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Average elastic-plastic behaviour of polylayered composite

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A POLYLAYERED composite is an aggregate constituted by a collection of grains. Each grain represents a periodically layered superposition of two phases. Each phase is described by an isotropic elastic perfectly plastic model; both phases have the same elastic properties and they differ only by their plastic yield stress. The aim of the paper is to derive the constitutive equations of such materials using different homogenization approaches.

1. Introduction

THE COMPOSITE investigated is an aggregate of periodically layered grain (Fig. 1). This situation corresponds, for example, to a polycrystal in which every grain has developed a dislocation structure with soft cells limited by hard walls. Such structure is developed, in particular, by plastic cyclic excitations on metallic materials [1, 2]. The material defined in this way is called a polylayered composite.

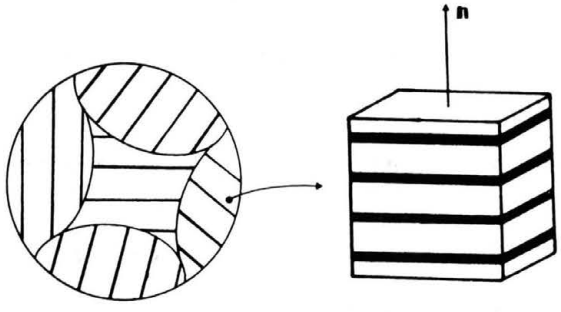

FIG. 1. A polylayered composite.

The purpose of the present work is to present and illustrate, on such an aggregate, a simplified elastic-plastic homogenization approach improving the classical Voigt approximation and which takes advantage of the layered structure of each grain. So it will, in some averaged sense, take into account the anisotropic and heterogeneous behaviour of each grain.

The individual behaviour of each grain has been analyzed in [3]. Each grain is assumed to represent a periodical layered superposition of two phases. Each phase is described by an isotropic elastic-perfectly plastic model obeying the Von Mises plastic criterion; both phases have the same elastic properties and they differ only by their plastic yield stress Y. The homogenization of such a layered structure is a simple problem with explicit localization conditions leading to uniform stress and strain in each constituent. The resulting transverse isotropic behaviour of the grain can then be completely analyzed and is characterized by a strong anisotropy.

These results, however, cannot be directly compared to experiments on polycrystals in which this anisotropy is smoothened and averaged through the intergranular heterogeneities. Thus, the polylayered aggregate will provide a better understanding of the elastic-plastic behaviour of polycrystals.

2. The homogenized behaviour of a two phases layered composite

This composite represents a periodical layered superposition of two different materials A and B with volume fractions f and $(1-f)$, respectively. Materials A and B are described by an isotropic elastic-perfectly plastic model obeying Von Mises yield condition $(|\boldsymbol{\sigma}^D| - \sqrt{(2/3)}Y = 0$, where $|\boldsymbol{\sigma}^D|$ denotes the stress deviator intensity $(|\boldsymbol{\sigma}^D|^2 = \boldsymbol{\sigma}^D : \boldsymbol{\sigma}^D = \sigma^D{}_{ij}\sigma^D{}_{ij})$. The two phases have the same elastic properties defined by the usual elastic constants E (Young modulus) and ν (Poisson ratio) and they differ by their plastic yield limit Y.

The homogenized behaviour of such a composite has been presented and discussed in [3] using Voigt, Reuss or layered homogenization. Let us briefly present here the essential results. Denoting by capital letters the global macroscopic quantities and by small letters the microscopic ones corresponding to the different phases, the macroscopic mean stress $\boldsymbol{\Sigma}$ and strain $\boldsymbol{\epsilon}$ are defined by:

$$\boldsymbol{\Sigma} = f\boldsymbol{\sigma}^A + (1-f)\boldsymbol{\sigma}^B, \quad \boldsymbol{\epsilon} = f\boldsymbol{\varepsilon}^A + (1-f)\boldsymbol{\varepsilon}^B.$$

The localization conditions derived from continuity and equilibrium can be written [3]

$$\boldsymbol{\sigma}_V = \boldsymbol{\Sigma}_V, \quad \boldsymbol{\epsilon}_H = \boldsymbol{\varepsilon}_H,$$

where, for a given symmetric second order tensor $\mathbf{X}$, the projections $\mathbf{X}_V$ and $\mathbf{X}_H$ are explicitly defined by

$$\mathbf{X}_V = 2(\mathbf{X}\mathbf{n})^S - (\mathbf{X} : \mathbf{n})\mathbf{n}, \quad \mathbf{X}_H = \mathbf{X} - \mathbf{X}_V, \quad \mathbf{n} = \vec{n} \otimes \vec{n},$$

with $\vec{n}$ denoting the unit normal vector to the layers, $\otimes$ the tensor product and $\mathbf{Y}^S$ the symmetric part of any tensor $\mathbf{Y}$. Using a rectangular coordinate system with x_1, x_2 in the layer plane and x_3 in the $\vec{n}$ direction, these projections are

$$\mathbf{X}_H : \begin{bmatrix} X_{11} & X_{21} & 0 \\ X_{12} & X_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{X}_V : \begin{bmatrix} 0 & 0 & X_{31} \\ 0 & 0 & X_{32} \\ X_{13} & X_{23} & X_{33} \end{bmatrix}.$$

The homogenized elastic-plastic incremental behaviour law of this composite, derived in [3], can be written in the form

$$d\boldsymbol{\epsilon} = \frac{1+\nu}{E} d\boldsymbol{\Sigma} - \frac{\nu}{E}\,\mathrm{tr}(d\boldsymbol{\Sigma})\mathbf{1} + d\boldsymbol{\epsilon}^p, \tag{2.1}$$

$$d\boldsymbol{\epsilon}^p = f d\boldsymbol{\varepsilon}^{pA} + (1-f) d\boldsymbol{\varepsilon}^{pB}, \tag{2.2}$$

$$d\boldsymbol{\epsilon}^{pA} = \{[f h_{AA}\boldsymbol{\sigma}^{DA} + (1-f)h_{AB}\boldsymbol{\sigma}^{DB}] : d\boldsymbol{\Sigma}\}\boldsymbol{\sigma}^{DA}, \tag{2.3}$$

$$d\boldsymbol{\epsilon}^{pB} = \{[f h_{AB}\boldsymbol{\sigma}^{DA} + (1-f)h_{BB}\boldsymbol{\sigma}^{DB}] : d\boldsymbol{\Sigma}\}\boldsymbol{\sigma}^{DB} \tag{2.4}$$

with

$$\sigma^A = \Sigma - (1-f)\frac{E}{1+\nu}[(\varepsilon^{pA}{}_H - \varepsilon^{pB}{}_H) + \frac{\nu}{1-\nu}\operatorname{tr}(\varepsilon^{pA}{}_H - \varepsilon^{pB}{}_H)(\mathbf{1}-\mathbf{n})],$$

$$\sigma^B = \Sigma + f\frac{E}{1+\nu}[(\varepsilon^{pA}{}_H - \varepsilon^{pB}{}_H) + \frac{\nu}{1-\nu}\operatorname{tr}(\varepsilon^{pA}{}_H - \varepsilon^{pB}{}_H)(\mathbf{1}-\mathbf{n})].$$

The coefficients h_{AA}, h_{BB} and h_{AB}, vanishing in the elastic case, are given by

CASE 1. Only the soft phase A is plastic,

$$h_{AA} = (1+\nu)/[Ef(1-f)m_{AA}], \quad h_{BB} = h_{AB} = 0.$$

CASE 2. The two phases are plastic,

$$h_{BB} = (1+\nu)m_{BB}/[Ef(1-f)(m_{AA}m_{BB} - m_{AB}^2)],$$
$$h_{AA} = (1+\nu)m_{AA}/[Ef(1-f)(m_{AA}m_{BB} - m_{AB}^2)],$$
$$h_{AB} = (1+\nu)m_{AB}/[Ef(1-f)(m_{AA}m_{BB} - m_{AB}^2)],$$

with

$$m_{AA} = |\sigma^{DA}{}_H|^2 + [\nu/(1-\nu)]\operatorname{tr}(\sigma^{DA}{}_H),$$
$$m_{BB} = |\sigma^{DB}{}_H|^2 + [\nu/(1-\nu)]\operatorname{tr}(\sigma^{DB}{}_H),$$
$$m_{AB} = \sigma^{DA}{}_H : \sigma^{DB}{}_H + [\nu/(1-\nu)]\operatorname{tr}(\sigma^{DB}{}_H)\operatorname{tr}(\sigma^{DA}{}_H).$$

For a given excitation (for example Σ), the solution (ϵ) can be obtained through the numerical resolution of differential equations (2.1) to (2.4). Examples and more details will be found in [3].

3. The polylayered aggregate

3.1. Texture description

We now consider the polylayered aggregate consisting of a large number of identical layered grains with different orientations $\vec{n}$. This aggregate is therefore described by its texture, i.e. by the distribution function $f(\vec{n})$ of the orientation. This texture function is defined on the unit sphere Ω, even in $\vec{n}$, and it must satisfy the condition

$$\int_{\Omega} f(\vec{n})\, d\Omega = 1.$$

It will be convenient to use a Taylor expansion for the texture function [4],

$$f(\vec{n}) = \beta_{ij}{}^{(0)} + \beta_{ij}{}^{(1)}n_i n_j + \beta_{ijkl}{}^{(2)}n_i n_j n_k n_l + \dots .$$

The first order approximation $f = \beta^{(0)} = 1/4\pi$ corresponds to the isotropic texture. The second order approximation which can be written as

$$f(\vec{n}) = 1/4\pi + \beta_{ij}n_i n_j, \quad \beta_{kk} = 0$$

with a symmetric deviatoric texture tensor β corresponding to an orthotropic texture, while more complex anisotropy will be obtained for the higher order approximations. It should also be noted that the texture tensors $\beta^{(n)}$ can be obtained from the moments of the texture distribution function [5].

In the following we shall concentrate on the second order approximation and on a transverse isotropic texture around $\vec{t}$. The texture function $f(\vec{n})$ can then be written as

$$f(\vec{n}) = f(\theta) = (1 + 3b\cos^2\theta)/[4\pi(1+b)] \tag{3.1}$$

with a single scalar parameter b ($b = 0$ in the isotropic case), while θ is the angle $(\vec{n}, \vec{t})$.

In the context of cyclic plasticity for instance, this special case will correspond to the microstructural evolution under a tensile cyclic test in the direction **t**. The scalar parameter b can then be considered as an internal state variable describing the microstructure induced evolution of the plastic response.

3.2. Polylayered homogenization

The determination of the macroscopic mechanical response of the polylayered aggregate is a second level homogenization process which will be performed using Voigt and Reuss approximations. More precisely, when a macroscopic stress $\overline{\boldsymbol{\Sigma}}$ and strain $\overline{\boldsymbol{\epsilon}}$ is imposed on the aggregate, the resulting stress and strain in each grain with orientation n will be assumed uniform and depending only on $\vec{n}$; they will be defined by $\boldsymbol{\Sigma}(\vec{n})$ and $\boldsymbol{\epsilon}(\vec{n})$

$$\overline{\boldsymbol{\Sigma}} = \int_{\Omega} f(\vec{n})\boldsymbol{\Sigma}(\vec{n})\, d\Omega, \quad \overline{\boldsymbol{\epsilon}} = \int_{\Omega} f(\vec{n})\boldsymbol{\epsilon}(\vec{n})\, d\Omega, \tag{3.2}$$

where Ω is the surface defined by a unit sphere. The tensors $\boldsymbol{\Sigma}(\vec{n})$ and $\boldsymbol{\epsilon}(\vec{n})$ are, respectively, the stress and strain in the grains oriented by $\vec{n}$.

The Reuss approximation assumes that every grain is submitted to the same stress tensor.

$$\boldsymbol{\Sigma}(\vec{n}) = \overline{\boldsymbol{\Sigma}} \quad \forall n.$$

For a given $\overline{\boldsymbol{\Sigma}}$, Eqs. (2.1) to (2.4), allow for the determination of $\boldsymbol{\epsilon}(\vec{n})$ and thus of $\overline{\boldsymbol{\epsilon}}$.

Similarly, the Voigt approximation assumes that every grain is submitted to the same strain tensor,

$$\boldsymbol{\epsilon}(\vec{n}) = \overline{\boldsymbol{\epsilon}} \quad \forall \vec{n}$$

allowing for the determination of $\boldsymbol{\Sigma}(\vec{n})$ and $\overline{\boldsymbol{\Sigma}}$.

4. Application to elastic-plastic tensile behaviour

As an application, this second level homogenization process will be performed for a tensile test in the direction $\vec{t}$ defining the transverse isotropy of the texture function (3.1).

$$\overline{\boldsymbol{\Sigma}} = \overline{\Sigma}\mathbf{t}, \quad \mathbf{t} = \vec{t} \otimes \vec{t}. \tag{4.1}$$

4.1. Reuss approximation

Each grain is then submitted to an off-axis tensile test. Let us denote by $\vec{n}_1$ the normal to the plane $(\vec{n}, \mathbf{t})$ (Fig. 2). The matrix $[\boldsymbol{\epsilon}(\theta)]$ representing the response $\boldsymbol{\epsilon}(\vec{n})$ in the orthonormal triad $(\vec{n}_1, \vec{n}_2, \vec{t})$, is then

$$[\boldsymbol{\epsilon}(\theta)] : \begin{bmatrix} \epsilon_1(\theta) & 0 & 0 \\ 0 & \epsilon_2(\theta) & \epsilon_{23}(\theta) \\ 0 & \epsilon_{23}(\theta) & \epsilon_L(\theta) \end{bmatrix}.$$

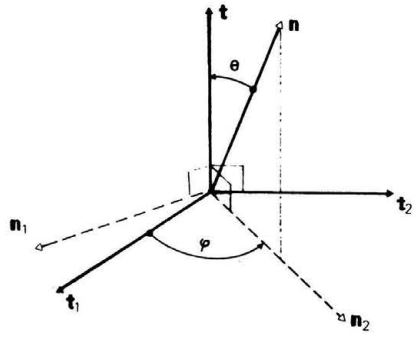

FIG. 2. Traction and layered directions.

The normal $\vec{n}$ can be defined in a fixed orthonormal triad $(\vec{t}_1, \vec{t}_2, \vec{t})$ by the two angles θ and φ (Fig. 2). The matrix $[\epsilon(\theta, \varphi)]$ representing $\epsilon(\vec{n})$ in the triad $(\vec{t}_1, \vec{t}_2, \vec{t})$ is obtained using the following transformation:

$$[\epsilon(\theta, \varphi)] = [\varphi]^T [\epsilon(\theta)][\varphi]$$

with

$$[\varphi] : \begin{bmatrix} \sin\varphi & -\cos\varphi & 0 \\ \cos\varphi & \sin\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

(4.2)

$$[\epsilon(\theta, \varphi)] : \begin{bmatrix} \epsilon_1(\theta)\sin^2\varphi + \epsilon_2(\theta)\cos^2\varphi & \sin 2\varphi[\epsilon_2(\theta) - \epsilon_1(\theta)]/2 & \cos\varphi\epsilon_{23}(\theta) \\ \sin 2\varphi[\epsilon_2(\theta) - \epsilon_1(\theta)]/2 & \epsilon_1(\theta)\cos^2\varphi + \epsilon_2(\theta)\sin^2\varphi & \sin\varphi\epsilon_{23}(\theta) \\ \cos\varphi\epsilon_{23}(\theta) & \sin\varphi\epsilon_{23}(\theta) & \epsilon_L(\theta) \end{bmatrix}.$$

Using (3.1) and (4.2) with

$$\int_{\Omega} x \, d\Omega = \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{2\pi} \sin\theta \, d\theta \, d\varphi$$

the mean strain $\bar{\epsilon}$ is obtained in the following form:

$$\bar{\epsilon} = \bar{\epsilon}_T(\vec{t}_1 \otimes \vec{t}_1 + \vec{t}_2 \otimes \vec{t}_2) + \bar{\epsilon}_L \vec{t} \otimes \vec{t}, \tag{4.3}$$

with

$$\bar{\epsilon}_L = 2\pi \int_0^{\pi} f(\theta)\epsilon_L(\theta)\sin\theta \, d\theta, \tag{4.4}$$

$$\bar{\epsilon}_T = \pi \int_0^{\pi} f(\theta)[\epsilon_1(\theta) + \epsilon_2(\theta)]\sin\theta \, d\theta. \tag{4.5}$$

These relations can be interpreted as defining some generalized mixture laws giving the macroscopic response from the elastic-plastic response of each grain in an off-axis tensile test.

4.2. Quasi-Voigt approximation

Another mixture law is obtained by assuming that each grain is submitted to an off-axis tensile test and to a uniform longitudinal strain,

$$\mathbf{\Sigma} = \Sigma(\theta)\mathbf{t}, \quad \epsilon_L = \bar{\epsilon}_L \quad \forall\theta.$$

Strictly speaking this is not the Voigt approximation but a certain mixed approximation using Voigt's assumption in the longitudinal direction only (quasi-Voigt approximation). We obtain then the same relations (4.1), (4.3) and (4.5) with

$$\bar{\Sigma} = 2\pi \int_0^{\pi} f(\theta)\Sigma_L(\theta)\sin\theta\, d\theta. \tag{4.6}$$

This defines another generalized mixture law.

4.3. Illustration

These two mixture laws give the elastic-plastic response of the aggregate from the average of the response of a layered grain in an off-axis tensile test. This response has been obtained and discussed in [3]. Application has been presented in [3] for the following numerical values (representation of the dislocation cell structures):

$$Y_A = 20.7 \text{ MPa}, \quad Y_B = 100 \text{ MPa},$$
$$E = 56700 \text{ MPa}, \quad n = 0.35, \quad f = 0.1.$$

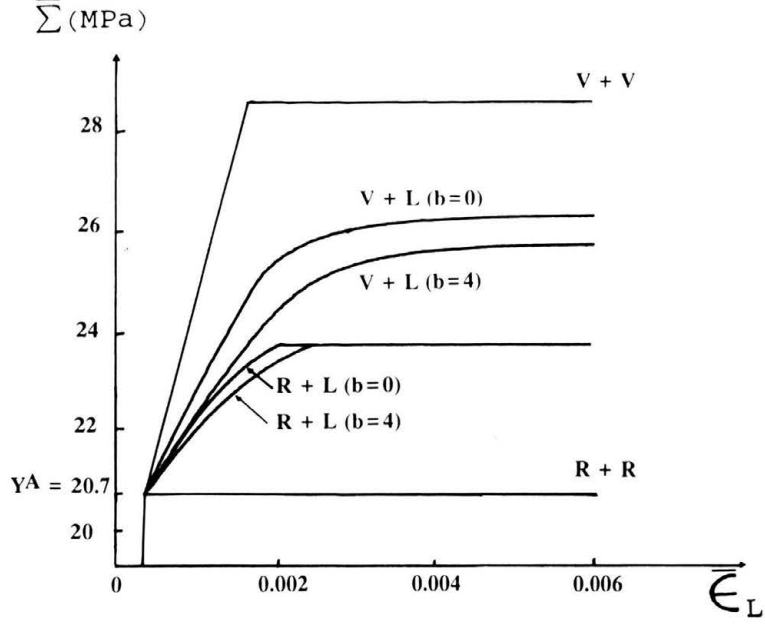

FIG. 3. Tensile test.

The corresponding polylayered responses are given in Fig. 3 in the isotropic case ($b = 0$) and in a strongly anisotropic case ($b = 4$) for different approximations:

$R + R$: Reuss homogenization for the grain [3] and for the polylayered response,

$V + V$: Voigt homogenization for the grain [3] and for the polylayered response,

$L + R$: Layered homogenization for the grain (relations (2.1) to (2.4)) and Reuss approximation for the polylayered response,

$L + V$: Layered homogenization for the grain (relations (2.1) to (2.4)) and Voigt approximation for the polylayered response.

It is clear that the layered homogenization which takes advantage of the anisotropic behaviour of the grain, is more reasonable that the Voigt or Reuss approximations.

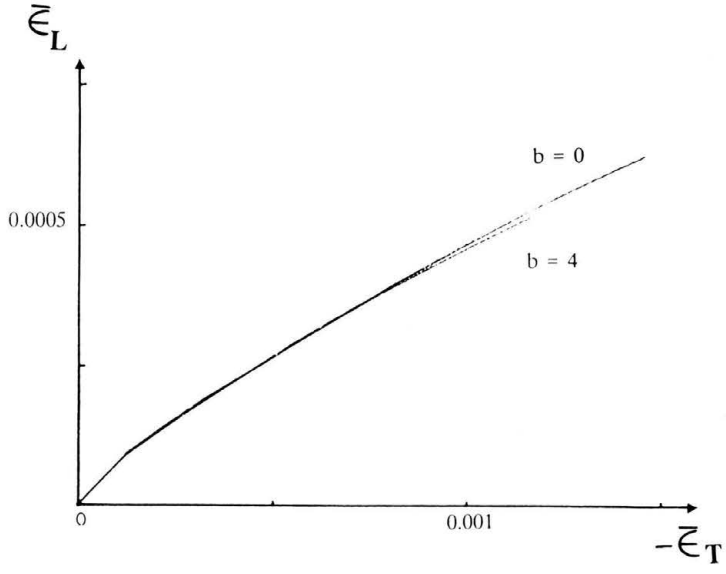

FIG. 4. Tensile test.

Figure 4 represents the evolution of $\bar{\epsilon}_T$ function of $\bar{\epsilon}_L$ for $L + R$ approximation (relations (4.4) and (4.5)). This shows a weak influence of the anisotropy.

5. Conclusion

It follows from this analysis and from the comparison between the complete Voigt or Reuss approximation and their polylayered counterparts that the layered structure and anisotropy of each brain may play a significant role in the macroscopic response, even if crudely averaged.

Much work remains to be done to obtain a complete description of the cyclic hardening in metallic materials, even in the simplest case analyzed here of tensile loading with the corresponding transverse isotropic structure evolution. More precisely, such a model would require the description of the texture evolution (i.e. for instance an evolution equation for the texture parameter b) and the combination of the homogenization procedure described here with an appropriate description of the intergranular hardening process, for instance along the line developed in [2].

The present work should be a significant step in this direction.

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