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Instability and bifurcation in the plane tension test

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THE BASIC PROBLEM of a block constrained to undergo a uniform in-plane tension is investigated. A wide class of materials, ranging from a time-independent rigid-plastic behavior to time-dependent viscoplasticity is considered. The transition from uniform to non-uniform deformation is shown to shift from a bifurcation phenomenon in the former to an instability phenomenon in the latter case. It is shown that a non-uniform deformation mode is uniquely determined from the time-dependent analysis, whereas no such unique mode can be obtained from the time-independent analysis.

1. Introduction

THE STANDARD PROBLEM of a rectangular block which is constrained to plane deformations and is subjected to tension in one direction, as formulated by HILL and HUTCHINSON [3] is investigated. The sides of the block are traction-free, and it is elongated by a uniform shear-free relative displacement of its ends. The material is assumed to be incompressible, it is characterized by the simple power law: $\sigma_e^* = \mu d_e^{*m}$ where σ_e^* and d_e^* are the equivalent stress and strain-rate, and m is the strain-rate sensitivity; elasticity is neglected, and any strain-dependence of μ is ignored. When $m = 0$, the rate-independent non-hardening purely plastic material $\sigma = \mu$ is obtained. For $0 < m \leq 1$ the material is rate-dependent; it goes from a highly nonlinear viscous behavior when $m \to 0$, to the Newtonian fluid-like behavior of viscosity μ for $m = 1$. Thus, a wide range of material response is spanned, and the problem lends itself to a thorough examination of the possible transitions from uniform to non-uniform deformation; since no material-hardening is permitted, it is expected that the transition starts from the very beginning of the loading process.

A Lagrangian perturbation analysis is devised to illustrate the issue; it leads to a fourth-order partial differential equation with time-dependent coefficients. The perturbation modes that correspond to the elliptic or hyperbolic classification of that equation are described. Their evolution is governed by the boundary conditions. The extent to which such a framework is significant for the understanding of the instability and bifurcation phenomena is one of the prominent objectives of this paper.

2. Basic formulation

Since this work is aimed at describing the evolution of deformation instabilities that follow the overall material stretching, the Lagrangian framework seems more suitable and it is used hereafter. As usual, it is convenient to employ non-dimensional variables; unless otherwise stated, all length variables are scaled by the length L of the specimen in the reference configuration. The characteristic velocity used for scaling every velocity variable is the constant velocity V of the specimen tip relative to its rear end; the velocity gradients are scaled by the characteristic stretching rate V/L and the time by L/V. Stresses are scaled by the initial axial stress σ^{0*} of a sheet stretching uniformly at the rate V/L (see below 2–10). All field quantities are considered to be functions of the Lagrangian coordinates $\mathbf{a} = (a_1, a_2, a_3)$, which serve as particle labels, and of the time t. The position of the particle $\mathbf{a}$ relative to a fixed Cartesian frame at time t is specified by the Eulerian coordinates $\mathbf{x} = (x_1, x_2, x_3)$. Let $\mathbf{v} = (v_1, v_2, v_3)$ be the velocity; then

$$x_i = a_i + \int_0^t v_i(\mathbf{a}, \tau)\, d\tau .$$

The attention is restricted to in-plane plane strain deformation of the sheet: all quantities are independent of a_3, and no velocity in the a_3-direction is allowed. The velocity boundary conditions are

$$v_1 = 0 \quad \text{at} \quad a_1 = 0, \qquad v_1 = 1 \quad \text{at} \quad a_1 = 1 \tag{2.1}$$

with the symmetry condition

$$v_2 = 0 \quad \text{at} \quad a_2 = 0 . \tag{2.2}$$

At any instant t and position a_1, the outer surface of the sheet is symmetrically disposed about its midplane at $a_2 = 0$, according to $a_2 = \pm h$ with $h = H/L$ ($2H$ is the thickness of the section); it is further assumed that the stress vector $\mathbf{t}$ vanishes on the outer surface,

$$\mathbf{t} = \boldsymbol{\sigma} \cdot \boldsymbol{\nu} = \mathbf{n} \cdot \boldsymbol{\nu}_0 = 0 \quad \text{at} \quad a_2 = h . \tag{2.3}$$

In addition, it is required that the shear stress be zero on the rest of the edges,

$$\sigma_{12} = 0 \quad \text{at} \quad a_2 = 0, \quad a_1 = 0 \quad \text{and} \quad a_1 = 1 . \tag{2.4}$$

$\boldsymbol{\sigma}$ and $\mathbf{n}$ denote respectively the Cauchy stress-tensor and the transpose of the nominal non-symmetric stress-tensor (Boussinesq tensor); $\boldsymbol{\nu}$ is the outer normal in the current state of a material element that had the orientation $\boldsymbol{\nu}_0$ in the reference configuration. These stress tensors are related to the force $\mathbf{df}$ transmitted across a material element by

$$\mathbf{df} = \boldsymbol{\sigma} \cdot \boldsymbol{\nu} ds = \mathbf{n} \cdot \boldsymbol{\nu}_0 ds_0 ,$$

where ds_0 and ds are the material element areas in the reference and current configurations; $\mathbf{F} = (\partial x_i / \partial a_j)$ being the deformation gradient, these tensors are linked by the relationship

$$\mathbf{n} = J \boldsymbol{\sigma} \cdot {}^t F^{-1}, \qquad J = \det \mathbf{F} . \tag{2.5}$$

Using the Boussinesq tensor $\mathbf{n}$, the Lagrangian equilibrium equations are written as

$$\begin{aligned} \frac{\partial n_{11}}{\partial a_1} + \frac{\partial n_{12}}{\partial a_2} &= 0 , \\ \frac{\partial n_{21}}{\partial a_1} + \frac{\partial n_{22}}{\partial a_2} &= 0 . \end{aligned} \tag{2.6}$$

The material is assumed to be incompressible; thus

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0, \qquad J = 1 . \tag{2.7}$$

Let us denote by $\mathbf{D}$ the strain rate tensor, while $\mathbf{s}$ stands for the Cauchy stress-tensor deviator. Let us use in addition the equivalent stress: $\sigma_e = (2\mathbf{s} : \mathbf{s})^{1/2}$ and the equivalent strain-rate $d_e = (1/2\mathbf{D} : \mathbf{D})^{1/2}$ (The symbol (:) stands for the double contracted product between tensors). Elasticity is neglected, and the viscoplastic behavior is specified by

$$\mathbf{s} = \Lambda \mathbf{D}, \quad \Lambda = 1/2\sigma_e/d_e, \quad \sigma_e = d_e^m, \tag{2.8}$$

where m denotes the material strain rate sensitivity. The relation: $\sigma_e^* = \mu d_e^{*m}$, with $\mu = \sigma^{0*}/(V/L)^m$, is the dimensional counterpart of $(2.8)_3$. Under in-plane plane strain conditions, the relations $(2.8)_1$ reduce to

$$\begin{aligned} \sigma_{11} - \sigma_{22} &= 2s_{11} = 2\Lambda \frac{\partial v_1}{\partial x_1}, \\ \sigma_{12} = s_{12} &= \frac{\Lambda}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right). \end{aligned} \tag{2.9}$$

It is straightforward to show that the fundamental stretching solution for Eqs. (2.5) and (2–6 to 2–9) is the unsteady uniaxial extension:

$$\begin{gathered} x_1^0 = a_1/\varepsilon', \quad x_2^0 = a_2\varepsilon', \quad \varepsilon' = 1/(1+t), \\ v_1^0 = a_1, \quad v_2^0 = -\varepsilon'^2 a_2, \\ \sigma_{11}^0 = -p^0 + \sigma^0, \quad \sigma_{12}^0 = 0, \quad \sigma_{22}^0 = -p^0, \\ \sigma_{33}^0 = (\sigma_{11}^0 + \sigma_{22}^0)/2, \quad \sigma^0 = \varepsilon'^m, \end{gathered} \tag{2.10}$$

for which the stream function $\psi = -\varepsilon' a_1 a_2$ can be defined. The stability of that fundamental stretching solution is now investigated by using a linear perturbation analysis.

3. Linear perturbation analysis

Let us neglect higher order terms and search for perturbed solutions in the form

$$f = f_0 + \delta f, \tag{3.1}$$

where f stands for every variable x_i, v_i, σ_{ij}, ψ. The substitution of these variable fields in Eqs. (2.5), (2.6), (2.7) and (2.9), substraction of the terms belonging to the fundamental solution (2.10) and retention of the first order terms lead to linear differential equations; in this way, the linearized equilibrium equations are

$$\begin{aligned} \frac{\partial \delta n_{11}}{\partial a_1} + \frac{\partial \delta n_{12}}{\partial a_2} &= 0, \\ \frac{\partial \delta n_{21}}{\partial a_1} + \frac{\partial \delta n_{22}}{\partial a_2} &= 0, \end{aligned} \tag{3.2}$$

where the disturbances in the Boussinesq stresses are related to the Cauchy stresses by (3.3)

$$\begin{aligned} \delta n_{11} &= \varepsilon'\left(\delta\sigma_{11} - \varepsilon'\sigma_{11}^0 \frac{\partial \delta x_1}{\partial a_1}\right), \\ \delta n_{12} &= \frac{1}{\varepsilon'}\left(\delta\sigma_{12} - \varepsilon'\sigma_{11}^0 \frac{\partial \delta x_2}{\partial a_1}\right), \\ \delta n_{21} &= \varepsilon'\left(\delta\sigma_{12} - \frac{1}{\varepsilon'}\sigma_{22}^0 \frac{\partial \delta x_1}{\partial a_2}\right), \\ \delta n_{22} &= \frac{1}{\varepsilon'}\left(\delta\sigma_{22} - \frac{1}{\varepsilon'}\sigma_{22}^0 \frac{\partial \delta x_2}{\partial a_2}\right). \end{aligned} \tag{3.3}$$

The incompressibility condition $(2.7)_1$ is linearized either as

$$\varepsilon' \frac{\partial \delta x_1}{\partial a_1} + \frac{1}{\varepsilon'} \frac{\partial \delta x_2}{\partial a_2} = 0 \tag{3.4}$$

or, after a time derivation, as

$$\varepsilon' \frac{\partial}{\partial a_1}(\delta v_1 - \varepsilon' \delta x_1) + \frac{1}{\varepsilon'} \frac{\partial}{\partial a_2}(\delta v_2 + \varepsilon' \delta x_2) = 0\,. \tag{3.5}$$

Using the relation (3.5), we introduce the perturbed stream-function $\delta\psi$ such that

$$\varepsilon'(\delta v_1 - \varepsilon' \delta x_1) = -\frac{\partial \delta\psi}{\partial a_2}, \qquad \delta v_2 + \varepsilon' \delta x_2 = \varepsilon' \frac{\partial \delta\psi}{\partial a_1}\,. \tag{3.6}$$

Since $\delta v_i = \partial \delta x_i / \partial t$, these relations are differential equations for the disturbances δx_1, δx_2 in the Eulerian coordinates, the solution of which is

$$\delta x_1 = \frac{1}{\varepsilon'}\left(\delta x_1^0 - \int\limits_0^t \frac{\partial \delta\psi}{\partial a_2} d\tau\right), \qquad \delta x_2 = \varepsilon'\left(\delta x_2^0 + \int\limits_0^t \frac{\partial \delta\psi}{\partial a_1} d\tau\right), \tag{3.7}$$

where δx_1^0 and δx_2^0 are initial values. Introducing the following notations

$$\delta\sigma_{11} = -\delta p + \delta\sigma, \qquad \delta\sigma_{22} = -\delta p, \qquad \delta\sigma_{12} = \delta\tau \tag{3.8}$$

and perturbing the constitutive relations (2.9), one obtains

$$\delta\sigma = m\varepsilon'^{m-1} \frac{\partial^2 \delta\psi}{\partial a_1 \partial a_2}, \qquad \delta\tau = 1/4\varepsilon'^{m-1}\left(\varepsilon'^2 \frac{\partial^2}{\partial a_1^2} - \frac{1}{\varepsilon'^2}\frac{\partial^2}{\partial a_2^2}\right)\delta\psi\,. \tag{3.9}$$

Substituting (3.3) into (3.2), making use of (3.4), (3.8) and (3.9), cross-differentiating the subsequent linearized equilibrium equations and eliminating δp leads to the following fourth-order partial differential equation which governs the evolution of $\delta\psi$.

$$\left(\varepsilon'^4 \frac{\partial^4}{\partial a_1^4} - 2(1-2m)\frac{\partial^4}{\partial a_1^2 \partial a_2^2} + \frac{1}{\varepsilon'^4}\frac{\partial^4}{\partial a_2^4}\right)\delta\psi = 0\,. \tag{3.10}$$

The linearization of the velocity boundary conditions (2.1) and their expression using the stream function lead to

$$\delta\psi = 0 \quad \text{at} \quad a_1 = 0, \qquad \delta\psi = 0 \quad \text{at} \quad a_1 = 1\,, \tag{3.11}$$

which also enforces the zero shear stress condition (2.4) at the ends; similarly, the midplane symmetry conditions (2.2), (2.4) for the velocity and shear stress imply

$$\delta\psi = 0, \qquad \partial^2/\partial a_2^2 \delta\psi = 0 \quad \text{at} \quad a_2 = 0\,. \tag{3.12}$$

From (2–3), the linearized stress-free boundary condition is

$$\delta n_{12} = \delta n_{22} = 0 \quad \text{at} \quad a_2 = h\,. \tag{3.13_1}$$

Using (3.3), (3.8) and (3.9) one obtains from this

$$\left(\varepsilon'^2 \frac{\partial^2}{\partial a_1^2} - \frac{1}{\varepsilon_2'^2}\frac{\partial^2}{\partial a_2^2}\right)\delta\psi = 4\varepsilon'^2 \frac{\partial \delta x_2}{\partial a_1} \quad \text{at} \quad a_2 = h \tag{3.13_2}$$

and

$$\left((4m-1)\varepsilon'^2 \frac{\partial^2}{\partial a_1^2} + \frac{1}{\varepsilon'^2}\frac{\partial^2}{\partial a_2^2}\right)\frac{\partial^2}{\partial a_2}\delta\psi = 0 \quad \text{at} \quad a_2 = h\,,$$

where δx_2 is given by (3.7). In order to integrate Eq. (3.10) subjected to the boundary conditions (3.11), (3.12) and (3.13), we need to specify values for $\delta\psi$ at the initial time $t = 0$. In this problem, initial conditions are not constrained whatsoever, with the exception of their amplitude which has to remain small enough to ensure the validity of the method. Since symmetry with respect to both axes is assumed, only symmetrical modes of non-uniform deformation are looked for; this is not unreasonable at the onset of necking.

Although the problem is expressed in terms of the perturbed stream-function $\delta\psi$, the main interest here is in the growth of the non-uniformity in the Eulerian thickness $r = x_2(a_1, h, t)$. Let $\delta r = r^0 - r$ be the non-uniformity in the thickness, and let $a = \delta r/r^0$ measure the size of the non-uniformity relative to the evolving thickness. The relative measure is more meaningful when the change in the thickness r^0 is large; taking the time derivative a', it is found that the rate of a is related to the rate of δr by:

(3.14)
$$a'/a = \delta r'/\delta r + \varepsilon' .$$

Since $\varepsilon' > 0$, it is seen that a decreasing disturbance ($\delta r'/\delta r < 0$) can still lead to a relative growth ($a'/a > 0$); we therefore define the instantaneous linearized relative instability of the uniform flow by $a'/a > 0$. Using (3.7), it is found that the variable of interest a is related to the perturbed stream function $\delta\psi$ by

(3.15)
$$a = \frac{1}{h}\left(\delta x_2^0 + \int_0^t \frac{\partial}{\partial a_1}\delta\psi(a_1, h, \tau)d\tau\right).$$

Further discussion of the relative instability concept, and of the extent to which it is useful for understanding the localization of the plastic deformation can be found in FRESSENGEAS and MOLINARI [2].

4. Results and discussion

To gain perspective, we begin by recalling the results of previous investigations based upon the long-wavelength approximation.

4.1. Long wavelength approximation

The long-wavelength approximation was studied by HUTCHINSON and NEALE [4] under quasi-static conditions, and by FRESSENGEAS and MOLINARI [2] in the dynamic case. It is obtained as a limiting case of the momentum and constitutive equations by letting $\sigma_{12} = 0$ and $\partial/\partial a_2 \equiv \partial/\partial x_2 = 0$; incompressibility is preserved by allowing the cross-section area $A(t, a_1)$ to vary according to the equation

(4.1)
$$\frac{\partial v_1}{\partial a_1} + \frac{A(0, a_1)}{A} \cdot \frac{A'}{A} = 0 .$$

With this model, the instantaneous relative rate of growth a'/a is:

(4.2)
$$a'/a = \varepsilon'/m ,$$

irrespective of the perturbation wavelength (HUTCHINSON, NEALE) [4]). Accordingly, nonlinear viscosity damps uniformly all wavelengths; such a model gives poor estimates for the short wave-lengths evolution. The Newtonian case is more stabilizing; considering the rate-insensitive limit: $m \to 0$, it is seen that the disturbance rate of growth may become very large. All the following growth rates will be scaled by the long wavelength quasi-static initial perturbation growth rate $1/m$; we will thus consider the normalized relative rate of growth G,

$$G = ma'/a\,. \tag{4.3}$$

The integration of (4.2), (4.3) leads to the relative amplification $a(t)/a(0)$

$$\frac{a(t)}{a(0)} = \exp\left(\int_0^t \frac{\varepsilon'(\tau)}{m} d\tau\right) = \exp\left(\int_0^t \frac{G(\tau)}{m}\, d\tau\right) = (1+t)^{1/m}\,. \tag{4.4}$$

4.2. Classification of regimes

We now look for solutions of Eq. (3.10) of the form

$$\delta\psi = \eta(t)\sin(k_0 a_1)\exp(il_0\varepsilon'^2 a_2)\,, \tag{4.5}$$

where k_0 is the Lagrangian wavenumber of a longitudinal perturbation which stretches with the material, and l_0 is the initial wavenumber of a transversal perturbation. In order to satisfy the velocity boundary conditions (3.11), k_0 has to be a multiple of π. Substituting into (3.10), one obtains a fourth-order algebraic equation for l_0 with constant coefficients

$$l_0^4 - 2(1-2m)k_0^2 l_0^2 + k_0^4 = 0\,. \tag{4.6}$$

The equation (3.10) can be elliptic, hyperbolic or parabolic, according to the number of real roots of Eq. (4.6); the regime is found elliptic for $0 < m \leq 1$ (no real root), and hyperbolic in the rate insensitive case $m = 0$ (four real roots).

4.3. Elliptic regime

Let us first focus on the elliptic *non-Newtonian* regime ($0 < m < 1$); for each k_0 there exist four complex l_0 solutions and thus four models of the type (4.5). The substitution of a linear combination of these elementary solutions into the midplane boundary condition (3.12) leads to solutions of the form

$$\begin{aligned}\delta\psi = \eta(t)\sin(k_0 a_1)(C_1(t)\cos(\sqrt{1-m}\,la_2)sh(\sqrt{m}\,la_2) \\ +C_2(t)\sin(\sqrt{1-m}\,la_2)ch(\sqrt{m}\,la_2))\,,\end{aligned} \tag{4.7}$$

where we used the notation $l = k_0\varepsilon'^2$. Selecting the initial values for δx_2

$$\delta x_2 = \delta h\cos(k_0 a_1) \tag{4.8}$$

and substituting (4.7) into the stress-free boundary condition (3.13) yields C_1 and C_2

$$C_1 = -2\sqrt{m}\,l^3\sin(\sqrt{1-m}\,hl)sh(\sqrt{m}\,hl)\,,$$
$$C_2 = 2\sqrt{m}\,l^3\cos(\sqrt{1-m}\,hl)ch(\sqrt{m}\,hl)\,,$$

as well as the following integro-differential equation for $\eta(t)$:

$$\begin{aligned}\eta m l^4(\sqrt{m}\sin(2\sqrt{1-m}\,hl) + \sqrt{1-m}\,sh(2\sqrt{m}\,hl)) \\ = 2\varepsilon'\left(\delta h\varepsilon'^2 + 1\sqrt{m}\int_0^t \eta l^3\sin(2\sqrt{1-m}\,hl)d\tau\right)\,.\end{aligned} \tag{4.9}$$

Since we are primarily interested in the relative growth, we search for G as a function of η. Making use of (3.15), (4.3), (4.8), and of $\delta\psi(a_1, h, t) = m^{1/2}\eta(t)\sin(k_0 a_1)l^3\sin(2hl(1-m)^{1/2})$, the perturbation instantaneous relative growth rate G is found as

$$G = \frac{m\sqrt{m}\eta l^4 \sin(2\sqrt{1-m}hl)}{\delta h\varepsilon'^2 + \sqrt{m}l\int_0^t \eta l^3 \sin(2\sqrt{1-m}hl)d\tau}, \tag{4.10}$$

where η is readily eliminated by using (4.9); the result is

$$G = \frac{2m\varepsilon' \sin(2\sqrt{1-m}hl)}{\sqrt{m(1-m)}sh(2\sqrt{m}hl) + m\sin(2\sqrt{1-m}hl)}. \tag{4.11$_1$}$$

In the rate-intensive limit $m \to 0$, the relative rate of growth $(4.11)_1$ tends to

$$G = \frac{2\varepsilon' \sin(2hl)}{2hl + \sin(2hl)}. \tag{4.11$_2$}$$

The relative amplification is given by the second of the relations (4.4)

$$\frac{a(t)}{a(0)} = \exp\left(\int_0^t \frac{G(\tau)}{m}d\tau\right). \tag{4.4$_2$}$$

The *Newtonian* elliptic regime needs a special treatment; looking for a combination of elementary solutions of the form (4.5) compatible with the boundary condition (3.12) leads to

$$\delta\psi = \eta(t)\sin(k_0 a_1)(C_1(t)sh(la_2) + C_2(t)a_2 ch(la_2)) \tag{4.12}$$

instead of (4.7); following the lines of the foregoing method, $\eta(t)$ is now found to be a solution for the integro-differential equation

$$\eta(2hl + sh2hl) = 4\varepsilon'\left(\delta h\varepsilon'^2 + hl\int_0^t \eta d\tau\right). \tag{4.13}$$

Making use of (3.15), (4.3), (4.12) and (4.13), one obtains the disturbance time-dependent relative growth rate G as

$$G = \frac{a'}{a} = \frac{hl\eta}{\delta h\varepsilon'^2 + hl\int_0^t \eta d\tau} = \frac{4hl\varepsilon'}{2hl + sh(2hl)}, \tag{4.14}$$

as well as the relative amplification $a(t)/a(0)$ by using the integration $(4.4)_2$.

G is time-dependent; it is sketched in the dispersion curve (Fig. 1), for example at the initial time $t = 0$. In the short wavelength range it is seen that the non-uniformity growth rate is overestimated in the long-wavelength model. Let us pinpoint the mechanism involved: the average axial tensile stress on the cross-section of the neck is larger in the long wavelength approximation than in the biaxial model. This implies that, for the same resultant axial tensile force, the section area is larger in the biaxial model. Thus necking proceeds more quickly in the long wavelength approximation; in other words, the stabilizing effects of viscosity are more effective at short wavelengths. Of primary interest for our purposes is the occurrence of a maximum at the origin of the wavenumber axis, that is to say a zero critical wavenumber. Note in addition, as HUTCHINSON *et al.* [5] did in their Eulerian account of the issue, that in the approximate range $1.6 < k_0 h < 3.2$, G is negative and the relative size of the non-uniformity actually decays for a nonlinear

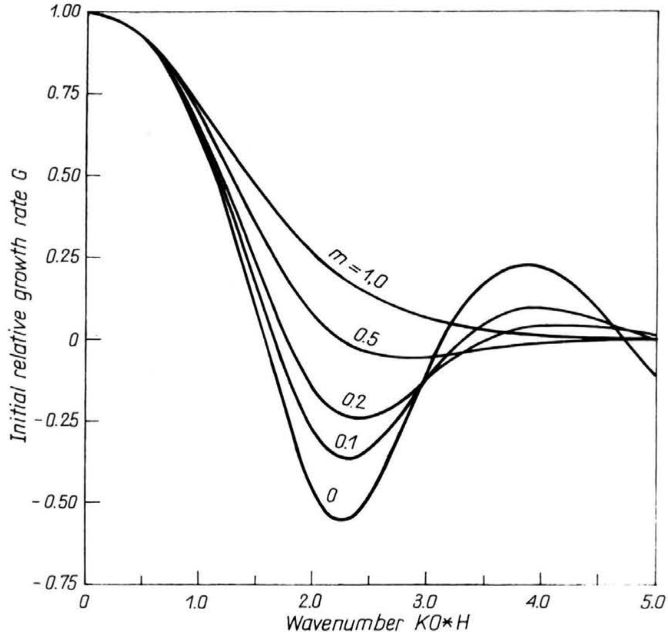

FIG. 1. Influence of rate-sensitivity on the initial dispersion curves (Relative growth rate G vs. wavelength k_0h); $m \to 0$, $m = 0.1, 0.2, 0.5, 1$.

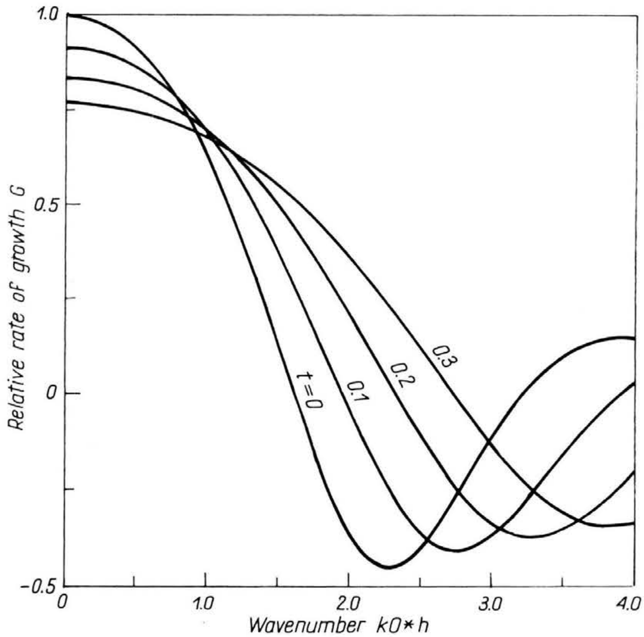

FIG. 2. Time evolution of the dispersion curve (Relative growth rate G vs. wavenumber k_0h); $m = 0.05$; $t = 0, 0.1, 0.2, 0.3$

viscosity. In the present approach, that surprising feature is viewed as initial behavior only; the time-evolution of the relative rate of growth G $(4.11)_1$, as sketched in Fig. 2, shows that G becomes positive in that range when a sufficiently long time has elapsed.

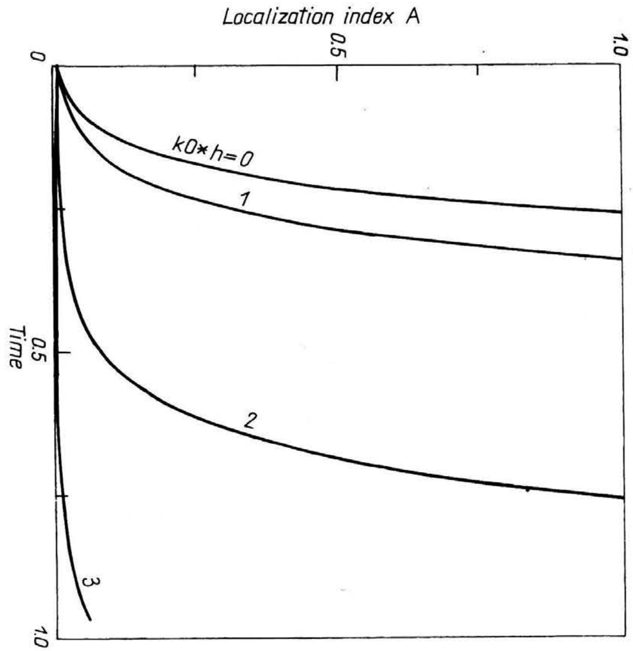

FIG. 3. Time evolution of the non-uniformity relative amplitude a; $m = 0.05, a(0) = 0.01; k_0h = 0, 1, 2, 3$.

The relative amplitude $(4.4)_2$ is integrated by a standard Gauss routine, with the initial condition $\delta h/h = 10^{-2}$; the results are plotted in Figs. 3, 4 and 5. The long wavelength time-evolution ($k_0 = 0$) is a monotonic increase (Fig. 4), although the relative rate of growth G decreases with time (Fig. 2). A dramatic increase in the amplification as the rate-sensitivity decreases from its Newtonian value $m = 1$ to the rate-insensitive limit $m = 0$ is observed in this limit, the instability is instantaneously completed. The larger the rate sensitivity, the more stabilizing the viscosity at long wavelengths. Nonlinear viscosity plays a far more complex role in the evolution of disturbances of shorter wavelengths (Figs. 3 and 5). For $0 < m < 1$, these perturbations have a monotonic relative growth only after a certain amount of time has elapsed; therefore, any conclusion of stability grounded on the early time decrease of some perturbations (for example: $1.6 < k_0h < 3.2$ in Fig. 2) overlooks the long-term tendency to relative instability of those perturbations. However, the shorter the wavelength, the smaller the corresponding amplification, and the later it appears (Fig. 3); therefore, the perturbation with the largest wavelength remains dominant throughout the phenomenon. In Fig. 5 are plotted the results for $k_0h = 2.3$; in the long term, the stabilizing effects are confirmed as rate-sensitivity increases. Yet, the trend is reversed at early times when $0 < m < 1$, which can be seen in Fig. 1 as well, at time $t = 0$: the larger the rate-sensitivity, the less stabilizing the viscosity. Later on, the usual damping influence sets in, but the growth of the short wavelength perturbations lags behind its long wavelength counterpart (Fig. 3).

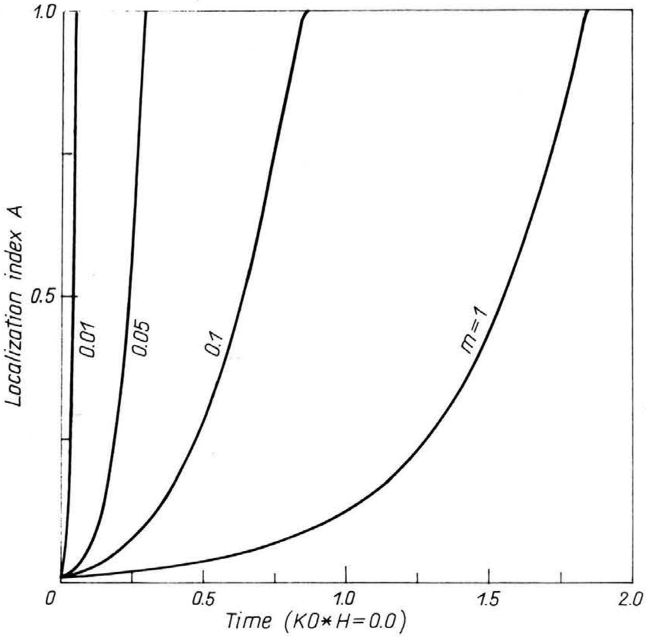

FIG. 4. Influence of non-linear viscosity on the evolution of the non-uniformity relative amplitude a; $k_0h = 0$. (Long wavelength approximation); $m = 0.01, 0.05, 0.1, 1$.

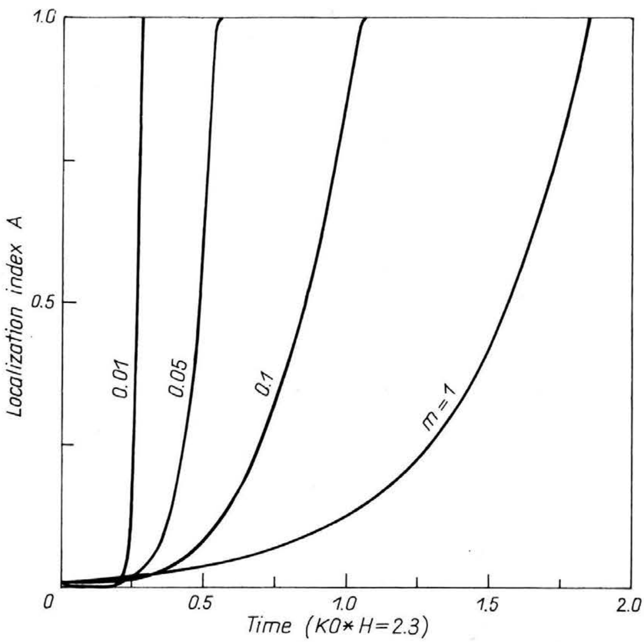

FIG. 5. Same as Fig. 4, with $k_0h = 2.3$, in the short wavelength range $1.6 < k_0h < 3.2$ with an initial decay in case of non-Newtonian behavior.

The origin of that effect can be pinpointed in the constitutive relationship $(2.8)_3$, which entails $\delta\sigma_e = m d_e^{m-1}\delta d_e$. Thus when $m = 1$, $\delta\sigma_{e1} = \delta d_e$ whereas $\delta\sigma_{e2} \cong m\delta d_e/d_e$ at small rate-sensitivities; with $d_e = \varepsilon'$, one obtains $\delta\sigma_{e2} \cong m\delta\sigma_{e1}/\varepsilon'$. Therefore $\delta\sigma_{e2} \ll \delta\sigma_{e1}$ at early times, and the variations δA in the cross-section area are such that $\delta A_2 \ll \delta A_1$, which indicates that necking proceeds more quickly in the Newtonian case. Later on, d_e decreases, and $\delta\sigma_{e2} > \delta\sigma_{e1}$ and $\delta A_2 > \delta A_1$; necking is then faster in low rate sensitive materials.

4.4. Hyperbolic regime

It is interesting to note that in the hyperbolic case $m = 0$ of a rigid-plastic material, the behavior of the perturbations is very different from what was obtained in the elliptic limit $m \to 0$. Following the lines of the foregoing method, one obtains solutions of the form

$$\delta\psi = \eta(t)\sin(k_0 a_1)\sin(l a_2)\,, \tag{4.15}$$

with a single pair of families of characteristics bisecting the $x_1 - x_2$ directions. Consequently it is found from the stress-free boundary condition (3.13) that

$$\forall t \quad \delta h + k_0 \int_0^t \eta(\tau)\sin(hl(\tau))d\tau = 0\,. \tag{4.16}$$

Taking the time-derivative

$$\forall t \quad \eta(t)\sin(hl(t)) = 0 \tag{4.17}$$

it is seen that η is zero, unless $hl = n\pi$ for which it is undetermined. Comparing the integrodifferential equations (4.9) and (4.13) in the elliptic regime with the algebraic relation (4.17) in the hyperbolic case, one can see that the formulation was transformed from an initial value problem to an eigenvalue problem. Since a series of values for l are available in the hyperbolic regime, the necking mode remains undetermined as well; it is only by allowing the material rate-sensitivity to approach zero in the elliptic regime, as done in $(4.11)_2$, that the preferred mode $l = 0$ is unequivocally pointed out. This singular behavior may be regarded as an abrupt transition from the uniform stretching mode to a different mode characteristic of necking. In this context, bifurcation is a more appropriate term that instability to qualify the transition.

5. Summary and concluding remarks

The present paper emphasizes some differences in the transition from uniform to non-uniform deformation between rate-dependent and rate-independent rigid-plastic materials; the issue is illustrated in the basic problem of the plane tension test. For this purpose, an incompressible power law material $\sigma_e^* = \mu d_e^{*m}$ is used. When $m \neq 0$, the material is time-dependent; the rate-independent non-hardening rigid-plastic material, $\sigma = \mu$, is obtained for $m = 0$. A very small plastic strain-rate sensitivity, which would go unnoticed in any low rate testing, is sufficient to alter significantly the predictions on the transition from uniform to non-uniform deformation; it is either a bifurcation ($m = 0$) or an instability ($m \neq 0$), depending on the mere existence of rate-sensitivity.

Revealing results are obtained for these simple material models by using a linear Lagrangian perturbation methods since the question at issue is merely the onset of necking, linear methods are sufficient. The analysis results in a fourth order partial differential equation with time-dependent coefficients. Depending on the elliptic (viscous) or hyperbolic (non-viscous) classification of that equation, different perturbation modes are obtained. In the former case, the boundary conditions provide an initial value problem. Owing to the stretching, the time-dependence of the perturbation modes is not merely exponential; it is rather the solution for an integro-differential equation. In the latter case the boundary conditions provide an eigenvalue equation, with non-unique solutions; thus, viscosity helps to select a unique time-evolving deformation mode, whereas no such mode can be uniquely determined through the rigid-plastic approach. That conclusion could possibly be altered if the material were strain-hardening.

The well-known retardation of long-wavelength instabilities due to material rate-sensitivity is noted; the role of nonlinear viscosity in the evolution of disturbances of shorter wavelength is also analyzed. The surprising prediction of HUTCHINSON, NEALE and NEEDLEMAN [5] that, in some short wavelength range, the nonuniformity actually decays in non-Newtonian materials is shown to pertain only to the early time evolution. Later on, the usual tendency to instability sets in, but the growth of short wavelength perturbations lags behind its long wavelength counterpart.

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