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INCOMPRESSIBLE MATERIALS IN THE THEORY OF ELASTICITY

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Incompressible materials in the theory of elasticity

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There exists extensive literature where incompressible materials are defined by the demand, that only isochoric motions are allowed. In the nature, however, exist compressible materials only. Therefore the theory based on such a geometrical definition not complemented by the theorems on the relationship between compressible and incompressible materials is of mathematical interest only. Moreover the geometrical definition allows some ambiguity. Namely the constraint imposed on the motion does not determine uniquely the additional degree of freedom in the stress-strain relation. This indeterminancy is usually avoided by assumption that the extra stress does not produce work on the isochoric / i.e. compatible with the constraint / deformations. Only in some special cases such assumption is justified.

In fact the only purpose of introducing the incompressible materials is to find approximate solution for the oryginal material, but with the aid of the equations simpler than the oryginal ones. Of the principal interest is therefore the difference between the solution of the boundary problem for the real compressible material and the solution for the hypotetic incompressible one.

In the present work the limit incompressible material is defined as the material for which the solution of the boundary problem equals the limit solution for the compressible materials if the compressibility tends to zero. Because it was possible to find in the linear theory more elegant

/ 1.8 / $$\lim_{\nu \to \infty} \left[c_{ijkl}(\nu) - \frac{1}{3} c_{ijrr}(\nu) \delta_{kl} \right] < \infty .$$

Under assumption that the stresses remain finite the condition / 1.7 / assures that the deformation is isochoric in the limit case, but allows the additional constraints. For example if all the $c_{ijkl} \to \infty$ the material tends to the rigid body. The possibility of the additional constraints is excluded by the condition / 1.8 /.

Assume the materials $M(\nu)$ defined by the relation

/ 1.9 / $$c_{ijkl} = a_{ijkl} + \nu b_{ijkl} ,$$

where a_{ijkl} and b_{ijkl} are constant tensors independent of ν. The materials $M(\nu)$ satisfy the conditions / 1.8 /,/ 1.9 / if and only if

/ 1.10 / $$c_{ijkl} = a_{ijkl} + \nu b_{ijrr} \delta_{kl} .$$

If the deformation is isochoric $\varepsilon_{rr} = 0$ and all the materials / 1.10 / have the same stress-strain relation.

For arbitrary a_{ijkl} and b_{ijkl} the relation / 1.10 / can describe some degenerate material. In order to exclude this possibility assume that the boundary problem in stresses for the material $M(\nu)$

$$\left(a_{ijkl} + \nu b_{ijrr} \delta_{kl}\right) \varepsilon_{kl,i} = -\rho f_j \quad \text{in } \Omega ,$$

/ 1.11 / $$\left(a_{ijkl} + \nu b_{ijrr} \delta_{kl}\right) \varepsilon_{kl} n_i = t_j \quad \text{on } S ,$$

$$e_{irs} e_{jpq} \varepsilon_{rp,sq} = 0 \quad \text{in } \Omega^c .$$

possesses the unique solution. It follows that the only solution of the corresponding homogeneous boundary problem is the null-solution $\varepsilon_{ij} = 0$. It can be proved that from the uniqueness of solution of the boundary problem / 1.11 / follows the uniqueness of solution of the boundary problem

$$
\begin{aligned}
& a_{ijkl}\,\varepsilon_{kl,i} + b_{ijrr}\,q_{,i} = -\varrho f_j \quad \text{in } \mathcal{V}, \\
& a_{ijkl}\,\varepsilon_{kl}\,n_i + b_{ijrr}\,q\,n_i = t_j \quad \text{on } \mathcal{S}, \\
& e_{irs}\,e_{jpq}\,\varepsilon_{rp,sq} = 0 \quad \text{in } \mathcal{V}, \\
& \varepsilon_{rr} = 0 \quad \text{in } \mathcal{V},
\end{aligned}
\tag{1.12}
$$

where q is an additional unknown. It follows that the only solution of the corresponding homogeneous boundary problem is the null-solution $\varepsilon_{ij} = 0$, $q = 0$.

Consider an auxiliary one parameter family $\mathcal{U}$ of the displacement fields $u_i(\lambda)$, $0 \leq \lambda < \infty$ of the class C^3 such that the corresponding deformation tensors $\varepsilon_{ij}(\lambda)$ and their gradients $\varepsilon_{ij,k}(\lambda)$ are finite in $\mathcal{V}$ and $\varepsilon_{ij}(\lambda)$ and $\varepsilon_{ij,k}(\lambda)$ are pointwise convergent to the finite limits $\bar{\varepsilon}_{ij}$ and $\bar{\varepsilon}_{ij,k}$ respectively. Assume additionally that the trace of $\bar{\varepsilon}_{ij}$ is equal zero

$$
\begin{aligned}
& \lim_{\lambda \to \infty} \varepsilon_{ij}(\lambda) = \bar{\varepsilon}_{ij} < \infty, \quad \bar{\varepsilon}_{rr} = 0, \\
& \lim_{\lambda \to \infty} \varepsilon_{ij,k}(\lambda) = \bar{\varepsilon}_{ij,k}.
\end{aligned}
\tag{1.13}
$$

Because $\varepsilon_{ij}(\lambda)$ is a symmetrized gradient of a displacement field there is

$$
\begin{aligned}
& e_{irs}\,e_{jpq}\,\varepsilon_{rp,sq}(\lambda) = 0 \quad \text{in } \mathcal{V}, \\
& e_{irs}\,e_{jpq}\,\bar{\varepsilon}_{rp,sq} = 0 \quad \text{in } \mathcal{V}.
\end{aligned}
\tag{1.14}
$$

In the material $M(\nu)$ the deformation $\varepsilon_{ij}(\lambda)$ produces the stress tensor

$$\tau_{ij}(\nu,\lambda) = a_{ijkl}\,\varepsilon_{kl}(\lambda) + \nu\, b_{ijrr}\,\varepsilon_{ss}(\lambda) \qquad /\ 1.15\ /$$

Because the first term of the right hand side of / 1.15 / is finite and $\lim\limits_{\lambda\to\infty} \varepsilon_{rr} = 0$, it is always possible to find at least one continuous monotonic function $\lambda(\nu)$, $\lim\limits_{\nu\to\infty} \lambda(\nu) = \infty$ such that $\nu\,\varepsilon_{rr}(\lambda(\nu))$ is finite for every ν and has once differentiable finite limit

$$\lim_{\nu\to\infty} \nu\,\varepsilon_{rr}(\lambda(\nu)) = p\,. \qquad /\ 1.16\ /$$

The function $\lambda(\nu)$ fixes the one-to-one correspondence between the materials $M(\nu)$ and the deformations $\varepsilon_{ij}(\lambda)$. The deformation $\varepsilon_{ij}(\lambda(\nu))$ is further denoted by $\varepsilon_{ij}(\nu)$ and the stress produced in the material $M(\nu)$ by the strain $\varepsilon_{ij}(\nu)$ is denoted by $\tau_{ij}(\nu)$

$$\tau_{ij}(\nu) = a_{ijkl}\,\varepsilon_{kl}(\nu) + \nu\, b_{ijrr}\,\varepsilon_{ss}(\nu)\,. \qquad /\ 1.17\ /$$

For a given fixed domain $\mathcal{V}$ bounded by a smooth surface $\mathcal{S}$ the stresses / 1.17 / are equilibrated by the body forces and surface tractions that according to / 1.3 /, / 1.4 / are

$$\begin{aligned} -\varrho f_j(\nu) &= a_{ijkl}\,\varepsilon_{kl,i}(\nu) + \nu\, b_{ijrr}\,\varepsilon_{ss,i}(\nu) \quad \text{in } \mathcal{V}, \\ t_j(\nu) &= a_{ijkl}\,\varepsilon_{kl}(\nu)\,n_i + \nu\, b_{ijrr}\,\varepsilon_{ss}(\nu)\,n_i \quad \text{on } \mathcal{S}. \end{aligned} \qquad /\ 1.18\ /$$

In accord with / 1.13 / and / 1.16 / for $\nu\to\infty$ the forces $\varrho f_j(\nu)$ and $t_j(\nu)$ posses finite limits equal respectively

$$-\varrho \bar{f}_j = a_{ijkl}\,\bar{\varepsilon}_{kl,i} + b_{ijrr}\,p_{,i} \quad \text{in } \mathcal{V},$$

/ 1.19 /

$$\bar{t}_j = a_{ijkl}\,\bar{\varepsilon}_{kl}\,n_i + b_{ijrr}\,p\,n_i \quad \text{on } S.$$

From the above assumptions it follows that $\bar{t}_j$ is twice differentiable on S, and $\varrho\bar{f}_j$ is once differentiable in $\mathcal{V}$.

After this auxiliary formulae consider the boundary problem / 1.11 / for fixed once differentiable in $\mathcal{V}$ body force $\varrho\bar{f}_i$ and fixed twice differentiable on S surface traction $\bar{t}_j$. For each material $M(\nu)$ the unique solution of this problem can be found. Denote this solution by $\delta_{ij}(\nu)$. There is

$$a_{ijkl}\,\delta_{kl,i}(\nu) + \nu\, b_{ijrr}\,\delta_{ss,i}(\nu) = -\varrho\bar{f}_j \quad \text{in } \mathcal{V},$$

/ 1.20 /

$$a_{ijkl}\,\delta_{kl}(\nu)\,n_i + \nu\, b_{ijrr}\,\delta_{ss}(\nu)\,n_i = \bar{t}_j \quad \text{on } S,$$

$$e_{irs}\,e_{jpq}\,\delta_{rp,sq}(\nu) = 0 \quad \text{in } \mathcal{V}.$$

Subtracting now / 1.20 / from / 1.18 / and taking into account / 1.14 /$_1$ we have

$$a_{ijkl}\big(\varepsilon_{kl}(\nu) - \delta_{kl}(\nu)\big)_{,i} + \nu\, b_{ijrr}\big(\varepsilon_{ss}(\nu) - \delta_{ss}(\nu)\big)_{,i} = -\varrho\big(f_j(\nu) - \bar{f}_j\big)$$

/ 1.21 / $$a_{ijkl}\big(\varepsilon_{kl}(\nu) - \delta_{kl}(\nu)\big)n_i + \nu\, b_{ijrr}\big(\varepsilon_{ss}(\nu) - \delta_{ss}(\nu)\big)n_i = t_j(\nu) - \bar{t}_j \quad \text{on } S$$

$$e_{irs}\,e_{jpq}\,\big(\varepsilon_{rp}(\nu) - \delta_{rp}(\nu)\big)_{,sq} = 0 \quad \text{in } \mathcal{V}$$

In the limit $\nu \to \infty$ it follows in accord with / 1.13 / and / 1.16 /

$$a_{ijkl}\left(\bar{\varepsilon}_{kl}-\lim_{\nu\to\infty}\delta_{kl}\right)_{,i}+b_{ijrr}\left(p-\lim_{\nu\to\infty}\nu\delta_{rr}\right)_{,i}=0 \text{ in } V,$$

$$/\ 1.22\ / \quad a_{ijkl}\left(\bar{\varepsilon}_{kl}-\lim_{\nu\to\infty}\delta_{kl}\right)n_i+b_{ijrr}\left(p-\lim_{\nu\to\infty}\nu\delta_{rr}\right)n_i=0 \text{ on } S,$$

$$e_{irs}\,e_{jpq}\left(\bar{\varepsilon}_{rp}-\lim_{\nu\to\infty}\delta_{rp}\right)_{,sq}=0 \quad \text{in } V,$$

$$\bar{\varepsilon}_{rr}-\lim_{\nu\to\infty}\delta_{rr}=0 \quad \text{in } V$$

In accord with / 1.12 / the unique solution of the boundary problem / 1.22 / is the null-solution

$$/\ 1.23\ / \qquad \bar{\gamma}_{ij}=\lim_{\nu\to\infty}\delta_{ij}=\bar{\varepsilon}_{ij}\,.$$

Because $\bar{\varepsilon}_{ij}$ satisfies the relations / 1.18 / and / 1.19 / the relation / 1.23 / establishes the theorem: <u>The limit solution for every fixed boundary problem / 1.20 / is the solution of the boundary problem</u>

$$a_{ijkl}\,\bar{\delta}_{kl,i}+b_{ijrr}\,p_{,i}=-\rho\bar{f}_j \quad \text{in } V,$$

$$/\ 1.24\ / \quad a_{ijkl}\,\bar{\delta}_{kl}\,n_i+b_{ijrr}\,p\,n_i=\bar{t}_j \quad \text{on } S,$$

$$e_{irs}\,e_{jpq}\,\bar{\gamma}_{rp,sq}=0 \quad \text{in } V,$$

$$\bar{\gamma}_{rr}=0 \quad \text{in } V,$$

<u>where p is an additional function.</u> This theorem holds for sufficently smooth ρf_j and $\bar{t}_j$ as indicated above. In such formulation the existence of the auxiliary family of the displacements $u_i(\lambda)$ is not essential. It is evident that the problem / 1.24 / does not constitute the boundary problem of the elasticity theory for any of the materials of the family $\mathcal{M}$.

Given the boundary problem determined by $\bar{f}_j$, and M (ν) the unique solution $\delta_{ij}(\nu)$ can be found. After substitution $\delta'_{ij}(\nu) = u_{(i,j)}(\nu)$ this requires solving three second order partial differential equations with appriopriate boundary conditions. In the case, when it suffices to know the limit solution $\lim_{\nu \to \infty} \delta_{ij}(\nu)$ only, the system / 1.24 / must be solved. The latter is simpler than the former because one of the equations is of the first order and only first derivatives of the function p are involved.

Quite formal reasons make it convenient to introduce if possible the hypotetic material M / not belonging to the family $\mathcal{M}$ / for which the boundary problem of the theory of elasticity / 1.1 $/_2$– / 1.4 / reduces to the problem / 1.24 /. It follows that the reduction takes place if and only if for this material

$$/\ 1.25\ / \qquad \tau_{ij} = a_{ijkl}\,\varepsilon_{kl} + q.\,b_{ijrr}\,, \quad \varepsilon_{rr} = 0\,,$$

where q is an arbitrary scalar function. Because of the relation / 1.25 $/_2$ in the hypotetic material M only isochoric motions are allowed. <u>The solution of the boundary problem for the material M (ν) for fixed body forces and surface tractions tends in the limit $\nu \to \infty$ to the solution of the same boundary problem for the hypotetic material M / 1.25 /</u>. This material will be called further the limit incompressible material. From / 1.25 / it follows: <u>The extra stress $q.\,b_{ijrr}$ in general is not the spherical tensor.</u> This stress produces work on isochoric deformations. Because in general elasticity the stored energy function does not exist this fact does not contradict the basic principles of mechanics.

Consider now the hyperelastic material. Because for each material M (ν) of the family / 1.10 / the elastic constants must posses the symmetry / 1.6 / there is

/ 1.26 / $\quad c_{ijkl} = a_{ijkl} + \nu b \hat{c}_{ij} \hat{c}_{kl}$,

/ 1.27 / $\quad a_{ijkl} = a_{klij}$

where b is a scalar. The relation / 1.25 /$_1$ reduces now to

/ 1.28 / $\quad \tau_{ij} = a_{ijkl} \varepsilon_{kl} + q \hat{c}_{ij}$

where q is an arbitrary scalar function. It follows: <u>The extra stress in the hyperelastic materials is the spherical tensor</u>. Because in the linear theory every isotropic elastic material is hyperelastic it follows: <u>In isotropic elastic materials the extra stress is the spherical tensor.</u>

2. Non-linear elasticity.

Identify the material points by their coordinates X^α in the natural state B_X . In the deformed state B_x the points X^α occupy the positions $x^i(X^\alpha)$. Denote the metric tensors of the coordinate systems in B_X and B_x by $g_{\alpha\beta}$ and g_{ij} respectively. The derivatives $\partial x^i/\partial X^\alpha$ constitute the deformation gradient $F^i{}_\alpha$. There exists the unique multiplicative decomposition of $F^i{}_\alpha$ into the orthogonal tensor $R^{i\alpha}$ and positive definite symmetric tensor $U_{\alpha\beta}$. The quadrat of this tensor is the right Cauchy-Green deformation tensor $C_{\alpha\beta}$ / cf. [2] /

/ 2.1 / $\quad F^i{}_\alpha = R^{i\varkappa} U_{\varkappa\alpha} , \quad C_{\alpha\beta} = U_{\alpha\varkappa} U^{\varkappa}{}_{\beta} = F^i{}_\alpha F_{i\beta}$

If the deformation is isochoric

/ 2.2 / $\det C_{\varkappa}{}^{\cdot 3} = 1, \quad (C^{-1})_{\alpha 3} d\, C^{\alpha 3} = 0.$

The basic equations of the non-linear elasticity are

$$\tau^{ij}{}_{,i} - \varrho_{x} f^{j} = 0 \quad \text{in } \mathcal{V}, \qquad \text{/ 2.3 /}$$
$$\tau^{ij} n_i = t^{j} \quad \text{on } \mathcal{S},$$

$$\tau^{ij} = F^{i}{}_{\alpha} F^{j}{}_{\beta}\, h^{\alpha\beta}(C_{\varkappa\varsigma}), \qquad \text{/ 2.4 /}$$

$$C_{\alpha\beta} = F^{i}{}_{\alpha} F_{i\beta}, \qquad \text{/ 2.5 /}$$

where $h^{\alpha\beta}$ is the material function. The density $\varrho_{\varkappa}$ and the unit normal n_i in the state $B_{\varkappa}$ are uniquely determined by the density ϱ_{X} and the normal N_{α} in B_{X} and the deformation gradient $F^{i}{}_{\alpha}$. The appriopriate relations are given in [2] .

Assume one parameter family $\mathcal{M}$ of materials $M(\nu)$, $0 \leq \nu < \infty$. Similary as in the linear theory define that the materials tend to the incompressible material without additional constraints if and only if, for given finite and fixed $C_{\alpha\beta}$

i/ the stresses $\tau^{ij}(\nu)$ produced in the material $M(\nu)$ by the deformation $C_{\alpha\beta}$ tend to infinity if $\det C_{\varkappa}{}^{\beta} \neq 1$,

ii/ the stresses $\tau^{ij}(\nu)$ produced in the material $M(\nu)$ by the deformation $C_{\alpha\beta}$ remain finite if $\det C_{\alpha}{}^{\beta} = 1$.

Consider the particular family $\mathcal{M}$ defined by the relation

$$h^{\alpha\beta}(C_{\varkappa\varsigma}) = a^{\alpha\beta}(C_{\varkappa\varsigma}) + \nu\, b^{\alpha\beta}(C_{\varkappa\varsigma}), \qquad \text{/ 2.6 /}$$

where $a^{\alpha\beta}$ and $b^{\alpha\beta}$ are fixed independent of ν

functions of $C_{\alpha\beta}$. The materials $M(\nu)$ satisfy the above conditions i/ and ii/ if and only if

i/ $a^{\alpha\beta}(C_{\varkappa\varsigma})$ and $b^{\alpha\beta}(C_{\varkappa\varsigma})$ remain finite for arbitrary finite $C_{\varkappa\varsigma}$,

ii/ $b^{\alpha\beta}(C_{\varkappa\varsigma}) = 0$ if $\det C_\alpha{}^\beta = 1$.

Let $x^i(X^\alpha, \lambda)$ constitute the family $\propto$ of finite deformations such that $x^i(X^\alpha, \lambda)$ and $x^i{}_{,\alpha}(X^\alpha, \lambda)$ are pointwise convergent to the finite limits $\bar{x}^i(X^\alpha)$ and $\bar{x}^i{}_{,\alpha}(X^\alpha)$ respectively. Assume additionally that the limit deformation $\bar{x}^i(X^\alpha)$ is isochoric

$$
/\,2.7\,/ \qquad
\begin{aligned}
&\lim_{\lambda\to\infty} x^i(X^\alpha, \lambda) = \bar{x}^i(X^\alpha),\\
&\lim_{\lambda\to\infty} x^i{}_{,\alpha}(X^\alpha, \lambda) = \bar{x}^i{}_{,\alpha}(X^\alpha),\\
&\lim_{\lambda\to\infty} \det C_\alpha{}^\beta(X^\alpha, \lambda) = 1.
\end{aligned}
$$

In accord with / 2.7 /$_{1,2}$ there exist finite limits $\bar{C}_{\alpha\beta}$, $\bar{R}^i{}_\alpha$, $\bar{n}_i$ and $\bar{\varsigma}_x$. Because of / 2.7 /$_3$ it is possible to find at least one monotonic function $\lambda(\nu)$, $\lim_{\nu\to\infty} \lambda(\nu) = \infty$ such that $\nu\, b^{\alpha\beta}[C_{\varkappa\varsigma}(\lambda(\nu))]$ is finite and has finite limit

$$
/\,2.8\,/ \qquad \lim_{\nu\to\infty} \nu\, b^{\alpha\beta}[C_{\varkappa\varsigma}(\lambda(\nu))] = \mu\, m^{\alpha\beta}.
$$

The function $\lambda(\nu)$ fixes the one-to-one correspondence between the family of the materials $M(\nu)$ and the family of deformations $x^i(X^\alpha, \lambda)$. Denote the quantities corresponding to $\lambda(\nu)$ by ν , e.g. $C_{\alpha\beta}(\lambda(\nu)) = C_{\alpha\beta}(\nu)$

On the basis of / 2.1 /-/ 2.6 / the stress produced by the deformation $x^i(X^\alpha, \nu)$ in the material $M(\nu)$ and the equilibrating body force $f^d(\nu)$ and surface tractions $t^d(\nu)$ can be found to be respectively / region $\mathcal{V}$ changes/

$$
/\,2.9\,/ \qquad \tau^{ij}(\nu) = F^i{}_\alpha(\nu)\, F^j{}_\beta(\nu) \left[a^{\alpha\beta}(C_{\varkappa\varsigma}(\nu)) + \nu\, b^{\alpha\beta}(C_{\varkappa\varsigma}(\nu)) \right],
$$

$$/\,2.10\,/ \quad -\varrho_x(\nu) f^j(\nu) = \left\{ F^i_{\ \alpha}(\nu) F^j_{\ ;\beta}(\nu) \left[a^{\alpha\beta}\left(C_{\varkappa\varrho}(\nu)\right) + \nu b^{\alpha\beta}\left(C_{\varkappa\varrho}(\nu)\right)\right]\right\}_{,i} ,$$

$$t^j(\nu) = F^i_{\ \alpha}(\nu) F^j_{\ ;\beta}(\nu) \left[a^{\alpha\beta}\left(C_{\varkappa\varrho}(\nu)\right) + \nu b^{\alpha\beta}\left(C_{\varkappa\varrho}(\nu)\right)\right] n_i(\nu) .$$

Because according to the above assumptions there exist finite limits of $F^i_{\ \alpha}(\nu)$ and $\nu\, b^{\alpha\beta}(C_{\varkappa\varrho}(\nu))$ there exist also finite limits of $\varrho_x(\nu)$, $f^j(\nu)$ and $t^j(\nu)$

$$/\,2.11\,/ \quad \lim_{\nu\to\infty} \varrho_x(\nu) f^j(\nu) = \bar\varrho_x \bar f^j , \quad \lim_{\nu\to\infty} t^j(\nu) = \bar t^j ,$$

where

$$-\bar\varrho_x \bar f^j = \left\{ \bar F^i_{\ \alpha} \bar F^j_{\ ;\beta} \left[a^{\alpha\beta}\left(\bar C_{\varkappa\varrho}\right) + p\, m^{\alpha\beta}\right]\right\}_{,i} ,$$

$$/\,2.12\,/ \quad \bar t^j = \bar F^i_{\ \alpha} \bar F^j_{\ ;\beta} \left[a^{\alpha\beta}\left(\bar C_{\varkappa\varrho}\right) + p\, m^{\alpha\beta}\right] \bar n_i ,$$

$$\det \bar C_{\alpha}^{\ \beta} = 1 .$$

The limit deformation $\bar x^i(X^\alpha)$ is a solution of the boundary problem / 2.12 /. This boundary problem is not the elastic boundary problem for any material $M(\nu)$. Quite formal reasons make it convenient to introduce the hypotetic material for which the boundary problem of finite elasticity / 2.3 /-/ 2.6 / reduces to the boundary problem / 2.12 /. This holds if and only if for the hypotetic material

$$/\,2.13\,/ \quad \tau^{ij} = F^i_{\ \alpha} F^j_{\ ;\beta} \left[a^{\alpha\beta}\left(C_{\varkappa\varrho}\right) + q\, m^{\alpha\beta}\right] , \quad \det C_{\alpha}^{\ \beta} = 1 ,$$

where q is an arbitrary function. Because of the relation / 2.13 /$_2$ this material allows isochoric motions only. In accord with its properties call this material the limit incompressible material.

The limit deformation $x^i(X^\alpha_\infty)$ is a solution of the boundary problem for the material / 2.13 /. In general however the definition / 2.13 / is not self consistent. Namely the definition of the limit incompressible material must depend on the family $\mathcal{M}$ of the materials $M(\nu)$ only, and be independent of the particular family of the deformations α^{ν} used. If for two families of deformations $\alpha_{(1)}$ and $\alpha_{(2)}$ that tend to the same limit $\bar{C}_{\alpha\beta}$, det $\bar{C}_\alpha{}^\beta = 1$ the limits $m^{\alpha\beta}_{(1)}$ and $m^{\alpha\beta}_{(2)}$ / 2.8 / are different it is impossible to define the limit incompressible material M. Therefore: <u>In general in the non-linear theory the limit incompressible material does not exist.</u>

Assume that $b^{\alpha\beta}$ is a continuous function of $C_{\varkappa\varsigma}$. In this case for all the families α that tend to the same limit deformation $\bar{C}_{\alpha\beta}$, det $\bar{C}_\alpha{}^\beta = 1$ we obtain the same $m^{\alpha\beta}$. For families α' with other limit deformation $\bar{C}'_{\alpha\beta}$ we obtain in general $m'^{\alpha\beta} \neq m^{\alpha\beta}$. Assume therefore that $m^{\alpha\beta}$ is not a constant tensor, but depends on $C_{\varkappa\varsigma}$, $m^{\alpha\beta} = m^{\alpha\beta}(C_{\varkappa\varsigma})$ and is defined on the hypersurface det $C_\alpha{}^\beta = 1$. It follows: <u>If $b^{\alpha\beta}$ is continuous function of $C_{\varkappa\varsigma}$ the limit incompressible material always exists</u> and is defined by / 2.13 / where $m^{\alpha\beta}$ is a function of $C_{\varkappa\varsigma}$ defined on the hypersurface det $C_\alpha{}^\beta = 1$. For such a modified definition it follows: <u>There exists the set of the solutions for compressible materials, the limit of which is the solution for the limit incompressible material</u>. Because in the finite elasticity the uniqueness fails, there can exist other solutions. From / 2.13 / it follows: <u>The extra stress in general elasticity is not the spherical tensor</u>.

Let pass to the hyperelastic material. For such material there is / cf. [1] /

$$/\ 2.14\ / \qquad h^{\alpha\beta}(C_{\varkappa\varsigma}) = \frac{\partial \sigma(C_{\varkappa\varsigma})}{\partial C_{\alpha\beta}} ,$$

where $\sigma(C_{\varkappa\varsigma})$ is the stored energy function. Because the form of the function $h^{\alpha\beta}$ is preserved for every ν there is

$$/\ 2.15\ / \qquad a^{\alpha\beta} = \partial\sigma_a / \partial C_{\alpha\beta} , \quad b^{\alpha\beta} = \partial\sigma_b / \partial C_{\alpha\beta} ,$$

where σ_a and σ_b are scalar functions, such that $\sigma_a + \nu\sigma_b$ is the stored energy. In accord with the demands following / 2.6 / there is

$$/\ 2.16\ / \qquad b^{\alpha\beta}(C_{\varkappa\varsigma}) = 0 \quad \text{if} \quad \det C_{\varkappa}{}^{\varsigma} = 1 .$$

It follows that on the whole hypersurface $\det C_{\varkappa}{}^{\varsigma} = 1$ there is $\sigma_b = \text{const.}$

In order to find the limit / 2.8 / represent σ_b as a Taylor series in the neighbourhood of $\overset{*}{C}_{\alpha\beta}$, $\det \overset{*}{C}_{\alpha}{}^{\beta} = 1$

$$/2.17/ \qquad \sigma_b = \sigma_b\big|_* + \frac{\partial\sigma_b}{\partial C_{\alpha\beta}}\Big|_* (C_{\alpha\beta} - \overset{*}{C}_{\alpha\beta}) + \frac{1}{2}\frac{\partial^2\sigma_b}{\partial C_{\alpha\beta}\partial C_{\varkappa\varsigma}}\Big|_* (C_{\alpha\beta} - \overset{*}{C}_{\alpha\beta})(C_{\varkappa\varsigma} - \overset{*}{C}_{\varkappa\varsigma}) +$$

In accord with / 2.16 / to the first order in $C_{\alpha\beta} - \overset{*}{C}_{\alpha\beta}$ there is / the first term only influence the limit $\lim\limits_{\nu\to\infty} \nu\, b^{\alpha\beta}(C_{\varkappa\varsigma}(\nu))$ /

$$/\ 2.18\ / \qquad b^{\alpha\beta}(C_{\varkappa\varsigma}) = \frac{\partial^2\sigma_b}{\partial C_{\alpha\beta}\partial C_{\varkappa\varsigma}}\Big|_* (C_{\varkappa\varsigma} - \overset{*}{C}_{\varkappa\varsigma}) .$$

Consider infinitesimal difference $C_{\varkappa\varsigma} - \overset{*}{C}_{\varkappa\varsigma}$. The tensor $b^{\alpha\beta}$ is equal to zero if this difference satisfies / 2.2 /$_2$. It follows that

$$/\ 2.19\ / \qquad b^{\alpha\beta}(C_{\varkappa\varsigma}) = h\,(C^{-1})^{\alpha\beta}(C^{-1})^{\varkappa\varsigma}(C_{\varkappa\varsigma} - \overset{*}{C}_{\varkappa\varsigma}) ,$$

for arbitrary point on the hypersurface det $C_\alpha{}^\beta = 1$. The scalar function h depends on $C_{\alpha\beta}$. Therefore

$$\lim_{\nu\to\infty} \nu b^{\alpha\beta}(C_{\mathcal{H}\xi}(\nu)) = h\,(C^{-1})^{\alpha\beta}, \tag{2.20}$$

$$\tau^{ij} = F^i_{\ \alpha} F^j_{\ \beta}\, a^{\alpha\beta}(\bar{C}_{\mathcal{H}\xi}) + \mu g^{ij}. \tag{2.21}$$

It follows: In the hyperelastic materials the extra stress is the spherical tensor.

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