AMAS Course on Reliability-Based Design and Optimisation
RBO'03 – (pp.241–323) – Warsaw, September 15-18, 2003

Five lectures on reliability applications of Rice's formula for the intensity of level crossings

I. RYCHLIK

Centre for Mathematical Sciences
Lund University
Box 118, S-221 00 Lund, Sweden
igor@maths.lth.se

1. Lecture I: Maximum of a Gaussian load

The celebrated Rice formula for the expected number of times a stationary process $X(s)$, $s \in [0,t]$, "crosses" a fixed level u has found application in various engineering problems, especially in safety analysis of structures interacting with the environment, for example through wind pressure, ocean waves or temperature variations. The safety of a structure may depend on extreme and rare events such as loads which exceed the strength of a component, or on everyday load variability that may cause changes in the properties of the material, e.g. cracking (fatigue) or other types of aging processes. In the first case, the number of rare events that occur in time or in space is often modeled as a Poisson process. Then, the Rice formula is used to compute the intensity of events, and hence gives the parameters in the Poisson model. In the second case, the aging process may depend both on frequencies of some events as well as their magnitudes. A magnitude of an event is called a "mark".

Let us start by demonstrating in a few typical examples some applications of the Rice's formula in engineering problems.

1.1. Design load – crossing intensity

A structure should sustain the everyday variability of load during its service time. Usually the stronger (and safer structure) tends to be more

expensive, hence one needs methods to quantify risks in order to achieve some balance between strength and cost.

Still at the design stage, one has to choose the so-called "design loads" and to plan the strength of the components so that, with high probability, those will sustain the loads. Exceedance of the design load limits, by the real load, results in structural damage. So it is important to know how often those potentially dangerous conditions may happen. A convenient measure of load severity is the so-called "return period" which equals the inverse of the frequency of exceedances over a certain limit. Let us assume that a design load limit was set-up to be u and that the future load can be described mathematically, as a continuous function $x(s)$, $s \geq 0$. Let us also denote by $n_t(u)$ the number of times the function $x(s)$ takes the value u, i.e

$$n_t(u) = \text{number of times } x(s) = u, \quad 0 \leq s \leq t. \tag{1.1}$$

With some abuse of terminology, we will call $n_t(u)$ number of *level crossings*. If x is also differentiable, the number of *upcrossings* of the level u, may be defined as

$$n_t^+(u) = \text{number of times } x(s) = u,\ x'(s) > 0, \quad 0 \leq s \leq t. \tag{1.2}$$

The crossing, upcrossing - intensity of the level u by the load x may be defined as:

$$n(u) = \lim_{t\to\infty} \frac{n_t(u)}{t}, \qquad n^+(u) = \lim_{t\to\infty} \frac{n_t^+(u)}{t}, \tag{1.3}$$

provided the limits exist. Then the *return period* is given by $T = 1/n^+(u)$.

Often in practice one has only one historical record of the load $x(s), s \in [0,t]$. If $n_t^+(u) > 0$ then the return period can be estimated by means of $t/n_t^+(u)$. Under the assumption that the future load will demonstrate the same kind of "behavior" as the historical one, the estimated return period may be used for safety evaluation. However, it often happens, especially when $t < T$, that $n_t^+(u) = 0$. In such situations, one needs to employ some mathematical models in order to extrapolate $n_t^+(u)$ for t approaching infinity. Two methods are commonly employed:

- Statistical algorithms (fitting of extreme value distribution, POT) are used to estimate the tail of the cumulative probability distribution of $x(s)$. Then the return period is computed using the fitted model for the tail.
- A family of random processes is used to model the load, whose parameters are estimated by means of the historical data. Then the return period is computed for the chosen family. The Gaussian processes are often employed as useful models.

In these notes we will only consider the second method.

Suppose that the function x is modeled as a random function. By this we mean that $x(s)$ is a realization of the random process $X(s)$. Functionals defined on x become random variables and according to our notational convention are denoted by capital letters. Hence we write $N_t(u)$, $(N_t^+(u))$ for the number of crossings (upcrossings) of the process X in the interval $[0,t]$.

Remark 1. The crossing intensity $n(u)$ is defined as the limit, $n_t(u)/t$ as t approaches infinity. It is not obvious that such a limit exists for all random functions x. However, if X is a stationary process then the sequences $N_t(u)/t$ and $N_t^+(u)/t$ converge as t tends to infinity, with probability one. Still though, the limiting values (crossing intensities) denoted by $N(u)$ and $N^+(u)$ respectively, may vary for different random functions x, see the following Example 1.

For stationary loads the upcrossing intensity is random $N^+(u)$ and hence the return period $1/N^+(u)$ is also random. To avoid using a random variable in order to measure the risk of exceedance of the design load, we modify the above definition to

$$T = \frac{1}{\mathsf{E}(N^+(u))}.$$

The expected values $\mathsf{E}(N(u))$, $\mathsf{E}(N^+(u))$ will be denoted by $\mu(u)$, $\mu^+(u)$ respectively. The important property is that

$$\mathsf{E}(N_t(u)) = t\mu(u), \quad \mathsf{E}(N_t^+(u)) = t\mu^+(u). \tag{1.4}$$

Often the expectations $\mu(u)$, $\mu^+(u)$ can be computed using the Rice's formula, given in the following section. The formula gives us the tools to solve the following inverse problem; *given the return period T find the design load u*, i.e. solve

$$\mu^+(u) = \frac{1}{T}, \tag{1.5}$$

for u.

Example 1. Consider a simple Gaussian wave $X(t) = \sigma R\cos(\lambda t + \theta)$ where R, θ are independent r.v's distributed as Rayleigh and uniform respectively. The variable R has probability density $f(r) = re^{-r^2/2}$, $r \geq 0$. It is easy to see that the upcrossing intensity $N^+(u)$ is a random variable given by

$$N^+(u) = \begin{cases} \frac{\lambda}{2\pi}, & \text{if } u \in (-\sigma R, \sigma R), \\ 0, & \text{otherwise.} \end{cases}$$

The expected upcrossing intensity $\mu^+(u) = \mathsf{E}(N^+(u))$ is easy to compute, viz.

$$E(N^+(u)) = \frac{\lambda}{2\pi}P(u \in (-\sigma R, \sigma R)) = \frac{\lambda}{2\pi}P(R > u/\sigma) = \frac{\lambda}{2\pi}e^{-\frac{u^2}{2\sigma^2}}. \tag{1.6}$$

Using Rice's formula we shall demonstrate that the last equation holds for any differentiable stationary Gaussian processes $X(t)$ having expectation $\mathsf{E}(X(t)) = 0$, variance $\mathsf{V}(X(t)) = \sigma^2$ and the mean angular frequency $\hat{\lambda}$ equal to λ, see (1.15).

Suppose that we model the future load by means of a random process X. If the load is a simple Gaussian wave then for a particular random function $x(t)$ (yet unknown) the load will exceed (almost immediately) the design level u or will stay for ever under it. So the safety is zero or one depending on which of these two possibilities will happen. The return period defined as $1/\mu^+(u)$ can be still used to measure the uncertainty whether the design load will be exceeded or not.

In the following we will always assume that the process is ergodic. If X is a stationary and ergodic process, $N(u) = \mu(u)$, $N^+(u) = \mu^+(u)$ with probability one. This means that crossing intensities are constant and equal to the expected crossing intensities.

1.2. Extreme loads – safety

The return period is a very simple measure of the severity of a load. Often one wishes to know the probability that the structure will experience load exceeding the design load u during a service time S, say. In other words one wishes to find probability that $M_S(X) = \max_{0\le s\le S} X(s) > u$. Finding this probability is an important problem both in probability theory and in applications.

As we will demonstrate next, finding the probability $P(M_S > u)$ is equivalent to estimation of $P(N_S^+(u) > 0)$, since

$$\begin{aligned} P(M_S > u) &= P(X(0) > u) + P(X(0) \le u, N_S^+(u) > 0) \\ &\approx P(N_S^+(u) > 0) \le \mathsf{E}(N_S^+(u)) = S\mu^+(u) = \frac{S}{T}, \end{aligned} \tag{1.7}$$

where T is the return period. The approximation in the second line can be motivated when the probability that the load exceeds the design value at $t = 0$ is negligible. However, if S is very short then one should rather use the following bound

$$\begin{aligned} P(M_S > u) &= P(X(0) > u) + P(N_S^+(u) > 0 | X(0) \le u) P(X(0) \le u) \\ &\le P(X(0) > u) + \mathsf{E}(N_S^+(u) | X(0) \le u) P(X(0) \le u). \end{aligned} \tag{1.8}$$

The Rice's formula for the intensity of upcrossings of the level u will be employed to compute bounds (1.7).

Example 2. *Design of sea-walls.* When designing protection against the high sea level, one speaks about 100-year or 10 000-year storms which means that in average one have to wait 100 or 10 000 years, to observe a storm stronger than the 100-year or 10 000-year storms. We will discuss two examples of design of sea-walls: at Ribe in Denmark (at the North Sea) and in the Netherlands. In Denmark, one chose in the 1970s a design load with a return period of 200 years. (The old level was 30-45 years.) In the Netherlands, after disastrous floods in the early 1950s, the decision was taken to design the sea-walls against return storms of 20 000 years. Let neglect the sesonal variability of strength of storms, use of Eq. (1.7) gives the probability of catastrophical floods in the following 50 years, i.e. at least one flood, $p \leq \frac{t}{T}$, which, in case of Ribe in Denmark, gives a considerable risk with likelihood 1/4. Due to this risk, it is worth to have some alarm system to warn the inhabitants of the possibility of a flood. Such systems are installed. In the Netherlands the chance is negligible if all computations and constructions have been done properly.

However, aspects not known at the time of the analysis have obviously not been taken into account. Wave climate in the Atlantic Sea may change, knowledge about the impact of ice melting at the poles is uncertain. Besides this "model type" uncertainties we need to acknowledge that we have also statistical uncertainty due the fact that one wishes to find properties of storms that are very rare. Consequently our estimates will be uncertain values too. For example the storm, which we consider as 20 000 years storm may have return period of 1000 years or less. All this means that the calculated values of risk for rarely occurring catastrophes should not be treated as exact values.

Finally, note that we have not said how to find the size of the sea-walls which will sustain storm with return period of 50 or 20 000 years, or equivalently how to find parameters describing those storms. □

Finding the probability $P(N_S^+(u) > 0)$ is a very difficult problem. The following two asymptotic results are often useful: under some assumptions on the process X, e.g. X is Gaussian with covariance function $r_X(t) \cdot \ln t \to 0$ as $t \to \infty$, see Leadbetter *et al.* (1983) for more detailed discussion, one can demonstrate that

- *as t and u go to infinity in a way that $E(N_t^+(u))$ is constant (say equal to μ) then $P(N_t^+(u) = 0)$ converges to $e^{-\mu}$.*
- *for fixed u, $t^{-1/2}\left(N_t^+(u) - t\mu^+(u)\right)$ is asymptotically (as t tends to infinity) normally distributed.*

The first asymptotic result is often used to approximate $P(N_S^+ > 0)$ in the case when both u and S take large values. Then, by taking $\mu = \mathsf{E}(N_S^+(u))$,

one can derive the following approximation

$$P(M_S > u) \approx P(N_S^+(u) > 0) \approx 1 - e^{-E(N_S^+(u))}. \tag{1.9}$$

If $E(N_S^+(u))$ is small then $1 - e^{-E(N_S^+(u))} \approx E(N_S^+(u)) = S\mu^+(u)$. Consequently if the service time S of the structure is much shorter then the return period for the design load then

$$P\Big(\max_{0\le s\le S} X(t) > u\Big) \approx \frac{S}{T}.$$

Accuracy of this approximation for moderate levels u will be studied in Sec. 1.3.1. In the case when u is not very high and t very long then the second asymptotic result could be applied. Namely, for a fixed level u, $N_t^+(u)$ is asymptotically normally distributed. In order to be able to use this result one needs to be able to compute (or estimate) the variance $\mathsf{V}(N_t^+(u))$. The approximation is applicable when t is large ($\mathsf{E}(N_t^+(u))$ is large and $\mathsf{V}(N_t^+(u)) \neq \mathsf{E}(N_t^+(u))$. We mention the last condition since even Poisson asymptotic may lead to the Gaussian approximation. More precisely, if both t and u are large and $\mu^+ = \mathsf{E}(N_t^+(u))) > 10$ then $N_t^+(u)$ is approximately $N(\mu, \mu)$-distributed.

In the following example the design load is the endurance limit for fatigue.

Example 3. *Multiaxial fatigue - extreme stresses.* Multiaxial fatigue failure criteria are now widely used design tools to evaluate design margins against long term fatigue for metallic structures subjected to multiaxial periodic loads. The criteria are generalizations of the uniaxial endurance limit, defined as the smallest stress level S_e leading to the initiation of a macroscopic crack, which generally appears after $N_e = 10^6 - 10^7$ cycles under an alternating sine stress of constant amplitude. Even if its existence is being called into question, the endurance limit is generally used design tool even for variable amplitude loading, i.e. if a component is subjected to fluctuating stresses never exceeding the stress level S_e, a component is supposed to have an infinite fatigue life. The extension of this concept to multiaxial stress states is often done by separation of the stress space into two parts, the unsafe one and the safe one. We shall illustrate the approach using the so-called Crossland's criterion, which will be given next. (Crossland's criterion belongs to the category of global approaches which are based on stress tensor invariants, see Crossland (1956). Comparison studies of multiaxial criteria, see e.g. Papadopoulos *et al.* (1999), report that Crossland's criterion can lead to a deviation error between predictions and experimental values ranging from -30% and $+15\%$.)

Let us consider a material point $\mathbf{q}$ in a structure subjected to biaxial stress fields where $s_x(t)$, $s_y(t)$ and $s_{xy}(t)$ are the normal and shear stresses expressed in the local coordinate system $(\vec{x}, \vec{y})$. Crossland's criterion can be formulated as follows: if

$$\frac{\max_{0\le t\le S} |s_c(t)|/\sqrt{3} + \alpha_c \max_{0\le t\le S} p(t)}{\beta_c} \le 1 \quad \text{for all } \mathbf{q}, \tag{1.10}$$

then no fatigue crack will initiate before the time $S \cdot N_e$; otherwise, a fatigue crack is expected to appear at every point of the structure where the condition is not satisfied. Clearly service time should be shorter than $S \cdot N_e$.

Here $s_c(t)$ is the von Mises stress that is defined in the biaxial case by

$$s_c^2(t) = s_x^2(t) + s_y^2(t) - s_x(t)s_y(t) + 3s_{xy}^2(t), \tag{1.11}$$

while $p(t)$ is the hydrostatic pressure defined as the first invariant of the stress tensor $s_{ij}(t)$ and which can be written as follows

$$p(t) = \frac{1}{3}(s_x(t) + s_y(t)), \tag{1.12}$$

with α_c and β_c determining material properties

$$\begin{cases} \alpha_c = (t_{-1} - f_{-1}/\sqrt{3})/(f_{-1}/3), \\ \beta_c = t_{-1}, \end{cases}$$

where f_{-1} and t_{-1} are the endurance limits for a reversed tension stress and a reversed torsion stress respectively.

Clearly if the applied load is random then also $\mathbf{s}(t) = (s_x(t), s_y(t), s_{xy}(t))$ is a vector of random processes. (Since the stress tensor is also dependent on location $\mathbf{q}$ it can also be considered as random vector valued field.) In Fig. 1 we give simulated records of the stress tensor under the assumption that the load is stationary Gaussian process.

In safety analysis one wishes to compute the probability that Crossland's criterion fails or, slightly easier problem, to check if

$$\max_{\mathbf{q}} \mathsf{E}\left(\frac{\max_{0\le t\le S} |s_c(t)|/\sqrt{3} + \alpha_c \max_{0\le t\le S} p(t)}{\beta_c}\right) \le 1. \tag{1.13}$$

In following we will assume that the load is a zero mean stationary Gaussian process and hence $p(t)$ will be a Gaussian process too, while $s_c^2(t)$ is a χ^2 process. In order to compute the expectations in (1.13) we will compute the crossing intensities $\mu_p^+(u)$, $\mu_{s_c^2}^+(u)$ and use the approximation (1.9) of $P(M_S > u)$. (Note that $\mathsf{E}(M_S) \approx \int_0^{+\infty} P(M_S > u)\, du$.) We return to the computations of the expectation $\mathsf{E}(M_S(p)), \mathsf{E}(M_S(s_c^2))$ later on.

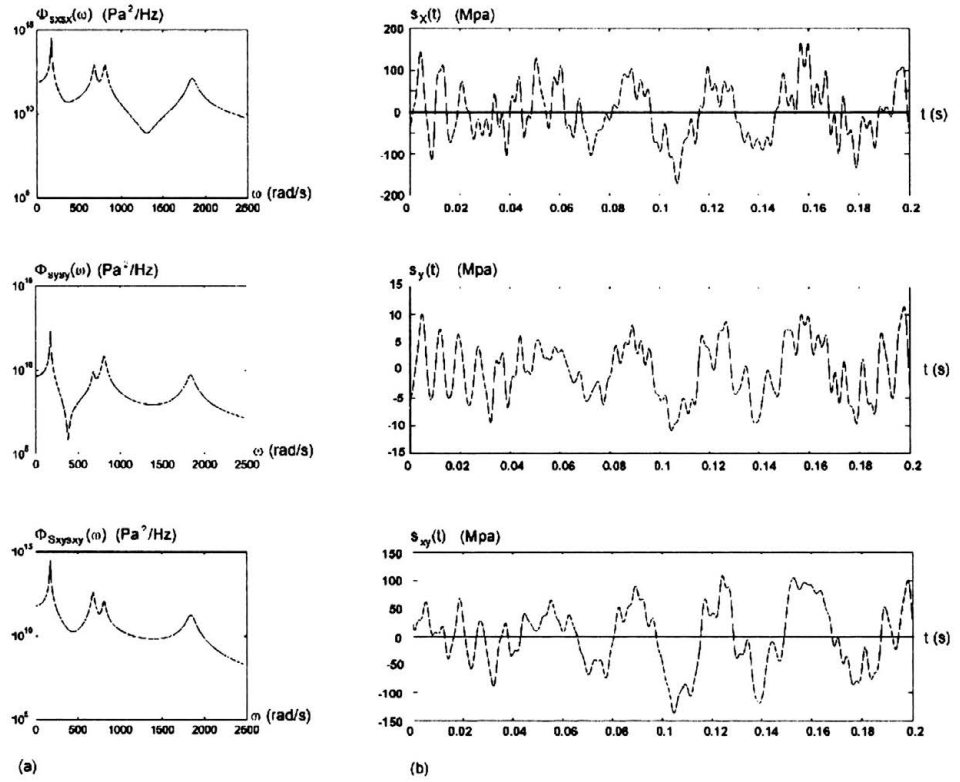

FIGURE 1. (a) The power spectral densities of s_x, s_y, s_{xy}. (b) The simulated stresses.

1.3. Rice's formula-Gaussian processes

Let $X(t)$ be a stationary Gaussian process with mean m and finite zero and second order spectral moments, $\lambda_0, \lambda_2 < \infty$. Such a process possesses a derivative $\dot{X}(t)$ and we have

$$m = E(X(t)), \quad V(X(t)) = \lambda_0, \quad V(\dot{X}(t)) = \lambda_2.$$

For the process $X(t)$, the expected crossing intensity is given by the Rice's formula

$$\mu(u) = \frac{1}{\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}e^{-(u-m)^2/2\lambda_0}. \tag{1.14}$$

Since on average there are equally many up- and down-crossings, $\mu(u) = 2\mu^+(u)$, and hence the intensity of upcrossings of the mean level,

$$\mu^+(m) = \frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}} = \frac{\hat{\lambda}}{2\pi}. \tag{1.15}$$

The parameter $\hat{\lambda} = \sqrt{\lambda_2/\lambda_0}$ is called the *mean angular frequency* of the process. Clearly the intensity can now be written as

$$\mu^+(u) = \frac{\hat{\lambda}}{2\pi} e^{-(u-m)^2/2\lambda_0}. \tag{1.16}$$

For the simple Gaussian wave in Example 1, we have that $\lambda_0 = \sigma^2$, $m = 0$, while $\hat{\lambda} = \lambda$ and we have verified that the Rice's formula holds for the simple Gaussian wave.

Example 4. *Significant wave height.* In oceanography one often assumes that under relatively short period of time, between 20 minutes and a few hours, the sea surface elevation at a fixed point can be, with sufficient accuracy, described using a stationary Gaussian process $X(t)$, say. The exact shape of the spectral density $S(\lambda)$ is usually not known and one is characterizing the sea conditions by means of the so-called significant wave-height H_s and zero-crossing wave period T_z, i.e. the inverse upcrossing intensity of the still water level. The mean m, of $X(t)$ is called the still water level and is often taken to be zero. (Note that for non Gaussian sea models the still water level can be defined as the level most frequently crossed by $X(t)$.) The variance and the angular frequency of the process $X(t)$ are defined by the parameters H_s, T_z as follows

$$\lambda_0 = \frac{H_s^2}{16}, \qquad \hat{\lambda} = \frac{2\pi}{T_z}.$$

Obviously, for Gaussian sea the parameters m, H_s and T_z define the crossing intensity

$$\mu^+(u) = \frac{1}{T_z} e^{-8\frac{(u-m)^2}{H_s^2}}.$$

In the analysis of frequency of occurrence of extreme waves in an interval of length S, one is sometimes using the following parameter

$$N = \frac{\hat{\lambda} S}{2\pi}, \tag{1.17}$$

which is equal to the expected number of waves in $[0, S]$.

We turn now to our main example and compute $\mathsf{E}(M_S(p(t))$.

Example 5. *Multiaxial fatigue continuation of Example 3.* If the load applied on the specimen is stationary and ergodic Gaussian process then (assuming the linear theory) the stresses $s_x(t), s_y(t), s_{xy}(t)$ at a fixed position $\mathbf{q}$ represent a Gaussian vector valued process. Consequently the hydrostatic pressure $p(t)$, see (1.12) is a Gaussian process too. Knowing the mean and

spectral density (measure) of the load one can, by means of the finite element program, estimate the mean m and spectral moments λ_0, λ_2 for $p(t)$. Obviously the parameters will vary between the different locations in the structure. For simplicity only let us assume that $m = 0$. Now for a fixed S we want to derive an approximation for $\mathsf{E}(\max_{0\le t\le S} p(t))$.

As before let us denote by $M_S(p) = \max_{0\le t\le S} p(t)$. As we have mentioned earlier for extreme levels u and long periods T one can use the Poisson approximation, i.e.,

$$P(M_S > u) \approx 1 - e^{-\mathsf{E}(N_S^+(u))} = 1 - e^{-S\mu^+(u)}.$$

Clearly for long periods S and low positive values u $S\mu^+(u)$ is large and hence $P(M_S > u) \approx 1$. From this we are drawing the conclusion that for long periods S the last formula is an accurate approximation for any positive u. Consequently, since for Gaussian processes $S\mu^+(u) = Ne^{-\frac{u^2}{2\lambda_0}}$, where $N = \frac{\hat{\lambda}}{2\pi}S$ is the expected number of waves (upcrossing of mean), we have

$$\mathsf{E}(M_S) \approx \int_0^\infty P(M_S > u)\, du \approx \int_0^\infty 1 - e^{-Ne^{-\frac{(u-m)^2}{2\lambda_0}}}\, du. \tag{1.18}$$

The last integral can be computed numerically.

However, for Gaussian processes an alternative asymptotic result has been proved. Namely that M_S is asymptotically Gumbel distributed as u and S tend to infinity.

Let $X(t)$, $0 \le t \le S$, with $E(X(t)) = m$ be a stationary Gaussian process having expected number of waves $N = \frac{\hat{\lambda}}{2\pi}S$, where where $\hat{\lambda}$ is the average angular frequency, see (1.15). Now, under suitable assumptions on the covariance function, e.g. $r_X(t) \cdot \ln t \to 0$ as $t \to \infty$, we have, as $S \to \infty$,

$$P(M_S \le u) \approx \exp\left\{-e^{-(u-b_N)/a_N}\right\},$$

where

$$a_N = \frac{\sqrt{\lambda_0}}{\sqrt{2\ln N}},$$

and

$$b_N = m + \sqrt{\lambda_0}\sqrt{2\ln N}.$$

Now the Gumbel distributed variable X, say, with parameters a, b has the mean

$$\mathsf{E}(X) = b + 0.5572a.$$

Let, for the process $p(t)$, N_p be the expected number of waves in $[0, S]$ while $\lambda_0^p = \mathsf{V}(p(t))$, then

$$\mathsf{E}(M_S(p_c)) = \sqrt{\lambda_0^p}\left(\sqrt{2\ln N_p} + \frac{0.5772}{\sqrt{2\ln N_p}}\right). \tag{1.19}$$

Finally one can ask how the two formulas (1.18) (1.19) are interrelate. This will be shown next. For simplicity only we assume that $m = 0$, then

$$\begin{aligned} P(M_S \leq u) &\approx e^{-\mathsf{E}(N_S^+(u))} = e^{-S\mu^+(u)} \\ &= \exp\left\{-Ne^{-u^2/2\lambda_0}\right\} = \exp\left\{-e^{-u^2/2\lambda_0 + \ln(N)}\right\} \\ &= \exp\left\{-e^{-\frac{u^2-b_N^2}{2\lambda_0}}\right\} = \exp\left\{-e^{-\frac{(u-b_N)(u+b_N)}{2\lambda_0}}\right\} \\ &\approx \exp\left\{-e^{-\frac{u-b_N}{2\lambda_0/2b_N}}\right\} = \exp\left\{-e^{-\frac{u-b_N}{a_N}}\right\}, \end{aligned} \tag{1.20}$$

since $u \approx b_N$. Consequently for very long times S both approaches are equivalent.

Conclusion of the last example is that assuming that hydrostatic pressure is Gaussian then knowing mean value m, variance λ_0 and the mean angular frequency $\hat{\lambda}$ is sufficient to approximate the expected value of the $M_S(p) = \max_{0\leq t\leq S} p(t)$. Since we proposed to use a Gumbel distribution to compute $\mathsf{E}(M_S(p))$ the accuracy of this approximation is satisfactory only when $\hat{\lambda}T/2\pi$ (expected number of waves) is large.

However, it is not easy, for a particular process $X(t)$ to give conditions when $N = \hat{\lambda}S/2\pi$ (average number of waves) is sufficiently large so that using asymptotic results gives accurate approximations. The name *asymptotic approximation* is motivated by a general result that, under some assumptions on how fast the covariance function $r(t)$ decreases to zero as $t \to \infty$, the distribution of a suitably normalized points $\{t \in [0, S] : X(t) = u, \dot{X}(t) > 0\}$ converges to a Poisson process as the level u and S tend to infinity. (This fact also holds for some non-Gaussian processes).

The Poisson approximation can be very accurate even for moderately high levels u if the process has a broad-band spectrum, e.g. the covariance function $r(t)$ converges rapidly to zero as t goes to infinity. For a narrow band process, upcrossings may occur in clusters or "clumps", and then the level u has to be very high to assure an acceptable accuracy of the asymptotic approximation, as we shall see in the following subsection.

1.3.1. Maximum of a narrow-band Gaussian process. Let $X(t)$ be the displacement of a damped harmonic oscillator driven by a Gaussian white noise process, i.e. the solution to the equation

$$\ddot{X}(t) + 2\zeta\dot{X}(t) + X(t) = \sigma W(t), \qquad t \geq 0,$$

where $W(t)$ is a Gaussian white noise process. (The relative damping is ζ, $0 < \zeta < 1$.) Assuming that $\sigma^2 = 4\zeta$ then $\dot{X}(0), X(0)$ are independent and standard Gaussian variables. The process $X(t)$ has mean $m = 0$, mean angular frequency $\hat{\lambda} = 1$ and the zero spectral moment $\lambda_0 = 1$. The spectral density of $X(t)$ is given by

$$S(\lambda) = \frac{2\zeta/\pi}{(1-\lambda^2)^2 + (2\zeta\lambda)^2}. \tag{1.21}$$

For the process $X(t)$ (due to the particular choice of constants), the crossing intensity is simply

$$\mu^+(u) = \frac{1}{2\pi} e^{-u^2/2}.$$

By choosing the damping parameter ζ close to zero we obtain a narrow band spectrum while for $\zeta > 0.2$ the process becomes broad-band. In Fig. 2(a) and (b) we have parts of a simulated process $X(k \cdot \Delta t)$ where $\Delta t = 0.1$, for $\zeta = 0.01$ and $\zeta = 0.4$. The narrow band character is very clearly seen in Fig. 2(a).

We take $u = 3$ (three standard deviations from the mean) and find the distances between consecutive upcrossings of the level $u = 3$ in a sample of $X(t)$, $0 \leq t \leq S$. Here we have used $S = 10^5$ and hence by Rice's formula we have an expected number of $\frac{S}{2\pi} e^{-u^2/2} = 176.8$, u-upcrossings in the entire interval, and the mean length between the upcrossings (return period) is $2\pi e^{4.5} = 565.6$ seconds. Since the average number of upcrossings is finite we can order all the points in $\{t \in [0, S] : X(t) = u, \dot{X}(t) > 0\}$ and let denote them by t_i.

In the asymptotic Poisson approximation the intercrossing distances $t_i - t_{i-1}$ are independent exponential variables. From Fig. 2(a) it is obvious that this model can not be correct for a narrow band process with $\zeta = 0.01$, since after an upcrossing u there is a high probability that the next upcrossing is located approximately in a period 2π. This can also be seen on the Weibull probability paper in Fig. 4(a). Often the maximum of a narrow-band Gaussian process is approximated by the maximum of its envelope process, see Fig. 6 for the illustration of the relationship between the process and its envelope. The Poisson approximation is then applied to crossings of the

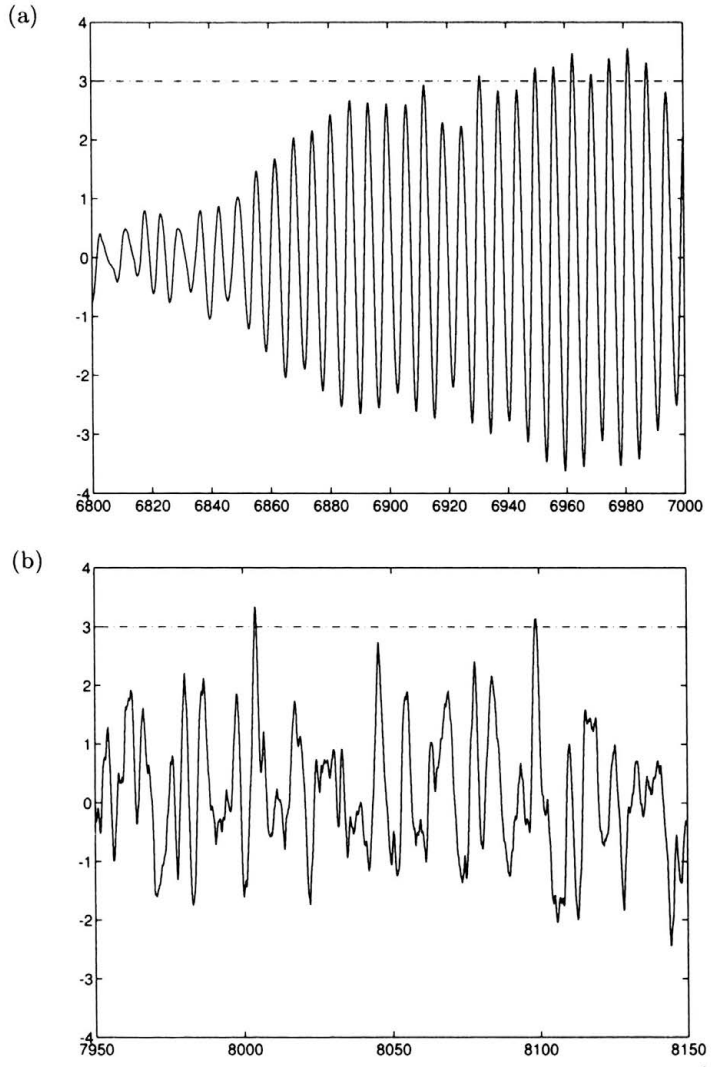

FIGURE 2. Narrow- and broad-band linear oscilator: (a) $\zeta = 0.01$, (b) $\zeta = 0.4$.

envelope, see Ditlevsen and Lindgren (1988) for more detailed presentation. Consequently, one can approximate

$$P(\max_{0\le t\le S} X(t) > u) \approx 1 - e^{-S\mu_E^+(u)},$$

where crossing intensity of the envelope process $\mu_E^+(u)$ is given by (2.9). The intuitive motivation is that the envelope crosses less frequently the level u than the process X and hence would give better approximation. This is not

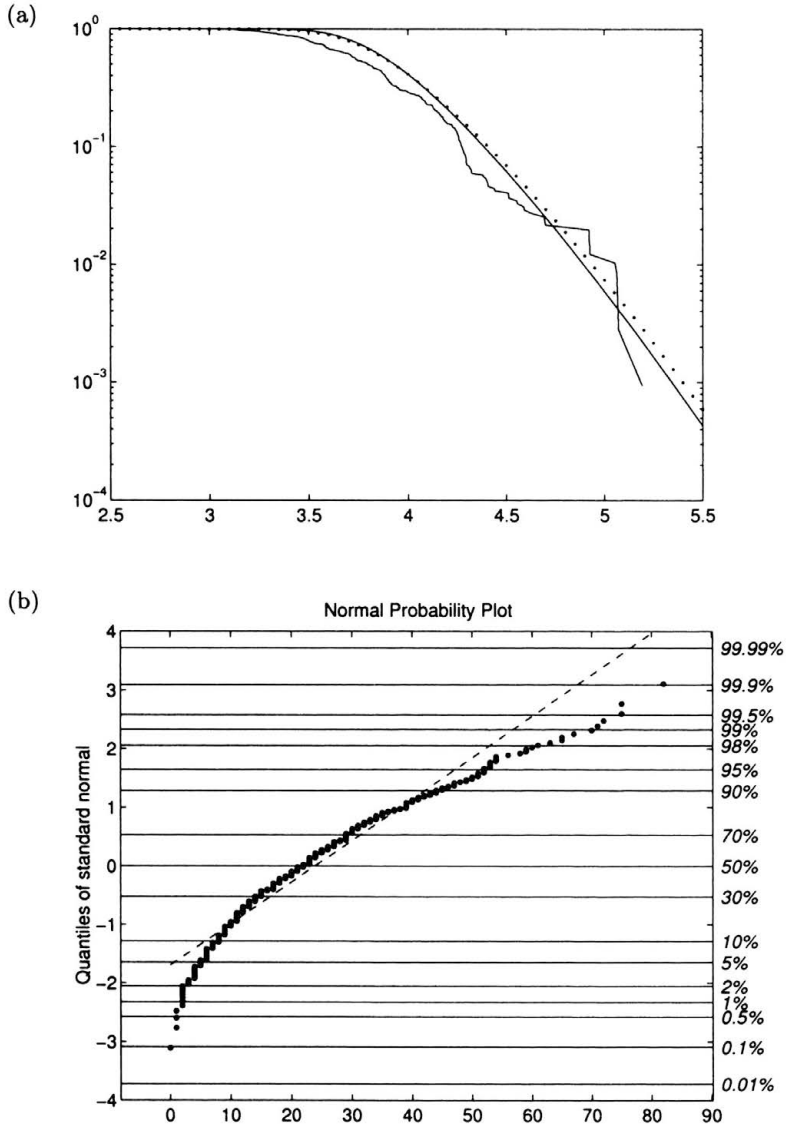

FIGURE 3. (a) Comparison between the empirical distribution of $M_S(X)$, $\zeta = 0.01$ and $S = 10^4$ s, (irregular line) with the asymptotic Poisson approximation $P(M_S(X) > u) \approx 1 - e^{-S\mu^+(u)}$ solid line and using the envelope $P(M_S(X) > u) \approx 1 - e^{-S\mu_E^+(u)}$ dots. (b) The simulated number of upcrossings of the level u by the process X plotted on the normal probability paper.

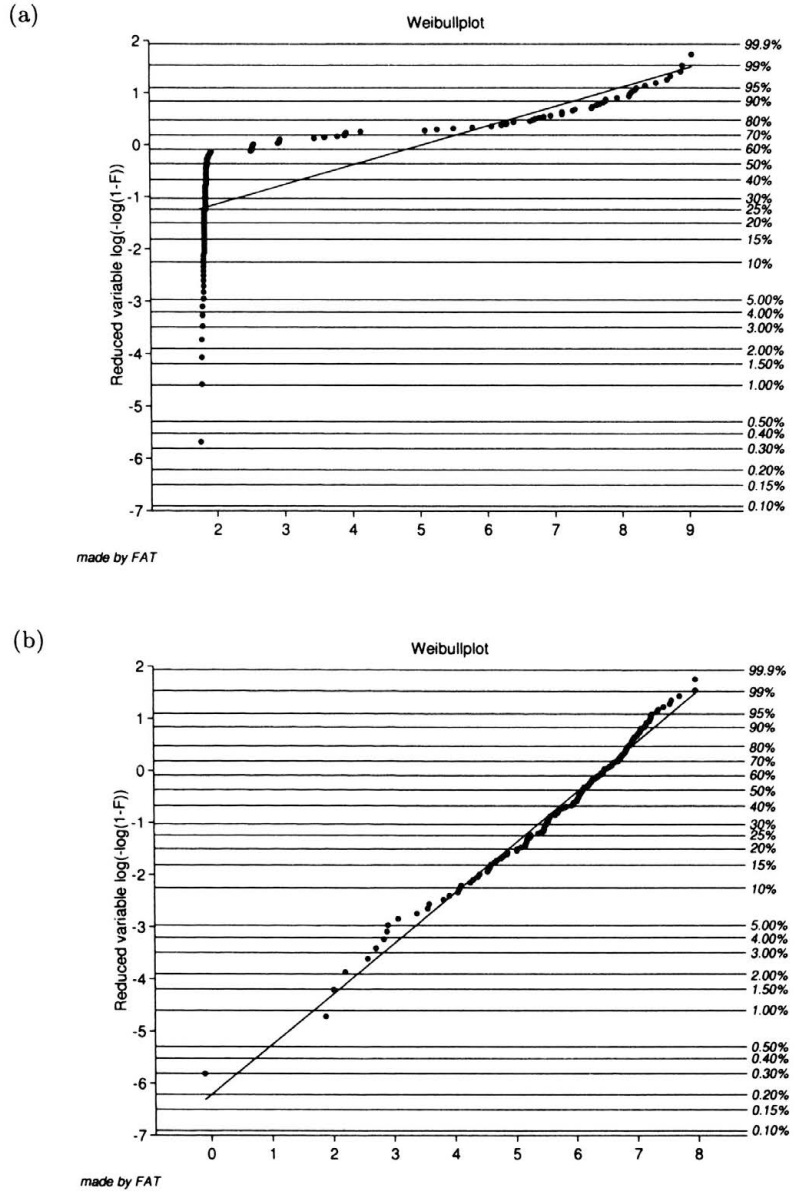

FIGURE 4. Illustration of the asymptotic Poisson approximation for upcrossings of narrow- and broad-band damped linear oscillator. Empirical distribution of upcrossing distances on Weibull paper: (a) $\zeta = 0.01$, (b) $\zeta = 0.4$.

the case for the narrow-band process studied in this example, see Fig. 3(a). Figure 3(b) shows $N_S^+(u)$, $S = 10^4$, the number of upcrossings of the level $u = 3$ plotted on normal probability paper. Clearly $N_S^+(u)$ is not normally distributed indicating that the time period S, corresponding to 1600 waves, is still too short for the asymptotic results to be valid. The same conclusion can be drawn from the Fig. 3(a). We conclude that approximating the maximum distribution for the narrow-band process studied in this example is a very complicated problem.

For a broad-band process the asymptotic approximation works well for the level u equal to three standard deviations from the mean. In Fig. 4(b) we show the distribution of $t_i - t_{i-1}$ on Weibull probability paper. The estimated parameters are $k = 0.97$ and $a = 624.3$ which are very close to the exponential density, i.e. $k = 1$ and $a = 565.6$. Note, that we have only 168 observations, and the expected number of u-upcrossings in $[0, 10^5]$ equals 176.8 by Rice's formula.

Finally, in Fig. 5 one can see that the Poisson asymptotic result gives a very accurate approximation of the distribution of the maximum M_S, for the broad-band process X. Note that $S = 100\,\mathrm{s}$ is very short, one has in average only $N = 16$ waves in simulated paths. The envelope approximation is not applicable at all for this case since it has much higher maxima than the process X.

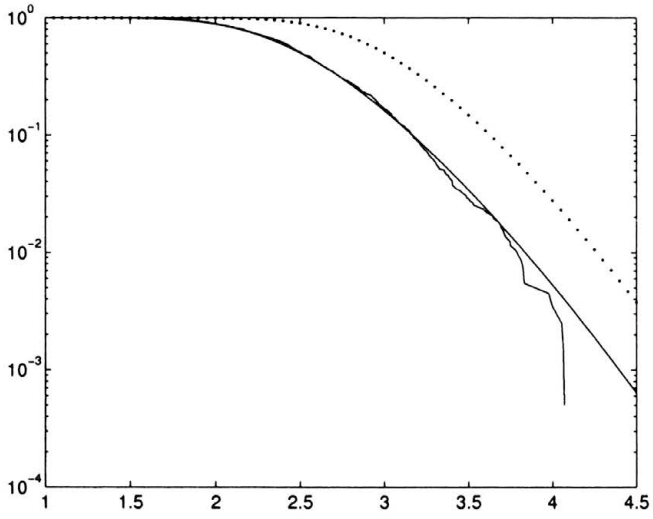

FIGURE 5. Comparison between the empirical disstribution of $M_S(X)$, $\zeta = 0.4$ and $S = 100\,\mathrm{s}$, (irregular line) and the asymptotic Poisson approximation $P(M_S(X) > u) \approx 1 - e^{-S\mu^+(u)}$ (solid line) and using the envelope $P(M_S(X) > u) \approx 1 - e^{-S\mu_E^+(u)}$ (dots).

2. Lecture II: Maximum of a χ^2-load

2.1. Rice's formula for non-Gaussian processes

Rice (1944, 1945) and Kac (1943) obtained (1.14) for a certain class of Gaussian processes and polynomials with random coefficients, respectively. The formula is often written in the following alternative way

$$\mu(u) = \int_{-\infty}^{+\infty} |\dot{x}|\, f(\dot{x}, u)\, d\dot{x}, \tag{2.1}$$

$$\mu^+(u) = \int_{0}^{\infty} \dot{x}\, f(\dot{x}, u)\, d\dot{x}, \tag{2.2}$$

where $f(\dot{x}, x)$ is the density of $\dot{X}(0), X(0)$. In the following example we will demonstrate that (1.14) follows from (2.2).

Remark 2. We know that for a stationary Gaussian process X, the process value $X(t)$ and the derivative $\dot{X}(t)$ at the same time are independent, so the joint density is just the product of the densities

$$f(\dot{x}, x) = f(\dot{x})f(x).$$

Furthermore, if $X(t) \in N(m, \lambda_0)$, then $\dot{X}(t) \in N(0, \lambda_2)$. So, we can write

$$\mu^+(u) = f(u) \int_{0}^{\infty} \dot{x} f(\dot{x})\, d\dot{x} = f(u)\mathsf{E}(\dot{X}(t)^+),$$

where $x^+ = \max(0, x)$. Now, for any Gaussian variable Y, if $Y \in N(m, \sigma^2)$, then $\mathsf{E}(Y^+) = \sigma\Psi(-m/\sigma)$, see (A.4). Using the last property we get

$$\mu^+(u) = f(u)\sqrt{\lambda_2}\Psi(0).$$

This is just the formula (1.14) since

$$\Psi(0) = \frac{1}{\sqrt{2\pi}} \quad \text{and} \quad f(u) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\lambda_0}} e^{-(u-m)^2/2\lambda_0}.$$

It has been proved that (2.1), (2.2) hold even for many non-Gaussian processes X, see e.g. Leadbetter (1966), Marcus (1977) for sufficient conditions. In engineering literature formulas (2.1), (2.2) are used to compute the expectations $E(N_S(u))$, $E(N_S^+(u))$ for non-Gaussian processes as long as the

joint density of $\dot{X}(0), X(0)$ is available. Generally the Rice's formula always holds if the expectations $E(N_S(u))$, $E(N_S^+(u))$ and the integrals in (2.1-2.2) are continuous functions of u (proof of continuity of $E(N_S(u))$, $E(N_S^+(u))$ can be technical).

The difficulty in evaluating the Rice's formula for non-Gaussian processes is that the integral $\int_0^\infty \dot{x} f(\dot{x}, u)\, d\dot{x}$, usually, can not be evaluated analytically and that for many important classes of processes X, the explicit expression for the joint density $f(\dot{x}, x)$ of $\dot{X}(0), X(0)$ is not known.

We shall illustrate these problems by considering the class of non-central χ^2 processes which find many applications in engineering. We give first a motivating example.

Example 6. Suppose we are interested in modeling the response of a structure to a wind load. Also, suppose that the structure is linear and can be approximated by means of the second order oscillator. Now the wind load $F(t) = cV(t)^2$ where $V(t)$ is the wind velocity process and c denotes a constant. We assume that $V(t)$ is a stationary Gaussian process with the mean $\mathsf{E}(V(t)) = v$ and the spectrum $S(\lambda)$. Now let $Y(t) = V(t) - v$, i.e. $Y(t)$ is a zero mean Gaussian process describing fluctuations of the wind velocity around its mean value, then the response $X(t)$ of the structure is defined by the following differential equation

$$\ddot{X}(t) + 2\zeta\lambda_p\dot{X}(t) + \lambda_p^2 X(t) = \frac{2cv}{M}Y(t) + \frac{c}{M}Y(t)^2,$$

where M is the total mass, λ_p is the natural frequency of the system, ζ is the relative damping. (Note that we have not included in the model the constant load cv^2.) Since the differential equation describing the structure is linear, the response process can be written as a sum

$$X(t) = X_l(t) + X_q(t),$$

where $X_l(t)$ is the zero mean Gaussian process defined by

$$\ddot{X}_l(t) + 2\zeta\lambda_p\dot{X}_l(t) + \lambda_p^2 X_l(t) = \frac{2cv}{M}Y(t),$$

while $X_q(t)$ is the quadratic correction for non-linearity and is defined by

$$\ddot{X}_q(t) + 2\zeta\lambda_p\dot{X}_q(t) + \lambda_p^2 X_q(t) = \frac{c}{M}Y(t)^2$$

Suppose now that the process $Y(t)$ can be well approximated by means of a sum of N independent Gaussian waves defined in Example 1, i.e.,

$$Y(t) = \sum_{j=1}^{N} \sigma_j R_j \cos(\lambda_j t + \theta_j), \tag{2.3}$$

which means that Y has a discrete spectrum with atoms at λ_j. Here σ_j^2 is the average energy of the wave with angular frequency λ_j. In engineering, one usually assumes that $Y(t)$ has continuous spectral measure with one-sided spectral density $S_Y(\lambda)$, $\lambda \geq 0$, say. Often a cut-off frequency λ_c is chosen, so that $S_Y(\lambda) = 0$, for $\lambda > \lambda_c$. This implies that the process is analytical (all spectral moments are finite). Now the process can be approximated by the process defined in (2.3) with $\sigma_j = \sqrt{S_Y(\lambda_j)\Delta\lambda}$, where $\lambda_j = j\lambda_c/N$ and $\Delta\lambda = \lambda_c/N$. Actually it is easier to write the defining equation for the nonlinear response $X_q(t)$ using complex random variables. This will be done next:

Let $\lambda_0 = 0$ and $\sigma_0 = 0$ and define $U_j = R_j \cos(\theta_j)$ and $V_j = R_j \sin(\theta_j)$. It is well known that U_j, V_j, $j = 1, \ldots .N$ are independent $N(0,1)$ variables. Defining $\sigma_{-j} = \sigma_j$, $U_{-j} = U_j$ $V_{-j} = -V_j$, the process $Y(t)$ defined in (2.3) can be written in the following alternative way

$$Y(t) = \sum_{j=-N}^{N} \frac{\sigma_j}{2}(U_j - iV_j)e^{i\lambda_j t},$$

which is the so-called spectral representation of the process Y, see formula (A.23). Since the transfer function of the structure (oscillator) is equal to

$$H(\lambda) = \frac{1}{\lambda_p^2 - \lambda^2 + 2i\zeta\lambda_p\lambda},$$

the Gaussian response is given by

$$X_l(t) = \frac{2cv}{M} \sum_{j=-N}^{N} \frac{\sigma_j}{2} H(\lambda_j)(U_j - iV_j)e^{i\lambda_j t}. \tag{2.4}$$

Next, since

$$Y(t)^2 = \sum_{j=-N}^{N} \sum_{k=-N}^{N} 2\frac{\sigma_j}{2}\frac{\sigma_k}{2}(U_j - iV_j)(U_k - iV_k)e^{i(\lambda_j+\lambda_k)t}$$

(properties of the $Y(t)^2$ process were studied already in Rice (1944)) we have that the quadratic correction term is now given by

$$X_q(t) = \frac{c}{2M} \sum_{j=-N}^{N} \sum_{k=-N}^{N} \sigma_j\sigma_k H(\lambda_j+\lambda_k)(U_j - iV_j)(U_k - iV_k)e^{i(\lambda_j+\lambda_k)t}. \tag{2.5}$$

As before, if one wishes to study the extreme responses of the structure, one needs to first compute the upcrossing intensity $\mu^+(u)$. However, we face the

problem that in general, one can not find analytical expression for the density $f(\dot{x}, x)$ of the process. In Machado (2002), Section 4.2, a detailed discussion is presented.

For the chosen parameters, the author demonstrated that the quadratic correction term $X_q(t)$ can not be neglected if one is interested in the distribution of maximum response $P(M_S(X) > u)$.

In the last example we have introduced the process (2.5), which describes the non-Gaussian properties of the response process. Through matrix diagonalization and some matrix algebra, see Machado (2002) for detailed presentation, we can rewrite the equations (2.4), (2.5) so that the response process $X(t) = X_l(t) + X_q(t)$ is a non-central χ^2 process as defined next.

Definition 1: *The process $X(t)$ will be called non-central χ^2 if*

$$X(t) = m + \sum_{j=1}^{n} (\beta_j Z_j(t) + \gamma_j Z_j(t)^2) = m_0 + \sum_{j=1}^{n} \gamma_j \left(Z_j(t) + \frac{\beta_j}{2\gamma_j} \right)^2, \quad (2.6)$$

where $\mathbf{Z}(t) = (Z_1(t), \ldots, Z_n(t))$ is a vector-valued stationary Gaussian process, such that for each t, $Z_j(t) \in N(0,1)$ and the variables $Z_j(t), Z_k(t)$ are independent. If $\dot{\mathbf{Z}}(t) = (\dot{Z}_1(t), \ldots, \dot{Z}_n(t))$, then $(\mathbf{Z}(t), \dot{\mathbf{Z}}(t)) \in N(\mathbf{0}, \Sigma)$, where

$$\sigma = \begin{bmatrix} I & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix}, \quad (2.7)$$

where I is the identity matrix (note that matrices Σ_{12}, Σ_{22} do not need to be identity matrices). Finally the constant $m_0 = m - \sum_{j=1}^{n} \frac{\beta_j^2}{4\gamma_j}$.

Many processes can be transformed into this form, for example the von Misses stress, envelope process, Stoke's waves, motion of a moored floating structure, second-order sea-surface and many other. In order to study the global maximum of the process X, one needs to compute the crossing intensity $\mu^+(u)$. As we have already mentioned, an analytical expression for the density $f(\dot{x}, x)$, except for some special cases, is not known at present and hence one can not use the Rice's formula to compute $\mu^+(u)$.

An important property of the process, defined by (2.6), is that one can compute the so-called moment generating function of $(X(0), \dot{X}(0))$, $M(s,t)$, defined by

$$M(s,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{sx+t\dot{x}} f(x, \dot{x})\, d\dot{x}\, dx = \mathsf{E}\left(e^{sX(0)+t\dot{X}(0)} \right).$$

Theorem 1: *Suppose that $X(t)$ is a non-central χ^2 process, specified in Definition 1. Then the moment generating function is given by*

$$M(s,t) = \frac{1}{\sqrt{\det(A)}} \exp\left(ms + \frac{t^2}{2}\beta^T V \beta + \frac{1}{2}\mathbf{t}^T A^{-1}\mathbf{t}\right),$$

where

$$A = I - 2s\Gamma - 2t(\Gamma\Sigma_{12}^T + \Sigma_{12}\Gamma) - 4t^2\Gamma V\Gamma,$$

$$\mathbf{t} = (sI + t\Sigma_{12} + 2t^2\Gamma V)\beta,$$

$$V = \Sigma_{22} - \Sigma_{12}^T\Sigma_{12}.$$

Here Γ is a diagonal matrix with $\gamma_1, \ldots, \gamma_n$ on the main diagonal and $\beta = (\beta_1, \ldots, \beta_n)^T$.

Knowing the moment generating function, the density of $X(0), \dot{X}(0)$ can be obtained by means of the two-dimensional Fourier inverse

$$f(x,\dot{x}) = -\frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} M(s,t) e^{-(sx+t\dot{x})}\, ds\, dt.$$

Consequently, one could compute numerically the last integral and then use the Rice's formula to compute the crossing intensity $\mu^+(u)$. This can be done, but the numerical algorithms are often slow and can be unstable. Another approach is to approximate the integral

$$\mu^+(u) = \frac{1}{(2\pi)^2} \int_0^{\infty} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} M(s,t)\dot{x} e^{-(sx+t\dot{x})}\, ds\, dt\, d\dot{x}$$

using the so-called saddle-point approximation, introduced in statistics by Daniels (1954). *For the process X the saddle point approximation of the up-crossing intensity $\mu_X^+(u)$ will be denoted by $\bar{\mu}_X^+(u)$.* Note that for the Gaussian X, $\bar{\mu}_X(u) = \mu_X^+(u)$. The algorithm is presented in Butler *et al.* (2003) and contains the following steps:

- Compute the cumulant generating function $K(s,t) = \ln(M(s,t))$ and its derivative $K_{10}(s,0) = \partial K(s,0)/\partial s$.
- For a fixed u find s such that $K_{10}(s,0) = u$.
- For s, compute at $(s,0)$ the following derivatives: K_{20}, K_{02}, K_{30}, K_{12}, K_{22}, K_{04}, K_{20}. The final approximation is an explicit function of the derivatives.

The program is included in the toolbox WAFO, see Brodtkorb *et al.* (2000), that can be downloaded, free of charge, from:
`http://www.maths.lth.se/matstat/wafo`

2.1.1. Crossings of the envelope process. An important process that can be written in the form (2.6) is the square of the envelope of a Gaussian process. More precisely, let $X(t)$ be a zero mean stationary Gaussian process having finite spectral moments $\lambda_0, \lambda_1, \lambda_2$. If $\hat{X}(t)$ denotes the Hilbert transform of X, see Cramér and Leadbetter (1967), then the envelope process is defined by

$$E(t) = \sqrt{X(t)^2 + \hat{X}(t)^2}.$$

In engineering literature $E(t)$ is often called Cramér and Leadbetter envelope.

Remark 3. In this remark we shall write $E(t)^2$ in the form (2.6). First, one can show that $Z_1(t) = X(t)/\sqrt{\lambda_0}$ and $Z_2(t) = \hat{X}(t)/\sqrt{\lambda_0}$ are independent $N(0,1)$. Then the covariance matrix Σ, defined in Definition 1, consists of the following blocks

$$\Sigma_{12} = \begin{bmatrix} 0 & \lambda_1/\lambda_0 \\ -\lambda_1/\lambda_0 & 0 \end{bmatrix}, \qquad \Sigma_{22} = \begin{bmatrix} \lambda_2/\lambda_0 & 0 \\ 0 & \lambda_2/\lambda_0 \end{bmatrix}.$$

Consequently $E(t)^2 = \lambda_0 Z_1(t)^2 + \lambda_0 Z_2(t)^2$ is in the form (2.6) with $n = 2$, $\beta_1 = \beta_2 = 0$, $\gamma_1 = \gamma_2 = \lambda_0$ and $m = 0$. Since both β coefficients are zero therefore $E(t)^2$ is actually a χ^2 process. By Theorem 1 the moment generating function $M(s,t)$ is given by

$$M(s,t) = \det(A)^{-1/2} = \frac{1}{1 - 2\lambda_0 s - 4t^2\lambda_0\lambda_2(1-\alpha_1^2)}, \tag{2.8}$$

where $\alpha_1 = \lambda_1/\sqrt{\lambda_0\lambda_2}$ is called the groupness parameter.

The process $E(t)$ has found several applications in engineering. The one which is of special interest, is in the study of wave groups in the sea. In ocean engineering for example, the sea surface is usually modeled by a stationary Gaussian field with a special form of the spectrum which follows from the dispersion of water waves.

From a formal standpoint, wave groups are difficult objects to study. They can be observed only under special circumstances when the sea surface is narrow-band and long-crested. For a confused sea there is no noticeable organized movement of waves and thus each large extreme wave can be taken as a crest of a wave group.

In his pioneering work, Longuet–Higgins (1957) has introduced the decomposition of traveling random waves into the envelope (low frequency varying amplitude) and the carrier (high frequency oscillations). The simplest (one-dimensional) record in which wave groups can be observed, consists of

two sinusoidal waves traveling together, given by

$$W(x,t) = \cos(\lambda_1 t + \frac{\lambda_1^2}{g}x + \phi_1) + \cos(\lambda_2 t + \frac{\lambda_2^2}{g}x + \phi_2)$$
$$= 2\cos\left(\frac{\lambda_2 - \lambda_1}{2}t + \frac{\lambda_2^2 - \lambda_1^2}{2g}x + \frac{\phi_2 - \phi_1}{2}\right)$$
$$\cdot \sin\left(\frac{\lambda_2 + \lambda_1}{2}t + \frac{\lambda_2^2 + \lambda_1^2}{2g}x + \frac{\phi_2 + \phi_1}{2}\right).$$

The sinus in the last formula represents individual waves while the cosine which modulates their amplitudes describes the motion of a wave package. The group velocity is given by $v_G = g/(\lambda_2 + \lambda_1)$ and the wave velocity by $v_W = g(\lambda_2 + \lambda_1)/(\lambda_2^2 + \lambda_1^2)$. In the "narrow band" case when $\lambda_1 \approx \lambda_2$ we have $v_W \approx g/\lambda_1 \approx 2v_G$, thus individual waves travel approximately twice as fast as wave groups.

More generally, the envelope is a positive process that is always higher than the sea elevation process. For narrow-band processes, the envelope is passing close to local maximum of the process and hence can be used to describe the evolution of wave groups (see Fig. 6). Often, one uses the so-called groupness parameter $\alpha_1 = \frac{\lambda_1}{\sqrt{\lambda_0 \lambda_2}}$ which is a measure of how broad-band the power spectrum is. For α_1 close to one, the process is considered

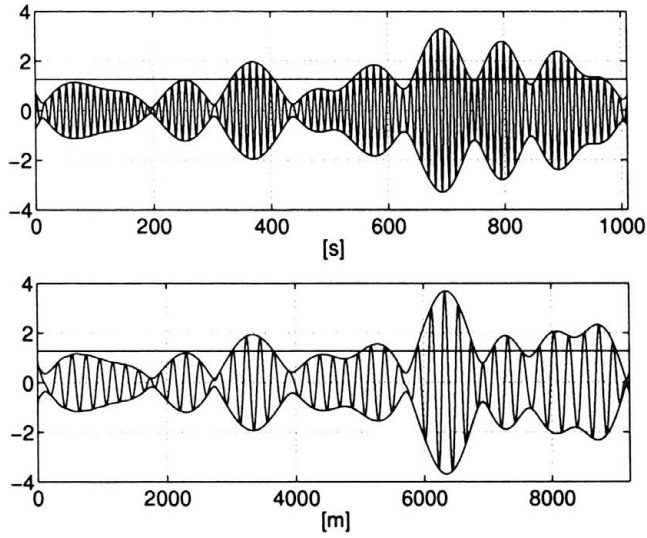

FIGURE 6. Envelope and wave groups in the narrow-band case. Records: in time (top), in space (bottom).

narrow-band and one can observe grouping of waves. As we have mentioned before, the other two important parameters describing sea spectrum are: the mean wave period T_z that equals $2\pi/\hat{\lambda}$, and the significant wave height H_s, defined as $4\sqrt{\lambda_0}$. The three parameters $\alpha_1, \hat{\lambda}$ and H_s define the upcrossings intensity $\mu_E^+(u)$ of the envelope process, namely

$$\mu_E^+(u) = \frac{\hat{\lambda}}{\sqrt{2\pi}}\sqrt{1-\alpha_1^2}\frac{u}{\sqrt{\lambda_0}}e^{-\frac{u^2}{2\lambda_0}}. \tag{2.9}$$

Note that $\bar{\mu}_E^+(u)$, the saddle point approximation of the crossing intensity $\mu_E^+(u)$, is exact.

Often, a wave group is defined as a part of a space or time record that lies between two consecutive upcrossings of the level u_0, where the level u_0 is the one at which the envelope attains the highest intensity of crossings. For the intensity of crossings $\mu_E^+(u)$, see (2.9), the maximal intensity is attained at the level $u_0 = \sqrt{\lambda_0}$. In Fig. 6, this level is marked by a horizontal line. Changing the level would lead in average to smaller number of wave groups. Finally, the average length of the wave group can be computed by combining the following two results: $P(E(0) > u_0)$ is equal to the fraction of the time (distance) the envelope spends above the level u and $\mu_E^+(u_0)$ is the intensity of wave groups. Hence the average length of a wave group is equal to

$$\frac{P(E(0) > u_0)}{\mu_E^+(u_0)} = \frac{\sqrt{2\pi}}{\hat{\lambda}\sqrt{1-\alpha_1^2}}.$$

Similarly one can compute the average length the process X stays above the level u_0 by means of $\frac{P(X(0)>u_0)}{\mu_X^+(u_0)}$. Consequently, the fraction

$$\frac{P(E(0) > u_0)/\mu_E^+(u_0)}{P(X(0) > u_0)/\mu_X^+(u_0)} = \frac{e^1\sqrt{2\pi}\Phi(-1)}{\sqrt{1-\alpha_1^2}} = \frac{1.081}{\sqrt{1-\alpha_1^2}},$$

could be taken as an approximation of the expected number of waves in a wave-group.

For the responses of narrow- and broad-band linear oscillator with $\zeta = 0.01$, $\zeta = 0.4$, respectively, discussed in Section 1.3.1, the parameter α_1 is equal 0.995, 0.85, and hence the average number of waves per group is approximately 11, for the narrow-band case, and 2 for the broad-band one. This example motivates why α_1 is called the groupness parameter.

We finish this subsection by proving formula (2.9).

Example 7. *Proof of (2.9).* Let $R(t) = E^2(t)/\lambda_0 = Z_1(t)^2 + Z_2(t)^2$, where the processes Z_1, Z_2 are defined in Remark 3. Since $\mu_E^+(u) =$

$\mu_R^+(u^2/\lambda_0)$, we only need to show that

$$\mu_R^+(r) = \frac{\hat{\lambda}}{\sqrt{2\pi}}\sqrt{1-\alpha_1^2}\sqrt{r}e^{-\frac{r}{2}}. \tag{2.10}$$

Denote by $f(\dot{r}, r)$ the joint density of $\dot{R}(0), R(0)$. Then by Rice's formula

$$\mu_R^+(r) = \int_0^\infty \dot{r} f(\dot{r}, r)\, d\dot{r}.$$

Consequently we need to find the joint density $f(\dot{r}, r)$. Since $Z_1(0), Z_2(0)$ are independent $N(0,1)$ we know that $2R(0)$ is exponentially distributed and hence the marginal density of $R(0)$ is given by

$$f(r) = \frac{1}{2}e^{-\frac{r}{2}}.$$

Next we write $f(\dot{r}, r) = f_{\dot{R}|R}(\dot{r}|r)f_R(r)$, $r > 0$, where $f(\dot{r}|r)$ is the conditional density of $\dot{R}(0)$ given that $R(0) = r$. Consequently we get

$$\mu_R^+(r) = f(r)\int_0^\infty \dot{r} f(\dot{r}|r)\, d\dot{r} = f(r)\int_0^\infty y\tilde{f}(y)\, dy,$$

since r is fixed, we suppressed it in the notation. It can be shown, by a simple change of variable that the conditional density $\tilde{f}(y) \in N(0, \sigma^2(r))$, where $\sigma^2(r) = 4r\hat{\lambda}^2(1-\alpha_1^2)$. Consequently,

$$\int_0^\infty y\tilde{f}(y)\, dy = \sigma(r)\Psi(0) = \sigma(r)\frac{1}{\sqrt{2\pi}}$$

by means of (A.4). Since $\mu_R^+(r) = f(r)\sigma(r)/\sqrt{2\pi}$ the formula (2.10) has been derived. What remains is to show that the conditional density $\tilde{f}(y) \in N(0, \sigma^2(r))$.

Let introduce the additional variable θ defined by the relation $\sin(\theta) = Z_1(0)/\sqrt{R(0)}$. Obviously, $Z_1(0) = \sqrt{R(0)}\sin(\theta)$ and $Z_2(0) = \sqrt{R(0)}\cos(\theta)$.

Let $\mathbf{z} = (z_1, z_2)^T$ and $r = z_1^2 + z_2^2$. Now using (A.20), (A.21) we have that the conditional density of $\dot{Z}_1(0)$, $\dot{Z}_1(0)$ given $Z_1(0) = z_1$, $Z_2(0) = z_2$ is a two-dimensional Gaussian with mean and variance given by

$$\mathbf{m}(\mathbf{z}) = \Sigma_{21}\mathbf{z} = \frac{\lambda_1}{\lambda_0}\begin{bmatrix} -z_2 \\ z_1 \end{bmatrix},$$

and the conditional covariance matrix has the following form

$$\Sigma_{\dot{\mathbf{Z}}|\mathbf{Z}} = \Sigma_{22} - \Sigma_{21}\Sigma_{12} = \frac{1}{\lambda_0}\begin{bmatrix} \lambda_2 - \lambda_1^2/\lambda_0 & 0 \\ 0 & \lambda_2 - \lambda_1^2/\lambda_0 \end{bmatrix}.$$

Consequently the conditional density of $\dot{R} = 2(Z_1\dot{Z}_1 + Z_2\dot{Z}_2)$ with $Z_1 = z_1$, $Z_2 = z_2$ is Gaussian with mean

$$2\frac{\lambda_1}{\lambda_0}(z_1(-z_2) + z_2 z_1) = 0,$$

and variance

$$4(x^2 + y^2)(\lambda_2 - \frac{\lambda_1^2}{\lambda_0})/\lambda_0 = \sigma^2(r).$$

Since the conditional density does not depend on the angle θ it is easy to demonstrate that the conditional density of $\dot{R}(0)$ given $R = r$ is zero mean normal with variance $\sigma^2(r)$.

2.1.2. Narrow-band Stoke's waves. Let us consider the sea containing only one Gaussian cosine wave, i.e. $X(t) = \sigma R\cos(\lambda t + \phi)$, where R is a standard Rayleigh distributed variable and ϕ is a uniformly distributed random phase, which is independent of R. As before σ^2 is the energy of the wave. The Hilbert transform of the cosine wave is $\hat{X}(t) = \sigma R\sin(\lambda t + \phi)$. The Stoke's wave can then be written as follows:

$$\begin{aligned} W(t) &= \sigma R\cos(\lambda t + \phi) + \frac{\lambda^2}{2g}\sigma^2 R^2\cos(2\lambda t + 2\phi) \\ &= X(t) + \frac{\lambda^2}{2g}(X(t)^2 - \hat{X}(t)^2). \end{aligned} \tag{2.11}$$

We shall now generalize Eq. (2.11) and let $X(t) = \sqrt{\lambda_0}Z_1(t)$ be a zero-mean, stationary Gaussian process with spectral moments $\lambda_0, \lambda_1, \lambda_2 < \infty$. As in the previous subsection let $\hat{X}(t) = \sqrt{\lambda_0}Z_2(t)$ be its Hilbert transform. Define Gaussian Stoke's wave as follows

$$W(t) = \sqrt{\lambda_0}Z_1(t) + \lambda_0\frac{\lambda_p^2}{2g}(Z_1(t)^2 - Z_2(t)^2), \tag{2.12}$$

where λ_p is the peak angular frequency of the spectrum $S(\lambda)$. Note that λ_p is usually close to the mean frequency $\hat{\lambda}$. Clearly the process $W(t)$ is a non-central χ^2 process written in a standard form (2.6).

In oceanography the process Eq. (2.12) is sometimes used to model sea surface elevation when the power spectrum is concentrated around the peak frequency λ_p. Such sea is called *narrow-band*.

Although the process $W(t)$ is given by only slightly more complex formula (2.12) than the squared envelope process, however the explicit analytical formula for the upcrossing intensity $\mu_W^+(u)$ is not known. In Butler *et al.* (2003), the intensity $\mu_W^+(u)$ is given as a one-dimensional integral that can be computed numerically with very high accuracy. So, computed values are used to check the accuracy of the saddle-point approximation $\bar{\mu}^+(u)$ of the crossing intensity $\mu_W^+(u)$. The relative error is presented in Fig. 7. The approximation $\bar{\mu}^+(u)$ is performing very well giving a relative error below 10%. As can be see in Fig. 7, taking only the linear part will give fewer crossings, leading to high relative errors.

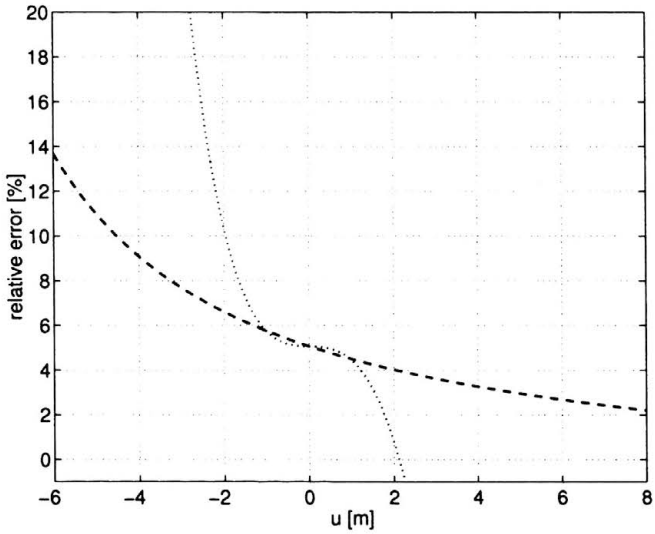

FIGURE 7. Relative errors of the saddle-point approximation $\bar{\mu}^+(u)$ (dashed line) and the crossing intensity of the linear (Gaussian) part of the sea $\sqrt{\lambda_0} Z_1(t)$ (dotted line).

2.1.3. Crossings of general χ2-processes. We turn now to a more general χ^2-process:

$$R(t) = \sum_{i=1}^{n} X_i(t)^2, \tag{2.13}$$

where $\mathbf{X}(t) = (X_1(t), X_2(t), ..., X_n(t))$ is a stationary, vector-valued, Gaussian process that can also be assumed zero mean, though the variables $X_i(t)$, $X_j(t)$ can be correlated. Note that in the case of an envelope process $X_1(t)$, $X_2(t)$ were independent. Obviously, $(\mathbf{X}(0), \dot{\mathbf{X}}(0))$ is a Gaussian vector, i.e.

$(\mathbf{X}(0), \dot{\mathbf{X}}(0)) \in N(\mathbf{0}, \Sigma)$ where the covariance matrix has the following block structure

$$\Sigma = \begin{bmatrix} \Sigma_{XX} & \Sigma_{X\dot{X}} \\ \Sigma_{\dot{X}X} & \Sigma_{\dot{X}\dot{X}} \end{bmatrix}.$$

We shall now write the process $R(t)$ in the form (2.6). First, we shall introduce an important variable transformation $\mathbf{Y}^T = A\mathbf{X}^T$, where A is an n-dimensional matrix.

Suppose $\mathbf{X} \in \mathbf{N}(\mathbf{m}, \mathbf{\Sigma})$ *is* n*-dimensional Gaussian vector such that* $\det(\Sigma) > 0$. *Let* A *be the* (n, n) *matrix whose rows are the eigenvectors of* Σ. *The eigenvectors are normalized to have unit length, i.e.* $AA^T = A^TA = I$. *Denote the corresponding eigenvalues by* γ_j *and let* Γ *be a diagonal matrix with the* γ_j *as the elements on the main diagonal, then*

$$\Sigma = A^T\Gamma A, \qquad \Gamma = A\Sigma A^T.$$

Consequently

$$\mathbf{Y}^T = A\mathbf{X}^T \in N(A\mathbf{m}, \mathbf{\Gamma}), \tag{2.14}$$

i.e. $\mathbf{Y}$ *is a vector of independent Gaussian variables with variances equal to eigenvalues, i.e.* $\mathsf{V}(Y_j(t)) = \gamma_j$.

Suppose that $\det(\Sigma_{XX}) > 0$ and let A be the above defined transformation, then

$$R(t) = \mathbf{X}(t)\mathbf{X}(t)^T = \mathbf{Y}(t)AA^T\mathbf{Y}(t)^T = \mathbf{Y}(t)\mathbf{Y}(t)^T,$$

where $Y_j(t)$ are independent $N(0, \gamma_j)$ variables. Consequently, with $Z_j(t) = Y_j(t)/\sqrt{\gamma_j}$, the process $R(t)$ defined by (2.13), can be rewritten in the standardized form of Definition 1, as follows

$$R(t) = \gamma_1 Z_1(t)^2 + \ldots + \gamma_n Z_n(t)^2,$$

where $Z_i(t)$ and $Z_j(t)$ are independent $N(0, 1)$. Since $\mathbf{Z}(t)^T = \Gamma^{-1/2}A\mathbf{X}(t)^T$, the covariance matrix Σ, defined in Definition 1, consists of the following blocks

$$\Sigma_{12} = \Gamma^{-1/2}A^T\Sigma_{X\dot{X}}A\Gamma^{-1/2}, \quad \Sigma_{22} = \Gamma^{-1/2}\left(A^T\Sigma_{\dot{X}\dot{X}}A\right)\Gamma^{-1/2}.$$

Here $\Gamma^{-1/2}, \Gamma^{-1}$ are diagonal matrices with elements $1/\sqrt{\gamma_j}$, $1/\gamma_j$ on the main diagonals, respectively.

Example 8. *Multiaxial fatigue continuation of Example 3.* We turn now to computation of the approximation of the expectation of the second term in the Crossland's criterion, namely

$$M_S(|s_c|) = \max_{0 \le t \le S} |s_c(t)|.$$

Since $\mathsf{E}(M_S(|s_c|)) = \int_0^\infty P(\max_{0\le t\le S}|s_c(t)| > u)\,du$ we will next approximate the probability $P(M_S(|s_c|) > u)$. First we shall demonstrate that s_c^2 is a χ^2-process. Note that after the variable change:

$$X_1(t) = \frac{s_x(t)}{\sqrt{2}}, \qquad X_2(t) = \frac{s_y(t)}{\sqrt{2}},$$

$$X_3(t) = \frac{s_x(t) - s_y(t)}{\sqrt{2}}, \qquad X_4(t) = \sqrt{3}s_{xy}(t),$$

we get $s_c(t)^2 = X_1(t)^2 + \ldots + X_4(t)^2$. Although the covariance matrix Σ of $\mathbf{X}(0) = (X_1(0), X_2(0), X_3(0), X_4(0))$ is degenerated, we still can find the transformation A where one of the eigenvalues is zero. Let us change rows in the matrix A so that $\gamma_1 \ge \gamma_2 \ge \gamma_3 \ge \gamma_4$ and define $\mathbf{Y}(t)^T = A\mathbf{X}(t)^T$. If some of the eigenvalues are very small relatively to γ_1 we can replace them by zero. Now for the j for which $\gamma_j > 0$ we define $Z_j(t) = Y_j(t)/\sqrt{\gamma_j}$. Suppose that there are $k \le 3$ positive eigenvaleus γ_j, then

$$s_c^2(t) = \gamma_1 Z_1(t)^2 + \ldots + \gamma_k Z_k(t)^2.$$

We see that $s_c^2(t)$ is represented in the standard form of Definition 1 and we can apply the saddle-point method to approximate the crossing intensity $\mu^+_{s_c^2}(u)$. As before denote the saddle-point approximation be $\bar{\mu}^+(u)$. Since $|s_c(t)| = \sqrt{s_c(t)^2}$ hence

$$\mu^+_{|s_c|}(u) = \mu^+_{s_c^2}(u^2) \approx \bar{\mu}^+(u^2).$$

Consequently, the expected maximum value of $|s_c(t)|$ on the interval $[0, S]$, is approximated by

$$\mathsf{E}(M_S(|s_c|)) \approx \int_0^\infty 1 - e^{-S\bar{\mu}^+(u^2)}\,du.$$

The function $\bar{\mu}^+(u^2)$ can be computed using a program in WAFO. We observe that an alternative method to compute $\mu^+_{s_c^2}(u^2)$, which uses Hasofer–Linds safety index, was proposed in Breitung (1988).

The simplest approximation of the crossing intensity of $s_c^2(t)$ is obtained by letting the eigenvalues $\gamma_2 = \gamma_3 = \gamma_4 = 0$. In that case $|s_c(t)|$ is an absolute value of the Gaussian process with mean zero and $\lambda_0 = \gamma_1$. Now for zero-mean Gaussian processes, under some assumptions, one can demonstrate that up- and down-crossings of level u, $-u$, respectively, converge to

independent Poisson processes. Consequently, asymptotically $M_S(|s_c|)$ has the same distribution as $M_{2S}(s_c)$.

Let V be the eigenvector corresponding to γ_1 and let Σ_{22} be the covariance matrix of $\dot{\mathbf{X}}(0)$, then the second order spectral moment of s_c, $\lambda_2 = V^T\Sigma_{22}V$ and hence the expected number of waves in $[0, 2S]$ is given by

$$N_c = \frac{S}{\pi}\sqrt{\frac{V^T\Sigma_{22}V}{\gamma_1}},$$

and, similarly to (1.19),

$$\mathsf{E}(\max_{0\le t\le S}|s_c(t)|) = \sqrt{\gamma_1}\left(\sqrt{2\ln N_c} + \frac{0.5772}{\sqrt{2\ln N_c}}\right). \tag{2.15}$$

Combining (1.19), (2.15) we derive an approximation of the expected value of the Crossland's criterion at a fixed position $\mathbf{q}$:

$$\frac{\mathsf{E}(M_S(|s_c|))/\sqrt{3} + \alpha_c\mathsf{E}(M_S(p))}{\beta_c}.$$

The approximation depends only on the parameters γ_1, λ_0^p and expected number of waves N_p, N_c. The parameters are easily derived from the spectrum of the stress tensor. For the particular structure and loading presented in Fig. 1, the resulting expected value is presented in Fig. 8. The regions where the expectation exceeds one, one predicts that cracks will start. In the left plot we have the estimate of the expectation based on a long simulation and in the

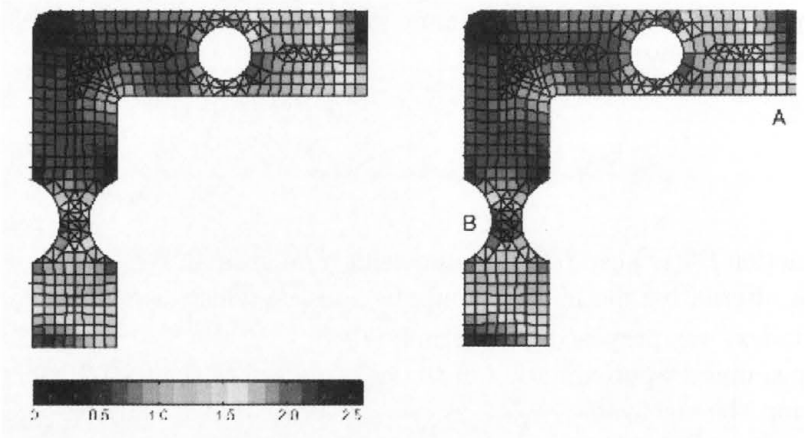

FIGURE 8. The estimated from simulations (left) and approximated (right) expected value of the Crosslands criterion.

right the approximated one in the way discussed above. Since the expected value is exceeding one at several locations, the conclusion is that the time S should be shortened. Note that the expectation is basically a linear function of $\sqrt{\ln S}$.

3. Lecture III: Crossings of intervals

In the previous two lectures we have presented means to compute the crossing (upcrossing) intensities for Gaussian and χ^2 processes. The upcrossing intensity was then used to find the return period for a load to exceed the design load u and to bound the distribution for the highest value (global maximum) of the load during the service period S, see formulas (1.5) and (1.7). The crossing intensity is an important characteristic describing the variability of a process. One additional property, for a stationary $X(t)$, is that

$$\int_{-\infty}^{+\infty} \mu(u)\, du = \mathsf{E}(|\dot{X}(0)|)$$

is equal to the average total variation of the process on interval $[0,1]$, $\mathsf{E}(\int_0^1 |\dot{X}(t)|\, dt)$. If X is Gaussian then $\mathsf{E}(|\dot{X}(0)|) = \sqrt{\lambda_2}\sqrt{\frac{2}{\pi}}$. The total variation is a simple measure of severity of the load for the fatigue damage. We recall that fatigue is a process of initiation and growth of cracks in a material, usually metal, due to variable stresses. However, often one needs a more complete description of the loads horizontal variability and such parameter is the following function, called *oscillation-intensity*, defined as follows.

Definition 2: *Suppose that $X(s)$, $s \in R$, is a stationary and ergodic random process with continuous sample paths. The oscillation intensity $\mu^{osc}(u,v)$ of the interval $[u,v]$ is defined by*

$$\mu^{osc}(u,v) = \lim_{t\to\infty} \frac{1}{t} \textit{ (number of upcrossings of } [u,v] \textit{ by } X(s),\ s \in [0,t]),$$

Note that the number of upcrossings of $[u,v]$ by a continuous function $x(t)$, $0 \le s \le t$, is equal to the largest index n such that there are times $0 \le s_1 < t_1 < s_2 < \ldots < s_n < t_n$, satisfying $x(s_i) < u \le v < x(t_i)$ and $s_n \le t$. The number of crossings of an interval is well defined even for very irregular functions, which may have infinite total variation and hence infinitely many crossings of u in any finite interval. The oscillation intensity can be defined using the concept of downcrossings of $[u,v]$, viz. (see Fig. 9):

$$\mu^{osc}(u,v) = \lim_{t\to\infty} \frac{1}{t} \text{ (number of downcrossings of } [u,v] \text{ by } X(s),\ s \in [0,t]).$$

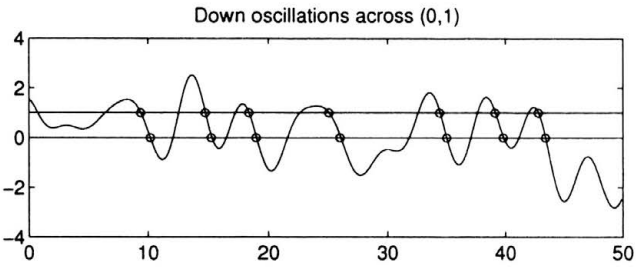

FIGURE 9. Oscillations between levels $v = 1$ and $u = 0$.

Obviously, $\mu^{osc}(u, v)$ is equal to the average rate of oscillations between sets $(-\infty, u)$ and $(v, +\infty)$ by $X(t)$, which motivates the name. The crossing intensity is difficult to compute. The explicit formulas which exist are for processes with Markov structure, e.g. Ornstein-Uhlenbeck process having $\lambda_2 = \infty$, see Rychlik (1996).

There are, however, the following explicit bounds:

$$\mu^{osc}(u, u) = \mu^{+}(u), \quad \mu^{osc}(u, v) \leq \min_{u \leq z \leq v} \mu^{+}(z), \tag{3.1}$$

since one has to cross, at least once, all levels in the interval $[u, v]$ in order to cross the interval.

3.1. Wave crest distribution

Design of offshore structures and safety of offshore operations both depend on accurate prediction of frequencies of occurrences of high waves (for given sea conditions). In this subsection we shall demonstrate that the crossing intensity is the most important "simple" characteristic of sea conditions relevant to the wave height prediction.

Let $X(t)$ be the height of the sea level at a fixed point as a function of time. In oceanographic applications $X(t)$, is often viewed as a sequence of "apparent waves". There is no general agreement about the formal definition of a wave. Often one uses the so-called upcrossing waves, where the apparent individual wave is the part of the record between two consecutive upcrossings of the so-called still water level m, say, see Fig. 10. (The still water level is generally unknown. It is often estimated by the mean of the recorded sea elevation X or is chosen to be equal to the level most frequently crossed by X.)

The intensity of waves is by definition equal to the intensity of upcrossings of the still water level m by the process X. Assume that the level m is known.

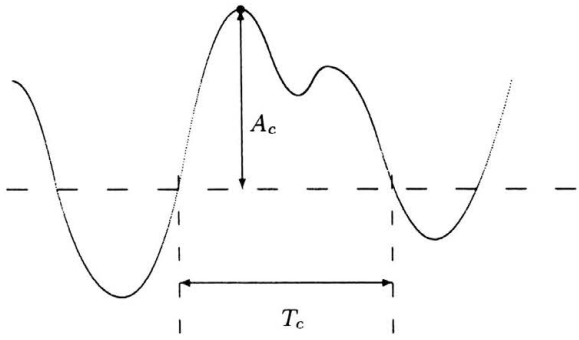

FIGURE 10. Some characteristic wave parameters: A_c (crest amplitude) and T_c (crest period), $m = 0$.

By Rice's formula the intensity of waves is equal to $\mu^+(m)$, given by (2.2), while the intensity of waves with crest higher than v is more difficult to compute. An important relation linking the crest height distribution and the oscillation intensity is that

$$\text{(intensity of waves with crest higher than } v) = \mu^{osc}(m, v). \tag{3.2}$$

We turn now to an important upper bound for the $P^m(A_c > v)$ which is defined as the proportion of waves with crest higher than v that is observed in $X(s)$, $0 \leq s \leq t$ as t tends to infinity. For ergodic sea $P^m(A_c > v) = \mu^{osc}(m, v)/\mu^+(m)$. Since $\mu^{osc}(m, v) \leq \mu^+(v)$ and hence

$$P^m(A_c > v) = \frac{\mu^{osc}(m, v)}{\mu^+(m)} \leq \frac{\mu^+(v)}{\mu^+(m)} \quad v \geq m. \tag{3.3}$$

Consequently, if we assume that the sea elevation is well modeled by a Gaussian process then, using Rice's formula, we obtain

$$P(A_c - m > h) \leq e^{-8\left(\frac{h}{H_s}\right)^2}. \tag{3.4}$$

In oceanography one often approximates the distribution of $A_c - m$ by the distribution of $\frac{H_s}{4}R$, where R is a standard Rayleigh distributed variable. That actually it is always a conservative bound, is less known.

Example 9. Assume that at a location of a buoy the sea has $T_z = 10$ s, and $H_s = 10$ m. Then the probability that a crest height is above 10 meters is less than e^{-8}. Practically, if one assumes that the sea state rest for 1 hour then it will be in average 360 waves passing the buoy and of them less than $100e^{-8} \approx 3.3\%$ (12 waves) will have crests higher than 10 meters.

The Rayleigh approximation for the crest height distribution in Gaussian sea is well established in oceanography. The deep water sea with moderate waves (not too steep) is often well described using Gaussian fields based on the linear wave theory. However, for severe sea states considerable asymmetry is observed; troughs are shallower while crests are higher than predicted by the Gaussian model. Since the crests can be up to 20% higher, these effects can not be neglected. In the literature one is often modeling the observed wave asymmetry by adding a random quadratic correction to the Gaussian model term, i.e $X(t) = X_l(t) + X_q(t)$ similarly to the example of response of a wind loaded structure. The proposed model is a noncentral χ^2 process, see Machado and Rychlik (2003). For this model, the upcrossing intensity $\mu^+(u)$ can be approximated by $\bar{\mu}^+(u)$, computed using the saddle-point method.

Example 10. Consider a deep water location having JONSWAP spectrum, see Hasselmann *et al.* (1973), with $H_s = 7\,\mathrm{m}$, peak period $T_p = 11$ [sec] and peak-shape parameter $\gamma = 2.385$. In Fig. 11 the accuracy of the approximation of the conditional distribution of the crest height given that crests are higher than one meter is presented, i.e.,

$$P^m(A_c > h | A_c > 1) = \frac{P^m(A_c > h)}{P^m(A_c > 1)} \approx \frac{\bar{\mu}^+(h)}{\bar{\mu}^+(1)}.$$

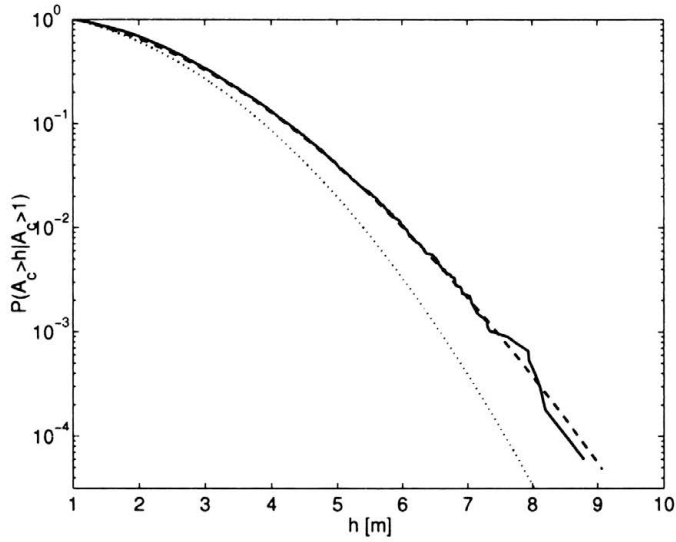

FIGURE 11. Conditional distribution of crest amplitudes A_c, i.e. $P(A_c > h | A_c > 1)$, h m. Irregular line: empirical distribution estimated from simulated 24-hour-sea elevation, dotted line: Gaussian approximation ($X(t) = X_l(t)$), i.e. the distribution of A_c given in Eq. (3.4) and dashed line: the approximation based on saddle-point approximation $\bar{\mu}^+(u)$.

(The reason for considering only the waves with crest above one meter is to exclude small waves that can not be well approximated by means of bound (3.3). Such small waves are less important in safety analysis of offshore structures.) In the figure we can see excellent agreement between the approximation and the empirical distribution obtained from simulated sea surface elevation.

We shall quantify the difference between the linear- and quadratic-model by giving the size of the so-called *10 days crest height*, i.e if the sea had the same JONSWAP spectrum for 10 days, then (on average) one wave in 10 days would have crest higher than the *10 days crest*. The 10 days crest height is predicted to be 8.4 m and 9.8 m, if we model the surface by means of a Gaussian linear sea X_l and the second-order sea, respectively.

3.2. Fatigue of metals

The first systematic quantitative investigation of fatigue damage was performed by August Wöhler (1870), and resulted in the widely known Wöhler curve, which shows the relationship between the amplitude $S = v - u$ and mean $m = (u + v)/2$ of the applied stress function

$$x(\tau) = \frac{u+v}{2} + \frac{v-u}{2}\sin(\lambda\tau), \quad \tau \geq 0, \tag{3.5}$$

and the number of periods to fatigue failure. Wöhler discovered that the number of periods to failure depend mostly on v, u and that it is almost independent of the frequency λ. In practical tests, it is hard to obtain stresses that are exactly sinusoidal, however the particular shape of the function x is of secondary importance as long as the local maxima are approximately v high, while local minima have height u. This property is called *rate independence*.

Although the tested specimens are almost identical and the loading function x is the same for all specimens, one observes a large variability of times to failure of a specimen due to variability of material properties. The number of cycles to failure, $sn(u, v)$ is used as a measure of material strength against fatigue. Often a simple regression model is fitted to the observed data and the strength of material is modeled as

$$sn(u, v) = K \cdot f(u, v), \qquad K = \exp(\sigma X), \tag{3.6}$$

where X is a standard normal variable, while $f(u, v)$ is a deterministic (median) strength. In fatigue codes one often uses

$$f(u, v) = k(v - u)^{-\alpha}, \tag{3.7}$$

where $\alpha \geq 1$, k and σ are material dependent constants. In the experiments one is observing that σ often depends on (u, v); however, this dependence is usually neglected. The parameters are estimated by fitting regression line to the logarithmed data.

Obviously real loads are seldom so simple as a constant amplitude load. Consequently one needs to find methods to characterize variability of a load relevant for fatigue life prediction. The approach presented here is based on a concept of linear damage accumulation model proposed by Palmgren (1924) and Miner (1945). This approach is commonly used in engineering practice.

Traditionally there are two main approaches to define $D(x)$, say, the damage caused to material by a variable load x:

- Define a suitable damage accumulation rule for the constant amplitude load (3.5) and then extend it to the case of more complicated loads by means of *cycle counting schemes*.
- Specify a number of properties that the fatigue damage accumulation process should satisfy. This will define a suitable class of functionals $D(x)$ which are then calibrated using Wöhler curve.

We limit ourselves here to the first approach. Assume that x has finite intensity of local extremes. Let us begin with the damage caused by a simple load function $x(\tau)$, $\tau \in [0, t]$, defined by (3.5). Denote by N_t the number of local maxima of x ($N_t \approx t\lambda/2\pi$). Since damage $D(\omega)$ is rate independent then it is a function of u, v and N_t alone. Using (3.6) a (dimensionless) damage is defined by

$$D_t(x) = N_t / f(u, v).$$

By (3.6), we derive the following fatigue criterion:

$$\text{"the load } x \text{ is safe for fatigue" if } D_t < K, \tag{3.8}$$

where K is a random threshold with median one. Obviously, the condition $D_t < K$ is equivalent to $N_t < sn(u, v)$.

Now we shall write the damage $D_t(x)$ in a way that is easy to extrapolate to more complicated loads than (3.5). Let t_i be times of local maxima in x. Suppose that the damage is constant in the intervals (t_{i-1}, t_i) and increases at t_i by

$$D_i = \frac{1}{f(u, v)}, \text{ then } D_t(x) = \sum_{t_i \leq t} D_i.$$

We turn now to a general load x. Even in this more complicated situation accumulated damage $D(x)$ is a function of the sequence of local maxima and minima of x. As before let t_i be the time of ith local maximum with value $v_i = x(t_i)$ and the damage increment in the interval $(t_{i-1}, t_i]$

equal to D_i. For a constant amplitude load x, the damage increases by $D_i = 1/f(u_i, v_i)(= 1/f(u, v))$, where $u_i = \min_{t_{i-1}<s<t_i} x(s)$ is the value of the preceeding minimum. Since u_i, v_i are well defined for general load having finite number of local maxima one could use $D_i = 1/f(u_i, v_i)$ to model damage increase between the two consecutive local maxima. Such fatigue accumulation rule was used before, but since it often gives too small damage increments it has been abandoned. It was concluded that the damage increments depend on the sequence of the local extremes in a more complicated way, and not only on the last two extreme values (u_i, v_i).

At present, the most commonly used definition of the damage increment D_i is the so-called *rainflow* method proposed by Endo, the first paper in English is Matsuishi and Endo (1968). It is a complicated algorithm using hysteresis properties of metals to define D_i, see Brokate and Sprekels (1996, p. 76) for details. While original algorithm for the computation of D_i is complicated the damage functional $D_t(x) = \sum_{t_i \leq t} D_i$ is much simpler to compute, see the following theorem, proved in Rychlik (1987).

Theorem 2: *Let each local maximum $v_i = x(t_i)$ in x be paired with one particular local minimum u_k, determined as follows: from the ith local maximum (value v_i) one determines the lowest values in forward and backward directions between t_i and the nearest points at which $x(t)$ exceeds v_i. The larger (less negative) of those two values, denoted by u_i^{rfc}, is the rainflow minimum paired with v_i, i.e. u_i^{rfc} is the least drop before reaching the value v_i again on either side. Thus the ith rainflow pair is (u_i^{rfc}, v_i), see Fig. 12. The total damage $D_t^{rfc}(x)$ defined using the rainflow method of Endo is equal to*

$$D_t^{rfc}(x) = \sum_{t_i \leq t} \frac{1}{f(u_i^{rfc}, v_i)}.$$

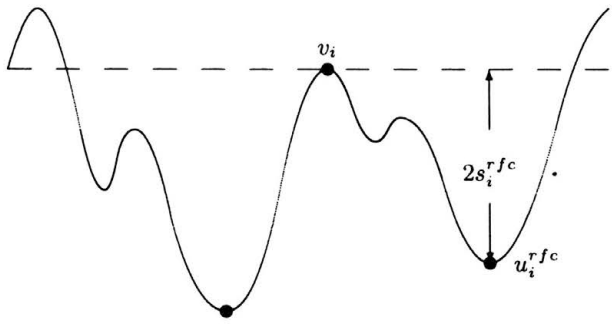

FIGURE 12. Definition of rainflow cycle.

As is evident the rainflow minimum does not necessarily immediately follow or preceede the corresponding maximum. This explains why it is difficult to compute the damage increase in the interval $(t_{i-1}, t_i]$, D_i.

The fatigue criterion (3.8) is reformulated as follows

$$\text{“}x \text{ is safe for fatigue”} \text{ if } D_t^{rfc}(x) < K. \tag{3.9}$$

Obviously, the environmental loads are not known in advance and hence one is modeling those by means of random processes. Suppose that $X(s)$, $s \in [0,t]$, is a random load, then the accumulated damage $D_t^{rfc}(X)$ is a random variable and hence one would like to compute the failure probability, i.e probability $P^f = 1 - P(D_t^{rfc}(X) < K)$. Since the computation of the failure probability is often very difficult thus, similarly as in the case of multiaxial fatigue, one is checking if the criterion is satisfied in average. More precisely, one is replacing, in (3.9), the random strength factor K by its median value that is 1 and the damage $D_t^{rfc}(X)$ by the expected damage $\mathsf{E}(D_t^{rfc}(X))$. However, the recent investigations by Johannesson *et al.* (2003) demonstrated that for non-sinusoidal loads K should be lognormally distributed variable with median smaller than one. It is not unusual that median is as low as 0.3, which expresses the fact that the variable load is more damaging that the equivalent [having the same expected damage $\mathsf{E}(D_t^{rfc})$] simple sinusoidal load is. This results in the following fatigue criterion

$$\text{“}X \text{ is safe for fatigue”} \text{ if } \mathsf{E}(D_t^{rfc}(X)) < 0.3. \tag{3.10}$$

For stationary load X there is a damage intensity d^{rfc}, such that $\mathsf{E}(D_t^{rfc}(X)) = t \cdot d^{rfc}$. If, in addition, the load is ergodic then the damage intensity is given by

$$d^{rfc} = \lim_{t\to\infty} \frac{D_t^{rfc}(X)}{t}.$$

Finally, if the service time of a component is denoted by S, then no fatigue failure is predicted if $S \cdot d^{rfc} < 1$ [or more conservatively if $S \cdot d^{rfc} < 0.3$].

Not many explicit general results are known about the rainflow damage intensity d^{rfc}, one of them is that in the special case when $\alpha = 1$ the damage intensity is proportional to the average variation of the process in the interval $[0,1]$, more precisely

$$d^{rfc} = \frac{1}{2k}\mathsf{E}(|\dot{X}(0)|) = \frac{1}{k}\int_{-\infty}^{\infty} \mu^+(u)\,du. \tag{3.11}$$

Note that the result is true for any stationary load. For $\alpha > 1$, which is the most important case in practice, it can be shown that

$$d^{rfc} = -\int\int \frac{\partial^2 1/f(u,v)}{\partial u \partial v} \mu^{osc}(u,v)\, du\, dv, \tag{3.12}$$

see Rychlik (1993). As we have mentioned before the oscillation intensity can be computed only in a few special cases. However, since $-\frac{\partial^2 1/f(u,v)}{\partial u \partial v} \geq 0$, thus by bounding $\mu^{osc}(u,v)$ by $\min(\mu^+(u), \mu^+(v))$ we obtain the conservative estimate of the rainflow damage intensity. In the following example we shall derive the bound for a Gaussian load.

Example 11. Let load $X(t)$, $0 \leq t \leq S$,be a zero-mean Gaussian process with variance λ_0 and average angular frequency $\hat{\lambda} = \sqrt{\lambda_2/\lambda_0}$. Further, let $N = S\frac{\hat{\lambda}}{2\pi}$ be the average number of zero-upcrossings of X during the service period S. The load is safe for fatigue if $S \cdot d^{rfc} < 1$. We shall demonstrate that, according to this criterion, X causes (in average) less damage to the material then the following simple Gaussian wave:

$$Y(t) = \sqrt{\lambda_0} R \cos(\hat{\lambda} t + \theta), \quad t \in [0, S],$$

where R is a standard Rayleigh distributed variable independent of the uniformly distributed θ. Since Y is a constant amplitude load we have

$$D_S(Y) = D_S^{rfc}(Y) = \frac{N}{k}(2\sqrt{\lambda_0})^\alpha R^\alpha.$$

Note that for $Y(t)$ one can compute the failure probability, i.e.,

$$P^f = 1 - P\left(-\alpha \ln(R) + \sigma U > \ln(N) + \alpha \ln(2\sqrt{\lambda_0}) - \ln(k)\right),$$

where $U \in N(0,1)$ is independent of $-\ln(R)$, which is Gumbel distributed. However, the computed failure probability is not necessarilly a conservative bound for the failure probability for the X load, see discussion following Example 1.

The damage intensity for the load Y is

$$d^{rfc}(Y) = \frac{\hat{\lambda}}{2\pi k}(2\sqrt{\lambda_0})^\alpha \mathsf{E}(R^\alpha). \tag{3.13}$$

We shall next demonstrate that $d^{rfc}(Y) > d^{rfc}(X)$. Using (3.12) and the bound

$$\mu^{osc}(u,v) \leq \frac{\hat{\lambda}}{2\pi} \min\left(e^{-\frac{u^2}{2\lambda_0}}, e^{-\frac{v^2}{2\lambda_0}}\right),$$

and some symmetries ($\mu^{osc}(u,v) = \mu^{osc}(-v,-u)$) we get

$$\begin{aligned} d^{rfc} &= -\int\int \frac{\partial^2 1/f(u,v)}{\partial u \partial v} \mu^{osc}(u,v)\, du\, dv \\ &\le -2\frac{\hat{\lambda}}{2\pi k} \int_{-\infty}^{0}\int_{-u}^{u} \alpha(\alpha-1)(v-u)^{\alpha-2} e^{-\frac{u^2}{2\lambda_0}}\, dv\, du \\ &= 2\frac{\hat{\lambda}}{2\pi k}\int_0^\infty \alpha(2u)^{\alpha-1} e^{-\frac{u^2}{2\lambda_0}}\, du = \frac{\hat{\lambda}}{2\pi k} 2^\alpha \int_0^\infty \alpha\, u^{\alpha-1} e^{-\frac{u^2}{2\lambda_0}}\, du \\ &= \frac{\hat{\lambda}}{2\pi k}(2\sqrt{\lambda_0})^\alpha \int_0^\infty r^{\alpha+1} e^{-r^2/2}\, dr = \frac{\hat{\lambda}}{2\pi k}(2\sqrt{\lambda_0})^\alpha \mathsf{E}(R^\alpha). \end{aligned}$$

Thus we have shown that $d^{rfc}(X) \le d^{rfc}(Y)$. The derived bound is often called *narrow-band approximation for Gaussian loads*. The bound is usually very accurate if the groupness parameter $\alpha_1 \approx 1$. Similar bounds can be derived even for non-Gaussian stationary loads when the upcrossing intensity is known or can be computed.

Note that for Gaussian process and $\alpha = 1$, the upper bound

$$\frac{\hat{\lambda}}{2\pi k}(2\sqrt{\lambda_0})^\alpha \mathsf{E}(R^\alpha) = \frac{1}{k}\sqrt{\lambda_2/2\pi} = \frac{1}{2k}\mathsf{E}(|\dot{X}(0)|),$$

is the exact value of damage intensity, see (3.11).

4. Lecture IV: Derivative and other marks at crossings

In the previous section we have discussed computation of the intensity of points t such that $X(t) = u$, where X is a random process while u is a fixed level. In many applications one would like to know more about the fine details of $X(t)$ near the points where it exceeds some level u, e.g. the slope or velocity by which it passes the level, or the duration and height of the excursion above u. Looking at local extremes, which are located at t where $\dot{X}(t) = 0$, one is interested in their heights, e.g. frequencies of t, such that $\dot{X}(t) = 0$ and $X(t) > u$. Such intensities are useful in evaluation of some characteristics of cycle amplitude used in fatigue analysis. In this section we will present methods to estimate intensities of the solutions to the equation $X(t) = u$ that satisfy some restrictions, see the following simple example.

Example 12. The upcrossing intensity is the intensity of t such that $X(t) = u$ and $\dot{X}(t) > 0$, i.e. we are not interested in all solutions to the equation $X(t) = u$ but only those that satisfy the additional condition $\dot{X}(t) > 0$.

4.1. Intensity of marked crossings

Let $Z(t)$ be a smooth process, such that the joint density $f(\dot{z}, z)$ of $\dot{Z}(t), Z(t)$ exists, and say we are interested in the intensity of t such that $Z(t) = u$. Suppose that, in addition to Z, one has a vector of processes $\mathbf{Y}(t) = (Y_1(t), \ldots, Y_n(t))$ such that $Z(t), \mathbf{Y}(t)$ are jointly stationary and ergodic. Consider now a statement A about the outcome of $\mathbf{Y}(t)$ and $\dot{Z}(t)$ for which we can say if it is true or not at any time t. Let

$$\mathcal{C}_t(u) = \{s \in [0,t] : Z(s) = u\},$$

and we shall write

$$\mathcal{C}_u = \mathcal{C}_1(u). \tag{4.1}$$

Now denote

$$N_t(A|u) = \text{number of } s \in \mathcal{C}_t(u) \text{ such that } A \text{ is true},$$

and let

$$\mu(A|s \in \mathcal{C}_u) = \lim_{t\to\infty} \frac{N_t(A|u)}{t}$$

be the intensity of the solutions of the equation $Z(s) = u$, such that $\mathbf{Y}(s), \dot{Z}(s)$ satisfy A. By assumed ergodicity of these processes, $\mu(A|s \in \mathcal{C}_u)$ is well defined (and deterministic). Obviously, $\mathsf{E}(N_t(A|u)) = t\mu(A|s \in \mathcal{C}_u)$. Using the introduced notation, $\mu(u) = \mu(s \in \mathcal{C}_u)$ is the intensity of the solutions $Z(s) = u$. Usually we will write $\mu(A|u)$, $\mu(u)$ instead of $\mu(A|s \in \mathcal{C}_u)$, $\mu(s \in \mathcal{C}_u)$, respectively.

Example 13. Let $Z(t) = \dot{X}(t)$, $Y(t) = X(t)$, and A be the statement $A =$"$Y(t) > u, \dot{Z}(t) < 0$", then $N_t(A|0)$ is the number of local maxima of X observed in $[0,t]$ exceeding level u while $\mu(A|s \in \mathcal{C}_0)$ is the intensity of such local maxima.

Now the "long run" probability of A observed at $s \in \mathcal{C}_u$, i.e. such that $Z(s) = u$, is defined as

$$P(A|s \in \mathcal{C}_u) = \lim_{t\to\infty} \frac{N_t(A|u)}{N_t(u)} = \frac{\mu(A|u)}{\mu(u)}. \tag{4.2}$$

Note that $P(A) = P((\mathbf{Y}(0), \dot{Z}(0)) \in A)$ (likelihood that A is true at time zero) is usually not equal $P(A|s \in \mathcal{C}_u)$. The reason for this is that values of $\mathbf{Y}(s), \dot{Z}(s)$ at points such that $Z(s) = u$ can have different distribution than $\mathbf{Y}(s), \dot{Z}(s)$ observed at $s = 1, 2, 3, \ldots$. We turn now to the computation of the intensity $\mu(A|u)$.

Theorem 3: *Under the general assumptions of this section, if the joint density* $f(\dot{z}, z)$ *of* $\dot{Z}(0), Z(0)$ *exists then*

$$\mu(A|u) = \int P(A|u, \dot{z})|\dot{z}|f(\dot{z}, u)\, d\dot{z}, \tag{4.3}$$

where $P(A|u, \dot{z})$ *is the conditional probability*

$$P\Big((\mathbf{Y}(0), \dot{Z}(0)) \in A|Z(0) = u, \dot{Z}(0) = \dot{z}\Big) = P\Big((\mathbf{Y}(0), \dot{z}) \in A|Z(0) = u, \dot{Z}(0) = \dot{z}\Big).$$

In addition, if also the density $f(\mathbf{y}, \dot{z}, z)$ *of* $\mathbf{Y}(0), \dot{Z}(0), Z(0)$ *exists, then*

$$\mu(A|u) = \int_A |\dot{z}|f(\mathbf{y}, \dot{z}, u)\, d\mathbf{y}\, d\dot{z}. \tag{4.4}$$

The proof of this theorem can be found in Leadbetter *et al.* (1983). In the Appendix we have introduced the concept of conditional probabilities; for any statement A there is a function $P(A|z, \dot{z})$ such that $P(A) = \int P(A|z, \dot{z}) f_{Z\dot{Z}}(z, \dot{z})\, dz\, d\dot{z}$. The formula (4.3) is a generalization of Rice's formula since it simplifies to (2.1) for A which is always true.

Studying the χ^2 processes we found that $Z(0), \dot{Z}(0)$ may have a density $f(z, \dot{z})$ whose explicit analytic form is not known. Now an additional complication arises if one also needs to find the conditional probability

$$P(A|z, \dot{z}) = P\Big((\mathbf{Y}(0), \dot{z}) \in A|Z(0) = z, \dot{Z}(0) = \dot{z}\Big).$$

Finding the conditional probability can be a very complicated problem if $Z(t)$ is a non-Gaussian process. There are only few examples of such analysis. The first one is given in Rice (1944) where the author was investigated the height of local maximum in the envelope process (Rice used another definition of the envelope than given in this presntation). Another one is presented in Ditlevsen and Lindgren (1988) where the authors studied frequencies of empty envelop excursions. In that work the process $Z(t) = E(t) = \sqrt{X(t)^2 + \hat{X}(t)^2}$ while $Y(t) = X(t + \tau)$, where $\tau \in R$ is a fixed constant. The statement A was a very complicated property of X called 'empty excursion", which means that the upcrossing of the level u by $E(t)$ is not followed by the upcrossing of the level by the process $X(t)$, and it needs to be approximated by some simpler statement. We will not go into details on this problem, see also Lindgren and Rychlik (1991) (and references

therein) for some additional examples of computations of marked crossings of non-Gaussian processes.

The problem of computing marked crossing intensities simplifies somewhat if $\mathbf{Y}(0)$, $Z(0)$, $\dot{Z}(0)$ is a Gaussian $n+2$ dimensional vector. If the condition (property) A is a complicated non-linear function of $\mathbf{Y}(t)$, then computation of $\mu(A|u)$ requires numerical integrations in $n+1$ dimensions.

4.2. Derivatives at crossings

Consider a stationary ergodic Gaussian process $Z(t)$. Suppose that $\mathsf{E}(Z(t)) = m$ and the spectral measure has the first two spectral moments finite, i.e. $\lambda_0, \lambda_2 < \infty$. It is well known that the derivative process $\dot{Z}(t)$ exists and is also Gaussian. Let us denote by A the statement that the observed derivative exceeds the level v. As we have mentioned before, the derivative of a stationary Gaussian process is a zero mean stationary Gaussian process with variance equal to the second order spectral moment λ_2, i.e. $\dot{Z}(t) \in N(0, \lambda_2)$. This means that the fraction of derivatives exceeding v, observed at times $t = 0, 1, 2, 3, \ldots$ is equal to

$$P(\dot{Z}(0) > v) = 1 - \Phi(v/\sqrt{\lambda_2}).$$

Now we shall demonstrate that the fraction of derivatives observed at points s, such that $Z(s) = u$ will have a different value, i.e. $P(\dot{Z}(0) > v) \neq P(\dot{Z}(s) > v | s \in \mathcal{C}_u)$.

Lemma 1: *Under the general assumptions of this section the distribution* $F^u(v) = P(\dot{Z}(s) \leq v | s \in \mathcal{C}_u)$ *has density*

$$f^u(v) = \frac{|v|}{2\lambda_2} e^{-\frac{v^2}{2\lambda_2}}. \tag{4.5}$$

Proof. For $A =$"$\dot{Z}(s) > v$", using (4.3), we have that the intensity $\mu(A|u)$ is given by

$$\begin{aligned} \mu(A|u) &= \int_{-\infty}^{\infty} P(\dot{Z}(0) > v \mid Z(0) = u, \dot{Z}(0) = \dot{z}) |\dot{z}| f(u, \dot{z})\, d\dot{z} \\ &= \int_{v}^{\infty} |\dot{z}| f(u, \dot{z})\, d\dot{z}, \end{aligned} \tag{4.6}$$

where $f(u, \dot{z})$ is the density of $(Z(0), \dot{Z}(0))$. By our assumptions $Z(0) \in N(m, \lambda_0)$ is independent of $\dot{Z}(0) \in N(0, \lambda_2)$. Consequently $f(u, \dot{z}) =$

$f(u)f(\dot{z})$ and hence

$$\mu(A|u) = f(u)\int_v^\infty |\dot{z}|f(\dot{z})\,d\dot{z} = 2f(u)\sqrt{\frac{\lambda_2}{2\pi}}\int_v^\infty \frac{|\dot{z}|}{2\lambda_2}e^{-\frac{\dot{z}^2}{2\lambda_2}}\,d\dot{z}.$$

Clearly $\mu(u) = \mu(A|u)$, for $v = -\infty$, and hence

$$\mu(u) = 2f(u)\sqrt{\frac{\lambda_2}{2\pi}}\int_{-\infty}^\infty \frac{|\dot{z}|}{2\lambda_2}e^{-\frac{\dot{z}^2}{2\lambda_2}}\,d\dot{z} = 2f(u)\sqrt{\frac{\lambda_2}{2\pi}}.$$

Consequently

$$P(\dot{Z}(s) > v|s \in \mathcal{C}_u) = \frac{\mu(A|u)}{\mu(u)} = \int_v^\infty \frac{|\dot{z}|}{2\lambda_2}e^{-\frac{\dot{z}^2}{2\lambda_2}}\,d\dot{z},$$

which after differentiation leads to (4.5). □

Before we turn to some more complicated example taken from naval engineering we shall first sketch a general approach to computation of $P((\mathbf{Y}(s), \dot{Z}(s)) \in A|s \in \mathcal{C}_u)$ for Gaussian processes $\mathbf{Y}(s), Z(s)$:

First write down a formula for the long-run distribution of the vector $\mathbf{Y}(s), \dot{Z}(s)$ *observed at* s *such that* $Z(s) = u$, *then find the density* $f^u(\mathbf{y}, \dot{z})$. *(Such density always exists if* $\mathbf{Y}(0), \dot{Z}(0), Z(0)$ *has non-degenerated Gaussian density.) Finally compute*

$$P((\mathbf{Y}(s), \dot{Z}(s)) \in A|s \in \mathcal{C}_u) = \int_A f^u(\mathbf{y}, \dot{z})\,d\mathbf{y}\,d\dot{z}. \tag{4.7}$$

In order to simplify notation one is often introducing variables $\mathbf{Y}_u, \dot{Z}_u$, *which have the density* $f^u(\mathbf{y}, \dot{z})$, *i.e. we can formally write*

$$P((\mathbf{Y}_u, \dot{Z}_u) \in A) = P((\mathbf{Y}, \dot{Z}) \in A|s \in \mathcal{C}_u) = \int_A f^u(\mathbf{y}, \dot{z})\,d\mathbf{y}\,d\dot{z}. \tag{4.8}$$

We observe the variables $\mathbf{Y}_u, \dot{Z}_u$ are the special case of the so-called Slepian model process, see Lindgren and Rychlik (1991) for detailed presentation.

Now one can ask the question about the long-run probability that the derivative observed at upcrossings of the level u exceeds level v, i.e. we wish to know the proportion of s such that $Z(s) = u$ and $\dot{Z}(s) > 0$ which also

satisfies the condition $\dot{Z}(s) > v$, $v \geq 0$. After a short reflection one can see that it is just the conditional probability

$$P(\dot{Z}_u > v | \dot{Z}_u > 0) = \frac{P(\dot{Z}(s) > v, \dot{Z}(s) > 0 | s \in \mathcal{C}_u)}{P(\dot{Z}(s) > 0 | s \in \mathcal{C}_u)} = e^{-\frac{v^2}{2\lambda_2}},$$

see Appendix for the definition of conditional probabilities. We conclude that the derivative at upcrossing is Rayleigh distributed. The distribution is independent of the level u.

The following lemma gives explicit formula for $Y_u, \dot{Z}_u$ for the one-dimensional $Y(s)$, i.e. when $\mathbf{Y}(s) = Y(s)$.

Lemma 2: *Suppose $Z(t), \dot{Z}(t), Y(t)$ is a stationary Gaussian vector valued process. Let $m_Z = \mathsf{E}(Z(t))$, $m_Y = \mathsf{E}(Y(t))$ and let the covariance matrix Σ of the vector $Z(t), \dot{Z}(t), Y(t)$ be given by*

$$\Sigma = \begin{bmatrix} \sigma_Z^2 & 0 & \sigma_Z \sigma_Y \rho_{ZY} \\ 0 & \sigma_{\dot{Z}}^2 & \sigma_{\dot{Z}} \sigma_Y \rho_{\dot{Z}Y} \\ \sigma_Z \sigma_Y \rho_{ZY} & \sigma_{\dot{Z}} \sigma_Y \rho_{\dot{Z}Y} & \sigma_Y^2 \end{bmatrix}. \tag{4.9}$$

The variables $Y_u, \dot{Z}_u$, with distribution defined by (4.8), have the following representation;

$$\begin{aligned} \dot{Z}_u &= \sigma_{\dot{Z}} R, \\ Y_u &= m_u + \sigma_Y \left(\rho_{\dot{Z}Y} R + \sqrt{1 - \rho_{ZY}^2 - \rho_{\dot{Z}Y}^2} U \right), \end{aligned} \tag{4.10}$$

where R, which is independent of $U \in N(0,1)$, has a double Rayleigh distribution, and hence the probability density $f(r) = \frac{|r|}{2} e^{-r^2/2}$. Here $m_u = m_Y + \frac{\sigma_Y}{\sigma_Z} \rho_{ZY}(u - m_Z)$ and ρ_{ZY}, $\rho_{\dot{Z}Y}$ are correlations between $Y(0)$ and $Z(0)$, $\dot{Z}(0)$, respectively. Obviously, the long-run distribution $F^u(y,v)$ of $Y(t), \dot{Z}(t)$ at points that satisfy $Z(s) = u$, i.e. $P(\dot{Z}(s) \leq v, Y(s)) \leq y | s \in \mathcal{C}_u)$ is given by

$$\begin{aligned} &P(Y_u \leq y, \dot{Z}_u \leq v) \\ &\qquad = P\left(\rho_{\dot{Z}Y} R + \sqrt{1 - \rho_{ZY}^2 - \rho_{\dot{Z}Y}^2} U \leq \frac{y - m_u}{\sigma_Y}, \ R \leq \frac{v}{\sigma_{\dot{Z}}} \right). \end{aligned}$$

Proof. We need to demonstrate that

$$P(Y(s) \leq y, \dot{Z}(s) \leq v | s \in \mathcal{C}_u) = \frac{\mu(A|u)}{\mu(u)},$$

where $A =$"$Y(t) \le y$ and $\dot{Z}(t) \le v$", is equal to

$$P\left(\rho_{\dot{Z}Y}R + \sqrt{1-\rho_{ZY}^2-\rho_{\dot{Z}Y}^2}U \le \frac{y-m_u}{\sigma_Y},\ R \le \frac{v}{\sigma_{\dot{Z}}}\right).$$

We begin by computating the intensity $\mu(A|u)$, i.e., the intensity of points s, such that $Z(s) = u$ and A is true. From formula (4.3) we know that

$$\mu(A|u) = \int_{-\infty}^{+\infty} P(Y(0) \le y, \dot{Z}(0) \le v \mid Z(0) = u, \dot{Z}(0) = \dot{z})|\dot{z}|f(u,\dot{z})\,d\dot{z},$$

where

$$P(Y(0) \le y, \dot{Z}(0) \le v|u,\dot{z}) = \begin{cases} P(Y(0) \le y|Z(0) = u, \dot{Z}(0) = \dot{z}) & \text{if } \dot{z} \le v \\ 0 & \text{if } \dot{z} > v. \end{cases}$$

Now we know, see Appendix, that the conditional density of $Y(0)$ given $Z(0) = z, \dot{Z}(0) = \dot{z}$ is Gaussian with the mean

$$\begin{aligned} m(z,\dot{z}) &= m_Y + \sigma_Y[\sigma_Z\rho_{ZY}\ \sigma_{\dot{Z}}\rho_{\dot{Z}Y}] \begin{bmatrix} \sigma_Z^2 & 0 \\ 0 & \sigma_{\dot{Z}}^2 \end{bmatrix}^{-1} \begin{bmatrix} z-m_Z \\ \dot{z} \end{bmatrix} \\ &= m_Y + \frac{\sigma_Y}{\sigma_Z}\rho_{ZY}(z-m_Z) + \frac{\sigma_Y}{\sigma_{\dot{Z}}}\rho_{\dot{Z}Y}\dot{z}, \end{aligned} \tag{4.11}$$

and the variance

$$\begin{aligned} \sigma^2 &= \sigma_Y^2 - \sigma_Y^2[\sigma_Z\rho_{ZY}\ \sigma_{\dot{Z}}\rho_{\dot{Z}Y}] \begin{bmatrix} \sigma_Z^2 & 0 \\ 0 & \sigma_{\dot{Z}}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sigma_Z\rho_{ZY} \\ \sigma_{\dot{Z}}\rho_{\dot{Z}Y} \end{bmatrix} \\ &= \sigma_Y^2(1-\rho_{ZY}^2-\rho_{\dot{Z}Y}^2). \end{aligned} \tag{4.12}$$

Consequently we have

$$\begin{aligned} \mu(A|u) &= \frac{1}{\sqrt{2\pi}\sigma_Z}e^{-\frac{(u-m_Z)^2}{2\sigma_Z^2}} \int_{-\infty}^{v} \Phi\left(\frac{y-m(u,\dot{z})}{\sigma}\right) \frac{|\dot{z}|}{\sqrt{2\pi}\sigma_{\dot{Z}}}e^{-(\dot{z}^2/2\sigma_{\dot{Z}}^2)}\,d\dot{z} \\ &= \frac{1}{\pi}\frac{\sigma_{\dot{Z}}}{\sigma_Z}e^{-\frac{(u-m_Z)^2}{2\sigma_Z^2}} \int_{-\infty}^{v} \Phi\left(\frac{y-m(u,\dot{z})}{\sigma}\right) \frac{|\dot{z}|}{2\sigma_{\dot{Z}}^2}e^{-(\dot{z}^2/2\sigma_{\dot{Z}}^2)}\,d\dot{z} \\ &= \frac{1}{\pi}\frac{\sigma_{\dot{Z}}}{\sigma_Z}e^{-\frac{(u-m_Z)^2}{2\sigma_Z^2}} \int_{-\infty}^{v} P\left(m_u + \frac{\sigma_Y}{\sigma_{\dot{Z}}}\rho_{\dot{Z}Y}\dot{z} + \sigma U < y\right) \frac{|\dot{z}|}{2\sigma_{\dot{Z}}^2}e^{-(\dot{z}^2/2\sigma_{\dot{Z}}^2)}\,d\dot{z} \\ &= \frac{1}{\pi}\frac{\sigma_{\dot{Z}}}{\sigma_Z}e^{-\frac{(u-m_Z)^2}{2\sigma_Z^2}} \int_{-\infty}^{v/\sigma_Z} P\left(m_u + \sigma_Y\rho_{\dot{Z}Y}r + \sigma U < y\right) \frac{|r|}{2}e^{-r^2/2}\,dr. \end{aligned} \tag{4.13}$$

Since $\mu(u) = \frac{1}{\pi}\frac{\sigma_{\dot{Z}}}{\sigma_Z} e^{-\frac{(u-m_Z)^2}{2\sigma_Z^2}}$, the lemma is proved. □

Note that the formula (4.13) can be computed using integration by parts, see the following lemma for the related result.

Lemma 3: *Let U, R be, independent, standard Gaussian and Rayleigh distributed variables. For $a > 0$ and any x*

$$P(aU + bR > x) = \Phi\left(-\frac{x}{a}\right) + e^{-\frac{x^2}{2\sigma^2}} \cdot \frac{b}{\sigma}\Phi\left(\frac{x}{\sigma}\frac{b}{a}\right), \tag{4.14}$$

where $\sigma = \sqrt{a^2 + b^2}$.

The last two lemmas give an explicit way to compute the distribution of $Y_u, \dot{Z}_u$, i.e. $F^u(y, \dot{z}) = P(Y_u \leq y, \dot{Z}_u \leq \dot{z})$. The representation, given in (4.10), helps to write the probabilities $P((Y(s), \dot{Z}(s)) \in A | s \in \mathcal{C}_u)$ in a more transparent way, see the example given in the following subsection. Similar results can be derived for vector valued process $\mathbf{Y}(t)$ and even for $\mathbf{Y}(t)$ containing infinitely many processes $Y_i(t)$, see Lindgren and Rychlik (1991).

Remark 4. Suppose $(Y(0), \dot{Z}(0))$ is Gaussian then

$$P(Y(0) \leq y, \dot{Z}(0) \leq v) = P\left(\rho_{\dot{Z}Y} R + \sqrt{1 - \rho_{\dot{Z}Y}^2}\, U \leq \frac{y - m_Y}{\sigma_Y},\ R \leq \frac{v}{\sigma_{\dot{Z}}}\right),$$

where U, R are independent standard Gaussian, i.e. $U, R \in N(0, 1)$. Consequently, if the process $Y(\cdot)$ is independent of $Z(\cdot)$ then the values of $Y(s)$ observed at points s where $Z(s) = u$, Y_u is the same as the distribution $Y(0)$.

4.2.1. Extreme stresses at slams. In modeling forces and movements of ships on random seas, an important safety factor is size of stresses due to slams, that occurs when a ship proceeds at certain speed in rough seas and the front part of the hull bottom sustains large forces resulting from impact with the sea surface. Following the approach of Leadbetter and Spaniolo (1998), let $Z(t)$ denote the relative vertical motion of the bow and wave height, each measured from their respective mean position so that $Z(t)$ has zero mean ($m_Z = 0$). The bow emerges from the water if $Z(t)$ exceeds the draft u of the vessel. Further, a slam occurs when $Z(t)$ crosses u from above to below with large enough speed, i.e. $Z(t) = u$ and $\dot{Z}(t) < c$, where $c < 0$ is some fixed threshold. The whipping stress at the instant t of the slam is often assumed to be proportional to $\dot{Z}(t)^2$. However, the vessel also experiences the so-called wave-induced stress $Y(t)$, say, caused by the wave interaction with the vessel. The total stress $S(t)$ at the slam instance is then

$$S(t) = Y(t) + k\dot{Z}(t)^2,$$

where k is a negative proportionality constant. In safety analysis of a vessel, one is interested in an estimate of

$$\mu(A|u) = \text{"intensity of } t \text{ such that } Z(t) = u \text{ and } \dot{Z}(t) \leq c,\ S(t) \leq s\text{"},$$

where $A =$"$\dot{Z}(t) \leq c,\ S(t) \leq s$" and s is a fixed (negative) stress.

We shall compute $\mu(A|u)$ using (4.2)

$$\mu(A|u) = P(\dot{Z}(t) \leq c, S(t) \leq s | t \in \mathcal{C}_u)\mu(u), \quad c < 0,$$

where $\mu(u) = \frac{1}{\pi}\frac{\sigma_{\dot{Z}}}{\sigma_Z} e^{-u^2/2\sigma_Z^2}$, while $P(\dot{Z}(t) \leq c, S(t) \leq s | t \in \mathcal{C}_u)$ will be computed by means of Lemma 2.

As we have mentioned before, the sea surface X evolving in time is usually modeled as a Gaussian random field. The simplified analysis of a ship motion assumes that responses are linear functionals of the sea and hence $(Z(t), \dot{Z}(t), Y(t))$ is a Gaussian vector valued process. By assumed stationarity, $Z(t), \dot{Z}(t)$ are independent but correlations ρ_{ZY}, $\rho_{\dot{Z}Y}$ can be non-zero in general. Suppose that the covariance matrix (4.9) has been computed using a suitable sea spectrum and the model for the vessel's motion. Assume further that $\mathsf{E}(Y(t)) = 0$. Note that the event that bow emerges from the water affects the ship motion in a nonlinear manner making use of the Gaussian model questionable, but we neglect this problem. Now using Lemma 2,

$$P(\dot{Z}(t) \leq c, S(t) \leq s | t \in \mathcal{C}_u) = P(\dot{Z}_u \leq c, Y_u + k\dot{Z}_u^2 \leq s).$$

Hence the intensity $\mu(A|u)$ can be written in a more explicit way:

$$\mu(A|u) = P\left(R < \frac{c}{\sigma_{\dot{Z}}},\ \rho_{\dot{Z}Y}R + \sigma U + k\frac{\sigma_{\dot{Z}}^2}{\sigma_Y}R^2 < \frac{s - u\frac{\sigma_Y}{\sigma_Z}\rho_{ZY}}{\sigma_Y}\right)\frac{1}{\pi}\frac{\sigma_{\dot{Z}}}{\sigma_Z}e^{-\frac{u^2}{2\sigma_Z^2}},$$

where U, R are independent, $U \in N(0,1)$ and $\sigma = \sqrt{1 - \rho_{ZY}^2 - \rho_{\dot{Z}Y}^2}$, while R has pdf $f(r) = \frac{|r|}{2}e^{-r^2/2}$. We conclude that the intensity of slamming events leading to high stresses can be estimated by computing a one-dimensional integral. Note we could not directly apply the lemma with $Y(t) = S(t)$ since Y is not a Gaussian process.

4.3. Crests and troughs in Gaussian processes

Let us consider a stationary Gaussian process $X(t)$ with mean m and spectral distribution having finite fourth spectral moment, i.e. $\lambda_0, \lambda_2, \lambda_4 < \infty$. This implies that the second order derivative $\ddot{X}(t)$ exists and $\ddot{X}(t) \in N(0, \lambda_4)$. Denote by $\mu_{\max}$ the intensity of local maxima. Following our

schema, let the process $Z(t) = \dot{X}(t)$. We know that, by stationarity, $Z(t) \in N(0, \lambda_2)$ is independent of $\dot{Z}(t) \in N(0, \lambda_4)$. Now the intensity of local maxima is given by

$$\begin{aligned}\mu_{\max} &= \mu(\dot{Z}(s) < 0 | s \in \mathcal{C}_0) \\ &= \int_{-\infty}^{\infty} P(\dot{Z}(0) < 0 | Z(0) = 0, \dot{Z}(0) = \dot{z}) |\dot{z}| f_{\dot{Z}(0), Z(0)}(\dot{z}, 0)\, d\dot{z} \\ &= f_{Z(0)}(0) \int_{-\infty}^{0} |\dot{z}| f_{\dot{Z}(0)}(\dot{z})\, d\dot{z} = \frac{1}{\sqrt{2\pi\lambda_2}} \sqrt{\lambda_4} \Psi(0) = \frac{1}{2\pi} \sqrt{\frac{\lambda_4}{\lambda_2}},\end{aligned}$$

since $P(\dot{Z}(0) < 0 | Z(0) = u, \dot{Z}(0) = \dot{z}) = 1$ if $\dot{z} < 0$ and zero otherwise. (The intensity of local minima is equal to $\mu_{\max}$).

Obviously, between two upcrossings of the mean level there is at least one local maximum. This fact is used as a motivation for the use of the following irregularity measure of the process X

$$\alpha_2 = \frac{\text{intensity of upcrossings of the mean E(X(0))}}{\text{intensity of local maxima}}. \tag{4.15}$$

For a stationary Gaussian process we have (by Rice's formula) that the intensity of upcrossings of the mean $E(X(0))$ is given by $\frac{1}{2\pi}\sqrt{\lambda_2/\lambda_0}$ and hence

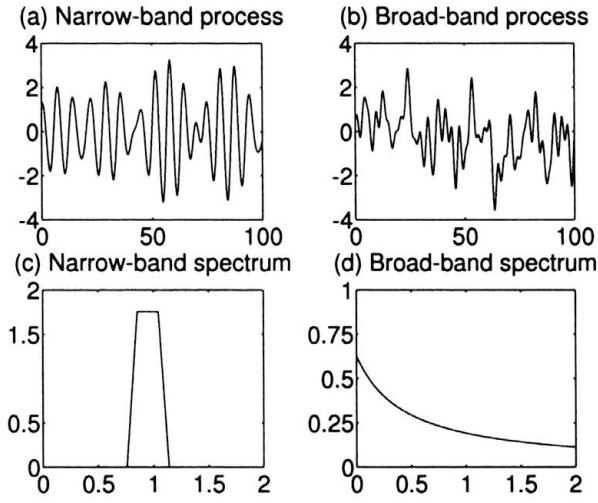

FIGURE 13. Examples of narrow- and broad-band processes.

the parameter α_2, defined by (4.15), is equal to

$$\alpha_2 = \frac{\lambda_2}{\sqrt{\lambda_0 \lambda_4}}.$$

Note that we have already introduced another irregularity measure $\alpha_1 = \frac{\lambda_1}{\sqrt{\lambda_0 \lambda_2}}$, called groupness measure.

From the definition of α_2 we have directly that $0 \leq \alpha_2 \leq 1$. Clearly, if $\alpha_2 \approx 1$ then the process $X(t)$ must be quite regular with one local maximum between the consecutive upcrossings of the mean level. Such process is usually called a narrow band process. Processes with $\alpha_2 \approx 0$ are in general very irregular and are called broad band; see Fig. 13 where two sample paths of Gaussian processes with their spectral densities are presented.

Remark 5. The parameter α_2 is a simple measure of irregularity of the process and one can easily construct processes, which in fatigue application should be considered as extremely narrow band but they have $\alpha_2 = 0$. Example of such process is presented in Fig. 14. In the figure we show two parts of a sample of the damped oscillator with spectral density (1.21) with $\zeta = 0.01$. At plot (left) we have a longer record of the process while on plot (right) we are zooming in on the neighborhood of a local maximum of the process. This process has $\alpha_2 = 0$ and is considered as extremely irregular (broad-band) since the intensity of the maxima $\mu_{\max}$ is infinite. Note that the parameter α_1 is close to one, $\alpha_1 = 0.995$. However, the maxima cluster in a negligible surrounding of a major top and in practical applications these small wiggles can be completely irrelevant and the process should in fact be called a narrow band process. For this process, the narrow-band approximation (3.13) will give very accurate approximations of the rainflow damage. Note that formally we have problems in defining the rainflow damage, since the discussed process has infinitely many local extremes (cycles) in any finite interval. However, for irregular processes one can use equation (3.12) to define the rainflow damage even for loads that have infinitely many cycles but the sum of their damages is finite, see Rychlik (1993) for discussion. As we have mentioned before the oscillation intensity is well defined even for very irregular processes.

We turn now to the long-run distribution of the height of local maxima, i.e. the distribution of $X(t)$ for t such that $\dot{X}(t) = 0$ and $\ddot{X}(t) < 0$ given by

$$P(\text{Max} \leq x) = \frac{P(X(s) \leq x; \ddot{X}(s) < 0 | s \in \mathcal{C}_0)}{P(\ddot{X}(s) < 0 | s \in \mathcal{C}_0)}. \tag{4.16}$$

We begin by computing $P(X(s) \leq x; \ddot{X}(s) < 0 | s \in \mathcal{C}_0)$.

In order to use the lemma define $Z(t) = \dot{X}(t)$ and $Y(t) = X(t)$ and let $u = 0$ in the condition $Z(s) = u$. We have $m_Y = m$, $m_Z = 0$ and the

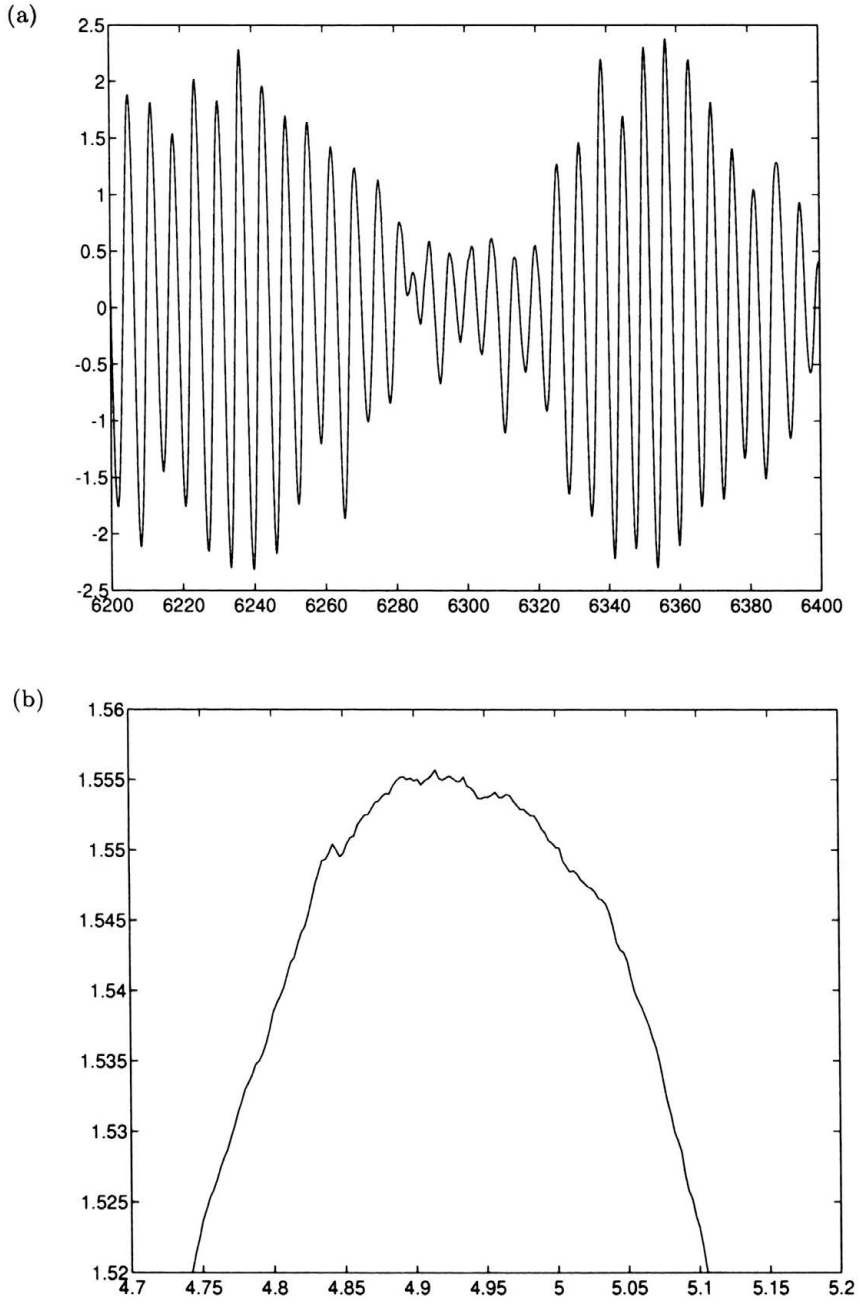

FIGURE 14. Narrow-band linear oscilator $\zeta = 0.01$: (a) simulated longer path of the process, (b) zoomed neighbourhood of a local maximum in the path.

covariance matrix

$$\Sigma = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_4 & -\lambda_2 \\ 0 & -\lambda_2 & \lambda_0 \end{bmatrix}.$$

Consequently $\sigma_Z^2 = \lambda_2$, $\sigma_{\dot{Z}}^2 = \lambda_4$, $\rho_{ZY} = 0$, $\rho_{\dot{Z}Y} = -\alpha_2$, $\sigma_Y^2 = \lambda_0$, and Lemma 2 gives:

$$\dot{Z}_0 = \sqrt{\lambda_4} R,$$

$$Y_0 = m + \sqrt{\lambda_0}\left(-\alpha_2 R + \sqrt{1-\alpha_2^2}U\right),$$

Note that the variable Y_0 is the height of local extreme (by Bulinskaya's lemma, Bulinskaya (1961), the solutions of the equation $\dot{X}(s) = 0$ are with probability one local maxima or minima) and not the height of the local maximum. The probability $P(\text{Max} \le x)$ can be derived as follows

$$P(\text{Max} \le x) = P(Y_0 \le x | \dot{Z}_0 < 0) = P(Y_0 \le x | R < 0).$$

This gives the final result

$$P(\text{Max} \le x) = P\left(m + \sqrt{\lambda_0}\left(\alpha_2 R + \sqrt{1-\alpha_2^2}U\right) \le x\right).$$

The long run distribution of height of a local maximum is the same as the distribution of the random variable $m + \sqrt{\lambda_0}\left(\alpha_2 R + \sqrt{1-\alpha_2^2}U\right)$, where U and R are independent random variables, U standard normal, and R standard Rayleigh, with density $f_R(r) = r\exp(-r^2/2), r \ge 0$. Probability $P(\text{Max} \le x)$ can be computed by means of the formula (4.14).

Remark 6. The intensity of local maxima with height above u, could also be used to bound the probability that $M_S(X) = \max_{0 \le s \le S} X(s) > u$. With $Z(s) = \dot{X}(s)$, $Y(s) = X(s)$, and $A =$"$Y(s) > u, \dot{Z}(s) < 0$" we have

$$\begin{aligned} N_S(A|u) &= \text{number of } s \in \mathcal{C}_S(u) \text{ such that } A \text{ is true} \\ &= \text{number of local maxima in } [0, S] \text{ with height above } u. \end{aligned}$$

Now, in a similar way as we derived (1.7), we can write

$$\begin{aligned} P(M_S > u) &= P(X(0) > u) + P(X(0) \le u, X(S) > u) \\ &\quad + P(X(0) \le u, X(S) \le u,\ N_S(A|u) > 0) \\ &\approx P(N_S(A|u) > 0) \le \mathsf{E}(N_S(A|u)) = S\mu(A|u). \end{aligned} \tag{4.17}$$

Obviously, since the number of upcrossings of the level u by X in the interval $[0, S]$, denoted by $N_S^+(u)$, is smaller than $N_S(A|u)$ the approximation (4.17) is less accurate (gives broader bound) than the one proposed in (1.7-1.8). However, if one is interested in the maximum value of the field then the method used in (4.17) can easily be generalized to be valid for the fields, while the method used in (1.7-1.8) can not be easily extended to this more complicated case. In addition, it can be easily proved, see Rice (1944), that

$$\mu(A|u) = \mu^+(u) + o(e^{-\frac{u^2}{2\lambda_0}}),$$

where $o(e^{-\frac{u^2}{2\lambda_{00}}})$ means that $e^{\frac{u^2}{2\lambda_0}} o(e^{-\frac{u^2}{2\lambda_0}})$ goes to zero as u tends to infinity. Hint: use that $\mu(A|u) = \mu_{\max} P(Max > u)$ and that $1 - \Phi(x) = o(e^{-\frac{u^2}{2}})$. Consequently, the approximations (1.8) and (4.17) are equivalent. Finally, we give an intuitive motivation why the two approaches are equivalent for high values of u.

Define first $B ='' Y(s) > u, \dot{Z}(s) > 0''$ then we have that

$$\begin{aligned} N_S(B|u) &= \text{number of } s \in \mathcal{C}_S(u) \text{ such that } B \text{ is true} \\ &= \text{number of local minima in } [0, S] \text{ with height above } u. \end{aligned}$$

Now it is easy to see that

$$\begin{aligned} &P\big(X(0) \le u, X(S) \le u, N_S(A|u) > 0\big) \\ &\qquad = P\big(X(0) \le u, X(S) \le u, N_S(A|u) - N_S(B|u) > 0\big), \end{aligned}$$

and that $N_S^+(u) = N_S(A|u) - N_S(B|u)$. Consequently, since, for very high levels u, $\mu(B|u)$ is much smaller than $\mu(A|u)$ one has that $\mu^+(u) \approx \mu(A|u)$. Consequently the approximations (1.7) and (4.17) are close.

4.4. Oscillation intensity as intensity of marked crossings

We shall now show how one can express the oscillation intensity using the concept of marked crossings. Consider load $Z(t)$ with finite intensity $\mu(v)$. We will show that

$$\mu^{osc}(u, v) = \mu(v) P(Y(s) < u | s \in \mathcal{C}_v),$$

for suitably defined process $Y(s)$. We observe that $\mu^{osc}(u, v)$ can be computed for Ornstein-Uhlenbeck process for which $\mu(v) = \infty$ for all v. We turn now to the definition of the $Y(t)$ process.

Definition 3: *Let $Z(t)$, $t \in R$ be a differentiable process, i.e. $\dot{Z}(t)$ exists. For a fixed value t let $t^- = \sup\{s \leq t : Z(s) > Z(t)\}$, with $\sup \emptyset = -\infty$, be the beginning of the downward excursion ending at t. The depth of the excursion will be denoted by $Y(t)$, and defined by*

$$Y(t) = \inf\{Z(s) : t^- \leq s \leq t\}, \tag{4.18}$$

see Fig. 15.

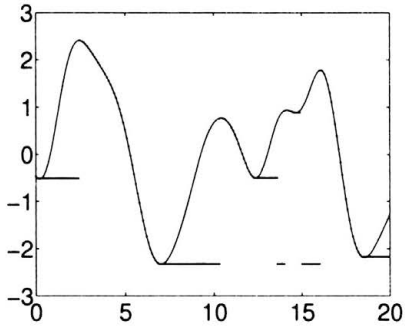

FIGURE 15. Simulation of $X(t)$ (solid line) and corresponding $Y(t)$ (dotted line).

Now, let A be the statement that observed Y is below the level u. Then it is easy to see that the oscillation intensity is equal to the intensity of t, such that $Z(t) = v$ such that $Y(t) < u$. Consequently

$$\mu^{osc}(u,v) = \mu(A|v) = \int_{-\infty}^{\infty} P(Y(0) < u | X(0) = v, \dot{X}(0) = \dot{x}) |\dot{x}| f(v,\dot{x})\, d\dot{x}$$

$$= \int_{0}^{\infty} P(Y(0) < u | X(0) = v, \dot{X}(0) = \dot{x}) \dot{x} f(v,\dot{x})\, d\dot{x}, \tag{4.19}$$

since, for $u \leq v$, $P(Y(0) < u | X(0) = v, \dot{X}(0) = \dot{x}) = 0$ for $\dot{x} < 0$.

As we have mentioned before, the sea level elevation $Z(t)$ is often modelled as a Gaussian process. The still water level $m = \mathsf{E}(Z(t)) = 0$ the significant wave height H_s defines $\lambda_0 = H_s^2/16$ while the average wave period $T_z = \hat{\lambda}/2\pi$. Actually the two parameters H_s and T_z are often the only information about the spectral density that is available. Since the intensity of waves with crest $A_c > v$ is equal to $\mu^{osc}(m,v)$ and hence one could use (4.19) to compute the intensity. The formula (4.19) is difficult to evaluate, since $Y(t)$ is a non-Gaussian process and that the conditional density of $Y(0)$ given probability

$Z(0), \dot{Z}(0)$ is not known. Very accurate approximations, based on numerical integrations do exists. For example program `spec2tpdf`, given in WAFO, computes the density $f_h(t)$, say, of T_z for waves with crest $A_c > h$, such that $P(A_c > h) = \int f_h(t)\,dt$, see Fig. 10 for definition of A_c, T_c. Consequently $\mu^{osc}(m, m+h) = \frac{1}{T_z} \int f_h(t)\,dt$.

Example 14. Consider a Gaussian sea $X(t)$ having the so-called Jonswap spectrum, see Hasselmann *et al.* (1973), with parameters $H_s = 7\,\mathrm{m}$ and $T_z = 8.5\,\mathrm{s}$ ($T_p = 11\,\mathrm{s}$). The still water level is zero, $m = 0$. In Fig. 16 we present the four densities $f_h(t)$ for thresholds $h = 0, 1, 2, 3$ meters. Note that for $h_1 > h_2$ we have that $f_{h_1}(t) \leq f_{h_2}(t)$ and hence $f_0(t)$, which is just the density of T_c is the highest curve. The densities are computed using WAFO. For example the $f_2(t)$ density can be computed and plotted using the following sequence of commands

```
f2=spec2tpdf(jonswap,0,[],[0 10 101],2,3); clf; pdfplot(f2)
```

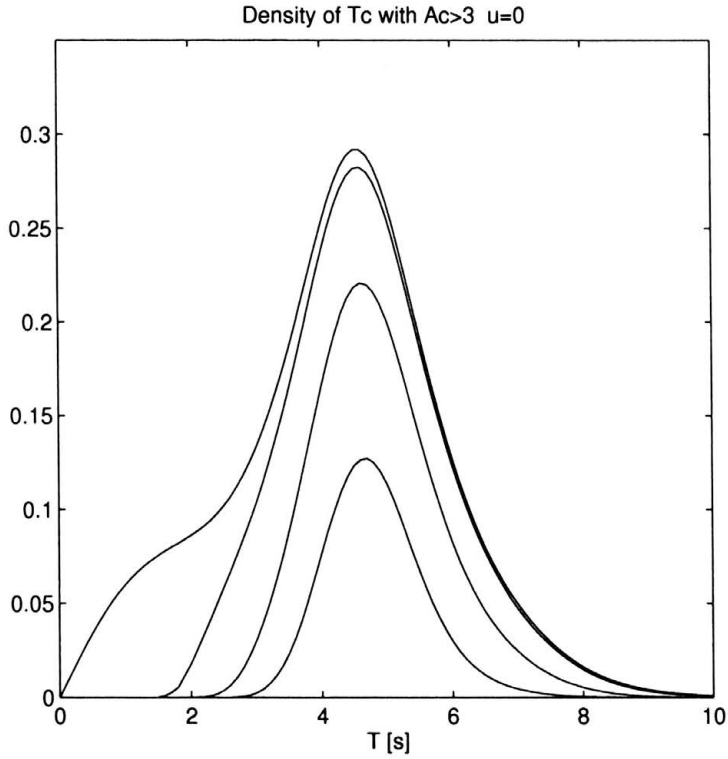

FIGURE 16. The density of T_c for waves with crest A_c above a threshold $h = 0, 1, 2, 3\,\mathrm{m}$ for a Gaussian sea with zero still water level $m = 0$, and Jonswap spectrum $H_s = 7\,\mathrm{m}$ and $T_z = 8.5\,\mathrm{s}$.

Finally, we compare the probabilities $P(A_c > h) = \int f_h(t)\,dt$ with the upper bound (3.4) for the probability $(e^{-8\left(\frac{h}{H_s}\right)^2})$. For $h = 1, 2, 3$ the integrals give the following probabilities 0.806, 0.510, 0.228 while the bound (3.4) gives 0.849, 0.521, 0.230, respectively. Note that for non-Gaussian sea the Rayleigh bound is unconservative, see Fig. 11.

5. Lecture V: Marks on contours

Stochastic processes serve as probability models of phenomena observed in some continuum, for example in time or in space or sometimes in both. It is often assumed that observed realizations produced by a model or, as they are often called, *sample paths* contain all information about the model (ergodic property). Vice versa, the theoretical model can provide with formulas for the statistical distributions extracted from sample paths. However, the relation between the sample path distributions and the theoretical distributions describing the model is not always straightforward. For example, complications can arise from the effect of *sampling bias*. The name refers to a change of sample distribution for the same quantity due to a different method of collecting its values.

Let first illustrate the concept of unbiased sampling distributions. Suppose $\mathbf{Y}(\mathbf{q})$ is a stationary vector valued random field. The probability that this field has a (measurable) property A is given by $P(\mathbf{Y}(\mathbf{0}) \in A)$. This probability has a natural statistical interpretation through the ergodic theorem which is stated below in the special case when $\mathbf{q} \in R^2$.

Let $X_{ij} = 1$ if $\mathbf{Y}(i,j) \in A$ and zero otherwise. If at least one of the integers N and K diverges to infinity, then the sequence of random variables

$$\frac{1}{NK}\sum_{i=1}^{N}\sum_{j=1}^{K} X_{ij},$$

converges with probability one to a random variable V such that $\mathsf{E}(V) = P(\mathbf{Y}(\mathbf{q}) \in A)$. In the special ergodic case, the limiting variable V is constant and equal to $P(\mathbf{Y}(\mathbf{q}) \in A)$. Because the distribution is obtained by sampling at non-random and equally spaced points, it is referred to as the *unbiased sampling distribution.*

5.1. Distributions arising from biased sampling

As we have discussed in the previous sections one is often interested in long run probabilities of $(\mathbf{Y}(t), \dot{Z}(t)) \in A$ for random locations t defined implicitly as the solution to an equation $Z(t) = u$. We have shown that, usually,

the long-run distribution differ from the probability $P((\mathbf{Y}(0), \dot{Z}(0)) \in A)$. Simply by observing $\mathbf{Y}(t), \dot{Z}(t)$ at random times t may affect the distribution of $\mathbf{Y}, \dot{Z}$.

An analogous question is what is the long-run probability of $(\mathbf{Y}(\mathbf{q}), \dot{\mathbf{Z}}(\mathbf{q})) \in A$ for random locations $\mathbf{q}$ defined implicitly as solutions to an equation $\mathbf{Z}(\mathbf{q}) = \mathbf{u}$. In order to discuss such a "biased" sampling distribution, first we should answer the question of how many points $\mathbf{q}$ satisfying $\mathbf{Z}(\mathbf{q}) = \mathbf{u}$ reside in a bounded domain. We turn to this problem next.

Suppose $\mathbf{Z}(\mathbf{q})$, $\mathbf{q} \in R^k$, is taking values in R^n. Assume that $n \leq k$ and from now on treat them as fixed. Let $\mathcal{V}$ be the relative volume in R^k of the dimension $k - n$. For example if $k = 3$, then $\mathcal{V}$ measures the length of a set if $n = 2$, the area if $n = 1$, and $\mathcal{V}$ simply counts points in a set in the special case $n = 3$. For $\mathbf{S} \subset R^k$ define the contour

$$\mathcal{C}_{\mathbf{S}}(\mathbf{u}) = \{\mathbf{q} \in \mathbf{S} : \mathbf{Z}(\mathbf{q}) = \mathbf{u}\} \quad \text{and let} \quad \mathcal{C}_{\mathbf{u}} = \mathcal{C}_{[0,1]^k}(\mathbf{u}). \tag{5.1}$$

Using this notation, define the intensity

$$\mu(\mathbf{u}) = \mathsf{E}\left[\mathcal{V}\left(\mathcal{C}_{\mathbf{u}}\right)\right]. \tag{5.2}$$

We called $\mu(\mathbf{u})$ an intensity since, by homogeneity,

$$\mathsf{E}\left[\mathcal{V}\left(\mathcal{C}_{\mathbf{S}}(\mathbf{u})\right)\right] = |\mathbf{S}|\mu(\mathbf{u}),$$

where $|\mathbf{S}|$ is the volume (size) of $\mathbf{S}$. In a similar way as in the previous section where A was a statement about $\mathbf{Y}(t), \dot{Z}(t)$, let here A be a statement about the outcome of a vector $\mathbf{Y}(\mathbf{q})$ and the matrix $\dot{\mathbf{Z}}(\mathbf{q})$, then

$$\mu(A|\mathbf{u}) = \mathsf{E}\left[\mathcal{V}\left\{\mathbf{q} \in \mathcal{C}_{\mathbf{u}} : (\mathbf{Y}(\mathbf{q}), \dot{\mathbf{Z}}(\mathbf{q})) \in A\right\}\right]. \tag{5.3}$$

Finally the long-run distribution $P((\mathbf{Y}(\mathbf{q}), \dot{\mathbf{Z}}(\mathbf{q})) \in A|\mathbf{q} \in \mathcal{C}_{\mathbf{u}})$ is defined as follows

$$P\big((\mathbf{Y}(\mathbf{q}), \dot{\mathbf{Z}}(\mathbf{q})) \in A|\mathbf{q} \in \mathcal{C}_{\mathbf{u}}\big) = \frac{\mu(A|\mathbf{u})}{\mu(\mathbf{u})}.$$

The significance of the long run distribution follows from its interpretation, which is along the same argument as in the one-dimensional case. The distribution can be defined as the average size of the part of $\mathcal{C}_{\mathbf{u}}$ on which statement A about $\mathbf{Y}(\mathbf{q})$ and $\dot{\mathbf{Z}}(\mathbf{q})$ is true divided by the average size of the entire $\mathcal{C}_{\mathbf{u}}$. By the ergodic theorem, the so-defined distribution coincides with the limiting statistical distribution of $\mathbf{Y}(\mathbf{q})$, $\dot{\mathbf{Z}}(\mathbf{q})$, when $\mathbf{q}$ is sampled on the contour $\mathcal{C}_{\mathbf{S}}(\mathbf{u})$ in the region $\mathbf{S} = [0, X_1] \times [0, X_2] \times [0, X_3]$ for large X_1 or X_2, X_3, if $k = 3$.

Example 15. The sampling interpretation in the multivariate case is illustrated using the model of sea surface $W(\mathbf{q},t) = W(x,y,t)$, which is discussed in the Appendix. The time variable t is considered to be fixed and thus by homogeneity it can be set to zero. Suppose that we are interested in the distributions of $\mathbf{V}^{gr}$ the velocity in the gradient direction. The velocity $\mathbf{V}^{gr}(\mathbf{q})$, presented as vectors in Fig. 17, is defined as follows

$$\mathbf{V}^{gr}(\mathbf{q}) = \begin{bmatrix} W_x(\mathbf{q},0) & W_y(\mathbf{q},0) \\ -W_y(\mathbf{q},0) & W_x(\mathbf{q},0) \end{bmatrix}^{-1} \begin{bmatrix} -W_t(\mathbf{q},0) \\ 0 \end{bmatrix}. \tag{5.4}$$

In Fig. 17(a), an unbiased sample of velocities recorded on the entire field is presented, that could be used to estimate $P(\mathbf{V}^{gr}(\mathbf{q}) \in A)$. In Fig. 17(b), an example of a biased sample is presented by velocities sampled along the contour $\mathcal{C}_{\mathbf{S}}(0) = \{\mathbf{q} \in \mathbf{S} : W(\mathbf{q},0) = 0\}$, $\mathbf{S} = [0,50] \times [100,150]$ meters, of the fixed (zero) sea level. The sample distribution of the velocity vectors obtained along this contour represents (approximately if the area is large enough) the biased sampling distribution that could be used to estimate $P(\mathbf{V}^{gr}(\mathbf{q}) \in A | \mathbf{q} \in \mathcal{C}_0)$. The two distributions of $\mathbf{V}^{gr}$ can be different.

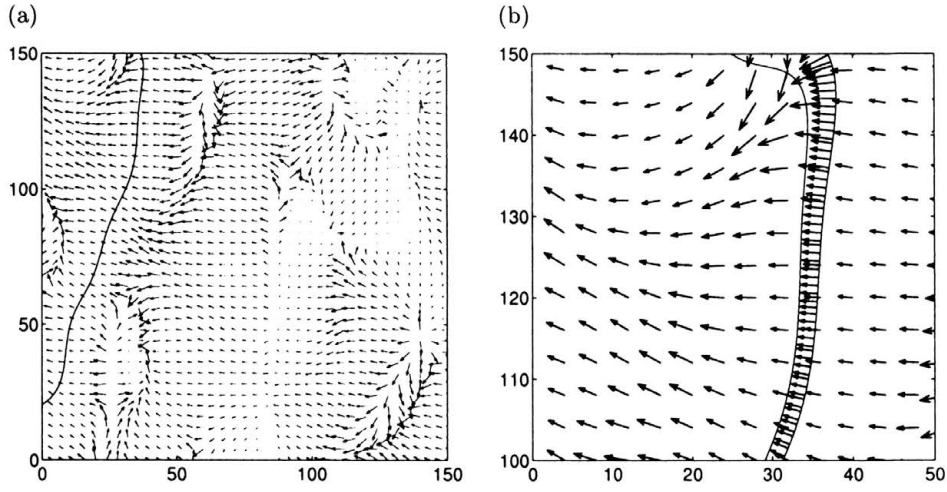

FIGURE 17. (a) Unbiased sample of velocities on the sea surface. (b) Biased sample of velocities along the level crossing contours vs. unbiased sample. Here the level crossing contour is presented at the initial time 0 and then at the time dt. (The scale of axes on both the pictures is in meters, velocities are rescaled for clarity.)

We turn now to the formulas for the intensities $\mu(\mathbf{u}), \mu(A|\mathbf{u})$ which are given by means of the generalized Rice's formula. We observe that vast literature is available on the various generalizations of the Rice formula, see:

Brillinger (1972), Marcus (1977), Adler (1981), Leadbetter *et al.* (1983), Zähle (1984), to mention just a few of many presentations.

Suppose that the joint density of $\mathbf{Y}$, $\mathbf{Z}$, $\dot{\mathbf{Z}}$ is available, which is always the case in this presentation, then

$$\mu(A|\mathbf{u}) = \int P\big((\mathbf{Y}(\mathbf{0}),\, \dot{\mathbf{Z}}(\mathbf{0})) \in A|\dot{\mathbf{Z}}(\mathbf{0}) = \dot{\mathbf{z}}, \mathbf{Z}(\mathbf{0}) = \mathbf{u}\big)$$

$$\cdot f_{\dot{\mathbf{Z}},\mathbf{Z}}(\dot{\mathbf{z}}, \mathbf{u})\sqrt{\det(\dot{\mathbf{z}}\dot{\mathbf{z}}^T)}\, d\dot{\mathbf{z}} = \int_A f_{\mathbf{Y},\dot{\mathbf{Z}},\mathbf{Z}}(\mathbf{y}, \dot{\mathbf{z}}, \mathbf{u}) \cdot \sqrt{\det(\dot{\mathbf{z}}\dot{\mathbf{z}}^T)}\, d\dot{\mathbf{z}}\, d\mathbf{y}. \quad (5.5)$$

Remark 7. The formula (5.5) is a generalization of the formula (4.4). Since, if $k = 1$ then $\dot{\mathbf{z}} = \dot{z}$ is a real number and hence $\sqrt{\det(\dot{\mathbf{z}}\dot{\mathbf{z}}^T)} = |\dot{z}|$.

By taking $A = R^n$ in (5.5) we derive the following formula for intensity $\mu(\mathbf{u})$

$$\mu(\mathbf{u}) = \int f_{\dot{\mathbf{Z}},\mathbf{Z}}(\dot{\mathbf{z}}, \mathbf{u}) \cdot \sqrt{\det(\dot{\mathbf{z}}\dot{\mathbf{z}}^T)}\, d\dot{\mathbf{z}}. \quad (5.6)$$

In the following examples we give explicit formulas for $\mu(\mathbf{u})$. Both cases were studied by Longuet–Higgins (1957). In the first example $\mathcal{C}_u$ is a line in $[0,1]^2$ $(k = 2, n = 1)$.

Example 16. Let $W(\mathbf{q})$, $\mathbf{q} = (x, y)$, be a homogenous Gaussian field with mean m and variance $\mathsf{V}(W(\mathbf{q})) = \lambda_{00}$, see Appendix for the definition of the spectral moments λ_{ij}.

The homogeneity of the field $W(x, y)$ does not imply that it is invariant under rotation. In fact a rotation affects its structure. For example let $W_A(\mathbf{q}) = W(A\mathbf{q}^T)$, where A is a 2×2 rotation matrix. Then the covariance matrix of the partial derivatives of $W_A(\mathbf{q})$ differs from the covariance matrix of the derivatives of $W(\mathbf{q})$. Actually one can rotate the plane in such a way that the partial spatial derivatives of the field are uncorrelated. We choose the rotation so that the derivative in the x-direction has larger variance than the one in the y-direction (the x-direction corresponds to the vector with the largest eigenvalue in the covariance matrix of the spatial derivatives). Since the length of a contour is independent of the rotation we choose to study the length of a contour in the rotated coordinate system. In particular we have

$$\mathsf{V}(W_x(\mathbf{q})) = \lambda_{20} \geq \lambda_{02} = \mathsf{V}(W_y(\mathbf{q})),$$

which for Gaussian fields, implies that the intensity of zero-crossings along any line passing through the origin attains its maximum on the x-axis and its minimum on the y-axis. (In the degenerate case when W is a model for the so-called long-crested sea we have that $\lambda_{02} = 0$ which means that there are no zero-crossings along the y-axis.)

Let us define by $Z(\mathbf{q}) = W(\mathbf{q})$, then the gradient vector is given by

$$\dot{Z}(\mathbf{q}) = (\dot{Z}_1(\mathbf{q}), \dot{Z}_2(\mathbf{q})) = (W_x(\mathbf{q}), W_y(\mathbf{q}))$$

which is a zero-mean Gaussian vector with covariance matrix

$$\Sigma_{\dot{Z}\dot{Z}} = \begin{bmatrix} \lambda_{20} & 0 \\ 0 & \lambda_{02} \end{bmatrix}. \tag{5.7}$$

Since, for stationary Gaussian field the value of the field $Z(\mathbf{q})$ is independent of the gradient vector $\dot{Z}(\mathbf{q})$ we have that the average length of the contour $\mathcal{C}_u$, given by (5.6), can be computed as follows

$$\begin{aligned}
\mu(u) &= \int f_{\dot{Z},Z}(\dot{\mathbf{z}}, u) \cdot \sqrt{\det(\dot{\mathbf{z}}\dot{\mathbf{z}}^T)}\, d\dot{\mathbf{z}} \\
&= \int f_{\dot{Z}}(\dot{\mathbf{z}}) f_Z(u) \cdot \sqrt{\dot{z}_1^2 + \dot{z}_2^2}\, d\dot{\mathbf{z}} \\
&= \frac{1}{\sqrt{2\pi\lambda_{00}}} e^{-\frac{(u-m)^2}{2\lambda_{00}}} \mathsf{E}\left(\sqrt{\dot{Z}_1(\mathbf{0})^2 + \dot{Z}_2(\mathbf{0})^2}\right).
\end{aligned}$$

Now, with $1/\gamma = \sqrt{\lambda_{20}/\lambda_{02}}$, which is often used as a measure of long-crestnedness of the sea ($\gamma = 0$ implies that all waves are propagating along the x-axis), we get

$$\mathsf{E}\left(\sqrt{\dot{Z}_1(\mathbf{0})^2 + \dot{Z}_2(\mathbf{0})^2}\right) = \sqrt{\frac{2(\lambda_{20} + \lambda_{02})}{\pi(1+\gamma^2)}} E(\sqrt{1-\gamma^2}),$$

where $E(k)$ is Legendre's elliptic integral of the first kind:

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2(\alpha)}\, d\alpha,$$

see Longuet–Higgins (1957, p. 346) for the proof.

We turn now to the case when the contour contains only isolated points, for example positions of local extremes of a field ($k = 2, n = 2$).

Example 17. In this example we shall discuss the average number of specular points of the field $W(\mathbf{q}) = W(x, y)$, satisfying the assumptions of the previous example. The point $\mathbf{q}$ is called a specular point if $W_x(\mathbf{q}) = u_1$ and $W_y(\mathbf{q}) = u_2$ and hence, for a fixed $\mathbf{u} = (u_1, u_2)$, the contour $\mathcal{C}_\mathbf{u}$ contains only isolated points. Let us compute the intensity of specular points, i.e. how many, in average, there are $\mathbf{q} \in [0,1]^2$ such that $W_x(\mathbf{q}) = u_1$ and $W_y(\mathbf{q}) = u_2$.

In order to use (5.6) define $\mathbf{Z}(\mathbf{q}) = (W_x(\mathbf{q}), W_y(\mathbf{q}))$, then $\dot{\mathbf{Z}}(\mathbf{q})$ is the following matrix

$$\dot{\mathbf{Z}}(\mathbf{q}) = \begin{bmatrix} W_{xx}(\mathbf{q}) & W_{xy}(\mathbf{q}) \\ W_{yx}(\mathbf{q}) & W_{yy}(\mathbf{q}) \end{bmatrix}.$$

Since the vector $\mathbf{Z}(\mathbf{q})$ is independent of the matrix $\dot{\mathbf{Z}}(\mathbf{q})$ we have that the average size of $\mathcal{C}_{\mathbf{u}}$ is given by (5.6):

$$\begin{aligned} \mu(\mathbf{u}) &= \int f_{\dot{\mathbf{Z}},\mathbf{Z}}(\dot{\mathbf{z}}, \mathbf{u}) \cdot \sqrt{\det(\dot{\mathbf{z}}\dot{\mathbf{z}}^T)}\, d\dot{\mathbf{z}} \\ &= \int f_{\dot{\mathbf{Z}}}(\dot{\mathbf{z}}) f_{\mathbf{Z}}(\mathbf{u}) |\det(\dot{\mathbf{z}})|\, d\dot{\mathbf{z}} \\ &= \frac{1}{2\pi\sqrt{\lambda_{20}\lambda_{02}}} e^{-\frac{(u_1)^2}{2\lambda_{20}} - \frac{(u_2)^2}{2\lambda_{02}}} \mathsf{E}\left(|\det(\dot{\mathbf{Z}}(\mathbf{0}))|\right). \end{aligned}$$

Note that we use the coordinate system where $W_x(\mathbf{q}), W_y(\mathbf{q})$ are independent. The computation of the expectation $\mathsf{E}\left(|\det(\dot{\mathbf{Z}}(\mathbf{0}))|\right)$ is quite lengthy and a full account of the derivation can be found in Longuet–Higgins (1957, pp. 351-353). Here, for reasons of completeness, we shall give the final step. Consider the matrix

$$\mathbf{H} = \begin{bmatrix} \frac{1}{2}\lambda_{22} & -\lambda_{31} & \frac{1}{2}\lambda_{40} \\ \frac{1}{2}\lambda_{13} & -\lambda_{22} & \frac{1}{2}\lambda_{31} \\ \frac{1}{2}\lambda_{04} & -\lambda_{13} & \frac{1}{2}\lambda_{22} \end{bmatrix}, \tag{5.8}$$

and let $l = eig(\mathbf{H})$, with $l_3 \leq l_2 < 0 < l_1$. Then

$$\mathsf{E}\left(|\det(\dot{\mathbf{Z}}(\mathbf{0}))|\right) = \frac{4}{\pi}\sqrt{l_2 l_3}\left(\sqrt{\frac{l_2 - l_1}{l_2}}E(\kappa, \frac{\pi}{2}) - \sqrt{\frac{l_2}{l_2 - l_1}}F(\kappa, \frac{\pi}{2})\right), \tag{5.9}$$

where $\kappa^2 = \frac{l_1}{l_3}\frac{l_3 - l_2}{l_1 - l_2}$ and K and F are the Legendre elliptic integrals of first and second kind.

Finally, since the local extremes are the specular points for $\mathbf{u} = \mathbf{0}$ and since, in average, each fourth extreme point is a local maximum we have the following important result: intensity of local maxima of the field $W(x, y)$ is given by

$$\mu = \frac{1}{2\sqrt{\lambda_{20}\lambda_{02}}}\frac{1}{\pi^2}\sqrt{l_2 l_3}\left(\sqrt{\frac{l_2 - l_1}{l_2}}E(\kappa, \frac{\pi}{2}) - \sqrt{\frac{l_2}{l_2 - l_1}}F(\kappa, \frac{\pi}{2})\right). \tag{5.10}$$

5.2. Distributions on contours in R^2, $k = 2$ and $n = 1$.

Now we return to the problems discussed in the Lecture IV namely distribution of variables observed at crossing times. The difference is that now the crossings are no longer isolated points but contour lines. The formulas are still quite similar in the structure but more complicated since the derivative at crossing is a gradient vector. We shall prove a Slepian type lemma that is an extension of Lemma 2. The results will be then used to study velocities at crossings of moving fields.

Suppose that $Z(\mathbf{q})$, $\mathbf{q} = (x, y) \in R^2$, is a homogenous Gaussian field, then the derivative

$$\dot{Z}(\mathbf{q}) = (\dot{Z}_1(\mathbf{q}), \dot{Z}_2(\mathbf{q}))$$

(the gradient) has zero-mean Gaussian components. Let $\mathbf{Y}(\mathbf{q}) \in R^m$ be a Gaussian vector-valued field. For a fixed u, consider the contour

$$\mathcal{C}_u = \{(x, y) \in [0, 1]^2 : Z(x, y) = u\}.$$

We begin by studying of the distribution of $\mathbf{Y}(\mathbf{q}), \dot{Z}(\mathbf{q})$ on the contour $\mathcal{C}_u$. As in the formula (4.8), we shall now define the variables $\mathbf{Y}_u, \dot{Z}_u$ which have the distribution

$$P(\mathbf{Y}_u \leq \mathbf{y}, \dot{\mathbf{Z}}_u \leq \dot{\mathbf{z}}) = P(\mathbf{Y}(\mathbf{q}) \leq \mathbf{y}, \dot{Z}(\mathbf{q}) \leq \dot{\mathbf{z}} | \mathbf{q} \in \mathcal{C}_u). \tag{5.11}$$

In the following lemma we shall consider only the case when $n = 1$, i.e. $\mathbf{Y}(\mathbf{q})$ is a scalar valued field denoted by $Y(\mathbf{q})$. The more general case for $n > 1$ can be treated in the similar way. Some of the distributions, presented in the following lemma, were already given by Longuet–Higgins (1957), see also Adler (1981).

Lemma 4: *Suppose $Z(\mathbf{q}), \dot{Z}(\mathbf{q}), Y(\mathbf{q})$ is a vector-valued (in R^4) stationary Gaussian field. Let $m_Z = \mathsf{E}(Z(\mathbf{q}))$, $m_Y = \mathsf{E}(Y(\mathbf{q}))$ and let the covariance matrix Σ of the vector $Z(\mathbf{0}), \dot{Z}(\mathbf{0}), Y(\mathbf{0})$ be given by*

$$\Sigma = \begin{bmatrix} \sigma_Z^2 & \Sigma_{Z\dot{Z}} & \sigma_Z \sigma_Y \rho_{ZY} \\ \Sigma_{Z\dot{Z}}^T & \Sigma_{\dot{Z}\dot{Z}} & \Sigma_{Y\dot{Z}}^T \\ \sigma_Z \sigma_Y \rho_{ZY} & \Sigma_{Y\dot{Z}} & \sigma_Y^2 \end{bmatrix}, \tag{5.12}$$

where $\Sigma_{Z\dot{Z}}, \Sigma_{Y\dot{Z}}$ are (1,2)-matrices containing covariances between $Z(\mathbf{0})$, $Y(\mathbf{0})$ and $\dot{Z}(\mathbf{0})$, respectively, while $\Sigma_{\dot{Z}\dot{Z}}$ is the covariance matrix of the vector $\dot{Z}(\mathbf{0})$. For $-\pi < \beta \leq \pi$, let $\mathbf{n}(\beta) = (\cos(\beta), \sin(\beta))$ and

$$\mathbf{s}(\beta) = \left(\mathbf{n}(\beta)\Sigma_{\dot{Z}\dot{Z}}^{-1}\mathbf{n}(\beta)^T\right)^{-1/2}\mathbf{n}(\beta)$$

be a vector parallel to $\mathbf{n}(\beta)$ *with length* $s(\beta) = \left(\mathbf{n}(\beta)\Sigma_{\dot{Z}\dot{Z}}^{-1}\mathbf{n}(\beta)^T\right)^{-1/2}$. *The variables* $Y_u, \dot{\mathbf{Z}}_u$ *with distribution defined by (5.11), have the following representation*

$$\begin{aligned} \dot{\mathbf{Z}}_u &= R\,\mathbf{s}(\beta_u), \\ Y_u &= m_u + \Sigma_{Y\dot{Z}}\Sigma_{\dot{Z}\dot{Z}}^{-1}\dot{\mathbf{Z}}_u + \sigma\,U = m_u + m(\beta_u)s(\beta_u)\,R + \sigma\,U, \end{aligned} \tag{5.13}$$

where R, β_u, U *are independent random variables with the following densities:*

$$f_{\beta_u}(\beta) = c\,s(\beta)^3, \qquad f_R(r) = \sqrt{\frac{2}{\pi}}r^2e^{-r^2/2}, r \geq 0, \tag{5.14}$$

$-\pi < \beta \leq \pi$ *and* c *is the normalization constant, while* $U \in N(0,1)$. *Furthermore,*

$$\begin{aligned} m_u &= m_Y + \frac{\sigma_Y}{\sigma_Z}\rho_{ZY}(u - m_Z) \\ m(\beta) &= \Sigma_{Y\dot{Z}}\Sigma_{\dot{Z}\dot{Z}}^{-1}\mathbf{n}(\beta)^T \\ \sigma^2 &= \sigma_Y^2(1-\rho_{ZY}^2) - \Sigma_{Y\dot{Z}}\Sigma_{\dot{Z}\dot{Z}}^{-1}\Sigma_{Y\dot{Z}}^T. \end{aligned}$$

Proof. We need to demonstrate that

$$P(Y(\mathbf{q}) \leq y, \dot{Z}(\mathbf{q}) \leq \mathbf{v} | \mathbf{q} \in \mathcal{C}_u) = \frac{\mu(A|u)}{\mu(u)},$$

(where $A =$"$Y(\mathbf{q}) \leq y$ and $\dot{Z}(\mathbf{q}) \leq \mathbf{v}$"), is equal to

$$P\left(m(\beta_u)s(\beta_u)\,R + \sigma\,U \leq y - m_u,\ \ R\,\mathbf{s}(\beta_u) \leq \mathbf{v}\right).$$

We begin by computing the intensity $\mu(A|u)$, i.e. the intensity of points $\mathbf{q}$, such that $Z(\mathbf{q}) = u$ and A is true. From formula (5.5) with $\dot{\mathbf{z}} = (\dot{z}_1, \dot{z}_2)$, we get

$$\sqrt{\det(\dot{\mathbf{z}}\dot{\mathbf{z}}^T)} = \sqrt{\dot{z}_1^2 + \dot{z}_2^2} = |\dot{\mathbf{z}}|,$$

and hence

$$\begin{aligned} \mu(A|u) &= \int_{-\infty}^{+\infty} P(Y(\mathbf{0}) \leq y, \dot{Z}(\mathbf{0}) \leq \mathbf{v} \mid Z(\mathbf{0}) = u, \dot{Z}(\mathbf{0}) = \dot{\mathbf{z}})|\dot{\mathbf{z}}|f(u,\dot{\mathbf{z}})\,d\dot{\mathbf{z}} \\ &= \int_{-\infty}^{+\infty} P(Y(\mathbf{0}) \leq y, \dot{\mathbf{z}} \leq \mathbf{v} \mid Z(\mathbf{0}) = u, \dot{Z}(\mathbf{0}) = \dot{\mathbf{z}})|\dot{\mathbf{z}}|f(u,\dot{\mathbf{z}})\,d\dot{\mathbf{z}} \\ &= \int_{-\infty}^{\mathbf{v}} P(Y(\mathbf{0}) \leq y \mid Z(\mathbf{0}) = u, \dot{Z}(\mathbf{0}) = \dot{\mathbf{z}})|\dot{\mathbf{z}}|f(u,\dot{\mathbf{z}})\,d\dot{\mathbf{z}}. \end{aligned}$$

Now we know, see Appendix, that the conditional density of $Y(\mathbf{0})$ given $Z(\mathbf{0}) = u, \dot{Z}(\mathbf{0}) = \dot{\mathbf{z}}$ is Gaussian with the mean

$$m(u, \dot{\mathbf{z}}) = m_Y + \frac{\sigma_Y}{\sigma_Z}\rho_{ZY}(u - m_Z) + \Sigma_{Y\dot{Z}}\Sigma_{\dot{Z}\dot{Z}}^{-1}\dot{\mathbf{z}}^T = m_u + \Sigma_{Y\dot{Z}}\Sigma_{\dot{Z}\dot{Z}}^{-1}\dot{\mathbf{z}}^T.$$

Here we have used the fact that $Z(\mathbf{0})$ is independent of $\dot{Z}(\mathbf{0})$ and that $\mathsf{E}(\dot{Z}(\mathbf{0})) = \mathbf{0}$. We turn now to the conditional variance σ^2, say, which is given by

$$\sigma^2 = \sigma_Y^2(1 - \rho_{ZY}^2) - \Sigma_{Y\dot{Z}}\Sigma_{\dot{Z}\dot{Z}}^{-1}\Sigma_{Y\dot{Z}}^T.$$

Let $f(u) = \frac{1}{\sqrt{2\pi}\sigma_Z}e^{-(u-m_Z)^2/2\sigma_Z^2}$ be the density of $Z(\mathbf{0})$ while the density $f(\dot{\mathbf{z}})$ of $\dot{Z}(\mathbf{0})$ is given by

$$f(\dot{\mathbf{z}}) = \frac{1}{2\pi\sqrt{\det(\Sigma_{\dot{Z}\dot{Z}})}}e^{-\frac{1}{2}\dot{\mathbf{z}}\Sigma_{\dot{Z}\dot{Z}}^{-1}\dot{\mathbf{z}}^T}.$$

Now, since $Z(\mathbf{0})$ and $\dot{Z}(\mathbf{0})$ are independent we have

$$\begin{aligned}\mu(A|u) &= f(u)\int_{-\infty}^{\mathbf{v}} \Phi\left(\frac{y - m(u, \dot{\mathbf{z}})}{\sigma}\right)|\dot{\mathbf{z}}|f(\dot{\mathbf{z}})\,d\dot{\mathbf{z}} \\ &= f(u)\int_{-\infty}^{\infty} P\left(m_u + \Sigma_{Y\dot{Z}}\Sigma_{\dot{Z}\dot{Z}}^{-1}\dot{\mathbf{z}}^T + \sigma U < y,\ \dot{\mathbf{z}} \le \mathbf{v}\right)|\dot{\mathbf{z}}|f(\dot{\mathbf{z}})\,d\dot{\mathbf{z}}.\end{aligned}$$

Let us write the gradient vector $\dot{\mathbf{z}} = (\dot{z}_1, \dot{z}_2)$ in polar coordinates with $r^2 = \dot{z}_1^2 + \dot{z}_2^2$, $r \ge 0$, and $\cos(\beta) = \dot{z}_1/r$, $\sin(\beta) = \dot{z}_2/r$, hence $\dot{\mathbf{z}} = r\mathbf{n}(\beta)$. Using the introduced variables the last integral can be rewritten as follows

$$\begin{aligned}\mu(A|u) &= f(u)\int_{-\infty}^{\infty} P\left(m_u + r\Sigma_{Y\dot{Z}}\Sigma_{\dot{Z}\dot{Z}}^{-1}\mathbf{n}(\beta)^T + \sigma U < y,\ r\mathbf{n}(\beta) \le \mathbf{v}\right) rf(\dot{\mathbf{z}})\,d\dot{\mathbf{z}} \\ &= cf(u)\int_{\pi}^{\pi}\int_{0}^{\infty} P\left(m_u + rm(\beta) + \sigma U < y,\ r\mathbf{n}(\beta) \le \mathbf{v}\right) r^2 e^{-\frac{r^2}{2s(\beta)^2}}\,dr\,d\beta \\ &= cf(u)\int_{\pi}^{\pi}\int_{0}^{\infty} P\left(r\mathbf{n}(\beta) \le \mathbf{v},\ m_u + rm(\beta) + \sigma U < y\right) s(\beta)^3 \\ &\qquad \cdot\sqrt{2/\pi}\frac{r^2}{s(\beta)^3}e^{-\frac{r^2}{2s(\beta)^2}}\,dr\,d\beta \\ &= cf(u)\int_{\pi}^{\pi} P\left(R\mathbf{s}(\beta) \le \mathbf{v},\ m_u + Rs(\beta)m(\beta) + \sigma U < y\right) s(\beta)^3\,d\beta,\end{aligned}$$

where c is a constant which may change values between the lines. Now, it is easy to see that $f_{\beta_u}(\beta) = cf(u)s(\beta)^3/\mu(u)$ and the lemma is proved. □

Denote by $|\dot{\mathbf{Z}}_u|$ the length of the vector $\dot{\mathbf{Z}}_u$, i.e. $|\dot{\mathbf{Z}}_u| = \sqrt{\dot{\mathbf{Z}}_u \dot{\mathbf{Z}}_u^T}$, and let β_u be the angle between the x axis and the vector $\dot{\mathbf{Z}}_u$; obviously $\dot{\mathbf{Z}}_u = |\dot{\mathbf{Z}}_u|(\cos(\beta_u), \sin(\beta_u))$. Now it is a simple consequence of the lemma that β_u has the density $f_{\beta_u}(\beta)$ given by (5.14) while $|\dot{\mathbf{Z}}_u|$, conditionally that the angle $\beta_u = \beta$, has the density $s(\beta)R$, where the density of R is also specified in (5.14).

In the following subsection we shall use the lemma to compute the biased distribution of the velocity $\mathbf{V}^{gr}$ defined by (5.4), i.e. the probability $P(\mathbf{V}^{gr}(\mathbf{q}) \leq \mathbf{v} | \mathbf{q} \in \mathcal{C}_0)$.

5.2.1. Velocity of a level-contour. In this subsection we shall discuss velocities of level-contours, by which we mean velocities of points on contour. We assume that $W(x, y, t)$ is a smooth, stationary, Gaussian field with a known power spectral density function such that the spectral moments λ_{200}, λ_{020} and λ_{002}, which are the variances of the components in the gradient vector (W_x, W_y, W_t), respectively, are finite. We assume that the field is given in the coordinate system (x, y, t) so that the partial derivatives on x and y are independent, see Example 16 for detailed discussion.

Now at time $t = 0$, a contour $\mathcal{C}_u$ is considered as the boundary of an excursion set (possibly dangerous concentration of pollutants or region of high loads). We wish to study the velocity with which this contour moves. There is no unique approach to define dynamics of a random surface. One possibility is to define the velocity in the direction of gradient $\mathbf{V}^{gr}(\mathbf{q})$, see Baxevani *et al.* (2003) for alternative definitions of velocities, and then study the distribution of $\mathbf{V}^{gr}(\mathbf{q})$ for points $\mathbf{q}$ on the u-level contour. We shall employ Lemma 4 to derive this distribution.

Let us define $Z(\mathbf{q}) = W(\mathbf{q}, 0)$, $Y(\mathbf{q}) = W_t(\mathbf{q}, 0)$ and the gradient vector

$$\dot{Z}(\mathbf{q}) = (\dot{Z}_1(\mathbf{q}), \dot{Z}_2(\mathbf{q})) = (W_x(\mathbf{q}, 0), W_y(\mathbf{q}, 0))$$

which is a zero-mean Gaussian vector with the covariance matrix

$$\Sigma_{\dot{Z}\dot{Z}} = \begin{bmatrix} \lambda_{200} & 0 \\ 0 & \lambda_{020} \end{bmatrix}. \tag{5.15}$$

We also need the following covariances

$$\Sigma_{Z\dot{Z}} = [0\ 0], \quad \Sigma_{Y\dot{Z}} = [\lambda_{101}\ \lambda_{011}] \quad \sigma_Z^2 = \lambda_{000}, \quad \sigma_Y^2 = \lambda_{002}, \quad \rho_{YZ} = 0,$$

see Appendix for the definitions of spectral moments λ_{ijk}. Since $m_Z = 0$ and $u = 0$ the constants in Lemma 4 can now be written:

$$m_u = 0, \qquad s(\beta) = \frac{\sqrt{\lambda_{200}\lambda_{020}}}{\sqrt{\lambda_{020}\cos^2(\beta) + \lambda_{200}\sin^2(\beta)}},$$

$$m(\beta) = \frac{\lambda_{101}}{\lambda_{200}}\cos(\beta) + \frac{\lambda_{011}}{\lambda_{020}}\sin(\beta), \qquad \sigma^2 = \lambda_{002} - \frac{\lambda_{101}^2}{\lambda_{200}} - \frac{\lambda_{011}^2}{\lambda_{020}}.$$

Now, let V^{gr} be the speed of the velocity $\mathbf{V}^{gr}$, i.e. $\mathbf{V}^{gr} = V^{gr}\mathbf{n}(\beta)$. The speed can be computed as follows

$$V^{gr}(\mathbf{q}) = -\frac{Y(\mathbf{q})}{\sqrt{\dot{Z}_1(\mathbf{q})^2 + \dot{Z}_2(\mathbf{q})^2}}.$$

Since on the zero level contour, $\sqrt{\dot{Z}_1(\mathbf{q})^2 + \dot{Z}_2(\mathbf{q})^2}$ has the same distribution as the length of the vector $\dot{\mathbf{Z}}_u$ which is $s(\beta_u)R$, and $Y(\mathbf{q})$ as $Y_u = m(\beta_u)s(\beta_u)\,R + \sigma\,U$, the speed has the following distribution

$$V_u^{gr} = -\frac{Y_u}{s(\beta_u)R} = -m(\beta_u) + \frac{\sigma}{s(\beta_u)}\frac{U}{R} = -m(\beta_u) + \frac{\sigma}{s(\beta_u)}T, \qquad (5.16)$$

where $\sqrt{3}T$ is t-distributed with three degrees of freedom and is independent of β_u which has density (5.14). Note that the distribution of V_u^{gr} is independent of the level u.

Example 18. Let $W(x, y, t)$ be the sea surface observed at time t at the location $\mathbf{q} = (x, y)$. As it is common in oceanography we shall assume that $W(x, y, t)$ is a homogenous Gaussian field with mean zero and some directional power spectrum, see Baxevani *et al.* (2003) for definition of the spectrum. Let $u = 0$ then the contour represents the edges of the wave crests. The formula (5.16) gives us the means to study the velocity the crest moves.

For a typical spectrum, presented in Fig. 18(a) [the same as used to simulate velocities in Fig. 17], the contours of the density of (β_u, V_u^{gr}) are given in Fig. 18(b). From the plot of spectrum we can see that the main direction of wave propagation (most energy) is parallel to the x axis. This is the reason why the joint density has its highest values for $\beta_u = 0$. Next the directional spectrum is concentrated for azimuth close to angle 0 which means that waves travels from that direction and hence the speeds are negative. This can also be seen in Fig. 17. The velocity of individual waves has practical implications for wave surfing, both for pleasure and in analysis of the safety of vessels sailing in the following sea.

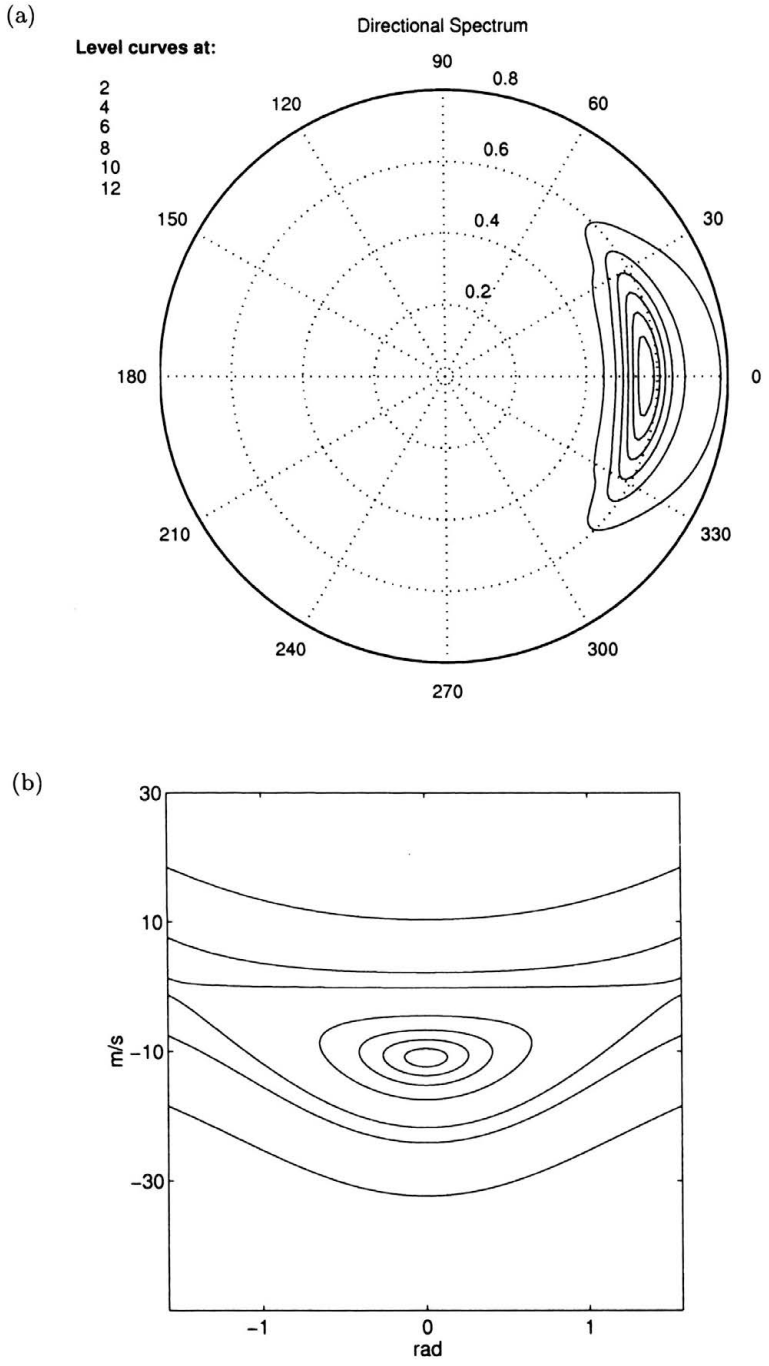

FIGURE 18. The directional spectrum with $H_s = 7$ meters (a), and contour lines of the joint density of velocity direction β_u and the speed V_u^{gr} (b).

Acknowledgements

These notes use material from papers written by Anastassia Baxevani, Georg Lindgren, Ulla Machado, Xavier Pitsonet, Krzysztof Podgórski, Jesper Rydén and myself. I would like also to thank Anastassia Baxevani for comments and careful reading of the manuscript.

References

1. ADLER, R.J., *The geometry of random fields*, Wiley, New York, 1981.
2. BAXEVANI, A., PODGÓRSKI, K. and RYCHLIK, I., Velocities for moving random surfaces. *Probabilistic Engineering Mechanics*, Vol.18, pp.251-271, 2003.
3. BREITUNG, K., Asymptotic crossing rates for stationary Gaussian vector processes, *Stochastic Processes and their Applications*, **29**, pp.195-207, 1988.
4. BRILLINGER, D.R., On the number of solutions of random equations, *Ann. Math. Statist.*, Vol.43, pp.534-540, 1972.
5. BRODTKORB, P.A., JOHANNESSON, P., LINGREN, G., RYCHLIK, I., RYDÉN, J., and SJÖ, E., WAFO – a Matlab toolbox for analysis of random waves and loads, *Proceedings of the 10th International Offshore and Polar Engineering Conference*, Vol.3, pp.343-350, 2000.
6. BROKATE, M., SPREKELS, J., *Hysteresis and Phase Transitions*, Springer Verlag, New York, 1996.
7. BULINSKAYA, E. V., On the mean number of crossings of a level by a stationary Gaussian process, *Theory Probab. Appl.*, Vol.6, pp.435-438, 1961.
8. BUTLER R., MACHADO, U. and RYCHLIK, I., Distribution of wave crests in non-Gaussian sea, *Proceedings of the 11th ISOPE conference, Honolulu*, 2003.
9. CRAMÉR H. and LEADBETTER M.R., *Stationary and Related Stochastic Processes*, Wiley, New York, 1967.
10. CROSSLAND, B., Effects of large hydrostatic pressures on the torsional fatigue strength of an alloy steel, in: *International conference on fatigue of metals*, London, 1956.
11. DITLEVSEN, O. and LINDGREN, G., Empty envelope excurssions in stationary Gaussian processes, *J. Sound Vib.*, Vol.122, pp.571–587, 1988.
12. DANIELS, H.E., Saddlepoint approximations in statistics, *Ann. Math. Statist.*, Vol.25, pp.631-50, 1954.
13. HASSELMANN, K., BARNETT, T.P., BOUWS, E., CARLSON, H., CARTWRIGHT, D.E., ENKE, K., EWING, J.A., GIENAPP, H., HASSELMANN, D.E., KRUSEMAN, P., MEERBURG, A., MÜLLER, P., OLBERS, D.J., RICHTER, K., SELL, W. and WALDEN, H., Measurements of Wind Wave Growth and Swell Decay During the Joint North Sea Wave Project (JONSWAP), *Deutsches Hydro. Zeit.*, Reich A12, pp.1-95, 1973.
14. JOHANNESSON, P., SVENSSON, T. and DE MARÉ, J., Fatigue life prediction based on variable amplitude tests-methodology, *Chalmers Tech. Rep.*, pp.1-22, 2003.

15. KAC, M., On the average number of real roots of a random algebraic equation, *Bull. Amer. Math. Soc.*, Vol.30, pp.1215-1228, 1943.

16. KREÉ, P. and SOIZE, C., *Mathematics of random phenomena. Random vibrations of mechanical structures*, Mathematics and its Applications, Dordrecht: Reidel, 1986.

17. LEADBETTER, M. R., On crossings of levels and curves by a wide class of stochastic processes, *Ann. Math. Statist.*, Vol.37, pp.260-267, 1966.

18. LEADBETTER, M.R., LINDGREN, G. and ROOTZÉN, H., *Extremes and Related Properties of Random Sequences and Processes*, Springer-Verlag, 1983.

19. LEADBETTER, M.R. and SPANIOLO, G.V., On whipping and wave-iduced stresses at slams, 1998.

20. LINDGREN, G. and RYCHLIK, I., Slepian models and regression approximations in crossings and extreme value theory, *International Statistical Review*, Vol.59, pp.195-225, 1991.

21. LONGUET–HIGGINS, M.S., The statistical analysis of a random, moving surface, *Phil. Trans. R. Soc. A*, Vol.249, pp.321-387, 1957.

22. MACHADO, U., A new approach for predicting the extreme response of structures, *Applied Ocean Research*, Vol.24, pp.321-330, 2002.

23. MACHADO, U.B. AND RYCHLIK, I., Wave statistics in nonlinear random sea, *Extremes*, Vol.6, pp.125–146, (2003).

24. MARCUS, M. B., Level crossings of a stochastic process with absolutely continuous sample paths, *The Annals of Probability*, Vol.5, pp.52-71, 1977.

25. MATSUISHI, M. and ENDO, T., Fatigue of metals subject to varying stress, Paper presented to: *Japan Soc. Mech. Engrs.*, Jukvoka, Japan, 1968.

26. MINER, M.A., Cumulative damage in fatigue, *J. Appl. Mech.*, Vol.12, pp.A159-A164, 1945.

27. PALMGREN, A., Die Lebensdauer von Kugellagern, *VDI Zeitschrift*, Vol.68, pp.339-341, 1924.

28. PAPADOPOULOS, I.V., DAVOLI, P., GORLA, C., FILLIPINI, M. and BERNASCONI, A., A comparative study of multiaxial high-cycle fatigue criteria for metals, *Int. J. Fatigue*, Vol.19, pp.219-235, 1997.

29. LORD RAYLEIGH, On resultant of a large number of vibrations of the same pitch and arbitrary phase, *Phil. Mag.*, Vol.10, pp.73-78, 1880.

30. RICE, S.O., The mathematical analysis of random noise, *Bell Syst. Techn. J.*, Vol.23, pp.282-332, 1944.

31. RICE, S.O., The mathematical analysis of random noise, II,, *Bell Syst. Techn. J.*, Vol.24, pp.46-156, 1945.

32. RYCHLIK I., A new definition of the rainflow cycle counting method, *Int. J. Fatigue*, Vol.9, pp.119-121, 1987.

33. RYCHLIK, I., Note on cycle counts in irregular loads, *Fatigue Fract. Engng. Mater. and Struc.*, Vol.16, 377-390, 1993.

34. RYCHLIK, I., On the "narrow-band" approximation for expected fatigue damage, *Probabilistic Engineering Mechanics*, Vol.8, pp.1-4, 1993.

35. RYCHLIK I., Fatigue and stochastic loads, *Scandinavian Journal of Statistics*, Vol.23, pp.387-404, 1996.

36. RYCHLIK, I. and LEADBETTER, M.R., Analysis of ocean waves by crossing and oscillation intensities, *International Journal of Offshore and Polar Engineering* Vol.10, pp.282-289, 2000.

37. RYCHLIK, I., On some reliability applications of Rice formula for intensity of level crossings, *Extremes*, Vol.3, No.4, pp.331-348, 2000.

38. WÖHLER, A., Über die Festigheits-Versuche mit Eisen und Stahl. *Zeitschrift für Bauwesen*, Vol.20, pp.73-106, 1870.

39. ZÄHLE, U., A general Rice formula, Palm measures, and horizontal-window conditioning for random fields, *Stochastic Process. Appl.*, Vol.17, pp.265-283, 1984.

A. Appendix

A.1. Random Variables and their properties

Often in engineering and physical sciences, outcomes of random experiments are numbers associated with some physical quantities. We shall often consider random experiments with numerical outcomes. Such experiments will be denoted by capital letters, e.g. U, X, Y, N, K. The set $\mathcal{S}$ of possible values of a random variable is a sample space which can be all real numbers, all integer numbers, or subsets thereof. Statements about random variables have truth sets (events) which are subsets of $\mathcal{S}$.

A statement of the type "$X \leq x$" for any fixed real value x, plays an important role in computation of probabilities for statements on random variables. More precisely, denote by

$$F_X(x) = P(X \leq x), \quad x \in R,$$

and call the function $F_X(x)$ the *probability distribution* or *cumulative distribution function* (cdf).

The importance of the probability distribution function lies in the following fact:

Theorem 4: *The probability of any statement about random variable X is computable (at least in theory) when the distribution function $F_X(x)$ is known.* □

For simplicity we write $F(x)$ for $F_X(x)$. Now, if the distribution function $F(x)$ is differentiable then the derivative

$$f(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x},$$

called *probability density function* (pdf), has the following interpretation $P(X \approx x) \approx f(x)\,\mathrm{d}x$.

The density $f(x)$ is not only a property of the distribution but it can also be used to define a distribution function, since any non-negative function that integrates to one is a density of some distribution. Actually, the distribution of a standard Gaussian or standard *normal* random variable is defined by means of its density function. The density of a standard normal variable has its own symbol $\phi(x)$ and is given by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}. \tag{A.1}$$

The r.v. X having this density is often denoted as $X \in N(0,1)$. The distribution function of the variable, $F(x)$, has its own symbol $\Phi(x)$,

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-t^2/2}\,\mathrm{d}t. \tag{A.2}$$

Another useful function is Ψ defined as follows

$$\Psi(x) = \int_{x}^{+\infty} (1 - \Phi(z))\,dz = \phi(x) - x(1 - \Phi(x)). \tag{A.3}$$

Note that both Φ and Ψ functions can not be computed analytically. Very precise approximations exist and they are included in most of numerical toolboxes. In the special case when $x = 0$, we have that $\Phi(0) = 1/2$ and $\Psi(0) = 1/\sqrt{2\pi}$. Using the introduced functions we can write the following formulas: Let $f(x)$ be the normal density with mean m and variance σ^2, then

$$\begin{aligned} &\int_{0}^{+\infty} x f(x)\,dx = \sigma \Psi\left(-\frac{m}{\sigma}\right), \\ &\int_{-\infty}^{+\infty} |x| f(x)\,dx = \sigma\left(\Psi\left(-\frac{m}{\sigma}\right) + \Psi\left(\frac{m}{\sigma}\right)\right). \end{aligned} \tag{A.4}$$

A.2. Dependent observations

When the outcome of an experiment is numerical we call it a random variable. Obviously, for one and the same outcome, many properties can be measured, for example at a meteorological station weather is described by

temperature, pressure, wind speed etc., i.e. is a vector of random variables $X_1, \ldots, X_n$, say, defined on the same outcome. For simplicity only, let us consider first the case $n = 2$. The function

$$F_{X_1,X_2}(x_1, x_2) = P(X_1 \leq x_1 \text{ and } X_2 \leq x_2),$$

is called the distribution function of a pair of random variables. The probability of any statement about X_1, X_2 can be computed (at least in theory) if the distribution function $F_{X_1,X_2}(x_1, x_2)$ is known, for example by means of formula (A.10). The distribution of a vector of n random variables is defined in a similar way.

We say that two events (statements) A, B about the same outcome are independent if

$$P(A \cap B) = P(A)P(B).$$

For random variables X_1 and X_2, we say that, if any statement about X_1 is independent of a statement about X_2, then they are independent. The following theorem gives the conditions for independence:

Theorem 5: *The variables X_1, X_2 with distributions $F_{X_1}(x), F_{X_2}(x)$, respectively, are independent if for all values x_1, x_2:*

$$P(X_1 \leq x_1, X_2 \leq x_2) = F_{X_1}(x_1) \cdot F_{X_2}(x_2).$$

Similarly for any n, we have that the distribution of n independent r.v. should satisfy

$$\begin{aligned} F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) &= P(X_1 \leq x_1, \ldots, X_n \leq x_n) \\ &= F_{X_1}(x_1) \cdot F_{X_2}(x_2) \cdot \ldots \cdot F_{X_n}(x_n). \end{aligned} \quad \text{(A.5)}$$

Theorem 6: ***Law of large numbers:*** *Let $X_1, \ldots, X_k$ be a sequence of iid (independent identically distributed) variables all having the distribution $F_X(x)$. Denote by $\bar{X}$ the average of X_i, i.e.*

$$\bar{X} = \frac{1}{k}(X_1 + X_2 + \cdots + X_k). \quad \text{(A.6)}$$

(Obviously, $\bar{X}$ is a random variable itself.) Let us also introduce a constant called the expected value of X, defined by

$$\mathsf{E}(X) = \int_{-\infty}^{+\infty} x f_X(x)\, \mathrm{d}x, \quad \text{(A.7)}$$

if the density $f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x)$ *exists, or*

$$\mathsf{E}(X) = \sum_x x P(X = x),$$

where summation is over those x *for which* $P(X = x) > 0$. *If the expected value of* X *exists and is finite then, as* k *increases (we are averaging more and more variables),* $\bar{X} \approx \mathsf{E}(X)$ *with equality when* k *approaches infinity.*

For the most common distributions, the expectations have been calculated and can be found in tables.

A.3. Some properties of two-dimensional distributions

In this section, we shall assume that we have only two random variables, $n = 2$, and, in order to simplify the notation, we shall denote X_1, X_2 by X, Y. The distribution function $F_{X_1,X_2}(x_1, x_2)$ will be also denoted by

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y),$$

which we shall often simplify to $F(x, y)$. The distributions of the variables X and Y will be denoted by $F(x) = P(X \leq x)$ and $F(y) = P(Y \leq y)$, respectively. From the definition of $F(x, y)$, it follows immediately that

$$F(x) = F(x, +\infty), \quad F(y) = F(+\infty, y).$$

If the distribution $F(x, y)$ is differentiable with respect to x and y, the derivative

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}$$

is called the probability density function (pdf). If $f(x, y)$ is known, then

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(\widetilde{x}, \widetilde{y})\, \mathrm{d}\widetilde{x}\, \mathrm{d}\widetilde{y}.$$

Any non-negative function $f(x, y)$ that integrates to one

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\, \mathrm{d}x\, \mathrm{d}y = 1$$

is a density of some random variables (X, Y). Often one specifies the density and computes the distribution function. The one-dimensional densities of X,

Y can be computed from the joint density by means of the following integrals

$$f(x) = \int_{-\infty}^{+\infty} f(x, \widetilde{y})\, \mathrm{d}\widetilde{y}, \qquad f(y) = \int_{-\infty}^{+\infty} f(\widetilde{x}, y)\, \mathrm{d}\widetilde{x}.$$

It is easy to prove that for independent X, Y

$$f(x, y) = f(x)f(y). \tag{A.8}$$

Example 19. *Two-dimensional Gaussian distribution.* Suppose that X and Y are Gaussian r.v., with distributions $N(m_X, \sigma_X^2)$, $N(m_Y, \sigma_Y^2)$, respectively. This means that their probability density functions are written

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \mathrm{e}^{-\frac{1}{2\sigma_X^2}(x-m_X)^2}, \qquad f(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} \mathrm{e}^{-\frac{1}{2\sigma_Y^2}(y-m_Y)^2}.$$

If X and Y are independent, their joint probability density $f(x, y)$ is given by

$$f(x, y) = f(x)f(y) = \frac{1}{2\pi\sigma_X\sigma_Y} \mathrm{e}^{-\frac{1}{2}\left(\frac{(x-m_X)^2}{\sigma_X^2} + \frac{(y-m_Y)^2}{\sigma_Y^2}\right)}.$$

The variables X and Y can also be dependent. Then, there is a parameter $-1 \leq \rho \leq 1$, called correlation (to be introduced later on), that measures the degree of dependence between X and Y. If $\rho = 0$ then X and Y are independent. Consequently, five parameters define the two-dimensional Gaussian distribution. These are m_X, m_y, σ_X^2, σ_Y^2, and ρ, and the statement that X, Y is Gaussian,

$$X, Y \in N(m_X, m_y, \sigma_X^2, \sigma_Y^2, \rho),$$

means that the joint density of X, Y is given by

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \mathrm{e}^{-\frac{1}{2}\left(\frac{(x-m_X)^2}{\sigma_X^2} + \frac{(y-m_Y)^2}{\sigma_Y^2} - 2\rho\frac{(x-m_X)}{\sigma_X}\frac{(y-m_Y)}{\sigma_Y}\right)}. \tag{A.9}$$

An illustration of the two-dimensional Gaussian distribution will now be given. Let $m_X = 3$, $\sigma_X = 0.5$, $m_Y = -2$, $\sigma_Y = 1$. First, let $\rho = 0$. In Fig. 19(a), the density function is shown, and in Fig. 19(b), the corresponding contour lines are shown. Introducing a dependence between the variables by $\rho = 0.2$, the corresponding plots are shown in Fig. 20. □

Finally, for any events A, B

$$P(X \in A \text{ and } Y \in B) = \int_A \int_B f(x, y)\, \mathrm{d}y\, \mathrm{d}x, \tag{A.10}$$

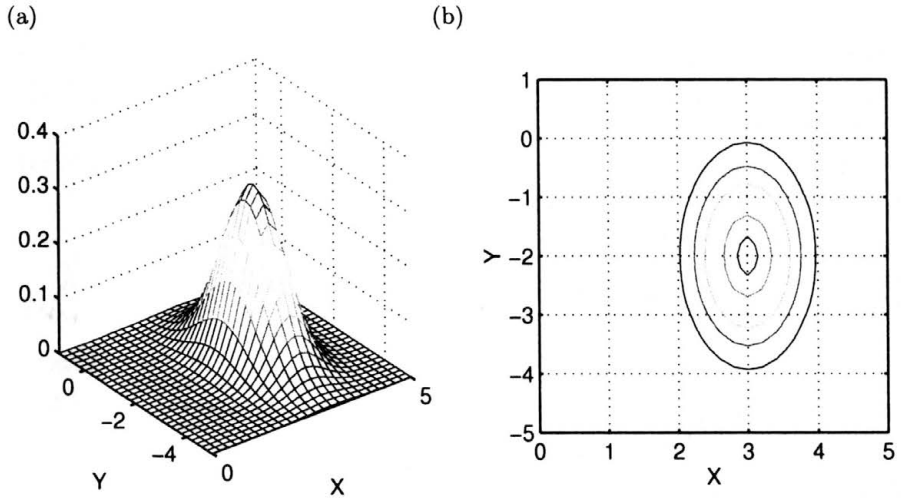

FIGURE 19. Two-dimenaional Gaussian distribution: $m_X = 3$, $\sigma_X = 0.5$, $m_Y = -2$, $\sigma_Y = 1$. Independent variables: $\rho = 0$. (a) Density function. (b) Contour lines.

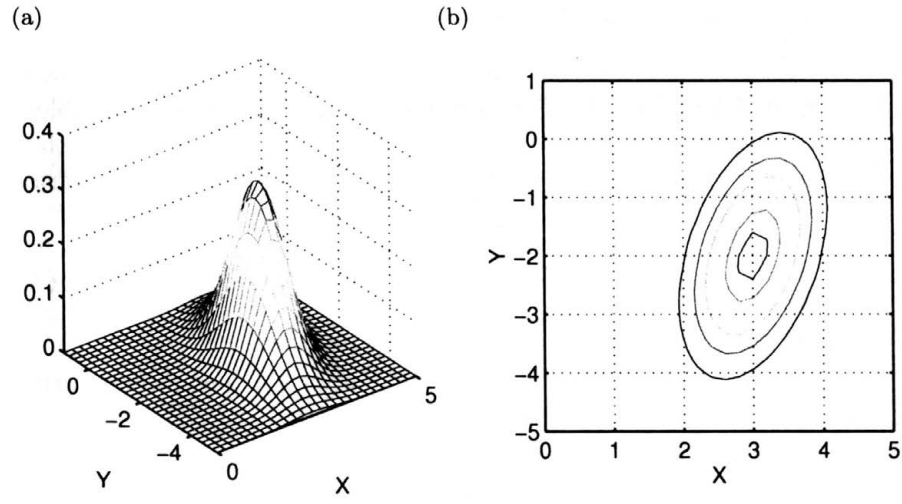

FIGURE 20. Two-dimenaional Gaussian distribution: $m_X = 3$, $\sigma_X = 0.5$, $m_Y = -2$, $\sigma_Y = 1$. Dependent variables: $\rho = 0.2$. (a) Density function. (b) Contour lines.

Quite often, the last formula has to be computed numerically, even for simple sets A, B. For example, this is the case when X, Y are Gaussian.

Consider a function $z = h(x, y)$. Define a new random variable Z as a value of h computed at results of a random experiment X, Y, i.e. $Z = h(X, Y)$. For example, $Z = X + Y$ or $Z = X \cdot Y$. We want to find the expected value of Z. Obviously if the distribution of Z was known, we could use (A.7) to compute the expectation of Z. However, it can also be done directly by means of the following formulas:

$$\mathsf{E}(Z) = \mathsf{E}(g(X,Y)) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{A.11}$$

or

$$\mathsf{E}(Z) = \mathsf{E}(g(X,Y)) = \sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty} g(j,k)\,p_{jk},$$

where $p_{jk} = P(X = j, Y = k)$.

A.3.1. Covariance and correlation It is also easy to check (Eqs. (A.8) and (A.11)) that for independent variables X and Y,

$$\mathsf{E}(X \cdot Y) = \mathsf{E}(X) \cdot \mathsf{E}(Y).$$

Actually, it can happen that the last equality holds even for dependent variables. Such variables are called *uncorrelated*. (All independent variables are uncorrelated but not conversely.) Now, if the equation does not hold, the difference between the terms is a measure of dependence between the variables X and Y. This measure is called *covariance* and is defined by

$$\mathrm{Cov}(X,Y) = \mathsf{E}(X \cdot Y) - \mathsf{E}(X) \cdot \mathsf{E}(Y), \tag{A.12}$$

obviously $\mathrm{Cov}(Y, Y) = \mathsf{V}(Y)$.

When one has two random variables, one often represents their variances and covariances in form of a matrix

$$\mathrm{Cov}(X,Y;X,Y) = \begin{bmatrix} \mathsf{V}(X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(X,Y) & \mathsf{V}(Y) \end{bmatrix}.$$

Since $\mathrm{Cov}(aX, bY) = ab \cdot \mathrm{Cov}(X, Y)$, this means that by changing the units in which the variables X and Y are measured, one can make the covariance very close to zero. This could be misinterpreted as X and Y being only weakly dependent. Consequently, one is often scaling the covariance so

that it becomes independent of the units in which the variables are measured. Such a scaled covariance is called *correlation* and is defined as follows

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\mathsf{D}(X)\mathsf{D}(Y)}, \tag{A.13}$$

where $\mathsf{D}(X) = \sqrt{\mathsf{V}(X)}$. The correlation is always between one and minus one.

Theorem 7: *Let X, Y be two random variables such that $|\rho_{XY}| = 1$. Then there are constants a, b (both not equal zero) such that $aX + bY = 0$ with probability one.*

However, not all functionally dependent variables X, Y are perfectly correlated ($|\rho_{XY}| = 1$). For example, for $X \in \mathrm{N}(0,1)$, define $Y = X^3$. Obviously, if we know the outcome of the random experiment, then $X = x$ while $Y = x^3$. Now the correlation between X, Y is given by $\rho_{XY} = 3/\sqrt{15} < 1$.

A.4. Conditional distributions and densities

Suppose we are told that the event A, such that $P(A) > 0$, has occurred, then the sample space of possible outcomes of an experiment reduces from $\mathcal{S}$ to A and the probability that B occurs, given that A has occurred, is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Consider now variables X, Y having continuous distributions. We wish to find the conditional distribution $F(x|Y = y) = P(X \leq x|Y = y)$. However, we face a problem since for continuous variables Y, $P(Y = y) = 0$ for all y. An easy solution to this problem can be found if X, Y have the density $f(x,y)$. In such a case we can define

$$f(x|y) = \frac{f(x,y)}{f(y)}. \tag{A.14}$$

Since, for a fixed value y, $f(x|y)$ as a function of x integrates to one, it is a probability density function. Let denote by $F(x|y)$ a distribution having as density $f(x|y)$, i.e. for any x

$$F(x|y) = \int_{-\infty}^{x} f(\widetilde{x}|y)\,\mathrm{d}\widetilde{x}. \tag{A.15}$$

Now, combination of Eqs. (A.10), (A.14) and (A.15) leads to the following important result

$$P(X \leq x) = F(x) = \int_{-\infty}^{+\infty} F(x|y) f(y)\,\mathrm{d}y. \tag{A.16}$$

The last equation is a special case of the law of total probability given next.

Theorem 8: *Assume that a random experiment renders values of a r.v. Y and that we, in addition, are interested in any statement B, say, about the outcomes of the experiment. Then, for each y, there exists a probability $P(B|Y = y)$ (for different y the probabilities $P(B|Y = y)$ may take different values), such that*

$$P(B) = \int_{-\infty}^{+\infty} P(B|Y = y) f_Y(y)\,\mathrm{d}y. \tag{A.17}$$

If X, Y have joint density $f(x, y)$, then

$$P(B|Y = y) = \int_B f(x|y)\,\mathrm{d}x,$$

where $f(x|y)$ is the conditional probability density defined by (A.14).

A.5. Some properties of Gaussian vectors

Consider a vector of r.v. $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$, which consists of n different random variables. We say that $\mathbf{X}$ is a Gaussian vector if

$$Y = \sum_{i=1}^{n} a_i X_i, \tag{A.18}$$

where $a_1, a_2, \ldots, a_n$ are arbitrary real constants, has a Gaussian distribution.

Let denote the means by $m_i = \mathsf{E}(X_i)$, the variances by $\sigma_{ii} = \mathrm{Cov}(X_i, X_i)$ and the pairwise covariances by $\sigma_{ij} = \mathrm{Cov}(X_i, X_j)$. Obviously for any i, $X_i \in N(m_i, \sigma_{ii})$.

Let us introduce the notation

$$\mathbf{m} = (m_1, m_2, \ldots, m_n)^T, \qquad \Sigma = \begin{pmatrix} \sigma_{11} & \ldots & \sigma_{1n} \\ \vdots & \sigma_{ij} & \vdots \\ \sigma_{n1} & \ldots & \sigma_{nn} \end{pmatrix}.$$

If the determinant $\det(\Sigma) > 0$, then the distribution $\mathbf{X}$ has a density given by

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \frac{1}{\det(\Sigma)^{1/2}} \mathrm{e}^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \Sigma^{-1}(\mathbf{x}-\mathbf{m})}, \tag{A.19}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$. The statements "$\mathbf{X}$ is a Gaussian vector" or "f is a Gaussían density" will be written as $\mathbf{X} \in \mathbf{N}(\mathbf{m}, \boldsymbol{\Sigma})$, $f(\mathbf{x}) \in \mathbf{N}(\mathbf{m}, \boldsymbol{\Sigma})$, respectively.

From (A.18) it follows that for any real-valued (k, n)-matrix A the vector $\mathbf{Y} = A\mathbf{X}$ is a k-dimensional Gaussian vector too. If $\mathbf{X} \in \mathbf{N}(\mathbf{m}, \boldsymbol{\Sigma})$ then $\mathbf{Y} \in N(A\mathbf{m}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\mathbf{T}})$.

One more useful result about the conditional densities. Let $\mathbf{X}$ be factorized into two vectors $\mathbf{Z} = (X_1, \ldots, X_k)^T$ and $\mathbf{Y} = (X_{k+1}, \ldots, X_n)^T$. With obvious notation we write

$$\mathbf{m} = \begin{bmatrix} \mathbf{m}_Z \\ \mathbf{m}_Y \end{bmatrix}, \qquad \Sigma = \begin{pmatrix} \Sigma_{ZZ} & \Sigma_{ZY} \\ \Sigma_{YZ} & \Sigma_{YY} \end{pmatrix}.$$

Now the conditional density $f_{\mathbf{Z}|\mathbf{Y}}(\mathbf{z}|\mathbf{y})$ of $\mathbf{Z}$ given $\mathbf{Y} = \mathbf{y}$ is Gaussian with the mean

$$\mathbf{m}_{Z|Y} = \mathbf{m}_Z + \Sigma_{ZY}\Sigma_{YY}^{-1}(\mathbf{y} - \mathbf{m}_Z), \tag{A.20}$$

and covariance function

$$\Sigma_{Z|Y} = \Sigma_{ZZ} - \Sigma_{ZY}\Sigma_{YY}^{-1}\Sigma_{YZ}, \tag{A.21}$$

i.e. $f_{\mathbf{Z}|\mathbf{Y}}(\mathbf{z}|\mathbf{y}) \in N(\mathbf{m}_{Z|Y}, \Sigma_{Z|Y})$.

A.6. Infinitely many r.v. – random processes

A stochastic process is a family of random variables $\{X(t),\ t \in \mathcal{A}\}$, defined on the same sample space $\mathcal{S}$, i.e. infinitely many functions defined on the outcome of an experiment. The parameter t can be one-dimensional, and is then often called "time", but it can also be two- or multidimensional. A stochastic process with multidimensional parameter is called a stochastic field. A random function is a realization $x(t)$, $t \in \mathcal{A}$ of a stochastic process $\{X(t),\ t \in \mathcal{A}\}$. We shall also use the vector-valued processes which will be denoted by $\{\mathbf{X}(t),\ t \in \mathcal{A}\}$, i.e. we have a collection of random processes.

Since every value of a stochastic process is a random variable $X(t)$, one can study its *moments*, like expectation, variance, and these generally depend on the specific time point t at which we observe the process:

- mean function: $m(t) = \mathsf{E}(X(t))$,

- covariance function: $r(s,t) = \mathrm{Cov}(X(s), X(t))$,
- variance function: $v(t) = \mathsf{V}(X(t)) = r(t,t)$.

Similarly as there are random variables for which expected value (and variance) do not exist, e.g. $Z = X/Y$, where X, Y are independent $N(0,1)$ variables, not all processes have finite moments. However, here we shall only discuss processes with finite variance.

The moments only partially specify the properties of the random process, e.g. in the case of random variables one may have two variables having the same expectation and variance but different distributions. In order to specify the distribution of a stochastic process one needs to specify the joint distribution of all finite collections of variables that constitute the process, e.g. $X(t_1), X(t_2), \ldots, X(t_n)$. In particular one needs the *marginal distributions*, i.e. the distribution of each $X(t)$. This is very difficult in general.

In the following we shall mostly study processes and hence we assume that $\mathcal{A} = R$. (The tools needed for study vector valued processes or random fields can be introduced in a similar way.) We shall assume that the process is smooth, i.e. the probability of each of the statements "*random function has continuous n-th derivative*", is one. The value n will be taken as high as one needs for convenient discussion of particular applications. Discussion of regularity of random functions as well as existence of the distribution of the random process can be found in Cramér and Leadbetter (1967).

The process is called stationary if for any τ the vectors $X(t_1), X(t_2), \ldots, X(t_n)$ and $X(t_1 + \tau), X(t_2 + \tau), \ldots, X(t_n + \tau)$ have the same distribution. Obviously, for stationary processes, the mean and variance functions are constant $m(t) = m$, $v(t) = \sigma^2$ and there is a function $r_X(\tau)$ such that $r(t,s) = r_X(s-t)$. Obviously $r_X(0) = \sigma^2$.

Now if $r_X(0) > 0$ and the function $r_X(\tau)$ is continuous at zero then one can define the so-called spectral measure $\sigma(\lambda)$, such that

$$r_X(\tau) = \int e^{i\tau\lambda}\, d\sigma(\lambda).$$

If the spectral measure is absolutely continuous, then there exist the spectral density $S(\lambda)$ such that $r_X(\tau) = \int e^{i\tau\lambda} S(\lambda)\, d\lambda$. Some properties of the process can be expressed using the so-called spectral moments which are defined as follows

$$\lambda_i = 2\int_0^\infty \lambda^i\, d\sigma(\lambda) = 2\int_0^\infty \lambda^i S(\lambda)\, d\lambda,$$

where the last equation is true if the spectral density exists. In this notation we have that $\mathsf{V}(X(0)) = \lambda_0$, $\mathsf{V}(\dot{X}(0)) = \lambda_2$ and $\mathsf{V}(\ddot{X}(0)) = \lambda_4$.

Here we shall study a special class of processes, namely Gaussian processes and their functions. Only the Gaussian processes have the property that every linear operation produces a normal random variable, and one can use this fact to provide with a formal definition of a Gaussian process.

Definition 4: *A stochastic process $X(t)$ is called a Gaussian process (or normal process) if all linear combinations*

$$a_1X(t_1) + a_2X(t_2) + \ldots + a_nX(t_n)$$

of the process values at fixed but arbitrary time points, have a normal distribution. Here $t_1, t_2, \ldots, t_n$ are arbitrary time points and $a_1, a_2, \ldots, a_n$ arbitrary real constants.

We shall now generalize the presented results for the case that t is multivariate.

A.7. Gaussian fields

Let $\boldsymbol{\tau} = (\mathbf{p}, t) = (x, y, t)$ be a point in R^3, and let, for every $\boldsymbol{\tau}$, a real valued random variable $X(\boldsymbol{\tau})$ be given. Such a collection $\{X(\boldsymbol{\tau}), \boldsymbol{\tau} \in R^3\}$ of random variables is referred to as a random field. A random field is called Gaussian if all its finite dimensional distributions, i.e. distributions of $(X(\boldsymbol{\tau}_1), \ldots, X(\boldsymbol{\tau}_n))$, $n \in N$, $\boldsymbol{\tau}_i \in R^3$, are multivariate Gaussian (normal). We call $\{X(\boldsymbol{\tau}), \boldsymbol{\tau} \in R^3\}$ homogeneous if its finite dimensional distributions are invariant under shift in $\boldsymbol{\tau}$. The covariance function $R(\boldsymbol{\tau}) = \mathrm{Cov}(X(\boldsymbol{\tau}_0 + \boldsymbol{\tau}), X(\boldsymbol{\tau}_0))$ of a homogeneous random field is positively definite and thus it follows from Bochner's theorem that it can be written in the form

$$R(\boldsymbol{\tau}) = \int_{R^3} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau}) d\sigma(\boldsymbol{\lambda}), \tag{A.22}$$

where $\sigma(\boldsymbol{\lambda})$ is a finite measure on Borel sets of R^3 which is called the *spectral measure* of $X(\boldsymbol{\tau})$.

It follows from the theory of Hilbert spaces that the field $X(\boldsymbol{\tau})$ has the following spectral representation

$$X(\boldsymbol{\tau}) = \int_{R^3} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau}) d\zeta(\boldsymbol{\lambda}), \tag{A.23}$$

where the field $\zeta(\boldsymbol{\lambda})$ is complex valued with orthogonal increments defined up to an additive constant. If this is fixed by $\zeta(-\infty, -\infty, -\infty) = 0$, we also have $E(\zeta(\boldsymbol{\lambda})) = 0$, and such that $E(|\zeta(\boldsymbol{\lambda})|^2) = \sigma(-\infty, \boldsymbol{\lambda}]$, $E(|\zeta(\mathbf{I})|^2) = \sigma(\mathbf{I})$,

where $\mathbf{I} \subset R^3$ is an interval. The spectral measure is also symetric with respect to the origin of R^3; see, for example, Cramér and Leadbetter (1966) or Kreé and Soize (1986) for reviews.

Example 20. Assume that the spectral measure σ is discrete, i.e. has discrete support at points $\{\boldsymbol{\lambda}_j\} = \{(\lambda_{1j}, \lambda_{2j}, \lambda_{3j})\}$, $j \in N$, of which none is equal to zero, with masses $\sigma(\boldsymbol{\lambda}_j)$. For a real valued random field the support of σ has to be symmetric, i.e. if $\boldsymbol{\lambda}_j$ is in the support, then also $-\boldsymbol{\lambda}_j$ is in the support and both frequencies have equal masses $\sigma(\boldsymbol{\lambda}_j) = \sigma(-\boldsymbol{\lambda}_j)$. It follows from the spectral representation (A.23) that

$$X(\boldsymbol{\tau}) = \sum_{\boldsymbol{\lambda}_j \in \Lambda_+} \sqrt{2\sigma(\boldsymbol{\lambda}_j)} R_j \cos(\boldsymbol{\lambda}_j^T \boldsymbol{\tau} + \epsilon_j),$$

where Λ_+ is any set in R^3 such that $-\Lambda_+ \cap \Lambda_+$ has volume zero and $-\Lambda_+ \cup \Lambda_+ = R^3$. For example, one possible choice of Λ_+ is

$$\left\{(x_1, x_2, x_3) \in R^3 : x_3 \geq 0\right\}.$$

Moreover, (R_j) and (ϵ_j) are two independent sequences of independent identically distributed random variables, the first one distributed according to the Rayleigh density

$$f(r) = re^{-r^2/2}, \quad r > 0. \tag{A.24}$$

This density was derived by Rayleigh (1880) in connection with some problem in acoustics. The second one is distributed uniformly on $[0, 2\pi]$.

By (A.22), the covariance of this field is a sum of cosines

$$R(\boldsymbol{\tau}) = \sum_{\boldsymbol{\lambda}_j \in R^3} \sigma(\boldsymbol{\lambda}_j) \cos(\boldsymbol{\lambda}_j^T \boldsymbol{\tau}) = 2 \sum_{\boldsymbol{\lambda}_j \in \Lambda_+} \sigma(\boldsymbol{\lambda}_j) \cos(\boldsymbol{\lambda}_j^T \boldsymbol{\tau}).$$

If the covariance function $R(\boldsymbol{\tau})$ decreases sufficiently fast at infinity, so that $\int_{R^3} |R(\boldsymbol{\tau})| d\boldsymbol{\tau} < \infty$, then σ has the density $S(\boldsymbol{\lambda})$ and the covariance function can be represented as its Fourier integral

$$R(\boldsymbol{\tau}) = \int_{R^3} \exp(i\boldsymbol{\lambda}^T \boldsymbol{\tau}) S(\boldsymbol{\lambda}) d\boldsymbol{\lambda}. \tag{A.25}$$

The spectral density $S(\boldsymbol{\lambda})$ is real, non-negative, bounded and symmetric, i.e. $S(\boldsymbol{\lambda}) = S(-\boldsymbol{\lambda})$ for all $\boldsymbol{\lambda} \in R^3$.

The *spectral moments* λ_{ijk}, if they are finite, are defined as

$$\lambda_{ijk} = 2 \int_{\Lambda^+} \lambda_1^i \lambda_2^j \lambda_3^k \, d\sigma(\boldsymbol{\lambda}), \tag{A.26}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$. The variance of the field can be expressed in terms of spectral moments as the zero moment λ_{000} increased by the weight of the spectral measure at zero $\sigma(\mathbf{0})$. Higher moments are used to compute the covariances between the process and its derivatives. For example,

$$\mathsf{E}\left[\frac{\partial X(x,y,t)}{\partial x}\frac{\partial X(x,y,t)}{\partial y}\right] = \int_{R^3} i\lambda_1(-i\lambda_2)\,d\sigma(\boldsymbol{\lambda}) = \lambda_{110},$$

where the last condition follows from the fact that because of the symmetry of σ we have $\lambda_{ijk} = \int_{R^3} \lambda_1^i \lambda_2^j \lambda_3^k \, d\sigma(\boldsymbol{\lambda})$, whenever $i + j + k$ is even. This and similar formulas for the covariances between the derivatives follow from the spectral representation of the Hilbert space spanned by $X(\boldsymbol{\tau}), \boldsymbol{\tau} \in R^3$. Namely, if $Y = \int_{R^3} f(\boldsymbol{\lambda}) d\zeta(\boldsymbol{\lambda})$, $Z = \int_{R^3} g(\boldsymbol{\lambda}) d\zeta(\boldsymbol{\lambda})$, where f and g are both complex functions square integrable with respect to σ, then

$$\mathrm{Cov}(Y, Z) = \int_{R^3} f(\boldsymbol{\lambda}) \cdot \bar{g}(\boldsymbol{\lambda}) d\sigma(\boldsymbol{\lambda}).$$

Often it is customary to write the covariances using the spectral moments λ_{ijk} as defined by (A.26). We give two examples: The covariances between the first order derivatives are given in the covariance matrix of the gradient vector $\dot{\mathbf{X}} = (X_x, X_y)$:

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_{200} & \lambda_{110} \\ \lambda_{110} & \lambda_{020} \end{bmatrix}. \tag{A.27}$$