APPROXIMATE AND EXACT SOLUTIONS OF THE FIRST-PASSAGE PROBLEM FOR STOCHASTIC OSCILLATORS

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1. Introduction and motivation

In problems that cover a wide range of engineering applications, a reference mode of the system operation is associated with motion within an admissible domain G; exit from this domain can be in some sense catastrophic. Examples are the loss of integrity and the destruction of structures, the loss of stability, etc. The mean time to escape from a region of desired operations is one of the most basic reliability measures in stochastic dynamics.

Formally, the mean escape time $\mathbf{E}\tau$ can be found as a solution to the boundary problem for a relevant Fokker-Plank equation [1]. Whilst there are a small number of cases for which closed-form solutions are available, in most cases one must attempt numerical approximation, and so one is limited to only low dimensional problems. Even putting aside the restriction of numerical methods to low dimensions, one would prefer an explicit formula for $\mathbf{E}\tau$ since they have many other uses. In particular, beyond simply identifying the rate of decay, an analytic expression for $\mathbf{E}\tau$ can be used to characterize the most likely way to escape and/or to choose an efficient control strategy.

Over last decades, much attention has been given to weakly perturbed systems with weak dissipation, e.g. [2] – [4]. The purpose of this paper is to discuss recent results concerning explicit solutions of the first-passage problem for two opposite classes of stochastic oscillators, a multidimensional non-dissipative oscillator with non-small additive noise and a weakly perturbed oscillator with significant dissipation.

2. Stochastic models

As a first model, we consider a multidimensional non-dissipative oscillatory system excited by additive white noise. The equations of motion are written in the Lagrangian form

(1) $$\frac{d}{dt}\frac{\partial L(q,\dot{q})}{\partial \dot{q}} - \frac{\partial L(q,\dot{q})}{\partial q} = \sigma\dot{w}(t)\,,\ q,\ \dot{q}\ \in G \in R^{2n}$$

where $q \in R^{n}$ is the vector of generalized coordinates; $w(t)$ is standard Wiener process in R^{m}; $L(q,\dot{q}) = T(q,\dot{q}) - U(q)$ is the Lagrangian of the system, $T(q,\dot{q})$ is the kinetic energy, $U(q)$ is the potential energy. The diffusion matrix σ yields a symmetric positive definite matrix $A = \sigma\sigma^{T}$. In general, the reference domain G is considered as a connected open bounded set in R^{2n} with smooth boundary Γ and compact closure $\bar{G}$.

Escape time for system (1) is defined as the time needed to reach a critical level of energy H^{*} from the initial state with the energy H^{0}. This implies that

(2) $$G\colon \{H^{0} \le H < H^{*}\}\ ,\ \Gamma\colon \{H = H^{*}\}$$

In sharp contrast to the great majority of stochastic problems, for the non-dissipative oscillator (1) one can obtain a precise analytic solution of the Fokker-Plank equation. As shown in [5], the mean time to escape from the domain (2) is defined as

(3) $$\mathbf{E}\tau = \frac{H^* - H^0}{\mathbf{Tr}A}.$$

In order to take proper account of the effect of dissipation, we consider an opposing model. The equations of motion are written as

(4) $$\frac{d}{dt}\frac{\partial L(q,\dot{q})}{\partial \dot{q}} - \frac{\partial L(q,\dot{q})}{\partial q} + B(q,\dot{q})\dot{q} = \varepsilon\sigma(q,\dot{q})\dot{w}(t),\ q,\ \dot{q}\ \in G \in R^{2n},$$

where $B(q, \dot{q})$ is the matrix of dissipation forces; the small parameter $\varepsilon > 0$ implies the weak effect of noise compared to the effect of potential and dissipation forces. The matrices $B(q, \dot{q})$ and $A(q, \dot{q}) = \sigma(q, \dot{q})\sigma^T(q, \dot{q})$ are assumed to be symmetric positive definite in $\overline{G}$.

Taking the noiseless system asymptotically stable, we construct a closed-form logarithmic asymptotics of $\mathbf{E}\tau^\varepsilon$ as $\varepsilon \to 0$ by means of the large deviations techniques [2], [3]. It has been shown [6] that the asymptotic estimate can be represented as a sum of two terms associated with the kinetic and potential energy, respectively. The first term can be found explicitly; the second term satisfies a linear PDE. The explicit solution of the latter equation can be obtained for several classes of systems.

Finally, we use the solution of the above problem to develop a control strategy ensuring a noise-independent escape rate in the controlled system. This result is exploited to design a stabilizing control for a gimbal suspensions gyroscope.

3. Acknowledgement

The work has been carried out with financial assistance of the Russian Foundation for Basic Research (grant 08-01-00068).

4. References

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