{f}\overline{g}=\frac{1}{E}(1-\nu^2)=\frac{1}{2\mu}(1-\nu)=\frac{1}{4\mu}\frac{1}{1-\varkappa},$$

$$(4.15)\qquad 2[1+\overline{\mathcal{H}}^{-1}(\overline{f}-\overline{B}_1)]+4s^2\overline{\mathcal{K}}^{-1}\overline{\mathcal{E}}=2\left(1+\left\{\frac{1}{1+k\nu}[k\nu(1+\nu)-(hk^2\nu^2-1)]\times\left(1+2\frac{1}{1-\widehat{\Omega}}\right)-2\frac{1}{\widehat{\Omega}(1-\widehat{\Omega})}\right\}\left[k\nu^2\frac{hk-1}{1+k\nu}+\frac{2-\widehat{\Omega}}{\widehat{\Omega}}\right]^{-1}\right),$$

$$\overline{f}(\overline{B}\,\overline{f}\overline{g})=\frac{hk\nu}{8\mu^2(1-\varkappa)},\qquad \overline{B}_1\rho^{-1}\left(\frac{s}{p}\right)^2=\frac{hk^2\nu^2-1}{2\widehat{\mu}^2(1+k\nu)\widehat{\Omega}},$$

$$\overline{B}\,\overline{B}_1\overline{f}\overline{g}=\frac{hk^2\nu^2-1}{8\widehat{\mu}\mu(1-k\nu)(1-\varkappa)},\qquad \varkappa=\frac{1-2\nu}{2-2\nu},\qquad \widehat{\Omega}=\frac{C_R^2}{\widehat{c}_2^2},\qquad \Omega=\frac{C_R^2}{c_2^2},$$

whence we obtain

$$(4.16)\qquad \left\{\overline{B}\,\overline{f}\overline{g}+\overline{\mathcal{H}}^{-1}\left[\overline{f}(\overline{B}\,\overline{f}\overline{g})+\overline{B}_1\left[\rho^{-1}\left(\frac{s}{p}\right)^2-\overline{B}\,\overline{f}\overline{g}\right]\right]\right\} =\frac{1}{4\mu}\frac{1}{(1-\varkappa)}\left\{1+\left[\frac{k\nu(1+\nu)}{1+k\nu}+\frac{4h(hk^2\nu^2-1)(1-\varkappa)}{\widehat{\Omega}(1+\nu)}-\frac{hk^2\nu^2-1}{1+k\nu}\right]\right.$$

$$\times\left[k\nu^2\frac{hk-1}{1+k\nu}+\frac{2-\widehat{\Omega}}{\widehat{\Omega}}\right]^{-1}\Bigg\}, \tag{4.17 [cont.]}$$

$$2\rho p^2\frac{1}{4\mu}=s^2\rho\left(\frac{p}{s}\right)^2\frac{1}{2\mu}=\frac{1}{2}s^2\frac{C_R^2}{c_2^2}=\frac{1}{2}s^2\Omega\,. \tag{4.17}$$

In view of the relations (4.16), (4.17), the term in braces appearing in Eq. (4.13) may be rewritten in the form

$$2\rho p^2\{\ldots\}=\frac{1}{2}s^2\Omega\frac{1}{(1-\varkappa)}\{1+[\cdots][---]^{-1}\}, \tag{4.18}$$

with the notations

$$[\cdots]=\frac{k\nu(1+\nu)}{1+k\nu}+\frac{4h(hk^2\nu^2-1)(1-\varkappa)}{\widehat{\Omega}(1+\nu)}-\frac{hk^2\nu^2-1}{1+k\nu},$$

$$[---]=k\nu^2\frac{hk-1}{1+k\nu}+\frac{2-\widehat{\Omega}}{\widehat{\Omega}}.$$

On substituting $(4.14)_{1,2,3}$, $(4.15)_2$ and (4.18) in (4.13) we obtain the final form of the equation for the stress function β in a transversely isotropic homogeneous body. This equation may easily be transformed to deduce the simplified equation for the homogeneous "weakly anisotropic" medium by following the considerations presented in Secs. 4.2 and 4.3.

4.5. Isotropic nonhomogeneous body

To conclude the discussion of particular cases of the Eq. (3.31) let us present the equation governing the surface stress wave propagation in an isotropic, nonhomogeneous semi-space. The form of the equation obtained is similar to Eq. (4.11), though the coefficients are much simpler. Equation (3.31) is now represented in the form

$$\begin{aligned}(D\widehat{\mathcal{K}}^{-1}(z)D-1)(-\widehat{\mathcal{A}}(z)[D^2-s^2(1-\widehat{\mathcal{R}}(z))]\widehat{\beta}(z)&\\ +(2[1+\widehat{\mathcal{H}}^{-1}(z)(\widehat{f}(z)-\widehat{B}_1(z))]+4s^2D\widehat{\mathcal{K}}^{-1}(z)D\widehat{\mathcal{E}}(z))\widehat{\beta}(z)&=0,\end{aligned} \tag{4.19}$$

with the notations

$$\begin{aligned}\widehat{\mathcal{K}}(z)&=s^2(1-\Omega(z)),\\ \widehat{\mathcal{A}}(z)&=-\frac{\Omega(z)}{(1-\varkappa)(2-\Omega(z))},\\ \widehat{\mathcal{R}}(z)&=\varkappa\Omega(z),\\ \widehat{\mathcal{H}}(z)&=\frac{1}{2\mu(z)}\frac{2-\Omega(z)}{\Omega(z)},\\ 2[1+\widehat{\mathcal{H}}^{-1}(z)(\widehat{f}(z)-\widehat{B}_1(z))]&=4\frac{1}{2-\Omega(z)},\\ \widehat{\mathcal{E}}(z)=\widehat{\mathcal{H}}^{-1}\left[\widehat{f}(z)-\widehat{B}_1(z)-\frac{1}{2}\rho^{-1}\left(\frac{s}{p}\right)^2\right]&=-\frac{1-\Omega(z)}{2-\Omega(z)}.\end{aligned} \tag{4.20}$$

Inserting Eqs. (4.20) into Eq. (4.19) we obtain for $\widehat{\beta}(z)$ the following equation

$$\text{(4.21)}\qquad \left(\frac{1}{s^2}D\frac{1}{1-\Omega}D-1\right)\frac{\Omega}{(1-\varkappa)(2-\Omega)}[D^2-s^2(1-\varkappa\Omega)]\widehat{\beta}(z) +4\left(\frac{1}{2-\Omega}D^2-D\frac{1}{1-\Omega}D\frac{1-\Omega}{2-\Omega}\right)\widehat{\beta}(z)=0.$$

Amplitudes α and β determined by Eqs. (3.20) and (3.28) assume the form

$$\text{(4.22)}\qquad \begin{aligned} \alpha(z) &= -\frac{1}{s^2(2-\Omega)}(s^2\Omega+2D^2)\widehat{\beta}(z),\\ 2is\gamma(z) &= \frac{1}{s^2(1-\Omega)}D\left\{\frac{\Omega}{(1-\varkappa)(2-\Omega)}[D^2-s^2(1-\varkappa\Omega)]\widehat{\beta}(z)-4s^2\frac{1-\Omega}{2-\Omega}\widehat{\beta}(z)\right\}.\end{aligned}$$

In such a case function $\widehat{\beta}(z)$ is required to satisfy the conditions

$$\widehat{\beta}(0)=\widehat{\beta}(\infty)=0.$$

$$\text{(4.23)}\qquad D\left\{\frac{\Omega}{2-\Omega}\frac{1}{1-\varkappa}[D^2-s^2(1-\varkappa\Omega)]\widehat{\beta}(z)-4s^2\frac{1-\Omega}{2-\Omega}\widehat{\beta}(z)\right\}_{\substack{z=0\\ z=\infty}}=0.$$

The following notations have been introduced in Eqs. (4.21)–(4.23)

$$\text{(4.24)}\qquad \begin{aligned} \varkappa(z) &= \frac{1-2\nu(z)}{2(1-\nu(z))}, & \nu(z) &= \frac{1-2\varkappa(z)}{2(1-\varkappa(z))},\\ \Omega=\Omega(z) &= \frac{C_R^2}{c_2^2(z)}, & c_2(z) &= \left(\frac{\mu(z)}{\rho}\right)^{\frac{1}{2}}.\end{aligned}$$

Equations (4.21)–(4.23) are known from the literature; they were originally derived and presented by IGNACZAK in his dissertation "Problem of completeness for the equations of motion in linear elasticity written in terms of stresses" (1963); see, in particular, Chapter II "Rayleigh waves in a nonhomogeneous, isotropic elastic halfspace", [1] or [2].

5. Method of solution of the equations derived for the anisotropic and nonhomogeneous elastic medium

The results of investigations made thus far in the fields of surface stress waves in isotropic nonhomogeneous media, based on the application of the equations of motion written in terms of stresses (see IGNACZAK [1], RAO [15], ROŻNOWSKI [11, 12], KLECHA [16]) may now be used to outline the method of solution of Eq. (3.31) in the case of a transversely isotropic nonhomogeneous body.

Certain analogies between the Eqs. (3.31) and (4.21) suggest the similarity of the corresponding solutions, what may facilitate the choice of the suitable solution methods. In paper [11] the present author considered an equation similar to Eq. (4.21); the problem of surface wave propagation in an isotropic halfspace of "small nonhomogeneity" was solved under the two following assumptions:

a. Variation of the elastic shear modulus is expressed by a monotone function of depth coordinate z,

$$\text{(5.1)}\qquad \mu(z)=\frac{\mu_0\mu_\infty}{\mu_0-(\mu_0-\mu_\infty)\exp[-2\varepsilon z]},\qquad \varepsilon>0,$$

where μ_0, μ_∞ denote the shear moduli at the boundary of the halfspace, $z = 0$, and at infinity, $z = \infty$, respectively, and ε denotes the nonhomogeneity parameter.

b. The term in Eq. (4.21) which vanishes identically in a homogeneous halfspace is disregarded, i.e. the term

$$4\left[\frac{1}{2-\Omega}D^2 - D\frac{1}{1-\Omega}D\frac{1-\Omega}{2-\Omega}\right]\widehat{\beta}(z). \tag{5.2}$$

In such a case a closed form solution for the function $\widehat{\beta}(z)$ was obtained; the dispersion equation has been discussed under the assumption that the nonhomogeneity parameter ε was large enough to justify the asymptotic expansions used.

The general problem of Eq. (4.21) with conditions (4.23) under the assumed nonhomogeneity (5.1) was considered by the author in [12]. The results obtained earlier in [11] enabled the solution by means of the method of successive approximations.

Owing to the fact that the equations derived in the present paper (3.31), (4.4), (4.11), (4.12), (4.13) are of a similar type as those considered in [11] and [12], let us follow the way of reasoning presented in those papers and use the differential operators introduced there. Equation (3.31) is written in the form

$$L_1^2\Gamma(x)L_2^2\beta(x) = F(x), \tag{5.3}$$

where

$$\begin{aligned} L_1^2 &= Df_1(x)D - 1, \\ L_2^2 &= D^2 - s^2(1 - f_2(x)), \\ F(x) &= P(x) + G(x), \\ P(x) &= [\pi_1(x)D^2 + D4s^2 f_1(x)D\pi_2(x)]\beta(x), \\ G(x) &= g_1(x)\beta(x). \end{aligned} \tag{5.4}$$

functions $f_1(x)$, $f_2(x)$, $\pi_1(x)$, $\pi_2(x)$, $g_1(x)$, $\Gamma(x)$ being defined by Eq. (3.31). Introducing the notation

$$v(x) = \Gamma(x)L_2^2\beta(x) \tag{5.5}$$

and substituting Eq. (5.5) in Eq. (5.3), we obtain

$$L_1^2 v(x) = F(x) \tag{5.6}$$

and

$$L_2^2\beta(x) = \Gamma^{-1}(x)v(x). \tag{5.7}$$

Assume now that

$$\begin{aligned} v(x) &\overset{\text{df}}{=} D\varphi(x), \\ F(x) &\overset{\text{df}}{=} Df_1(x)q(x). \end{aligned} \tag{5.8}$$

Substituting Eq. $(5.8)_1$ in (5.6) and using Eq. $(5.4)_1$, we obtain

$$\begin{aligned} L_1^2 v(x) = L_1^2 D\varphi(x) &= (Df_1(x)D - 1)D\varphi(x) \\ &= Df_1(x)\{D^2 - h(x)\}\varphi(x) = Df_1(x)q(x). \end{aligned} \tag{5.9}$$

From Eq. (5.9) it follows immediately that

$$Df_1(x)\{D^2\varphi(x) - h(x)\varphi(x) - q(x)\} = 0,$$

and then

$$D^2\varphi(x) - h(x)\varphi(x) = q(x), \qquad h(x) = (f_1(x))^{-1}, \qquad f_1(x) \neq 0 \tag{5.10}$$

so if $\varphi(x)$ satisfies Eqs. (5.10), the function $(5.8)_1$ will also satisfy the Eq. (5.6).

Comparison of Eqs. (5.3)–(5.10) with Eqs. (2.10)–(2.14) in paper [12] yields the analogy enabling us to use the method of solution applied in [12] in solving the Eq. (3.31) and its simplified versions. Thus the solution of the equation governing the stress function in an anisotropic nonhomogeneous medium may be determined by following the procedure demonstrated in [12], i.e. by solving the set of Eqs.

$$\begin{aligned} D^2\varphi(x) - h(x)\varphi(x) &= q(x), \\ L_2^2\beta(x) &= \Gamma(x)^{-1}D\varphi(x) \end{aligned} \tag{5.11}$$

under the boundary conditions

$$\begin{aligned} \beta(0) = \beta(\infty) &= 0, \\ D\{\mathcal{H}^{-1}(x)B_1(x)[D^2 - s^2(1-\mathcal{R}(x))] - 2s^2\mathcal{E}(x)\}\beta(x)\big|_{\substack{x=0\\x=\infty}} &= 0. \end{aligned} \tag{5.12}$$

A more detailed analysis of function $q(x)$ is given in the paper [12].

Particular attention should be paid to Eq. (4.12); in that case $F(x) \equiv 0$ and suitable construction of functions $\mathcal{K}^{-1}(x)$, $\mathcal{A}(x)$, $\mathcal{R}(x)$ (written in notation introduced in Eq. (5.3)) makes it possible to use the consecutive stages of solution presented in [11]. In that case Eqs. (5.1), (5.4) yield the following relations:

$$\begin{aligned} D^2\varphi(x) - h(x)\varphi(x) &= 0, \\ L_2^2\beta(x) &= (\Gamma(x))^{-1}D\varphi(x), \end{aligned} \tag{5.13}$$

with the notations

$$\begin{aligned} h(x) &= (\mathcal{K}^{-1}(x))^{-1} = \mathcal{K}(x) = s^2\left[1 - 2B(x)\rho\left(\frac{p}{s}\right)^2\right], \\ \Gamma(x) &= -\mathcal{A}(x) = -2\mathcal{H}^{-1}(x)B_1(x), \\ f_2(x) &= \mathcal{R}(x) = \rho\left(\frac{p}{s}\right)^2[(B_1(x))^{-1}f^2(x) + B(x)f(x)g(x)]. \end{aligned} \tag{5.14}$$

The same method may also be used to propose the physical interpretation of the problem considered.

The cognitive value of Eqs. (3.30) or (3.31) derived here and applicability of the method of solution based on Eqs. (5.11), (5.12) will be presented on the example of a simplified equation (4.12) governing the stress function in a "weakly anisotropic" medium of "small nonhomogeneity" in the paper [22].

6. Conclusions

1. Problem of the Rayleigh-type surface waves in a transversely isotropic semi-space, nonhomogeneous in the vertical plane of isotropy, may be analyzed as the eigenproblem of the fourth-order differential equation with variable coefficients — Eq. (3.30) or (3.31) with the corresponding boundary conditions.

2. Solving the equation by means of the method considered in Sec. 5, Eqs. (5.3)–(5.12), we obtain the set of two ordinary second order differential equations with vari-

able coefficients, with the necessary boundary conditions. In order to determine the solution $\beta = \beta(z)$, the functions describing the variation of material characteristics of the halfspace must be assumed (Young's moduli, Poisson's ratios), and the equations must be reduced to the form enabling the qualitative and quantitative analysis, see e.g. E. Kamke [21].

3. Once the function $\beta = \beta(z)$ is determined, the dispersion equation may be obtained from the condition of vanishing of the shear stress amplitudes at the free surface of the halfplane.

4. Of special interest is the approximate equation (4.12) derived for the "weakly anisotropic" halfspace with a "small nonhomogeneity", since it is much simpler than the accurate equation (3.31) and it contains all the material coefficients determining the properties of the transversely isotropic, nonhomogeneous medium.

5. The equations derived enhance the possibility of theoretical analysis of continua exhibiting certain properties of anisotropy and nonhomogeneity, the anisotropic and nonhomogeneous surface layer in particular.

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