AMAS Course on Reliability-Based Design and Optimisation
RBO'03 – (pp.325–344) – Warsaw, September 15-18, 2003

Bounds on system failure probability by linear programming

J. SONG and A. DER KIUREGHIAN

Department of Civil and Environmental Engineering
University of California at Berkeley
CA, USA

Bounds on system reliability employing marginal or joint component probabilities are useful approximations when exact solutions are costly or unavailable. Linear Programming (LP) can provide bounds for any series, parallel, or general system with any level or type of information available on the component probabilities. For given information, the LP bounds are the narrowest possible bounds that are not affected by the ordering of the components. The bounds can even utilize an incomplete set of component probabilities or any linear equality/inequality constraints on component probabilities. Numerical examples in this paper demonstrate the methodology and show how LP bounds can be used to estimate and improve the system reliability of electrical substations.

1. Introduction

This paper presents a summary of a new method for reliability analysis of systems by use of Linear Programming (LP), which is recently developed by the authors (Song and Der Kiureghian, 2003a). The method provides the narrowest possible bounds on the system failure probability for any given information on the marginal or joint component failure probabilities. The method is applicable to general systems, for which no bounding formulas exist. For series systems, the method provides narrower bounds than existing bounding formulas based on bi-component and tri-component failure probabilities. After briefly describing the method, examples to series and general systems are presented to demonstrate the method.

The state of a system composed of a set of components in general can be expressed as a Boolean or logical function of the component states. Considering two-state components and systems, let E_i denote the event of failure of component i, $(i = 1, \ldots, n)$, and $\bar{E}_i$ denote its complement, the survival of the component. Likewise, let E_{system} denote the event of failure of the system and $\bar{E}_{\text{system}}$ denote its survival. For three classes of systems, the relations between the component states and the system state are as follows:

$$\text{Series systems:} \qquad E_{\text{series system}} = \bigcup_{i=1}^{n} E_i, \tag{1.1}$$

$$\text{Parallel systems:} \qquad E_{\text{parallel system}} = \bigcap_{i=1}^{n} E_i, \tag{1.2}$$

$$\text{General systems:} \qquad E_{\text{system}} = \bigcup_{k=1}^{K} \bigcap_{i\in C_k} E_i. \tag{1.3}$$

In the case of "general," i.e., non-series and non-parallel, systems, the system failure event is defined in terms of cut sets C_k, $(k = 1, \ldots, K)$, where each cut set is a set of components, whose joint failure constitutes failure of the system. In this case, the system can be represented as a series of parallel sub-systems. An equivalent formulation is also possible, whereby the general system is represented as a parallel system of series sub-systems called link sets.

Once the system state is formulated as in (1.1)-(1.3), the system failure probability, or its complement, the system reliability, can be expressed as the probability of the logical function. The exact computation of this probability is often costly or unavailable due to the complexity of the system or lack of complete probability information. For these reasons, theoretical bounding formulas have been derived in terms of marginal or joint component probabilities. These include uni-component bounds for series and parallel systems (Boole, 1854), and bi-component (Kounias, 1968; Hunter, 1976; Ditlevsen, 1979) and multi-component (Hohenbichler and Rackwitz, 1983; Zhang, 1993) bounds for series systems. The latter bounds can also be used for parallel systems after such systems are converted to series systems by use of de Morgan's rule. No theoretical bounding formulas are available for general systems.

For the purpose of later comparison, here we present the bi- and tri-component bounding formulas for series systems. The bi-component bounds by Kounias (1968), Hunter (1976) and Ditlevsen (1979) are

$$P_1 + \sum_{i=2}^{n} \max\left(0\,,\, P_i - \sum_{j=1}^{i-1} P_{ij}\right) \leqslant P\left(E_{\text{series system}}\right) \leqslant P_1 + \sum_{i=2}^{n}\left(P_i - \max_{j<i} P_{ij}\right), \tag{1.4}$$

and the tri-component bounds by Hohenbichler and Rackwitz (1983), and Zhang (1993) are

$$P(E_{\text{series system}}) \geqslant P_1 + P_2 - P_{12} + \sum_{i=3}^{n} \max\left(0,\ P_i - \sum_{j=1}^{i-1} P_{ij} + \max_{k\in\{1,2,\cdots,i-1\}} \sum_{\substack{j=1\\ j\neq k}}^{i-1} P_{ijk}\right), \quad (1.5a)$$

$$P(E_{\text{series system}}) \leqslant P_1 + P_2 - P_{12} + \sum_{i=3}^{n} \left[P_i - \max_{\substack{k\in\{2,3,\cdots,i-1\}\\ j<k}} \left(P_{ik} + P_{ij} - P_{ijk}\right)\right]. \quad (1.5b)$$

In the above, $P_i = P(E_i)$ denotes the failure probability of component i (uni-component failure probability), $P_{ij} = P(E_{ij})$ is the joint failure probability of components i and j (bi-component failure probability), and $P_{ijk} = P(E_i E_j E_k)$ is the joint failure probability of components i, j and k (tri-component failure probability). See Song and Der Kiureghian (2003a) for a detailed review of these and other theoretical bounding formulas.

Although the uni-component bounds are guaranteed to be the narrowest possible bounds if only the marginal component probabilities are known (Fréchet, 1935), they are often too wide to be of practical use. The bi- and tri-component bounds are widely used in reliability analysis of series systems due to their relative narrowness. However, these bounds depend on the ordering of the component events. Thus, in order to obtain the narrowest bounds, one must consider all $n!$ ordering alternatives of the components. Furthermore, as shown below, the ordering with the narrowest bounds does not necessarily produce the narrowest possible bounds for the specified probability information.

The idea of using LP for computing bounds on system failure probability was first explored by Hailperin (1965). Specialized versions of this approach were employed in operations research (Prékopa, 1988). However, it appears that this approach has never been used in the field of structural reliability or civil engineering.

LP bounds are applicable to any type of system and any level of information regarding the component probabilities. Equally important, these bounds are the narrowest possible bounds that one can obtain for any specified information regarding the marginal or joint component probabilities. Although the resulting LP problem can be large for a system with many components, with the enormous increase in the speed and capacity of computers in recent years, we believe the LP approach is viable and a powerful tool for many system reliability problems. Furthermore, on-going work shows that a large LP bounding problem can be decomposed into a number of smaller LP problems with a drastic reduction in size.

After briefly describing the formulation of the LP bounds, we consider the reliability of a truss structure represented as a series system in order to compare probability bounds obtained by LP with those obtained by the theoretical bounding formulas in (1.4) and (1.5). The comparison demonstrates that these theoretical bounding formulas do not necessarily produce the narrowest possible bounds. As an application to a general system, we consider the seismic reliability of an electrical substation previously reported in Song and Der Kiureghian (2003b). The electrical substation is an important element within the power transmission lifeline. It consists of a complex set of interconnected equipment items, such as circuit breakers, transformers, disconnect switches, surge arresters, etc. The continued functioning of an electrical substation and the power network after a major earthquake is essential for rescue and recovery operations. Major damages to electrical substation equipment and systems were caused by recent earthquakes in the United States, Japan and elsewhere. As a result, there is heightened interest in assessing and improving the seismic reliability of these systems.

LP bounds can be useful for assessing the reliability of electrical substations, because these systems are usually too complex to be analyzed analytically and the probability information on equipment items is often incomplete or unavailable. This paper demonstrates the use of LP bounds for estimating and improving the system reliability of example electrical substations. The first example is a single-transmission-line substation, which is modeled as a series system. The influence of the reliability of a critical component on the system reliability is investigated. The second example explores the effectiveness of adding redundancy to the weakest component of the series system in order to improve the system reliability. The third example deals with a two-transmission-line substation, designed to provide more redundancy. This is a general system and is formulated by use of cut sets. For this example we explore the case of incomplete probability information. In each case, the LP bounds are computed assuming knowledge of bi-component or bi- and tri-component probabilities. These results are compared with Monte Carlo simulation results assuming complete probability information to demonstrate the accuracy of the bounds.

2. Linear programming theory

LP solves the problem of minimizing (maximizing) a linear function, whose variables are subject to linear equality or inequality constraints. LP gained worldwide interests after G. B. Dantzig developed the simplex method in 1947 (Dantzig, 1951). Since then, encouraged by dramatic improvements in computing technology, many powerful algorithms have been developed and a profound mathematical understanding of the problem has been gained.

Among various equivalent forms, the compact formulation of LP appropriate for our analysis has the form:

$$\text{minimize (maximize):}\quad \mathbf{c}^{\mathrm{T}}\mathbf{p}, \tag{2.1a}$$

$$\text{subject to:}\quad \mathbf{a}_1\mathbf{p} = \mathbf{b}_1, \tag{2.1b}$$

$$\mathbf{a}_2\mathbf{p} \geqslant \mathbf{b}_2. \tag{2.1c}$$

In the above, $\mathbf{p} = (p_1, p_2, \ldots)$ is the column vector of "decision" or "design" variables, $\mathbf{c}^{\mathrm{T}}\mathbf{p}$ with $\mathbf{c}$ a vector of coefficients is the linear "objective" or "cost" function, and $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{a}_2$ and $\mathbf{b}_2$ are coefficient matrices and vectors that respectively define equality and inequality constraints. In (2.1c), the inequality between the vectors must be interpreted component-wise. A vector $\mathbf{p}$ is called "feasible" if it satisfies all the given constraints. The solution of the LP problem is a feasible $\mathbf{p}$ that minimizes (maximizes) the objective function.

According to a key theorem in linear programming (Bertsimas and Tsitsiklis, 1997), the optimal solution of LP, if it exists, is located at one of the extreme points (vertices) of the polyhedron defined by the given linear constraints. This fact led to the development of the simplex algorithm (Dantzig, 1951), which moves from one vertex to another under a certain pivoting rule, until the requirements for the optimal solution are met. Since the simplex algorithm appeared, LP has flourished and numerous algorithms (interior point method, ellipsoid method, etc.) have been developed, dramatically increasing our ability to solve large-scale problems.

3. Bounds by linear programming

Hailperin (1965) showed that the problem of finding bounds on the probability of a Boolean function is a LP problem. He first divided the sample space of the n component events into 2^n mutually exclusive and collectively exhaustive (MECE) events, each consisting of a distinct intersection of the component events E_i and their complements $\bar{E}_i$ $(i = 1, \ldots, n)$. We name these the 'basic' MECE events and denote them by e_i $(i = 1, \ldots, 2^n)$. As an example, Fig. 1 shows the basic MECE events for $n = 3$ components. For example, $e_3 = E_1\bar{E}_2E_3$.

Let $p_i = P(e_i)$ $(i = 1, \ldots, 2^n)$, denote the probabilities of the basic MECE events. These probabilities serve as the design variables in the LP problem to be formulated.

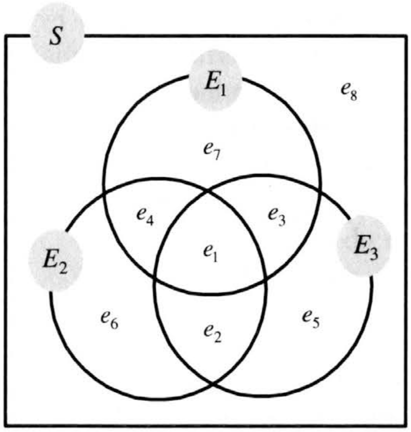

FIGURE 1. Basic MECE events e_i for 3-event sample space.

According to the basic axioms of probability theory, p_i $(i = 1, \ldots, 2^n)$ should satisfy the following linear constraints:

$$\sum_{i=1}^{2^n} p_i = 1, \tag{3.1}$$

$$p_i \geqslant 0, \quad \forall i. \tag{3.2}$$

The constraint (3.1) is analogous to (2.1b) with $\mathbf{a}_1$ being a row vector of 1's and $\mathbf{b}_1 = 1$, whereas (3.2) is analogous to (2.1c) with $\mathbf{a}_2$ being an identity matrix of size 2^n and $\mathbf{b}_2$ a 2^n-vector of 0's.

Due to mutual exclusivity of the basic MECE events, the probability of any subset made of these events is the sum of the corresponding probabilities. In particular, the probability of any component event E_i is the sum of the probabilities of the basic MECE events that constitute that component event. Similarly, the probability of any intersection of the component events is given as the sum of the probabilities of the basic MECE events that constitute the intersection event. Therefore, we can write

$$P(E_i) = P_i = \sum_{r:\ e_r \subseteq E_i} p_r, \tag{3.3a}$$

$$P(E_i E_j) = P_{ij} = \sum_{r:\ e_r \subseteq E_i E_j} p_r, \tag{3.3b}$$

$$P(E_i E_j E_k) = P_{ijk} = \sum_{r:\ e_r \subseteq E_i E_j E_k} p_r, \tag{3.3c}$$

and so on.

In most system reliability problems, the uni-, bi- and sometimes tri-component probabilities (P_i, P_{ij}, and P_{ijk}) are known or can be computed. In that case, the above expressions provide linear equality constraints on the variables $\mathbf{p}$ in the form of (2.1b) with $\mathbf{a}_1$ a matrix having elements of 0 or 1 and $\mathbf{b}_1$ a vector listing the known component probabilities. If, instead, inequality constraints on component probabilities are given, such as $P_i \leqslant 0.01$, $0.01 \leqslant P_{ij} \leqslant 0.03$ or $P_i \geqslant P_j$, then the above expressions provide linear inequality constraints on the variables $\mathbf{p}$ in the form of (2.1c).

Any Boolean function of the component events can also be considered as being composed of a subset of the basic MECE events. It follows that the probability of the system event E_{system} can be written in the form $P(E_{\text{system}}) = \mathbf{c}^{\mathrm{T}}\mathbf{p}$, where $\mathbf{c}$ is a vector whose elements are either 0 or 1.

The lower bound of the system probability is obtained by minimizing the objective function, and the upper bound is obtained by maximizing the same function. For a system with n component events, the number of design variables is 2^n; one equality and 2^n inequality constraints result from the probability axioms (3.1) and (3.2), respectively, n equality or inequality constraints result from knowledge of uni-component probabilities or their bounds as in (3.3a), $n!/[2!(n-2)!]$ equality or inequality constraints result from knowledge of bi-component probabilities or their bounds as in (3.3b), and so on. Obviously the size of the LP problem quickly grows with the number of component events.

The bounds by LP have many advantages over the existing theoretical bounding formulas. First, LP is guaranteed to provide the narrowest possible bounds, if a feasible solution exists for the given constraints (Hailperin, 1965). This is not the case for the theoretical bounds for series systems based on the multi-component probabilities, even for the best ordering of the component events. (Note that the LP formulation is independent of the ordering of the component events.) Second, the LP formulation is uniformly applicable to all systems, including general systems characterized by unions and intersections of component events (and their complements). Third, the LP formulation can incorporate general forms of information about the component probabilities. Specifically, any linear equality or inequality expression involving uni- or multi-component probabilities can be used. Finally, it is not necessary to have the complete set of probabilities for all the components at a particular, e.g., uni-, bi- or tri-component, level. Any partial set of the component probabilities can be used. Of course, incomplete information will lead to wider bounds.

The main drawback of the LP approach is that the size of the problem increases exponentially with the number of component events. For example, with $n = 17$ components, one has to solve for about 10^5 design variables.

There are a number of advanced LP algorithms for such large problems (see Chapter 6 in Bertsimas and Tsitsiklis, 1997). Moreover, with rapidly advancing speed and capacity of computers, these purely computational issues may not be a major hindrance in application of the approach to many systems reliability problems, even with large n. Finally, with the LP approach, it is easy to consider sub-systems of a large system as equivalent components, and thereby reduce the actual number of components that one has to work with. We note, in passing, that for $n = 17$ the number of orderings of the component events is $17! = 3.56 \cdot 10^{14}$. In using the bounding formulas in (1.4) and (1.5), it may not be possible to check all the possible orderings of the component events. In that case, bounds computed by these theoretical formulas can potentially be far from the narrowest possible bounds.

In summary, while the LP approach may become computationally demanding for systems with large number of component events, it has the following important advantages:

- it provides the narrowest possible bounds for any given level of information on component probabilities,
- it is independent of the ordering of the component events,
- it can incorporate general forms of information about component events,
- it is uniformly applicable to all types of systems,
- general-purpose software is widely available for solving the problem.

Computational limitations of the approach are expected to diminish with increasing speed and memory capacity of computers.

4. Application 1: Truss as a series system

Consider the truss in Fig. 2, which is adopted from Ditlevsen (1979) and Song and Der Kiureghian (2003a). Since this is a statically determinate structure, the failure of any member constitutes failure of the truss. Therefore, the truss is a series system with its members representing the components. Let L denote the load acting on the truss. Neglecting the buckling failure mode, let X_i, $i = 1, 2, \ldots, 7$, denote the tensile/compressive strength of the i-th member. Based on the distribution of internal forces shown in Fig. 2, the failure events of the individual members are $E_i = \{X_i \leqslant L/(2\sqrt{3})\}$ for $i = 1$ and 2, and $E_i = \{X_i \leqslant L/\sqrt{3}\}$ for $i = 3, 4, \ldots, 7$. Suppose the load has the deterministic value $L = 100$ and the member strengths X_i, $i = 1, 2, \cdots, 7$, are jointly normally distributed random variables with X_1 and X_2 having means 100 and standard deviations 20 and $X_3 - X_7$ having means 200 and standard deviations 40. Under these conditions, the members have equal probabilities

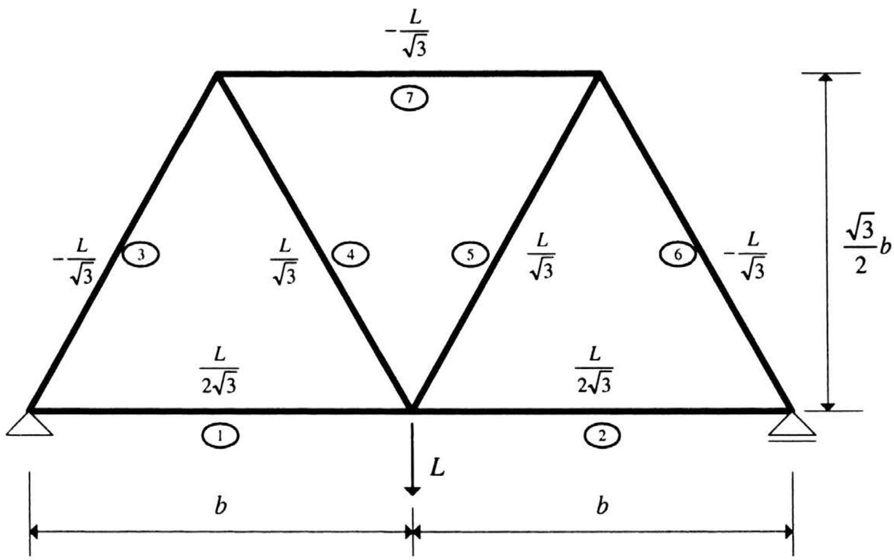

FIGURE 2. Statically determinate truss as a series system

of failure given by

$$P_i = P(E_i) = \Phi\left(\frac{100/\sqrt{3} - 200}{40}\right) = 1.88 \cdot 10^{-4}, \quad i = 1, 2, \ldots, 7, \quad (4.1)$$

where $\Phi(\cdot)$ denotes the standard normal cumulative probability. Further, suppose X_i's have a Dunnet-Sobel (D-S) class correlation matrix, which is specified as $\rho_{ij} = r_i r_j$ for $i \neq j$ and $\rho_{ii} = 1$. In that case, the m-component joint failure probabilities can be computed by the one-dimensional integral (Dunnett and Sobel, 1955):

$$P_{12\cdots m} = P\left(\bigcap_{i=1}^{m} E_i\right) = \Phi_m(u_1, \ldots, u_m; \mathbf{R})$$
$$= \int_{-\infty}^{\infty} \left[\varphi(t) \prod_{i=1}^{m} \Phi\left(\frac{u_i - r_i t}{\sqrt{1 - r_i^2}}\right)\right] dt, \quad (4.2)$$

where $\Phi_m(u_1, \ldots, u_m; \mathbf{R})$ is the m-variate standard normal cumulative probability function with correlation matrix $\mathbf{R} = [\rho_{ij}]$ at coordinates $u_i = (100/\sqrt{3} - 200)/40$ and $\varphi(\cdot)$ denotes the one-dimensional standard normal probability density function.

First consider the case $r_1 = 0.90$, $r_2 = 0.96$, $r_3 = 0.91$, $r_4 = 0.95$, $r_5 = 0.92$, $r_6 = 0.94$ and $r_7 = 0.93$. The bi- and tri-component probabilities, P_{ij} and P_{ijk}, respectively, computed using (4.2) are $(\times 10^{-5})$:

$$
\begin{array}{llll}
P_{12} = 5.73, & P_{13} = 4.35, & P_{14} = 5.42, & P_{15} = 4.59, \\
P_{16} = 5.13, & P_{17} = 4.85, & P_{23} = 6.08, & P_{24} = 7.79, \\
P_{25} = 6.47, & P_{26} = 7.42, & P_{27} = 6.87, & P_{34} = 5.75, \\
P_{35} = 4.86, & P_{36} = 5.43, & P_{37} = 5.14, & P_{45} = 6.10, \\
P_{46} = 6.88, & P_{47} = 6.48, & P_{56} = 5.76, & P_{57} = 5.44, \\
P_{67} = 6.11, & P_{123} = 2.81, & P_{124} = 3.58, & P_{125} = 2.99, \\
P_{126} = 3.37, & P_{127} = 3.17, & P_{134} = 2.66, & P_{135} = 2.25, \\
P_{136} = 2.52, & P_{137} = 2.38, & P_{145} = 2.82, & P_{146} = 3.18, \\
P_{147} = 2.99, & P_{156} = 2.67, & P_{157} = 2.52, & P_{167} = 2.83, \\
P_{234} = 3.80, & P_{235} = 3.17, & P_{236} = 3.58, & P_{237} = 3.37, \\
P_{245} = 4.04, & P_{246} = 4.58, & P_{247} = 4.30, & P_{256} = 3.80, \\
P_{257} = 3.58, & P_{267} = 4.04, & P_{345} = 2.99, & P_{346} = 3.37, \\
P_{347} = 3.18, & P_{356} = 2.83, & P_{357} = 2.67, & P_{367} = 3.00, \\
P_{456} = 3.58, & P_{457} = 3.37, & P_{467} = 3.81, & P_{567} = 3.18.
\end{array}
$$

These uni-, bi- and tri-component probabilities are used to compute the bounds on the series system probability by use of the theoretical formulas in (1.4) and (1.5). The same information is used to compute the bounds by LP. The LP formulation involves $2^7 = 128$ design variables, 7 equality constraints for the uni-component probabilities, 21 for bi-component probabilities, and 35 for tri-component probabilities.

Table 1 compares the theoretical bounds and the bounds by LP. For the theoretical bounds all the possible $7! = 5\,040$ ordering alternatives are considered and the range of results for each bound is shown with the best result in bold. Considerable variation in the theoretical bounds is observed, depending on the ordering of the components. It is seen that for the bi-component bounds, the best lower theoretical bound obtained from (1.4) is

TABLE 1. Bounds for series system with equal component probabilities.

Bounds $(\times 10^{-3})$		*Lower*	*Upper*
Bi-component	Equation 4	0.344 - **0.459**	**0.912** - 0.961
	LP	0.477	0.912
Tri-component	Equation 5	0.605 - **0.631**	**0.809** - 0.833
	LP	0.631	0.796

smaller than the lower LP bound. For the tri-component bounds, the best upper theoretical bound obtained from (1.5) is found to be greater than the upper LP bound. These clearly show that the theoretical bounding formulas in (1.4) and (1.5) may not produce the narrowest possible bounds, even if all possible component orderings are considered.

As a further example, consider the case where $r_i = \sqrt{\rho}$ in the D-S correlation model, such that $\rho_{ij} = \rho$ for $i \neq j$. This is the case of a series system with equi-probability and equi-correlated components. It is well known that this is the least favorable condition for bi- or tri-component bounds. Tables 2 and 3 list the computed bi- and tri-component probabilities and the corresponding bounds by LP and the theoretical formulas in (1.4) andand (1.5) for $\rho = 0.2$, 0.4, 0.6, 0.8 and 0.9. Obviously the theoretical bounds are independent of the ordering of the components in this case. It is seen that in several cases (highlighted in bold) the theoretical bounds are inferior to the LP bounds. This further demonstrates that the theoretical bounding formulas may not produce the narrowest possible bounds for a given probability information.

TABLE 2. Bi-component bounds for series system with equal component probabilities and correlations.

ρ	P_{ij} $(1 \leqslant i < j \leqslant 7)$	*Bounds* $(\times 10^{-3})$ LP	Equation 4
0.2	$4.11 \cdot 10^{-7}$	1.3062 - 1.3063	1.3062 - 1.3063
0.4	$2.56 \cdot 10^{-6}$	1.26 - 1.30	1.26 - 1.30
0.6	$1.10 \cdot 10^{-5}$	1.08 - 1.25	1.08 - 1.25
0.8	$3.87 \cdot 10^{-5}$	**0.606** - 1.08	**0.552** - 1.08
0.9	$7.20 \cdot 10^{-5}$	**0.405** - 0.883	**0.347** - 0.883

TABLE 3. Tri-component bounds for series system with equal component probabilities and correlations.

ρ	P_{ijk} $(1 \leqslant i < j < k \leqslant 7)$	Bounds $(\times 10^{-3})$ LP	Equation 5
0.2	$3.86 \cdot 10^{-9}$	1.3062 - **1.3063**	1.3062 - **1.3103**
0.4	$1.68 \cdot 10^{-7}$	1.26 - **1.27**	1.26 - **1.29**
0.6	$2.26 \cdot 10^{-6}$	1.12 - **1.16**	1.12 - **1.20**
0.8	$1.72 \cdot 10^{-5}$	0.761 - **0.927**	0.761 - **0.975**
0.9	$4.47 \cdot 10^{-5}$	**0.514 - 0.720**	**0.493 - 0.746**

5. Application 2: Seismic reliability of electrical substation

In this section, three numerical examples are used to demonstrate the application of the LP bounds to estimating the seismic reliability of electrical substation systems. The first example deals with a single-transmission-line system with 5 equipment items, which is modeled as a series system with $n = 5$ components. The system is subjected to an earthquake ground motion with random intensity and local soil effects. Each component is assumed to have an uncertain capacity to base acceleration. For this system, the uni-, bi- and tri-component bounds are estimated by LP and are compared with simulation results. The effect of varying the capacity of a critical component on the system reliability is investigated. The second example deals with the same system with the critical component replaced with a parallel subsystem, hence introducing redundancy with respect to the state of the critical component. Systems with different number of redundant components are investigated by use of the cut-set formulation. LP bounds based on uni-, bi- and tri-component probabilities are compared with simulation results. The final example deals with a two-transmission-line system, which is a variation of the first example with system redundancy. The system is modeled through a cut-set formulation. The uni-, bi- and tri-component bounds are computed by LP and compared with simulation results. The simplex algorithm and the primal-dual algorithm implemented in Matlab Optimization Toolbox are used to solve all the LP problems.

For all the examples in this paper, let A denote the bed-rock peak ground acceleration (PGA) in the vicinity of the substation and S_i denote a factor representing the local site response for equipment i, such that AS_i is the actual PGA experienced by the i-th equipment item. Assume A is a lognormal random variable with mean $0.15g$ (in units of gravity acceleration, g) and coefficient of variation (C.O.V.) 0.5, and S_i, $i = 1, \ldots, n$, are independent lognormal random variables, also independent of A, with means 1.0 and C.O.V. 0.2. Also let R_i denote the capacity of the i-th equipment item with respect to base acceleration in units of g, and assume it has the lognormal distribution. The means and C.O.V.'s of the equipment capacities are assumed as follows: disconnect switch (DS) $\sim (0.4g, 0.3)$, circuit breaker (CB) $\sim (0.3g, 0.3)$, power transformer (PT) $\sim (0.5g, 0.5)$, drawout breaker (DB) $\sim (0.4g, 0.3)$, feeder breaker (FB) $\sim (1.0g, 0.3)$ and tie breaker (TB) $\sim (1.0g, 0.3)$. The capacities of equipment items within each category are assumed to be equally correlated with a correlation coefficient of 0.3 except 0.5 for PT's. Equipment capacities in different categories are assumed to be statistically independent. The statistics assumed above are rough approximations based on Bayesian analyses of observed data on the performance of electrical substation equipment in past earthquakes (Der Kiureghian, 2002).

5.1. Example 1: Single-transmission-line substation

Consider the single-transmission-line substation system in Fig. 3, which is adopted from Brown (2002). The failure of any equipment item constitutes failure of the substation. Therefore, the single-transmission-line substation is a series system with its equipment items representing the components.

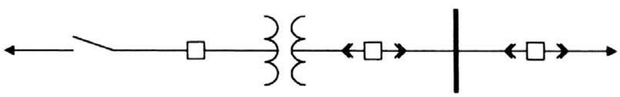

FIGURE 3. A single-transmission-line system.

The failure events of the individual equipment items are formulated as

$$E_i = \{\ln R_i - \ln A - \ln S_i \leqslant 0\}, \quad i = 1, \dots, 5. \tag{5.1}$$

Since the logarithm of a lognormal random variable is normal, $v_i = \ln R_i - \ln A - \ln S_i$ has the normal distribution. Therefore, the uni-component probabilities are given by

$$P_i = \Phi\left(-\frac{\mu_i}{\sigma_i}\right), \quad i = 1, \dots, 5, \tag{5.2}$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function and μ_i and σ_i are the mean and standard deviation of v_i, respectively, which are easily computed in terms of the statistics of A, S_i and R_i given above. Furthermore, for any pair of components i and j, the random variables v_i and v_j are jointly normal and the bi-component joint probabilities can be computed from (Ditlevsen and Madsen 1996)

$$P_{ij} = \Phi(u_i)\,\Phi(u_j) + \int_0^{\rho_{ij}} \varphi(u_i, u_j, \rho)\, d\rho, \tag{5.3}$$

where $u_i = -\mu_i/\sigma_i$, ρ_{ij} denotes the correlation coefficient between v_i and v_j, and $\varphi(\cdot, \cdot, \rho)$ represents the bi-normal probability density function with zero means, unit standard deviations and correlation coefficient ρ.

The above uni- and bi-component probabilities are used to compute the bounds on the series-system probability by use of LP. The LP formulation involves $2^5 = 32$ design variables, 5 equality constraints for the uni-component probabilities, 10 for bi-component, and 10 for tri-component probabilities. The uni-component bounds on the system failure probability are 0.0925 and 0.202. The bi-component bounds are 0.122 and 0.147. The tri-component bounds are 0.139 and 0.142. To check the accuracy of these results, Monte Carlo simulation is performed and the system failure probability is estimated as 0.138 with a 1% coefficient of variation. This result is bracketed by both the uni-, bi- and tri-component LP bounds.

One may ask the need for LP bounds when Monte Carlo simulation can be performed. The point is that the Monte Carlo simulation method can be impractical when the failure probability is small, whereas LP bounds are not affected by the magnitude of the failure probability. In this application, by nature of the problem, the probability of failure is high. This, however, is not the case with all systems, including all electrical substations.

An important observation to be derived from the above result is that the seismic reliability of a single-transmission-line substation is quite low, i.e., the failure probability is high. This is partly due to the vulnerability of the circuit breaker, which is a top-heavy item with tendency to fail by fracture of its ceramic bushings or oil leakage through its gaskets. In order to investigate the influence of the capacity of the circuit breaker on the reliability of the system, the failure probability of the single-transmission-line substation is computed for a range of mean values of its capacity, while maintaining a constant C.O.V. Table 4 shows the assumed mean values of the capacity of the circuit breaker and the corresponding component and system failure probabilities. Uni-, bi- and tri-component LP bounds as well as Monte Carlo simulation results are listed. Figure 4 presents the same results in a graphical

TABLE 4. Failure probabilities of circuit breaker and corresponding system failure probabilities.

$E[R_{CB}]$	P_{CB}	Uni-comp. LP	Bi-comp. LP	Tri-comp. LP	M.C. c.o.v. = 0.01
0.1	0.704	0.704 - 0.813	0.7048 - 0.7053	0.7052 - 0.7052	0.701
0.2	0.261	0.261 - 0.371	0.272 - 0.284	0.2818 - 0.2824	0.280
0.3	0.0925	0.0925 - 0.202	0.122 - 0.147	0.139 - 0.142	0.138
0.4	0.0349	0.0393 - 0.144	0.0853 - 0.114	0.0989 - 0.103	0.0997
0.5	0.0142	0.0393 - 0.124	0.0805 - 0.0980	0.0886 - 0.0908	0.0901
0.6	0.00621	0.0393 - 0.116	0.0805 - 0.0927	0.0858 - 0.0868	0.0869
0.7	0.00288	0.0393 - 0.112	0.0805 - 0.0908	0.0850 - 0.0855	0.0858

form. It is seen that a reduced capacity for the circuit breaker[1)] drastically increases the failure probability of the system, whereas increasing the mean capacity of the circuit breaker to $0.4g$ significantly enhances the reliability of the system. Further increases in the mean capacity of the circuit breaker, however, have little influence on the reliability of the system. This is because another component in the series system becomes the "weakest link."

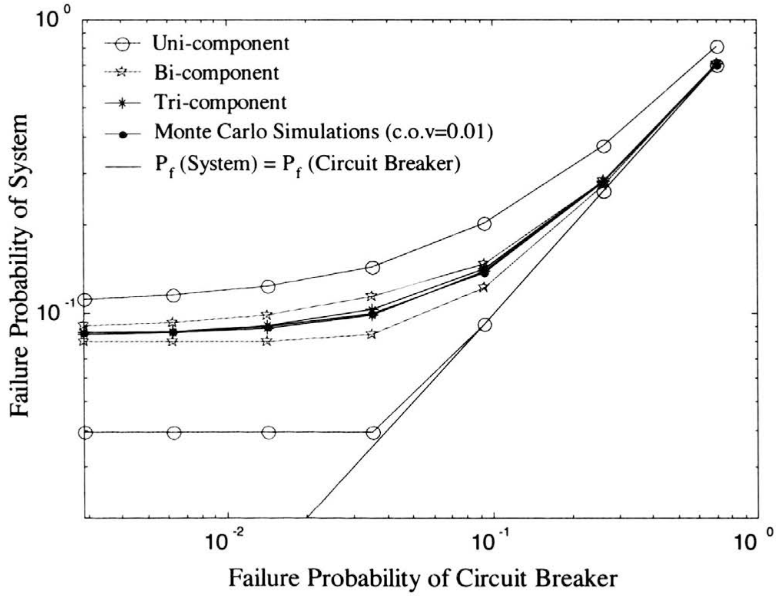

FIGURE 4. System versus circuit-breaker failure probability.

5.2. Example 2: Single-transmission-line with a parallel sub-system of circuit breakers

Another way to enhance the reliability of the single-transmission-line substation is to install several circuit breakers in parallel. This provides redundancy to the system, such that one or more circuit breakers can be taken out of service without affecting the operation of the substation (ASCE 1999).

As shown in Fig. 5, this example replaces the single circuit breaker in the previous example with a parallel sub-system of k circuit breakers. As mentioned earlier, the capacities of the circuit breakers are equally correlated with a coefficient of variation of 0.3. Numbering the components from left to right in Fig. 5, the system failure event is described by the following cut-set

[1)] Many circuit breakers in operation actually have mean capacities around $0.2g$.

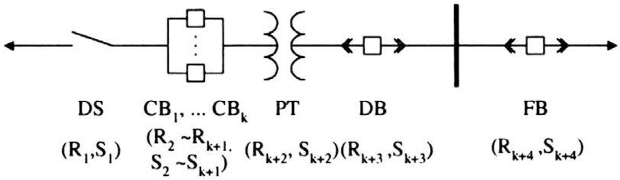

FIGURE 5. A single-transmission-line substation with a parallel sub-system of circuit breakers.

formulation:

$$E_{\text{system}} = E_1 \cup (E_2 E_3 \cdots E_{k+1}) \cup E_{k+2} \cup E_{k+3} \cup E_{k+4}. \tag{5.4}$$

For a system with k circuit breakers in parallel, the LP formulation has 2^{k+4} design variables, $k+4$ equality constraints for the uni-component probabilities, $(k+3)(k+4)/2$ equalities for the bi-component, and $(k+2)(k+3)(k+4)/6$ for the tri-component probabilities. Table 5 lists the uni-, bi-, and tri-component LP bounds as well as Monte Carlo simulation results for selected numbers, k, of the circuit breakers in parallel. It can be seen that adding a second circuit breaker in parallel to the first one significantly enhances the reliability of the single-transmission-line system. However, addition of further circuit breakers in parallel does not provide significantly more improvement in the reliability of the system.

TABLE 5. Failure probabilities of single-transmission-line substation with parallel sub-system of k correlated circuit breakers.

k	Uni-comp. LP	Bi-comp. LP	Tri-comp. LP	M.C. c.o.v. = 0.01
1	0.0925 - 0.202	0.122 - 0.147	0.139 - 0.142	0.138
2	0.0393 - 0.202	0.0805 - 0.130	0.0992 - 0.109	0.104
3	0.0393 - 0.202	0.0805 - 0.122	0.0874 - 0.104	0.0950
4	0.0393 - 0.202	0.0805 - 0.120	0.0847 - 0.100	0.0892

TABLE 6. Failure probabilities of single-transmission-line substation with parallel sub-system of k uncorrelated circuit breakers.

k	Uni-comp. LP	Bi-comp. LP	Tri-comp. LP	M.C. c.o.v. = 0.01
1	0.0925 - 0.202	0.122 - 0.147	0.139 - 0.142	0.138
2	0.0393 - 0.202	0.0805 - 0.125	0.0957 - 0.105	0.100
3	0.0393 - 0.202	0.0805 - 0.116	0.0847 - 0.100	0.0916
4	0.0393 - 0.202	0.0805 - 0.114	0.0847 - 0.0961	0.0864

In the above example, the circuit breakers were assumed to be positively correlated. Such correlation is present when circuit breakers are of the same model or from the same manufacturer. One can increase the reliability of a parallel sub-system by reducing positive correlation between the components. To investigate this effect, the above example is repeated, while assuming the circuit breaker capacities are uncorrelated. In practice, such a case might be achieved by assembling circuit breakers of different make or model. The results in Table 6 show that this modification improves the reliability of the system, but only by a small amount. The reason is that the common random variable A still causes strong correlation between the component failure events.

5.3. Example 3: Two-transmission-line substation

Another way to increase the redundancy of the substation system is to add one or more transmission lines, such that the system has alternative paths for electric flow. Consider the two-transmission-line substation system shown in Fig. 6. As before, we assume a correlation coefficient of 0.3 between equipment capacities within each category (except 0.5 for PT's), and statistical independence between equipment capacities in different categories.

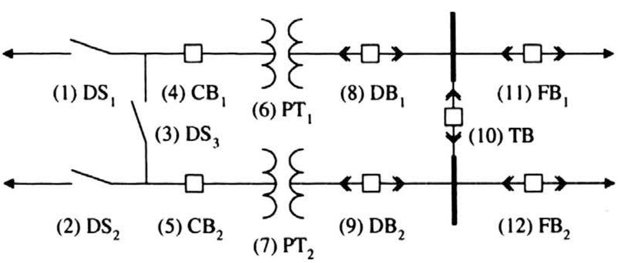

FIGURE 6. A two-transmission-line substation system.

Using the component identification numbers shown in parenthesis in Fig. 6, the 25 minimum cut sets of the system are identified as follows: (1,2), (4,5), (4,7), (4,9), (5,6), (6,7), (6,9), (5,8), (7,8), (8,9), (11,12), (1,3,5), (1,3,7), (1,3,9), (2,3,4), (2,3,6), (2,3,8), (4,10,12), (6,10,12), (8,10,12), (5,10,11), (7,10,11), (9,10,11), (1,3,10,12), (2,3,10,11). The LP problem has $2^{12} = 4\,096$ design variables. The uni-component probabilities introduce 12 equality constraints and the bi- and tri-component probabilities introduce additional 66 and 220 equality constraints, respectively.

The uni-, bi- and tri-component bounds obtained by LP, as well as the Monte Carlo simulation result, are listed in Table 7 as "Case 1." Compared to the single-transmission-line substation, we observe a significant reduction in the tri-component LP bounds. The simulation result also confirms the improvement in reliability in account of the added system redundancy.

TABLE 7. Failure probabilities of two-transmission-line substation system.

Case	Uni-comp. LP	Bi-comp. LP	Tri-comp. LP	M.C. c.o.v. = 0.01
1	$1.13 \cdot 10^{-12}$ - 0.202	0.0436 - 0.146	0.0616 - 0.0942	0.0752
2	$4.69 \cdot 10^{-9}$ - 0.202	0.0436 - 0.146	0.0615 - 0.0943	NA
3	$1.26 \cdot 10^{-9}$ - 0.202	0.0267 - 0.147	0.0395 - 0.1360	NA
4	$5.19 \cdot 10^{-9}$ - 0.120	0.0267 - 0.0995	0.0395 - 0.0701	NA

To further demonstrate the usefulness of the LP bounds, suppose no information is available on one of the equipment items in the substation, say the disconnect switch DS_3 (component 3) that connects/disconnects the two transmission lines. In that case, the uni-component probability and all the joint-component probabilities involving this equipment item are not available. With the LP bounds, we only need to remove the equality constraints corresponding to these unknown probabilities. For the present example, the result obtained by removing the equality constraints involving the disconnect switch $DS_3$is shown in Table 7 as "Case 2." No appreciable change in the bi- or tri-component bounds of the system is observed for this case. This implies that DS_3 may not have a critical role in the system. Note that with incomplete probability information, Monte Carlo simulation cannot be performed and, for that reason, "NA" (not applicable) is indicated in the last column of Table 7. If, instead of DS_3, the equipment item CB_1 is assumed to lack probability information, the result for "Case 3" in Table 7 is obtained. The tri-component LP bounds show a significant widening of the bounds in comparison to Case 1.

Now suppose that the equipment item CB_1 (component 4 in the system), which has a marginal failure probability of 0.0925 (see Table 4, third row), is strengthened and it is estimated that its marginal probability of failure after strengthening is less than 0.01. Suppose no information on joint-component probabilities between this and other equipment items is available. The LP solution for this case, denoted "Case 4" in Table 7, is obtained by removing all equality constraints involving this component and adding an inequality constraint of the form $P_4 \leqslant 0.01$. The result in Table 7 indicates a reduction in the upper bound, but no change in the lower bound.

6. Summary and conclusions

LP can provide the best possible bounds on a system probability for any level of information on marginal or joint component probabilities. The information can be in the form of equality or inequality expressions involving uni- or multi-component probabilities. The method is applicable to any system state defined as a logical expression of component states. This includes series and parallel systems, as well as general systems that are characterized in terms of cut sets or link sets.

The LP methodology is applied to assessing the seismic reliability of three example electrical substation systems. For a single-transmission-line substation modeled as a series system, it is shown that the LP bounds bracket the system failure probability with a width dependent on the level of available probability information. No ordering of the components is necessary to achieve the narrowest possible bounds. The influence of the mean capacity of a critical component system reliability is investigated. The second example involves a single-transmission-line system with a parallel sub-system of circuit breakers. The uni-, bi- and tri-component probability bounds using LP are computed and compared with Monte Carlo simulation results for different numbers of circuit breakers in the parallel sub-system. It is found that adding one circuit breaker in parallel significantly increases the reliability, but further additions are not effective. The influence of the correlation between equipment capacities is also investigated. The third example is a two-transmission-line substation, which is endowed with redundancy in the electric flow paths. The system is modeled with 12 components and 25 cut sets. The uni-, bi- and tri-component bounds are computed by LP and compared with the Monte Carlo simulation result. This example also shows how the LP bounds can be obtained when the information on a certain component is incomplete or in the form of an inequality.

7. Acknowledgment

The first author was supported by a grant from the Lifelines Program of the Pacific Earthquake Engineering Research Center. Partial support was also provided by the Earthquake Engineering Research Centers Program of the National Science Foundation under Award No. EEC-9701568. These supports are gratefully acknowledged.

References

1. ASCE, (1999), *Guide to Improved Earthquake Performance of Electric Power System*, Electric Power and Communications Committee Technical Council on Lifeline Earthquake Engineering (prepared), A.J. Schiff (Ed.), Reston, ASCE.

2. BERTSIMAS, D. and TSITSIKLIS, J.N., (1997), *Introduction to Linear Optimization*, Athena Scientific, Belmont.

3. BOOLE, G., (1854), *Laws of Thought*, Amer. reprint of 1854 ed., Dover, New York.

4. BROWN, R.E., (2002), *Electric Power Distribution Reliability*, Marcel Dekker Inc., New York.

5. DANTZIG, G.B., (1951), Application of the simplex method to a transportation problem, *Activity analysis of production and allocation*, T.C. Koopmans (Ed.), pp.359-373.

6. DER KIUREGHIAN, A., (2002), Bayesian methods for seismic fragility assessment of lifeline components, *Acceptable Risk Processes – Lifelines and Natural Hazards*, ASCE, pp.61-77.

7. DITLEVSEN, O., (1979), Narrow reliability bounds for structural systems, *Journal of Structural Mechanics*, Vol.7, No.4, pp.453-472.

8. DITLEVSEN, O. and MADSEN, H.O., (1996), *Structural Reliability Methods*, J. Wiley & Sons, New York.

9. DUNNET, C.W. and SOBEL, M., (1955), Approximations to the probability integral and certain percentage points of a multivariate analogue of Student's t-distribution. *Biometrika*, Vol.42, pp.258-260.

10. FRÉCHET, M., (1935), Généralizations du théorème des probabilités totales, *Fund. Math.*, Vol.25, pp.379-387.

11. HAILPERIN, T., (1965), Best possible inequalities for the probability of a logical function of events, *American mathematical monthly*, Vol.72, No.4, pp.343-359.

12. HOHENBICHLER, M., and RACKWITZ, R., (1983), First-order concepts in system reliability, *Structural Safety*, Vol.1, No,3, pp.177-188.

13. HUNTER, D., (1976), An upper bound for the probability of a union, *Journal of Applied Probability*, Vol.13, pp.597-603.

14. KOUNIAS, E.G., (1968), Bounds for the probability of a union, with applications, *Annals of Mathematical Statistic*, Vol.39, No.6, pp.2154-2158.

15. PRÉKOPA, A., (1988), Boole-Bonferroni inequalities and linear programming, *Operations Research*, Vol.36, No.1, pp.145-162.

16. SONG, J. and DER KIUREGHIAN, A., (2003a), Bounds on system reliability by linear programming, *Journal of Engineering Mechanics, ASCE*, Vol.129, No.6, pp.627-636.

17. SONG, J. and DER KIUREGHIAN, A., (2003b), Bounds on system reliability by linear programming and applications to electrical substations, *Proc. 9th International Conference on Applications of Statistics and Probability in Civil Engineering*, San Francisco, CA, July 6-9, Volume I, pp.111-118.

18. ZHANG, Y.C., (1993), High-order reliability bounds for series systems and application to structural systems, *Computers and Structures*, Vol.46, No.2, pp.381-386.