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Elasto-plasticity of slackened systems

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THE PAPER concerns the so-called "slackened" systems, i.e. systems with gaps (clearances) at the joints between finite elements. Fundamental theoretical problems of mechanics of such systems made of the elastic-plastic material have been presented. Only quasi-static processes, within the framework of the geometrically linear theory, are considered, and all friction effects are neglected. Several theorems concerning the problems of analysis and synthesis have been derived. Particular attention has been paid to problems of uniqueness of solutions. A mathematical model proposed in the work can describe the behaviour of locking elastic-plastic systems. This model covers a considerably wide class of time-independent materials. The theory allows us to describe the frictionless cases of unilateral constraints in the frame of small deformations. Results obtained in the paper can be applied in structural and solid mechanics.

1. Introduction

THE PRESENCE of gaps (clearances) at structural connections leads to numerous unexpected effects occurring in the behaviour of structures during the loading process. The general theory of such structures, called here "slackened" systems, will be derived and discussed. The problem is not quite new, nevertheless, very few works are known to deal with problems of slackened systems. It will be also shown that the theory of slackened systems has an essential significance for the theory of elastic-plastic continua.

In a "macro" scale the behaviour of rigid-slackened systems is identical with that of the ideal locking material (cf. W. PRAGER [1]). However, the behaviour of rigid-slackened systems does not follow from the physical properties of the material; it is due to variable boundary conditions of the system elements. In both cases the corresponding mathematical model of the problems appears to be the same. Thus, in fact, the slackened systems made of elastic-plastic material can be treated as locking-elastic-plastic ones. A pioneering work in this domain was done by L. CORRADI and G. MAIER [2], where elastic-locking structures were considered. Note that the locking of material was not interpreted there as a clearance effect. Further research has been focused on a more general class of the so-called "conditional joints" in works of S. KALISZKY [3] and M. KURUTZ [4–6]. The problem of slackened systems belongs to mechanics of systems with unilateral constraints. In the last decade many papers in this domain have appeared. Of a particular significance are the papers of G. DUVAUT and J.L. LIONS [7] and P.D. PANAGIOTOPOULOS [9–11], where unilateral constraint effects have been considered. All these papers are of mathematical character. The influence of mathematics has been expressed in a new, more general, terminology of the subject, by introducing the so-called "subdifferential connections" where, among others, plasticity and contact effects are taken into account.

In the present paper the major theoretical results for the slackened systems obtained by the author in the last years are summarized and discussed. All the considerations are carried out according to the following fundamental assumptions:

• the system is assembled of deformable (linear elastic-plastic) structural elements and of indestructible and undeformable connecting elements (joints) of very small dimensions

• the relative motion (due to the presence of gaps at the connections) between the structural element and connecting element is permitted,

• displacements and clearances are so small that the application of the geometrically linear theory is justified,

• all friction effects are completely neglected,

• only quasi-static processes are considered,

• the "ideal" structure, i.e. the reference structure without gaps, is geometrically stable.

The assumption on the frictionless motion requires some additional comments. It is obvious that a mechanical model, in which the friction forces are taken into consideration, more precisely describes the behaviour of real deformable systems. However, the formulation of such a model is much more complicated. The problems of the existence and uniqueness of solutions in this case appear to be very difficult, and only quite recently some essentially important results in this domain have been obtained by mathematicians, cf., for example, J.J. TELEGA [12]. The solutions of contact problems with friction are not unique in general, also in the case of Coulomb's friction. The uniqueness is assured if the coefficient of friction is sufficiently small. The non-uniqueness of solutions results from the fact that the friction law is not associated with the friction condition. In the present paper it will be shown that, even in the simplest case of frictionless deformations, the mechanics of slackened systems provides a sufficiently large number of new, non-trivial problems.

The fundamental assumptions specified above, together with some additional simplifications consisting in a linear approximation of constraints imposed on the plastic and clearance strains, lead to problems of convex analysis which can be solved by means of linear and quadratic programming methods. The usefulness of these methods will be widely illustrated in the present paper.

2. Specific features of slackened systems

2.1. Ideal configuration

One should be aware of the fact that the presence of clearances at connections can induce a geometric instability of the system. Therefore, it is necessary to establish an "ideal configuration" chosen from all the kinematically admissible configurations of the unloaded slackened system. The displacements of connecting elements will be related to the ideal configuration. Otherwise, the kinematical quantities would be not uniquely determined. This problem will be also considered in Sec. 4.6.

2.2. Mathematical description of slackened connections

Consider in detail a model of connection with clearances shown in Fig. 1b as a part of a plane slackened system of Fig. 1a. Two bars are joined by five bolts attached to the connecting element (connecting plate). Due to the presence of gaps between the bolts (treated here as points) and the corresponding holes drilled in the end (rigid) parts of structural elements, a relative constrained motion of the bar element and connecting plate can occur. The relative displacements play here a role of concentrated "generalized clearance strains" that appear within the "clearance region". This region is bounded by the so-called "clearance surface" corresponding to Prager's locking surface.

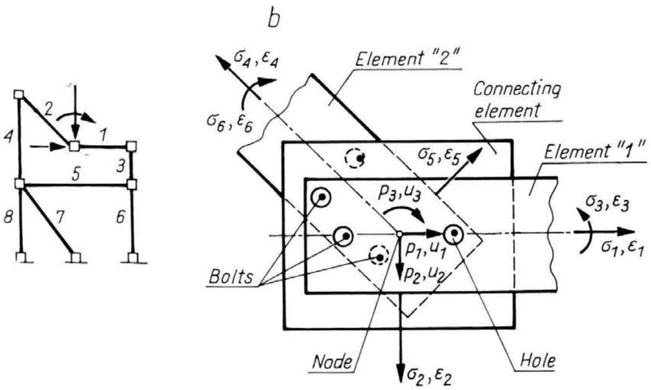

FIG. 1. Model of system with slackened connection; a) plane slackened system, b) slackened connection.

Thus, as in the theory of plastic structures, an idea of a "generalized clearance hinge" can be introduced. Note that the clearance surface can be constructed for any connection with clearances which can occur in real structures. As an example, Fig. 2a illustrates the clearance surface for the clearance hinge of element "1" (cf. Fig. 1b and [23]).

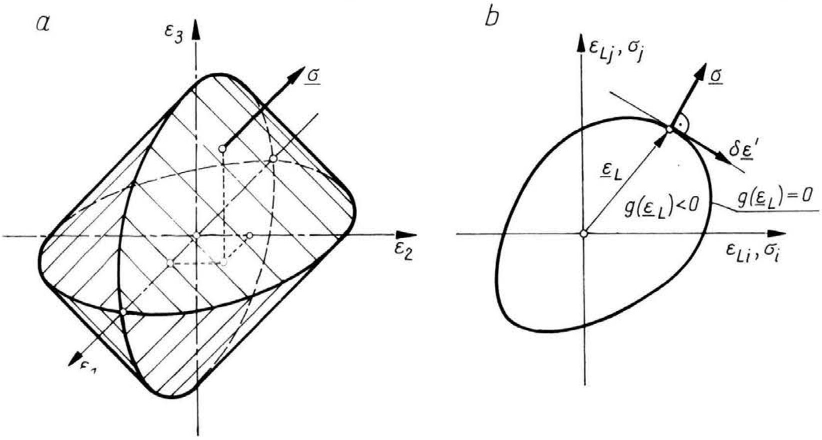

FIG. 2. Clearance surface and normality law; a) clearance surface for hinge at element 1, b) geometric interpretation of normality law.

In order to derive the fundamental relations for slackened connections, consider the equilibrium of a system composed of the connecting plate and element 1 (Fig. 3a). The system is loaded by generalized stress components σ_1, σ_2 and σ_3, denoting the normal force, shear force and bending moment in the bar element 1, respectively. The generalized stresses are in equilibrium with contact forces $\mathbf{S}^{(j)}$ both in element 1 and in the connecting plate (see Fig. 3b,c). The contact forces $\mathbf{S}^{(j)}$ act at particular points of the hole boundary of element 1 (j denotes here the number of the corresponding bolt). Since no friction forces are assumed, the contact forces have to be perpendicular to the hole boundaries.

Consider now a virtual motion of the connecting plate, assuming that no loss of existing contacts occurs (Fig. 3c).

The virtual motion is described by the generalized clearance strain variations: $\delta\varepsilon_1'$, $\delta\varepsilon_2'$, $\delta\varepsilon_3'$ associated with variations of displacement of particular bolts, $\delta\mathbf{v}^{(j)}$. Note that

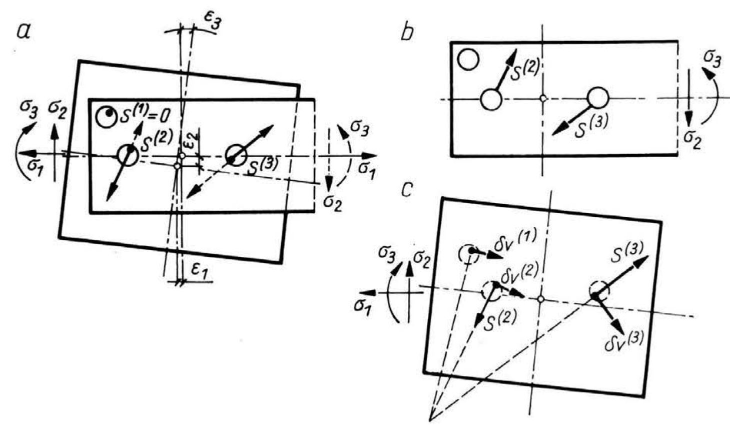

FIG. 3. Connecting plate-structural element system; a) equilibrium of "connecting plate-element" system, b) equilibrium of structural element, c) equilibrium and virtual motion of connecting plate.

$\delta \mathbf{v}^{(j)}$ results from geometric compatibility equations for the rigid body (i.e. connecting plate) motion. Thus, $\delta\varepsilon_i'$ and $\delta\mathbf{v}^{(j)}$ represent the kinematically admissible system. Since $\mathbf{S}^{(j)}$ and σ_i are statically admissible, one can use the virtual work equation for rigid body:

$$\sum_j \mathbf{S}^{(j)}\delta\mathbf{v}^{(j)} + \sum_{i=1}^{a} \sigma_i \delta\varepsilon_i' = 0. \tag{2.1}$$

On the other hand $\mathbf{S}^{(j)}\delta\mathbf{v}^{(j)} = 0$, because the non-vanishing contact forces $\mathbf{S}^{(j)}$ are always orthogonal to $\delta\mathbf{v}^{(j)}$. Thus

$$\boldsymbol{\sigma}^T\delta\boldsymbol{\varepsilon}' = 0, \tag{2.2}$$

where

$$\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3]^T, \qquad \delta\boldsymbol{\varepsilon}' = [\delta\varepsilon_1', \delta\varepsilon_2', \delta\varepsilon_3']^T$$

and superscript T denotes the operator of matrix transposition.

It is clearly seen that $\delta\varepsilon_i'$ can be interpreted as components of a vector tangent to the clearance surface. This situation is explained in Fig. 2b. Thus, we can conclude that the generalized stress vector $\boldsymbol{\sigma}$ is orthogonal to the clearance surface and, therefore, the following relation is valid:

$$\boldsymbol{\sigma} = \psi \cdot (\partial g/\partial \boldsymbol{\varepsilon}_L). \tag{2.3}$$

In Eq. (2.3) $(\partial g/\partial\boldsymbol{\varepsilon}_L)$ is the clearance surface gradient vector, ε_L denotes the vector of clearance strain, whereas $g(\boldsymbol{\varepsilon}_L) = 0$ is the equation of the clearance surface. The clearance region is described by

$$g(\boldsymbol{\varepsilon}_L) \le 0, \tag{2.4}$$

and ψ is a non-negative (due to unilateral constraints) stress multiplier that satisfies the relations

$$\begin{aligned} &\psi \ge 0 \quad \text{if} \quad g = 0 \quad \text{and} \quad \dot{g} = 0, \\ &\psi = 0 \quad \text{if} \quad g < 0, \quad \text{and also if} \quad g = 0 \quad \text{and} \quad \dot{g} < 0, \end{aligned} \tag{2.5}$$

where the dot denotes the differentiation with respect to time.

Relations (2.5) are equivalent to the conditions

$$\psi \cdot g = 0, \qquad \psi \cdot \dot{g} = 0. \tag{2.6}$$

If the virtual motion requires the loss of the existing contacts, $\mathbf{S}^{(j)}\delta\mathbf{v}^{(j)} \geq 0$, and then, according to Eq. (2.1), $\sigma^T \delta\varepsilon' \leq 0$. This case corresponds to the corners of the clearance surface.

Non-vanishing generalized stresses can occur only if the corresponding generalized clearance strain point lies on the clearance surface. The uniqueness of the clearance strain state is guaranteed if the clearance region is strictly convex. Only in this case a given stress vector uniquely determines the clearance strain vector. Convexity of the clearance region essentially depends on the shapes and dimensions of the element holes. The non-convexity of clearance region appears, for example, in the particular case of one bolt and one hole of a non-convex shape.

The problem of uniqueness of the clearance strain, discussed here, is identical with that of stress uniqueness in the theory of plasticity. It is known that the stress state uniqueness for a given strain rate field occurs only if the yield condition is strictly convex. If the yield condition is weakly convex (described by linear inequalities) and the strain rate vector is orthogonal to a flat portion of the limit surface, the stress state can not be uniquely determined. In the case of non-convex clearance region the clearance strains can be non-uniquely determined, in general. Furthermore, the non-convexity of clearance region leads to a much more complicated problem of non-convex analysis.

It should be pointed out that the model of the slackened connection considered above has been used only in order to explain the mechanical meaning of the slackening of the system. The FEM-oriented formulation presented in the work permits us to describe more complicated systems, e.g. plates, shells and continuum systems. It requires, however, a suitable discretization of the system and methods for the construction of clearance regions.

3. Mathematical model of elastic-plastic-slackened systems

3.1. Fundamental relations and their physical interpretation

According to the assumption specified in Sec. 1., a given system consists of deformable (elastic-plastic) structural elements and ideal rigid connecting elements of very small dimensions. In the interior of each connecting element a certain point called "node" is distinguished, and the external load can be applied only at the nodes. In order to construct the mathematical model of the slackened systems we recall the well-known matrix description used by G. Maier and his co-workers (cf. [13–15]), widely applied in contemporary mechanics. The complete system of relations which describes the mathematical model of elastic-plastic-slackened systems can be presented as follows:

GENERAL RELATIONS

(3.1)

1) $\mathbf{Cu} - \boldsymbol{\varepsilon} = \mathbf{0}$ geometric compatibility,

2) $\mathbf{C}^T\boldsymbol{\sigma} - \mathbf{p} = \mathbf{0}$ equilibrium.

STRAIN DECOMPOSITION

3) $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_L + \boldsymbol{\varepsilon}_E + \boldsymbol{\varepsilon}_P.$

LINEAR ELASTICITY

(3.1) [cont.] 4) $\boldsymbol{\varepsilon}_E = \mathbf{E}^{-1}\boldsymbol{\sigma}$.

PLASTICITY

5) $\mathbf{f} = \mathbf{N}^T\boldsymbol{\sigma} - \mathbf{H}\boldsymbol{\lambda} - \mathbf{k} \leq \mathbf{0}$	elastic region,
6) $\dot{\boldsymbol{\varepsilon}}_P = \mathbf{N}\dot{\boldsymbol{\lambda}}$	normality law,
7) $\dot{\boldsymbol{\lambda}} \geq \mathbf{0}$	nonnegativity requirement,
8) $\dot{\boldsymbol{\lambda}}^T\mathbf{f} = 0$	orthogonality condition,
9) $\dot{\boldsymbol{\lambda}}^T\dot{\mathbf{f}} = 0$	orthogonality condition.

SLACKENING

10) $\mathbf{g} = \mathbf{M}^T\boldsymbol{\varepsilon}_L - \mathbf{l} \leq \mathbf{0}$	clearance region,
11) $\boldsymbol{\sigma} = \mathbf{M}\boldsymbol{\psi}$	normality law,
12) $\boldsymbol{\psi} \geq \mathbf{0}$	nonnegativity requirement,
13) $\boldsymbol{\psi}^T\mathbf{g} = 0$	orthogonality condition,
14) $\boldsymbol{\psi}^T\dot{\mathbf{g}} = 0$	orthogonality condition.

In Eqs. (3.1) $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are supervectors that collect all the generalized stresses and strains, respectively, of all the structural elements; **p** and **u** denote the respective supervectors of generalized loads and generalized displacements of all the connecting elements; **C** is the geometric compatibility matrix that depends only on geometry and boundary conditions of all the structural elements.

Equations $(3.1)_1$ and $(3.1)_2$ are valid for any structures or solids that deform according to the geometrically linear theory. In particular, they hold also true for ideally rigid systems. It is important to mention that the geometric compatibility matrix **C** refers to the ideal structure without clearances, and the assumption of geometric stability of the ideal structure corresponds to the requirement

$$\det[\mathbf{C}^T\mathbf{C}] \neq 0. \tag{3.2}$$

Since the problem is considered within the framework of small deformations, the total generalized strain ε can be assumed as the sum of elastic ε_E, clearance $\boldsymbol{\varepsilon}_L$ and plastic $\boldsymbol{\varepsilon}_P$ components (see $(3.1)_3$). The clearance and plastic components represent here the concentrated strains at the generalized clearance and plastic hinges.

Matrix equation $(3.1)_4$ expresses the essence of generalized Hooke's law, where **E** is a strictly positive definite square and symmetric matrix of elasticity. So, the common elastic structures or solids are completely described by matrix equations $(3.1)_1$ – $(3.1)_4$ assuming $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_E$ and $\boldsymbol{\varepsilon}_L \equiv \boldsymbol{\varepsilon}_P \equiv \mathbf{0}$.

The next five matrix formulae $(3.1)_5$ – $(3.1)_9$ represent additional relations necessary to account for the effects of plastic deformations. The yield condition is assumed to be piecewise-linear in the form of a set of linear inequalities $(3.1)_5$, where **N** is a rectangular matrix that collects all the external normals of the yield hyperpolyhedron of all the structural elements, **H** is a square matrix of linear plastic hardening, $\dot{\boldsymbol{\lambda}}$ represents a vector of plastic strain multiplier rates and components of **k** denote the distances from the respective sides to the origin. The normality rule for the generalized plastic strain rates is described by $(3.1)_6$. Algebraic equations $(3.1)_8$ and $(3.1)_9$ represent the so-called orthogonality conditions.

Relations $(3.1)_1 - (3.1)_9$ for $\boldsymbol{\varepsilon}_L \equiv \mathbf{0}$ correspond to the well-known formulation of linear elastic-plastic solids or structures, (G. MAIER, [14]).

The mathematical model of elastic-plastic-slackened systems is supplemented by five matrix relations: $(3.1)_{10} - (3.1)_{14}$. It is assumed that the clearance surface can be approximated by a convex hyperpolyhedron, and the clearance region is described by Eq. $(3.1)_{10}$, where $\mathbf{M}$ is a rectangular matrix that collects all the external normals of the clearance hyperpolyhedron of all the elements, and $\mathbf{l}$ contains the distances from the individual sides of the clearance hyperpolyhedron to the origin. The normality rule for the generalized stresses is given by $(3.1)_{11}$, where $\boldsymbol{\psi}$ is a supervector of the generalized stress multipliers. Orthogonality conditions $(3.1)_{13}$ and $(3.1)_{14}$ correspond to those previously derived for the individual structural element, (2.6).

Mathematical model (3.1) describes a relatively wide class of materials and structures behaviour. In particular, relations $(3.1)_1 - (3.1)_2$ and $(3.1)_{10} - (3.1)_{14}$ for $\boldsymbol{\varepsilon} \equiv \varepsilon_L$, describe the problem of ideal locking systems. All the elastic systems with unilateral displacement boundary conditions are described by $(3.1)_1 - (3.1)_4$ and $(3.1)_{10} - (3.1)_{14}$ assuming that $\boldsymbol{\varepsilon}_P \equiv \mathbf{0}$. The formulation of rigid-plastic body consists of relations $(3.1)_1$, $(3.2)_2$ and $(3.1)_5 - (3.1)_9$ together with $\varepsilon_E \equiv \boldsymbol{\varepsilon}_L \equiv \mathbf{0}$. Rigid-plastic-slackened systems are expressed by $(3.1)_1 - (3.1)_3$ and $(3.1)_{10} - (3.1)_{14}$, where $\boldsymbol{\varepsilon}_E \equiv \mathbf{0}$.

Important relations for the whole structure can be obtained directly from the orthogonality conditions. Condition $(3.1)_8$ connected with the presence of plastic deformations gives

$$\dot{\boldsymbol{\lambda}}^T\mathbf{f} = \dot{\boldsymbol{\lambda}}^T[\mathbf{N}^T\boldsymbol{\sigma} - \mathbf{H}\boldsymbol{\lambda} - \mathbf{k}] = \boldsymbol{\sigma}^T\dot{\boldsymbol{\varepsilon}}_P - \dot{\boldsymbol{\lambda}}^T\mathbf{H}\boldsymbol{\lambda} - \dot{\boldsymbol{\lambda}}^T\mathbf{k} = 0,$$

(3.3) or

$$\boldsymbol{\sigma}^T\dot{\boldsymbol{\varepsilon}}_P = \dot{\boldsymbol{\lambda}}^T\mathbf{H}\boldsymbol{\lambda} + \dot{\boldsymbol{\lambda}}^T\mathbf{k}.$$

W. PRAGER [16] introduced the kinematic hardening concept that corresponds to the following definition of matrix $\mathbf{H}$:

$$\mathbf{H} = h\mathbf{N}^T\mathbf{N}, \tag{3.4}$$

where h is a positive scalar multiplier of plastic hardening. Assuming that $\boldsymbol{\lambda}(0) = \mathbf{0}$, the plastic strain vector can be expressed as

$$\boldsymbol{\varepsilon}_P = \int\limits_0^t \dot{\boldsymbol{\varepsilon}}_P(t')\,dt' = \mathbf{N}\int\limits_0^t \dot{\boldsymbol{\lambda}}(t')\,dt' = \mathbf{N}\boldsymbol{\lambda}, \qquad \boldsymbol{\lambda} = \boldsymbol{\lambda}(t).$$

Using definition (3.4) and Eq. $(3.1)_6$ we obtain

$$D_P = \boldsymbol{\sigma}^T\dot{\boldsymbol{\varepsilon}}_P = \dot{\boldsymbol{\lambda}}^T\mathbf{k} + h\dot{\boldsymbol{\varepsilon}}_P^T\boldsymbol{\varepsilon}_P. \tag{3.5}$$

The plastic dissipation is usually non-negatively defined, excluding incidentally large plastic deformations when $h\dot{\boldsymbol{\varepsilon}}_P^T\boldsymbol{\varepsilon}_P < -\lambda^T\mathbf{k} < 0$. In the case of ideal plasticity ($\mathbf{H} = \mathbf{0}$) Eq. (3.5) becomes

$$D_P = \sigma^T\dot{\boldsymbol{\varepsilon}}_P = \dot{\boldsymbol{\lambda}}^T\mathbf{k} \geq 0. \tag{3.6}$$

The inequality sign results from the fact that $\mathbf{k}$ is always positive definite since the state of $\sigma = \mathbf{0}$ does not violate the yield condition.

The second orthogonality condition applied to plastic deformations, Eq. $(3.1)_9$, expresses the assumption of stable behaviour of the structure material, namely

$$\dot{\boldsymbol{\lambda}}^T \dot{\mathbf{f}} = \dot{\boldsymbol{\lambda}}^T \mathbf{N}^T \dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\lambda}}^T \mathbf{H} \dot{\boldsymbol{\lambda}} = 0, \tag{3.7}$$

hence

$$\dot{\sigma}^T \dot{\boldsymbol{\varepsilon}}_P = \dot{\boldsymbol{\lambda}}^T \mathbf{H} \dot{\boldsymbol{\lambda}}.$$

If **H** is positive definite, the material behaves as a stable one. Note that W. Prager's plastic hardening rule leads to the stable behaviour of the material

$$\dot{\boldsymbol{\sigma}}^T \dot{\boldsymbol{\varepsilon}}_P = h \dot{\boldsymbol{\varepsilon}}_P^T \dot{\boldsymbol{\varepsilon}}_P > 0, \quad \text{if} \quad \dot{\boldsymbol{\varepsilon}}_P \neq \mathbf{0}. \tag{3.8}$$

In the case of slackened systems Eq. $(3.1)_{13}$ is equivalent to the following

$$\boldsymbol{\psi}^T \mathbf{g} = \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}_L - \boldsymbol{\psi}^T \mathbf{l} = 0, \tag{3.9}$$

hence

$$W_L = \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}_L = \boldsymbol{\psi}^T \mathbf{l}, \tag{3.10}$$

where W_L represents the non-negative definite "clearance work", if $\mathbf{l} \geq \mathbf{0}$.

After differentiation of Eq. (3.9) with respect to time, and using Eq. $(3.1)_{14}$, we obtain $\dot{\boldsymbol{\psi}}^T \mathbf{g} = \boldsymbol{\psi}^T \dot{\mathbf{g}} = 0$. Hence

$$D_L = \boldsymbol{\sigma}^T \dot{\boldsymbol{\varepsilon}}_L = 0, \tag{3.11}$$

and

$$\dot{\boldsymbol{\sigma}}^T \boldsymbol{\varepsilon}_L = \dot{\boldsymbol{\psi}}^T \mathbf{l}. \tag{3.12}$$

In view of Eq. (3.11) we can conclude that the "clearance dissipation" always is equal to zero.

Finally, it should be mentioned that fundamental relations (3.1) enable us to describe arbitrary boundary conditions for particular structural elements. Any change of the type of structure affecting the degree of statical indeterminacy can be realized in the natural way by means of a proper choice of the components of the clearance modulae vector **l**. In particular, the case of $\mathbf{l} = \mathbf{0}$ corresponds to the ideal structure with bilateral constraints at all connections.

3.2. General properties of the model

In order to recognize the general properties of the mathematical model we can use the linear equations which describe the elastic structure with imposed strains (distortions). The problem can be formulated by four matrix equations $(3.1)_1$ – $(3.1)_4$. Starting from Eq. $(3.1)_2$ and using Eqs. $(3.1)_4$ and $(3.1)_1$, we have

$$\mathbf{p} = \mathbf{K}\mathbf{u} - \mathbf{C}^T \mathbf{E} \boldsymbol{\varepsilon}_D, \quad \boldsymbol{\varepsilon}_D = \boldsymbol{\varepsilon}_L + \boldsymbol{\varepsilon}_P, \quad \mathbf{K} = \mathbf{C}^T \mathbf{E} \mathbf{C}, \tag{3.13}$$

where $\boldsymbol{\varepsilon}_D$ denotes the given distortion vector and **K** is the common stiffness matrix for linear elastic body without distortions. **K** is a strictly positive definite, square, symmetric matrix which includes the boundary conditions. From Eqs. (3.13) one immediately obtains

$$\mathbf{u} = \mathbf{K}^{-1}\mathbf{p} + \mathbf{K}^{-1}\mathbf{p}_d = \mathbf{u}_e + \mathbf{u}_d. \tag{3.14}$$

In Eq. (3.14) $\mathbf{u}_e$ denotes the vector of generalized displacements for the purely elastic body subjected to the external load **p**, and $\mathbf{u}_d$ includes the influence of the presence of distortions. The additional term $\mathbf{p}_d$ represents a fictitious "distortion" load given by

$$\mathbf{p}_d = \mathbf{C}^T \mathbf{E} \boldsymbol{\varepsilon}_D. \tag{3.15}$$

Making use of Eqs. (3.13), $(3.1)_3$ and $(3.1)_4$ we arrive at

$$\boldsymbol{\sigma} = \mathbf{ECK}^{-1}\mathbf{p} + \mathbf{Z}\boldsymbol{\varepsilon}_D = \boldsymbol{\sigma}_e + \boldsymbol{\sigma}_d. \tag{3.16}$$

In Eq. (3.16) $\boldsymbol{\sigma}_e$ and $\boldsymbol{\sigma}_d$ are stress vectors in purely elastic system and a corrective stress vector due to the presence of distortions, respectively. $\mathbf{Z}$ denotes a singular, symmetric, square and semi-negative definite matrix of distortion influence given by (cf. G. MAIER [14] and A. BORKOWSKI [17])

$$\mathbf{Z} = \mathbf{ECK}^{-1}\mathbf{C}^T\mathbf{E} - \mathbf{E}, \tag{3.17}$$

with

$$\mathbf{ZC} \equiv \mathbf{0} \quad \text{and} \quad \mathbf{C}^T\mathbf{Z} \equiv \mathbf{0}. \tag{3.18}$$

Relation $(3.18)_1$ expresses the fact that any compatible distortions (i.e. $\boldsymbol{\varepsilon}_D = \mathbf{Cu}_d$) do not induce additional stresses (i.e. $\boldsymbol{\sigma}_d = \mathbf{0}$). On the other hand, from Eq. $(3.18)_2$ it follows that non-vanishing $\boldsymbol{\sigma}_d$ is in equilibrium with a zero-value external load, namely

$$\mathbf{C}^T\boldsymbol{\sigma}_d = \mathbf{0}. \tag{3.19}$$

Further properties of the mathematical model connected with effects of plasticity and slackening will be considered in next sections.

4. Problems of analysis

4.1. Problem of the original structure

The most important requirement in the incremental analysis is to know the initial (unloaded) state of the system. Suppose that a given system is geometrically unstable due to a slackening. Now, a fundamental non-trivial problem arises: "find a non-zero stiffness structure that can carry prescribed external loads $\mathbf{p}_0$". The solution of this problem corresponds to the conversion of the mechanism into the so-called "original structure", depending on the given load vector $\mathbf{p}_0$, cf. A. GAWĘCKI [18]. The original structure is completely determined by the generalized displacement vector $\mathbf{u}_0$ that describes positions of all the connecting elements. Usually, the original structure is statically determinate and does not depend on physical properties of the material.

The complete system of relations describing the original structure problem consists of Eqs. $(3.1)_1$, $(3.1)_2$, $(3.1)_{10}$ –$(3.1)_{13}$ and can be presented in the form

$$\begin{array}{llll} & 1)\ \ \mathbf{Cu}_0 = \boldsymbol{\varepsilon}_{L0}, & & 4)\ \ \mathbf{g}_0 = \mathbf{M}^T\boldsymbol{\varepsilon}_{L0} - \mathbf{l} \le \mathbf{0}, \\ (4.1) & 2)\ \ \mathbf{C}^T\boldsymbol{\sigma}_0 = \mathbf{p}_0, & & 5)\ \ \boldsymbol{\psi}_0 \ge \mathbf{0}, \\ & 3)\ \ \boldsymbol{\sigma}_0 = \mathbf{M}\boldsymbol{\psi}_0, & & 6)\ \ \boldsymbol{\psi}_0^T\mathbf{g} = 0. \end{array}$$

Treating these relations as Kuhn–Tucker's conditions we can formulate the suitable dual extremum principles in the framework of linear programming,

$$\begin{array}{llll} & [F' = \mathbf{u}_0^T\mathbf{p}_0] & \Rightarrow & \max \ \ |\mathbf{M}^T\mathbf{Cu}_0 - \mathbf{l} \le \mathbf{0}, \\ (4.2) & & & \\ & [F'' = \boldsymbol{\psi}_0^T\mathbf{l}] & \Rightarrow & \min \ \ |\mathbf{C}^T\mathbf{M}\boldsymbol{\psi}_0 - \mathbf{p}_0 = 0, \quad \boldsymbol{\psi}_0 \ge \mathbf{0}. \end{array}$$

One has to be aware of the following possibilities which can be met when a linear programming method is applied:

a) the solution is unique and corresponds to the finite value of the objective function,

b) the solution is non-unique but the corresponding magnitude of the objective function is unique and finite,

c) the form of constraints does not allow to reach any finite value of the objective function,

d) the solution does not exist due to the constraint contradiction.

Cases c) and d) cannot occur, but appearance of the case b) is quite possible. The non-uniqueness of the solution may appear if a certain contour-line of the objective function coincides with the constraint. It corresponds to the displacement non-uniqueness in the primary problem. In this case F' remains constant, whereas the difference between solutions $\Delta\mathbf{u}_0$ is orthogonal to $\mathbf{p}_0$. Such a situation is observed when the clearance strain vector corresponds to flat parts of the clearance surface. Another possibility arises if, at particular nodes, p_{0i} are equal to zero. Then, for any kinematically admissible u_{0i}, the product $p_{0i}u_{0i}$ vanishes and does not affect the value of the objective function F'. The non-uniqueness of the stress multiplier vector $\boldsymbol{\psi}_0$ can be noted in the dual problem. There are two possibilities. The first one takes place if contact appears simultaneously at least in two different elements, and then the original structure is statically indeterminate. The second possibility can occur if the point of the clearance strain state lies at the vertex of the clearance hyperpolyhedron where the number of the sides is larger than the dimension of the stress space. Then the stress state is unique but it can be expressed by different vector $\boldsymbol{\psi}_0$.

After solving the original structure problem one can divide $\mathbf{g}_0$, $\boldsymbol{\psi}_0$ and $\mathbf{M}$ into the active and passive parts:

$$
\begin{aligned}
&\mathbf{g}_0 = [\mathbf{g}_a, \mathbf{g}_p]^T: \qquad \mathbf{g}_a = \mathbf{0}, \qquad \mathbf{g}_p < 0, \\
&\boldsymbol{\psi}_0 = [\boldsymbol{\psi}_a, \boldsymbol{\psi}_p]^T, \qquad \mathbf{M} = [\mathbf{M}_a, \mathbf{M}_p].
\end{aligned} \tag{4.3}
$$

This observation will be utilized in further considerations.

EXAMPLE I

In order to illustrate the original structure problem and the stress state non-uniqueness, consider a slackened truss with longitudinal gaps at the end parts of the elements (Fig. 4a).

The limit values of clearance strains are:

$$\mathbf{L} = [l_1^-,\ l_1^+,\ l_2^-, \ldots, l_6^-,\ l_6^+]^T = [1,\ 2,\ 1,\ 0,\ 1,\ 0.5,\ 2,\ 1,\ 2,\ 2,\ 3.4,\ 2]^T \ \text{[mm]}.$$

The reference load vector is given by

$$\mathbf{p}_0 = [p_1,\ p_2,\ p_3,\ p_4]^T = [1,\ 3,\ 1,\ -1]^T \ \text{[kN]}.$$

The solution of the primary problem is unique:

$$\mathbf{u}_0 = [u_1,\ u_2,\ u_3,\ u_4]^T = [-1,\ 1.5,\ 2,\ -2.8]^T \ \text{[mm]}.$$

The solution of the dual problem is not unique:

$$
\begin{aligned}
\boldsymbol{\sigma}_0' &= [-1.25,\ -3.75,\ 0, -1.25,\ -1.75,\ 0]^T \ \text{[kN]},\\
\boldsymbol{\sigma}_0'' &= [-1.25,\ -3.75,\ 0,\ 0,\ \ -0.25,\ 1.25]^T \ \text{[kN]},
\end{aligned}
$$

where

$$\boldsymbol{\sigma}_0 = [S_1,\ S_2,\ S_3,\ S_4,\ S_5,\ S_6]^T,$$

and the corresponding original structures are shown in Fig. 4a and Fig. 4b, respectively. The dashed lines indicate the elements of zero-value normal forces.

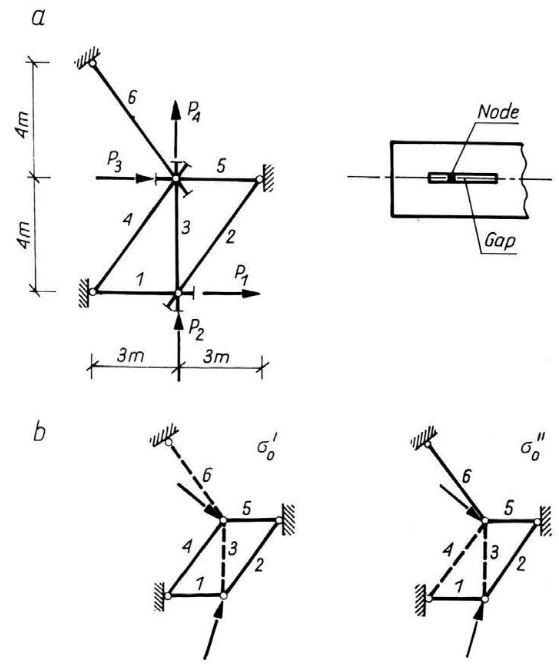

FIG. 4. Original structure problem for a slackened truss; a) geometry and load of the truss with longitudinal gaps, b) non-unique stress states: σ_0', σ_0''.

4.2. Elastic-slackened systems

The mathematical model for elastic-slackened systems consists of relations $(3.1)_1$–$(3.1)_4$ and $(3.1)_{10}$–$(3.1)_{14}$, where $\boldsymbol{\varepsilon}_P \equiv \mathbf{0}$.

THEOREM 1. *If the clearance region is convex, then the stress* $\boldsymbol{\sigma}$ *and the strain* $\boldsymbol{\varepsilon}$ *are unique for a given displacement vector* $\mathbf{u}$.

Proof

If the displacement vector $\mathbf{u}$ is known, the total strains $\boldsymbol{\varepsilon}$ can be determined directly from geometric equations $(3.1)_2$, and on the basis of Hooke's law, one obtains

$$\boldsymbol{\sigma} = \mathbf{E}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_L). \tag{4.4}$$

Assume now that there are two different stress states $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, associated with clearance strains $\boldsymbol{\varepsilon}_{L_1}$ and $\boldsymbol{\varepsilon}_{L_2}$, respectively. Then the difference between $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ can be expressed as

$$\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 = -\mathbf{E}(\boldsymbol{\varepsilon}_{L_1} - \boldsymbol{\varepsilon}_{L_2}). \tag{4.5}$$

The convexity of the clearance region yields

$$(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^T(\boldsymbol{\varepsilon}_{L_1} - \boldsymbol{\varepsilon}_{L_2}) \geq 0. \tag{4.6}$$

After substituting Eq. (4.5) into Eq. (4.6) we obtain

$$(\boldsymbol{\varepsilon}_{L_1} - \boldsymbol{\varepsilon}_{L_2})^T\mathbf{E}(\boldsymbol{\varepsilon}_{L_1} - \boldsymbol{\varepsilon}_{L_2}) \leq 0. \tag{4.7}$$

Since $\mathbf{E}$ is strictly positive definite, the clearance strains are the same, i.e. $\boldsymbol{\varepsilon}_{L_1} = \boldsymbol{\varepsilon}_{L_2}$.

Thus, the elastic strain state as well as the stress state are also unique, i.e. $\varepsilon_{E_1} = \varepsilon_{E_2}$ and $\sigma_1 = \sigma_2$.

THEOREM 2. *For a given external load vector* $\mathbf{p}$, *the stress state* σ *is uniquely determined if the clearance region is convex. The uniqueness of displacements* $\mathbf{u}$ *and clearance strains* ε_L *is guaranteed if the clearance region is strictly convex.*

Proof

Assume that $\mathbf{p}$ is in equilibrium with two different stress states σ_1 and σ_2. The convexity of clearance region yields (cf. Eq. (4.6)):

$$(\sigma_1 - \sigma_2)^T(\varepsilon_{L_1} - \varepsilon_{L_2}) \geq 0.$$

On the other hand

$$(\sigma_1 - \sigma_2)^T(\varepsilon_{L_1} - \varepsilon_{L_2}) = (\varepsilon_{L_1} - \varepsilon_{L_2})^T\mathbf{Z}(\varepsilon_{L_1} - \varepsilon_{L_2}) \leq 0, \tag{4.8}$$

because $\mathbf{Z}$ is semi-negative definite. Thus

$$(\sigma_1 - \sigma_2)^T(\varepsilon_{L_1} - \varepsilon_{L_2}) = 0. \tag{4.9}$$

Using Eqs. $(3.1)_1$, $(3.1)_3$ and (3.18), Eq. (4.9) may be rewritten in the form

$$\begin{aligned}(\sigma_1 - \sigma_2)^T(\varepsilon_{L_1} - \varepsilon_{L_2}) &= (\sigma_1 - \sigma_2)^T(\mathbf{C}\mathbf{u}_1 - \varepsilon_{E_1} - \mathbf{C}\mathbf{u}_2 + \varepsilon_{E_2}) \\ &\quad -(\sigma_1 - \sigma_2)^T(\varepsilon_{E_1} - \varepsilon_{E_2}) = -(\varepsilon_{E_1} - \varepsilon_{E_2})^T\mathbf{E}(\varepsilon_{E_1} - \varepsilon_{E_2}) = 0,\end{aligned} \tag{4.10}$$

hence $\varepsilon_{E_1} = \varepsilon_{E_2}$ and the stress state σ appears to be unique.

If the clearance region is strictly convex, then the uniqueness of a stress state implies the uniqueness of clearance strains and displacements. This statement is valid for the nodes with non-zero external loads. Otherwise, kinematically admissible clearance strains and displacements can occur.

If the clearance region is weakly convex — as we have assumed in the model proposed herein — the non-uniqueness of clearance strains can occur. In this case a difference between two clearance strain states corresponds to a certain displacement vector $\Delta\mathbf{u}$ that is kinematically admissible,

$$\Delta\varepsilon_L = \varepsilon_{L_1} - \varepsilon_{L_2} = \mathbf{C}\Delta\mathbf{u}. \tag{4.11}$$

Such a situation is illustrated in Fig. 5.

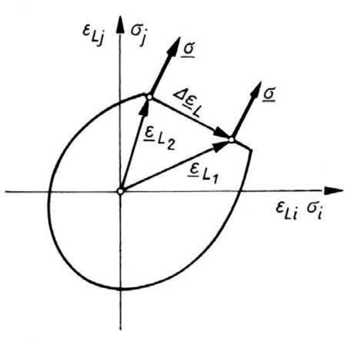

FIG. 5. Geometric interpretation of non-uniqueness of clearance strain.

As was mentioned, the uniqueness of the stress state does not lead to the uniqueness of stress multipliers $\boldsymbol{\psi}$. For two different values of the clearance strains the following relations hold

$$\begin{aligned} \boldsymbol{\psi}_1^T \mathbf{g}_1 &= 0, \quad \boldsymbol{\psi}_2^T \mathbf{g}_2 = 0, \\ \boldsymbol{\psi}_1^T \mathbf{g}_2 &\leq 0, \quad \boldsymbol{\psi}_2^T \mathbf{g}_1 \leq 0. \end{aligned} \tag{4.12}$$

The inequality signs result from the fact that $\boldsymbol{\psi}_i \geq \mathbf{0}$ and $\mathbf{g}_i \leq \mathbf{0}$ ($i = 1, 2$). Using Eqs. $(3.1)_{10}$ and $(3.1)_{11}$ we obtain

$$\begin{aligned} \boldsymbol{\psi}_1^T (\mathbf{g}_1 - \mathbf{g}_2) &= \boldsymbol{\sigma}^T (\boldsymbol{\varepsilon}_{L_1} - \boldsymbol{\varepsilon}_{L_2}) \geq 0, \\ \boldsymbol{\psi}_2^T (\mathbf{g}_1 - \mathbf{g}_2) &= \boldsymbol{\sigma}^T (\boldsymbol{\varepsilon}_{L_1} - \boldsymbol{\varepsilon}_{L_2}) \leq 0. \end{aligned} \tag{4.13}$$

Hence

$$\boldsymbol{\sigma}^T (\boldsymbol{\varepsilon}_{L_1} - \boldsymbol{\varepsilon}_{L_2}) = 0, \tag{4.14}$$

and

$$\mathbf{p}^T (\mathbf{u}_1 - \mathbf{u}_2) = 0. \tag{4.15}$$

Equation (4.14) can be rewritten in the form

$$\boldsymbol{\psi}_1^T (\mathbf{g}_1 + \mathbf{l}) = \boldsymbol{\psi}_2^T (\mathbf{g}_2 + \mathbf{l}). \tag{4.16}$$

Since $\boldsymbol{\psi}_1^T \mathbf{g}_1 = \boldsymbol{\psi}_2^T \mathbf{g}_2 = 0$, we obtain an equality

$$\boldsymbol{\psi}_1^T \mathbf{l} = \boldsymbol{\psi}_2^T \mathbf{l}, \tag{4.17}$$

expressing the fact that the stress multiplier vector $\boldsymbol{\psi}$ has a non-unique representation at vertices of the clearance region boundary.

In view of Eqs. (4.14), (4.15) and (4.17) we can conclude that the difference between two admissible clearance strains does not affect the work done by external loads. In other words, the uniqueness of the clearance work is noted, independently of the non-unique representation of the stress multipliers.

The mathematical model of elastic-slackened systems can be reduced to the form

$$\begin{aligned} \mathbf{C}^T \mathbf{M} \boldsymbol{\psi} - \mathbf{p} &= \mathbf{0}, \\ \mathbf{g} = \mathbf{M}^T \mathbf{C} \mathbf{u} - \mathbf{G} \boldsymbol{\psi} - \mathbf{l} &\leq \mathbf{0}, \end{aligned} \tag{4.18}$$

together with the requirements

$$\boldsymbol{\psi} \geq \mathbf{0}, \quad \boldsymbol{\psi}^T \mathbf{g} = 0, \tag{4.19}$$

where

$$\mathbf{G} = \mathbf{M}^T \mathbf{E}^{-1} \mathbf{M} \tag{4.20}$$

is a semi-positive definite square *matrix of clearance compliance*. Problem (4.18), (4.19) corresponds to the well-known mini-max formulation, (cf. BORKOWSKI [17]):

$$F(\mathbf{u}_*, \boldsymbol{\psi}_*) = \min_{\mathbf{u}} \max_{\boldsymbol{\psi} \geq \mathbf{0}} F(\mathbf{u}, \boldsymbol{\psi}), \tag{4.21}$$

where $\mathbf{u}_*$ and $\boldsymbol{\psi}_*$ are the saddle-point coordinates, and

$$F(\mathbf{u}, \boldsymbol{\psi}) = \frac{1}{2} \boldsymbol{\psi}^T \mathbf{G} \boldsymbol{\psi} + \mathbf{u}^T \mathbf{C}^T \mathbf{M} \boldsymbol{\psi} - \mathbf{u}^T \mathbf{p} - \boldsymbol{\psi}^T \mathbf{l} \tag{4.22}$$

Using Legendre's transformation we arrive at the following extremum principles that correspond to dual quadratic programming problems:

$$F'(\mathbf{u}, \boldsymbol{\varepsilon}_E)\ \frac{1}{2}\boldsymbol{\varepsilon}_E^T \mathbf{E} \boldsymbol{\varepsilon}_E - \mathbf{u}^T \mathbf{p} \Rightarrow \min, \tag{4.23}$$

subjected to the constraints: $\mathbf{M}^T(\mathbf{C}\mathbf{u} - \boldsymbol{\varepsilon}_E) - \mathbf{l} \leq \mathbf{0}$ and

$$F''(\boldsymbol{\psi}) = -\frac{1}{2}\boldsymbol{\psi}^T \mathbf{G} \boldsymbol{\psi} - \boldsymbol{\psi}^T \mathbf{l} \Rightarrow \max, \tag{4.24}$$

subjected to the constraints: $\mathbf{C}^T \mathbf{M} \boldsymbol{\psi} - \mathbf{p} = \mathbf{0}, \quad \boldsymbol{\psi} \geq \mathbf{0}$.

The saddle point is a true solution. In this case the values of the objective functions are the same, i.e. $F = F' = F''$. This fact can be used to derive the following relation:

$$\mathbf{p}^T \mathbf{u} = W_L + W_E + W_S \geq 0, \tag{4.25}$$

where

$$W_L = \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}_L \geq 0, \quad W_E = \frac{1}{2}\boldsymbol{\varepsilon}_E^T \mathbf{E} \boldsymbol{\varepsilon}_E = W_S = \frac{1}{2}\boldsymbol{\sigma}^T \mathbf{E}^{-1} \boldsymbol{\sigma} \geq 0. \tag{4.26}$$

Thus, a "conventional" work of the external loads, $\mathbf{p}^T \mathbf{u}$, is equal to the sum of the clearance work and two elastic energies: W_E and W_S.

EXAMPLE II

The meaning and graphical interpretation of Eq. (4.25) are explained in an example of the three-bar slackened truss with a longitudinal gap in the bar number 3 (see Fig. 6a). The horizontal force P, ($0 \leq P \leq 150$ kN) applied at the truss node produces a displacement with horizontal component Δ. Figure 6b illustrates the bi-linear elastic $P - \Delta$ relation as well as the conventional work decomposition into three parts W_S, W_L and W_E. Note that the essentially nonlinear behaviour of the truss occurs in the range of small displacements, and therefore the application of geometrically linear theory is really justified.

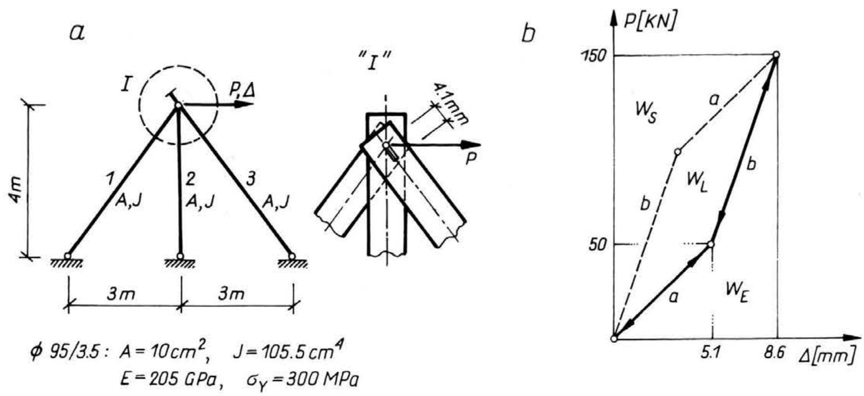

FIG. 6. Conventional work decomposition for a three-bar truss; a) truss layout, gap in bar 3, b) $P - \Delta$ diagram.

It should be pointed out that elastic-slackened systems behave always holonomically if the clearance surface is convex, and therefore the incremental analysis is not required. Moreover, it is impossible to describe the dual extremum principles using only increments of the state variables. The current non-negative components of the stress multiplier vector

Using Legendre's transformation we arrive at the following extremum principles that correspond to dual quadratic programming problems:

$$F'(\mathbf{u}, \boldsymbol{\varepsilon}_E)\ \frac{1}{2}\boldsymbol{\varepsilon}_E^T \mathbf{E} \boldsymbol{\varepsilon}_E - \mathbf{u}^T \mathbf{p} \Rightarrow \min, \tag{4.23}$$

subjected to the constraints: $\mathbf{M}^T(\mathbf{Cu} - \boldsymbol{\varepsilon}_E) - \mathbf{l} \leq \mathbf{0}$ and

$$F''(\boldsymbol{\psi}) = -\frac{1}{2}\boldsymbol{\psi}^T \mathbf{G} \boldsymbol{\psi} - \boldsymbol{\psi}^T \mathbf{l} \Rightarrow \max, \tag{4.24}$$

subjected to the constraints: $\mathbf{C}^T \mathbf{M} \boldsymbol{\psi} - \mathbf{p} = \mathbf{0}, \ \ \boldsymbol{\psi} \geq \mathbf{0}$.

The saddle point is a true solution. In this case the values of the objective functions are the same, i.e. $F = F' = F''$. This fact can be used to derive the following relation:

$$\mathbf{p}^T \mathbf{u} = W_L + W_E + W_S \geq 0, \tag{4.25}$$

where

$$W_L = \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}_L \geq 0, \quad W_E = \frac{1}{2}\boldsymbol{\varepsilon}_E^T \mathbf{E} \boldsymbol{\varepsilon}_E = W_S = \frac{1}{2}\boldsymbol{\sigma}^T \mathbf{E}^{-1} \boldsymbol{\sigma} \geq 0. \tag{4.26}$$

Thus, a "conventional" work of the external loads, $\mathbf{p}^T\mathbf{u}$, is equal to the sum of the clearance work and two elastic energies: W_E and W_S.

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The meaning and graphical interpretation of Eq. (4.25) are explained in an example of the three-bar slackened truss with a longitudinal gap in the bar number 3 (see Fig. 6a). The horizontal force P, $(0 \leq P \leq 150$ kN) applied at the truss node produces a displacement with horizontal component Δ. Figure 6b illustrates the bi-linear elastic $P - \Delta$ relation as well as the conventional work decomposition into three parts W_S, W_L and W_E. Note that the essentially nonlinear behaviour of the truss occurs in the range of small displacements, and therefore the application of geometrically linear theory is really justified.

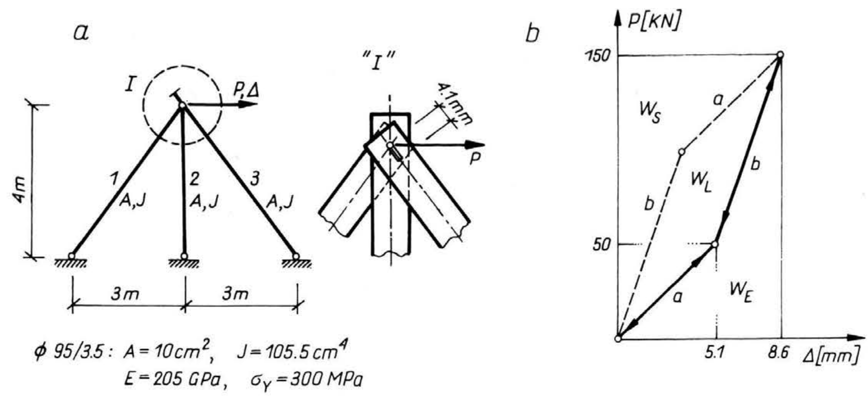

FIG. 6. Conventional work decomposition for a three-bar truss; a) truss layout, gap in bar 3, b) $P - \Delta$ diagram.

It should be pointed out that elastic-slackened systems behave always holonomically if the clearance surface is convex, and therefore the incremental analysis is not required. Moreover, it is impossible to describe the dual extremum principles using only increments of the state variables. The current non-negative components of the stress multiplier vector be modified to the following form:

$$
(4.32)\qquad
\begin{array}{ll}
1)\ \mathbf{C}\Delta u = \Delta\boldsymbol{\varepsilon}_L + \Delta\boldsymbol{\varepsilon}_E + \Delta\boldsymbol{\varepsilon}_P, & 6)\ \mathbf{g}_a = \mathbf{M}_a^T(\boldsymbol{\varepsilon}_L + \Delta\boldsymbol{\varepsilon}_L) - \mathbf{l} \le \mathbf{0},\\
2)\ \mathbf{C}^T\Delta\boldsymbol{\sigma} = \Delta\mathbf{p}, & 7)\ (\boldsymbol{\psi}_a + \Delta\boldsymbol{\psi}_a)^T(\mathbf{g}_a + \Delta\mathbf{g}_a) = \mathbf{0},\\
3)\ \Delta\boldsymbol{\sigma} = \mathbf{E}\Delta\boldsymbol{\varepsilon}_E, & 8)\ \Delta\boldsymbol{\varepsilon}_P = \mathbf{N}_a\Delta\boldsymbol{\lambda}_a,\\
4)\ \mathbf{M}_a\Delta\boldsymbol{\psi}_a = \mathbf{E}\Delta\boldsymbol{\varepsilon}_E, & 9)\ \Delta\boldsymbol{\lambda}_a \ge \mathbf{0},\\
5)\ (\boldsymbol{\psi} + \Delta\boldsymbol{\psi})_a \ge \mathbf{0}, & 11)\ \Delta\boldsymbol{\lambda}_a^T(\mathbf{f}_a + \Delta\mathbf{f}_a) = \mathbf{0}.\\
\multicolumn{2}{l}{10)\ \mathbf{f}_a + \Delta\mathbf{f}_a = \mathbf{N}_a^T(\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}) - \mathbf{H}_a(\boldsymbol{\lambda}_a + \Delta\boldsymbol{\lambda}_a) - \mathbf{k} \le \mathbf{0},}
\end{array}
$$

In Eqs. (4.32) the symbol a indicates the active parts of the particular matrices.

The corresponding saddle function can be expressed as

$$
\begin{aligned}
(4.33)\qquad & F(\Delta\boldsymbol{\lambda}_a, \Delta\mathbf{u}, \boldsymbol{\psi}_a + \Delta\boldsymbol{\psi}_a)\\
&= \frac{1}{2}\Delta\boldsymbol{\lambda}_a^T\mathbf{H}_a\Delta\boldsymbol{\lambda}_a - \frac{1}{2}(\boldsymbol{\psi}_a + \Delta\boldsymbol{\psi}_a)^T\mathbf{G}(\boldsymbol{\psi}_a + \Delta\boldsymbol{\psi}_a) - \Delta\boldsymbol{\lambda}_a^T\mathbf{N}_a^T\mathbf{M}_a(\boldsymbol{\psi}_a + \Delta\boldsymbol{\psi}_a)\\
&\quad + \Delta\mathbf{u}^T\mathbf{C}^T\mathbf{M}_a(\boldsymbol{\psi}_a + \Delta\boldsymbol{\psi}_a) + \Delta\boldsymbol{\lambda}_a^T\mathbf{f}_* - \Delta\mathbf{u}^T\mathbf{p}_* + (\boldsymbol{\psi}_a + \Delta\boldsymbol{\psi}_a)^T\mathbf{g}_*,
\end{aligned}
$$

where

$$
(4.34)\qquad
\begin{aligned}
\mathbf{f}_* &= \mathbf{N}_a^T\mathbf{M}_a\boldsymbol{\psi}_a = \mathbf{N}_a^T\boldsymbol{\sigma},\\
\mathbf{p}_* &= \mathbf{p} + \Delta\mathbf{p},\\
\mathbf{g}_* &= \mathbf{G}_a\boldsymbol{\psi}_a = \mathbf{M}_a^T\boldsymbol{\varepsilon}_E.
\end{aligned}
$$

Note that $\mathbf{F}$ is concave with respect to $(\boldsymbol{\psi}_a + \Delta\boldsymbol{\psi}_a)$ and convex with respect to $\Delta\boldsymbol{\lambda}_a$ and $\Delta\mathbf{u}$. Using Legendre's transformation, we arrive at

$$
F' = \frac{1}{2}\Delta\boldsymbol{\lambda}_a^T\mathbf{H}_a\Delta\boldsymbol{\lambda}_a + \frac{1}{2}(\boldsymbol{\varepsilon}_E + \Delta\boldsymbol{\varepsilon}_E)^T\mathbf{E}(\boldsymbol{\varepsilon}_E + \Delta\boldsymbol{\varepsilon}_E) + \Delta\boldsymbol{\lambda}_a^T\mathbf{f}_* - \Delta\mathbf{u}^T\mathbf{p}_* \Rightarrow \min,
$$

subjected to the constraints

$$
\mathbf{M}_a^T[\mathbf{C}\Delta\mathbf{u} - \mathbf{N}_a\Delta\boldsymbol{\lambda}_a - (\boldsymbol{\varepsilon}_E + \Delta\boldsymbol{\varepsilon}_E)] + \mathbf{g}_* \le 0, \qquad \Delta\boldsymbol{\lambda}_a \ge \mathbf{0},
$$

(4.35) and

$$
\begin{aligned}
F''(\Delta\boldsymbol{\lambda}_a, \boldsymbol{\psi}_a + \Delta\boldsymbol{\psi}_a) = -\frac{1}{2}\Delta\boldsymbol{\lambda}_a^T\mathbf{H}_a\Delta\boldsymbol{\lambda}_a - \frac{1}{2}(\boldsymbol{\psi}_a + \Delta\boldsymbol{\psi}_a)^T\mathbf{G}(\boldsymbol{\psi}_a + \Delta\boldsymbol{\psi}_a) +\\
(\boldsymbol{\psi}_a + \Delta\boldsymbol{\psi}a_a)^T\mathbf{g}_* \Rightarrow \max
\end{aligned}
$$

subjected to the constraints

$$
(4.36)\qquad
\begin{aligned}
\mathbf{H}_a\Delta\boldsymbol{\lambda}_a - \mathbf{N}_a^T\mathbf{M}_a(\boldsymbol{\psi}_a + \Delta\boldsymbol{\psi}_a) + \mathbf{f}_* \ge \mathbf{0}, \quad \mathbf{C}^T\mathbf{M}_a(\boldsymbol{\psi}_a + \Delta\boldsymbol{\psi}_a) - \mathbf{p}_* = \mathbf{0},\\
(\boldsymbol{\psi}_a + \Delta\boldsymbol{\psi}_a) \ge \mathbf{0}.
\end{aligned}
$$

THEOREM 3. *For a given external load increment* $\Delta\mathbf{p}$ *the corresponding stress increment* $\Delta\boldsymbol{\sigma}$ *is unique if the clearance region is convex.*

Proof

Assume that $\Delta\mathbf{p}$ is in equilibrium with two different stress increments $\Delta\boldsymbol{\sigma}_1$ and $\Delta\boldsymbol{\sigma}_2$. Then

$$
(4.37)\qquad \Delta\boldsymbol{\sigma}_1 - \Delta\boldsymbol{\sigma}_2 = \boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}_1 - (\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}_2)
$$

or

$$
(4.38)\qquad \Delta\boldsymbol{\sigma}_1 - \Delta\boldsymbol{\sigma}_2 = \mathbf{Z}(\Delta\boldsymbol{\varepsilon}_{D_1} - \Delta\boldsymbol{\varepsilon}_{D_2}),
$$

where

$$\Delta\varepsilon_{D_1} = \Delta\varepsilon_{L_1} + \Delta\varepsilon_{P_1}, \qquad \Delta\varepsilon_{D_2} = \Delta\varepsilon_{L_2} + \Delta\varepsilon_{P_2}.$$

Since $\mathbf{Z}$ is semi-negative definite, it follows from Eq. (4.38)

$$(4.39) \qquad (\Delta\sigma_1 - \Delta\sigma_2)^T(\Delta\varepsilon_{D_1} - \Delta\varepsilon_{D_2}) = (\Delta\varepsilon_{D_1} - \Delta\varepsilon_{D_2})^T\mathbf{Z}(\Delta\varepsilon_{D_1} - \Delta\varepsilon_{D_2}) \leq 0.$$

Using Eq. (4.37), we obtain

$$(4.40) \qquad (\Delta\sigma_1 - \Delta\sigma_2)^T(\Delta\varepsilon_{D_1} - \Delta\varepsilon_{D_2}) = (\Delta\varepsilon_{P_1} - \Delta\varepsilon_{P_2})^T(\Delta\sigma_1 - \Delta\sigma_2) + \\ [(\varepsilon_L + \Delta\varepsilon_{L_1} - \varepsilon_L - \Delta\varepsilon_{L_2})]^T[(\sigma + \Delta\sigma_1) - (\sigma + \Delta\sigma_2)] \geq 0.$$

The former right-hand term is non-negative due to convexity of the yield condition, the latter term is also non-negative due to convexity of the clearance region. Thus we can conclude that

$$(\Delta\sigma_1 - \Delta\sigma_2)^T(\Delta\varepsilon_{D1} - \Delta\varepsilon_{D2}) = 0,$$

or

$$(4.41) \qquad (\Delta\sigma_1 - \Delta\sigma_2)^T(\Delta\varepsilon_{D_1} - \Delta\varepsilon_{D_2}) = (\Delta\sigma_1 - \Delta\sigma_2)^T(\mathbf{C}\Delta\mathbf{u}_2 - \Delta\varepsilon_{E_1} - \mathbf{C}\Delta\mathbf{u}_2 + \Delta\varepsilon_{E_2}) \\ = -(\Delta\sigma_1 - \Delta\sigma_2)^T(\Delta\varepsilon_{E_1} - \Delta\varepsilon_{E_2}) = -(\Delta\varepsilon_{E_1} - \Delta\varepsilon_{E_2})^T\mathbf{E}(\Delta\varepsilon_{E_1} - \Delta\varepsilon_{E_2}) = 0.$$

Since $\mathbf{E}$ is strictly positive definite, $\Delta\varepsilon_{E_1} = \Delta\varepsilon_{E_2}$ and $\Delta\sigma_1 = \Delta\sigma_2$. Thus, the stress increments are unique. Nevertheless, the increments of kinematical quantities can be non-uniquely determined. From Eq. (3.18), (4.39) and (4.41) we obtain

$$(4.42) \qquad (\Delta\varepsilon_{D_1} - \Delta\varepsilon_{D_2})^T\mathbf{Z}(\Delta\varepsilon_{D_1} - \Delta\varepsilon_{D_2}) = 0.$$

Equation (4.42) is satisfied if

$$(4.43) \qquad (\Delta\varepsilon_{D_1} - \Delta\varepsilon_{D_2}) = \mathbf{C}\Delta\mathbf{u}.$$

In general $\mathbf{C}\Delta\mathbf{u} \neq \mathbf{0}$, namely: in the case of ideal plastic material ($\mathbf{H} = \mathbf{0}$) $\Delta\varepsilon_P$ can be non-unique, and in the case of slackened systems $\Delta\varepsilon_{L_1} - \Delta\varepsilon_{L_2} \neq \mathbf{0}$ due to the weak convexity of clearance region. Thus, we can conclude that the increments of kinematical quantities are unique only in the case when the material exhibits positive plastic hardening and the clearance region is strictly convex.

4.4. Holonomic behaviour of elastic-plastic-slackened systems

The holonomic behaviour of the system takes place when final states of the stress and strain do not depend on the deformation history. It means that the plastic unloading does not occur and, in the case of slackened system, convexity of the clearance region is additionally required. The holonomic behaviour is usually observed for the proportional simple loading. Mathematical model (3.1) can also describe the case of holonomic behaviour if the rate-quantities are replaced by their final values. For the holonomic case the following dual extremum principles can be derived, A. GAWĘCKI [19]:

$$\left\{F' = \frac{1}{2}\boldsymbol{\lambda}^T\mathbf{H}\boldsymbol{\lambda} + \frac{1}{2}\varepsilon_E^T\mathbf{E}\varepsilon_E + \boldsymbol{\lambda}^T\mathbf{k} - \mathbf{u}^T\mathbf{p}\right\} \Rightarrow \min,$$

(4.44) subjected to the constraints

$$\mathbf{M}^T(\mathbf{C}\mathbf{u} - \mathbf{N}\boldsymbol{\lambda} - \varepsilon_E) - \mathbf{l} \leq \mathbf{0}, \qquad \boldsymbol{\lambda} \geq \mathbf{0},$$

and

$$\left\{F'' = -\frac{1}{2}\boldsymbol{\lambda}^T\mathbf{H}\boldsymbol{\lambda} - \frac{1}{2}\boldsymbol{\psi}^T G\boldsymbol{\psi} - \boldsymbol{\psi}^T\mathbf{l}\right\} \Rightarrow \max,$$

(4.45) subjected to the constraints

$$\mathbf{H}\boldsymbol{\lambda} - \mathbf{N}^T\mathbf{M}\boldsymbol{\psi} + \mathbf{k} \geq 0, \quad \mathbf{C}^T\mathbf{M}\boldsymbol{\psi} - \mathbf{p} = \mathbf{0}, \quad \boldsymbol{\psi} \geq \mathbf{0}.$$

The problem of uniqueness of the solution is similar to that considered in the incremental analysis: uniqueness of the stress and strain states is assured if the material exhibits positive plastic hardening and the clearance region is strictly convex. Otherwise, non-uniqueness of kinematical quantities can occur.

4.5. Limit load problem

It is known that the presence of clearance strongly influences the elastic strength of structures, A. GAWĘCKI [20]. Therefore a fundamental question arises: "does the connection slackening affect the ultimate limit load?"

The common limit load problem for the prescribed reference load vector $\mathbf{p}_0$ consists in the determination of a scalar load multiplier μ and plastic displacement rate vector $\dot{\mathbf{u}}$ describing the corresponding plastic flow mechanism. The limit load problem can be formulated by means of dual linear programming method as (BORKOWSKI [17]):

$$[F' = \boldsymbol{\lambda}^T\mathbf{k}] \Rightarrow \min\,|\boldsymbol{\lambda} \geq \mathbf{0}, \;\; \mathbf{u}^T\mathbf{p}_0 = 1, \tag{4.46}$$

$$[F'' = \mu] \Rightarrow \max\,| - \mathbf{N}^T\boldsymbol{\sigma} + \mathbf{k} \geq \mathbf{0}, \;\; \mathbf{C}^T\boldsymbol{\sigma} - \mu\mathbf{p}_0 = \mathbf{0}, \tag{4.47}$$

where Eqs. (4.46) and (4.47) correspond to the kinematical and statical theorem, respectively.

In the case of slackened systems, made of the perfectly plastic material, the described above formulation should be somewhat modified. The mathematical model may then be written in the following form:

$$\begin{array}{ll}
1)\;\; \mathbf{C}\dot{\mathbf{u}} = \dot{\boldsymbol{\varepsilon}}, & 7)\;\; \dot{\mathbf{g}}_a = \mathbf{M}_a^T\dot{\boldsymbol{\varepsilon}}_L \leq \mathbf{0},\\
2)\;\; \mathbf{C}^T\boldsymbol{\sigma} = \mu\mathbf{p}_0, & 8)\;\; \boldsymbol{\psi}_a^T\mathbf{g}_a = 0,\\
3)\;\; \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}_L + \dot{\boldsymbol{\varepsilon}}_P, & 9)\;\; \dot{\boldsymbol{\varepsilon}}_P = \mathbf{N}\dot{\boldsymbol{\lambda}},\\
4)\;\; \dot{\mathbf{u}}^T\mathbf{p}_0 = 1, & 10)\;\; \dot{\boldsymbol{\lambda}} \geq \mathbf{0},\\
5)\;\; \boldsymbol{\sigma} = \mathbf{M}_a\boldsymbol{\psi}_a, & 11)\;\; \mathbf{f} = \mathbf{N}^T\boldsymbol{\sigma} - \mathbf{k} \leq \mathbf{0},\\
6)\;\; \boldsymbol{\psi}_a \geq \mathbf{0}, & 12)\;\; \dot{\boldsymbol{\lambda}}^T\mathbf{f} = 0,
\end{array} \tag{4.48}$$

where subscript a indicates the active submatrices determined on the basis of the original structure solution problem. It can be shown (cf. GAWĘCKI [21]) that the solution of system (4.48) is equivalent to solutions of the following dual linear programming problems:

$$F' = \dot{\boldsymbol{\lambda}}^T\mathbf{k} \Rightarrow \min\,|\mathbf{M}_a^T(\mathbf{C}\mathbf{u} - \mathbf{N}\boldsymbol{\lambda}) \leq \mathbf{0}, \;\; \mathbf{u}^T\mathbf{p}_0 = 1, \;\; \boldsymbol{\lambda} \geq \mathbf{0}, \tag{4.49}$$

$$F'' = \mu \Rightarrow \max\,| - \mathbf{N}^T\mathbf{M}_a^T\boldsymbol{\psi}_a + \mathbf{k} \geq \mathbf{0}, \quad \mathbf{C}^T\mathbf{M}_a^T\boldsymbol{\psi}_a - \mu\mathbf{p}_0 = \mathbf{0}, \quad \boldsymbol{\psi}_a \geq \mathbf{0}. \tag{4.50}$$

Solutions of both problems allow us to determine the kinematical quantities $(\dot{\boldsymbol{\varepsilon}}_L, \dot{\boldsymbol{\varepsilon}}_P, \dot{\mathbf{u}})$ and statical ones $(\boldsymbol{\sigma}, \mu)$. In general, these solutions are different from those obtained for the respective structure without clearances, namely: the load multiplier appears to be less than that for the ideal structure and a different plastic flow mechanism develops. This situation corresponds to the so-called "sublimit" plastic flow mechanism.

Assume now that the kinematical quantities are uniquely determined (it does not always occur!) and the uniform motion of the system is noted. Then, after integrating with respect to time, we obtain

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_0 + t\dot{\mathbf{u}}, \\ \boldsymbol{\varepsilon}_L &= \boldsymbol{\varepsilon}_{L_0} + t\dot{\boldsymbol{\varepsilon}}_L, \\ \boldsymbol{\varepsilon}_P &= \boldsymbol{\varepsilon}_{P_0} + t\dot{\boldsymbol{\varepsilon}}_P, \end{aligned} \tag{4.51}$$

where $\mathbf{u}_0$ and $\boldsymbol{\varepsilon}_{L_0}$ are related to the ideal structure configuration and t denotes the "time" measured from the moment when the plastic flow has begun. A further problem consists in the determination of $t = t^*$, when the flow mechanism stops due to the appearance of a new contact at a slackened connection. To do it, we utilize equations of contact at the passive sides of the clearance hyperpolyhedron

$$\mathbf{M}_p^T(\boldsymbol{\varepsilon}_{L_0} + t\dot{\boldsymbol{\varepsilon}}_L) - \mathbf{l}_p = \mathbf{0} \tag{4.52}$$

together with

$$t \geq 0. \tag{4.53}$$

Problem (4.52), (4.53) is extremely simple and consists in the determination of a smallest non-negative root of the linear equation system (4.52) with one unknown, t. For a given value of $t = t^*$ one can obtain a configuration of a new original structure which is described by

$$\mathbf{u}_0' = \mathbf{u}_0 + t^*\dot{\mathbf{u}}, \qquad \boldsymbol{\varepsilon}_{L_0}' = \boldsymbol{\varepsilon}_{L_0} + t^*\dot{\boldsymbol{\varepsilon}}_L. \tag{4.54}$$

The next step is to do a new matrix partition into the active and passive parts. Then the above procedure should be repeated again.

Now the most important question arises: "when will the final limit load be reached?". Since the clearance region represents the bounded set, the clearance strain rate $\dot{\boldsymbol{\varepsilon}}_L$ must eventually vanish. It means that the ultimate plastic flow mechanism is reached. After substituting $\dot{\boldsymbol{\varepsilon}}_L = \mathbf{0}$ into relations (4.48) we arrive at principles (4.46) and (4.47) that are valid for the ideal structure. Thus, we can formulate the following theorem:

THEOREM 4. *In the case of convex and bounded clearance region, the ultimate limit load and the corresponding plastic flow mechanism are identical with those obtained for the respective ideal structure (without clearances).*

It is worth to notice, however, that this theorem has been derived on the basis of geometrically linear theory. A more realistic approach should be based on the nonlinear theory of post-yield behaviour where geometry changes as well as dynamic effects are taken into account. This remark relates first of all to optimal plastic structures, where sometimes the geometry effects lead to the statement that the true initial plastic flow mechanism is quite different from that predicted by the geometrically linear theory (Z. MRÓZ and A. GAWĘCKI [22]).

EXAMPLE III

Consider the limit load problem for a frame with constrained rotations at hinges. The ideal frame (i.e. the frame without rotation clearances) corresponds to that considered by Borkowski [1985]. The slackened frame layout, loads and limit clearance modulae are presented in Fig. 7a.

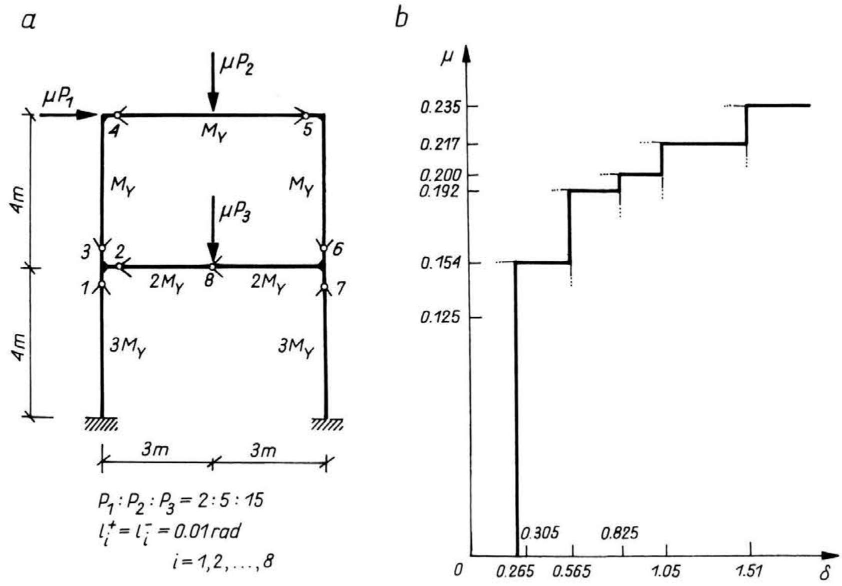

FIG. 7. Limit load problem for a slackened frame; a) frame layout, loads, rotation constraints, b) "load multiplier-weighted displacement" diagram.

The behaviour of the frame is very complicated and it will be discussed elsewhere. We will present only the diagram of load multiplier variations as a function of a "weighted" displacement $\delta = \Sigma P_i u_i$ (Fig. 7b). Note that the frame deformations at the final yield point load are different from zero — as it is observed in common structures — but the maximum value of load multiplier as well as the plastic flow mechanism are identical with the known solution for the structure without clearances given by A. BORKOWSKI [17].

4.6. Ideal configuration and uniqueness of solution

As was mentioned in Sec. 2.1, the ideal configuration can be arbitrarily chosen from all the kinematically admissible configurations which do not violate the clearance constraints. Thus, the following problem should be considered: "are the solutions invariant with respect to the choice of an ideal configuration?"

To avoid the non-uniqueness due to a linear approximation of clearance constraints, we assume that the clearance region is strictly convex. Then the stress state uniquely determines the coordinates of the clearance vector. All the considerations — without any loss of generality — will be restricted to the elastic-slackened systems (i.e. $\varepsilon = \varepsilon_L + \varepsilon_E$). It allows us to neglect other sources of non-uniqueness due to the presence of plastic deformations.

Let us assume that calculations are carried out for the same elastic-slackened structure loaded by the same external load **p** but for two different ideal configurations. Since the theory is geometrically linear, the matrices **C**, **E**, **K**, **Z** in both the problems are the same. The difference between the ideal configurations is described by the kinematically admissible displacement vector $\Delta\mathbf{u}_0$ such that

$$\mathbf{C}\Delta\mathbf{u}_0 = \Delta\boldsymbol{\varepsilon}_{L_0}. \tag{4.55}$$

The stress vectors can be expressed as

$$\begin{aligned}\boldsymbol{\sigma}_1 &= \mathbf{ECK}^{-1}\mathbf{p} + \mathbf{Z}\boldsymbol{\varepsilon}_{L_1},\\ \boldsymbol{\sigma}_2 &= \mathbf{ECK}^{-1}\mathbf{p} + \mathbf{Z}\boldsymbol{\varepsilon}_{L_2},\end{aligned} \tag{4.56}$$

where 1 and 2 indicate the number of the solution. A geometric interpretation of stress and clearance strain states is presented in Fig. 8a.

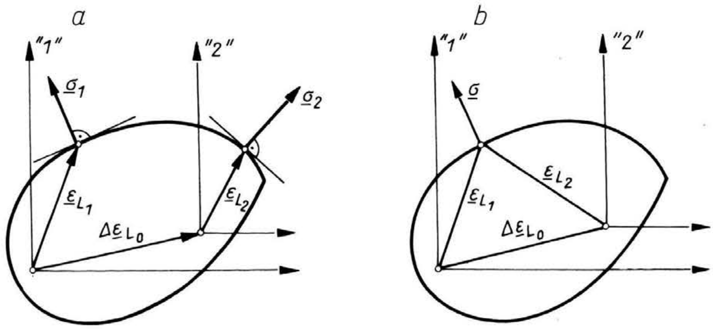

FIG. 8. Non-uniqueness of solution; a) non-unique solutions for clearance strains and stresses, b) uniqueness of solution with respect to stress state.

From Eqs. (4.56) we obtain $\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 = \mathbf{Z}(\boldsymbol{\varepsilon}_{L_1} - \boldsymbol{\varepsilon}_{L_2})$ and

$$(\boldsymbol{\varepsilon}_{L_1} - \boldsymbol{\varepsilon}_{L_2})^T(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) = (\boldsymbol{\varepsilon}_{L_1} - \boldsymbol{\varepsilon}_{L_2})^T\mathbf{Z}(\boldsymbol{\varepsilon}_{L_1} - \boldsymbol{\varepsilon}_{L_2}) \leq 0. \tag{4.57}$$

The inequality sign follows from the fact that $\mathbf{Z}$ is semi-negative definite. On the other hand, the convexity of clearance region gives

$$(\boldsymbol{\varepsilon}_{L_1} - \boldsymbol{\varepsilon}_{L_2})^T(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \geq 0. \tag{4.58}$$

The comparison of Eqs. (4.57) and (4.58) leads to

$$(\boldsymbol{\varepsilon}_{L_1} - \boldsymbol{\varepsilon}_{L_2})^T(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) = 0. \tag{4.59}$$

Since $\boldsymbol{\varepsilon}_{L_i} = \mathbf{Cu}_i - \boldsymbol{\varepsilon}_{E_1}$ and $\boldsymbol{\sigma}_i = \mathbf{E}\boldsymbol{\varepsilon}_{E_i}$ $(i = 1, 2)$, Eq. (4.59) is equivalent to

$$(\boldsymbol{\varepsilon}_{E_1} - \boldsymbol{\varepsilon}_{E_2})^T\mathbf{E}(\boldsymbol{\varepsilon}_{E_1} - \boldsymbol{\varepsilon}_{E_2}) = 0. \tag{4.60}$$

Thus, $\boldsymbol{\varepsilon}_{E_1} = \boldsymbol{\varepsilon}_{E_2} = \boldsymbol{\varepsilon}_E$ and the stress state is unique (i.e. $\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}_E$) because $\mathbf{E}$ is strictly positive definite. The uniqueness of the stress state means that the active points on the strictly convex clearance surface for both solutions are the same. This situation is explained in Fig. 8b.

Using the geometric compatibility relation we obtain

$$\mathbf{C}(\mathbf{u}_1 - \mathbf{u}_2) = \boldsymbol{\varepsilon}_{L_1} - \boldsymbol{\varepsilon}_{L_2} = \Delta\boldsymbol{\varepsilon}_{L_0}. \tag{4.61}$$

So, the clearance strains are not uniquely determined, but the difference between these strains is kinematically admissible. Moreover, as we can see in Figs. 8a,b, this difference corresponds to the mutual shifting of both ideal configurations. As $\det[\mathbf{C}^T\mathbf{C}] \neq 0$, from Eq. (4.61) it follows

$$\Delta\mathbf{u}_0 = \mathbf{u}_1 - \mathbf{u}_2 = (\mathbf{C}^T\mathbf{C})^{-1}\mathbf{C}^T\Delta\boldsymbol{\varepsilon}_{L_0}. \tag{4.62}$$

Thus, we can state that the stress state is independent of the ideal configuration choice whereas the difference of final displacements is equal to the initial relative shifting of the ideal configurations.

5. Selected problems of synthesis

5.1. Elastic strength

The concept of an elastic region in the external load space plays a significant role in the design philosophy of mechanical systems. Most important becomes the answer to the question whether the elastic region is convex or not. The convexity of this region occurs for the common linear elastic-plastic bodies. However, in the case of other constitutive laws and non-standard systems, there are no general theorems on this problem known to the author.

Let us consider a body loaded by an external load $\mathbf{p}$. According to Eq. (3.15) at a given load level, the stress vector can be decomposed into two parts, $\boldsymbol{\sigma}_e$ and $\boldsymbol{\sigma}_d$, where $\boldsymbol{\sigma}_e$ denotes the stress vector which would appear in the linear elastic reference system if it were subjected to the same load as the actual one. The remaining part $\boldsymbol{\sigma}_d$ is a corrective self-equilibrated stress vector due to a deviation from the linear elastic behaviour.

The elastic region S can be described by the following matrix inequality

$$\mathbf{N}^T\boldsymbol{\sigma} - \mathbf{k} \leq \mathbf{0}. \tag{5.1}$$

In the case of the linear elastic system with imposed distortions $\boldsymbol{\varepsilon}_D$, inequality (5.1) can be rewritten in the form

$$\mathbf{N}^T(\mathbf{ECK}^{-1}\mathbf{p} + \mathbf{Z}\boldsymbol{\varepsilon}_D) - \mathbf{k} \leq \mathbf{0}. \tag{5.2}$$

Let us assume that $\mathbf{p}'$ and $\mathbf{p}''$ are elements of the set S, i.e.

$$\begin{aligned} &\mathbf{N}^T(\mathbf{ECK}^{-1}\mathbf{p}' + \mathbf{Z}\boldsymbol{\varepsilon}'_D) - \mathbf{k} \leq \mathbf{0},\\ &\mathbf{N}^T(\mathbf{ECK}^{-1}\mathbf{p}'' + \mathbf{Z}\boldsymbol{\varepsilon}''_D) - \mathbf{k} \leq \mathbf{0}, \end{aligned} \tag{5.3}$$

where $\boldsymbol{\varepsilon}'_D$ and $\boldsymbol{\varepsilon}''_D$ denote distortions connected with $\mathbf{p}'$ and $\mathbf{p}''$, respectively. If $\mathbf{p} = \beta\mathbf{p}' + (1-\beta)\mathbf{p}''$, $\beta \in (0,1)$, belongs to S, then S is convex. From (5.3) one immediately obtains

$$\mathbf{N}^T\{\mathbf{ECK}^{-1}\mathbf{p} + \mathbf{Z}[\beta\boldsymbol{\varepsilon}'_D + (1-\beta)\boldsymbol{\varepsilon}''_D]\} - \mathbf{k} \leq \mathbf{0}. \tag{5.4}$$

Assume now that $\boldsymbol{\varepsilon}_D$ is a steady plastic strain vector independent of $\mathbf{p}'$ and $\mathbf{p}''$. Then $\boldsymbol{\varepsilon}'_D = \boldsymbol{\varepsilon}''_D = \boldsymbol{\varepsilon}_P$ and (5.4) takes the form

$$\mathbf{N}^T(\mathbf{ECK}^{-1}\mathbf{p} + \mathbf{Z}\boldsymbol{\varepsilon}_P) - \mathbf{k} \leq \mathbf{0}. \tag{5.5}$$

Thus, the elastic region is convex in this case. It corresponds, for example, to a linear elastic-plastic structure that shakes down over any load path p within the convex region S.

In the case of a linear elastic-plastic-slackened system with a steady plastic strain vector $\boldsymbol{\varepsilon}_P$ and clearance strain vectors $\boldsymbol{\varepsilon}'_L$, $\boldsymbol{\varepsilon}''_L$, depending on loads $\mathbf{p}'$ and $\mathbf{p}''$, one obtains

$$\mathbf{N}^T\{\mathbf{ECK}^{-1}\mathbf{p} + \mathbf{Z}[\boldsymbol{\varepsilon}_P + \beta\boldsymbol{\varepsilon}'_L + (1-\beta)\boldsymbol{\varepsilon}''_L]\} - \mathbf{k} \leq \mathbf{0}. \tag{5.6}$$

In general, $\boldsymbol{\varepsilon}_L \neq \beta\boldsymbol{\varepsilon}'_L + (1-\beta)\boldsymbol{\varepsilon}''_L$, and therefore elastic region (5.6) can be non-convex, unless $\boldsymbol{\varepsilon}_L$, $\boldsymbol{\varepsilon}'_L$ and $\boldsymbol{\varepsilon}''_L$ are kinematically admissible (i.e. $\mathbf{Z}\boldsymbol{\varepsilon}_L = \mathbf{Z}\boldsymbol{\varepsilon}'_L = \mathbf{Z}\boldsymbol{\varepsilon}''_L = \mathbf{0}$). Usually, the elastic-plastic-slackened system may behave elastically for load paths contained within the non-convex domain.

The conclusion obtained here is also valid for any nonlinear elastic-plastic systems. In such cases the distortion strain vector does not satisfy the superposition principle, $\boldsymbol{\varepsilon}_D \neq \beta\boldsymbol{\varepsilon}'_D + (1-\beta)\boldsymbol{\varepsilon}''_D$. Thus, we can formulate the following theorem [23]:

THEOREM 5. *The elastic region S for a linear elastic-plastic-slackened system and for a system made of a nonlinear elastic-plastic material can be non-convex.*

EXAMPLE IV

In order to illustrate the problem let us consider an elastic-perfectly plastic portal frame with rotation constraints at the midspan hinge. Fig. 9a presents the geometry of the frame, loads and rotation constraints. The ultimate yield surface and the elastic surface are constructed assuming the constant, ideal I-cross section for all the frame elements and treating the normal and shear forces as reactions. It is clearly seen in Fig. 9b that the elastic surface is non-convex. Note that the shape and dimensions of the elastic region strongly depend on the values of limit rotations.

The behaviour of the frame is far from that observed for the common elastic-plastic structures without clearances. This statement is illustrated diagrammatically by $\mu - \delta$ relations plotted for two different load paths (μ-load multiplier, δ-weighted displacement, cf. Example III). On load path I (cf. Fig. 9b,c), segments OA and AB correspond to nonlinear elastic behaviour of the frame and segment BC corresponds to elastic-plastic behaviour. At point C a limit load associated with the combined flow mechanism is achieved. Along load path II structural behaviour is quite different. On segment OD, linear elastic response is noted. At point D, a yield sublimit load for the three-hinged frame is reached, and a combined flow mechanism develops up to a displacement for which the mutual rotation at point 2 is equal to the limit rotation l_2^+. Then, the elastic-plastic behaviour and an increase in load multiplier μ is noted. At point F, the ultimate load is reached and plastic flow develops according to the sway mechanism.

5.2. Geometric interpretation of load-displacement diagrams

Consider the case of elastic-perfectly plastic-slackened systems which behave holonomically. In this case the dual extremum principles take the form:

$$\{F' = \frac{1}{2}\boldsymbol{\varepsilon}_E^T\mathbf{E}\boldsymbol{\varepsilon}_E + \boldsymbol{\lambda}^T\mathbf{k} - \mathbf{u}^T\mathbf{p}\} \Rightarrow \min,$$

(5.7) subjected to the constraints

$$\mathbf{M}^T(\mathbf{Cu} - \mathbf{N}\boldsymbol{\lambda} - \boldsymbol{\varepsilon}_E) - \mathbf{l} \leq 0, \qquad \boldsymbol{\lambda} \geq \mathbf{0},$$

and

$$\{F'' = -\frac{1}{2}\boldsymbol{\psi}^T\mathbf{G}\boldsymbol{\psi} - \boldsymbol{\psi}^T\mathbf{l}\} \Rightarrow \max$$

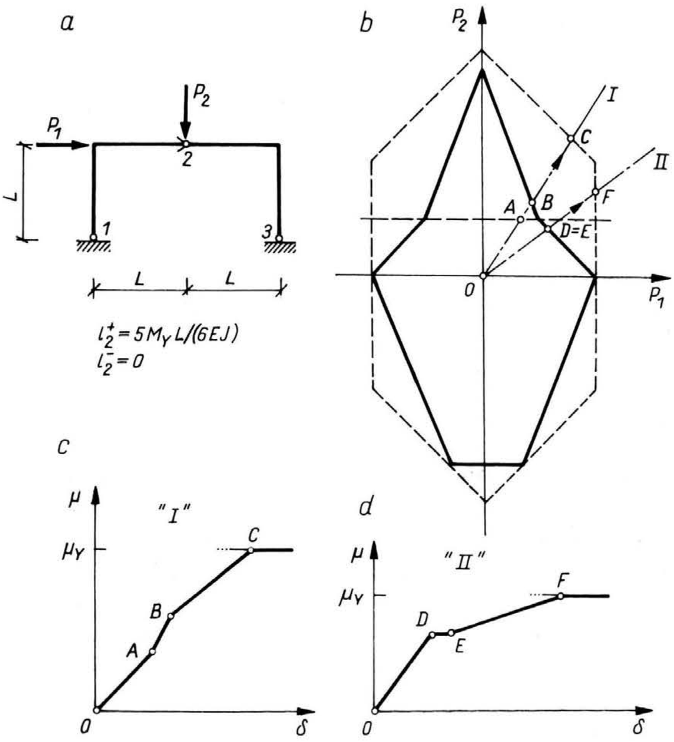

FIG. 9. The three-hinged frame with rotation constraints at the midspan hinge; a) frame, load and rotation constraints, b) load paths, elastic and yield surfaces, c) $\mu - \delta$ diagram for load path I, d) $\mu - \delta$ diagram for load path II.

(5.8) subjected to the constraints

$$\mathbf{N}^T\mathbf{M}\boldsymbol{\psi} + \mathbf{k} \geq \mathbf{0}, \quad \mathbf{C}^T\mathbf{M}\boldsymbol{\psi} - \mathbf{p} = \mathbf{0}, \quad \boldsymbol{\psi} \geq \mathbf{0}.$$

Assume that the load-displacement relation is presented in the form of diagram $\mu(\delta)$, where μ denotes the load factor for a given reference load vector $\mathbf{p}_0$, i.e. $\mathbf{p} = \mu\mathbf{p}_0$, and δ is the weighted displacement expressed by $\delta = \mathbf{p}_0^T\mathbf{u}$. Thus, the conventional work of external loads can be presented as follows

$$\mathbf{p}^T\mathbf{u} = \mu\mathbf{p}_0^T\mathbf{u} = \mu\delta = W_\varepsilon + W_\sigma. \tag{5.9}$$

In Eq. (5.9) W_ε and W_σ are the total strain and stress works, respectively. To determine these works let us integrate the respective increments throughout the deformation process:

$$W_\varepsilon = \int_0^u \mathbf{p}^T\,d\mathbf{u} = \int_0^t \mathbf{p}^T\dot{\mathbf{u}}\,dt = \int_0^t \boldsymbol{\sigma}^T(\dot{\boldsymbol{\varepsilon}}_L + \dot{\boldsymbol{\varepsilon}}_E + \dot{\boldsymbol{\varepsilon}}_P)\,dt, \tag{5.10}$$

where t denotes the "time" measured from the initial natural state up to the final state. Since

$$\boldsymbol{\sigma}^T\dot{\boldsymbol{\varepsilon}}_L \equiv 0, \quad \boldsymbol{\sigma}^T\dot{\boldsymbol{\varepsilon}}_P = \boldsymbol{\lambda}^T\mathbf{k}, \quad \boldsymbol{\sigma}^T\dot{\boldsymbol{\varepsilon}}_E = \boldsymbol{\varepsilon}_E^T\mathbf{E}\dot{\boldsymbol{\varepsilon}}_E = \frac{1}{2}(\boldsymbol{\varepsilon}_E^T\mathbf{E}\boldsymbol{\varepsilon}_E)^{\cdot},$$

together with the initial conditions: $\lambda(0) = \mathbf{0}$, $\varepsilon_E(0) = \mathbf{0}$, one obtains

$$W_\varepsilon = \frac{1}{2}\varepsilon_E^T\mathbf{E}\varepsilon_E + \lambda^T\mathbf{k} = W_E + W_P. \tag{5.11}$$

Here W_E and W_P represent the elastic and plastic strain works, respectively.

Similarly, the complementary energy can be written as follows

$$W_\varepsilon = \int_0^p \mathbf{u}^T\,d\mathbf{p} = \int_0^t \mathbf{u}^T\dot{\mathbf{p}}\,dt = \int_0^t \dot{\sigma}^T(\varepsilon_E + \varepsilon_L + \varepsilon_P)\,dt. \tag{5.12}$$

Equation (3.14) and the symmetry of $\mathbf{E}$ give

$$\dot{\sigma}^T\varepsilon_L = \dot{\psi}^T\mathbf{l}, \quad \dot{\sigma}^T\varepsilon_E = \frac{1}{2}(\sigma^T\mathbf{E}^{-1}\sigma)^{\cdot}.$$

The last term of the integral expression requires some supplementary considerations. For holonomic behaviour, $\lambda^T\mathbf{f} = 0$. Differentiation with respect to time leads to the result

$$\dot{\lambda}^T\mathbf{f} + \lambda^T\dot{\mathbf{f}} = 0. \tag{5.14}$$

Since $\dot{\lambda}^T\mathbf{f} = 0$ is always true (also for the holonomic behaviour) we conclude that

$$\lambda^T\dot{\mathbf{f}} = \dot{\sigma}^T\varepsilon_P = 0. \tag{5.15}$$

Finally, if $\mathbf{l} \geq 0$, $\sigma(0) = 0$, $\psi(0) = 0$, we obtain

$$W_\sigma = \frac{1}{2}\sigma^T\mathbf{E}^{-1}\sigma + \psi^T\mathbf{l} = W_S + W_L, \tag{5.16}$$

where W_S and W_L are the elastic and clearance stress works, respectively.

On the other hand the extremum principles at the saddle point ($F' = F''$) provide the following relation:

$$\mathbf{p}^T\mathbf{u} = \frac{1}{2}\varepsilon_E^T\mathbf{E}\varepsilon_E + \frac{1}{2}\sigma^T\mathbf{E}^{-1}\sigma + \lambda^T\mathbf{k} + \psi^T\mathbf{k} = W_\varepsilon + W_\sigma, \tag{5.17}$$

which additionally confirms the results obtained herein. Note that the elastic stress and strain works are equal to each another, i.e. $W_E = W_S$. A geometrical interpretation of the total work decomposition is shown in Fig. 10a, where μ^* corresponds to the limit load multiplier μ_Y. In a particular case of common elastic-plastic structure (without clearances), $W_L = 0$. Then the area below the $\mu - \delta$ diagram consists of three parts: plastic work W_P, "recovered" elastic strain work W_{E_r} and "hidden" (stored) elastic strain work W_{E_h} that remains in the body due to the presence of kinematically non-admissible plastic strains ε_P. This case is illustrated in Fig. 10b. The interpretation presented here seems to be rather new. It could be derived by means of the concept of system slackening.

5.3. Elastic strength maximization

Let us consider in detail two identical elastic-plastic-slackened systems of different clearance moduli $\mathbf{l}_1$ and $\mathbf{l}_2$, which behave holonomically, subjected to the same load ($\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}$). Assume also that, at a final state, the displacement vectors of both systems are the same, i.e. $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}$. Then from Eq. (3.18) it follows

$$\sigma_1 - \sigma_2 = \mathbf{Z}(\varepsilon_{L_1} + \varepsilon_{P_1} - \varepsilon_{L_1} - \varepsilon_{P_2}). \tag{5.18}$$

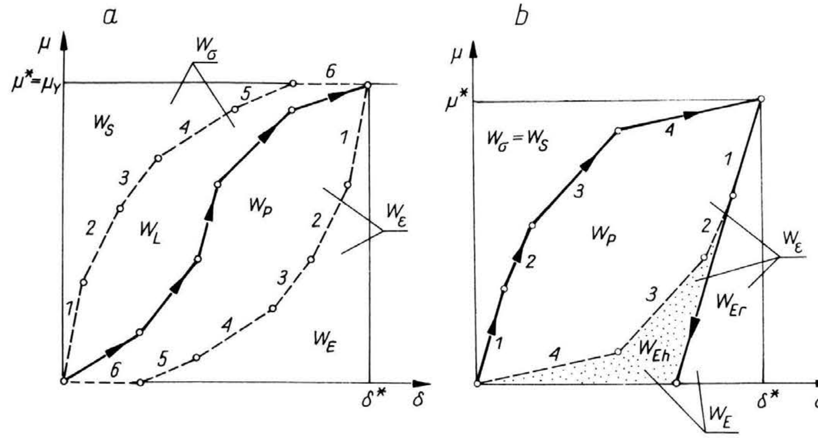

FIG. 10. Conventional work decomposition; a) elastic-plastic slackened system, b) elastic-plastic system.

In view of Eq. (5.18) it is seen that $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2$ and $\boldsymbol{\varepsilon}_1 = \boldsymbol{\varepsilon}_2$ if $\boldsymbol{\varepsilon}_{L_1} + \boldsymbol{\varepsilon}_{P_1} = \boldsymbol{\varepsilon}_{L_2} + \boldsymbol{\varepsilon}_{P_2} + \mathbf{C}\Delta\mathbf{u}$. This result can be used to maximize the elastic strength of the slackened structure.

According to Theorem 4 the ultimate limit load $\mathbf{p}_Y$ does not depend on the presence of clearances and their distribution, so that the state where $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}_Y$ and $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}$ can always be attained. Assume that the structure 1 is the common elastic-plastic one ($\mathbf{l}_1 = \mathbf{0}$) and its plastic strains at the yield point load are described by $\boldsymbol{\varepsilon}_{P_1}$. The condition for the structure to remain in a purely elastic state requires that the plastic strains should be equal to zero ($\boldsymbol{\varepsilon}_{P_2} = \mathbf{0}$). Then, according to Eq. (5.18), $\boldsymbol{\varepsilon}_{L_2} = \boldsymbol{\varepsilon}_{P_1} + \mathbf{C}\Delta\mathbf{u}$ and the yield load may be attained elastically.

Identical considerations can be carried out for any load vector $\mathbf{p} \leq \mathbf{p}_Y$. Thus, the following theorem holds:

THEOREM 6. *Any holonomic elastic-perfectly plastic structure can be associated with an elastic-slackened structure, which for the same load exhibits the same stress, displacement and total strain states.*

It is proper to add that, in general, such an attractive solution is difficult to achieve in practice, especially in the case when a clearance interaction occurs. Nevertheless, some results have been already obtained in the optimization of trusses, beams and frames for non-interacting clearance strains (cf. GAWĘCKI [23]). In those cases the non-zero clearance moduli have been assumed to be identical with plastic strains of the reference structure without clearances.

EXAMPLE V

The optimization procedure which maximizes the elastic strength of the system will be illustrated for a truss shown in Fig. 11a.

The cross-sections and the material (elastic-perfectly plastic with yield stress σ_Y) for all the truss members are the same. To determine the unknown dimensions of longitudinal gaps in particular members (cf. Example I) one has to solve the problem of the ideal truss without clearances. In this case the $\mu - \delta$ diagram for $p_1/p_2 = 0.65$ is shown in Fig. 11b (dashed line). At the yield point load the plastic strains are: $\varepsilon_{Y_4} = 9.74\sigma_Y/E$ [m], $\varepsilon_{Y_5} = -7.20\sigma_Y/E$ [m], where E denotes Young's modulus. Thus, the clearances modulae have to be assumed as

$$\mathbf{l} = [l_1^+,\ l_1^-,\ l_2^+,\ l_2^-, \ldots, l_6^-]^T = [0\ 0\ 0\ 0\ 0\ 0\ 9.74\ 0\ 0\ 7.20\ 0\ 0]^T(\sigma_Y/E).$$

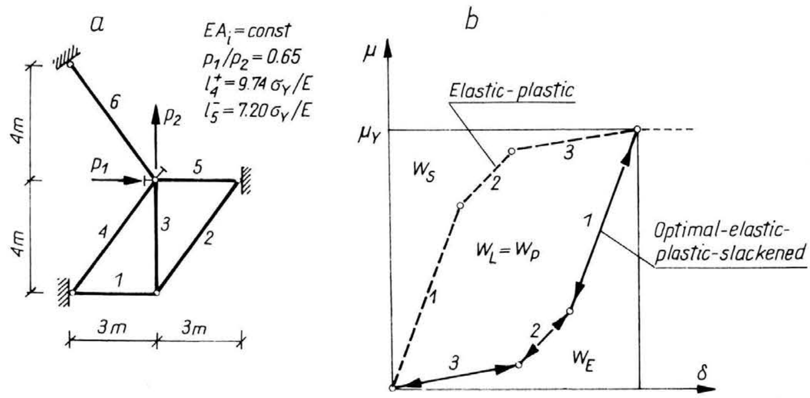

FIG. 11. Elastic strength maximization for a slackened truss; a) truss and loads, b) "load multiplier-weighted displacement" diagrams.

The optimal distribution of longitudinal gaps leads to pure elastic behaviour of the truss up to ultimate limit load and the $\mu - \delta$ relation corresponds to a piece-wise linear curve (solid line in Fig. 11b). Thus, the elastic strength of the slackened truss reaches its maximum. Note that $\mu - \delta$ diagram consists of the same segments as in the case of the elastic-plastic truss without clearances, but they are located in the strictly inverted sequence. It is interesting that the response of optimal slackened structure exhibits a qualitative similarity to that of biomaterials which behave as locking-elastic systems. The behaviour of optimal systems consists in a consecutive incorporation of particular structural elements as the external load increases when it is really necessary. So, we can suppose that the nature prefers the "maximum reserve" or "minimum effort" principle.

6. Final remarks

Fundamental theoretical problems of quasi-static frictionless behaviour of slackened systems within the framework of geometrically linear theory have been presented in the paper. From both the theoretical and practical points of view the problems of slackened systems appear to be very rich and wide. It seems that only the idea of slackening enables us to construct a consistent theory of time-independent materials and systems. A convincing interpretation and better understanding of the energy division for deformable systems can be made by applying this idea to elastic-plastic structures without clearances. Several important theorems on various problems and the corresponding extremum principles have been derived. Particular attention has been paid to problems of uniqueness of solution which are of practical significance for the numerical methods. It should be pointed out that the theorems and conclusions presented herein are also valid for many unilateral problems that can be met in structural and solid mechanics. The considerations correspond to problems of analysis and synthesis, including the optimization of clearance moduli with respect to the elastic strength maximization. It is interesting to note that the behaviour of optimally slackened structures appears to be very similar to that observed in biomechanical materials. The results obtained in the paper may already be applied in structural mechanics, fatigue strength and robotics. Some applications of the presented theory to problems in soil and damage mechanics can be anticipated.

However, many questions remain open in the fundamental theory of slackened systems. Such topics as shakedown, the evolution of clearance regions, and thermal, rheological and friction effects are particularly noteworthy. Furthermore, the stability, dynamics and all the geometrically nonlinear problems should be formulated and solved.

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