GENERALIZED FPK EQUATIONS FOR NON-LINEAR DYNAMICAL SYSTEMS UNDER GENERAL STOCHASTIC EXCITATION

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We address the problem of determining the probabilistic structure of the response of a dynamical system governed by a differential system of quite general form, with arbitrary polynomial non-linearities, subject to a general stochastic excitation. The latter is assumed to be distributed in accordance with a known probability measure, defined over the Borel $\sigma-$algebra of continuous functions. The goal of this paper is to derive appropriate equations for determining the probability density function of the response, in terms of the probabilistic characteristics of the stochastic excitation. Since the excitation is not a delta correlated process, the response does not obey the Markov property.

Consider, for simplicity, the two-equation differential system

$$\begin{aligned} \dot{x}_n(t) &= \sum_{q_1,q_2} A^{(n)}_{q_1 q_2}(t)\cdot x_1^{q_1}(t) x_2^{q_2}(t) + \sum_{Q_1,Q_2} B^{(n)}_{Q_1 Q_2}(t)\cdot y_1^{Q_1}(t;\theta) y_2^{Q_2}(t;\theta), \\ x_n(t_0) &= x_{n0}(\theta), \qquad n=1,2, \end{aligned} \tag{1}$$

where $A^{(n)}_{q_1 q_2}(t)$ and $B^{(n)}_{Q_1 Q_2}(t)$ are known deterministic functions, $y_1(t;\theta)$, $y_2(t;\theta)$ are given stochastic functions, both defined on the common domain $\mathbb{T}\times\Theta = [t_0, T]\times\Theta$, Θ being the sample space, $x_0(\theta)$ is a given stochastic variable, and q_1, q_2, Q_1, Q_2 are non-negative integers, each one non-exceeding a bounded maximum value. Clearly, both linear and quadratic stochastic excitation are included in the excitation term of the above equation. A special case of particular interest is a linear system driven by quadratic (colored) noise, already examined by Luczka (1986), using methods different from the one presented herewith. See also Luczka, Hänggi & Gadomski (1995).

A general technique to deal with problem (1) is to derive and study the infinite system of moment equations. After truncation this system can become closed (by means of appropriate closure schemes) and solved, providing us with useful (yet restricted) information about the probabilistic characteristics of the response process.

Another general approach to treat problem (1), initiated by Hopf (1952) in the context of his *statistical approach to turbulence*, is to consider the characteristic functional of the response process and find equations governing its evolution (see also Monin & Yaglom 1971, 1975). Due to the complexity of these Functional Differential Equations (FDEs), solutions are known only to very specific cases. An alternative approach, developed by Kotulski & Sobczyk (1984), is to directly construct the characteristic functional of the response process, exploiting the differential equations for sample functions. The latter method, as well as some similar works by Budini & Caseres (e.g., 2004), seem to be generally applicable only to linear problems.

An extension to Hopf's method was presented by Lewis & Kraichnan (1962), who introduced the joint, response-excitation, characteristic functional, and found the corresponding FDEs (see, also, Beran 1968). In the present study we follow the latter idea, appropriately adapted to problem (1). After deriving the FDEs, we combine them and project appropriately to finite dimensions. The projection is implemented by substituting the general arguments in the characteristic functional by delta functionals, when appropriate. After lengthy calculations we are able to get an equation for the joint, response-excitation (4-dimensional) characteristic function $\varphi_{x_1(t)x_2(t)y_1(s)y_2(s)}(u_1,u_2,\upsilon_1,\upsilon_2)$. Applying Fourier transform, we finally derive an equation governing the evolution of the corresponding joint, response-excitation (4-dimensional) probability density function $f_{x_1(t)x_2(t)y_1(s)y_2(s)}(\alpha_1,\alpha_2,\beta_1,\beta_2)$:

$$\left.\frac{\partial}{\partial t} f_{x_1(t)x_2(t)y_1(s)y_2(s)}(\alpha_1,\alpha_2,\beta_1,\beta_2)\right|_{s\to t} + \mathscr{L}_{\alpha_1\alpha_2}\left[f_{x_1(t)x_2(t)y_1(t)y_2(t)}(\alpha_1,\alpha_2,\beta_1,\beta_2)\right] + \\ + \sum_{n=1,2}\sum_{\varrho_1,\varrho_2} B^{(n)}_{\varrho_1\varrho_2}\beta_1^{\varrho_1}\beta_2^{\varrho_2}\frac{\partial}{\partial a_n} f_{x_1(t)x_2(t)y_1(t)y_2(t)}(\alpha_1,\alpha_2,\beta_1,\beta_2) = 0 \tag{2}$$

where $\mathscr{L}_{\alpha_1\alpha_2}[\bullet]$ is a first-order differential operator with respect to α_1,α_2, defined by

$$\mathscr{L}_{\alpha_1\alpha_2}[\bullet] = \sum_{n=1,2}\sum_{q_1,q_2} A^{(n)}_{q_1q_2}\alpha_1^{q_1}\alpha_2^{q_2}\left(\alpha_n^{-1}q_n + \frac{\partial}{\partial\alpha_n}\right)[\bullet]. \tag{3}$$

[For $q_n = 1$ the term α_n^{-1} should be replaced by 1]. Equation (2) should be supplemented by appropriate initial and marginal-compatibility conditions. It generalizes a similar result for a single scalar equation, presented in Sapsis & Athanassoulis (2008).

The validity of this new equation is assured by showing that the infinite system of the moment equations can be directly derived from it. Further, it should be noted that equation (2) applies to any kind of stochastic excitation, with continuous sample functions. No specific simplifying assumptions, concerning either the correlation structure or the distributions of the stochastic data, are needed. Because of the generic nature of the excitation, the response is non-Markovian. Thus, equation (2) can be considered as a generalization of the FPK equation to a broad class of stochastic dynamical systems, exhibiting non-Markovian responses. Generalizations of the FPK equation for specific systems exhibiting non-Markovian responses have also been presented by many authors. See, e.g., Luczka, Hänggi & Gadomski (1995) and the survey by Luczka (2005).

Equation (2) does not belong to any, already studied, type of partial differential equations. Its solvability theory and appropriate methods for its effective numerical solution should by developed. A particular method for its numerical solution is under development and will be presented.

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