STOCHASTIC HOMOGENIZATION FOR CHAOTIC AND QUASI-PERIODIC MASONRY STRUCTURES

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1. Introduction

Ancient masonry structures are often characterized by a chaotic distribution of stones in the walls. In other cases, the shape of the stones and their disposition are such that the structure is called quasi-periodic. In these circumstances, it has been recognised that the best way for defining the mechanic characteristics of the structure is the stochastic one, through the use of the random fields. In fact, the deterministic homogenization, often used for the classical periodic masonry structures, fails for chaotic and quasi-periodic masonry structures and the stochastic homogenization must be considered. In the literature, the stochastic homogenization is referred to the first order statistics, both for the mechanic properties and for the response quantities (displacements and internal forces). It can be applied by using some different approaches: the stochastic convergence approach [1], the polarization tensor approach coupled with the Hashin-Shtrikman variational principle [2] and the concentration tensor approach coupled with the Eshelby equivalence principle. Some approaches in literature consider higher order statistics, but they cannot be considered as homogenization approaches. They are referred as Stochastic Finite Element (SFE) approaches; the most used are: the stochastic perturbation methods [3] and the series expansion approaches, among which the most known is the Karunen-Love series method coupled with the polynomial chaos approach [4].

In the present work a stochastic homogenization approach based on the second order statistics is presented. It is founded on the extension to the second order analyses of the Moving Window Method (MWM), that has been used in the first order stochastic homogenization approach [1,5]. In particular, the extensions of the Voigt and Reuss limits and of the Hill theorem will be considered.

2. Basic formulation of the first order stochastic homogenization

Under the assumptions of stochastic homogeneous and ergodic medium, the average elastic constitutive equation can be written as:

(1) $$\mathsf{E}\left[\sigma_{ij}(\mathbf{x})\right]=\mathsf{E}\left[C_{ijkl}(\mathbf{x})\varepsilon_{ij}(\mathbf{x})\right]$$

where $\mathsf{E}\left[(\ )\right]$ indicates the mean of $(\ )$, while σ_{ij}, ε_{ij} and C_{ijkl} are the stress, strain and stiffness tensors. The classical homogenization approach searches for that ideal materials for which the following relationship holds:

(2) $$\mathsf{E}\left[\sigma_{ij}(\mathbf{x})\right]=C_{ijkl}^{(h)}\mathsf{E}\left[\varepsilon_{ij}(\mathbf{x})\right]$$

If the strains are assumed constant, $\varepsilon_{ij}(\mathbf{x})=\varepsilon_{ij}^{(o)}$, then eq.(1) gives

(3) $$\mathsf{E}\left[\sigma_{ij}(\mathbf{x})\right]=\mathsf{E}\left[C_{ijkl}(\mathbf{x})\right]\varepsilon_{ij}^{(o)} \quad\Rightarrow\quad C_{ijkl}^{(h)}\equiv\mathsf{E}\left[C_{ijkl}(\mathbf{x})\right]\equiv C_{ijkl}^{(V)}$$

$C_{ijkl}^{(V)}$ being the Voigt interpretation of the homogenized stiffness tensor. On the contrary, if the stresses are assumed constant, $\sigma_{ij}(\mathbf{x}) = \sigma_{ij}^{(o)}$, then

$$\mathrm{E}\left[\varepsilon_{ij}(\mathbf{x})\right] = \mathrm{E}\left[D_{ijkl}(\mathbf{x})\right]\sigma_{ij}^{(o)} \quad \Rightarrow \quad C_{ijkl}^{(h)} \equiv \left(\mathrm{E}\left[D_{ijkl}(\mathbf{x})\right]\right)^{-1} \equiv C_{ijkl}^{(R)} \tag{4}$$

D_{ijkl} being the point compliance tensor and $C_{ijkl}^{(R)}$ the Reuss interpretation of the homogenized stiffness tensor. The Hill theorem ensures that $\left|C_{ijkl}^{(R)}\right| \le \left|C_{ijkl}^{(h)}\right| \le \left|C_{ijkl}^{(V)}\right|$, the equality sign being strictly verified only when the reference volume of the structure is infinite.

For two finite reference volume $\Omega_1 < \Omega_2$, indicated with $C_{ijkl}^{(a,\Omega_i)}$ and $C_{ijkl}^{(b,\Omega_i)}$ the average stiffness tensors obtained with constant strains and stresses, respectively, then it has been shown that the following fundamental relationship holds [6]:

$$\left|C_{ijkl}^{(b,\Omega_1)}\right| \le \left|C_{ijkl}^{(b,\Omega_2)}\right| \le \left|C_{ijkl}^{(h)}\right| \le \left|C_{ijkl}^{(a,\Omega_2)}\right| \le \left|C_{ijkl}^{(a,\Omega_1)}\right| \tag{5}$$

This last relationship is fundamental in the estimation of the homogenized stiffness tensor in terms of mean values. An effective approach for the evaluation of $C_{ijkl}^{(a,\Omega_i)}$ and $C_{ijkl}^{(b,\Omega_i)}$ is the MWM.

3. Proposed approach

The aim of the present work is the extension of the result summarized in the previous section to the second order statistics. This means that one will work in terms of the correlation function

$$\mathrm{R}_{\substack{ijkl\\mnpq}}^{(C)}(\mathbf{x}_2 - \mathbf{x}_1) = \mathrm{E}\left[C_{ijkl}(\mathbf{x}_1)C_{mnpq}(\mathbf{x}_2)\right] - \mathrm{E}\left[C_{ijkl}(\mathbf{x}_1)\right]\mathrm{E}\left[C_{mnpq}(\mathbf{x}_2)\right] \tag{6}$$

For example, in these terms, the homogenized stiffness tensor will be that tensor $C_{ijkl}^{(h2)}$ satisfying the following relationship:

$$\mathrm{R}_{\substack{ij\\mn}}^{(\sigma)}(\mathbf{x}_2 - \mathbf{x}_1) = \mathrm{R}_{\substack{ijkl\\mnpq}}^{(C^{(h2)})}(\mathbf{x}_2 - \mathbf{x}_1)\mathrm{R}_{\substack{kl\\pq}}^{(\varepsilon)}(\mathbf{x}_2 - \mathbf{x}_1) \tag{7}$$

that is an extension of eq.(2). In a similar way, it will be shown as the results given into eqs(3-5) are extended to the second order statistics.

4. References

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