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ON THERMODYNAMIC FOUNDATIONS OF VISCOPLASTICITY

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ON THERMODYNAMIC FOUNDATIONS OF VISCOPLASTICITY *)

INTRODUCTION

In agreement with J.Meixner's well founded opinion [1] three different approaches to a thermodynamic theory of continuum can be distinguished. These approaches differ from each other by the foundamental postulates, on which the theory is based. All of them are characterized by the same foundamental requirement that the results should be obtained without having recourse to statistical or kinetic theories. None of these approaches is concerned with the atomic structure of the material. Therefore, they represent a pure phenomenological approach.

The principal postulates of the first approach, initiated by Onsager's works and usually called the classical thermodynamics of irreversible processes, are as follows (cf. S.R.De Groot and P.Mazur [2]): 1. The principle of local state is assumed to be valid. 2. The Gibbs' relation is satisfied. 3. The equation of entropy balance is assumed to involve a term expressing the entropy production which can be represented as a sum of products of fluxes and forces. This term is zero for a state of equilibrium and positive for an irreversible process. 4. The fluxes are function of forces, not necessarily linear. However, the Onsager-Casimir reciprocity relations concern only coefficients of the linear terms of the series expansions. Using methods of this approach, a thermodynamic description of elastic, rheologic and plastic materials was obtained. Let us mention the works by M.A.Biot [3], D.C.Drucker [4], H.Ziegler [5-14] , A.A.Vakulenko [15,16],

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A.E.Green and J.E.Adkins [17], O.W.Dillon [18], G.A.Kluitenberg [19-25], S.L.Koh and A.C.Eringen [26], J.Kestin [27] and J.F.Besseling [28].

The second approach, called the thermodynamic theory of materials with memory, was initiated by the work of B.D.Coleman and W.Noll [29]. The foundamental postulates of this approach are as follows: 1. The temperature and entropy functions are assumed to exist for non-equilibrium states. 2. The principal restriction imposed on the constitutive equations is the Clausius-Duhem inequality. 3. The notion of the thermodynamic state is modified by assuming that the state of a given particle at time t is characterized, in general, by the time history of the local configuration of that particle. It should be emphasised, however, that in particular cases the history of the local configuration of a particle can be determined by giving the actual values of this configuration and its time derivatives (cf., for instance, B.D.Coleman and V.J. Mizel [30]). 4. No limitations are introduced for the processes considered. The constitutive equations are in general nonlinear. Within the frame-work of this approach, thermodynamic foundations of rheologic materials were established cf. B.D.Coleman [31,32], B.D.Coleman and V.J.Mizel [30], C.Truesdell and W.Noll [33] and R.Christensen and P.M.Naghdi [34]. The same was done for plastic materials also (cf. A.E.Green and P.M.Naghdi [35,36] and O.W.Dillon [37]).

The third approach has been developed by J.Meixner [38-40] and is called the thermodynamic theory of passive systems. It is based on the following postulates: 1. The introduction of the notion of entropy is avoided for non-equilibrium states and the principle of local state is not assumed. 2. The Clausius-Duhem inequality is replaced by an inequality expressing the fundamental property of passivity. This inequality follows from the second law of

thermodynamics and the condition of thermodynamic stability. Further the inequality is known to have sense only for states of equilibrium. 3. The temperature is assumed to exist for non-equilibrium states. 4. As a consequence of the foundamental inequality the class of processes under consideration is limited to processes in which deviations from the equilibrium conditions are small. This enables full linearization of the constitutive equations. An important feature of this approach is the clear physical interpretation of all the quantities introduced.

Each of the three approaches above has its weaknesses and none is commonly accepted*). The first is subject to excessive limitations in the form of the assumptions of the Onsager-Casimir relations. Its present development does not appear to be promising for the overcoming of the difficulties that are encountered in nonlinear mechanics**) . The second approach is criticized principally from the point of view of physical foundations (cf. J.Meixner [1]) . Indeed, we must agree with the opinion, that the problem of physical interpretation of quantities such as the temperature or entropy has not found a detailed treatment within the frame-work of this approach. The advantages of the second approach are the mathematical foundations which are very well developed and offer a possibility of analysis of many interesting processes. They can also be used for the description of nonlinear materials. It is also worth mentioning that the theories of

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*) This fact was pointed out by a detailed discussion at the IUTAM Symposium on Irreversible Aspects of Continuum Mechanics in Vienna, June 1966 (cf. J.Meixner [1] and C.Truesdell [41]).

**) A detailed critical analysis of this approach can be found in the monograph of C.Truesdell and R.Toupin [42].

elastic and viscoelastic materials can be obtained as particular cases of the theory of materials with memory (cf.B.D. Coleman [31,32]) . This theory enables the description of many important mechanical phenomena, such as elastic instability and phenomena accompanying wave propagation (cf. C.Truesdell [41]) . The applicability of the methods of the third approach is, on the other hand, limited to linear problems. It does not seem likely that further generalization to nonlinear problems is possible within the framework of the assumptions of this approach. The results obtained concern problems of linear viscoelasticity only (cf. J.Meixner [38-40,43]).

It is worth mentioning that recent works concerned with axiomatic foundations of continuum thermodynamics have shown in a clear manner the correctness of the conception of the second approach*). Although they have not removed the objections against the physical foundations of the theory, they have formulated in a mathematically accurate manner the applicability conditions of the methods of rational thermodynamics.

The aim of the present paper is to discuss the thermodynamic foundations of the theory of viscoplasticity.The essential feature of which is the simultaneous description of rheologic and plastic effects of a material. The necessity of simultaneous consideration of viscoelastic and plastic properties of a material is indicated by the results of experimental investigations of dynamic loads. These investigations show clearly that during dynamic load-

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*) The axiomatic foundations of continuum thermodynamics have been presented in the papers by M.E.Gurtin and W.O.Williams [44,45]. They are a generalization to thermodynamics problems of the earlier conceptions of W.Noll [46,47] concerning the purely mechanical theory of a continuum.

ing of a test piece the plastic and viscoelastic effects are coupled and play roles equal importance[x]. The viscous properties of the material introduce a time dependence of the states of stress and strain. The plastic properties, on the other hand, make these states depend on the deformation path. Different results will be obtained for different deformation paths and for different time durations of the process.

It appears that by investigating thermodynamic processes in viscoplastic materials, characterized by a nonlinearity resulting from dependence on the time and the path, their description can be obtained within the assumptions of the second approach. We shall try to show that the methods of thermodynamics of materials with memory can be used for the establishment of the thermodynamic foundations of viscoplasticity.

It will be proposed to describe a viscoplastic material as a material with memory, for which the history of the local configuration depends on the time as well as the path. Viscoplastic materials of the rate type, for which the path-dependency is characterized by different sets of constitutive equations, for the loading and unloading process, will be discussed in more detail. As a particular case of an elastic-viscoplastic material of the rate type, a rate sensitive plastic material will be discussed.

Finite deformations of a body during the general thermodynamic process will be considered. All the constitutive equations obtained will be invariant under a change of the reference frame.

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x) A discussion of the results of experimental investigations in the domain of dynamic loads acting on metals can be found in the paper [48].

PRELIMINARY CONSIDERATIONS

In general we shall use a similar notation as in monograph of C.Truesdell and W.Noll [33]. The motion of the body $\mathcal{B}$ with points $\underset{\sim}{X}$ is described by the equation

$$\underset{\sim}{x} = \underset{\sim}{\chi}(\underset{\sim}{X}, t), \tag{1}$$

where $\underset{\sim}{x}$ denotes the spatial position occupied by the material point $\underset{\sim}{X}$ at time t [x]. The gradient $\underset{\sim}{F}$ of $\underset{\sim}{\chi}$ with respect to $\underset{\sim}{X}$, i.e.,

$$\underset{\sim}{F} = \partial \underset{\sim}{\chi}(\underset{\sim}{X}, t)/\partial \underset{\sim}{X} \tag{2}$$

is the deformation gradient at the points $\underset{\sim}{X}$ relative to the reference configuration. It is assumed that $\det \underset{\sim}{F} \neq 0$. By the polar decomposition of $\underset{\sim}{F}$, viz.,

$$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U} \tag{3}$$

we define the orthogonal rotational tensor $\underset{\sim}{R}$ and positive definite and symmetric right stretch tensor $\underset{\sim}{U}$. Similarly, the relation

$$\underset{\sim}{C} = \underset{\sim}{F}^{T}\underset{\sim}{F} = \underset{\sim}{U}^{2}, \tag{4}$$

where $\underset{\sim}{F}^{T}$ denotes the transpose of $\underset{\sim}{F}$, defines the right Cauchy-Green tensor.

If $\underset{\sim}{F}$ is replaced by the relative deformation gradient $\underset{\sim}{F}_{(t)}$, then the notations $\underset{\sim}{R}_{(t)}$, $\underset{\sim}{U}_{(t)}$ and $\underset{\sim}{C}_{(t)}$ are used respectively for the corresponding relative rotation tensor relative stretch tensor, and relative Cauchy-Green tensor. The expressions $\underset{\sim}{W}(t) = \dot{\underset{\sim}{R}}_{(t)}(t)$ and $\underset{\sim}{D}(t) = \dot{\underset{\sim}{U}}_{(t)}(t)$ define the spin and the stretching tensor, re-

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x) We identify the material point X with its position $\underset{\sim}{X}$ in the reference configuration.

spectively.

Let us denote the Cauchy stress tensor by $\underset{\sim}{T}(t)$. We shall introduce, after W.Noll [46], the following co-rotational stress rate:

$$\overset{\circ}{\underset{\sim}{T}} = \dot{\underset{\sim}{T}} - \underset{\sim}{W}\underset{\sim}{T} + \underset{\sim}{T}\underset{\sim}{W}, \tag{5}$$

where $\underset{\sim}{W}$ denotes spin.

A thermodynamic process in $\mathcal{B}$ is described by eight functions $\{\underset{\sim}{\chi}, \underset{\sim}{T}, \underset{\sim}{b}, \varepsilon, \eta, \vartheta, \underset{\sim}{q}, r\}$ of $\underset{\sim}{X}$ and time t. The values of these functions have the following physical interpretation (cf., B.D.Coleman and W.Noll [29]). The function $\underset{\sim}{\chi}(\underset{\sim}{X},t)$ describes the motion of the body $\mathcal{B}$ and is called the deformation function; $\underset{\sim}{T}(\underset{\sim}{X},t)$ is the symmetric stress tensor; $\underset{\sim}{b}(\underset{\sim}{X},t)$ is the body force per unit mass; $\varepsilon(\underset{\sim}{X},t)$ denotes specific internal energy per unit mass; $\eta(\underset{\sim}{X},t)$ the specific entropy and $\vartheta(\underset{\sim}{X},t)$ is the local absolute temperature; $\underset{\sim}{q}(\underset{\sim}{X},t)$ is the heat flux vector and $r(\underset{\sim}{X},t)$ the heat supply per unit mass and unit time.

The set of eight functions $\{\underset{\sim}{\chi}, \underset{\sim}{T}, \underset{\sim}{b}, \varepsilon, \eta, \vartheta, \underset{\sim}{q}, r\}$ is called a thermodynamic process if, and only if, it is compatible with the condition for the balance of linear momentum (Cauchy's first law of motion) *)

$$\operatorname{div}\underset{\sim}{T} - \varrho\ddot{\underset{\sim}{\chi}} = -\varrho\underset{\sim}{b}, \tag{6}$$

and the law of balance of energy (the first law of thermodynamics)

$$\operatorname{tr}\{\underset{\sim}{T}\underset{\sim}{L}\} - \operatorname{div}\underset{\sim}{q} - \varrho\dot{\varepsilon} = -\varrho r, \tag{7}$$

where ϱ denotes the mass density, $\underset{\sim}{L} = \partial\dot{\underset{\sim}{\chi}}/\underset{\sim}{\chi}$ and

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*) Cauchy's second law of motion requires $\underset{\sim}{T} = \underset{\sim}{T}^{T}$, which we assumed in advance. Couple stresses and body couples are assumed to be absent.

the trace operator is denoted by tr .

In order to define a thermodynamic process, it suffices to prescribe the six functions $\{\underset{\sim}{\chi}, \underset{\sim}{T}, \varepsilon, \eta, \vartheta, \underset{\sim}{q}\}$. The two remaining functions $\underset{\sim}{b}$ and r are then uniquely determined by Eqs.(6) and (7).

Two thermodynamic processes $\{\underset{\sim}{\chi}, \underset{\sim}{T}, \varepsilon, \eta, \vartheta, \underset{\sim}{q}\}$ and $\{\underset{\sim}{\chi}^*, \underset{\sim}{T}^*, \varepsilon^*, \eta^*, \vartheta^*, \underset{\sim}{q}^*\}$ are equivalent if they are related only by a change of a reference frame.

We assume that all constitutive equations describing the physical properties of the material satisfy the following principle of material frame-indifference (cf. W.Noll [46] and C.Truesdell and W.Noll [33]) : If process $\{\underset{\sim}{\chi}, \underset{\sim}{T}, \varepsilon, \eta, \vartheta, \underset{\sim}{q}\}$ is compatible with a system of constitutive equations, then all processes $\{\underset{\sim}{\chi}^*, \underset{\sim}{T}^*, \varepsilon^*, \eta^*, \vartheta^*, \underset{\sim}{q}^*\}$ equivalent to it must be compatible with the same system of constitutive equations.

A thermodynamic process in $\mathcal{B}$, compatible with the constitutive equations at each point X of $\mathcal{B}$ and for all time t , will be called an admissible process in $\mathcal{B}$.

Thus, the principle of material frame-indifference states, that if a thermodynamic process $\{\underset{\sim}{\chi}, \underset{\sim}{T}, \varepsilon, \eta, \vartheta, \underset{\sim}{q}\}$ is admissible in $\mathcal{B}$, then also all thermodynamic processes $\{\underset{\sim}{\chi}, \underset{\sim}{T}, \varepsilon, \eta, \vartheta, \underset{\sim}{q}\}$ must be admissible in $\mathcal{B}$. The physical meaning of this principle is simply that the material properties of a body should not depend on the observer, irrespective of how he moves.

We assume the following postulate: For every admissible thermodynamic process in a body $\mathcal{B}$ the production of entropy must be nonnegative.

Under suitable smoothness assumptions we can write

$$(8)\quad -\dot{\psi} + \frac{1}{\varrho} tr\{\underset{\sim}{F}^{-1}\underset{\sim}{T}\dot{\underset{\sim}{F}}\} - \eta\dot{\vartheta} - \frac{1}{\varrho\vartheta}\underset{\sim}{q}\cdot \operatorname{grad}\vartheta \geq 0,$$

where $\psi = \varepsilon - \vartheta\eta$ is the specific free energy function. This is a local form of Clausius-Duhem inequality which is a mathematical statement of the second law of thermonamics*).

TIME AND PATH DEPENDENT MATERIALS

The non-linear material with memory is defined by a system of constitutive equations (cf. B.D.Coleman [31,32], C.Truesdell and W.Noll [33] and C.-C.Wang and R.M.Bowen [49])

$$\Pi(t) = \overset{t}{\underset{\tau=-\infty}{\mathcal{F}}}(\Psi(\tau)), \tag{9}$$

in which $\Pi(t)$ represents the actual values of some state functions such as the stress tensor $\underline{T}$, the specific free energy ψ, the heat flux $\underline{q}$ and the specific entropy η. $\Psi(\tau)$ is a function which represents the history of the local configuration of a material. It is assumed that the history of the local configuration $\Psi(\tau)$ can be characterized by the history of the local deformation gradient $\underline{F}(\tau)$ with $\tau \in (-\infty, t]$; the history of the local temperature $\vartheta(\tau)$ with $\tau \in (-\infty, t]$, and the local temperature gradient, grad ϑ **). Thus we have

$$\Pi(t) = \{\underline{T}(t), \psi(t), \underline{q}(t), \eta(t)\}, \tag{10}$$

$$\Psi(\tau) = \{\underline{F}(\tau), \vartheta(\tau), \operatorname{grad}\vartheta\}, \quad \tau \in (-\infty, t]. \tag{11}$$

The response of the material is characterized by the functional $\mathcal{F}$, called the constitutive functional. The functional $\mathcal{F}$ must satisfy an invariance requirement rela-

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*) For the discussion of the conditions under which a local form of the Clausius-Duhem inequality [8] is valid see M.E.Gurtin and W.O.Williams [45].

**) Different assumptions have been discussed by M.E.Gurtin [50].

tive to a change of the observer, some requirements of the symmetry and special smoothness requirements (cf. B.D.Coleman [31,32]).

We shall now concern ourselves with simultaneous description of the elastic, rheologic and plastic properties of a material. To this end, let us accurately define the meaning of the plastic behaviour of a material. Unloading is an important feature which distinguishes the behaviour of a plastic material from that of a nonlinear material with memory. We shall introduce the distinction between /unloading, neutral state, and loading phenomena.

In inviscid plasticity, it is assumed that the material deforms elastically until the state of what is called an initial yield surface or the loading surface is reached. Unloading neutral state and loading in this theory are defined as follows (cf. A.E.Green and P.M.Naghdi [35]) . An unloading process has taken place if the deformation from an existing elastic-plastic state takes place elastically so that the stress point lies inside the yield surface. Neutral state has taken place if no additional plastic strain is produced when the stress point lies on the yield surface. Similarly a loading process has taken place, if due to additional deformation the stress point reaches a subsequent yield surface.

The determination of the yield condition for a nonlinear material with memory at finite deformations is very difficult and has not yet been achieved. Thus, in the formulation of the general constitutive equations of an elastic-viscoplastic material, we do not use the yield condition. We shall assume that the material of a body $\mathcal{B}$ will show plastic effects from the initial configuration χ_{t_o} This initial configuration χ_{t_o} will be called the configuration of yielding.

Thus, as a result of simultaneous consideration of rheologic and plastic properties of a material a descript-

ion of the actual state functions involves the history of the local configuration expressed as a function of time and of path. To make this idea clear, let us introduce the definition of path in the ten-dimensional space of deformation and temperature as follows*)

$$s(t) = \int_{-\infty}^{t} [\mathrm{tr}(\dot{\underline{F}}\dot{\underline{F}}^{T}) + \dot{\vartheta}^{2}]^{1/2} d\tau' . \tag{12}$$

We shall introduce the history of the local configuration $\Xi(t,s)$ in the following form

$$\Xi(t,s) = \{\Psi(\tau), s\}, \qquad \begin{matrix} \tau \in (-\infty, t], \\ s \in [0, s(t)]. \end{matrix} \tag{13}$$

The constitutive equations for time and path dependent material can be defined by the system of the form

$$\Pi(t) = g(\Xi(\tau,s)) \qquad \tau \in (-\infty, t),\ s \in [0, s(t)] \tag{14}$$

ELASTIC-VISCOPLASTIC MATERIAL OF THE RATE TYPE

Unloading from an elastic-viscoplastic state follows a path in the deformation and temperature space different from that of loading. We assume the following definition.

A thermodynamic process represents unloading if the condition

$$\frac{1}{\varrho}\,\mathrm{tr}\{\underline{T}\,\underline{L}\} - \eta\dot{\vartheta} < 0 \tag{15}$$

is satisfied, i.e., if the rate of work of the generalized stress is negative. The case

$$\frac{1}{\varrho}\,\mathrm{tr}\{\underline{T}\,\underline{L}\} - \eta\dot{\vartheta} = 0 \tag{16}$$

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*) This idea is similar to that of A.C.Pipkin and R.S.Rivlin [51].

is called neutral state, whereas the positive rate of work of the generalized stress

$$\frac{1}{\varrho}\,\mathrm{tr}\{\underset{\sim}{T}\underset{\sim}{L}\} - \eta\dot{\vartheta} > 0 \tag{17}$$

determines the loading process*).

Let us introduce the following definition.

A non-linear material has plastic properties if its behaviour is described by different constitutive equations for loading and unloading.

This formal definition shows that the loading and unloading phenomena for the same material are characterized by different features.

We shall not bother to introduce the distinction between elastic, viscous and plastic deformations. Total deformation will be treated as a combined result of elastic, rheologic and plastic effects.

We now intend to obtain, on the basis of the thermodynamic theory of the rate type material, the general constitutive equations of an elastic-viscoplastic material**). We shall use only the first order differential equations. This, of course, implies some restrictions concerning the memory of a material. We assume following definition.

An elastic-viscoplastic material is a simple material of the rate type of the first order which is characterized by different properties during the loading and unloading processes.

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*) During the isothermal process, the criteria of unloading, neutral state and loading take the respective forms $\mathrm{tr}\{\underset{\sim}{T}\underset{\sim}{L}\} \lessgtr 0$, i.e., they coincide with those first introduced by A.E.Green [52].

**) Cf. the previous papers of the author [53,54].

Thus, in the thermodynamic process in a body $\mathcal{B}$, which represents loading i.e., the condition (17) is satisfied, we postulate for an elastic-viscoplastic material the following system of the constitutive equations

$$\dot{\pi}(t) = \bar{f}\left(\pi(t), \psi(t), \dot{\psi}(t)\right) \tag{18}$$

with initial values

$$\pi(t_0) = \left\{\psi(t_0), \eta(t_0), \underline{T}(t_0), \underline{q}(t_0)\right\}. \tag{19}$$

For the thermodynamic process in a body $\mathcal{B}$, which is unloading i.e., satisfying the condition (15), we assume the following system of the constitutive equations

$$\dot{\pi}(t) = \hat{f}\left(\pi(t), \psi(t), \dot{\psi}(t)\right). \tag{20}$$

The response functions $\bar{f}$ and $\hat{f}$ must satisfy the following condition for a neutral state

$$\bar{f} = \hat{f} \quad \text{if} \quad \frac{1}{\varrho}\operatorname{tr}\{\underline{T}\,\underline{L}\} - \eta\vartheta = 0. \tag{21}$$

This condition is the continuity condition for the first derivatives $\dot{\psi}$, $\dot{\eta}$, $\dot{\underline{T}}$ and $\dot{\underline{q}}$. Additionally, we have to assume the continuity condition for the functions ψ, η, $\underline{T}$ and $\underline{q}$. This assumption determines the initial values for the system (20).

It is worth noting, that the path dependence in the constitutive equations of an elastic-viscoplastic material of the rate type is expressed by the following fact. The properties of this material are described by a different system of equations for the loading and for the unloading path.

After satisfying the principle of material frame-indifference, we can write the systems of constitutive

equations (18) and (20) respectively in the following reduced and explicit form

$$
\begin{aligned}
\dot{\psi} &= \bar{p}\,(\psi, \underset{\sim}{D}^*, \underset{\sim}{U}, \vartheta, \dot{\vartheta}, \underset{\sim}{R}^T \operatorname{grad}\vartheta),\\
\dot{\eta} &= \bar{h}\,(\eta, \underset{\sim}{D}^*, \underset{\sim}{U}, \vartheta, \dot{\vartheta}, \underset{\sim}{R}^T \operatorname{grad}\vartheta),\\
\overset{\circ}{\underset{\sim}{T}}{}^* &= \bar{\underset{\sim}{T}}\,(\underset{\sim}{T}^*, \underset{\sim}{D}^*, \underset{\sim}{U}, \vartheta, \dot{\vartheta}, \underset{\sim}{R}^T \operatorname{grad}\vartheta),\\
\underset{\sim}{R}^T \overset{\circ}{\underset{\sim}{q}} &= \bar{\underset{\sim}{q}}\,(\underset{\sim}{R}^T \underset{\sim}{q}, \underset{\sim}{D}^*, \underset{\sim}{U}, \vartheta, \dot{\vartheta}, \underset{\sim}{R}^T \operatorname{grad}\vartheta),
\end{aligned}
\tag{22}
$$

and

$$
\begin{aligned}
\dot{\psi} &= \hat{p}\,(\psi, \underset{\sim}{D}^*, \underset{\sim}{U}, \vartheta, \dot{\vartheta}, \underset{\sim}{R}^T \operatorname{grad}\vartheta),\\
\dot{\eta} &= \hat{h}\,(\eta, \underset{\sim}{D}^*, \underset{\sim}{U}, \vartheta, \dot{\vartheta}, \underset{\sim}{R}^T \operatorname{grad}\vartheta),\\
\overset{\circ}{\underset{\sim}{T}}{}^* &= \hat{\underset{\sim}{T}}\,(\underset{\sim}{T}^*, \underset{\sim}{D}^*, \underset{\sim}{U}, \vartheta, \dot{\vartheta}, \underset{\sim}{R}^T \operatorname{grad}\vartheta),\\
\underset{\sim}{R}^T \overset{\circ}{\underset{\sim}{q}} &= \hat{\underset{\sim}{q}}\,(\underset{\sim}{R}^T \underset{\sim}{q}, \underset{\sim}{D}^*, \underset{\sim}{U}, \vartheta, \dot{\vartheta}, \underset{\sim}{R}^T \operatorname{grad}\vartheta),
\end{aligned}
\tag{23}
$$

where $\underset{\sim}{T}^* = \underset{\sim}{R}^T \underset{\sim}{T} \underset{\sim}{R}$, $\overset{\circ}{\underset{\sim}{T}}{}^* = \underset{\sim}{R}^T \overset{\circ}{\underset{\sim}{T}} \underset{\sim}{R}$ and $\underset{\sim}{D}^* = \underset{\sim}{R}^T \underset{\sim}{D} \underset{\sim}{R}$

We assume that the constitutive equations of an elastic-viscoplastic material in both the loading and the unloading ranges satisfy the thermodynamic postulate. By (8) we have the following inequalities:

$$
-\bar{p}(\psi, \underset{\sim}{D}^*, \underset{\sim}{U}, \vartheta, \dot{\vartheta}, \underset{\sim}{R}^T \operatorname{grad}\vartheta) + \frac{1}{\varrho} \operatorname{tr}\{\underset{\sim}{T}\underset{\sim}{L}\} - \eta\dot{\vartheta} - \frac{1}{\varrho\vartheta}\underset{\sim}{q}\cdot\operatorname{grad}\vartheta \geqq 0 \tag{24}
$$

for

$$
\frac{1}{\varrho} \operatorname{tr}\{\underset{\sim}{T}\underset{\sim}{L}\} - \eta\dot{\vartheta} > 0, \tag{25}
$$

and

$$
-\hat{p}(\psi, \underset{\sim}{D}^*, \underset{\sim}{U}, \vartheta, \dot{\vartheta}, \underset{\sim}{R}^T \operatorname{grad}\vartheta) + \frac{1}{\varrho} \operatorname{tr}\{\underset{\sim}{T}\underset{\sim}{L}\} - \eta\dot{\vartheta} - \frac{1}{\varrho\vartheta}\underset{\sim}{q}\cdot\operatorname{grad}\vartheta \geqq 0, \tag{26}
$$

for

$$(27) \qquad \frac{1}{\varrho} \operatorname{tr}\{\underline{T}\underline{L}\} - \eta \dot{\vartheta} < 0 .$$

The inequalities (24) - (27) represent the basis restrictions imposed on the constitutive equations for an elastic-viscoplastic material.

RATE SENSITIVE PLASTIC MATERIAL

Recent theoretical and experimental research in the domain of the dynamical properties of materials has shown the significant sensitivity of some materials to the rate of deformation. This effect is disregarded in the inviscid theory of plasticity. The influence of strain rate may, however, be taken into account, within the framework of assumptions of an elastic-viscoplastic material of rate type.

Every material displays more or less definite viscous properties. For many materials, however, these properties are more pronounced after the plastic state has been reached. In these cases it may be assumed that material displays viscous properties in the plastic range only.

General foundations for the study of problems connected with rate sensitive plastic material were given by K.Hohenemser and W.Prager [55]. Further development of this idea is contained in the papers [48, 56 - 61].

The basic assumption in the theory of rate sensitive plastic materials is the additivity of the elastic and inelastic parts of the rate of deformation tensor

$$(28) \qquad \underline{D} = {}^{e}\underline{D} + {}^{i}\underline{D}$$

where ${}^{e}\underline{D}$ and ${}^{i}\underline{D}$ are the elastic and inelastic parts of the stretching tensor, respectively.

To obtain the elastic response from rate type material we should assume that the constitutive equation (22)$_3$

is invariant under a change of time-scale and is independent on the stretch tensor $\underset{\sim}{U}$. Thus, the constitutive equation of a rate type for elastic response has form similar to that of hypoelastic material

$$(29) \qquad \overset{\circ}{\underset{\sim}{T}} = \underset{\sim}{H}_1(\underset{\sim}{T}^*, \vartheta)[{}^e\underset{\sim}{D}] + \underset{\sim}{H}_2(\underset{\sim}{T}^*, \vartheta)\dot{\vartheta}$$

After substituting (28) into this equation we have following result

$$(30) \qquad \overset{\circ}{\underset{\sim}{T}} = \underset{\sim}{H}_1(\underset{\sim}{T}^*, \vartheta)[\underset{\sim}{D} - {}^i\underset{\sim}{D}] + \underset{\sim}{H}_2(\underset{\sim}{T}^*, \vartheta)\dot{\vartheta}.$$

Since the material has no viscous properties in the elastic region, the choice of an adequate yield criterion will be much simpler than in the case of an elastic-viscoplastic material. The initial yield condition, which will be called the static yield criterion, will not differ from the known condition of the inviscid theory of plasticity at finite strains.

In order to keep our considerations sufficiently general, we now introduce a static yield function in the form

$$(31) \qquad \mathcal{F}(\underset{\sim}{T}^*, {}^i\underset{\sim}{E}, \vartheta) = \frac{f(\underset{\sim}{T}^*, {}^i\underset{\sim}{E}, \vartheta)}{\varkappa(\underset{\sim}{T}^*, {}^i\underset{\sim}{E}, \vartheta)} - 1,$$

where the function $f(\underset{\sim}{T}^*, {}^i\underset{\sim}{E}, \vartheta)$ depends on the state of stress $\underset{\sim}{T}$ and the state of inelastic strain ${}^i\underset{\sim}{E}$ and temperature ϑ . The strain tensor $\underset{\sim}{E}$ is defined by the relation $2\underset{\sim}{E} = \underset{\sim}{C} - \underset{\sim}{1}$ and it is assumed that $\underset{\sim}{E} = {}^e\underset{\sim}{E} + {}^i\underset{\sim}{E}$ *). The work-hardening parameter $\varkappa$ is defined by the expression (cf. A.E.Green and P.M.Naghdi [35])

*) We understand this assumption as a very restrictive postulate.

$$(32) \qquad \dot{\varkappa} = \mathrm{tr}\{\underset{\sim}{N}(\underset{\sim}{T}^*, {}^i\underset{\sim}{E}, \vartheta)\, {}^i\underset{\sim}{D}^*\}$$

where $\underset{\sim}{N}$ is a tensor function. The flow surface, $\overline{F} = 0$, in the ten-diemensional stress and temperature space is assumed regular and convex.

We propose for the inelastic part of the rate of deformation tensor the equation

$$(33) \qquad {}^i\underset{\sim}{D} = \gamma(\vartheta)\langle\phi(\overline{F})\rangle \underset{\sim}{M}(\underset{\sim}{T}^*, \vartheta, {}^i\underset{\sim}{E}),$$

where the tensor function $\underset{\sim}{M}$ satisfies the relation

$$(34) \qquad \underset{\sim}{M} = \underset{\sim}{M}^T,$$

$\gamma(\vartheta)$ denotes a viscosity coefficient and the symbol $\langle\phi(\overline{F})\rangle$ is defined as follows:

$$(35) \qquad \langle\phi(\overline{F})\rangle = \begin{cases} 0 & \text{for } \overline{F} \leq 0, \\ \varphi(\overline{F}) & \text{for } \overline{F} > 0. \end{cases}$$

The function $\phi(\overline{F})$ may be chosen to represent the results of tests on the behaviour of metals under dynamic loading. The proper choice of $\phi(\overline{F})$ at the same time enables a description of the influence of the rate of deformation and the temperature on the yield limit of the material.

By (30) and (33) we have

$$(36) \qquad \overset{\circ}{\underset{\sim}{T}} = \underset{\sim}{H}_1(\underset{\sim}{T}^*, \vartheta)\left[\underset{\sim}{D} - \gamma(\vartheta)\langle\phi(\overline{F})\rangle \underset{\sim}{M}(\underset{\sim}{T}^*, \vartheta, {}^i\underset{\sim}{E})\right] + \underset{\sim}{H}_2(\underset{\sim}{T}^*, \vartheta)\,\dot{\vartheta}.$$

This constitutive equation involves the assumption that the inelastic part of the rate of deformation tensor is a function of excess stresses above the static yield criterion. This function of stress above the static yield criterion generates the inelastic rate of deformation tensor according to a viscosity law of the Maxwell type.

It can easily be seen that the constitutive equation (36) leads to the following dynamic yield condition

$$(37)\quad \psi(\underline{T}^*, {}^i\underline{E}, \vartheta) = \varkappa(\underline{T}^*, {}^i\underline{E}, \vartheta)\left\{1 + \Phi^{-1}\left[\frac{(\mathrm{tr}\,{}^i\underline{D}^2)^{1/2}}{\gamma(\vartheta)}(\mathrm{tr}\,\underline{M}^2)^{-1/2}\right]\right\}.$$

This relation determines the change of the actual yield surface during the inelastic deformation process. This change is caused by isotropic and anisotropic work-hardening effects and by influence of the rate of deformation tensor and temperature on the yield point of the material.

The full system of constitutive equations for a rate sensitive plastic material has been postulated in the following form*)

$$(38)\quad \begin{aligned}
\dot{\psi} &= p_1(\psi, \vartheta)\,\mathrm{tr}(\underline{D} - {}^i\underline{D}) + p_2(\psi, \vartheta)\,\dot{\vartheta},\\
\eta &= n_1(\eta, \vartheta)\,\mathrm{tr}(\underline{D} - {}^i\underline{D}) + n_2(\eta, \vartheta)\,\dot{\vartheta},\\
\overset{\circ}{\underline{T}} &= \underline{H}_1(\underline{T}^*, \vartheta)[\underline{D} - {}^i\underline{D}] + \underline{H}_2(\underline{T}^*, \vartheta)\,\dot{\vartheta},\\
\underline{R}^T\overset{\circ}{\underline{q}} &= \underline{q}_1(\underline{R}^T\underline{q}, \vartheta, \underline{R}^T\mathrm{grad}\,\vartheta)[\underline{D} - {}^i\underline{D}] + \underline{q}_2(\underline{R}^T\underline{q}, \vartheta, \underline{R}^T\mathrm{grad}\,\vartheta)\,\dot{\vartheta},\\
{}^i\underline{D} &= \gamma(\vartheta)\langle\Phi(\mathcal{F})\rangle\,\underline{M}(\underline{T}^*, \vartheta, {}^i\underline{E}),
\end{aligned}$$

*) A different approach for establishing the constitutive equations for a rate sensitive plastic material has been recently presented in the paper [81].

with initial values

$$\psi(t_0)\ ,\ \eta(t_0),\ \underline{T}(t_0)\ ,\ \underline{q}(t_0). \tag{39}$$

The constitutive equations must satisfy the thermodynamic postulate

$$-\rho_1(\psi;\vartheta)\,\mathrm{tr}\{\underline{D} - \gamma(\vartheta)\langle\Phi(F)\rangle\underline{M}(\underline{T}^*,\vartheta;{}^{i}E)\} - \rho_2(\psi,\vartheta)\dot{\vartheta}$$
$$+\frac{1}{\varrho}\mathrm{tr}\{\underline{T}\underline{L}\} - \eta\dot{\vartheta} - \frac{1}{\varrho\vartheta}\underline{q}\cdot\mathrm{grad}\,\vartheta \geqq 0. \tag{40}$$

After assuming the infinitesimal deformations (cf. the definition in the monograph by C.Truesdell and R.Toupin [42]) and perfectly plastic material with Huber-Mises initial yield condition we obtain from $(38)_3$ and $(38)_5$ the constitutive equations for a temperature and rate sensitive plastic material which were discussed in the paper [59]. In that paper a detailed analysis of some particular cases of the constitutive equations and a comparison of theoretical and experimental results for metals can be found. Reference [59] also presents a complete discussion of the problem of an appropriate selection of the temperature dependent coefficients (cf. also the review paper [48]).

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Streszczenie

Termodynamiczne podstawy lepkoplastyczności

Celem obecnej pracy jest dyskusja termodynamicznej metody jednoczesnego traktowania zjawisk reologicznych i plastycznych oraz konstrukcja termodynamicznej teorii nieliniowych materiałów lepkoplastycznych, która może być użyta do opisu zachowania się metali pod obciążeniami dynamicznymi.

W pierwszej części pracy przedyskutowano trzy różne podejścia termodynamiczne do ośrodka ciągłego. Wykazano, że termodynamiczne podstawy lepkoplastyczności mogą być zbudowane w ramach koncepcji materiałów z pamięcią. Nieliniowy materiał z pamięcią jest zdefiniowany za pomocą układu równań konstytutywnych, w którym pewne funkcje stanu takie jak tensor naprężenia, energia wewnętrzna, strumień ciepła itd. są określone przez funkcjonały zależne od funkcji, która reprezentuje historię lokalnej konfiguracji materialnej cząsteczki. Jako rezultat jednoczesnego uwzględnienia sprężystych, lepkich i plastycznych właściwości materiału opis funkcji aktualnego stanu wprowadza historię lokalnej konfiguracji materialnej cząsteczki jako funkcję czasu i drogi.

Zostały zanalizowane ograniczenia jakie nakłada drugie prawo termodynamiki oraz zasada obiektywności.

W drugiej części pracy przedyskutowano pewne szczególne przypadki równań konstytutywnych. Między innymi został zbadany materiał lepkoplastyczny typu prędkościowego i materiał wrażliwy na prędkość odkształcenia.