Arch. Mech., 44, 1, pp. 15–29, Warszawa 1992

On localization phenomena in thermo-elasto-plasticity

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THE LOCALIZATION OF DEFORMATION in the form of shear bands is understood as an instability in the macroscopic description of inelastic deformation. The effects of thermomechanical couplings in the development of these shear bands are analyzed. Conditions for the onset of localization when these couplings are taken into account are given both inside a body and at its boundary. These conditions are illustrated by a particular set of constitutive equations. Destabilizing effects due to couplings are underlined.

1. Introduction

LOCALIZATION OF DEFORMATION in narrow zones or shear bands is very common in practice for many solids. A common approach to this phenomenon is to understand it as an instability in the macroscopic description of inelastic deformation of the material. More precisely, this phenomenon is usually described for rate-independent elastic-plastic materials as a bifurcation into modes involving discontinuities of the velocity gradient.

While the basic principles for the analysis of shear banding were established by HADAMARD [1] and then applied by THOMAS [2], HILL [3] and MANDEL [4] in various contexts, the full application to localization phenomena is due to RICE [5, 6] who gave the necessary conditions for the occurrence of continuous localization for rate-independent materials by considering only the linear comparison solid corresponding to the loading branch of the constitutive relation. RUDNICKI and RICE [7], taking into account unloadings, furnished later the necessary conditions for the appearance of discontinuous localization for incrementally bi-linear solids. Recently, BORRÉ and MAIER [8] showed that the above conditions are actually necessary and sufficient, for both continuous and discontinuous localizations. Recently, BENALLAL, BILLARDON and GEYMONAT [9] introduced the boundary conditions and gave the necessary and sufficient conditions for a singular surface of the velocity gradient to appear or to reach the boundary of a solid. In [10], the same authors considered also the interactions of shear bands with interfaces.

The complete analysis of the rate problem, at least for incrementally linear solids, suggests the possibility of emergence of other types of localization phenomena such as surface modes or interfacial modes which can indeed be interpreted respectively as localization of the deformation at either the boundaries or interfaces of solids.

The general analysis of the linear rate problem, based on results of modern theory of linear elliptic boundary value problems was given by BENALLAL, BILLARDON and GEYMONAT [10, 11, 12], who showed that for linear solids, the three types of localization modes described above correspond exactly to ill-posedness of this rate problem. Indeed, ill-posedness occurs *if and only if* one of the three following conditions fails:

the ellipticity condition, related to shear band modes, and the mechanical interpretation of which is the appearance of stationary acceleration waves [3, 4, 6];

the boundary complementing condition, related to surface modes and to the emergence of stationary surface waves such as Rayleigh waves for instance [13];

the interfacial complementing condition, related to interfacial modes and to the existence of stationary interfacial waves such as Stoneley waves [10].

The authors also remarked that when these last three conditions are satisfied, a finite number of linearly independent solutions (possibly uniqueness) exist at most; these solutions depend moreover continuously on the data and represent diffuse modes of deformation.

The same general analysis is not yet available for nonlinear materials and even for the incrementally bi-linear materials to be considered in this paper. However, it is possible for these last materials, to seek necessary and sufficient conditions for the appearance of singular surfaces of the velocity gradient. While it is possible to consider the corresponding linear problem and to define localization modes in presence of thermal effects in the same lines as above, we only study here the role of thermal effects on shear banding. We note however that, to the failure of the complementing conditions underlined before and to the corresponding deformation modes, are associated temperature fields localized either at the boundary or the interfaces of the solid. These latter fields are readily given by the heat equation when the velocity field is determined.

This paper examines one further aspect of the theory of localization for general rate-independent elastic-plastic solids. For sake of simplicity, we limit ourselves to the small strain range and to quasi-static situations; both large strains and dynamic effects can easily be incorporated and will be given in a forthcoming paper. The aspect considered here is the analysis of the roles of thermal properties and thermo-mechanical couplings on shear band localization modes. General constitutive equations are considered here and include most of the models existing in the literature and the results obtained are applicable to various localization phenomena in the thermomechanical behavior of materials.

In the usual analysis of shear banding, the velocity field is assumed to be and remain continuous at the instant of localization. When introducing thermal effects, similar bifurcation modes from a continuous state of temperature lead to temperature fields which may remain or not continuous. Two types of shear banding modes are thus possible depending on whether the temperature field is assumed to remain continuous or not.

The outline of the paper is as follows: in Sec. 2, the constitutive equations and related details are presented with particular emphasis to the heat equation; in Sec. 3, the conditions for localization including thermal effects are given both inside the body and at its boundary. In the last part, the results are applied to a more specific set of constitutive equations which allows to highlight some of the consequences of thermal effects on localization phenomena.

2. Constitutive relations

2.1. Constitutive equations

We consider in the following a general class of rate-independent (elastic-plastic, damageable,...) materials in the small strain range. The reversible behavior of such materials is given by the knowledge of the specific free energy $\Psi(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}_i, T)$ depending on the strain tensor $\boldsymbol{\varepsilon}$, the absolute temperature T and a set of internal variables $\boldsymbol{\alpha}_i$ assumed to describe various physical mechanisms. The stress tensor $\boldsymbol{\sigma}$, the thermodynamical forces $\mathbf{A}_i$ associated to the internal variables and the entropy s are obtained by the state laws:

$$\boldsymbol{\sigma} = \rho\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}}, \quad \mathbf{A}_i = -\rho\frac{\partial\Psi}{\partial\boldsymbol{\alpha}_i}, \quad s = -\frac{\partial\Psi}{\partial T}, \tag{2.1}$$

where ρ denotes the mass density. The irreversible behavior is characterized by a reversibility domain (inside which no irreversibility occurs) in the forces space and defined by

$$f(\mathbf{A}_j, \boldsymbol{\alpha}_j, T) \le 0. \tag{2.2}$$

The evolution of the internal variables is determined by the potential of irreversibility $F(\mathbf{A}_j, \boldsymbol{\alpha}_j, T)$ through the following normality rule

$$\dot{\boldsymbol{\alpha}}_i = \lambda\frac{\partial F}{\partial \mathbf{A}_i}, \quad \lambda \ge 0, \quad f(\mathbf{A}_j, \boldsymbol{\alpha}_j, T) \le 0, \quad \lambda\dot{f}(\mathbf{A}_j, \boldsymbol{\alpha}_j, T) = 0. \tag{2.3}$$

The heat flux $\mathbf{q}$ is taken to obey generalized Fourier's law of heat conduction

$$\mathbf{q} = -\mathbf{K}(\mathbf{A}_j, \boldsymbol{\alpha}_j, T)\mathbf{g} \tag{2.4}$$

where $\mathbf{g}$ is the temperature gradient and $\mathbf{K}$ the heat conduction matrix assumed to be positive definite in this paper.

When λ is positive, it is computed by the consistency condition $\dot{f} = 0$ as

$$\lambda = \left\langle \frac{-\dfrac{\partial f}{\partial \mathbf{A}_j} \bullet \boldsymbol{\Lambda}_j : \dot{\boldsymbol{\varepsilon}} + \left[\dfrac{\partial f}{\partial T} - \dfrac{\partial f}{\partial \mathbf{A}_j} \bullet \boldsymbol{\Gamma}_j\right]\dot{T}}{h} \right\rangle, \tag{2.5}$$

where we have set:

$$h = \frac{\partial f}{\partial \mathbf{A}_i} \bullet \boldsymbol{\Pi}_{ij} \bullet \frac{\partial F}{\partial \mathbf{A}_j} - \frac{\partial f}{\partial \boldsymbol{\alpha}_k} \bullet \frac{\partial F}{\partial \mathbf{A}_k}, \tag{2.6}$$

$$\boldsymbol{\Lambda}_j = \rho\frac{\partial^2\Psi}{\partial\boldsymbol{\alpha}_j\partial\boldsymbol{\varepsilon}}, \quad \boldsymbol{\Pi}_{ij} = \rho\frac{\partial^2\Psi}{\partial\boldsymbol{\alpha}_i\partial\boldsymbol{\alpha}_j}, \quad \boldsymbol{\Gamma}_j = \rho\frac{\partial^2\Psi}{\partial T\partial\boldsymbol{\alpha}_j}. \tag{2.7}$$

The symbol "$\bullet$" is the adequate contraction on the nature of the internal variables $\boldsymbol{\alpha}_j$; ":" is the double tensor contraction and $\langle\cdot\rangle$, the McAuley bracket. Summation convention on the indices is used.

With the former constitutive relations, the heat equation securing local conservation of energy is written as

$$\rho c\dot{T} = T\boldsymbol{\Delta} : \dot{\boldsymbol{\varepsilon}} + (T\boldsymbol{\Gamma}_j + \mathbf{A}_j) \bullet \dot{\boldsymbol{\alpha}}_j + r - \operatorname{Div}\mathbf{q}, \tag{2.8}$$

where r is the internal heat supply, c the specific heat at constant strain and internal state and we have set

$$c = -T\frac{\partial^2\Psi}{\partial T^2}, \tag{2.9}$$

$$\mathbf{D} = \mathbf{E} + \frac{T}{\rho c}\boldsymbol{\Delta} \otimes \boldsymbol{\Delta}, \tag{2.10}$$

$$\mathbf{E} = \rho\frac{\partial^2\Psi}{\partial\boldsymbol{\varepsilon}\partial\boldsymbol{\varepsilon}}, \quad \boldsymbol{\Delta} = \rho\frac{\partial^2\Psi}{\partial T\partial\boldsymbol{\varepsilon}}, \quad \mathbf{P}_j = \rho\frac{\partial^2\Psi}{\partial\boldsymbol{\varepsilon}\partial\boldsymbol{\alpha}_j}. \tag{2.11}$$

Using the energy equation (2.8) in the consistency condition (2.5) allows to get the plastic multiplier λ

$$\lambda = \left\langle \frac{\mathbf{v} : \mathbf{D} : \dot{\boldsymbol{\varepsilon}} - \beta}{H} \right\rangle, \tag{2.12}$$

where we have set again

$$\mathbf{v} : \mathbf{D} = -\frac{\partial f}{\partial \mathbf{A}_j} \bullet \mathbf{P}_j + \frac{T}{\rho c}\left\{\frac{\partial f}{\partial T} - \frac{\partial f}{\partial \mathbf{A}_j} \bullet ; \boldsymbol{\Gamma}_j\right\}\boldsymbol{\Delta}, \tag{2.13}$$

$$\beta = -\frac{T}{\rho c}\left[\frac{\partial f}{\partial T} - \frac{\partial f}{\partial \mathbf{A}_j} \bullet \boldsymbol{\Gamma}_j\right](r - \operatorname{div} \mathbf{q}), \tag{2.14}$$

$$H = h - \frac{T}{\rho c}\left[\frac{\partial f}{\partial T} - \frac{\partial f}{\partial \mathbf{A}_j} \bullet \boldsymbol{\Gamma}_j\right]\left[\left(\boldsymbol{\Gamma}_k + \frac{\mathbf{A}_k}{T}\right) \bullet \frac{\partial F}{\partial \mathbf{A}_k}\right]. \tag{2.15}$$

Using the state law $(2.1)_1$, the heat equation (2.8), the evolution laws $(2.3)_1$ and the formula (2.12), the stress rate may be written as

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{L}} : \dot{\boldsymbol{\varepsilon}} + \mathbf{m}, \tag{2.16}$$

where $\boldsymbol{\mathcal{L}}$ and $\mathbf{m}$ take respectively the forms

$$\boldsymbol{\mathcal{L}} = \begin{cases} \mathbf{D} = \mathbf{E} + \dfrac{T}{\rho c}\boldsymbol{\Delta} \otimes \boldsymbol{\Delta} \\ \qquad\qquad \text{if} \quad f < 0 \quad \text{or} \quad f = 0 \quad \text{and} \quad \mathbf{v} : \mathbf{D} : \dot{\boldsymbol{\varepsilon}} < \beta, \\ \mathbf{H} = \mathbf{D} - \dfrac{(\mathbf{D} : \mathbf{u}) \otimes (\mathbf{v} : \mathbf{D})}{H} \quad \text{if} \quad f = 0 \quad \text{and} \quad \mathbf{v} : \mathbf{D}\dot{\boldsymbol{\varepsilon}} \geq \beta, \end{cases} \tag{2.17}$$

$$\mathbf{m} = \begin{cases} \mathbf{d} = \dfrac{1}{\rho c}(r - \operatorname{div} \mathbf{q})\boldsymbol{\Delta} \\ \qquad\qquad \text{if} \quad f < 0 \quad \text{or} \quad f = 0 \quad \text{and} \quad \mathbf{v} : \mathbf{D} : \dot{\boldsymbol{\varepsilon}} < \beta, \\ \mathbf{h} = \dfrac{1}{\rho c}(r - \operatorname{div} \mathbf{q})\left[\boldsymbol{\Delta} + \dfrac{1}{H}\left\{\dfrac{\partial f}{\partial T} - \dfrac{\partial f}{\partial \mathbf{A}_j} \bullet \boldsymbol{\Gamma}_j\right\}\mathbf{D} : \mathbf{u}\right] \\ \qquad\qquad \text{if} \quad f = 0 \quad \text{and} \quad \mathbf{v} : \mathbf{D} : \dot{\boldsymbol{\varepsilon}} \geq \beta, \end{cases} \tag{2.18}$$

with

$$\mathbf{D} : \mathbf{u} = -\boldsymbol{\Lambda}_j \bullet \frac{\partial F}{\partial \mathbf{A}_j} - \frac{T}{\rho c}\left[\left\{\boldsymbol{\Gamma}_j + \frac{\mathbf{A}_j}{T}\right\} \bullet \frac{\partial F}{\partial \mathbf{A}_j}\right]\boldsymbol{\Delta}. \tag{2.19}$$

In the above formulae, $\boldsymbol{\mathcal{L}}$ denotes the adiabatic tangent modulus, $\mathbf{D}$ the adiabatic elastic modulus and $\mathbf{E}$ the isothermal elastic modulus. It is convenient for the following to define also the isothermal tangent modulus $\mathbf{L}$ (relating the stress rate to the strain rate in isothermal conditions) as

$$\mathbf{L} = \mathbf{E} - \frac{(\mathbf{E} : \mathbf{a}) \otimes (\mathbf{b} : \mathbf{E})}{h}, \tag{2.20}$$

with

$$\mathbf{E} : \mathbf{a} = -\left[\boldsymbol{\Lambda}_j \bullet \frac{\partial F}{\partial \mathbf{A}_j}\right], \quad \mathbf{b} : \mathbf{E} = \left[-\frac{\partial f}{\partial \mathbf{A}_j} \bullet \mathbf{P}_j\right]. \tag{2.21}$$

We assume in the following that $h > 0$, $H \neq 0$, $\mathbf{E}$ and $\mathbf{D}$ are positive definite.

3. Conditions for localization

3.1. The jump conditions

The conditions for localization which will be derived later express merely the appearance of singular surfaces across which the velocity gradient suffers jumps. In the uncoupled problem, these conditions were given by RICE [5, 6], RUDNICKI and RICE [7], BORRE and MAIER [8] for localisation inside the body and by BENALLAL, BILLARDON and GEYMONAT [9] for localization at the boundary. Conditions for singular surfaces of the velocity gradient to reach interfaces can also be established (see BENALLAL, BILLARDON and GEYMONAT [10]). As the rate constitutive equations given in (16) are bi-linear (the tangent modulus is dependent on the strain rate), two types of localization are usually recognized: continuous localization when the body is under plastic loading on each side of the singular surface and discontinuous localization when the body is under loading in one side and unloading on the other side.

Let us suppose that a singular surface $(\mathcal{S})$ of the velocity gradient lies in the body. This surface divides the body Ω into two parts Ω_1 and Ω_2, where mechanical and thermal fields exist denoted by subscripts 1 and 2, respectively. Let us also call $\mathbf{n}$ the unit normal to $(\mathcal{S})$ directed from Ω_1 to Ω_2. In the presence of such surfaces, it is necessary to consider jump conditions across the surface, imposed by the mechanical as well as the thermal equations.

If the velocity field $\mathbf{v}$ is assumed to be continuous, then it is necessary according to the Maxwell kinematical compatibility relation that the jump of the velocity gradient be in the form

$$[\varepsilon(\mathbf{v})] = \varepsilon(\mathbf{v}_1) - \varepsilon(\mathbf{v}_2) = \frac{1}{2}[\mathbf{a} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{a}], \tag{3.1}$$

where $\mathbf{a}$ is an arbitrary vector for the time being. The balance of linear momentum requires that the traction rate must be continuous across $(\mathcal{S})$ and therefore we have

$$[\dot{\sigma} \cdot \mathbf{n}] = \dot{\sigma}_1 \cdot \mathbf{n} - \dot{\sigma}_2 \cdot \mathbf{n} = \mathbf{0}. \tag{3.2}$$

Finally, the conservation of energy leads to the continuity of the heat flux across the singular surface $(\mathcal{S})$, which will be written as

$$[\mathbf{q} \cdot \mathbf{n}] = \mathbf{q}_1 \cdot \mathbf{n} - \mathbf{q}_2 \cdot \mathbf{n} = 0. \tag{3.3}$$

In the following we will give conditions for localization in two circumstances. We will consider the case where no restriction is imposed on the temperature field and the case where this field is assumed to remain continuous. These cases will be termed case I and case II, respectively. In case I, there will be no restriction on the temperature rate; in case II however, the jump condition (3.3) combined with the continuity of the temperature requires that the temperature rate should be continuous across $(\mathcal{S})$ so that

$$[\dot{T}] = \dot{T}_1 - \dot{T}_2 = 0. \tag{3.4}$$

In what follows, the necessary and sufficient conditions will be given for the singular surfaces of the velocity gradient to appear inside or at the boundary of a solid.

3.2. Localization inside the body

CASE I. The necessary and sufficient condition for localization inside the body is

$$\det(\mathbf{n} \cdot \mathbf{H} \cdot \mathbf{n}) \leq 0\,. \tag{3.5}$$

CASE II. The necessary and sufficient conditions for localization inside the body are

$$\det(\mathbf{n} \cdot \mathbf{H} \cdot \mathbf{n}) \leq 0\,, \tag{3.5}$$

$$(\mathbf{n} \cdot \mathbf{\Delta}) \cdot [\mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n}]^{-1} \cdot (\mathbf{n} \cdot \mathbf{D} : \mathbf{u}) + \left(\mathbf{\Gamma}_j + \frac{\mathbf{A}_j}{T}\right) \bullet \frac{\partial F}{\partial \mathbf{A}_j} = 0\,, \tag{3.6}$$

or equivalently

$$\det(\mathbf{n} \cdot \mathbf{L} \cdot \mathbf{n}) \leq 0\,, \tag{3.7}$$

$$(\mathbf{n} \cdot \mathbf{\Delta}) \cdot [\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n}]^{-1} (\mathbf{n} \cdot \mathbf{E} : \mathbf{a}) + \left(\mathbf{\Gamma}_j + \frac{\mathbf{A}_j}{T}\right) \bullet \frac{\partial F}{\partial \mathbf{A}_j} = 0\,. \tag{3.8}$$

The proof of (3.5) is similar to that of the uncoupled case (see BORRÉ and MAIER [8]). In Case II, condition (3.6) is obtained by computing the jump of the temperature rate via the heat equation (2.8) and setting it to zero according to the jump condition (3.4). Finally, the equivalence contained in Case II follows if we notice that conditions (3.6) and (3.8) are equivalent which can be easily checked by using relations (2.13), (2.19), (2.21) and the expressions of **D** and **E**. In conditions (3.5) and (3.6), the equality refers to continuous localization, whereas the inequality is related to discontinuous localization.

It has to be recalled at this stage that condition (3.7) is exactly the necessary and sufficient condition for localization in the uncoupled problem. This will allow comparison between coupled and uncoupled situations.

In the general case, the localization is governed by the adiabatic tangent modulus **H**. It has also to be emphasized that the heat conduction properties play no role in the type of instabilities investigated here. As it can be seen from relations (2.17), (2.13) and (2.19), the adiabatic tangent modulus is generally unsymmetric even when the mechanical behavior is such that $f = F$ (associated plasticity); from results of RUDNICKI and RICE [14], it is then expected that thermomechanical couplings may lead to destabilizing effects; this will be shown in the next section through a simple example.

As mentioned in the introduction, localization of deformation is linked to acceleration waves [3, 6] in the uncoupled approach. When thermal effects are considered, acceleration waves (in the sense of TRUESDELL and NOLL [15]) are linked to Case II localization modes where temperature is assumed to remain continuous. One should also remark that in this last case, the loss of ellipticity of the equilibrium equations is not sufficient for localization; another condition is necessary but can also be interpreted in the same manner as the loss of ellipticity of the field equations of a linear solid, the tangent modulus of which is given by

$$\mathbf{T} = \mathbf{D} + \frac{(\mathbf{D} : \mathbf{u}) \otimes \mathbf{\Delta}}{\left(\mathbf{\Gamma}_j + \frac{\mathbf{A}_j}{T}\right) \bullet \frac{\partial F}{\partial \mathbf{A}_j}} = \mathbf{E} + \frac{(\mathbf{E} : \mathbf{a}) \otimes \mathbf{\Delta}}{\left(\mathbf{\Gamma}_j + \frac{\mathbf{A}_j}{T}\right) \bullet \frac{\partial F}{\partial \mathbf{A}_j}}\,.$$

Notice also that at this type of localization, the loss of ellipticity of the adiabatic tangent moduli is equivalent to the loss of ellipticity of the isothermal tangent moduli.

3.3. Localization at the boundary

At a point P of the boundary where only traction $\mathbf{F}$ and temperature θ are imposed, the necessary and sufficient conditions for continuous localization are

CASE I. The necessary and sufficient conditions for continuous localization at the boundary are:

$$\det(\mathbf{n}\cdot\mathbf{H}\cdot\mathbf{n}) = 0\,, \tag{3.5}$$

$$(\mathbf{m}\cdot\mathbf{D}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n})^{-1}\cdot(\mathbf{n}\cdot\mathbf{D}\cdot\mathbf{u}) = (\mathbf{m}\cdot\mathbf{D}:\mathbf{u})\,, \tag{3.9}$$

$$\exists\dot{\boldsymbol{\varepsilon}}_0 \quad \text{such that } \mathbf{m}\cdot\mathbf{L}:\dot{\boldsymbol{\varepsilon}}_0 = \dot{F} - \frac{\dot{\theta}}{h}\left[\frac{\partial f}{\partial T} - \frac{\partial f}{\partial \mathbf{A}_j}\bullet\boldsymbol{\Gamma}_j\right]\mathbf{m}\cdot\boldsymbol{\Lambda}_j\bullet\frac{\partial F}{\partial \mathbf{A}_j} - \dot{\theta}\mathbf{m}\cdot\boldsymbol{\Delta}\,. \tag{3.11}$$

CASE II. The necessary and sufficient conditions for continuous localization at the boundary are:

$$\det(\mathbf{n}\cdot\mathbf{H}\cdot\mathbf{n}) = 0\,, \tag{3.5}$$

$$(\mathbf{n}\cdot\boldsymbol{\Delta})\cdot[\mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n}]^{-1}\cdot(\mathbf{n}\cdot\mathbf{D}:\mathbf{u}) + \left(\boldsymbol{\Gamma}_j + \frac{\mathbf{A}_j}{T}\right)\bullet\frac{\partial F}{\partial \mathbf{A}_j} = 0\,, \tag{3.6}$$

$$(\mathbf{m}\cdot\mathbf{D}\cdot\mathbf{n})\cdot(\mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n})^{-1}\cdot(\mathbf{n}\cdot\mathbf{D}:\mathbf{u}) = (\mathbf{m}\cdot\mathbf{D}:\mathbf{u})\,, \tag{3.9}$$

(3.10) $\exists\dot{\boldsymbol{\varepsilon}}_0$ such that

$$\mathbf{m}\cdot\mathbf{L}:\dot{\boldsymbol{\varepsilon}}_0 = \dot{F} - \frac{\dot{\theta}}{h}\left[\frac{\partial f}{\partial T} - \frac{\partial f}{\partial \mathbf{A}_j}\bullet\boldsymbol{\Gamma}_j\right]\mathbf{m}\cdot\boldsymbol{\Lambda}_j\bullet\frac{\partial F}{\partial \mathbf{A}_j} - \dot{\theta}\mathbf{m}\cdot\boldsymbol{\Delta}\,.$$

or equivalently

$$\det(\mathbf{n}\cdot\mathbf{L}\cdot\mathbf{n}) = 0\,, \tag{3.7}$$

$$(\mathbf{n}\cdot\boldsymbol{\Delta})\cdot[\mathbf{n}\cdot\mathbf{E}\cdot\mathbf{n}]^{-1}\cdot(\mathbf{n}\cdot\mathbf{E}:\mathbf{a}) + \left(\boldsymbol{\Gamma}_j + \frac{\mathbf{A}_j}{T}\right)\bullet\frac{\partial F}{\partial \mathbf{A}_j} = 0\,, \tag{3.8}$$

$$(\mathbf{m}\cdot\mathbf{E}\cdot\mathbf{n})\cdot(\mathbf{n}\cdot\mathbf{E}\cdot\mathbf{n})^{-1}\cdot(\mathbf{n}\cdot\mathbf{E}:\mathbf{a}) = (\mathbf{m}\cdot\mathbf{E}:\mathbf{a})\,, \tag{3.10}$$

$$\exists\dot{\boldsymbol{\varepsilon}}_0 \quad \text{such that } \mathbf{m}\cdot\mathbf{L}:\dot{\boldsymbol{\varepsilon}}_0 = \dot{\mathbf{F}} - \frac{\dot{\theta}}{h}\left[\frac{\partial f}{\partial T} - \frac{\partial f}{\partial \mathbf{A}_j}\bullet\boldsymbol{\Gamma}_j\right]\mathbf{m}\cdot\boldsymbol{\Lambda}_j\bullet\frac{\partial F}{\partial \mathbf{A}_j} - \dot{\theta}\mathbf{m}\cdot\boldsymbol{\Delta}\,. \tag{3.11}$$

Slightly different conditions are available for discontinuous localization. They are omitted here for simplicity and can be found in BENALLAL [13].

The proof of conditions in Case I are an extension of that given for uncoupled approach in BENALLAL, BILLARDON and GEYMONAT [9] if we write the stress rate as

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}:\dot{\boldsymbol{\varepsilon}} + \boldsymbol{\Lambda}_j\bullet\dot{\boldsymbol{\alpha}}_j + \boldsymbol{\Delta}\dot{T}$$

so that the traction rate at the boundary leads to

$$\mathbf{m}\cdot\mathbf{E}:\dot{\boldsymbol{\varepsilon}} = \dot{F} - \mathbf{m}\cdot\boldsymbol{\Lambda}_j\bullet\dot{\boldsymbol{\alpha}}_j - \dot{\theta}\mathbf{m}\cdot\boldsymbol{\Delta}\,.$$

In Case II, the equivalence and the supplementary conditions are obtained by the same arguments as above.

4. An example

This section is devoted to a detailed analysis of localization for a particular set of constitutive equations to which the formerly presented conditions are applied. The model considered here is based upon two internal variables (the plastic strain $\boldsymbol{\varepsilon}^p$ and the cumulated plastic strain p) describing an elastic-plastic material with isotropic hardening, and corresponds in the framework described before to a free energy, an elastic domain and a plastic potential defined respectively by

$$(4.1)\qquad \rho\Psi = \frac{1}{2}\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)]^2 + \mu(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) - [3\lambda + 2\mu]\alpha(T - T_0)\,\mathrm{Tr}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + l(T)f(p) - bTLn\left(\frac{t}{T_0}\right),$$

$$(4.2)\qquad f = F = \overline{\sigma} + R - k\,,$$

λ and μ are the Lamé constants, $\boldsymbol{\alpha}$ the thermal expansion coefficient, T_0 the reference temperature, $g(p)$ being an arbitrary function of p; besides, the von Mises equivalent stress is defined by $\overline{\sigma} = \sqrt{\mathbf{s} : \mathbf{s}}$, R is the thermodynamical force associated to p, $\mathbf{s}$ is the stress deviator and k the yield stress. b is a constant which represents heat capacity when all the material properties are temperature-independent.

The acoustic tensor is easily computed (for details see Appendix 1) as

$$(4.3)\qquad \mathbf{n}\cdot\mathbf{H}\cdot\mathbf{n} = A(\mathbf{n}\cdot\mathbf{n})\mathbf{1} + B\mathbf{n}\otimes\mathbf{n} + C\mathbf{s}\cdot\mathbf{n}\otimes s\cdot\mathbf{n} + D\mathbf{n}\otimes\mathbf{s}\cdot\mathbf{n} + E\mathbf{s}\cdot\mathbf{n}\otimes\mathbf{n}$$

with

$$(4.4)\qquad A = \mu\,,$$

$$(4.5)\qquad B = \lambda + \mu + \frac{\mathrm{Tr}^2}{\rho c} - \frac{up}{H}\,,$$

$$(4.6)\qquad C = \frac{Ts^2}{\rho c} - \frac{vq}{H}\,,$$

$$(4.7)\qquad D = \frac{Trs}{\rho c} - \frac{uq}{H}\,,$$

$$(4.8)\qquad E = \frac{Trs}{\rho c} - \frac{vp}{H}\,.$$

The expressions of c, r, s, u, v, p, q and H are given for convenience in Appendix 1.

We limit the analysis here to continuous localization only. If we denote by Σ the deviatoric normal stress on the plane of localization and by S the shear stress on the same plane, the localization condition (3.5) reduces to

$$(4.9)\qquad AC\Sigma^2 + \{C(B + A) - ED\}S^2 + A(E + D)\Sigma + A^2 + AB = 0$$

and represents in general a conical curve in the (Σ, S) plane; on other hand (3.7) and (3.8) represent, respectively, an ellipse and a straight line. The analysis is first carried out when all the thermal as well as the mechanical properties are temperature-independent.

4.1. The role of thermo-mechanical couplings

In this part, the model described above is used to show the consequences of taking into account the thermo-mechanical couplings. To separate the effects of these couplings, all the mechanical as well as the thermal properties of the material are assumed to be temperature-independent. In particular, no thermal softening is considered ($l(T) = 1$), and the thermoelastic properties and the yield stress are kept constant. The expressions given above and Appendix 1 are simplified. Indeed, the isothermal and adiabatic tangent moduli are easily obtained by

$$\text{(4.10)} \qquad \mathbf{L}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\left\{\frac{2\mu}{\overline{\sigma}}s_{ij}\right\} \cdot \left\{\frac{2\mu}{\overline{\sigma}}s_{kl}\right\}}{2\mu + g''},$$

$$\text{(4.11)} \qquad \mathbf{H}_{ijkl} = \left[\lambda + \frac{T(3\lambda + 2\mu)^2\alpha^2}{b}\right]\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\left\{\frac{2\mu}{\overline{\sigma}}s_{ij} + \frac{(3\lambda + 2\mu)\alpha k}{b}\delta_{ij}\right\} \cdot \left\{\frac{2\mu}{\overline{\sigma}}s_{kl}\right\}}{2\mu + g''}.$$

It is worthwhile to notice here that the last expression is formally similar to the tangent modulus used by RUDNICKI and RICE [14] in their analysis of localization for pressure-sensitive dilatant materials. The material obtained here belongs to this kind of material with Lamé constants given by

$$\text{(4.12)} \qquad \mu^* = \mu, \qquad \lambda^* = \lambda + \frac{T(3\lambda + 2\mu)^2\alpha^2}{b}$$

and frictional ω^* and dilatational π^* parameters (the parameters μ and β of [14]) given respectively by

$$\text{(4.13)} \qquad \omega^* = 0, \qquad \pi^* = \frac{3k\alpha}{b + 3T(3\lambda + 2\mu)\alpha^2}.$$

This remark emphasizes to some extent the analogy between thermal effects and those of pressure and dilatancy. It allows also to transfer to our analysis some of the results obtained by RUDNICKI and RICE [14]. We prefer here to use another approach which can be used in the more general case where the mechanical and thermal properties are temperature-dependent. This approach is essentially geometrical.

Let us denote in the following by Σ^* the deviatoric normal stress on the plane of localization and by S^* the shear stress on this same plane; these stresses are respectively defined by

$$\text{(4.14)} \qquad \Sigma^* = \mathbf{n} \cdot \mathbf{s} \cdot \mathbf{n}, \qquad (S^*)^2 = (\mathbf{sn}) \cdot (\mathbf{sn}) - (\mathbf{n} \cdot \mathbf{s} \cdot \mathbf{n})^2$$

and by Σ and S their normalized values with respect to the von Mises equivalent stress $\overline{\sigma}$.

From (4.10), (4.11), localization conditions (3.5), (3.7) and (3.8) given above become respectively:

$$\text{(4.15)} \qquad 2\mu + g'' + \frac{4\mu^2 Z^2}{\lambda^* + 2\mu} = \frac{4\mu^2}{\lambda^* + 2\mu}(\Sigma + Z)^2 + 4\mu S^2,$$

$$\text{(4.16)} \qquad 2\mu + g'' = 4\mu^2\left\{\frac{S^2}{\mu} + \frac{\Sigma^2}{\lambda + 2\mu}\right\},$$

(4.17)
$$\Sigma = \frac{(\lambda + 2\mu)k}{2\mu\alpha T(3\lambda + 2\mu)},$$

where Z is defined by

(4.18)
$$Z = \frac{(3\lambda + 2\mu)k\alpha}{2b\mu}.$$

It is then interesting to carry on the analysis in the "reduced Mohr's plane" $\left(\Sigma = \frac{\Sigma^*}{\overline{\sigma}}, S = \frac{S^*}{\overline{\sigma}}\right)$ depicted in Fig.1 where relation (4.15) and (4.16) represent two ellipses (EI) and (EII) whereas relation (4.17) is a straight line (L) parallel to the S axis. The size of the ellipse is linked to the "isothermal" hardening modulus $-h = g''(p)$; the larger this hardening modulus, the larger are the ellipses.

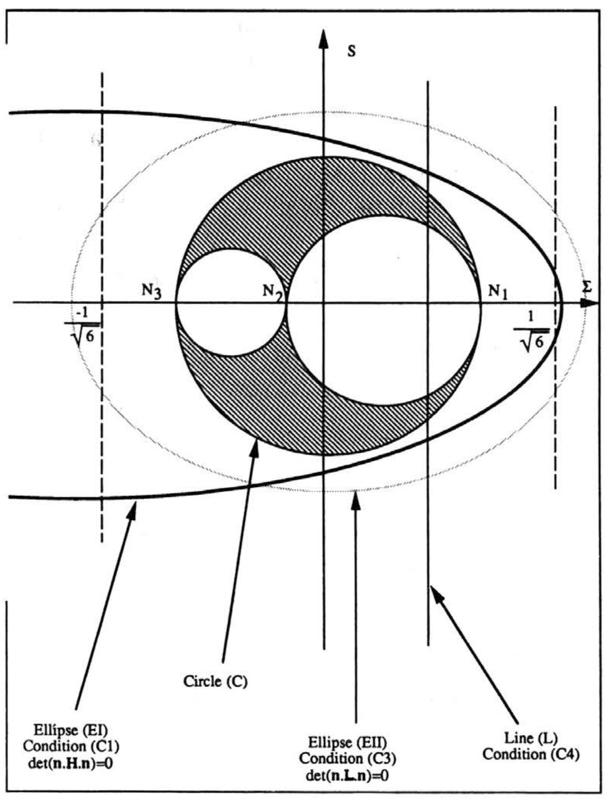

FIG. 1.

From elementary Mohr analysis, it is concluded that solutions **n** exist for the localization problem for a given state of stress and a given temperature if and only if:

the ellipse (EI) intersects the dashed area or its boundary in Case I;

the ellipse (EII) intersects the line (L) inside or at the boundary of this dashed area in Case II.

Let the state of stress $\boldsymbol{\sigma}$ and the temperature T be fixed and assume for simplicity that the principal stresses are distinct. The former remarks allow for the following conclusions:

in Case I, the largest hardening modulus h_{I} leading to localization is obtained when the ellipse (EI) is tangential to the largest Mohr circle (C) and is given by

$$\frac{h_{\mathrm{I}}}{2\mu} = \frac{\mu Z^2}{2(\lambda^* + 2\mu)(\lambda^* + \mu)} - \frac{\mu Z}{\lambda^* + \mu} N_2 - \frac{(3\lambda^* + 2\mu)}{2(\lambda^* + \mu)}(N_2)^2 ; \tag{4.19}$$

in Case II, the maximum hardening modulus h_{II} for localization is obtained when the line (L) and ellipse (EII) intersect on the circle (C). It is clear from Fig. 1 that this is possible only when

$$\frac{(\lambda + 2\mu)k}{2\mu\alpha T(3\lambda + 2\mu)} \leq N_1 , \tag{4.20}$$

in such a case, this modulus is given by

$$\frac{h_{\mathrm{II}}}{2\mu} = -2\left[N_2 + \frac{(\lambda + 2\mu)k}{2\mu(3\lambda + 2\mu)\alpha T}\right]^2 - \frac{(\lambda + 2\mu)k^2}{8\mu^2(3\lambda + 2\mu)\alpha^2 T^2} . \tag{4.21}$$

The maximum hardening h for localization in the uncoupled isothermal problem is obtained when the ellipse (EII) is tangential to the circle (C), which gives

$$\frac{h}{2\mu} = -\frac{(3\lambda + 2\mu)}{2(\lambda + \mu)}(N_2)^2 . \tag{4.22}$$

In the above relations, $N_1 = s_1/\overline{\sigma}$, $N_2 = s_2/\overline{\sigma}$ and $N_3 = s_3/\overline{\sigma}$ are the reduced deviatoric principal stresses and s_1, s_2 and s_3 are the deviatoric principal stresses, i.e. the eigenvalues of the stress deviator. These eigenvalues are arranged in such a way that $s_1 \geq s_2 \geq s_3$.

When the principal stresses are distinct, it follows then that the normal to the plane of localization is perpendicular to the s_2-direction. If two of the principal stresses are equal, only the angle between the normal to the critical plane and the third of the principal axes is uniquely determined. Conclusions similar to those of RUDNICKI and RICE [14] are drawn.

The normal $\mathbf{n}$ to the surface of localization, which lies in the extreme principal plane of stresses can also be obtained by a simple Mohr analysis and the angle θ between $\mathbf{n}$ and the greatest principal stress is given by

$$(\operatorname{tg} 2\theta)^2 = \frac{S^2}{\left(\Sigma - \dfrac{N_1 + N_3}{2}\right)^2} \tag{4.23}$$

which leads in the two cases to the results

CASE I

$$(\operatorname{tg} 2\theta)^2 = \frac{\left[\dfrac{N_1 - N_3}{2}\right]^2 - \left[\dfrac{\mu}{\lambda^* + \mu}\right]^2\left[Z + \left(\dfrac{N_1 + N_3}{2}\right)\right]^2}{\left[\dfrac{\mu}{\lambda^* + \mu}\left[Z + \left(\dfrac{N_1 + N_3}{2}\right)\right]\right]^2} . \tag{4.24}$$

CASE II

$$(4.25)\qquad (\operatorname{tg} 2\theta)^2 = \frac{\left[\dfrac{N_1 - N_3}{2}\right]^2 - \left[\dfrac{(\lambda + 2\mu)k}{2\mu(3\lambda + 2\mu)\alpha T} - \left(\dfrac{N_1 + N_3}{2}\right)\right]^2}{\left[\dfrac{(\lambda + 2\mu)k}{2\mu(3\lambda + 2\mu)\alpha T} - \dfrac{N_1 + N_3}{2}\right]^2}.$$

A simple comparison of relations (4.22) and (4.19) reveals the destabilizing effects of thermomechanical couplings; indeed, uncoupled (or isothermal) analysis gives a negative critical hardening modulus while this modulus may become positive when using a coupled approach. This is mainly due to the asymmetry of the adiabatic tangent modulus which governs this localization, as already mentioned earlier and expected also from the results of RUDNICKI and RICE [14]. There are cases where uncoupled analysis do not predict localization whereas the coupled approach does for the same state of stress and temperature. It is to be emphasized also that these destabilizing effects are due to the only fact of taking into account thermomechanical properties which were introduced in the example. Of course, these destabilizing effects must be evaluated and assessed in order to see their practical importance.

Relation (4.19) exemplifies the roles of each of the constitutive parameters on the critical hardening modulus for a given stress state and a given temperature. For a given temperature, this modulus has a parabolic variation with stress state through the parameter N_2; for a fixed state of stress, the role of the temperature is involved through Z and the adiabatic Lamé parameter λ^*; however, for many materials, the adiabatic elastic constants are only slightly different from the isothermal ones. Also of some importance is the role of thermal expansion and the yield stress k included in Z: the larger the yield stress, the larger the critical hardening modulus.

Now, as regards to localization at the boundary, the complete analysis is not possible here and will be given elsewhere. Let us just mention the role of boundary conditions in the simple case of a tension test. Condition (3.9) leads to

$$(4.26)\qquad (\operatorname{tg}\theta)^2 = \frac{\dfrac{\sqrt{3}k\alpha}{b + 3T(3\lambda + 2\mu)\alpha^2} + \dfrac{3\lambda + 4\mu + \dfrac{3T(3\lambda + 2\mu)^2\alpha^2}{b}}{3\lambda + 2\mu + \dfrac{3T(3\lambda + 2\mu)^2\alpha^2}{b}}}{1 - \dfrac{\sqrt{3}k\alpha}{b + 3T(3\lambda + 2\mu)\alpha^2}}.$$

For typical steels, $\lambda^* \approx \lambda$ (the adiabatic elastic modulus is slightly different from the isothermal elastic modulus) and the ratio $k\alpha/b$ is of the order 10^{-3} so that formula (4.26) reduces to

$$(4.27)\qquad (\operatorname{tg}\theta)^2 = \frac{2 - \nu}{1 + \nu}$$

which is exactly the relation given in [9] and which corresponds to condition (3.10). This means that thermomechanical couplings have a small influence on the localization directions at the boundary, at least for these materials and for this state of stress. ν is the Poisson ratio.

4.2. The role of thermal softening

In this part, the localization conditions are analysed by adding to thermo-mechanical couplings only the thermal softening. This thermal softening can be due to the decrease of elastic moduli or yield properties with temperature. We assume in the following, for the sake of simplicity, temperature dependence of the isotropic hardening (through the function $l(T)$) and the yield stress (through $k(T)$) only. The thermoelastic properties are considered to be constant. In this case, we have again $s = 0$ and (see Appendix 1), the adiabatic tangent modulus becomes

$$(4.28)\quad \mathbf{H}_{ijkl} = \left[\lambda + \mu + \frac{T(3\lambda + 2\mu)^2\alpha^2}{b + lg''}\right]\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$-\frac{\left\{\dfrac{2\mu}{\overline{\sigma}}s_{ij} + \dfrac{(3\lambda + 2\mu)\alpha T\left(l'g' + \dfrac{k}{T}\right)}{b + l''g}\delta_{ij}\right\}\cdot\left\{\dfrac{2\mu}{\overline{\sigma}}s_{kl} + \dfrac{(3\lambda + 2\mu)\alpha T(l'g' + k')}{b + l''g}\delta_{kl}\right\}}{2\mu + lg'' + \dfrac{T}{b + l''g}\left[l'g' + \dfrac{k}{T}\right][l'g' + k']}.$$

A similar analysis to that of Sec. 4.1 can be carried out and the same qualitative conclusions can be reached. In particular, the critical hardening modulus at localization under adiabatic conditions is given by

$$(4.29)\quad \frac{h_{\mathrm{I}}}{2\mu} = \frac{3\lambda^* + 2\mu}{4(\lambda^* + 2\mu)}(\omega - \beta)^2 - \frac{(3\lambda^* + 2\mu)}{8(\lambda^* + \mu)}\left[2N + \frac{1}{3}(\omega + \beta)\right]^2$$
$$- \frac{(3\lambda + 2\mu)\alpha T}{2\mu(3\lambda^* + 2\mu)}\frac{\left[\dfrac{k}{T} + l'g'\right][k' + l'g']}{b + lg''},$$

$$\omega = \frac{3\lambda + 2\mu}{3\lambda^* + 2\mu}\frac{\alpha T}{b + lg''}[k' + l'g'],$$

$$\beta = \frac{3\lambda + 2\mu}{3\lambda^* + 2\mu}\frac{\alpha T}{b + lg''}\left[\frac{k}{T} + l'g'\right],$$

from where one can easily see the destabilizing effects of thermal softening (compare with relation (4.19)).

4.3. The general case

When all the mechanical as well as the thermal properties are assumed to vary arbitrarily with temperature, the analysis is much more involved and is not yet completed. It has not been considered here.

5. Conclusions

We have analysed the role of thermal effects and thermomechanical couplings on the development of shear bands and their interactions with boundaries of solids. Their interactions with interfaces in presence of thermal effects can also be incorporated in a simple way. It has been emphasized that for the type of analysis used here, heat conduction properties have no influence on this development and that the adiabatic properties determine the onset of these shear bands. The effects of heat conduction can be incorporated by a stability analysis and will be addressed elsewhere.

We have illustrated the destabilizing effects of thermomechanical couplings and underlined the differences between the coupled and uncoupled approaches. Destabilizing effects due to thermal expansion and thermal softening were underlined.

Although for practical reasons, Case I is more important than Case II, the simple example analysed shows that the latter one is possible for given stress and temperature ranges. The conditions for localization given before show that an uncoupled analysis furnishes a conservative prediction for Case II while this is not necessarily true for the Case I by the example.

Finally, the localization conditions obtained can be interpreted as local failure criteria or at least as indicators of rupture under general thermomechanical loadings which is still an open problem. For isothermal problems, this has already been suggested by RUDNICKI and RICE [14] and BILLARDON and DOGHRI [16].

Appendix 1

We give in this appendix the results of the various computations involved in the application of the localization conditions to the model described in Sec. 4.

$$a = -(3\lambda + 2\mu)\alpha(T - T_0),$$

$$\mathbf{\Delta} = r\mathbf{1} + s\mathbf{s},$$

$$r = a' - \frac{a(3\lambda' + 2\mu')}{3\lambda + 2\mu} + \left\{\frac{2(\mu\lambda' - \lambda\mu')}{2\mu(3\lambda + 2\mu)} + \frac{\mu'}{3\mu}\operatorname{Tr}\boldsymbol{\sigma}\right\},$$

$$s = \frac{\mu'}{\mu},$$

$$\mathbf{D} : \mathbf{u} = u\mathbf{1} + v\mathbf{s},$$

$$u = -\frac{T}{\rho c}\left(r + \frac{s}{3}\operatorname{Tr}\boldsymbol{\sigma}\right)\left(l'g' + \frac{k}{T} - s\overline{\sigma}\right),$$

$$v = \frac{2\mu}{\overline{\sigma}} - \frac{Ts}{\rho c}\left(l'g' + \frac{k}{T} - s\overline{\sigma}\right),$$

$$\mathbf{v} : \mathbf{D} = p\mathbf{1} + q\mathbf{s},$$

$$p = -\frac{T}{\rho c}\left(r + \frac{s}{3}\operatorname{Tr}\boldsymbol{\sigma}\right)(l'g' + k' - s\overline{\sigma}),$$

$$q = \frac{2\mu}{\overline{\sigma}} - \frac{Ts}{\rho c}\left(l'g' + \frac{k}{T} - s\overline{\sigma}\right),$$

$$h = 2\mu + lg'',$$

$$H = 2\mu + lg'' + \frac{T}{\rho c}\left\{l'g' + \frac{k}{T} - s\overline{\sigma}\right\}\{l'g' + k' - s\overline{\sigma}\}.$$

In these relations, the prime (′) denotes derivative of a function of a single variable with respect to this variable. Double prime (″) is the second derivative. **s** is the stress deviator, **1** the unit tensor and Tr denotes the trace operator.

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Received September 3, 1991.