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PROPERTIES OF SOLUTIONS OF ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS DESCRIBING THE MOTION OF MECHANICAL SYSTEMS

31/1983

WARSZAWA 1983

ISSN 0208-5658

Praca wpłynęła do Redakcji dnia 29 marca 1983 r.

Praca wykonana w ramach problemu międzyresortowego I-23
koordynowanego przez Instytut Podstawowych Problemów
Techniki PAN

N a   p r a w a c h   r ę k o p i s u

Instytut Podstawowych Problemów Techniki PAN

Nakład 140 egz. Ark.wyd. 5. Ark.druk. 8,25 .
Oddano do drukarni w lipcu 1983 r.
Nr zamówienia 456/83   M-24 .

Warszawska Drukarnia Naukowa, Warszawa,
ul.Śniadeckich 8

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PROPERTIES OF SOLUTIONS OF ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS DESCRIBING THE MOTION OF MECHANICAL SYSTEMS (Differential and integral inequalities and Liapunov's function methods)(*)

1. Ordinary differential equation of the first order.

1.1 The method of comparison equation.

Let us consider the following ordinary non-linear equation of the first order in the matrix form

$$\dot{y} = R(t, y) \quad , \quad y(t_0) = y_0 \tag{1.1}$$

where $y = col\,[y_1, \dots, y_n]$, $R = col\,[R_1, \dots, R_n]$ are the column matrices and t denotes the time.

The function R is assumed to satisfy the conditions of existence and uniqueness of the solutions of the formulated initial value problem in the interval $[t_0, \infty)$.

We shall construct a comparison equation for the equation (1.1), which from general point of view ought to be easier to investigate than equation (1.1). A very convenient situation we obtain in the case, when we can either solve the comparison equation in an explicit form or at least estimate its solutions.

A) The first method of construction of the comparison equation.

Let us represent the function R in the form of a sum

(*) This paper is connected with the Problem I-23 of two components $\mathcal{P}$ and F and let us construct the comparison equation with the aid of the function $\mathcal{P}$ that is

$$\mathcal{R}(t,y) = \mathcal{P}(t,y) + F(t,y)$$

$$\left.\begin{array}{llr} (1.2) & \dot{y} = \mathcal{P}(t,y) + F(t,y)\,, & y(t_0) = y_0 \\ (1.3) & \dot{x} = \mathcal{P}(t,x) & ,\quad x(t_0) = x_0 = y_0 \end{array}\right\}\text{Problem I}$$

B) The second method of construction of the comparison equation.

Let us denote the solution of the equation (1.1) by

$$y = \varphi(t;t_0,y_0)$$

Putting

$$y = \eta + \varphi$$

we get the equation (1.1) in the form

$$\dot{\eta} = \mathcal{R}(t,\eta+\varphi) - \mathcal{R}(t,\varphi) = \Phi(t,\eta)$$

Let us denote additionally

$$\left.\frac{\partial\Phi}{\partial\eta}\right|_{\eta=0} = A(t)\;,\quad F(t,\eta) = \Phi(t,\eta) - A(t)\eta$$

where $A(t)$ is a square matrix. With the aid of the matrix $A(t)$ we construct the comparison equation in the form of a linear differential equation

$$\left.\begin{array}{llr} (1.4) & \dot{\eta} = A(t)\eta + F(t,\eta)\,, & \eta(t_0) = \eta_0 \\ (1.5) & \dot{\xi} = A(t)\xi & ,\quad \xi(t_0) = \xi_0 = \eta_0 \end{array}\right\}\text{Problem II}$$

Remark 1. If the function $\mathcal{P}$ in the comparison equation (1.3) has the form

$$\mathcal{P}(t,x) = A(t)x$$

then the problem I is identic with the problem II.

Remark 2. If the function F has the form

$$F(t,\eta) = H(t,\eta) + p(t)$$

we construct the comparison equation in the following form

$$(1.6)\qquad \dot{\eta} = A(t)\eta + H(t,\eta) + p(t)$$

$$(1.7)\qquad \dot{\xi} = A(t)\xi + p(t)$$

According to the assumptions which shall be admit, one may transform the investigation of the problem I and II to the investigation of the non-linear integral inequalities of the so called 1-st and 2-nd kind.

1.2 The integral inequality of the 1-st kind for the problems I and II.

A) Problem I.

Let us consider the variational equation for the comparison equation (1.3)

$$\dot{z} = \frac{\partial P}{\partial x}\left[t, x(t; t_0, y_0)\right] z \tag{1.8}$$

Let $U(t; t_0, y_0)$ denotes the fundamental matrix of the solutions of this equation normed for $t = t_0$, that is $U(t_0; t_0, y_0) = E$ (where E is the unit matrix). One can prove that the equation (1.2) is equivalent to the following integral equation

$$y(t; t_0, y_0) = x(t; t_0, y_0) + \int_{t_0}^{t} U\big(t, s, y(s; t_0, y_0)\big) F\big(s, y(s; t_0, y_0)\big) ds \tag{1.9}$$

B) Problem II.

The variational equation for the comparison equation (1.5) has the form

$$\dot{z} = A(t) z$$

Let $X(t)$ denotes the fundamental matrix of the solutions of the equation (1.5). In this case we have

$$U(t; s, y) = X(t) X^{-1}(s)$$

Thus the equation (1.4) is equivalent to the following integral equation

$$\eta(t; t_0, \eta_0) = \xi(t; t_0, \eta_0) + \int_{t_0}^{t} X(t) X^{-1}(s) F\big(s, \eta(s; t_0, \eta_0)\big) ds \tag{1.10}$$

$$\xi(t; t_0, \eta_0) = X(t) X^{-1}(t_0) \eta_0$$

We admit the following assumptions ($\|\cdot\|$ - denotes a norm)

a) $\|U(t,s,y)\| \leq \sigma e^{-n(t-s)}$ for the problem I

$\|X(t)X^{-1}(s)\| \leq \sigma e^{-n(t-s)}$ for the problem II

for arbitrary $\|y\| < \infty$ and $t_0 \leq s \leq t < \infty$.

b) $\|F(t,y)\| \leq k\, g(\|y\|)$ for $\|y\| < \infty$ and $t \in [t_0, \infty)$

where $k \geqslant 0$, and g(u) is a continuous, non-negative and non--decreasing function with respect to $u \geqslant 0$

c) $\|x(t;t_0,y_0)\| \leq c(t_0,y_0) < \infty$, $\|y_0\| < \infty$, $t \in [t_0,\infty)$

$\|\xi(t;t_0,\eta_0)\| \leq c(t_0,\eta_0) < \infty$, $\|\eta_0\| < \infty$, $t \in [t_0,\infty)$

where

$$\lim_{\|\gamma\| \to 0} c(t_0,\gamma) = 0 \quad , \quad t_0 = const \geqslant 0$$

Let us denote

$$\|y(t;t_0,y_0)\| = u(t) \quad , \quad \|\eta(t;t_0,\eta_0)\| = u(t)$$

Taking into account the assumptions (a),(b),(c) we may reduce the integral equations (1.9) and (1.10) to the following non-linear inequality

$$u(t) \leq c + \sigma k \int_{t_0}^{t} e^{-n(t-s)} g(u(s))\, ds \tag{1.11}$$

which will be called the integral inequality of the 1-st kind

Remark. If the problem II is written in the form (1.6), (1.7) we have

$$\xi(t;t_0,\eta_0) = X(t)X^{-1}(t_0)\eta_0 + \int_{t_0}^{t} X(t)X^{-1}(s)\varphi(s)\, ds$$

In this case instead of the assumption (b), the following assumption is used

$$\|H(t,\eta)\| \leq k\, g(\|\eta\|)$$

The assumption (c) is satisfied when the assumption (a) is satisfied and when additionally, for instance

$$\|\varphi(t)\| \leq \beta = const < \infty$$

1.3. The integral inequality of the 2-nd kind for the problems I and II.

A) Problem I.

Let us suppose that the solutions of the comparison equation (1.3) satisfy the following inequality $(t \in [t_0, \infty))$

$$(1.12) \quad \|x(t;t_0,\varphi_1) - x(t;t_0,\varphi_2)\| \leq k(t_0)\|\varphi_1 - \varphi_2\| \exp\left(-\int_{t_0}^{t}\alpha(s)ds\right)$$

where φ_1 , φ_2 , $k(t)$, $\alpha(t)$ denote continuous functions and $k(t) \geq 0$, $\alpha(t) \geq 0$, for $t \geq t_0$. Additionally let us suppose that the solution of the equation (1.3) satisfies the following condition

$$x(t;t_0,0) \equiv 0$$

Let us denote

$$(1.13) \quad V(t,\varphi) = W(t,\varphi) = \sup_{\xi \geq 0} \|x(t+\xi, t, \varphi)\| \exp\left(\int_{t}^{t+\xi}\alpha(s)ds\right)$$

$$(1.14) \quad \dot{V}(t,\varphi) = \overline{\lim_{h \to 0^+}} \frac{1}{h}\left[V(t+h, x(t+h,t,\varphi)) - V(t,\varphi)\right]$$

$$(1.15) \quad \dot{W}(t,\varphi) = \overline{\lim_{h \to 0^+}} \frac{1}{h}\left[V(t+h, y(t+h,t,\varphi)) - V(t,\varphi)\right]$$

The formula (1.15) has the same form as the formula (1.14), only x is replaced by y .

One can prove, that the function V satisfies the following inequalities

$$(1.16) \quad \|\varphi\| \leq V(t,\varphi) \leq k(t)\|\varphi\|$$

(1.17) $$\dot{V}(t,\varphi) \leqslant -\alpha(t)\, V(t,\varphi)$$

for $\|\varphi\| < \infty$ and $t \in [t_0, \infty)$.

Theorem.

If the inequality (1.12) is satisfied, then for $t \in [t_0, \infty)$ the following inequality

(1.18) $$\dot{W}(t,\varphi) \leqslant -\alpha(t) W(t,\varphi) + k(t)\,\|F(t,\varphi)\|$$

is satisfied, where $\alpha(t)$ and $k(t)$ are the functions appearing in the formula (1.12), and F is the function which appeared in the equation (1.2) of the problem I.

Put

$$k(t)\,\|F(t,\varphi)\| \leqslant \omega(t, \|\varphi\|) \ , \ \|\varphi\| < \infty \ , \ t \in [t_0, \infty)$$

Let us consider the differential equation

(1.19) $$\dot{u} = -\alpha(t) u + \omega(t, u)$$

On the basis of the inequality (1.16) we have

$$\|\varphi\| \leqslant V(t,\varphi) = W(t,\varphi) \ , \ t \in [t_0, \infty)$$

and taking into account the inequalities (1.18) and (1.19) we have

$$\|\varphi\| \leqslant u(t) \qquad t \in [t_0, \infty)$$

Since φ is an arbitrary continuous function, that's why we may identify this function as a solution of the equation (1.2) of the problem I. Hence it follows

(1.20) $$\|y(t; t_0, y_0)\| \leqslant u(t) \ , \ t \in [t_0, \infty)$$

The function $u(t)$ satisfies the integral inequality which we obtain integrating the equation (1.19) and putting $c \geqslant u_0$ that means

$$(1.21) \quad u(t) \leq c + \int_{t_0}^{t} \left[-\alpha(s) u(s) + \omega(s, u(s)) \right] ds$$

B) Problem II.

Let us represent the matrix $A(t)$ in the equation (1.4) of the problem II as a following sum

$$A(t) = B(t) + C(t)$$

Thus the problem II may be written in the transformed form

$$(1.22) \quad \dot{\eta} = B(t)\eta + C(t)\eta + F(t, \eta) \quad , \quad \eta(t_0) = \eta_0$$

$$(1.23) \quad \dot{\xi} = B(t)\xi \qquad , \quad \xi(t_0) = \xi_0 = \eta_0$$

This transformation is useful in the case, when the comparison equation (1.23) with the matrix B may be investigated easier than the comparison equation with the matrix A.

Let $X(t)$ be the fundamental matrix of the solutions of the comparison equation (1.23). We introduce a new variable by the formula

$$z = X^{-1}(t)\eta(t) \qquad or \qquad \eta(t) = X(t)z(t)$$

Substituting this expression into the equation (1.22) we get

$$\dot{\eta} = \dot{X}z + X\dot{z} = B(t)Xz + C(t)Xz + F(t, Xz)$$

$$\dot{z} = X^{-1}CXz + X^{-1}F(t, Xz)$$

$$(1.24) \quad z(t) = z(t_0) + \int_{t_0}^{t} \left[X^{-1}(s)C(s)X(s)z(s) + X^{-1}(s)F(s, X(s)z(s)) \right] ds$$

Let us suppose that the following estimations are satisfied

(1.25) $\| X^{-1}(t) C(t) X(t) \| \leq \beta(t)$, $\| X^{-1}(t) F(t, Xz) \| \leq \omega(t, \|z\|)$

for $\|z\| < \infty$ and $t \geq t_0$, and let us denote

$$\|z(t)\| = u(t) \quad , \quad \|z(t_0)\| = c$$

In view of (1.24) we get the following integral inequality

$$(1.26) \quad u(t) \leq c + \int_{t_0}^{t} [\beta(s) u(s) + \omega(s, u(s))] ds$$

Remark. If the transformation of the considered problem is not necessary, that means if $C = 0$, $A = B$ we have $\beta = 0$ and the integral inequality (1.26) takes the form

$$(1.27) \quad u(t) \leq c + \int_{t_0}^{t} \omega(s, u(s)) ds$$

The integral inequalities obtained above for the problems I and II may be written in the form

$$(1.28) \quad u(t) \leq c + \int_{t_0}^{t} [w(s) u(s) + \omega(s, u(s))] ds$$

where

$$(1.29) \quad \begin{cases} w(s) = -\alpha(s) & \text{for the inequality (1.21)} \\ w(s) = \beta(s) & \text{for the inequality (1.26)} \\ w(s) = 0 & \text{for the inequality (1.27)} \end{cases}$$

The inequality (1.28) is called the integral inequality of the 2-nd kind.

In the further considerations we shall show the assumptions which are sufficient for the limitation or tending to zero of the function u(t) and of the corresponding solutions of problems I and II, taking into account the domain of the initial values from where the solution is starting.

On the basis of above considerations it follows, that the investigation of such properties of solutions of the equation (1.1) as, for instance, the limitation, rate of increase or decrease with time, tending to zero, stability and so on, after the transformation to the problems I or II, may be reduced to the investigation of the integral inequalities of the 1-st or 2-nd kind, depending on the admitted assumptions. These integral inequalities have the form

$$(1.30) \quad u(t) \leq c + \sigma k \int_{t_0}^{t} e^{-n(t-s)} g(u(s))\,ds \qquad \text{1-st kind}$$

$$(1.31) \quad u(t) \leq c + \int_{t_0}^{t} [w(s)u(s) + \omega(s, u(s))]\,ds \qquad \text{2-nd kind}$$

where $w(s)$ may take the values given by the formulas (1.29).

2. Ordinary differential equation of the second order.

2.1 The method of integral inequalities.

Let us consider the following ordinary differential equation in the matrix form

$$(2.1) \quad \ddot{y} + H(t)\dot{y} + K(t)y = f(t, y, \dot{y}), \quad y(t_0) = y_0, \quad \dot{y}(t_0) = \dot{y}_0$$

where

$$y = \mathrm{col}\,[y_1, \ldots, y_n]$$

$$H = \mathrm{diag}\,[h_{11}, \ldots, h_{nn}]$$

$$K = \mathrm{diag}\,[k_{11}, \ldots, k_{nn}]$$

$$f = \mathrm{col}\,[f_1, \ldots, f_n]$$

The functions $H(t)$ and $K(t)$ are assumed to be continuous and the function f is assumed to satisfy the conditions of existence and uniqueness of the solutions of the formulated initial value problem in the interval $[t_0, \infty)$.

From the viewpoint of mechanics, we can regard $H(t)$ as the damping matrix, $K(t)$ as elasticity matrix and $f(t, y, \dot{y})$ as external force or a permanent perturbation.

Let us change in Eq.(2.1) the dependent variable

$$y = L(t)x \tag{2.2}$$

where $L^{*} = diag[l_{11}, \dots, l_{nn}]$

is a limited non-singular matrix of class $C^{(2)}$ in the interval $[t_0, \infty)$.

We have

$$\dot{y} = \dot{L}x + L\dot{x} \quad , \quad \ddot{y} = \ddot{L}x + 2\dot{L}\dot{x} + L\ddot{x} \tag{2.3}$$

Substituting (2.2) and (2.3) into (2.1) we get

$$\ddot{L}x + 2\dot{L}\dot{x} + L\ddot{x} + H\dot{L}x + HL\dot{x} + KLx = f$$

$$L\ddot{x} + (2\dot{L} + HL)\dot{x} + (\ddot{L} + H\dot{L} + KL)x = f$$

$$\ddot{x} + (2L^{-1}\dot{L} + L^{-1}HL)\dot{x} + (L^{-1}\ddot{L} + L^{-1}H\dot{L} + L^{-1}KL)x = L^{-1}f \tag{2.4}$$

The satisfaction of the following relation is required

$$2L^{-1}\dot{L} + L^{-1}HL = 0 \quad , \quad 2\dot{L}L^{-1} = -H \tag{2.5}$$

that is

$$\frac{2\dot{l}_{\alpha\alpha}}{l_{\alpha\alpha}} = -h_{\alpha\alpha} \quad , \quad l_{\alpha\alpha} = e^{-\frac{1}{2}\int_{t_0}^{t} h_{\alpha\alpha}(s)ds}$$

Therefore

$$L(t) = diag\left[\exp\left(-\frac{1}{2}\int_{t_0}^{t} h_{11}(s)ds\right), \dots, \exp\left(-\frac{1}{2}\int_{t_0}^{t} h_{nn}(s)ds\right)\right] \tag{2.6}$$

Let us calculate the term within the last bracket of the equation (2.4)

$$\dot{L} = \mathrm{diag}\left[-\tfrac{1}{2}h_{11}e^{-\frac{1}{2}\int_{t_0}^{t}h_{11}(s)ds}, \dots, -\tfrac{1}{2}h_{nn}e^{-\frac{1}{2}\int_{t_0}^{t}h_{nn}(s)ds}\right]$$

$$\ddot{L} = \mathrm{diag}\left[\tfrac{1}{4}h_{11}^2 e^{-\frac{1}{2}\int_{t_0}^{t}h_{11}(s)ds} - \tfrac{1}{2}\dot{h}_{11}e^{-\frac{1}{2}\int_{t_0}^{t}h_{11}(s)ds}, \dots \right.$$

$$\left. \dots, \tfrac{1}{4}h_{nn}^2 e^{-\frac{1}{2}\int_{t_0}^{t}h_{nn}(s)ds} - \tfrac{1}{2}\dot{h}_{nn}e^{-\frac{1}{2}\int_{t_0}^{t}h_{nn}(s)ds}\right]$$

$$L^{-1}\ddot{L} = \mathrm{diag}\left[\tfrac{1}{4}h_{11}^2 - \tfrac{1}{2}\dot{h}_{11}, \dots, \tfrac{1}{4}h_{nn}^2 - \tfrac{1}{2}\dot{h}_{nn}\right]$$

$$L^{-1}H\dot{L} = \mathrm{diag}\left[-\tfrac{1}{2}h_{11}^2, \dots, -\tfrac{1}{2}h_{nn}^2\right]$$

$$L^{-1}KL = \mathrm{diag}\left[k_{11}, \dots, k_{nn}\right]$$

$$L^{-1}\ddot{L} + L^{-1}H\dot{L} + L^{-1}KL = \mathrm{diag}\left[\tfrac{1}{4}h_{11}^2 - \tfrac{1}{2}\dot{h}_{11} - \tfrac{1}{2}h_{11}^2 + k_{11}, \dots \right.$$

$$\left. \dots, \tfrac{1}{4}h_{nn}^2 - \tfrac{1}{2}\dot{h}_{nn} - \tfrac{1}{2}h_{nn}^2 + k_{nn}\right] = -\tfrac{1}{2}\dot{H}(t) - \tfrac{1}{4}H^2(t) + K(t) = P(t)$$

Hence the differential equation (2.4) takes the form

$$\ddot{x} + P(t)x = L^{-1}f(t, y, \dot{y}) = q(t, y, \dot{y}) \tag{2.7}$$

In view of (2.3) and (2.5) we have

$$\dot{y} = \dot{L}x + L\dot{x} \qquad , \quad \dot{L} = -\tfrac{1}{2}HL = -\tfrac{1}{2}LH$$

$$\dot{y} = L\dot{x} - \tfrac{1}{2}HLx = L(\dot{x} - \tfrac{1}{2}Hx) \tag{2.8}$$

Denoting

$$q(t, y, \dot{y})\Big|_{\substack{y = Lx \\ \dot{y} = L(\dot{x} - \frac{1}{2}Hx)}} = r(t, x, \dot{x}) \tag{2.9}$$

we get the equation (2.7) in the form

$$\ddot{x} + P(t)x = r(t, x, \dot{x}) \tag{2.10}$$

Let us determine the initial values for the variable
In view of (2.2) and (2.8) we have

$$(2.11)\qquad \begin{cases} x(t_0) = x_0 = L^{-1}(t_0)\, y_0 \\ \dot{x}(t_0) = \dot{x}_0 = L^{-1}(t_0)\, \dot{y}_0 + \frac{1}{2} H x_0 = L^{-1}(t_0)\, \dot{y}_0 + \frac{1}{2} H L^{-1}(t_0)\, y_0 = \\ \quad = L^{-1}(t_0)(\dot{y}_0 + \frac{1}{2} H y_0) \end{cases}$$

A) <u>The integral inequality of the 1-st kind.</u>

Let us transform the equation (2.10) by the method of varying the constants

$$(2.12)\qquad x(t) = X_1(t)\, a(t) + X_2(t)\, b(t)$$

where

$$(2.13)\qquad X(t) = \begin{bmatrix} X_1(t) , & X_2(t) \\ \dot{X}_1(t) , & \dot{X}_2(t) \end{bmatrix}$$

denote the fundamental matrix of the homogeneous equation corresponding to the equation (2.10)

$$(2.14)\qquad \ddot{x} + P(t)\, x = 0$$

We shall assume that the matrix (2.13) is normed for $t = t_0$, that is

$$(2.15)\qquad W(t) = \det X(t) = X_1 \dot{X}_2 - \dot{X}_1 X_2 = W(t_0) = 1$$

We have

$$(2.16)\qquad \dot{x} = \dot{X}_1 a + X_1 \dot{a} + \dot{X}_2 b + X_2 \dot{b}$$

Putting

$$(2.17)\qquad X_1 \dot{a} + X_2 \dot{b} = 0$$

we get (2.16) in the form

(2.18) $$\dot{x} = \dot{X}_1 a + \dot{X}_2 b$$

From (2.18) we have

(2.19) $$\ddot{x} = \ddot{X}_1 a + \dot{X}_1 \dot{a} + \ddot{X}_2 b + \dot{X}_2 \dot{b}$$

Substituting (2.12) and (2.19) into (2.10) we obtain

$$(\ddot{X}_1 + P X_1) a + (\ddot{X}_2 + P X_2) b + \dot{X}_1 \dot{a} + \dot{X}_2 \dot{b} = r(t, x, \dot{x})$$

that is

(2.20) $$\dot{X}_1 \dot{a} + \dot{X}_2 \dot{b} = r(t, x, \dot{x})$$

Solving the equations (2.17) and (2.20) with respect to $\dot{a}$ and $\dot{b}$ we have

(2.21) $$\begin{cases} \dot{a} = \dfrac{\begin{vmatrix} 0 & X_2 \\ r & \dot{X}_2 \end{vmatrix}}{W(t)} = -\dfrac{X_2 r}{W(t)} \\[2ex] \dot{b} = \dfrac{\begin{vmatrix} X_1 & 0 \\ \dot{X}_1 & r \end{vmatrix}}{W(t)} = \dfrac{X_1 r}{W(t)} \end{cases}$$

where in view of (2.15) $W(t) = X_1 \dot{X}_2 - \dot{X}_1 X_2 = W(t_0) = 1$

Hence the relations (2.21) imply the following equations

(2.22) $$\begin{cases} a(t) = x_0 - \displaystyle\int_{t_0}^{t} X_2(s)\, r\big(s, x(s), \dot{x}(s)\big)\, ds \\[2ex] b(t) = \dot{x}_0 + \displaystyle\int_{t_0}^{t} X_1(s)\, r\big(s, x(s), \dot{x}(s)\big)\, ds \end{cases}$$

Substituting (2.22) into (2.12) we arrive at a non-linear Volterra equation which is equivalent to the initial value problem (2.10),(2.11).

$$x = X_1(t)x_0 - \int_{t_0}^{t} X_1(t)X_2(s)\,r\left(s, x(s), \dot{x}(s)\right)ds +$$

$$+ X_2(t)\dot{x}_0 + \int_{t_0}^{t} X_2(t)X_1(s)\,r\left(s, x(s), \dot{x}(s)\right)ds$$

(2.23) $$x = X_1(t)x_0 + X_2(t)\dot{x}_0 +$$

$$+ \int_{t_0}^{t}\left(X_1(s)X_2(t) - X_1(t)X_2(s)\right) r\left(s, x(s), \dot{x}(s)\right)ds$$

Now we return to the starting-point, that is to the variable y ,using the formulas (2.2),(2.7),(2.9),(2.11). Thus we have

$$L^{-1}(t)y = X_1(t)L^{-1}(t_0)y_0 + X_2(t)L^{-1}(t_0)\left(\dot{y}_0 + \tfrac{1}{2}Hy_0\right) +$$

$$+ \int_{t_0}^{t}\left[X_1(s)X_2(t) - X_1(t)X_2(s)\right]L^{-1}(s)\,f\left(s, y(s), \dot{y}(s)\right)ds$$

(2.24) $$y = L(t)\left[X_1(t)L^{-1}(t_0)y_0 + X_2(t)L^{-1}(t_0)\left(\dot{y}_0 + \tfrac{1}{2}Hy_0\right)\right] +$$

$$+ \int_{t_0}^{t} L(t)\left[X_1(s)X_2(t) - X_1(t)X_2(s)\right]L^{-1}(s)\,f\left(s, y(s), \dot{y}(s)\right)ds$$

Let us assume that the following inequalities are satisfied

(2.25) $$\left\| L(t)\left[X_1(t)L^{-1}(t_0)y_0 + X_2(t)L^{-1}(t_0)\left(\dot{y}_0 + \tfrac{1}{2}Hy_0\right)\right]\right\| \le c(t_0, y_0) = \text{const} < \infty$$

(2.26) $$\left\| L(t)\left[X_1(s)X_2(t) - X_1(t)X_2(s)\right]L^{-1}(s)\right\| \le \sigma\, e^{-n(t-s)}, \quad t_0 \le s \le t < \infty$$

(2.27) $$\left\| f\left(t, y(t), \dot{y}(t)\right)\right\| \le k\, g\left(\| y(t)\|\right)$$

where $\sigma > 0$, $k \geq 0$ and $g(u)$ is a continuous, non-negative and non-decreasing function with respect to $u \geq 0$.

Taking the norm of the both sides of (2.24), denoting $\|y(t)\| = u(t)$ and taking into account (2.25), (2.26). (2.27), we obtain the integral inequality of the 1-st kind (see(1.11)).

(2.28) $$u(t) \leq c + \sigma k \int_{t_0}^{t} e^{-n(t-s)} g(u(s))\,ds$$

B) The integral inequality of the 2-nd kind.

In general, the fundamental matrix (2.13) of solutions of the equation (2.14) is very difficult to determine. To avoid this difficulty we add to the both sides of the equation (2.10) a term $Q(t)x$ choosen in such a way, that the fundamental matrix of the homogeneous equation

(2.29) $$\ddot{x} + [P(t) + Q(t)]x = 0$$

is known in an explicit form, or at least may be easily evaluated.

The differential equation (2.10) takes then the form

(2.30) $$\ddot{x} + [P(t) + Q(t)]x = Q(t)x + r(t, x, \dot{x})$$

Remark. If, for instance, we take the matrix $Q(t)$ in the form

$$Q(t) = \frac{1}{4}\ddot{P}(t)P^{-1}(t) - \frac{5}{16}\left[\dot{P}(t)P^{-1}(t)\right]^2$$

then denoting

$$V(t) = diag\left[p_{11}^{-\frac{1}{4}}(t), \ldots, p_{nn}^{-\frac{1}{4}}(t)\right]$$

$$\varphi_i(t) = \int_{t_0}^{t} p_{ii}^{\frac{1}{2}}(s)\,ds \quad , \quad i = 1, \ldots, n$$

$$N_1(t) = diag\left[\cos\varphi_1(t), \ldots, \cos\varphi_n(t)\right]$$

$$N_2(t) = diag\left[\sin\varphi_1(t), \ldots, \sin\varphi_n(t)\right]$$

we obtain the normed fundamental matrix (2.13) in the form

$$X_1(t) = V(t)\left[V^{-1}(t_0)N_1(t) - V(t_0)N_2(t)\right]$$

$$X_2(t) = V(t)V(t_0)N_2(t)$$

$$\dot{X}_1(t) = \dot{V}(t)\left[V^{-1}(t_0)N_1(t) - V(t_0)N_2(t)\right] - V^{-1}(t)\left[V^{-1}(t_0)N_2(t) + V(t_0)N_1(t)\right]$$

$$\dot{X}_2(t) = V(t_0)\left[\dot{V}(t)N_2(t) + V^{-1}(t)N_1(t)\right]$$

Let us transform the equation (2.30) by the method of varying the constants, using the formula (2.12), where now $X(t)$ denotes a known fundamental matrix of solutions of the equation (2.29). In the same way we obtain the following relations instead of (2.22)

$$a(t) = x_0 - \int_{t_0}^{t} X_2 Q x \, ds - \int_{t_0}^{t} X_2 \, r(s, x, \dot{x}) \, ds$$

$$b(t) = \dot{x}_0 + \int_{t_0}^{t} X_1 Q x \, ds + \int_{t_0}^{t} X_1 \, r(s, x, \dot{x}) \, ds$$

Using the formulas (2.12) and (2.18) we get

$$\text{(2.31)} \quad \begin{aligned} a(t) &= x_0 - \int_{t_0}^{t} X_2 Q(X_1 a + X_2 b) ds - \int_{t_0}^{t} X_2 \, r(s, X_1 a + X_2 b, \dot{X}_1 a + \dot{X}_2 b) ds \\ b(t) &= \dot{x}_0 + \int_{t_0}^{t} X_1 Q(X_1 a + X_2 b) ds + \int_{t_0}^{t} X_1 \, r(s, X_1 a + X_2 b, \dot{X}_1 a + \dot{X}_2 b) ds \end{aligned}$$

The relations (2.31) constitute a system of non-linear Volterra integral equations with respect to the functions $a(t)$ and $b(t)$.

Now we return to the starting point, that is to the equation (2.1). using the formulas (see (2.2), (2.7), (2.8), (2.9), (2.11), (2.12), (2.18)).

$$x_0 = L^{-1}(t_0) y_0$$

$$\dot{x}_0 = L^{-1}(t_0)\left(\dot{y}_0 + \tfrac{1}{2} H y_0\right)$$

$$r = q = L^{-1}(t) f(t, y, \dot{y})$$

$$y = Lx = L(X_1 a + X_2 b)$$

$$\dot{y} = L(\dot{x} - \tfrac{1}{2}Hx) = L(\dot{X}_1 a + \dot{X}_2 b - \tfrac{1}{2}HX_1 a - \tfrac{1}{2}HX_1 b) =$$

$$= L[(\dot{X}_1 - \tfrac{1}{2}HX_1)a + (\dot{X}_2 - \tfrac{1}{2}HX_2)b]$$

On this basis we obtain the relations (2.31) in the form

$$a(t) = L^{-1}(t_0)y_0 - \int_{t_0}^{t} X_2 Q(X_1 a + X_2 b)ds - \tag{2.32}$$

$$- \int_{t_0}^{t} X_2 L^{-1} f\left(s, L(X_1 a + X_2 b), L[(\dot{X}_1 - \tfrac{1}{2}HX_1)a + (\dot{X}_2 - \tfrac{1}{2}HX_2)b]\right)ds$$

$$b(t) = L^{-1}(t_0)[\dot{y}_0 + \tfrac{1}{2}H(t_0)y_0] + \int_{t_0}^{t} X_1 Q(X_1 a + X_2 b)ds + \tag{2.33}$$

$$+ \int_{t_0}^{t} X_1 L^{-1} f\left(s, L(X_1 a + X_2 b), L[(\dot{X}_1 - \tfrac{1}{2}HX_1)a + (\dot{X}_2 - \tfrac{1}{2}HX_2)b]\right)ds$$

Let us introduce the following notations

$$\left\{ \begin{aligned} c &= \|L^{-1}(t_0)\|\left(\|y_0\| + \|\dot{y}_0 + \tfrac{1}{2}H(t_0)y_0\|\right) \\ \xi(t) &= \max_t \left(\|X_1(t)\|, \|X_2(t)\|\right) \\ w(t) &= 2\,\xi^2(t)\|Q(t)\| \end{aligned} \right. \tag{2.34}$$

Taking the norm of both sides of the relations (2.32),(2.33) and making use of (2.34) we get

$$(2.35)\quad \|a\| \leq \|L^{-1}(t_0)\|\|y_0\| + \int_{t_0}^{t} \xi^2(s)\|Q(s)\|(\|a\|+\|b\|)ds +$$

$$+\int_{t_0}^{t} \xi(s)\|L^{-1}(s)\|\|f\|ds$$

$$(2.36)\quad \|b\| \leq \|L^{-1}(t_0)\|\|\dot{y}_0 + \tfrac{1}{2}H(t_0)y_0\| + \int_{t_0}^{t} \xi^2(s)\|Q(s)\|(\|a\|+\|b\|)ds +$$

$$+\int_{t_0}^{t} \xi(s)\|L^{-1}(s)\|\|f\|ds$$

Let us assume that the column matrix f is such, that the inequality

$$(2.37)\quad 2\xi(t)\|L^{-1}(t)\|\|f(t, L(X_1a+X_2b), L[(\dot{X}_1-\tfrac{1}{2}HX_1)a+(\dot{X}_2-\tfrac{1}{2}HX_2)b])\| \leq$$

$$\leq \omega(t, \|a\|+\|b\|)$$

holds for all $\|a\|+\|b\| < \infty$ and $t \geqslant t_0$. Adding side by side the inequalities (2.35),(2.36) and denoting

$$(2.38)\qquad u(t) = \|a(t)\| + \|b(t)\|$$

we obtain, on the basis of (2.34) and (2.37), the integral inequality of the 2-nd kind in the form (see (1.28))

$$(2.39)\qquad u(t) \leq c + \int_{t_0}^{t} [w(s)u(s) + \omega(s, u(s))]ds$$

After determining the function $u(t)$ satisfying the integral inequality (2.39), we have, on the basis of (2.12), (2.34) and (2.38)

$$\|x(t)\| \leq \xi(t)\big(\|a\| + \|b\|\big) = \xi(t)u(t) \tag{2.40}$$

Therefore in view of (2.2)

$$\|y(t)\| \leq \|L(t)\|\,\xi(t)\,u(t) \tag{2.41}$$

2.2. The method of the distance of Liapunov's function type.

Let us consider the following ordinary differential equation in the matrix form

$$\ddot{y} + 2H\dot{y} + Ky = f(t, y, \dot{y}), \quad y(t_0) = y_0, \quad \dot{y}(t_0) = \dot{y}_0 \tag{2.42}$$

where H and K are constant matrices. The notations here are the same as in Eq.(2.1). We assume, that

$$f(t, 0, 0) = 0 \tag{2.43}$$

which means, that Eq.(2.42) has the zero solution corresponding to the initial values being equal to zero.

We shall investigate the behaviour of solutions $y(t)$ of Eq.(2.42) corresponding to non-zero initial values, with respect to zero solution $y = 0$. To this end we assume, that the distance between solutions $y = y(t)$ and zero solution has the form

$$\varsigma = (\dot{y}, \dot{y}) + 2(Hy, \dot{y}) + (Ky, y) = \tag{2.44}$$

$$= \big((K - H^2)y, y\big) + \big((Hy + \dot{y}), (Hy + \dot{y})\big)$$

where $(\cdot\,, \cdot)$ denotes a scalar product. We assume that

$$K - H^2 > 0$$

Let us determine the time derivative of distance (2.44) for solutions of Eq.(2.42)

$$\dot{\varrho} = 2\left[\left((K-H^2)y, \dot{y}\right) + \left((Hy+\dot{y}), (H\dot{y}+\ddot{y})\right)\right] =$$

$$= 2\left[\left((K-H^2)y, \dot{y}\right) + \left((Hy+\dot{y}), (H\dot{y}-2H\dot{y}-Ky+f)\right)\right] =$$

$$= 2\left[\left((K-H^2)y, \dot{y}\right) + \left((Hy+\dot{y}), (-H\dot{y}-Ky+f)\right)\right]$$

$$\dot{\varrho} = 2\left[(Ky,\dot{y}) - (H^2y,\dot{y}) - (H^2y,\dot{y}) - (H\dot{y},\dot{y}) - (HKy,y) - (Ky,\dot{y})\right] + 2\left((Hy+\dot{y}), f\right)$$

(2.45) $\dot{\varrho} = -2\left[(H\dot{y},\dot{y}) + 2(H^2y,\dot{y}) + (HKy,y)\right] + 2\left((Hy+\dot{y}), f\right)$

Let us denote

(2.46) $\quad H = \min h_{ii} \quad , \; i = 1,\dots,n \quad , \quad h = mH$

Therefore the relation (2.45) takes the form of the inequality

(2.47) $\dot{\varrho} \leq -2h\left[(\dot{y},\dot{y}) + 2(Hy,\dot{y}) + (Ky,y)\right] + 2\left((Hy+\dot{y}), f\right)$

On the basis of (2.44) we have

(2.48) $\dot{\varrho} \leq -2h\varrho + 2\left((Hy+\dot{y}), f\right)$

Let us assume that

(2.49) $\quad 2\left((Hy+\dot{y}), f\right) \leq \gamma(t)\varrho + \omega(t,\varrho)$

where $\gamma = \gamma(t)$ is a continuous function, and $\omega(t, \varrho)$ is a continuous function being non-negative for all $t \geqslant t_0$, $\varrho \geqslant 0$.

Let us denote

(2.50) $$w(t) = \gamma(t) - 2h$$

From (2.48),(2.49),(2.50) we get the differential inequality of the form

(2.51) $$\dot{\varrho} \leq w(t)\varrho + \omega(t, \varrho)$$

Integration of both sides of (2.51) with respect to time provides the non-linear integral inequality of the 2-nd kind (see (1.28))

(2.52) $$\varrho \leq c + \int_{t_0}^{t} \left[w(s)\varrho(s) + \omega(s, \varrho(s)) \right] ds \quad , \quad c = \varrho(t_0)$$

The investigation of the behaviour of the solution $y = y(t)$ of the Eq.(2.42) consists on analysis at first the behaviour of distance $\varrho = \varrho(t)$ and construction of its estimation and next, the using of the obtained results to estimating $\|y(t)\|$.

3. The partial differential equations.

We shall investigate the stability in the Liapunov's sense of solutions of some partial differential equations describing the motion of the continuous systems. The stability shall be investigated with respect to the perturbations of initial values. The settlement of the Liapunov's stability with respect to the perturbations of initial values is, as it is well known, equivalent to the answer to the question if the problem is correctly set in the interval $[t_0, \infty)$.

In further considerations the term stability means the stability in the Liapunov's sense.

The main problem in the investigation of stability is the choice of the distance between the solutions.In particular, when the stability of the zero solution is investigated,it concerns the distance between an arbitrary solution and zero solution.It is well known that the same problem may be stable with respect to some distance and unstable with respect to the other.Therefore,the choice of the distance plays an essential role,because on the one side it is necessary to assure the possibility of mathematical investigations and on the other, the physical conclusions should be possible.

The second important circumstance,which must be taken into account by the investigation of the stability of the partial differential equations is the fact,that the initial values are the functions of the space variables, and only by relative high order of their regularity one can obtain so called classical solutions.If the initial values have too weak regularity it is possible to obtain only so called generalized solutions (distribution functions).It is necessary to take into account also these mentioned above facts,during formulation and investigation of the stability of solutions of partial differential equations,which depends non only on the form of the equation, but also on the regularity of the initial values.This induces the requirement of introducing the second distance being a measure of the difference of the initial values.The assumption of this second distance implies the admissible regularity of the initial values.The distances between the solutions and between the initial values may be assumed in the same or different forms.That's why the formulation and investigation of stability of the solutions of the partial differential equations appears with respect to the two distances.

The third circumstance which plays an important role in the investigation of the solutions of the partial differential equations is the appearance of the derivatives with respect to the space variables, which does not occur for the ordinary differential equations.

For this reason,the small differences of the initial values do not assure, in general, the small differences of the solutions, even for the linear partial differential equations with constant coefficients, as it occurs in the case of the ordinary differential equations. Therefore it is necessary to introduce some additional limitations for the space derivatives.

Let us consider, in more detail, this problem on the example of string vibration equation in the form

$$(3.1) \qquad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad , \quad 0 < x < \pi \quad , \quad t > 0$$

Let the boundary and initial conditions have the form

$$(3.2) \qquad u(0,t) = 0 \quad , \quad u(\pi,t) = 0$$

$$(3.3) \qquad u(x,0) = \varphi(x) \quad , \quad \frac{\partial u(x,0)}{\partial t} = \psi(x)$$

Using the method of the separation of the variables we get formally the solution of the formulated initial-boundary problem in the form

$$(3.4) \qquad u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos nt + B_n \sin nt \right) \sin nx$$

where the constants A_n, B_n can be determined on the basis of the initial values (3.3).

Let us introduce the distance between the perturbed solutions $u(x,t)$ and non-perturbed zero solution $u = 0$ in the form

$$(3.5) \qquad \rho = \int_0^{\pi} \left[u^2 + \left(\frac{\partial u}{\partial t} \right)^2 \right] dx =$$

$$= \frac{\pi}{2} \sum_{n=1}^{\infty} \left[\left(A_n \cos nt + B_n \sin nt \right)^2 + n^2 \left(A_n \sin nt - B_n \cos nt \right)^2 \right]$$

The limitation of ρ in the initial time $t = 0$ is therefore equivalent to the limitation of the initial values in this sense, that if $\rho|_{t=0}$ is small, the displacement $u(x,0)$ and velocity $\frac{\partial u(x,0)}{\partial t}$ of the points of the string are also small, and vice versa.

This kind of limitation of the initial state, in the form of limitation of the initial values, is very often assumed in the investigation of the stability in the Liapunov's sense for the ordinary differential equations. In the case of linear homogeneous ordinary differential equations with constant coefficients, this kind of limitation of the initial state implies the limitation of the displacements and velocities in an arbitrary time-point t. We shall show, that in the case of the string, the limitation of the displacements and velocities for $t=0$, that is of the distance ρ for $t=0$, does not imply, in general, the limitation of the displacements and velocities in an arbitrary time-point t, because it depends on the regularity of the initial values $\varphi(x)$ and $\psi(x)$.

Indeed, in view of (3.5) we have for $t=0$

$$\rho(0) = \frac{\pi}{2}\sum_{n=1}^{\infty}\left(A_n^2 + n^2 B_n^2\right)$$

Let us assume, that the initial functions have such a regularity, that the following relations are satisfied

$$A_n = \frac{\varepsilon}{n^{2/3}} \qquad nB_n = \frac{\varepsilon}{n^{2/3}} \tag{3.6}$$

Hence we have

$$\rho(0) = \frac{\varepsilon^2\pi}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n^{4/3}} + \frac{1}{n^{4/3}}\right) = \varepsilon^2\pi\sum_{n=1}^{\infty}\frac{1}{n^{4/3}} \tag{3.7}$$

This series is convergent because the series $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$. Hence it follows that $\rho|_{t=0} = \rho(0)$ is limited, and tends to zero for $\varepsilon \to 0$.

Nevertheless, the distance ρ given by formula (3.5) is not limited, because the series in this formula is divergent. Indeed, let us consider the series

$$J = \sum_{n=1}^{\infty} n^2 A_n^2 \sin^2 nt \tag{3.8}$$

being a part of the series (3.5). In view of (3.6) we have

$$(3.9)\qquad J = \sum_{n=1}^{\infty} n^2 \frac{\varepsilon^2}{n^{4/3}} \sin^2 nt = \varepsilon^2 \sum_{n=1}^{\infty} n^{2/3} \sin^2 nt$$

This series is divergent for each $t = t^* = \text{const}$, therefore the series (3.5), representing the distance ϱ, is also divergent.

This result unexpected from the first point of view may be explained from the mathematical, as well as from the physical point of view.

From the mathematical point of view, in the equation (3.1), besides the derivative with respect to time, appears the derivative with respect to x. But the distance (3.5) does not impose any limitation on the derivatives with respect to the space variable x. That's why the series representing these derivatives can be divergent, which is in the case of the generalized solutions.

From the physical point of view, the limitation of u and $\frac{\partial u}{\partial t}$ for $t = 0$, that is the limitation of the initial displacements and velocities, assured by the limitation of the distance in the form (3.5) for $t = 0$, does not assure the limitation of the initial potential energy, depending on the derivative $\frac{\partial u(x,0)}{\partial x}$. During the motion, the potential energy is transformed into the kinetic energy, and may provoke an unlimited increase of ϱ and also the displacements and velocities of the string. Obviously, it does not concern the real string, but it is only the consequence of the acceptation of its linear model. All these facts ought to be taken into account by the formulation and investigation of the stability, for obtaining the results having a physical sense.

In the case under consideration, the potential energy has the form

$$(3.10)\qquad E_p = \frac{1}{2} \int_0^{\pi} T_0 \left(\frac{\partial u}{\partial x}\right)^2 dx$$

where T_0 is the tension of the string. In view of (3.4) we have

$$(3.11) \qquad E_p = c \sum_{n=1}^{\infty} n^2 \left(A_n \cos nt + B_n \sin nt\right)^2$$

where $c = \frac{1}{2} T_0$ is a constant independent on n .For $t = 0$ we have

$$(3.12) \qquad E_p = c \sum_{n=1}^{\infty} n^2 A_n^2$$

hence for A_n satisfying (3.6) we get

$$(3.13) \qquad E_p = \varepsilon^2 c \sum_{n=1}^{\infty} n^{2/3}$$

This series is divergent for arbitrarily small $\varepsilon > 0$, therefore for the admitted regularity of the initial values, the potential energy is infinite for $t = 0$.

It is easy to verify, that in the case of so called classical solutions, the assumptions (3.6) cannot be admitted and have the form assuring the convergence of the series (3.12). Therefore in the classical case the potential energy is limited for $t = 0$,and the limitation of the initial displacements and velocities implies the limitation of these quantities in an arbitrary time-point t ,with respect to the assumed distance.

3.1. Formulation of the problem of the stability.

Let us assume, that the mathematical model of the motion of a continuous system has the form of the following equation

$$(3.14) \qquad [F]\left(u(P,t)\right) = 0$$

where $[F]$ denotes a linear or non-linear partial differential operator, the symbol P denotes the space variables and t denotes the time. We suppose, that the equation (3.14) describes a physical process in a domain Ω limited by a hypersurface Γ .

Let the boundary and initial conditions have the form

$$(3.15) \qquad [L_1](u(P,t))\Big|_{\Gamma} = 0$$

$$(3.16) \qquad [L_2](u(P,0)) = 0$$

where the notations (3.15) and (3.16) have a symbolic character, because they can represent a greater quantity of the conditions in which the derivatives can also appear.

For the further considerations we assume, that Eq. (3.14) has a zero solution, being equal to zero for the zero initial values. This zero solution we shall consider as an unperturbed solution, and all other solutions $u(P,t)$ as perturbed solutions. We shall investigate the stability of the unperturbed solution with respect to the perturbation of the initial values, assuming that the equation (3.14) and the boundary conditions remain unchanged.

If the function u does not depend on the space variables P, that is we have $u = u(t)$, then $[F]$ is an ordinary differential operator and the problem (3.14)-(3.16) takes the form of the Cauchy's problem for the ordinary differential equation.

Let us introduce the distance $\varrho = \varrho(u,t)$ which is a measure between the perturbed solutions and the unperturbed solution, satisfying the following conditions.

1) $\varrho(u,t) \geqslant 0$

2) $\varrho(0,t) = 0$

For each solution $u = u(P,t)$ the function $\varrho(u(P,t),t)$ considered as a function of the variable t, is continuous with respect to t.

It must be stressed here, that the distance $\varrho(u,t)$ must not satisfy the axioms of the metric space.

Simultaneously with the distance $\varrho(u,t)$ we introduce the distance $\varrho_L(u)$ which satisfies the conditions (1) and (2). This distance does not depend explicit on the time t, and using this distance, we shall limit the initial state. From among all possible initial states, the distance ϱ_i separates only these, for which it is limited.

Using different distances ϱ_i we introduce the initial perturbations of the different order of regularity. For a real object, the kind of perturbations follows from the character of the considered problem and the distances ϱ and ϱ_i for the solutions describing the behaviour of this object with respect to time, ought to be choosen from the physical point of view.

If two distances are considered, it is necessary to introduce one more condition for the distance ϱ that is

4) The distance $\varrho\,(u,t)$ is continuous with respect to the distance $\varrho_i\,(u)$ for $t = t_0$.

On the contrary, we do not assure that the distance ϱ_i is continuous with respect to the distance $\varrho\,(u,t)$ for $t = t_0$.

Let us consider, for instance, ϱ and ϱ_i in the form

$$\varrho = \int_{\Omega} u^2 d\Omega \quad , \quad \varrho_i = \int_{\Omega} \Big[u^2 + \sum_{k=1}^{n} \Big(\frac{\partial u}{\partial x_k} \Big)^2 \Big] d\Omega$$

In this case $\varrho_i \le \varepsilon$ implies $\varrho \le \varepsilon$, but $\varrho \le \varepsilon$ does not imply $\varrho_i \le \varepsilon$.

In the particular, but important case it can be

$$\varrho_i\,(u) = \big(\varrho(u,t) \big)_{t=t_0} \tag{3.17}$$

Now the initial distance is created by the distance ϱ for $t = t_0$. It means that the initial distance, and the distance in an arbitrary time, have the same form. In this case we say, that the stability is investigated with respect to the one distance.

In the further considerations we shall denote the distance $\varrho\,(u,t)$ for a determined function $u(P,t)$, by $\varrho = \varrho\,(t)$, and the initial distance $\varrho_i = \varrho_i\,(u)$ in the time-point $t = t_0$, by $\varrho_i = \varrho_i(t_0)$.

Definition (of the stability in Liapunov's sense with respect to the two distances).

The unperturbed solution $u=0$ is called stable in the Liapunov's sense with respect to the two distances ϱ and ϱ_i in the interval $[t_0, \infty)$ if

1)all perturbed solutions $u(P,t)$ are definite in the interval $[t_0, \infty)$

2) for each $\varepsilon > 0$ it exists such a $\eta(\varepsilon, t_0) > 0$, that for arbitrary solution $u(P,t)$ the inequality $\varrho_i \leqslant \eta(\varepsilon, t_0)$ implies the inequality $\varrho \leqslant \varepsilon$ for all $t \geqslant t_0$.

If moreover $\varrho \to 0$ for $t \to \infty$, the unperturbed solution $u=0$ is called asymptotically stable in the Liapunov's sense with respect to the distances ϱ and ϱ_i in the interval $[t_0, \infty)$.

The unperturbed solution $u=0$ is unstable, when it is not stable, that is when any of the conditions in the definition of the stability is not satisfied.

Sometimes it is possible to investigate the stability of a continuous system directly on the basis of the definition. These cases are rather very seldom and therefore we shall present only the, so called, direct Liapunov's method of the investigation of the stability of the solutions of partial differential equations. We shall also show, that the method of differential and integral inequalities can be useful for this aim.

3.2. The direct Liapunov's method of investigation of the stability of motion of the continuous systems.

We introduce a Liapunov's function $V(u,t)$. This function is for some given solution $u(P,t)$ the function of time, which we shall denote by $V(t)$. We assume, that $V(0,t) = 0$. For instance the distance $\varrho(t)$ is a function of this type and can be considered as a Liapunov's function and vice versa.

Definition 1. The function V is called a constant-positive-sign function or weak-positive function (a constant-negative-sign function or weak-negative function) with respect to the distance ϱ if $V \geqslant 0$ ($V \leqslant 0$) for each ϱ belonging to some domain $\varrho \leq H$, $t \geq t_0$.

Definition 2. The function V is called positive (negative) definite with respect to the distance ϱ, if $V \geq 0$ ($V \leq 0$) for $t \geq t_0$ and for each $\varepsilon > 0$ it exists such a $\delta = \delta(\varepsilon) > 0$, that the inequality $\varrho \geq \varepsilon$ implies the inequality $V \geq \delta(\varepsilon)$ ($V \leq -\delta(\varepsilon)$) for $t \geq t_0$.

Remark. The properties in the definitions 1 and 2 depends on the boundary conditions for the solution u(P,t). Let us consider, for instance, a function V in the form

$$V = \int_a^b \left(\varphi^2 + \varphi\frac{\partial\varphi}{\partial x}\right)dx = \int_a^b \varphi^2 dx + \frac{1}{2}\left[\varphi^2(b) - \varphi^2(a)\right]$$

If $\varphi(a) = 0$, then V is positive definite with respect to the distance $\varrho = \int_a^b \varphi^2\, dx$. If however $\varphi(a)$ may assume the arbitrary values, then V is not positive definite with respect to the distance ϱ.

Definition 3. The function V is called continuous with respect to the initial distance ϱ_i for $t = t_0$, if for each $\varepsilon > 0$ it exists such a $\delta(\varepsilon, t_0) > 0$, that the inequality $\varrho_i \leq \delta(\varepsilon, t_0)$ implies the inequality $|V| \leq \varepsilon$ for $t = t_0$. Moreover if $|V| \geq 0$, then $\varrho_i \geq \delta(\varepsilon, t_0)$.

Definition 4. Let G_V denote a manifold of functions u(P,t) for which it is $V > 0$. The function V is called limited in the manifold G_V if the following inequality is satisfied

$$|V| \leq N \quad , \quad N = \text{const} > 0$$

Definition 5. The function V has the time derivative positive definite in the manifold G_V if the derivative

$$\dot{V}\left(u(P,t), t\right) = \frac{dV}{dt}\left(u(P,t), t\right)$$

exists for the perturbed solutions u(P,t) in the entire manifold G_V, and for each $\varepsilon > 0$ it exists such a $\delta(\varepsilon) > 0$, that the inequality $V \geq 0$, implies the inequality $\frac{dV}{dt} \geq \delta(\varepsilon)$.

Theorem (about the stability and asymptotical stability).

The necessary and sufficient condition of the stability of the unperturbed solution $u=0$ with respect to the two distances, is the existence of a function V which must be:

1) positive definite with respect to the distance ϱ
2) continuous with respect to the initial distance ϱ_i for $t=t_0$
3) non-increasing for each perturbed solution $u(P,t)$ for all $t \geqslant t_0$.

If additionally it is $\lim\limits_{t \to \infty} V = 0$, then the unperturbed solution $u=0$ is asymptotically stable.

Let us consider some general problems connected with the investigation of the stability of the continuous systems, by the direct Liapunov's method. For the investigation of the stability of the unperturbed solution $u=0$ it is necessary to choose the function V and two distances ϱ and ϱ_i or one distance ϱ and $\varrho_i = (\varrho)_{t=t_0}$.

As regards the choice of V it does not exist an uniform method of construction of such a function, for a given problem. For the conservative processes it is often efficient to assume V in the form of total energy. In the case of the non-conservative processes we try to modify the form of the total energy in this way, that the condition (3) of the theorem of stability could be verified. The Liapunov's functions in this modified form are often called the energetical integrals. These modifications are mostly realized in the intuitive way.

In any case starting to investigate the stability of the unperturbed solution $u=0$ it is convenient to begin from the construction of the function V and next to verify the condition (3) of the theorem of stability. The next step is the choice of the distances ϱ and ϱ_i and the veryfying of the conditions (1) and (2) of the theorem of stability.

From the mathematical point of view it is most convenient to assume ϱ in the same form as the function V and to take $\varrho_i = (\varrho)_{t=t_0}$, because in this case, the conditions

(1) and (2) of the theorem of stability are automatically satisfied. This assumption may however appear unsuitable from the physical point of view for two reasons. At first, the assumption $\varrho_i = (\varrho)_{t=t_0} = (V)_{t=t_0}$ can impose too strong, or too weak limitations on the initial values, which ought to be taken from the physical point of view. At second, the assumption of ϱ in the same form as V implies in the case of stability the inequality $\varrho \leqslant \varepsilon$, which for complicated form of $\varrho = V$ may be few interesting and hardly interpretable from the physical viewpoint. The most interesting from the practical point of view is the case, when the inequality $\varrho \leqslant \varepsilon$ implies the limitation of the perturbed solution $|u| \leqslant M(t, \varepsilon, P_0, t_0)$. To arrive at such a result it is convenient to have ϱ in the form suitable for this aim, which is not necessary the same, as the form of the function V.

It is visible, that the choice of the Liapunov's function V and the distances ϱ and ϱ_i depends on different, often opposed, reasons of a mathematical as well as physical character. Therefore the success in the investigation of the stability of a continuous system depends on the suitable choice of V, ϱ, ϱ_i which often has to be a compromise between opposed requirements of the mathematical and physical type, connected with this choice.

3.3. The investigation of the stability of motion of some continuous systems by the direct Liapunov's method.

We shall consider a few problems of the dynamics (vibrations) of the elastic systems depending on one or two space variables. We shall realize the construction of the Liapunov's function on the basis of the total energy of the system, modifying it in this way, that it could be possible to verify the condition (3) of the theorem of stability. This method as we have mentioned above, is called the method of the energetic integrals. It should be stressed, that this method does not assure the construction of the best Liapunov's function, from the viewpoint of the optimal domain of the parameters, for which the unperturbed solution is stable.

3.3.1. The equation with constant coefficients of the linear vibrations of a string or of the longitudinal (torsional) vibrations of a bar.

Let us consider the differential equation in the form

$$(3.18)\qquad m\frac{\partial^2 u}{\partial t^2}+\mu\frac{\partial u}{\partial t}+r u = T_0\frac{\partial^2 u}{\partial x^2}\ ,\quad x\in(0,l),\ t\in(t_0,\infty)$$

where

m - linear density (mass of the unit of length)

μ - coefficient of exterior linear damping

r - coefficient of the linear elastic base

T_0 - tension of the string

u(x,t) - transversal displacement of the string

We can consider Eq.(3.18) as a mathematical model of the small (linear) free transversal vibrations of a string supported on the linear elastic base, taking into account the exterior linear damping.

Let the initial conditions have the form

$$(3.19)\qquad u(x,0)=\varphi(x)\quad ,\qquad \frac{\partial u(x,0)}{\partial t}=\psi(x)$$

and let the boundary conditions have the form

$$(3.20)\qquad u(0,t)=0\quad ,\qquad u(l,t)=0$$

Introducing the notations

$$\frac{\mu}{m}=2b\ ,\qquad \frac{r}{m}=c\ ,\qquad \frac{T_0}{m}=a$$

we get Eq.(3.18) in the form

$$(3.21)\qquad \frac{\partial^2 u}{\partial t^2}+2b\frac{\partial u}{\partial t}+cu=a\frac{\partial^2 u}{\partial x^2}$$

We shall investigate the stability in the Liapunov's sense of the unperturbed solution $u=0$. To this end let us assume, that the Liapunov's function has the form.

$$(3.22)\quad V(u)=\int_0^l\left[\left(\frac{\partial u}{\partial t}\right)^2+a\left(\frac{\partial u}{\partial x}\right)^2+2bu\frac{\partial u}{\partial t}+cu^2\right]dx=$$

$$=\int_0^l\left[(c-b^2)u^2+a\left(\frac{\partial u}{\partial x}\right)^2+\left(bu+\frac{\partial u}{\partial t}\right)^2\right]dx$$

where is assumed that

$$(3.23)\qquad c-b^2>0$$

Let us determine the time derivative of the function (3.22) for the solutions of Eq.(3.21)

$$\dot V=2\int_0^l\left[(c-b^2)u\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}+\left(bu+\frac{\partial u}{\partial t}\right)\left(b\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial t^2}\right)\right]dx=$$

$$=2\int_0^l\left[cu\frac{\partial u}{\partial t}-b^2u\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}+\left(bu+\frac{\partial u}{\partial t}\right)\left(-b\frac{\partial u}{\partial t}-cu+a\frac{\partial^2 u}{\partial x^2}\right)\right]dx=$$

$$=2\int_0^l\left[cu\frac{\partial u}{\partial t}-b^2u\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}-b^2u\frac{\partial u}{\partial t}-bcu^2+bau\frac{\partial^2 u}{\partial x^2}-\right.$$

$$\left.-b\left(\frac{\partial u}{\partial t}\right)^2-cu\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\right]dx$$

We apply integration by parts, using the boundary conditions (3.20) to calculate the following integrals:

$$(3.24)\quad \int_0^l\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dx=\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right]_0^l-\int_0^l\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}dx=-\int_0^l\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}dx$$

$$(3.25)\quad \int_0^l u\frac{\partial^2 u}{\partial x^2}dx=\left[u\frac{\partial u}{\partial x}\right]_0^l-\int_0^l\left(\frac{\partial u}{\partial x}\right)^2dx=-\int_0^l\left(\frac{\partial u}{\partial x}\right)^2dx$$

Thus we obtain

$$\dot V=-2b\int_0^l\left[\left(\frac{\partial u}{\partial t}\right)^2+2bu\frac{\partial u}{\partial t}+cu^2+a\left(\frac{\partial u}{\partial x}\right)^2\right]dx$$

Hence on the basis of (3.22) we have

$$\dot{V} = -2bV \tag{3.26}$$

This relation proves, that the function V is non-increasing with the time, for the solutions of Eq. (3.21), because its time derivative is negative. Moreover we have

$$V = Ae^{-2bt}\ , \qquad A = V\Big|_{t=0} > 0 \tag{3.27}$$

From (3.27) it follows that $V \to 0$ for $t \to \infty$.

It proves that the condition (3) of the theorem of stability is satisfied, but it gives even more, because this condition is proved on the basis of the equation (3.26), from which we have additionally the relation (3.27) and the following from its consequences.

Next it is necessary to choose the distances with respect to which we shall investigate the stability of the unperturbed solution $u = 0$.

Let us assume in the considered case one distance in the same form as the Liapunov's function, that is

$$\varrho(u) = \int_0^l \left[(c - b^2) u^2 + a\left(\frac{\partial u}{\partial x}\right)^2 + \left(bu + \frac{\partial u}{\partial t}\right)^2 \right] dx \tag{3.28}$$

$$\varrho_i = (\varrho)_{t=0} \tag{3.29}$$

Therefore the conditions (1) and (2) of the theorem of stability are automatically satisfied. On the basis of (3.27) we have

$$\varrho = \varrho_i e^{-2bt}\ , \qquad \varrho \to 0 \ \text{ for } \ t \to \infty \tag{3.30}$$

This proves that the unperturbed solution $u = 0$ is asymptotically stable, which means that the inequality $\varrho_i \leq \eta(\varepsilon)$ implies the inequality $\varrho \leq \varepsilon$ and $\varrho \to 0$ for $t \to \infty$.

It should be stressed here, that for the investigated problem we have obtained the more strong result, namely that the unperturbed solution is exponentially stable and that's why for $\rho_i \leq \eta(\varepsilon) = \varepsilon$ we have (see formulas (3.28) and (3.30))

$$\rho = \int_0^l \left[(c-b^2)u^2 + a\left(\frac{\partial u}{\partial x}\right)^2 + \left(bu + \frac{\partial u}{\partial t}\right)^2 \right] dx \leq \varepsilon e^{-2bt} \tag{3.31}$$

This result is hardly interpretable from the physical viewpoint and it would be more convenient to obtain a direct limitation of the perturbed solutions $u(x,t)$. Because in the considered problem appears only one space variable, therefore on the basis of (3.31) one can try to find such a limitation.

Indeed, if the inequality (3.31) is satisfied, then the inequality

$$\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx \leq \frac{\varepsilon}{a} e^{-2bt} \tag{3.32}$$

is even more satisfied. Using the Schwarz's inequality we get

$$u^2(x,t) = \left[\int_0^x \frac{\partial u}{\partial x} dx\right]^2 \leq \int_0^x 1^2 dx \int_0^x \left(\frac{\partial u}{\partial x}\right)^2 dx \leq l \int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx \tag{3.33}$$

Hence on the basis of (3.32) and (3.33) we have

$$u^2(x,t) \leq \frac{\varepsilon l}{a} e^{-2bt} \tag{3.34}$$

This inequality establishes the desirable limitation of the perturbed solution $u(x,t)$ and gives the rate of decrease in tending to the unperturbed solution $u = 0$ with the time.

In the particular case $b = 0$, the equation (3.21) takes the form

$$\frac{\partial^2 u}{\partial t^2} + cu = a \frac{\partial^2 u}{\partial x^2} \tag{3.35}$$

The Liapunov's function has in this case the form

$$V(u) = \int_0^l \left[\left(\frac{\partial u}{\partial t}\right)^2 + a\left(\frac{\partial u}{\partial x}\right)^2 + cu^2 \right] dx \tag{3.36}$$

If $b = 0$, $c = 0$, then the equation (3.21) takes the form

$$\frac{\partial^2 u}{\partial t^2} = a\,\frac{\partial^2 u}{\partial x^2} \tag{3.37}$$

The Liapunov's function has in this case the form

$$V(u) = \int_0^L \left[\left(\frac{\partial u}{\partial t}\right)^2 + a\left(\frac{\partial u}{\partial x}\right)^2 \right] dx \tag{3.38}$$

In the last two cases the function V represents the total energy of the vibrating string,and the relation (3.26) has the form.

$$\dot{V} = 0 \tag{3.39}$$

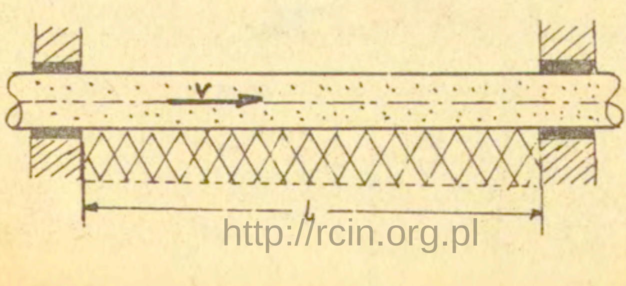
which expresses the law of the conservation of the total mechanical energy.In the last considered two cases,the unperturbed solution is stable, but not asymptotically.

The limitation (3.34) has then the form

(3.40) for $\rho_i \leq \eta(\varepsilon) = \varepsilon$ it is $u^2(x,t) \leq \dfrac{\varepsilon L}{a}$

3.3.2. The equation of the linear transversal vibrations of a string,supported on an elastic base taking into account the linear exterior damping and the flow of an ideal incompressible fluid in the interior of the string.

Let us consider an elastic conduit supported on an elastic base,fixed at both ends.Across this conduit flows an ideal and incompressible fluid with the constant velocity v .



We shall consider the small transversal vibrations of the conduit as the vibrations of a string, that is we shall neglect the bending stiffness of the conduit. We shall take into account the linear exterior damping and the linear elasticity of the base.

Let us introduce the notation

$$\dot{u} = \frac{du}{dt} = \frac{\partial u}{\partial x} v + \frac{\partial u}{\partial t} \tag{3.41}$$

Hence follows

$$\frac{d\dot{u}}{dt} = \frac{d}{dt}\left(\frac{du}{dt}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} v + \frac{\partial u}{\partial t}\right) v + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} v + \frac{\partial u}{\partial t}\right) = \tag{3.42}$$

$$= \frac{\partial^2 u}{\partial t^2} + 2v \frac{\partial^2 u}{\partial x \partial t} + v^2 \frac{\partial^2 u}{\partial x^2}$$

The differential equation of the transversal vibrations of the conduit has the form.

$$m_2 \frac{d\dot{u}}{dt} + m_1 \frac{\partial^2 u}{\partial t^2} + \mu \frac{\partial u}{\partial t} + r u = (T_0 - PA) \frac{\partial^2 u}{\partial x^2} \tag{3.43}$$

where

- m_2 - mass of the fluid in the unit of the length of the conduit
- m_1 - mass of the unit of the length of the conduit
- μ - coefficient of the exterior linear damping
- r - coefficient of the linear elasticity of the base
- T_0 - tension of the conduit
- P - pressure of the fluid in the conduit
- A - area of the transversal cross-section of the conduit
- v - velocity (constant) of the fluid in the conduit
- $u(x,t)$ - transversal displacement of the conduit

On the basis of (3.42) we have

$$(m_1 + m_2) \frac{\partial^2 u}{\partial t^2} + 2 v m_2 \frac{\partial^2 u}{\partial x \partial t} + \mu \frac{\partial u}{\partial t} + r u = \left[(T_0 - PA) - m_2 v^2\right] \frac{\partial^2 u}{\partial x^2} \tag{3.44}$$

Let us introduce the following notations

$$\frac{m_2}{m_1+m_2}v = w \;, \quad \frac{\mu}{m_1+m_2} = 2b \;, \quad \frac{\tau'}{m_1+m_2} = c$$

(3.45)

$$\frac{T_0 - PA}{m_1+m_2} = \alpha \;, \quad \frac{1}{m_1+m_2}\left[(T_0-PA) - m_2 v^2\right] = a$$

where w, b, c, α, a are positive constants.

It is visible, that the following relation

(3.46) $$a = \alpha - wv$$

is satisfied. Hence we may write the equation (3.44) in the form.

(3.47) $$\frac{\partial^2 u}{\partial t^2} + 2w\frac{\partial^2 u}{\partial x \partial t} + 2b\frac{\partial u}{\partial t} + cu = a\frac{\partial^2 u}{\partial x^2}$$

The initial and boundary conditions have the form (3.19) and (3.20). We will find the conditions which assure the asymptotical stability of the unperturbed solution $u = 0$.

It is visible, that in the case $v = 0$ the equation (3.47) takes the form of the equation (3.21).

Let us assume, that the Liapunov's function has the form

(3.48) $$V(u) = \int_0^l \left[\left(\frac{\partial u}{\partial t}\right)^2 + a\left(\frac{\partial u}{\partial x}\right)^2 + 2bu\frac{\partial u}{\partial t} + cu^2\right]dx =$$

$$= \int_0^l \left[(c-b^2)u^2 + a\left(\frac{\partial u}{\partial x}\right)^2 + \left(bu + \frac{\partial u}{\partial t}\right)^2\right]dx$$

where is assumed that

(3.49) $$c - b^2 > 0$$

Determining the time derivative of this function, we obtain in the same way as in the previous point, the following relation

(3.50) $$\dot{V} = -2bV + 4bw\int_0^l \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\,dx$$

This relation can be also written in the following form

$$\dot{V} = -2bV + 4bw\int_0^L \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}dx + 4b^2w\int_0^L u\frac{\partial u}{\partial x}dx$$

because

$$\int_0^L u\frac{\partial u}{\partial x}dx = \frac{1}{2}\int_0^L \frac{\partial(u^2)}{\partial x}dx = \frac{1}{2}\left[u^2(L) - u^2(0)\right] = 0$$

Hence we have

$$\dot{V} = -2bV + 4bw\int_0^L \frac{\partial u}{\partial x}\left(bu + \frac{\partial u}{\partial t}\right)dx \tag{3.51}$$

On the basis of the Cauchy's inequality we get

$$\dot{V} \leq -2bV + \frac{2bw}{\sqrt{a}}\int_0^L 2\left|\sqrt{a}\frac{\partial u}{\partial x}\left(bu + \frac{\partial u}{\partial t}\right)\right|dx \leq$$

$$\leq -2bV + \frac{2bw}{\sqrt{a}}\int_0^L \left[a\left(\frac{\partial u}{\partial x}\right)^2 + \left(bu + \frac{\partial u}{\partial t}\right)^2\right]dx$$

$$\dot{V} \leq -2bV + \frac{2bw}{\sqrt{a}}\int_0^L \left[(c-b^2)u^2 + a\left(\frac{\partial u}{\partial x}\right)^2 + \left(bu + \frac{\partial u}{\partial t}\right)^2\right]dx$$

$$\dot{V} \leq -2bV + \frac{2bw}{\sqrt{a}}V = 2b\left(\frac{w}{\sqrt{a}} - 1\right)V \tag{3.52}$$

To obtain the condition of the asymptotical stability it is sufficient to assume that

$$\frac{w}{\sqrt{a}} - 1 < 0 \tag{3.53}$$

Hence in view of (3.45) and (3.46) we have

$$w^2 < a = \alpha - wv = \frac{T_0 - PA}{m_1 + m_2} - \frac{m_2 v^2}{m_1 + m_2} \tag{3.54}$$

$$\left(\frac{m_2}{m_1 + m_2}\right)^2 v^2 < \frac{T_0 - PA}{m_1 + m_2} - \frac{m_2 v^2}{m_1 + m_2}$$

$$\frac{m_2^2 v^2}{m_1 + m_2} < T_0 - PA - m_2 v^2$$

$$\frac{m_2^2 v^2 + m_2 v^2 (m_1 + m_2)}{m_1 + m_2} < T_0 - PA$$

$$\frac{m_2^2 + m_1 m_2 + m_2^2}{m_1 + m_2} v^2 < T_0 - PA$$

$$\frac{m_2(m_1+2m_2)}{m_1+m_2}v^2 < T_0 - PA$$

$$(3.55)\quad m_2 v^2 < \frac{m_1+m_2}{m_1+2m_2}(T_0-PA) = \frac{1}{1+\frac{m_2}{m_1+m_2}}(T_0-PA)$$

It remains to choose the distance with respect to which we shall investigate the stability. Let us assume in the considered case

$$(3.56)\quad \varrho(u) = \int_0^l \left[\left(\frac{\partial u}{\partial t}\right)^2 + a\left(\frac{\partial u}{\partial x}\right)^2 + 2bu\frac{\partial u}{\partial t} + cu^2\right]dx$$

$$(3.57)\quad \varrho_i = (\varrho)_{t=0}$$

As the distance has the same form as the Liapunov's function, therefore the conditions (1) and (2) of the theorem of stability are automatically satisfied. The conditions $\dot{V} < 0$ and $V \xrightarrow[t\to\infty]{} 0$ were settled above. Hence from it follows, that the unperturbed solution is asymptotically (and exponentially) stable with respect to the distances (3.56) and (3.57) if

$$(3.58)\quad c - b^2 > 0 \quad , \quad m_2 v^2 < \frac{1}{1+\frac{m_2}{m_1+m_2}}(T_0-PA)$$

Because the distance (3.56) has the same form as V, then in view of (3.52) we have

$$(3.59)\quad \dot{\varrho} \leq -2b\left(1-\frac{w}{\sqrt{a}}\right)\varrho \quad , \quad 1-\frac{w}{\sqrt{a}} > 0$$

Thus we obtain

$$(3.60)\quad \varrho \leq B e^{-2b\left(1-\frac{w}{\sqrt{a}}\right)t} \quad , \quad B = \varrho\Big|_{t=0} = \varrho_i > 0$$

From (3.60) it follows that

$$(3.61)\quad \varrho \to 0 \quad \text{for} \quad t\to\infty$$

In the considered case we may obtain a result concerning the behaviour of the perturbed solutions $u(x,t)$ .Indeed,in view of (3.60) we have

$$a\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx \le B e^{-2b\left(1-\frac{w}{\sqrt{a}}\right)t}$$

Using the inequality (3.33) we get

$$u^2(x,t) \le \frac{Bl}{a} e^{-2b\left(1-\frac{w}{\sqrt{a}}\right)t} \tag{3.62}$$

3.3.3. The equation with constant coefficients of the linear transversal vibrations of a bar.

Let us consider the differential equation in the form

$$EJ\frac{\partial^4 u}{\partial x^4} + ru + \mu\frac{\partial u}{\partial t} + m\frac{\partial^2 u}{\partial t^2} = 0 \tag{3.63}$$

where

EJ - bending stiffness of the bar
r - coefficient of the linear elasticity of the base
μ - coefficient of exterior linear damping
m - linear density (mass of the unit of length)
$u(x,t)$ - transversal displacement of the bar

We can consider Eq.(3.63) as a mathematical model of the small (linear) free transversal vibrations of a bar, supported on the linear elastic base,taking into account the exterior linear damping.

Let the initial conditions have the form

$$u(x,0) = \varphi(x) \quad , \quad \frac{\partial u(x,0)}{\partial t} = \psi(x) \tag{3.64}$$

and let the boundary conditions have the form

$$\begin{aligned} &u(0,t) = 0 \quad , \quad u(l,t) = 0 \\ &\frac{\partial^2 u(0,t)}{\partial x^2} = 0 \quad , \quad \frac{\partial^2 u(l,t)}{\partial x^2} = 0 \end{aligned} \tag{3.65}$$

These boundary conditions mean that the bar has additionally two point supports on both ends.

Introducing the notations

$$\frac{EJ}{m} = q \quad , \qquad \frac{\tau}{m} = c \quad , \qquad \frac{\mu}{m} = 2b$$

we get Eq.(3.63) in the form

$$q\frac{\partial^4 u}{\partial x^4} + cu + 2b\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = 0 \tag{3.66}$$

We shall investigate the stability in the Liapunov's sense of the unperturbed solution $u = 0$. To this end, let us assume, that the Liapunov's function has the form

$$V(u) = \int_0^l \left[q\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 + 2bu\frac{\partial u}{\partial t} + cu^2 \right] dx = \tag{3.67}$$

$$= \int_0^l \left[(c-b^2)u^2 + \left(bu + \frac{\partial u}{\partial t}\right)^2 + q\left(\frac{\partial^2 u}{\partial x^2}\right)^2 \right] dx$$

where is assumed that

$$c - b^2 > 0 \tag{3.68}$$

Let us determine the time derivative of the function (3.67) for the solutions of Eq.(3.66).

$$\dot V = 2\int_0^l \left[(c-b^2)u\frac{\partial u}{\partial t} + \left(bu + \frac{\partial u}{\partial t}\right)\left(b\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2}\right) + q\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^2 \partial t} \right] dx$$

Substituting instead of $\frac{\partial^2 u}{\partial t^2}$ the corresponding terms from the equation (3.66) and performing analogous transformations as in the case of the string, we get this derivative in the form

$$\dot V = 2\int_0^l \left[-b\left(\frac{\partial u}{\partial t}\right)^2 - 2b^2 u\frac{\partial u}{\partial t} - bcu^2 - bqu\frac{\partial^4 u}{\partial x^4} - q\frac{\partial u}{\partial t}\frac{\partial^4 u}{\partial x^4} + q\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^2 \partial t} \right] dx$$

We apply integration by parts, using the boundary conditions to calculate the following integrals

$$\int_0^l \frac{\partial u}{\partial t}\frac{\partial^4 u}{\partial x^4} dx = \left[\frac{\partial u}{\partial t}\frac{\partial^3 u}{\partial x^3}\right]_0^l - \int_0^l \frac{\partial^3 u}{\partial x^3}\frac{\partial^2 u}{\partial x \partial t} dx = \tag{3.69}$$

$$= \left[-\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial x \partial t}\right]_0^l + \int_0^l \frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^2 \partial t} dx = \int_0^l \frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^2 \partial t} dx$$

$$(3.70)\quad \int_0^l u \frac{\partial^4 u}{\partial x^4} dx = \left[u \frac{\partial^3 u}{\partial x^3} \right]_0^l - \int_0^l \frac{\partial u}{\partial x} \frac{\partial^3 u}{\partial x^3} dx =$$

$$= \left[-\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} \right]_0^l + \int_0^l \left(\frac{\partial^2 u}{\partial x^2} \right)^2 dx = \int_0^l \left(\frac{\partial^2 u}{\partial x^2} \right)^2 dx$$

Thus we obtain

$$\dot{V} = -2b \int_0^l \left[\left(\frac{\partial u}{\partial t} \right)^2 + 2bu \frac{\partial u}{\partial t} + cu^2 + q \left(\frac{\partial^2 u}{\partial x^2} \right)^2 \right] dx$$

Hence on the basis of (3.67) we have

$$(3.71)\qquad \dot{V} = -2bV$$

This relation proves, that the function V is non-increasing with the time, for the solutions of Eq.(3.66), because its time derivative is negative. Moreover we have

$$(3.72)\qquad V = Ae^{-2bt} \quad , \quad A = V\big|_{t=0} > 0$$

From (3.72) it follows that $V \to 0$ for $t \to \infty$.

Next it is necessary to choose the distances with respect to which we shall investigate the stability of the unperturbed solution $u = 0$.

Let us assume in the considered case one distance, in the same form as the Liapunov's function, that is

$$(3.73)\quad \rho(u) = \int_0^l \left[(c - b^2) u^2 + \left(bu + \frac{\partial u}{\partial t} \right)^2 + q \left(\frac{\partial^2 u}{\partial x^2} \right)^2 \right] dx$$

$$(3.74)\quad \rho_i = (\rho)_{t=0}$$

As the distance ρ has the same form as the Liapunov's function V, therefore the conditions (1) and (2) of the theorem of stability are automatically satisfied. On the basis of (3.72) we have

$$(3.75)\qquad \rho = \rho_i e^{-2bt} \quad , \quad \rho \to 0 \quad \text{for} \quad t \to \infty$$

This proves, that the unperturbed solution $u = 0$ is asymptotically stable, which means that the inequality $\varrho_i \leq \eta(\varepsilon)$ implies the inequality $\varrho \leq \varepsilon$ and $\varrho \to 0$ for $t \to \infty$. In the considered case one can assume that $\eta(\varepsilon) = \varepsilon$.

It should be stressed here, that for the investigated problem we have obtained the more strong result, namely that the unperturbed solution is exponentially stable and that's why for $\varrho_i \leq \eta(\varepsilon) = \varepsilon$ we have (see formulas (3.73) and (3.75))

$$(3.76) \quad \varrho = \int_0^l \left[(c-b^2)u^2 + \left(bu + \frac{\partial u}{\partial t}\right)^2 + q\left(\frac{\partial^2 u}{\partial x^2}\right)^2 \right] dx \leq \varepsilon e^{-2bt}$$

This result is hardly interpretable from the physical viewpoint, so it is more convenient to obtain a direct limitation of the perturbed solutions $u(x,t)$. Because in the considered problem appears only one space variable, therefore on the basis of (3.76) one can try to find such a limitation.

Indeed, using the Picone's inequality

$$(3.77) \qquad \int_0^l \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx \geq \frac{\pi^2}{l^2} \int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx$$

in view of (3.76) we get

$$\int_0^l \left[(c-b^2)u^2 + \left(bu + \frac{\partial u}{\partial t}\right)^2 + q\frac{\pi^2}{l^2}\left(\frac{\partial u}{\partial x}\right)^2 \right] dx \leq \varepsilon e^{-2bt}$$

Hence we have

$$q\frac{\pi^2}{l^2} \int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx \leq \varepsilon e^{-2bt}$$

Using the inequality of Schwarz in the form (3.33) we obtain

$$(3.78) \qquad u^2(x,t) \leq \frac{\varepsilon l^3}{q\pi^2} e^{-2bt}$$

This inequality establishes the desirable limitation of the perturbed solution $u(x,t)$ and gives the rate of decrease in the tending to the unperturbed solution $u = 0$ with the time.

In the particular case $b = 0$, the equation (3.66) takes the form.

(3.79) $$q \frac{\partial^4 u}{\partial x^4} + cu + \frac{\partial^2 u}{\partial t^2} = 0$$

The Liapunov's function has in this case the form

(3.80) $$V(u) = \int_0^l \left[q \left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 + cu^2 \right] dx$$

If $b = 0$, $c = 0$ then the equation (3.66) takes the form

(3.81) $$q \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0$$

The Liapunov's function has in this case the form

(3.82) $$V(u) = \int_0^l \left[q \left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 \right] dx$$

In the last two cases, tha function V represents the total energy of the vibrating bar, and the relation (3.71) has the form

(3.83) $$\dot{V} = 0$$

which expresses the law of the conservation of the total mechanical energy. In the last two cases, the unperturbed solution is stable, but not asymptotically.

The limitation (3.78) has then the form

(3.84) for $\varrho_i \leq \eta(\varepsilon) = \varepsilon$ it is $u^2(x,t) \leq \dfrac{\varepsilon l^3}{q \pi^2}$

Let us consider the equation of linear transversal vibrations of a bar, without exterior damping and elastic base, but taking into account a longitudinal force compressing the bar.

In particular, it is interesting, if the investigation of the stability of motion of the unperturbed solution, will allow to find the Euler's critical force.

The equation of the vibrations has the form

$$EJ\frac{\partial^4 u}{\partial x^4}+P\frac{\partial^2 u}{\partial x^2}+m\frac{\partial^2 u}{\partial t^2}=0 \tag{3.85}$$

where P denotes the longitudinal compressive force.

Let the initial and boundary conditions have the form (3.64) and (3.65). Introducing the notations

$$\frac{EJ}{m}=q>0 \quad , \quad \frac{P}{m}=k>0$$

we get the equation (3.85) in the form

$$q\frac{\partial^4 u}{\partial x^4}+k\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial t^2}=0 \tag{3.86}$$

We shall investigate the stability in the Liapunov's sense of the unperturbed solution $u=0$. Let us assume the Liapunov's function in the form

$$V(u)=\int_0^L\left[q\left(\frac{\partial^2 u}{\partial x^2}\right)^2-k\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right]dx \tag{3.87}$$

On the basis of the inequality (3.77) we have

$$V(u)\geqslant\int_0^L\left[\left(q\frac{\pi^2}{L^2}-k\right)\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right]dx \tag{3.88}$$

Let us assume, that

$$q\frac{\pi^2}{L^2}-k>0 \quad \text{which means that} \quad P<P_{cr}=EJ\frac{\pi^2}{L^2} \tag{3.89}$$

Therefore if the longitudinal compressive force is less then the critical Euler's force P_{cr}, the Liapunov's function (3.87) is positive.

Let us determine the time derivative of the function (3.87) for the solutions of Eq. (3.86)

$$\dot V=2\int_0^L\left[q\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^2\partial t}-k\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}+\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}\right]dx=$$

$$=2\int_0^L\left[q\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^2\partial t}-k\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}-q\frac{\partial u}{\partial t}\frac{\partial^4 u}{\partial x^4}-k\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\right]dx$$

We apply integration by parts using the boundary conditions, to calculate the following integral

$$(3.90)\quad \int_0^l \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}dx = \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_0^l - \int_0^l \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}dx = -\int_0^l \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}dx$$

In view of (3.90) we have

$$(3.91)\quad \dot{V} = 2\int_0^l \left[q\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^2 \partial t} + k\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2} - q\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^2 \partial t} - k\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\right]dx = 0$$

Therefore the function V is non-increasing with the time.

Let us assume in the considered case one distance in the same form as the function V .

$$(3.92)\quad \varrho(u) = \int_0^l \left[q\left(\frac{\partial^2 u}{\partial x^2}\right)^2 - k\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]dx$$

$$(3.93)\quad \varrho_i = (\varrho)_{t=0}$$

Because the distance ϱ has the same form as the Liapunov's function V , therefore the conditions (1) and (2) of the theorem of stability are automatically satisfied. Besides we have settled above, that the condition (3) of the theorem of stability is also satisfied. Therefore the unperturbed solution $u = 0$ is stable but not asymptotically.

It means that if $\varrho_i \leq \eta(\varepsilon)$, then $\varrho \leq \varepsilon$ and in the considered case one can assume that $\eta(\varepsilon) = \varepsilon$. Therefore for $\varrho_i \leq \eta(\varepsilon) = \varepsilon$ we have on the basis of (3.88)

$$(3.94)\quad \int_0^l \left[\left(q\frac{\pi^2}{l^2} - k\right)\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]dx \leq \varrho \leq \varepsilon$$

In view of this inequality we have

$$\left(q\frac{\pi^2}{l^2} - k\right)\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx \leq \varepsilon$$

Using the inequality of Schwarz in the form (3.33) we get

$$(3.95)\quad u^2(x,t) \leq \frac{\varepsilon l}{q\frac{\pi^2}{l^2} - k}$$

3.3.4. The equation of the linear transversal vibrations of a bar supported on an elastic base taking into account the linear exterior damping and the flow of an ideal incompressible fluid in the interior of the bar.

Let us consider an elastic tube supported on an elastic base and additionally on two point supports on both ends. Inside of this tube flows an ideal incompressible fluid with the constant velocity v .

We shall consider the small vibrations of the tube as the vibrations of a bar, taking into account its bending stiffness. We shall also take into account the linear exterior damping and the linear elasticity of the base.

The differential equation of the transversal vibrations of the tube has the form

$$(3.96)\quad m_2 \frac{d\dot{u}}{dt} + m_1 \frac{\partial^2 u}{\partial t^2} + \mu \frac{\partial u}{\partial t} + r u + EJ \frac{\partial^4 u}{\partial x^4} = 0$$

where

m_2 - mass of the fluid in the unit of the length of the tube

m_1 - mass of the unit of the length of the tube

r - coefficient of the linear elasticity of the base

EJ - bending stiffness of the tube

$u(x,t)$ - transversal displacement of the tube

On the basis of (3.42) we have

$$(3.97)\quad EJ \frac{\partial^4 u}{\partial x^4} + (m_1 + m_2)\frac{\partial^2 u}{\partial t^2} + 2 v m_2 \frac{\partial^2 u}{\partial x \partial t} + \mu \frac{\partial u}{\partial t} + r u + m_2 v^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Let us introduce the following notations

$$(3.98)\quad \begin{cases} \dfrac{m_2}{m_1 + m_2} v = w \,, \quad \dfrac{\mu}{m_1 + m_2} = 2b \,, \quad \dfrac{r}{m_1 + m_2} = c \\[2ex] \dfrac{EJ}{m_1 + m_2} = q \,, \quad \dfrac{m_2}{m_1 + m_2} v^2 = k \end{cases}$$

where w, b, c, q, k are positive constants.

Therefore the equation (3.97) takes the form

$$(3.99)\quad q\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial t^2}+2w\frac{\partial^2 u}{\partial x\partial t}+2b\frac{\partial u}{\partial t}+cu+k\frac{\partial^2 u}{\partial x^2}=0$$

The initial and boundary conditions have the form (3.64) and (3.65).

We will find the conditions which assure the asymptotical stability of the unperturbed solution $u=0$.

It is visible that in the case $v=0$, the equation (3.99) takes the form of the equation (3.66).

Let us assume, that the Liapunov's function has the form

$$(3.100)\quad V(u)=\int_0^l\left[q\left(\frac{\partial^2 u}{\partial x^2}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2-k\left(\frac{\partial u}{\partial x}\right)^2+2bu\frac{\partial u}{\partial t}+cu^2\right]dx=$$

$$=\int_0^l\left[(c-b^2)u^2-k\left(\frac{\partial u}{\partial x}\right)^2+\left(bu+\frac{\partial u}{\partial t}\right)^2+q\left(\frac{\partial^2 u}{\partial x^2}\right)^2\right]dx$$

On the basis of the inequality (3.77) we have

$$(3.101)\quad V(u)\geqslant\int_0^l\left[\left(q\frac{\pi^2}{l^2}-k\right)\left(\frac{\partial u}{\partial x}\right)^2+(c-b^2)u^2+\left(bu+\frac{\partial u}{\partial t}\right)^2\right]dx$$

where is assumed, that

$$(3.102)\quad \alpha=q\frac{\pi^2}{l^2}-k>0\qquad,\qquad c-b^2>0$$

Let us determine the time derivative of the function V for the solutions of Eq.(3.99)

$$\dot V=2\int_0^l\left[(c-b^2)u\frac{\partial u}{\partial t}-k\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}+\left(bu+\frac{\partial u}{\partial t}\right)\left(b\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial t^2}\right)+q\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^2\partial t}\right]dx$$

$$\dot V=2\int_0^l\left[cu\frac{\partial u}{\partial t}-b^2u\frac{\partial u}{\partial t}-k\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}+\left(bu+\frac{\partial u}{\partial t}\right)\left(-b\frac{\partial u}{\partial t}-2w\frac{\partial^2 u}{\partial x\partial t}-\right.\right.$$

$$\left.\left.-cu-k\frac{\partial^2 u}{\partial x^2}-q\frac{\partial^4 u}{\partial x^4}\right)+q\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^2\partial t}\right]dx$$

$$\dot{V} = 2\int_0^l \Big[cu\frac{\partial u}{\partial t} - b^2 u\frac{\partial u}{\partial t} - k\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t} - b^2 u\frac{\partial u}{\partial t} - 2bwu\frac{\partial^2 u}{\partial x\partial t} - bcu^2 -$$

$$- bku\frac{\partial^2 u}{\partial x^2} - bqu\frac{\partial^4 u}{\partial x^4} - b\left(\frac{\partial u}{\partial t}\right)^2 - 2w\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x\partial t} - cu\frac{\partial u}{\partial t} - k\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2} -$$

$$- q\frac{\partial u}{\partial t}\frac{\partial^4 u}{\partial x^4} + q\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^2\partial t}\Big]dx$$

$$\dot{V} = 2\int_0^l \Big[-2b^2 u\frac{\partial u}{\partial t} - k\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t} - 2bwu\frac{\partial^2 u}{\partial x\partial t} - bcu^2 - b\left(\frac{\partial u}{\partial t}\right)^2 -$$

$$-2w\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x\partial t} - k\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2} - bku\frac{\partial^2 u}{\partial x^2} - bqu\frac{\partial^4 u}{\partial x^4} - q\frac{\partial u}{\partial t}\frac{\partial^4 u}{\partial x^4} + q\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^2\partial t}\Big]dx$$

We apply integration by parts,using the boundary conditions to calculate the following integrals

$$\int_0^l \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x\partial t}dx = \left[\left(\frac{\partial u}{\partial t}\right)^2\right]_0^l - \int_0^l \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x\partial t}dx$$

Hence we have

$$\int_0^l \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x\partial t}dx = 0$$

$$\int_0^l u\frac{\partial^2 u}{\partial x\partial t}dx = \left[u\frac{\partial u}{\partial t}\right]_0^l - \int_0^l \frac{\partial u}{\partial t}\frac{\partial u}{\partial x}dx = -\int_0^l \frac{\partial u}{\partial t}\frac{\partial u}{\partial x}dx$$

Besides we have already calculated the following integrals (see formulas (3.24),(3.25),(3.69),(3.70)).

$$\int_0^l \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}dx = -\int_0^l \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dx$$

$$\int_0^l u\frac{\partial^2 u}{\partial x^2}dx = -\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$\int_0^l u\frac{\partial^4 u}{\partial x^4}dx = \int_0^l \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx$$

$$\int_0^l \frac{\partial u}{\partial t}\frac{\partial^4 u}{\partial x^4}dx = \int_0^l \frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^2\partial t}dx$$

Therefore the time derivative $\dot{V}$ takes the form

$$\dot{V} = 2\int_0^l \Big[-2b^2 u\frac{\partial u}{\partial t} - k\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t} + 2bw\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} - bcu^2 - b\left(\frac{\partial u}{\partial t}\right)^2 +$$

$$+ k\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t} + bk\left(\frac{\partial u}{\partial x}\right)^2 - bq\left(\frac{\partial^2 u}{\partial x^2}\right)^2 - q\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^2\partial t} + q\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^2\partial t}\Big]dx$$

$$\dot{V} = -2b\int_0^l \left[q\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 - k\left(\frac{\partial u}{\partial x}\right)^2 + 2bu\frac{\partial u}{\partial t} + cu^2\right]dx +$$

$$+ 4bw\int_0^l \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\,dx$$

In view of (3.100) we have

$$\dot{V} = -2bV + 4bw\int_0^l \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\,dx \tag{3.103}$$

This relation has the same form as relation (3.50). Therefore, performing the same transformations as in the case of the relation (3.50), we obtain the inequality (3.52), that is

$$\dot{V} \le -2bV + \frac{2bw}{\sqrt{\alpha}}V = 2b\left(\frac{w}{\sqrt{\alpha}} - 1\right) \tag{3.104}$$

To obtain the condition of the asymptotical stability it is sufficient to assume, that

$$\frac{w}{\sqrt{\alpha}} - 1 < 0 \tag{3.105}$$

Hence in view of (3.98) and (3.102) we get

$$m_2 v^2 \le \frac{1}{1 + \frac{m_2}{m_1 + m_2}} EJ\frac{\pi^2}{l^2} \tag{3.106}$$

It remains to choose the distance, with respect to which we shall investigate the stability. Let us assume in the considered case

(3.107) $$\rho(u) = \int_0^l \left[q\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 - k\left(\frac{\partial u}{\partial x}\right)^2 + 2bu\frac{\partial u}{\partial t} + cu^2 \right] dx$$

(3.108) $$\rho_i = (\rho)_{t=0}$$

Since the distance has the same form as the Liapunov's function, therefore the conditions (1) and (2) of the theorem of stability are automatically satisfied. The conditions $\dot{V} < 0$ and $V \to 0$ for $t \to \infty$ were settled above. Hence from it follows, that the unperturbed solution is asymptotically (and exponentially) stable, with respect to the distances (3.107) and (3.108) if:

(3.109) $$c - b^2 > 0 \quad , \quad m_2 v^2 \leq \frac{1}{1 + \frac{m_2}{m_1 + m_2}} EJ \frac{\pi^2}{l^2}$$

Because the distance (3.107) has the same form as V, then in view of (3.104) we have

(3.110) $$\dot{\rho} \leq -2b\left(1 - \frac{w}{\sqrt{\alpha}}\right)\rho \quad , \quad 1 - \frac{w}{\sqrt{\alpha}} > 0$$

Thus we obtain

(3.111) $$\rho \leq B e^{-2b\left(1 - \frac{w}{\sqrt{\alpha}}\right)t} \quad , \quad B = \rho\Big|_{t=0} = \rho_i > 0$$

From (3.111) it follows

(3.112) $$\rho \to 0 \quad \text{for} \quad t \to \infty$$

In the considered case we may obtain a result concerning the behaviour of the perturbed solutions $u(x,t)$. Indeed, in view of (3.100), (3.101) and (3.111) we have

(3.113) $$\int_0^l \left[\alpha\left(\frac{\partial u}{\partial x}\right)^2 + (c - b^2)u^2 + \left(bu + \frac{\partial u}{\partial t}\right)^2 \right] dx \leq \rho \leq \rho_i e^{-2b\left(1 - \frac{w}{\sqrt{\alpha}}\right)t}$$

Therefore the following inequality is even more satidfied

$$\alpha \int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx \le S_i e^{-2b\left(\lambda - \frac{W}{\sqrt{\alpha}}\right)t} .$$

Using the inequality (3.33) we get

$$u^2(x,t) \le \frac{S_i l}{\alpha} e^{-2b\left(\lambda - \frac{W}{\sqrt{\alpha}}\right)t} \tag{3.114}$$

3.3.5. The equation with constant coefficients of the linear vibrations of a rectangular membrane.

Let us consider the differential equation in the form

$$m \frac{\partial^2 u}{\partial t^2} + \mu \frac{\partial u}{\partial t} + r u - T_0 \Delta u = 0 \tag{3.115}$$

where

- m - surface density (mass of the unit of area)
- μ - coefficient of exterior linear damping
- r - coefficient of the linear elasticity of the base
- T_0 - tension of the membrane
- $u(x,y,t)$ - transversal displacement of the membrane
- $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

We can consider Eq.(3.115) as a mathematical model of the small (linear) free transversal vibrations of a rectangular membrane, supported on the base of linear elasticity, taking into account the exterior linear damping.

Let the initial conditions have the form

$$u(x,y,0) = \varphi(x,y) \quad , \quad \frac{\partial u(x,y,0)}{\partial t} = \psi(x,y) \tag{3.116}$$

and let the boundary conditions have the form

$$\begin{cases} u(0,y,t) = 0 & , \quad u(a,y,t) = 0 \\ u(x,0,t) = 0 & , \quad u(x,b,t) = 0 \end{cases} \tag{3.117}$$

It means that the membrane is fixed along its edges having the lengths $[0,a]$ and $[0,b]$.

Introducing the notations

$$\frac{\mu}{m} = 2\beta \quad , \qquad \frac{\sigma}{m} = c \quad , \qquad \frac{T_0}{m} = \gamma$$

we get Eq.(3.115) in the form

$$\frac{\partial^2 u}{\partial t^2} + 2\beta \frac{\partial u}{\partial t} + cu - \gamma\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \tag{3.118}$$

We shall investigate the stability in the Liapunov's sense of the unperturbed solution $u = 0$. To this end let us assume, that the Liapunov's function has the form.

$$V(u) = \int_0^a\int_0^b \left[\left(\frac{\partial u}{\partial t}\right)^2 + 2\beta u \frac{\partial u}{\partial t} + cu^2 + \gamma\left(\frac{\partial u}{\partial x}\right)^2 + \gamma\left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy = \tag{3.119}$$

$$= \int_0^a\int_0^b \left[(c-\beta^2)u^2 + \left(\beta u + \frac{\partial u}{\partial t}\right)^2 + \gamma\left(\frac{\partial u}{\partial x}\right)^2 + \gamma\left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy$$

where is assumed that

$$c - \beta^2 > 0 \tag{3.120}$$

Let us determine the time derivative of the function (3.119) for the solutions of Eq.(3.118)

$$\dot{V} = 2\int_0^a\int_0^b \left[(c-\beta^2)u \frac{\partial u}{\partial t} + \left(\beta u + \frac{\partial u}{\partial t}\right)\left(\beta \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2}\right) + \right. \tag{3.121}$$

$$\left. + \gamma \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t} + \gamma \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y \partial t}\right] dx\,dy$$

Substituting instead of $\frac{\partial^2 u}{\partial t^2}$ the corresponding terms from the equation (3.118) we get this derivative in the form.

$$\dot{V} = 2\int_0^a\int_0^b \left[-\beta\left(\frac{\partial u}{\partial t}\right)^2 - 2\beta^2 u \frac{\partial u}{\partial t} - \beta c u^2 + \beta\gamma u \frac{\partial^2 u}{\partial x^2} + \beta\gamma u \frac{\partial^2 u}{\partial y^2} + \right.$$

$$\left. + \gamma \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial y^2} + \gamma \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t} + \gamma \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y \partial t}\right] dx\,dy$$

Applying integration by parts and using the boundary conditions (3.117) we obtain the following relations

$$\int_0^a\int_0^b u\,\frac{\partial^2 u}{\partial x^2}\,dx\,dy = -\int_0^a\int_0^b \left(\frac{\partial u}{\partial x}\right)^2 dx\,dy$$

$$\int_0^a\int_0^b u\,\frac{\partial^2 u}{\partial y^2}\,dx\,dy = -\int_0^a\int_0^b \left(\frac{\partial u}{\partial y}\right)^2 dx\,dy$$

$$\int_0^a\int_0^b \frac{\partial u}{\partial t}\,\frac{\partial^2 u}{\partial x^2}\,dx\,dy = -\int_0^a\int_0^b \frac{\partial u}{\partial x}\,\frac{\partial^2 u}{\partial x\,\partial t}\,dx\,dy$$

$$\int_0^a\int_0^b \frac{\partial u}{\partial t}\,\frac{\partial^2 u}{\partial y^2}\,dx\,dy = -\int_0^a\int_0^b \frac{\partial u}{\partial y}\,\frac{\partial^2 u}{\partial y\,\partial t}\,dx\,dy$$

On the basis of these relations we obtain the time derivative $\dot{V}$ in the form

$$\dot{V} = -2\beta \int_0^a\int_0^b \left[\left(\frac{\partial u}{\partial t}\right)^2 + 2\beta u\,\frac{\partial u}{\partial t} + c u^2 + \gamma\left(\frac{\partial u}{\partial x}\right)^2 + \gamma\left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy$$

Hence in view of (3.119) we have

$$\dot{V} = -2\beta V \tag{3.122}$$

Therefore the function V is non-increasing with time, for the solutions of Eq. (3.118), because its time derivative is negative. Moreover we have

$$V = A e^{-2\beta t}\ , \qquad A = V\big|_{t=0} > 0 \tag{3.123}$$

From (3.123) it follows that $V \to 0$ for $t \to \infty$.

Next it is necessary to choose the distances, with respect to which we shall investigate the stability of the unperturbed solution $u = 0$.

Let us assume in the considered case one distance in the form

$$\rho(u) = \int_0^a\int_0^b \left[(c-\beta^2)u^2 + \left(\beta u + \frac{\partial u}{\partial t}\right)^2 + \gamma\left(\frac{\partial u}{\partial x}\right)^2 + \gamma\left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy \tag{3.124}$$

$$\rho_i = (\rho)_{t=0} \tag{3.125}$$

Since the distance ρ has the same form as the Liapunov's function V, therefore the conditions (1) and (2) of the theorem of stability are automatically satisfied. Besides we have settled above, that the condition (3) of the theorem of stability is also satisfied. Therefore the unperturbed solution $u=0$ is stable in the Liapunov's sense, with respect to the distances (3.124) and (3.125). Additionally in view of (3.123) we have

$$(3.126)\quad \rho = \rho_i e^{-2\beta t}, \qquad \rho \to 0 \quad \text{for} \quad t \to \infty$$

and that's (or in view of (3.123)) why the unperturbed solution $u=0$ is asymptotically (and exponentially) stable with respect to the assumed distances.

It means, that if $\rho_i \leq \eta(\varepsilon)$, then $\rho \leq \varepsilon$ where in the considered case we can assume that $\eta(\varepsilon) = \varepsilon$. It should be stressed, that for the investigated problem we have obtained a better result, namely

$$(3.127)\quad \rho = \int_0^a\int_0^b \left[(c-\beta^2)u^2 + \left(\beta u + \frac{\partial u}{\partial t}\right)^2 + \gamma\left(\frac{\partial u}{\partial x}\right)^2 + \gamma\left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy \leq \varepsilon e^{-2\beta t}$$

This result is hardly interpretable from the physical viewpoint and it would be more convenient to obtain a direct limitation of the perturbed solutions $u(x,y,t)$. As in the considered problem appear two space variables x and y, it is not possible to obtain a direct limitation of the function $u(x,y,t)$. But one can obtain the limitation of the double integral of the square of the function $u(x,y,t)$. For this end it is sufficient to notice that if the inequality (3.127) is satisfied, then the inequality

$$(3.128)\quad \int_0^a\int_0^b u^2(x,y,t)\,dx\,dy \leq \frac{\varepsilon}{c-\beta^2} e^{-2\beta t}$$

is satisfied evenmore. However this inequality failes in the particular case $c=0$, $\beta=0$ and even when $c \neq 0$, $\beta \neq 0$ we do not know is the limitation (3.128) the best obtainable in the considered case. Therefore we shall use still another method which allows to obtain the inequality of the type (3.128).

The function $u^2(x,y,t)$ can be written in the form

$$u^2(x,y,t) = \int_0^y \frac{\partial}{\partial\eta} u^2(x,\eta,t)\,d\eta = \int_0^y 2u(x,\eta,t)\,\frac{\partial u(x,\eta,t)}{\partial\eta}\,d\eta$$

Hence we have

$$\int_0^a\!\!\int_0^b u^2(x,y,t)\,dx\,dy = \int_0^a\!\!\int_0^b \Big[\int_0^y 2u(x,\eta,t)\,\frac{\partial u(x,\eta,t)}{\partial\eta}\,d\eta\Big]\,dx\,dy$$

(3.129) $$\int_0^a\!\!\int_0^b u^2(x,y,t)\,dx\,dy = \int_0^a\Big\{\int_0^b\Big[\int_0^y 2u(x,\eta,t)\,\frac{\partial u(x,\eta,t)}{\partial\eta}\,d\eta\Big]dy\Big\}\,dx$$

Changing the succession of integration with respect to y and η, using the Dirichlet's formula in the form

(3.130) $$\int_0^b dy\int_0^y f(\eta,y)\,d\eta = \int_0^b d\eta\int_\eta^b f(\eta,y)\,dy$$

and noticing, that in the considered case, the function in formula (3.130) does not depend on the variable y, we get (3.129) in view of (3.130) in the form

$$\int_0^a\!\!\int_0^b u^2(x,y,t)\,dx\,dy = \int_0^a\Big\{\int_0^b\Big[\int_\eta^b 2u(x,\eta,t)\,\frac{\partial u(x,\eta,t)}{\partial\eta}\,dy\Big]d\eta\Big\}\,dx$$

Since the integrated function on the right-hand side does not depend on y, we have

$$\int_0^a\!\!\int_0^b u^2(x,y,t)\,dx\,dy = \int_0^a\!\!\int_0^b 2u(x,\eta,t)\,\frac{\partial u(x,\eta,t)}{\partial\eta}\,(b-\eta)\,dx\,d\eta$$

The variable η changes itself in the interval $0 \le \eta \le b$, therefore the inequality $b-\eta \le b$ takes place. Denoting η by y we get

$$\int_0^a\!\!\int_0^b u^2(x,y,t)\,dx\,dy \le 2b\int_0^a\!\!\int_0^b |u(x,y,t)|\left|\frac{\partial u(x,y,t)}{\partial y}\right|\,dx\,dy$$

Using the Schwarz's inequality we obtain

$$\int_0^a\!\!\int_0^b u^2(x,y,t)\,dx\,dy \le 2b\Big[\int_0^a\!\!\int_0^b u^2(x,y,t)\,dx\,dy\Big]^{\frac{1}{2}}\Big[\int_0^a\!\!\int_0^b\Big(\frac{\partial u(x,y,t)}{\partial y}\Big)^2 dx\,dy\Big]^{\frac{1}{2}}$$

Taking the square of the both sides of this inequality we get

$$\int_0^a\int_0^b u^2 dx\,dy \le 4b^2 \int_0^a\int_0^b \left(\frac{\partial u}{\partial y}\right)^2 dx\,dy \tag{3.131}$$

On the basis of (3.127) and (3.131) we have

$$\frac{1}{4b^2}\int_0^a\int_0^b u^2 dx\,dy \le \int_0^a\int_0^b \left(\frac{\partial u}{\partial y}\right)^2 dx\,dy \le \frac{\varepsilon}{\gamma} e^{-2\beta t}$$

Therefore it is

$$\int_0^a\int_0^b u^2 dx\,dy \le \frac{4b^2}{\gamma}\varepsilon e^{-2\beta t} \tag{3.132}$$

In the particular case $\beta = 0$, $c = 0$ we obtain the inequality

$$\int_0^a\int_0^b u^2 dx\,dy \le \frac{4b^2}{\gamma}\varepsilon \tag{3.133}$$

Finally we arrive to the following limitation of the perturbed solutions $u(x,y,t)$

$$\int_0^a\int_0^b u^2 dx\,dy \le M\varepsilon e^{-2\beta t} \tag{3.134}$$

where

$$M = \min\left(\frac{1}{c-\beta^2}, \frac{4b^2}{\gamma}\right) \tag{3.135}$$

and b denotes the shorter side of the membrane. The inequality is true for $t \in [0, \infty)$ and for $\beta \neq 0$, $c \neq 0$, as well as for $\beta = 0$, $c = 0$.

In the particular case $\beta = 0$ the equation (3.118) takes the form

$$\frac{\partial^2 u}{\partial t^2} + cu - \gamma\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \tag{3.136}$$

In this case the Liapunov's function has the form

$$V(u) = \int_0^a\int_0^b \left[\left(\frac{\partial u}{\partial t}\right)^2 + cu^2 + \gamma\left(\frac{\partial u}{\partial x}\right)^2 + \gamma\left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy \tag{3.137}$$

If $\beta = 0$, and $c = 0$, then the equation (3.118) takes the form

(3.138) $$\frac{\partial^2 u}{\partial t^2} - \gamma \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$

The Liapunov's function has in this case the form

(3.139) $$V(u) = \int_0^a \int_0^b \left[\left(\frac{\partial u}{\partial t} \right)^2 + \gamma \left(\frac{\partial u}{\partial x} \right)^2 + \gamma \left(\frac{\partial u}{\partial y} \right)^2 \right] dx\, dy$$

In the last two cases the function V represents the total energy of the vibrating membrane, and the relation (3.122) has the form

(3.140) $$\dot{V} = 0$$

which expresses the law of the conservation of the total mechanical energy. In the last considered two cases, the unperturbed solution is stable but not asymptotically.

The limitation (3.134) has then the form

(3.141) $$\int_0^a \int_0^b u^2 dx\, dy \leq M\varepsilon \qquad \text{for} \qquad \varrho_i \leq \eta(\varepsilon) = \varepsilon$$

3.3.6. The equation with constant coefficients of the linear vibrations of a rectangular plate.

Let us consider the differential equation in the form

(3.142) $$m \frac{\partial^2 u}{\partial t^2} + \mu \frac{\partial u}{\partial t} + \tau u + D \Delta\Delta u = 0$$

where
$$D = \frac{E h^3}{12 (1 - \nu^2)}$$

E - modulus of elasticity,

ν - Posisson's coefficient

h - thickness of the plate

$$\Delta\Delta u = \frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}$$

The other quantities have the same meaning as in the previous point.

We can consider Eq.(3.142) as a mathematical model of the small (linear) vibrations of a thin rectangular plate supported on the linear elastic base, taking into account the exterior linear damping.

We assume that the plate is fixed along its edges of lengths $[0,a]$, $[0,b]$.

Let the initial conditions have the form

$$(3.143) \quad u(x,y,0) = \varphi(x,y) \quad , \quad \frac{\partial u(x,y,0)}{\partial t} = \psi(x,y)$$

and let the boundary conditions have the form

$$u(0,y,t) = 0 \quad , \quad u(a,y,t) = 0$$

$$u(x,0,t) = 0 \quad , \quad u(x,b,t) = 0$$

$$\frac{\partial u(0,y,t)}{\partial x} = 0 \quad , \quad \frac{\partial u(a,y,t)}{\partial x} = 0$$

$$\frac{\partial u(x,0,t)}{\partial y} = 0 \quad , \quad \frac{\partial u(x,b,t)}{\partial y} = 0$$

Introducing the notations

$$(3.144) \quad \frac{\mu}{m} = 2\beta \quad , \quad \frac{\tau}{m} = c \quad , \quad \frac{D}{m} = \gamma$$

we get Eq.(3.142) in the form

$$(3.145) \quad \frac{\partial^2 u}{\partial t^2} + 2\beta u \frac{\partial u}{\partial t} + cu + \gamma\left(\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}\right) = 0$$

We shall investigate the stability in the Liapunov's sense of the unperturbed solution $u = 0$ .To this end let us assume, that the Liapunov's function has the form

$$(3.146) \quad V(u) = \int_0^a\int_0^b \left[\left(\frac{\partial u}{\partial t}\right)^2 + 2\beta u \frac{\partial u}{\partial t} + cu^2 + \gamma\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 2\gamma\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \gamma\left(\frac{\partial^2 u}{\partial y^2}\right)^2\right] dx\,dy =$$

$$= \int_0^a\int_0^b \left[(c-\beta^2)u^2 + \left(\beta u + \frac{\partial u}{\partial t}\right)^2 + \gamma\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 2\gamma\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \gamma\left(\frac{\partial^2 u}{\partial y^2}\right)\right] dx\,dy$$

where is assumed that

$$(3.147) \quad c - \beta^2 > 0$$

The further proceeding is the same as in the case of the membrane. We determine the time derivative of the function (3.146) for the solutions of the equation (3.145) and next calculate integrating by parts the adequate integrals, using the boundary conditions. In this way we arrive to the relation (3.132). Assuming one distance in the same form as the Liapunov's function we confirm, that the unperturbed solution $u = 0$ is asymptotically (and exponentially) stable and in the cases $\beta = 0$ or $\beta = 0$, $c = 0$ this solution is stable, but not asymptotically. In the particular cases $\beta = 0$ and $\beta = 0$, $c = 0$ we obtain the corresponding differential equations and Liapunov's functions putting these quantities into (3.145) and (3.146).

In the case of the plate the limitation (3.134) has the following form

(3.148) $$\int_0^a\int_0^b u^2\,dx\,dy \leq M\varepsilon e^{-2\beta t}$$

where

(3.149) $$M = \min\left(\frac{1}{c-\beta^2},\ \frac{(4b^2)^2}{\gamma}\right)$$

3.3.7. <u>The equation of the non-linear vibration of a string supported on the non-linear elastic base, taking into account the linear exterior damping.</u>

Let us consider the differential equation in the form

(3.150) $$\frac{\partial^2 u}{\partial t^2} + 2b\frac{\partial u}{\partial t} + \left[c + \gamma f(u)\right]u = a\frac{\partial^2 u}{\partial x^2}$$

where a, b, c, γ denote positive constants.

Let the initial and boundary conditions have the form

(3.151) $$\begin{cases} u(x,0) = \varphi(x) \quad , & \dfrac{\partial u(x,0)}{\partial t} = \psi(x) \\ u(0,t) = 0 \quad , & u(l,t) = 0 \end{cases}$$

The function f is assumed to satisfy the conditions of existence and uniqueness of the considered limit (initial-boundary) problem. Moreover, let this function satisfies the following condition.

(3.152) $$f(u) \geqslant 0$$

We shall investigate the stability in the Liapunov's sense of the unperturbed solution $u = 0$ with respect to the convenient choosen distances. To this end let us assume, that the Liapunov's function has the form.

(3.153) $$V(u) = \int_0^l \Big[\Big(\frac{\partial u}{\partial t}\Big)^2 + a\Big(\frac{\partial u}{\partial x}\Big)^2 + 2bu\frac{\partial u}{\partial t} + cu^2 + 2\gamma\int_0^u s f(s)\,ds \Big] dx =$$

$$= \int_0^l \Big[(c-b^2)u^2 + a\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(bu + \frac{\partial u}{\partial t}\Big)^2 + 2\gamma\int_0^u s f(s)\,ds \Big] dx$$

where is assumed that

(3.154) $$c - b^2 > 0$$

Let us determine the time derivative of the function (3.153) for the solutions of Eq. (3.150).

$$\dot V = 2\int_0^l \Big[(c-b^2)u\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t} + \Big(bu + \frac{\partial u}{\partial t}\Big)\Big(b\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2}\Big) + \gamma u f(u)\frac{\partial u}{\partial t} \Big] dx =$$

$$= 2\int_0^l \Big[cu\frac{\partial u}{\partial t} - b^2u\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t} + \Big(bu + \frac{\partial u}{\partial t}\Big)\Big(-b\frac{\partial u}{\partial t} - cu - \gamma u f(u) +$$

$$+ a\frac{\partial^2 u}{\partial x^2}\Big) + \gamma u f(u)\frac{\partial u}{\partial t} \Big] dx$$

$$\dot V = 2\int_0^l \Big[cu\frac{\partial u}{\partial t} - b^2u\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t} - b^2u\frac{\partial u}{\partial t} - bcu^2 - b\gamma u^2 f(u) + bau\frac{\partial^2 u}{\partial x^2} -$$

$$- b\Big(\frac{\partial u}{\partial t}\Big)^2 - cu\frac{\partial u}{\partial t} - \gamma u f(u)\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2} + \gamma u f(u)\frac{\partial u}{\partial t} \Big] dx$$

On the basis of (3.24) and (3.25) we have

$$\dot V = 2\int_0^l \Big[-b\Big(\frac{\partial u}{\partial t}\Big)^2 - ba\Big(\frac{\partial u}{\partial x}\Big)^2 - 2b^2u\frac{\partial u}{\partial t} - bcu^2 - b\gamma u^2 f(u) \Big] dx$$

Drawing out „- b" before the square bracket and next adding and substracting inside of this bracket the term $2\gamma\int_0^u sf(s)ds$ we obtain

$$\dot{V} = -2b\int_0^l \Big[\Big(\frac{\partial u}{\partial t}\Big)^2 + a\Big(\frac{\partial u}{\partial x}\Big)^2 + 2bu\frac{\partial u}{\partial t} + cu^2 + 2\gamma\int_0^u sf(s)ds -$$

$$-2\gamma\int_0^u sf(s)ds + \gamma u^2 f(u)\Big]dx$$

Thus in view of (3.153) we have

$$(3.155)\quad \dot{V} = -2bV - 2b\gamma\int_0^l \Big[u^2 f(u) - 2\int_0^u sf(s)ds\Big]dx$$

The asymptotical stability of the unperturbed solution is assured when the following condition is satisfied

$$(3.166)\qquad u^2 f(u) - 2\int_0^u sf(s)ds \geqslant 0$$

Hence we have

$$(3.167)\qquad uf'(u) \geqslant 0$$

Next it is necessary to choose the distances with respect to which we shall investigate the stability of the unperturbed solution $u = 0$.

Let us assume in the considered case one distance in the same form as the Liapunov's function, that is

$$(3.168)\ \varrho(u) = \int_0^l \Big[(c-b^2)u^2 + a\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(bu + \frac{\partial u}{\partial t}\Big)^2 + 2\gamma\int_0^u sf(s)ds\Big]dx$$

$$(3.169)\quad \varrho_i = (\varrho)_{t=0}$$

Therefore the conditions (1) and (2) of the theorem of stability are automatically satisfied. Besides we have proved above, that if the conditions (3.154) and (3.167) are satisfied, then the function V is non-increasing (in fact it is decreasing) with time. It means, that all conditions of the theorem of stability are satisfied, therefore the unperturbed solution $u = 0$ is stable in Liapunov's sense, with respect to the distance given by the formula (3.168).

Thus the conditions of stability of the zero solution $u = 0$ have the form

$$(3.170) \qquad c - b^2 > 0 \quad , \quad f(u) \geq 0 \quad , \quad u f'(u) \geq 0$$

This means, that the inequality

$$(3.171) \quad \rho_i = \int_0^l \Big[(c-b^2) u^2(x,0) + a \Big(\frac{\partial u(x,0)}{\partial x} \Big)^2 + \Big(b u(x,0) + \frac{\partial u(x,0)}{\partial t} \Big)^2 + 2\gamma \int_0^{u(x,0)} s f(s) ds \Big] dx \leq \eta(\varepsilon) = \varepsilon$$

implies the inequality

$$(3.172) \quad \rho = \int_0^l \Big[(c-b^2) u^2(x,t) + a \Big(\frac{\partial u(x,t)}{\partial x} \Big)^2 + \Big(b u(x,t) + \frac{\partial u(x,t)}{\partial t} \Big)^2 + 2\gamma \int_0^{u(x,t)} s f(s) ds \Big] dx \leq \varepsilon$$

Let us still find the limitation of the function $u(x,t)$ following from the inequality (3.172). On the basis of (3.172) we have

$$(3.173) \qquad a \int_0^l \Big(\frac{\partial u}{\partial x} \Big)^2 dx \leq \varepsilon$$

Using the inequality (3.33) we get

$$(3.174) \qquad u^2(x,t) \leq \frac{l}{a} \varepsilon$$

3.3.8. The investigation of the behaviour of solutions of a non-linear equation of the vibrations of a string.

Let us consider the differential equation in the form

$$\frac{\partial^2 u}{\partial t^2} + 2b\frac{\partial u}{\partial t} + hu = a\frac{\partial^2 u}{\partial x^2} + f\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) \tag{3.175}$$

where a,b,h denote positive constants.

Let the initial and boundary conditions have the form

$$\begin{cases} u(x,0) = \varphi(x) \quad , & \dfrac{\partial u(x,0)}{\partial t} = \psi(x) \\ u(0,t) = 0 \quad , & u(l,t) = 0 \end{cases} \tag{3.176}$$

The function f is assumed to satisfy the conditions of existence and uniqueness of the considered limit (initial-boundary) problem, and to have a form admitting the existence of the zero solution u = 0 of Eq. (3.175) with the initial values being equal to zero.

We shall investigate the behaviour of solutions u(x,t) corresponding to non-zero initial conditions, with respect to zero solution. To this end we assume, that the distance between solution u(x,t) and zero solution has the form.

$$\varrho(u) = \int_0^l \left[\left(\frac{\partial u}{\partial t}\right)^2 + a\left(\frac{\partial u}{\partial x}\right)^2 + 2bu\frac{\partial u}{\partial t} + hu^2\right]dx = \tag{3.177}$$

$$= \int_0^l \left[(h-b^2)u^2 + a\left(\frac{\partial u}{\partial x}\right)^2 + \left(bu + \frac{\partial u}{\partial t}\right)^2\right]dx$$

where is assumed that

$$h - b^2 > 0 \tag{3.178}$$

Let us determine the time derivative of the distance (3.177) for the solutions of Eq. (3.175). Performing the same transformations as for the function V (see (3.22)) in the point 3.3.1 we obtain instead of (3.26), the following relation.

(3.179) $$\dot{\varrho} = -2b\varrho + 2\int_0^L \left(bu + \frac{\partial u}{\partial t}\right) f\,dx$$

Let us assume that

(3.180) $$2\int_0^L \left(bu + \frac{\partial u}{\partial t}\right) f\,dx \le \gamma(t)\varrho + \omega(t,\varrho)$$

where $\gamma = \gamma(t)$ is a continuous function and $\omega(t,\varrho)$ is a continuous function being non-negative for all $t \geqslant 0, \varrho \geqslant 0$.

Let us denote

(3.181) $$w(t) = \gamma(t) - 2b$$

From (3.179),(3.180) and (3.181) we get the differential inequality of the form

(3.182) $$\dot{\varrho} \le w(t)\varrho + \omega(t,\varrho)$$

Integrating the both sides of (3.182) with respect to time, we arrive at the following non-linear integral inequality.

(3.183) $$\varrho \le c + \int_0^t \left[w(s)\varrho(s) + \omega(s,\varrho(s))\right] ds \quad , \quad c = \varrho(0)$$

This is the non-linear integral inequality of the 2-nd kind (see (1.28)).

At first it is necessary to analyze the behaviour of the distance $\varrho = \varrho(t)$ and to construct its estimation, which next allows to use the obtained results to estimate $u = u(x,t)$.

Indeed, let us suppose that we have determined a function $\Phi(t)$ which limits the distance (3.181).

(3.184) $$\varrho = \int_0^L \left[(h-b^2)u^2 + a\left(\frac{\partial u}{\partial x}\right)^2 + \left(bu + \frac{\partial u}{\partial t}\right)^2\right] dx \le \Phi(t)$$

From (3.184) it follows

(3.185) $$a\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx \le \Phi(t)$$

Applying the inequality (3.33) we obtain

(3.186) $$u^2(x,t) \leq \frac{L}{a}\,\Phi(t)$$

Boundedness, or rate of increase (decrease), or tending to zero of the function $u(x,t)$ follows from the properties of the function $\Phi(t)$. It is visible that the main problem is to investigate the inequality (3.182) or (3.183) and on the basis of this investigation to obtain the function $\Phi(t)$.

4. The investigation of integral inequalities of the 1-st and 2-nd kind.

4.1. The integral inequality of the 1-st kind.

Let us consider the integral inequality in the form

(4.1) $$u(t) \leq c + \sigma k \int_{t_0}^{t} e^{-n(t-s)} g(u(s))\,ds = v(t)$$

Determining the time derivative of the function $v(t)$ we obtain the differential inequality

(4.2) $$\dot{v} = -n\sigma k e^{-nt} \int_{t_0}^{t} e^{ns} g(u(s))\,ds + \sigma k e^{-nt} e^{nt} g(u(t)) =$$

$$= -n(v-c) + \sigma k\, g(u(t)) \leq nc - nv + \sigma k\, g(v(t))$$

Let us introduce the following change of variables

(4.3) $$\varrho = \frac{1}{c}v - 1 \;, \quad v = (1+\varrho)c \quad , \quad \tau = nt$$

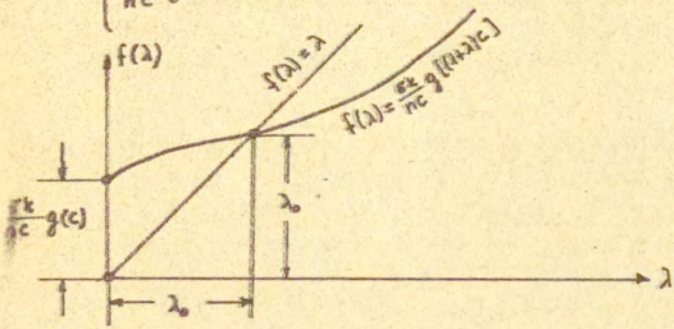
For $t = t_0$ it is $u = u_0 = c$ and that's why $\varrho_0 = \frac{u_0}{c} - 1 = 0$. Besides it is $\frac{d\tau}{dt} = n$. Therefore we obtain

$$\frac{dv}{dt} = c\,\frac{d\varrho}{d\tau}\,\frac{d\tau}{dt} \leq -nc\varrho + \sigma k\, g\left[(1+\varrho)c\right]$$

(4.4) $$\frac{d\varrho}{d\tau} \leq \frac{\sigma k}{nc}\, g\left[(1+\varrho)c\right] - \varrho$$

Let us assume that it exists a constant λ_0 satisfying the following relations

$$(4.5)\qquad \begin{cases} \frac{\sigma k}{nc} g[(1+\lambda_0)c] - \lambda_0 = 0 \\ \frac{\sigma k}{nc} g[(1+\lambda)c] - \lambda > 0 \quad , \quad \lambda \in [0, \lambda_0) \end{cases}$$



On the basis of (4.4) we obtain the following integral inequality

$$\int_0^{\varrho(\tau)} \frac{ds}{\frac{\sigma k}{nc} g[(1+s)c] - s} \leq \tau - \tau_0 \quad , \quad \tau \in [\tau_0, \tau^*)$$

As for $\lambda < \lambda_0$ it is $\frac{\sigma k}{nc} g[(1+\lambda)c] < \lambda_0$, therefore we may write the following inequality

$$(4.6)\qquad \int_0^{\varrho(\tau)} \frac{ds}{\lambda_0 - s} \leq \int_0^{\varrho(\tau)} \frac{ds}{\frac{\sigma k}{nc} g[(1+s)c] - s} \leq \tau - \tau_0 \quad , \quad \tau \in [\tau_0, \tau^*)$$

Let us suppose that for $\tau = \tau^* < \infty$ it is $\varrho(\tau^*) = \lambda_0$. Hence on the basis of (4.6) it follows

$$(4.7)\qquad \int_0^{\lambda_0} \frac{ds}{\lambda_0 - s} \leq \tau^* - \tau_0$$

However this inequality is impossible, because the integral is divergent and τ^* is finite. Therefore the following inequality

$$(4.8)\qquad \varrho(\tau) \leq \lambda_0 \quad , \quad \tau \in [\tau_0, \infty)$$

is satisfied in which the sign „ < " takes place for $\tau < \infty$ and „ = " for $\tau = \infty$. On the basis of (4.3) and (4.1) we have

$$(4.9) \qquad u(t) \le v(t) \le (1+\lambda_0)c$$

Let us write the first of the relations (4.5) in the form

$$(4.10) \qquad \lambda_0 = \frac{\sigma k}{n} \frac{1}{c} g[(1+\lambda_0)c]$$

If $c = \text{const} > 0$, then $\lambda_0 \to 0$ for $\frac{\sigma k}{n} \to 0$.

To consider the case $c \to 0$ we suppose additionally that

$$(4.11) \qquad \lim_{\vartheta \to 0} \frac{g(\vartheta)}{\vartheta} = \Omega = \text{const} < \infty$$

In view of (4.10) we have

$$c\lambda_0 = \frac{\sigma k}{n} g[(1+\lambda_0)c] \quad , \quad 1 = \frac{\sigma k}{n} \frac{g[(1+\lambda_0)c]}{(1+\lambda_0)c} + \frac{1}{1+\lambda_0}$$

For $c \to 0$ we obtain

$$(4.12) \qquad \lambda_0 = \frac{\frac{\sigma k}{n}}{1 - \frac{\sigma k}{n}\Omega}$$

From (4.12) it follows, that if $\frac{\sigma k}{n}\Omega < 1$, then also for $c \to 0$ we obtain $\lambda_0 \to 0$ for $\frac{\sigma k}{n} \to 0$.

4.2. The integral inequality of the 2-nd kind.

Let us consider the integral inequality in the form

$$(4.13) \quad u(t) \le c + \int_{t_0}^{t} [w(s)u(s) + \omega(s, u(s))]ds \ , \ t \ge t_0 \ , \ c > 0$$

It is necessary to remind here, that we have obtained this inequality in the part 1 (see (1.28)) for $w(t) = -\alpha(t)$ $\alpha > 0$, $w(t) = \beta(t)$, $\beta > 0$, and $w(t) = 0$. But the case $w(t) = -\alpha(t)$, $\alpha > 0$ was obtained only under assumption, that the inequality (1.12) for the comparison differential equation is satisfied and this inequality is rather difficult to establish. An another case in which we can arrive to the integral inequality (4.13) with $w(t) = -\alpha(t)$, $\alpha > 0$ is the case of the inequality (3.183) obtained in the part 3, by the investigation of the behaviour of solutions of the non-linear partial differential equation (3.175). Indeed, putting $\gamma(t) - 2b < 0$ we obtain $w(t) = -\alpha(t)$, $\alpha > 0$. But in this last case the integral inequality (3.183) was obtained on the basis of the differential inequality (3.182), which plays an essential role for further investigations.

It should be emphasized, that in the case $w(t) = -\alpha(t)$, $\alpha > 0$, it is necessary to be very cautious, because the results obtained for $w(t) \geqslant 0$ in general cannot be used without additional assumptions in the case $w(t) < 0$.

We shall begin the investigation of the integral inequality (4.13) assuming that $w(t) \geqslant 0$. The case $w(t) < 0$ will be considered separately.

Theorem 1. Let the functions $u(t)$, $w(t)$ be continuous and non-negative for all $t \geqslant t_0$ and function $\omega(t,u)$ continuous and non-negative for all $t \geqslant t_0$, $u \geqslant 0$. Let us assume that the inequality (4.13) holds and furthermore, there exists a continuous and non-negative function $v(t)$ for all $t \geqslant t_0$, and continuous non-negative and non-decreasing with respect to $u \geqslant 0$ function $g(u)$ (and there exists $u_0 \in (0,c]$ for which $g(u_0) > 0$) such, that the following inequality is satisfied.

$$(4.14) \quad \omega\left[t, z\exp\left(\int_{t_0}^{t} w(s)ds\right)\right]\exp\left(-\int_{t_0}^{t} w(s)ds\right) \leq v(t)g(z), \quad t \geqslant t_0, \ z \geqslant 0$$

Then for all $t \in [t_0, T)$ the following inequality holds

$$(4.15) \quad u(t) \leq G^{-1}\left[G(c) + \int_{t_0}^{t} v(s)ds\right]\exp\left(\int_{t_0}^{t} w(s)ds\right)$$

where G^{-1} is the function inverse to

$$(4.16) \qquad G(u) = \int_{u_0}^{u} \frac{ds}{g(s)}, \quad 0 < u_0 \leq c \leq u$$

and the value of T is determined by the relation

$$(4.17)\qquad \int_{u_0}^{\infty} \frac{ds}{g(s)} = G(c) + \int_{t_0}^{T} v(s)\,ds$$

If furthermore, there exists a constant $\tilde{c} > 0$ such that

$$(4.18)\qquad \int_{t_0}^{\infty} v(t)\,dt \leq \int_{\tilde{c}}^{\infty} \frac{ds}{g(s)}$$

then the inequality (4.15) holds in the interval $[t_0, \infty)$, that is for all $c \in (0, \tilde{c})$ we have $T = \infty$.

Remark. In the case $w = 0$ we obtain the familiar Bihari's theorem.

Proof. In view of (4.14) we have

$$\omega\left[t, z\exp\left(\int_{t_0}^{t} w(s)ds\right)\right] \leq v(t)\exp\left(\int_{t_0}^{t} w(s)ds\right) g(z)$$

Putting

$$z = u\exp\left(-\int_{t_0}^{t} w(s)ds\right)$$

we obtain

$$(4.19)\qquad \omega(t, u(t)) \leq v(t)\exp\left(\int_{t_0}^{t} w(s)ds\right) g\left[u\exp\left(-\int_{t_0}^{t} w(s)ds\right)\right]$$

On the basis of (4.13) and (4.19) we obtain the inequality, the right-hand side of which we denote by

$$R(t)\exp\left(\int_{t_0}^{t} w(s)\,ds\right)$$

$$(4.20)\qquad u(t) \leq c + \int_{t_0}^{t}\left[w(s)u(s) + v(s)\exp\left(\int_{t_0}^{s} w(\tau)d\tau\right) \times\right.$$
$$\left.\times\, g\left(u(s)\exp\left(-\int_{t_0}^{s} w(\tau)d\tau\right)\right)\right]ds = R(t)\exp\left(\int_{t_0}^{t} w(s)ds\right)$$

Differentiating with respect to t, we have

$$\dot{R}(t)\exp\left(\int_{t_0}^{t} w(s)ds\right) + R\,w(t)\exp\left(\int_{t_0}^{t} w(s)\,ds\right) =$$
$$= w(t)u(t) + v(t)\exp\left(\int_{t_0}^{t} w(s)ds\right) g\left(u(t)\exp\left(-\int_{t_0}^{t} w(s)ds\right)\right)$$

In view of (4.20) we get the inequality

$$\dot{R}\exp\left(\int_{t_0}^{t} w(s)ds\right) + R\,w(t)\exp\left(\int_{t_0}^{t} w(s)ds\right) \leq$$

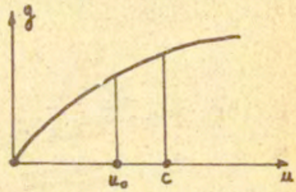
$$\leq w(t) R \exp\left(\int_{t_0}^{t} w(s)ds\right) + v(t)\exp\left(\int_{t_0}^{t} w(s)ds\right) g(R(t))$$

that is

(4.21) $$\dot{R} \leq v(t)\,g(R)$$

Since $c > 0$ therefore $g(c) > 0$ (because $g(u_0) > 0$) and hence in a vicinity of t_0 we can divide (4.21) by $g(R)$. Thus we have

(4.22) $$\int_{c}^{R(t)} \frac{ds}{g(s)} \leq \int_{t_0}^{t} v(s)ds$$



Since

$$\int_c^R = \int_c^{u_0} + \int_{u_0}^R = \int_{u_0}^R - \int_{u_0}^c$$

therefore in view of (4.16) the inequality (4.22) can be written in the form

(4.23) $$G(R) - G(c) \leq \int_{t_0}^{t} v(s)ds \quad \text{or} \quad G(R) \leq G(c) + \int_{t_0}^{t} v(s)ds$$

It should be emphasized, that in general the function $R(t)$ cannot be infinitely extended to the right, that is for a certain $t = T$ the value of R may be infinitely large. The corresponding value of T is determined by the formula (4.23), that is

(4.24) $$G(\infty) = \int_{u_0}^{\infty} \frac{ds}{g(s)} = G(c) + \int_{t_0}^{T} v(s)ds$$

Only in the interval $[t_0, T)$ the formula (4.23) determines tha function inverse to G . Therefore in this interval we have

(4.25) $$R(t) \leq G^{-1}\left[G(c) + \int_{t_0}^{t} v(s)ds\right] \quad \text{for} \quad t \in [t_0, T)$$

and on the basis of (4.20) we obtain

$$(4.26)\quad u(t) \le R(t)\exp\Big(\int_{t_0}^{t} w(s)ds\Big) \le G^{-1}\Big[G(c) + \int_{t_0}^{t} v(s)ds\Big]\exp\Big(\int_{t_0}^{t} w(s)ds\Big)$$

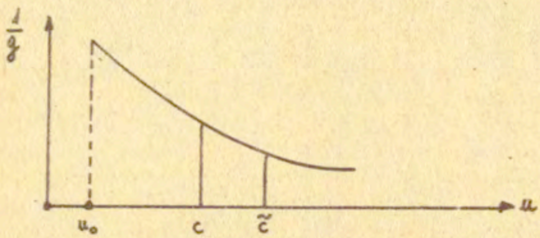
for $t \in [t_0, T)$.

The condition of continuability of the function $R(t)$ to infinity has, in accordance with (4.24), the form

$$G(c) + \int_{t_0}^{\infty} v(t)dt \le \int_{u_0}^{\infty} \frac{ds}{g(s)}$$

or

$$\int_{t_0}^{\infty} v(t)dt \le -\int_{u_0}^{c} \frac{ds}{g(s)} + \int_{u_0}^{\infty} \frac{ds}{g(s)} = \int_{c}^{\infty} \frac{ds}{g(s)}$$



This inequality is satisfied for each $c \in (0, \tilde{c})$ when the inequality (4.18) holds, since $c < \tilde{c}$ and therefore

$$\int_{t_0}^{\infty} v(t)dt \le \int_{\tilde{c}}^{\infty} \frac{ds}{g(s)} < \int_{c}^{\infty} \frac{ds}{g(s)}$$

Thus, in fact for $c \in (0, \tilde{c})$ we have $T = \infty$ if the inequality (4.18) is satisfied.

Q.E.D.

<u>Example for theorem 1.</u> Let $w(t) = 1$ and $\omega(t,u) = u^2$. Then the inequality (4.13) has the form

$$(4.27)\qquad u(t) \le c + \int_{t_0}^{t} [u(s) + u^2(s)]ds$$

The left-hand side of (4.14) can be written in the form

$$\omega\Big[t, z\exp\Big(\int_{t_0}^{t} w(s)ds\Big)\Big]\exp\Big(-\int_{t_0}^{t} w(s)ds\Big) = \omega[t, z\exp(t-t_0)]\exp(-(t-t_0)) =$$

$$= z^2\exp(2(t-t_0))\exp(-(t-t_0)) = z^2\exp(t-t_0)$$

Hence the functions $v(t)$ and $g(z)$ appearing in the inequality (4.14) can be taken in the form

$$v(t) = \exp(t-t_0) \quad , \qquad g(z) = z^2$$

The function G given by the formula (4.16) has in the considered case the following form

$$G(u) = \int_{u_0}^{u} \frac{ds}{s^2} = -\frac{1}{u} + \frac{1}{u_0} = \lambda$$

$$\frac{1}{u} = -\lambda + \frac{1}{u_0}$$

$$u = G^{-1}(\lambda) = \frac{1}{-\lambda + \frac{1}{u_0}}$$

Let us consider now the formula (4.15). At first we calculate

$$G(c) + \int_{t_0}^{t} v(s)ds = \int_{u_0}^{c} \frac{ds}{s^2} + \int_{t_0}^{t} \exp(s-t_0)ds = -\frac{1}{c} + \frac{1}{u_0} + \exp(t-t_0) - 1$$

hence

$$G^{-1}\left[G(c) + \int_{t_0}^{t} v(s)ds\right] = \frac{1}{-\left[G(c)+\int_{t_0}^{t} v(s)ds\right] + \frac{1}{u_0}} = \frac{1}{\frac{1}{c} - \frac{1}{u_0} - \exp(t-t_0) - 1 + \frac{1}{u_0}} =$$

$$= \frac{c}{1 + c\left[1 - \exp(t-t_0)\right]}$$

Consequently, the inequality (4.15) takes the form

$$u(t) \leq \frac{c \exp(t-t_0)}{1 + c\left[1 - \exp(t-t_0)\right]} \tag{4.28}$$

This inequality holds only in an interval $[t_0, T)$, the value of T being determined by equating to zero the denominator of (4.28). This yields

$$T = t_0 + \ln \frac{1+c}{c}$$

The same value of T can be determined directly from the formula (4.17)

$$\int_{u_0}^{\infty} \frac{ds}{s^2} = \int_{u_0}^{c} \frac{ds}{s^2} + \int_{t_0}^{T} \exp(s-t_0)ds$$

$$\frac{1}{u_0} = -\frac{1}{c} + \frac{1}{u_0} + \exp(T-t_0) - 1$$

yielding again

$$T = t_o + \ln \frac{1+c}{c}$$

Example for theorem 1. Let $w(t) = 1$ and $\omega(t,u) = u^2 \frac{1}{t^2} \exp(-t)$
Then the inequality (4.13) takes the form

(4.29) $$u(t) \leq c + \int_{t_o}^{t} \left[u(s) + u^2(s) \frac{1}{s^2} \exp(-s) \right] ds$$

The left-hand side of (4.14) can be written in the form

$$\omega\left[t, z \exp\left(\int_{t_o}^{t} w(s)ds\right)\right] \exp\left(-\int_{t_o}^{t} w(s)ds\right) = \omega[t, z\exp(t-t_o)] \exp[-(t-t_o)] =$$

$$= z^2 \exp(2(t-t_o)) \frac{1}{t^2} \exp(-t) \exp[-(t-t_o)] = z^2 \frac{1}{t^2} \exp(-t_o)$$

Hence the functions $v(t)$ and $g(z)$ appearing in the inequality (4.14) can be taken in the form

$$v(t) = \frac{1}{t^2} \exp(-t_o) \quad , \quad g(z) = z^2$$

The functions G and G^{-1} are the same as in the preceding example, hence we have

$$G(c) + \int_{t_o}^{t} v(s)ds = \int_{u_o}^{c} \frac{ds}{s^2} + \int_{t_o}^{t} \frac{e^{-t_o}}{s^2} ds = -\frac{1}{c} + \frac{1}{u_o} + \left(-\frac{1}{t} + \frac{1}{t_o}\right) \exp(-t_o)$$

$$G^{-1}\left[G(c) + \int_{t_o}^{t} v(s)ds\right] = \frac{1}{\frac{1}{c} - \frac{1}{u_o} - \left(-\frac{1}{t} + \frac{1}{t_o}\right)\exp(-t_o) + \frac{1}{u_o}} = \frac{c}{1 + c\left(\frac{1}{t} - \frac{1}{t_o}\right)\exp(-t_o)}$$

Thus, in the considered case the inequality (4.15) takes the form

$$u(t) \leq \frac{c \exp(t-t_o)}{1 + c\left(\frac{1}{t} - \frac{1}{t_o}\right)\exp(-t_o)} = \frac{c t t_o \exp(t-t_o)}{t t_o + c(t_o - t)\exp(-t_o)} =$$

$$= \frac{c t t_o \exp(t-t_o)}{t t_o + c t_o \exp(-t_o) - c t \exp(-t_o)} = \frac{c t t_o \exp(t-t_o)}{t[t_o - c\exp(-t_o)] + c t_o \exp(-t_o)}$$

(4.30) $$u(t) \leq \frac{c t t_o \exp(t)}{t[t_o \exp(t_o) - c] + c t_o}$$

Evidently, it holds in the interval $[t_0, \infty)$ if the term in the square brackets of the denominator is non-negative, that is for $c \in (0, \tilde{c})$ where $\tilde{c}$ is determined by the condition

$$\tilde{c} = t_0 \exp(t_0)$$

The same result can be obtained from the formula (4.18) that is

$$\int_{t_0}^{\infty} \frac{\exp(-t_0)}{t^2} dt \leq \int_{\tilde{c}}^{\infty} \frac{ds}{s^2} \quad , \qquad \frac{\exp(-t_0)}{t_0} \leq \frac{1}{\tilde{c}}$$

$$\tilde{c} \leq t_0 \exp(t_0)$$

This example proves, that the inequality (4.18) cannot be weakened, since then the above example would constitute a counter-example.

Theorem 2. Let us assume, that the assumptions of theorem 1 are satisfied and that the integrals

$$(4.31) \qquad \int_{t_0}^{\infty} w(t)\,dt < \infty \quad , \qquad \int_{u_0}^{\infty} \frac{ds}{g(s)} < \infty$$

are convergent. Moreover, let there exists a constant $\tilde{c} \in (0, \infty)$ such that

$$(4.32) \qquad \int_{t_0}^{\infty} v(t)\,dt = \int_{\tilde{c}}^{\infty} \frac{ds}{g(s)}$$

Then there exists a constant $M(c, t_0)$ such, that

$$u(t) \leq M(c, t_0) < \infty$$

for all $t \in [t_0, \infty)$ and $c \in (0, \tilde{c})$.

Proof. It follows from the assumption (4.32) that the condition (4.18) is satisfied, therefore the inequality (4.15) holds for $t \in [t_0, \infty)$ and $c \in (0, \tilde{c})$.

At first we prove, that under admitted assumptions the argument of the function G^{-1} is limited. Indeed, we have

$$G(c) + \int_{t_0}^{t} v(s)\,ds < G(\tilde{c}) + \int_{t_0}^{t} v(s)\,ds < G(\tilde{c}) + \int_{t_0}^{\infty} v(t)\,dt$$

$$G(\tilde{c}) + \int_{t_0}^{\infty} v(t)\,dt = \int_{u_0}^{\tilde{c}} \frac{ds}{g(s)} + \int_{t_0}^{\infty} v(t)\,dt = \int_{u_0}^{\infty} \frac{ds}{g(s)} + \int_{\infty}^{\tilde{c}} \frac{ds}{g(s)} + \int_{t_0}^{\infty} v(t)\,dt =$$

$$= \int_{u_0}^{\infty} \frac{ds}{g(s)} + \int_{t_0}^{\infty} v(t)\,dt - \int_{\tilde{c}}^{\infty} \frac{ds}{g(s)} = \int_{u_0}^{\infty} \frac{ds}{g(s)}$$

which follows in view of (4.23).Hence we have

$$G(c) + \int_{t_0}^{t} v(s)\,ds < \int_{u_0}^{\infty} \frac{ds}{g(s)} \quad , \quad 0 < c < \tilde{c}$$

The integral in the right-hand side of the last inequality is convergent in view of (4.31),and therefore on the basis of (4.31) and (4.15) it is

$$u(t) \leq M(c, t_0) < \infty$$

Q.E.D.

Example for theorem 2. Let $w(t) = \frac{1}{t^2}$ and

$$\omega(t,u) = \frac{1}{t^2} \exp\left[-\left(\frac{1}{t} - \frac{1}{t_0}\right) + u \exp\left(\frac{1}{t} - \frac{1}{t_0}\right)\right]$$

Then the inequality (4.13) takes the form

$$(4.33) \quad u(t) \leq c + \int_{t_0}^{t} \left(\frac{1}{s^2} u(s) + \frac{1}{s^2} \exp\left[-\left(\frac{1}{s} - \frac{1}{t_0}\right) + u(s) \exp\left(\frac{1}{s} - \frac{1}{t_0}\right)\right]\right) ds$$

The left-hand side of (4.14) can be written in the form

$$\omega\left[t, z \exp\left(\int_{t_0}^{t} w(s)\,ds\right)\right] \exp\left(-\int_{t_0}^{t} w(s)\,ds\right) = \omega\left(t, z \exp\left[-\left(\frac{1}{t} - \frac{1}{t_0}\right)\right]\right) \exp\left(\frac{1}{t} - \frac{1}{t_0}\right) =$$

$$= \frac{1}{t^2} \exp\left[-\left(\frac{1}{t} - \frac{1}{t_0}\right) + z \exp\left(-\left(\frac{1}{t} - \frac{1}{t_0}\right)\right) \exp\left(\frac{1}{t} - \frac{1}{t_0}\right)\right] \exp\left(\frac{1}{t} - \frac{1}{t_0}\right) =$$

$$= \frac{1}{t^2} \exp\left[-\left(\frac{1}{t} - \frac{1}{t_0}\right) + z\right] \exp\left(\frac{1}{t} - \frac{1}{t_0}\right) = \frac{1}{t^2} \exp(z)$$

Hence the functions $v(t)$ and $g(z)$ appearing in the inequality (4.14) can be taken in the form

$$v(t) = \frac{1}{t^2} \qquad , \qquad g(z) = \exp(z)$$

The function G given by the formula (4.16) has in the considered case the following form

$$G(u) = \int_{u_0}^{u} \exp(-s)\,ds = -\exp(-u) + \exp(-u_0)$$

$$-u = \ln[-\lambda + \exp(-u_0)]$$

$$u = G^{-1}(\lambda) = -\ln[-\lambda + \exp(-u_0)]$$

Let us consider now the formula (4.15).At first we calculate

$$G(c) + \int_{t_0}^{t} v(s)\,ds = \int_{u_0}^{c} \exp(-s)\,ds + \int_{t_0}^{t} \frac{ds}{s^2} = -\exp(-c) + \exp(-u_0) + \left(-\frac{1}{t} + \frac{1}{t_0}\right)$$

hence

$$G^{-1}\left[G(c) + \int_{t_0}^{t} v(s)\,ds\right] = -\ln\left[-\left(G(c) + \int_{t_0}^{t} v(s)\,ds\right) + \exp(-u_0)\right] =$$

$$= -\ln\left[\exp(-c) - \exp(-u_0) + \frac{1}{t} - \frac{1}{t_0} + \exp(-u_0)\right] =$$

$$= -\ln\left[\frac{1}{t} - \frac{1}{t_0} + \exp(-c)\right]$$

Consequently the inequality (4.15) takes the form

$$u(t) \le \exp\left[-\left(\frac{1}{t} - \frac{1}{t_0}\right)\right] \ln \frac{1}{\frac{1}{t} + \left[\exp(-c) - \frac{1}{t_0}\right]} \tag{4.34}$$

It is evident, that this inequality holds in the interval $[t_0, \infty)$ if the term in the square brackets of the denominator is non-negative, that is for $c \in (0, \tilde{c})$ where $\tilde{c}$ is determined by the condition

$$\exp(-\tilde{c}) - \frac{1}{t_0} = 0 \quad , \quad \tilde{c} = \ln t_0$$

It is also evident, that the function $u(t)$ is limited by a constant depending on c and t_0.

Let us verify the assumptions of theorem 2

$$\int_{t_0}^{\infty} w(t)\,dt = \frac{1}{t_0} < \infty \quad (t_0 \neq 0) \quad , \quad \int_{u_0}^{\infty} \frac{ds}{g(s)} = \exp(-u_0) < \infty$$

$$\int_{t_0}^{\infty} v(t)dt = \int_{\tilde{c}}^{\infty} \frac{ds}{g(s)} \quad , \quad \frac{1}{t_0} = \exp(-\tilde{c}) \quad , \quad \tilde{c} = \ln t_0$$

Theorem 3. Let us assume, that the assumptions of theorem are satisfied and that

$$(4.35) \quad \int_{t_0}^{\infty} [v(t) + w(t)]dt < \infty \quad , \quad \int_0 \frac{ds}{g(s)} < \int^{\infty} \frac{ds}{g(s)} = \infty$$

Then there exists a constant $M(c, t_0)$ such that

$$u(t) \leq M(c, t_0) < \infty$$

for all $t \in [t_0, \infty)$ and $c \in (0, \tilde{c})$, where $\tilde{c}$ can be arbitrarily large.

Proof. Since $\int_{t_0}^{\infty} v(t)dt < \infty$ and $\int^{\infty} \frac{ds}{g(s)} = \infty$, therefore the inequality (4.18) is satisfied and hence the inequality (4.15) is true in the interval $[t_0, \infty)$. Furthermore it follows from the inequality (4.18) that

$$\int_{t_0}^{\infty} v(t)dt \leq \int_{\tilde{c}}^{\infty} \frac{ds}{g(s)} = \int_{\tilde{c}}^{u_0} \frac{ds}{g(s)} + \int_{u_0}^{\infty} \frac{ds}{g(s)} = \int_{u_0}^{\infty} \frac{ds}{g(s)} - G(\tilde{c})$$

$$G(\tilde{c}) + \int_{t_0}^{\infty} v(t)dt \leq \int_{u_0}^{\infty} \frac{ds}{g(s)}$$

Since in view of (4.35) the integral $\int_{u_0}^{\infty} \frac{ds}{g(s)}$ is unlimited and $\int_{t_0}^{\infty} v(t)dt$ is limited, therefore $G(\tilde{c})$ can be arbitrarily large and that's why $\tilde{c}$ can also be taken arbitrarily large.

In order to prove that $u(t)$ is limited, it is sufficient to prove, that under admitted assumptions the argument of the function G^{-1} in the inequality (4.15) is limited. Indeed, we have

$$G(c) + \int_{t_0}^{t} v(s)ds < G(\tilde{c}) + \int_{t_0}^{\infty} v(t)dt = \int_{u_0}^{\tilde{c}} \frac{ds}{g(s)} + \int_{t_0}^{\infty} v(t)dt =$$

$$= \int_{u_0}^{c^*} \frac{ds}{g(s)} + \int_{c^*}^{\tilde{c}} \frac{ds}{g(s)} + \int_{t_0}^{\infty} v(t)dt$$

Since

$$\int_{u_0}^{c^*} = \int_{u_0}^{0} + \int_{0}^{c^*} = \int_{0}^{c^*} - \int_{0}^{u_0}$$

we obtain

$$G(c)+\int_{t_0}^{t}v(s)ds < \int_0^{c^*}\frac{ds}{g(s)} - \int_0^{u_0}\frac{ds}{g(s)} + \int_{t_0}^{\infty}v(t)dt + \int_{c^*}^{\tilde{c}}\frac{ds}{g(s)}$$

The first three integrals in the right-hand side of this inequality are limited in accordance with (4.35). The last integral in the right-hand side can be made arbitrarily small, by choosing a sufficiently large $c^* < \tilde{c}$. Hence we have

$$G(c)+\int_{t_0}^{t}v(s)ds < const < \infty$$

It means that

$$u(t) \leq M(c, t_0) < \infty$$

for $t \in [t_0, \infty)$ and $c \in (0, \tilde{c})$ where $\tilde{c}$ can be arbitrarily large.

Q.E.D.

Example for theorem 3. Let $w(t) = \frac{1}{t^2}$ and

$$\omega(t,u) = \frac{1}{t^2}u^{4/3}\exp\left(-\frac{2}{3t}\right)$$

Then the inequality (4.13) takes the form

$$(4.36)\quad u(t) \leq c + \int_{t_0}^{t}\left[\frac{1}{s^2}u(s) + \frac{1}{s^2}u^{4/3}(s)\exp\left(-\frac{2}{3s}\right)\right]ds$$

The left-hand side of (4.14) can be written in the form

$$\omega\left[t, z\exp\left(\int_{t_0}^{t}w(s)ds\right)\right]\exp\left(-\int_{t_0}^{t}w(s)ds\right) = \omega\left[t, z\exp\left(-\left(\tfrac{1}{t}-\tfrac{1}{t_0}\right)\right)\right]\exp\left(\tfrac{1}{t}-\tfrac{1}{t_0}\right) =$$

$$= \frac{1}{t^2}z^{4/3}\exp\left(-\frac{1}{3t}+\frac{1}{3t_0}\right)\exp\left(-\frac{2}{3t}\right)\exp\left(\frac{1}{t}-\frac{1}{t_0}\right) =$$

$$= \frac{1}{t^2}z^{4/3}\exp\left[\frac{1}{t}\left(-\frac{1}{3}-\frac{2}{3}+\frac{3}{3}\right)\right]\exp\left[\frac{1}{t_0}\left(\frac{1}{3}-\frac{2}{3}\right)\right] = \frac{1}{t^2}z^{4/3}\exp\left(-\frac{2}{3t_0}\right)$$

Hence the functions $v(t)$ and $g(z)$ appearing in the inequality (4.14) can be taken in the form

$$v(t) = \frac{1}{t^2}\exp\left(-\frac{2}{3t_0}\right), \qquad g(z) = z^{4/3}$$

In the considered case the function G given by the formula (4.16) has the following form

$$G(u) = \int_{u_0}^{u} s^{-1/3} ds = \frac{3}{2}(u^{2/3} - u_0^{2/3}) = \lambda$$

$$u^{2/3} = \frac{2}{3}\lambda + u_0^{2/3}$$

$$u = G^{-1}(\lambda) = \left(\frac{2}{3}\lambda + u_0^{2/3}\right)^{3/2}$$

Now let us consider the formula (4.15). At first we calculate

$$G(c) + \int_{t_0}^{t} v(s)ds = \int_{u_0}^{c} s^{-1/3} ds + \int_{t_0}^{t} \frac{1}{s^2} \exp\left(-\frac{2}{3t_0}\right) ds =$$

$$= \frac{3}{2}(c^{2/3} - u_0^{2/3}) + \left(-\frac{1}{t} + \frac{1}{t_0}\right)\exp\left(-\frac{2}{3t_0}\right)$$

hence

$$G^{-1}\left[G(c) + \int_{t_0}^{t} v(s)ds\right] = \left[\frac{2}{3}\left(G(c) + \int_{t_0}^{t} v(s)ds\right) + u_0^{2/3}\right]^{3/2} =$$

$$= \left[c^{2/3} - u_0^{2/3} - \frac{2}{3}\left(\frac{1}{t} - \frac{1}{t_0}\right)\exp\left(-\frac{2}{3t_0}\right) + u_0^{2/3}\right]^{3/2} =$$

$$= \left[c^{2/3} - \frac{2}{3}\left(\frac{1}{t} - \frac{1}{t_0}\right)\exp\left(-\frac{2}{3t_0}\right)\right]^{3/2}$$

Consequently the inequality (4.15) takes the form

$$u(t) \leq \left[c^{2/3} - \frac{2}{3}\left(\frac{1}{t} - \frac{1}{t_0}\right)\exp\left(-\frac{2}{3t_0}\right)\right]^{3/2} \exp\left[-\left(\frac{1}{t} - \frac{1}{t_0}\right)\right] =$$

$$= \left[\left(c^{2/3}\exp\left(\frac{2}{3t_0}\right) - \frac{2}{3}\left(\frac{1}{t} - \frac{1}{t_0}\right)\right)\exp\left(-\frac{2}{3t_0}\right)\right]^{3/2} \exp\left[-\left(\frac{1}{t} - \frac{1}{t_0}\right)\right] =$$

$$= \left[\left(c\exp\left(\frac{1}{t_0}\right)\right)^{2/3} - \frac{2}{3}\left(\frac{1}{t} - \frac{1}{t_0}\right)\right]^{3/2} \exp\left(-\frac{1}{t_0}\right)\exp\left[-\left(\frac{1}{t} - \frac{1}{t_0}\right)\right]$$

$$(4.37) \quad u(t) \leq \left[\left(c\exp\left(\frac{1}{t_0}\right)\right)^{2/3} - \frac{2}{3}\left(\frac{1}{t} - \frac{1}{t_0}\right)\right]^{3/2} \exp\left(-\frac{1}{t}\right)$$

It is evident, that this inequality holds in the interval $[t_0, \infty)$ and for $c \in (0, \tilde{c})$ where $\tilde{c}$ can be arbitrarily large. The constant $M(c, t_0)$ apeearing in theorem 3 is

$$M(c,t_0) = \left[\left(c \exp\left(\frac{1}{t_0}\right)\right)^{2/3} + \frac{2}{3t_0}\right]^{3/2}$$

Let us verify the assumptions of theorem 3.

$$\int_{t_0}^{\infty} [v(t) + w(t)]\,dt = \int_{t_0}^{\infty} \frac{1}{t^2} \exp\left(-\frac{2}{3t_0}\right) dt + \int_{t_0}^{\infty} \frac{dt}{t^2} =$$

$$= \frac{1}{t_0} \exp\left(-\frac{2}{3t_0}\right) + \frac{1}{t_0} < \infty \quad , \quad (t_0 \neq 0)$$

$$\int_0^A \frac{ds}{g(s)} = \int_0^A s^{-1/3}\,ds = \frac{3}{2} A^{2/3} < \infty$$

$$\int_A^{\infty} \frac{ds}{g(s)} = \int_A^{\infty} s^{-1/3}\,ds = \frac{3}{2} s^{2/3} \Big|_A^{\infty} = \infty$$

Theorem 4. Let us assume, that the assumptions of theorem 1 are satisfied and that

$$(4.38) \qquad \int_{t_0}^{\infty} [v(t) + w(t)]\,dt < \infty \quad , \quad \int^{\infty} \frac{ds}{g(s)} < \int_0 \frac{ds}{g(s)} = \infty$$

Moreover, let there exists a constant $\tilde{c} \in (c, \infty)$ such that

$$(4.39) \qquad \int_{t_0}^{\infty} v(t)\,dt = \int_{\tilde{c}}^{\infty} \frac{ds}{g(s)}$$

Then there exists a constant $M(c, t_0)$ such that

$$u(t) \leq M(c, t_0) < \infty$$

for all $t \in [t_0, \infty)$ and $c \in (0, \tilde{c})$ where

$$M(c, t_0) \to 0 \quad \text{for} \quad c \to 0 \quad , \quad t_0 = \text{const} \geq 0$$

The proof of this theorem can be carried out in the same way as the proofs of theorems 2 and 3 - therefore we omit it.

Example for theorem 4. Let $w(t) = \frac{1}{t^2}$ and $\omega(t,u) = u^2 \frac{1}{t^2} \exp\left(\frac{1}{t}\right)$

Then the inequality (4.13) takes the form

(4.40) $$u(t) \leq c + \int_{t_0}^{t} \left[\frac{1}{s^2} u(s) + u^2(s) \frac{1}{s^2} \exp\left(\frac{1}{s}\right) \right] ds$$

The left-hand side of (4.14) can be written in the form

$$\omega\left[t, z\exp\left(\int_{t_0}^{t} w(s)ds\right)\right]\exp\left(-\int_{t_0}^{t} w(s)ds\right) = \omega\left[t, z\exp\left(-\left(\frac{1}{t}-\frac{1}{t_0}\right)\right)\right]\exp\left(\frac{1}{t}-\frac{1}{t_0}\right) =$$

$$= \frac{1}{t^2}\exp\left(\frac{1}{t}\right) z^2 \exp\left(-\frac{2}{t}+\frac{2}{t_0}\right)\exp\left(\frac{1}{t}-\frac{1}{t_0}\right) =$$

$$= z^2 \frac{1}{t^2} \exp\left[\frac{1}{t}(1-2+1)\right]\exp\left[\frac{1}{t_0}(2-1)\right] =$$

$$= z^2 \frac{1}{t^2} \exp\left(\frac{1}{t_0}\right)$$

Hence the functions $v(t)$ and $g(z)$ appearing in the inequality (4.14) can be taken in the form

$$v(t) = \frac{1}{t^2}\exp\left(\frac{1}{t_0}\right) \quad , \quad g(z) = z^2$$

The functions G and G^{-1} are the same as in the example for the theorem 1. Let us consider now the formula (4.15). At first we calculate

$$G(c) + \int_{t_0}^{t} v(s)ds = \int_{u_0}^{c} \frac{ds}{s^2} + \int_{t_0}^{t} \frac{1}{s^2}\exp\left(\frac{1}{t_0}\right)ds =$$

$$= -\frac{1}{c} + \frac{1}{u_0} + \left(-\frac{1}{t}+\frac{1}{t_0}\right)\exp\left(\frac{1}{t_0}\right)$$

$$G^{-1}\left[G(c) + \int_{t_0}^{t} v(s)ds\right] = \frac{1}{-\left[G(c)+\int_{t_0}^{t} v(s)ds\right]+\frac{1}{u_0}} =$$

$$= \frac{1}{\frac{1}{c}-\frac{1}{u_0}-\left(-\frac{1}{t}+\frac{1}{t_0}\right)\exp\left(\frac{1}{t_0}\right)+\frac{1}{u_0}} = \frac{c}{1+c\left(\frac{1}{t}-\frac{1}{t_0}\right)\exp\left(\frac{1}{t_0}\right)}$$

Consequently the inequality (4.15) takes the form

(4.41) $$u(t) \leq \frac{c\exp\left[-\left(\frac{1}{t}-\frac{1}{t_0}\right)\right]}{1+c\left(\frac{1}{t}-\frac{1}{t_0}\right)\exp\left(\frac{1}{t_0}\right)}$$

It is evident, that this inequality holds in the interval $[t_0, \infty)$ and for $c \in (0, \tilde{c})$ where $\tilde{c}$ is determined from the condition

$$1 - \tilde{c}\,\frac{1}{t_0}\exp\left(\frac{1}{t_0}\right) = 0 \qquad , \qquad \tilde{c} = t_0 \exp\left(-\frac{1}{t_0}\right)$$

The constant $M(c, t_0)$ apeearing in theorem 4 is

$$M(c, t_0) = \frac{c\,t_0 \exp\left(\frac{1}{t_0}\right)}{t_0 - c\exp\left(\frac{1}{t_0}\right)}$$

It is evident, that $\lim\limits_{c \to 0} M(c, t_0) = 0$, for $t_0 = \text{const} > 0$ and $c \in (0, \tilde{c})$.

Let us verify the assumptions of theorem 4.

$$\int_{t_0}^{\infty} [v(t) + w(t)]\,dt = \int_{t_0}^{\infty} \frac{1}{t^2}\exp\left(\frac{1}{t_0}\right)dt + \int_{t_0}^{\infty}\frac{dt}{t^2} = \frac{1}{t_0}\exp\left(\frac{1}{t_0}\right) + \frac{1}{t_0} < \infty$$

$$\int_A^{\infty} \frac{ds}{s^2} = \frac{1}{A} < \infty$$

$$\int_0^A \frac{ds}{s^2} = \left[-\frac{1}{s}\right]_0^A = \infty$$

The value of $\tilde{c}$ can be also determined on the basis of the formula (4.39)

$$\int_{t_0}^{\infty} \frac{1}{t^2}\exp\left(\frac{1}{t_0}\right)dt = \int_{\tilde{c}}^{\infty} \frac{ds}{s^2} \qquad , \qquad \frac{1}{t_0}\exp\left(\frac{1}{t_0}\right) = \frac{1}{\tilde{c}}$$

$$\tilde{c} = t_0 \exp\left(-\frac{1}{t_0}\right)$$

Theorem 5. Let us assume, that the assumptions of theorem 1 are satisfied and that

$$(4.42) \qquad \int_{t_0}^{\infty} [v(t) + w(t)]\,dt < \infty \qquad , \qquad \int_0 \frac{ds}{g(s)} = \int^{\infty} \frac{ds}{g(s)} = \infty$$

Then there exists a constant $M(c, t_0)$ such that

$$u(t) \leq M(c, t_0) < \infty$$

for all $t \in [t_0, \infty)$ and $c \in (0, \tilde{c})$ where $\tilde{c}$ can be arbitrarily large. Moreover $M(c, t_0) \to 0$ for $c \to 0$ and $t_0 = \text{const} \geq 0$.

The proof of this theorem can be carried in the same way as the proofs of theorems 2 and 3 - therefore we omit it.

Example for theorem 5. Let $w(t)=\frac{1}{t^2}$ and $\omega(t,u)=\frac{1}{t^2}u$. Then the inequality (4.13) takes the form

(4.43) $$u(t)\leq c+\int_{t_0}^{t}\left[\frac{1}{s^2}u(s)+\frac{1}{s^2}u(s)\right]ds=c+\int_{t_0}^{t}\frac{2}{s^2}u(s)ds$$

The left-hand side of (4.14) can be written in the form

$$\omega\left[t,z\exp\left(\int_{t_0}^{t}w(s)ds\right)\right]\exp\left(-\int_{t_0}^{t}w(s)ds\right)=\omega\left[t,z\exp\left(-\left(\frac{1}{t}-\frac{1}{t_0}\right)\right)\right]\exp\left(\frac{1}{t}-\frac{1}{t_0}\right)=$$

$$=\frac{1}{t^2}z\exp\left[-\left(\frac{1}{t}-\frac{1}{t_0}\right)\right]\exp\left(\frac{1}{t}-\frac{1}{t_0}\right)=\frac{1}{t^2}z$$

Hence the functions $v(t)$ and $g(z)$ appearing in the inequality (4.14) can be taken in the form

$$v(t)=\frac{1}{t^2}\quad,\qquad g(z)=z$$

The function G given by the formula (4.16) has, in the considered case, the following form

$$G(u)=\int_{u_0}^{u}\frac{ds}{s}=\ln\frac{u}{u_0}=\lambda$$

$$u=G^{-1}(\lambda)=u_0\exp(\lambda)$$

Let us consider now the formula (4.15). At first we calculate

$$G(c)+\int_{t_0}^{t}v(s)ds=\int_{u_0}^{c}\frac{ds}{s}+\int_{t_0}^{t}\frac{ds}{s^2}=\ln\frac{c}{u_0}+\left(-\frac{1}{t}+\frac{1}{t_0}\right)$$

hence

$$G^{-1}\left[G(c)+\int_{t_0}^{t}v(s)ds\right]=u_0\exp\left[G(c)+\int_{t_0}^{t}v(s)ds\right]=$$

$$=u_0\exp\left[\ln\frac{c}{u_0}+\left(\frac{1}{t_0}-\frac{1}{t}\right)\right]=u_0\exp\left(\ln\frac{c}{u_0}\right)\exp\left(\frac{1}{t_0}-\frac{1}{t}\right)=$$

$$=c\exp\left(\frac{1}{t_0}-\frac{1}{t}\right)$$

Consequently the inequality (4.15) takes the form

$$u(t) \leq c \exp\left[2\left(\frac{1}{t_0} - \frac{1}{t}\right)\right] \tag{4.44}$$

It is evident, that this inequality holds in the interval $[t_0, \infty)$ and for $c \in (0, \tilde{c})$ where $\tilde{c}$ can be arbitrarily large. The constant $M(c, t_0)$ appearing in theorem 5 is

$$M(c, t_0) = c \exp\left(\frac{2}{t_0}\right) \quad , \quad t_0 > 0$$

Evidently $\lim_{c \to 0} M(c, t_c) = 0$ if $t_0 = \text{const} > 0$.
Let us verify the assumptions of theorem 5.

$$\int_{t_0}^{\infty} [v(t) + w(t)]dt = \int_{t_0}^{\infty} \frac{2}{t^2} dt = \frac{2}{t_0} < \infty$$

$$\int_0^A \frac{ds}{s} = \ln s \Big|_0^A = \infty$$

$$\int_A^{\infty} \frac{ds}{s} = \ln s \Big|_A^{\infty} = \infty$$

Now let us consider the case $w(t) = -\alpha(t)$, $\alpha > 0$ separately. To make advantage of the results obtained in the theorems 1-5, it is necessary to arrive at the integral inequality (4.13) for instance from the differential inequality in the form

$$\dot{u} \leq -\alpha(t)u + \omega(t, u) \tag{4.45}$$

Integrating the both sides of this inequality we get

$$u(t) \leq c + \int_{t_0}^{t} [-\alpha(s)u(s) + \omega(s, u(s))]ds \quad , \quad c = u(t_0) = u_0 \tag{4.46}$$

The differential inequality (4.45) constitutes here a kind of constraints on the functions $u(t)$ satisfying the integral inequality (4.46). Without these constraints, the results of theorem 1 cannot be proved for $w(t) = -\alpha(t)$, $\alpha > 0$ and therefore the results following this theorem may not be true.

In the considered case the assumption (4.14) in theorem 1 takes the form

$$(4.47)\quad \omega\left[t, z\exp\left(-\int_{t_0}^{t}\alpha(s)ds\right)\right]\exp\left(\int_{t_0}^{t}\alpha(s)ds\right) \leqslant v(t)g(z)\,,\quad t \geqslant t_0,\ z \geqslant 0$$

Putting

$$(4.48)\qquad u = R\exp\left(-\int_{t_0}^{t}\alpha(s)ds\right)$$

and substituting this relation into the differential inequality (4.45) we obtain

$$\dot{R}\exp\left(-\int_{t_0}^{t}\alpha(s)ds\right) - \alpha R\exp\left(-\int_{t_0}^{t}\alpha(s)ds\right) \leqslant$$

$$\leqslant -\alpha R\exp\left(-\int_{t_0}^{t}\alpha(s)ds\right) + \omega\left[t, R\exp\left(-\int_{t_0}^{t}\alpha(s)ds\right)\right]$$

Hence we get

$$\dot{R} \leqslant \omega\left[t, R\exp\left(-\int_{t_0}^{t}\alpha(s)ds\right)\right]\exp\left(\int_{t_0}^{t}\alpha(s)ds\right)$$

On the basis of (4.47) we arrive to the inequality

$$(4.49)\qquad \dot{R} \leqslant v(t)\,g(R)$$

which is the same as (4.21). Therefore all results which follows from the inequality (4.21) remain true, that is we obtain the inequality (4.25)

$$(4.50)\qquad R(t) \leqslant G^{-1}\left[G(c) + \int_{t_0}^{t} v(s)ds\right]\,,\qquad t \in [t_0, T)$$

In view of (4.48) and (4.50), instead of (4.26), we have the following inequality

$$(4.51)\quad u(t) \leqslant G^{-1}\left[G(c) + \int_{t_0}^{t} v(s)ds\right]\exp\left(-\int_{t_0}^{t}\alpha(s)ds\right)\,,\quad t \in [t_0, T)$$

Obviously, if the condition (4.18) is satisfied, the inequality (4.51) holds for $t \in [t_0, \infty)$.

The theorems 2 - 5 have now the form.

Theorem 2a. Let us assume that the assumptions of theorem 1 are satisfied ($w(t) = -\alpha(t)$, $\alpha > 0$) and that

$$\int_{u_0}^{\infty} \frac{ds}{g(s)} < \infty$$

Moreover let there exists a constant $\tilde{c} \in (0, \infty)$ such that

$$\int_{t_0}^{\infty} v(t)dt = \int_{\tilde{c}}^{\infty} \frac{ds}{g(s)}$$

Then there exists a constant $M(c,t_0) < \infty$ such that

$$u(t) \leq M(c,t_0) \exp\left(-\int_{t_0}^{t} \alpha(s)ds\right)$$

for all $t \in [t_0, \infty)$ and $c \in (0,\tilde{c})$.

Theorem 3a. Let us assume, that the assumptions of theorem 1 are satisfied ($w(t) = -\alpha(t)$, $\alpha > 0$) and that

$$\int_{t_0}^{\infty} v(t)dt < \infty \qquad , \qquad \int_0 \frac{ds}{g(s)} < \int^{\infty} \frac{ds}{g(s)} = \infty$$

Then there exists a constant $M(c,t_0) < \infty$ such that

$$u(t) \leq M(c,t_0) \exp\left(-\int_{t_0}^{t} \alpha(s)ds\right)$$

for all $t \in [t_0, \infty)$ and $c \in (0,\tilde{c})$, where $\tilde{c}$ can be arbitrarily large.

Theorem 4a. Let us assume, that the assumptions of theorem 1 are satisfied ($w(t) = -\alpha(t)$, $\alpha > 0$) and that

$$\int_{t_0}^{\infty} v(t)dt < \infty \qquad , \qquad \int^{\infty} \frac{ds}{g(s)} < \int_0 \frac{ds}{g(s)} = \infty$$

Moreover let there exists a constant $\tilde{c} \in (0, \infty)$ such that

$$\int_{t_0}^{\infty} v(t)dt = \int_{\tilde{c}}^{\infty} \frac{ds}{g(s)}$$

Then there exists a constant $M(c,t_0) < \infty$ such that

$$u(t) \leq M(c,t_0) \exp\left(-\int_{t_0}^{t} \alpha(s)ds\right)$$

for all $t \in [t_0, \infty)$ and $c \in (0,\tilde{c})$ where

$$M(c,t_0) \to 0 \quad \text{for} \quad c \to 0 \quad , \quad t_0 = \text{const} \geq 0$$

Theorem 5a. Let us assume, that the assumptions of theorem 1 are satisfied ($w(t) = -\alpha(t)$, $\alpha > 0$) and that

$$\int_{t_0}^{\infty} v(t)dt < \infty \qquad\qquad \int_0 \frac{ds}{g(s)} = \int^{\infty} \frac{ds}{g(s)} = \infty$$

Then there exists a constant $M(c,t_0) < \infty$ such that

$$u(t) \leq M(c,t_0) \exp\left(-\int_{t_0}^{t} \alpha(s)ds\right)$$

for all $t \in [t_0, \infty)$ and $c \in (0,\tilde{c})$ where $\tilde{c}$ can be arbitrarily large. Moreover

$$M(c,t_0) \to 0 \quad \text{for} \quad c \to 0 \quad , \quad t_0 = \text{const} \geq 0$$

It should be stressed once more, that the above results were obtained on the basis of the differential inequality (4.45) and not by the direct investigation of the integral inequality (4.46). These results may be not true if only the integral inequality (4.46) is satisfied without the differential inequality (4.45).

One can confirm this fact constructing a following example.

Let us consider directly the integral inequality (4.46) without taking into account that it follows from the differential inequality (4.45).

Let the integral inequality has the form

(4.52) $$u(t) \leq c - \int_0^t u(s)ds \quad , \quad c = u(0) = u_0$$

In the considered case is $\alpha(t) = 1$, $\omega(t,u) = 0$.

The left-hand side of (4.14) can be written in the form

$$0 \leq v(t)\, g(z)$$

Hence the functions $v(t)$ and $g(z)$ appearing in the inequality (4.14) can be taken in the form

$$v(t) = 0 \quad , \quad g(z) = 1$$

The function G given by the formula (4.16) has, in the considered case, the following form.

$$G(u) = \int_{u_0}^{u} \frac{ds}{1} = u - u_0 = \lambda$$

$$u = G^{-1}(\lambda) = \lambda + u_0$$

In the considered case we shall use the theorem 3a. Indeed, verifying the assumptions of this theorem we have

$$\int_{t_0}^{\infty} 0\, dt < \infty \quad , \quad \int_0^A \frac{ds}{1} = s\Big|_0^A = A < \int_A^{\infty} \frac{ds}{1} = s\Big|_A^{\infty} = \infty$$

Let us consider now the formula (4.15). At first we calculate

$$G(c) + \int_{t_0}^{t} v(s)\,ds = \int_{u_0}^{c} \frac{ds}{1} + \int_{t_0}^{t} 0\, dt = c - u_0$$

hence

$$G^{-1}\Big[G(c) + \int_{t_0}^{t} v(s)\,ds\Big] = \Big[G(c) + \int_{t_0}^{t} v(s)\,ds\Big] + u_0 = c - u_0 + u_0 = c = u(0)$$

Consequently the inequality (4.15) takes the form

(4.53) $$u(t) \leq c \exp\Big(-\int_{t_0}^{t} \alpha(s)\,ds\Big) = u(0)e^{-t}$$

We have here $M(c,t_0) = M(c,0) = c = u(0)$.

But this result is not true. We can easily verify this fact substituting into (4.52) the function

(4.54) $$u(t) = \frac{1}{(1+t)^2} \quad , \quad u(0) = c = 1$$

We obtain

$$\frac{1}{(1+t)^2} \leq 1 - \int_0^t \frac{ds}{(1+s)^2} = 1 + \frac{1}{1+s}\Big|_0^t = 1 + \frac{1}{1+t} - 1 = \frac{1}{1+t}$$

$$\frac{1}{1+t} \leq 1$$

(4.55) $$1+t \geqslant 1$$

It is evident, that the inequality (4.55) holds for each $t \geqslant 0$, therefore the function $u(t)$ given by the relation (4.54) satisfies the integral inequality (4.52). But this function does not belong to the class of the exponential functions of the type (4.53).

The function (4.54) becomes however eliminated if additionally the following corresponding differential inequality holds

$$\dot{u} \leq -u \tag{4.56}$$

which after integration with respect to time, gives the integral inequality (4.52). Indeed, on the basis of (4.56) it is

$$u(t) \leq u(0)\,e^{-t}$$

in accordance with (4.53).

5. The investigation of some problems of motion by the method of integral inequalities.

5.1. Perturbation of constrained motion of a mechanical system due to a change of holonomic constraints.

Let us consider n material points with Cartesian coordinates $\xi_1, \dots, \xi_{3n}$ binded by the holonomic, two-sided and perfect constraints of the form

$$f_\alpha(t, \xi_1, \dots, \xi_{3n}) = 0 \quad , \quad \alpha = 1, \dots, a < 3n \tag{5.1}$$

Let us introduce the number $k = 3n - a$ of independent generalized variables by the following relations

$$\xi_i = \xi_i(t, q_1, \dots, q_k) \tag{5.2}$$

These relations substituted into the equations of constraints (5.1) lead to identities.

The motion of the system will be described in canonical variables. Let

$$T = T(t, q_1, \dots, q_k, \dot{q}_1, \dots, \dot{q}_k)$$

denotes the kinetic energy of the considered system. Let us introduce the generalized momenta

$$p_\sigma = \frac{\partial T}{\partial \dot{q}_\sigma} \quad , \quad \sigma = 1, \dots, k \tag{5.3}$$

From the structure of the general formula for the kinetic energy of mechanical systems (quadratic form of generalized velocities) it follows, that the momenta p_σ ($\sigma = 1, \dots, k$)

are linear functions of the generalized velocities $\dot{q}_\sigma$ (σ = 1,...,k). Let us introduce the function

$$(5.4) \qquad K(t, q_1, \dots, q_k, \dot{q}_1, \dots, \dot{q}_k) = \sum_{\sigma=1}^{k} p_\sigma \tilde{\dot{q}}_\sigma - \tilde{T}$$

The symbol „ ~ " denotes, that the corresponding generalized velocities $\dot{q}_\sigma$ (σ = 1,...,k) are to be replaced by the generalized momenta p_σ (σ = 1,...,k) according to the equations (5.3).

Equations of motion of the considered mechanical system, written in the canonical variables q_σ , p_σ (σ = 1,...,k) have the form

$$(5.5) \qquad \begin{cases} \dot{q}_\sigma = \dfrac{\partial K}{\partial p_\sigma} = A_\sigma (t, q_1, \dots, q_k, p_1, \dots, p_k) \\ \dot{p}_\sigma = -\dfrac{\partial K}{\partial q_\sigma} + Q_\sigma = B_\sigma (t, q_1, \dots, q_k, p_1, \dots, p_k) \end{cases}$$

where $Q_\sigma(t, q_1, \dots, q_k, p_1, \dots, p_k)$, (σ = 1,...,k) denote the generalzed forces acting on the mechanical system.

Let us denote

$$q_1 = x_1, \dots, \quad q_k = x_k, \quad p_1 = x_{k+1}, \dots, \quad p_k = x_m$$

$$A_1 = P_1, \dots, \quad A_k = P_k, \quad B_1 = P_{k+1}, \dots, \quad B_k = P_m$$

where $m = 2k$.

Introducing the column matrices

$$\mathbf{x} = \mathrm{col}\,[x_1, \dots, x_m] \quad , \quad \mathbf{P} = \mathrm{col}\,[P_1, \dots, P_m]$$

we can write the equations of motion (5.5) in the form

$$(5.6) \qquad \dot{\mathbf{x}} = \mathbf{P}(t, \mathbf{x})$$

Now let us consider the motion of the same mechanical system in the same variables as in the constraints (5.1) and subject to the same generalized forces as before, though binded by a different system of two-sided, perfect and holonomic constraints having the form

$$(5.7) \qquad f^*_\alpha(t, \xi_1, \dots, \xi_{3n}) = 0 \quad , \quad \alpha = 1, \dots, a < 3n$$

A problem of this type may arise, for instance, in the case in which we try to perform a further idealization of the mathematical model and to replace the constraints (5.1) by simpler constraints (5.7) or, on the contrary, in which constraints (5.1) appear to be an exaggerated idealization of the mathematical model, which makes necessary to introduce a more accurate description by means of the constraints (5.7).

The motion of the system subject to identical forces but binded by different constraints (5.1) or (5.7) may tend to each other or not. We shall try to establish the conditions under which the motions of the system with the original constraints (5.1) and the modified constraints (5.7) tend to each other with $t \to \infty$.

Let us substitute the relations (5.2) into the equations (5.7). In the previously considered case the substitution of (5.2) into (5.1) provokes that the equations (5.1) become identities. Now this substitution gives a relation between the variables t, q_σ $(\sigma = 1,\dots,k)$ in the form

$$\varphi_\alpha(t, q_1,\dots,q_k) = 0 \quad , \quad \alpha = 1,\dots,a \tag{5.8}$$

Applying the method of the multipliers of Lagrange we can write the equations of motion of the system with constraints (5.7) in the form

$$(5.9)\quad \begin{cases} \dot{q}_\sigma = \dfrac{\partial K}{\partial p_\sigma} = A_\sigma(t, q_1,\dots,q_k, p_1,\dots,p_k) \\ \dot{p}_\sigma = -\dfrac{\partial K}{\partial q_\sigma} + Q_\sigma + \sum\limits_{l=1}^{a} \lambda_l \dfrac{\partial \varphi_l}{\partial q_\sigma} = B_\sigma(t, q_1,\dots,q_k, p_1,\dots,p_k) + \\ \qquad \sigma = 1,\dots,k \qquad\qquad + L_\sigma(t, q_1,\dots,q_k, p_1,\dots,p_k) \end{cases}$$

where

$$L_\sigma(t, q_1,\dots,q_k, p_1,\dots,p_k) = \sum_{l=1}^{a} \lambda_l \frac{\partial \varphi_l}{\partial q_\sigma} \quad , \quad \sigma = 1,\dots,k \tag{5.10}$$

Functions $\lambda_l = \lambda_l(t, q_1,\dots,q_k, p_1,\dots,p_k)$ will be considered as known. They can be determined in the following way. The second total time derivative of the function (5.8) has the form

$$(5.11)\quad \frac{d^2\varphi_\alpha}{dt^2} = \sum_{\sigma=1}^{k} \frac{\partial\varphi_\alpha}{\partial q_\sigma}\ddot{q}_\sigma + \sum_{\sigma=1}^{k}\sum_{\mu=1}^{k} \frac{\partial^2\varphi_\alpha}{\partial q_\sigma \partial q_\mu}\dot{q}_\sigma \dot{q}_\mu + 2\sum_{\sigma=1}^{k} \frac{\partial^2\varphi_\alpha}{\partial q_\sigma \partial t}\dot{q}_\sigma + \frac{\partial^2\varphi_\alpha}{\partial t^2} = 0$$

Let us substitute $\dot{q}_\sigma$ ($\sigma = 1,\dots,k$) from the equations (5.9) and then calculate $\ddot{q}_\sigma = \frac{d}{dt}\dot{q}_\sigma = \frac{d}{dt}A_\sigma$ ($\sigma=1,\dots,k$) and again instead of $\dot{p}_\sigma$ ($\sigma = 1,\dots,k$) substitute their values from the equations (5.9), and instead of $\dot{q}_\sigma$ ($\sigma = 1,\dots,k$) their values from the equations (5.9). In this way we obtain a system of linear equations for the unknowns $\lambda_1, \dots, \lambda_a$. It is known, that under sufficient regularity of the functions φ_α ($\alpha = 1,\dots,k$) the determinant of the linear system of equations is different from zero and the functions $\lambda_1, \dots, \lambda_a$ can be determined.

Let us denote

$$q_1 = y_1, \dots, q_k = y_k, \quad p_1 = y_{k+1}, \dots, p_k = y_m$$
$$R_1 = 0, \dots, R_k = 0, \quad L_1 = R_{k+1}, \dots, L_k = R_m$$

where $m = 2k$.

Introducing the column matrices

$$\mathbf{y} = \mathrm{col}\,[y_1, \dots, y_m], \quad \mathbf{R} = \mathrm{col}\,[R_1, \dots, R_m]$$

we can write the equations of motion (5.9) in the form

$$(5.12)\qquad \dot{\mathbf{y}} = \mathbf{P}(t, \mathbf{y}) + \mathbf{R}(t, \mathbf{y})$$

The differential equation (5.6) describes the motion of the system with the original constraints (5.1), the differential equation (5.12) describes the motion of the system with the modified constraints (5.7) and the column matrix $\mathbf{R}(t, \mathbf{y})$ expresses the influence of the modification of constraints from (5.1) to (57).

From the mathematical point of view the influence of the perturbations of the non-linear equation (5.6) due to the non-linear term $\mathbf{R}$ will be of particular interest to us.

It is evident, that depending on further assumptions, the considered problem can be reduced to the problem I or II examined in the part 1 and consequently to the investigation of the integral inequality of the 1-st or 2-nd kind.

5.2. A method for determining the forces to execute prescribed motion with a given accuracy.

The motion of a mechanical system with n - degrees of freedom can be described by the following differential equation in the matrix form.

(5.13) $$\dot{z} = \Gamma(t, z) \quad , \quad z(t_0) = z_0$$

where

$$z = \mathrm{col}\,[z_1, \dots, z_m] \quad , \quad \Gamma = \mathrm{col}\,[\Gamma_1, \dots, \Gamma_m] \quad , \quad m = 2n$$

We wish the motion of the considered system be close to the motion

(5.14) $$\eta = \psi(t; t_0, \eta_0) \quad , \quad t \in [t_0, \infty)$$

The motion (5.14) and its first derivative are assumed to be limited

(5.15) $$\| \psi(t; t_0, \eta_0) \| \leq \alpha \quad , \quad \| \dot{\psi}(t; t_0, \eta_0) \| \leq \alpha \quad , \quad 0 < \alpha < \infty$$

where $\| \cdot \|$ denotes the norm.

To ensure the solutions of the equation (5.13) in a vicinity of the function (5.14) we apply certain additional forces Π called program-forces. In this case, the motion of the mechanical system is described by the following differential equation

(5.16) $$\dot{\xi} = \Gamma(t, \xi) + \Pi(t, \xi) \quad , \quad \xi(t_0) = \xi_0 = z_0$$

We assume, that the functions Γ and Π satisfy the conditions of existence and uniqueness of solutions of the differential equations under consideration for $t \in [t_0, \infty)$.

Let us denote the solution of the equation (5.16) by

(5.17) $$\xi = \varphi(t; t_0, \xi_0)$$

The solving of the problem consits in finding the sufficient conditions under which the inequality

(5.18) $$\| \varphi(t; t_0, \xi_0) - \psi(t; t_0, \eta_0) \| \leq \varepsilon$$

holds where ε is given. To fulfil this inequality in the entire interval $[t_0, \infty)$ it is necessary, that

(5.19) $$\| \xi_0 - \eta_0 \| < \varepsilon$$

Let us denote

(.520) $$y = \varphi(t; t_0, \xi_0) - \psi(t; t_0, \eta_0)$$

Substituting (5.20) into (5.16) we obtain

(5.21) $$\dot{y} = \tilde{R}(t, y) + \tilde{P}(t, y) - \psi(t)$$

where

$$\tilde{R}(t, y) = \Gamma(t, y + \psi) = R(t, y) + f_1(\psi)$$
$$\tilde{P}(t, y) = \Pi(t, y + \psi) = P(t, y) + f_2(\psi)$$

Let us denote

(5.22) $$q(t) = f_1(\psi) + f_2(\psi) - \dot{\psi}$$

In view of (5.15) we have

(5.23) $$\| q(t) \| \leq \beta = \text{const} < \infty$$

Then the equation (5.15) takes the form

(5.24) $$\dot{y} = R(t, y) + P(t, y) + q(t), \quad y(t_0) = y_0 = \xi_0 - \eta_0$$

Now the solving of the problem consists in finding the sufficient conditions for

(5.25) $\qquad \|y\| \leq \varepsilon \qquad , \quad t \in [t_0, \infty)$

Let us consider simultaneously with equation (5.24) the following differential equation

(5.26) $\qquad \dot{x} = \mathcal{P}(t, x) \qquad , \quad x(t_0) = x_0 = y_0$

which is a comparison equation for the equation (5.24). The function $\mathcal{P}$ depends on the program-forces Π, therefore we can choose this function according to our wish. For instance, we can construct a function

(5.27) $$x = h(t; t_0, y_0)$$

and calculate its first derivative

(5.28) $$\dot{x} = \dot{h}(t; t_0, y_0)$$

Determining $y_0 = \omega(t; t_0, x)$ from (5.27) and substituting this relation into (5.28) we obtain

(5.29) $\qquad \dot{x} = \dot{h}[t; t_0, \omega(t; t_0, x)] = \mathcal{P}(t, x)$

In general, we choose the function $\mathcal{P}$ in such a way, that the assumptions, which allow to reduce the considered problem to the problem I or II, that is to the integral inequality of the 1-st or 2-nd kind are satisfied. Therefore the considered problem can be reduced to the problem I or II, examined in the part 1, and consequently to the investigation of the integral inequality of the 1-st or 2-nd kind.

5.3. The asymptotical behaviour of solutions of the differential equation of motion.

Let us consider the differential equation of motion in the following matrix form

$$(5.30)\qquad \dot{y} = A(t)y + f(t,y) \quad , \quad y(t_0) = y_0$$

where

$$y = col[y_1, \ldots, y_n] \quad , \quad f = col[f_1, \ldots, f_n]$$

and $A = \{a_{ik}\}$ is a square matrix $n \times n$,continuous with respect to time.

We assume,that the function f satisfies the conditions of existence and uniqueness of solutions of the differential equation (5.30) in the interval $[t_0, \infty)$.

Let us consider a comparison differential equation corresponding to the equation (5.30) in the form

$$(5.31)\qquad \dot{x} = A(t)x \quad , \quad x(t_0) = x_0 = y_0$$

Let us denote by $X(t)$ the fundamental matrix of solutions of the equation (5.31), normed for $t = t_0$.We want to establish the conditions allowing to represent the solution $y = y(t)$ of the equation (5.30) in the following form

$$(5.32)\qquad y(t) = X(t)[b + o(1)] \quad \text{for} \quad t \to \infty$$

where $b = col[b_1, \ldots, b_n]$ is a constant column matrix.

Let us assume,that the function f satisfies the following condition

$$(5.33)\qquad \| X^{-1}(t) f[t, X(t)u] \| \leq v(t)\, g(\|u\|)$$

where the functions v and g satisfy the conditions given in the theorem 1 of the part 4 (point 4.2).

We shall use the results of the part 4 (point 4.2) in the particular case $w = 0$.

Let us assume,that the assumptions of the theorem 2 (part 4,point 4.2) are satisfied and that the inequality (5.33) holds.Then for each solution $y = y(t)$ of the equation (5.30) it exists a constant column matrix b ,that the asymptotical representation (5.32) holds, if only $\|y_0\| \leqslant c^*$ where $c^* = \tilde{c}$ and $\tilde{c}$ is the constant appearing in the theorem 2.

Indeed, let us introduce a new variable in the following way

$$z(t) = X^{-1}(t)\, y(t) \tag{5.34}$$

In view of (5.30) and (5.31) we obtain

$$\dot{z} = X^{-1}(t)\, f\,[t, X(t)z]$$

Hence we have

$$z(t) = z(t_0) + \int_{t_0}^{t} X^{-1}(s)\, f\,[s, X(s)z(s)]\, ds \tag{5.35}$$

Taking the norm of both sides of the equation (5.35), in view of (5.33) we obtain

$$\|z(t)\| \leqslant \|z(t_0)\| + \int_{t_0}^{t} v(s)\, g(\|z(s)\|)\, ds$$

If $\|z(t_0)\| = c$ where $c \in (0, \tilde{c})$ and $\tilde{c}$ is determined by the relation (4,32), then there exists, on the basis of theorem 2, (part 4, point 4.2) a constant $M(c, t_0)$ such that

$$\|z(t)\| \leqslant M(c, t_0) < \infty \quad , \quad t \geqslant t_0$$

Therefore on the basis of (5.35) it follows, that the integral

$$\int_{t_0}^{\infty} X^{-1} f\,[t, X(t)z(t)]\, dt$$

exists and is convergent.

Let us represent the integral equation in the form

$$z(t) = z(t_0) + \int_{t_0}^{\infty} X^{-1}(t)\, f\,[t, X(t)z(t)]\, dt + \int_{\infty}^{t} X^{-1}(s)\, f\,[s, X(s)z(s)]\, ds$$

Denoting the first two terms at the right-hand side by b and taking into account that the third term is of the rank $o(1)$ for $t \to \infty$ we have

$$z(t) = b + o(1) \quad \text{for} \quad t \to \infty$$

Therefore in view of (5.34) we get

$$(5.36) \qquad y(t) = X(t)[b + o(1)] \quad \text{for} \quad t \to \infty$$

This relation takes place for all $\|z(t_0)\| \leq \tilde{c}$ and since $y = Xz$ that's why the obtained asymptotical representation takes place for all $\|y_0\| \leq c^*$ where $c^* = \tilde{c}\,\|X(t_0)\|$. Because we have assumed that the fundamental matrix $X(t)$ is normed for $t = t_0$, therefore it is $\|X(t_0)\| = 1$ and we have $c^* = \tilde{c}$.

One can prove also the following results.

Let us assume, that the assumptions of theorem 3 (part 4, p.4.2, $w = 0$) and the inequality (5.33) are satisfied. Then for each solution $y(t)$ of the equation (5.30) one can find such a constant column matrix b, that the asymptotical representation (5.36) takes place for all $\|y_0\| < \infty$.

Let us assume, that the assumptions of the theorem 4 (part 4, p.4.2, $w = 0$) and the inequality (5.33) are satisfied. Then for each solution $y(t)$ of the equation (5.30) one can find such a constant column matrix b, that the asymptotical representation (5.36) takes place for all $\|y_0\| \leq \tilde{c}$, where $\tilde{c}$ denotes the constant appearing in the theorem 4. Moreover it is $b \to 0$ for $c \to 0$.

Let us assume, that the assumptions of the theorem 5 (part 4, p.4.2, $w = 0$) and the inequality (5.33) are satisfied. Then for each solution $y(t)$ of the equation (5.30) one can find such a constant column matrix b, that the asymptotical representation (5.36) takes place for all $\|y_0\| < \infty$. Moreover it is $b \to 0$ for $c \to 0$.

The proofs of these results can be obtained in the same way as the formula (5.36) using respectively the theorems 3,4,5, proved in the part 4 point 4.2 (putting $w = 0$).

On the basis of the asymptotical representation (5.36) it is evident, that if all solutions of the linear comparison differential equation (5.31) are limited (or tend to zero) for $t \to \infty$, then also all solutions of the differential equation (5.30) are limited (or tend to zero) for $t \to \infty$.

Example 1. Let us consider a system of differential equations in the form

$$(5.37)\qquad \dot{y}_1 = -t y_1 + \alpha(t) y_1^{4/3} \quad , \quad \dot{y}_2 = y_1 - t y_2 + \alpha(t) y_1^{4/3}$$

where $\alpha(t) \geqslant 0$ is a continuous function. This system can be written in the following matrix form

$$(5.38)\qquad \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -t , & 0 \\ 1 , & -t \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \alpha(t) y_1^{4/3} \\ \alpha(t) y_1^{4/3} \end{bmatrix}$$

Let us notice, that all solutions of the system of differential equations

$$(5.39)\qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -t , & 0 \\ 1 , & -t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

are limited for $t \geqslant 1$ and tend to zero for $t \to \infty$.

Indeed, in the considered case we have

$$X(t) = \begin{bmatrix} e^{-t^2/2} , & 0 \\ t e^{-t^2/2} , & e^{-t^2/2} \end{bmatrix}$$

We shall now determine the functions v and g appearing in the inequality (5.33). To this end we calculate

$$X^{-1}(t) = \begin{bmatrix} e^{t^2/2} , & 0 \\ -t e^{t^2/2} , & e^{t^2/2} \end{bmatrix} , \qquad X(t)a = \begin{bmatrix} e^{-t^2/2} a_1 \\ t e^{-t^2/2} a_1 + e^{-t^2/2} a_2 \end{bmatrix}$$

$$f[t, X(t)a] = \begin{bmatrix} \alpha(t) e^{-t^2/6} a_1^{4/3} \\ \alpha(t) e^{-t^2/6} a_1^{4/3} \end{bmatrix}$$

$$X^{-1}(t)f[t, X(t)a] = \begin{bmatrix} \alpha(t)e^{t^2/3} a_1^{1/3} \\ (1-t)\alpha(t)e^{t^2/3} a_1^{1/3} \end{bmatrix}$$

Therefore in the considered case the inequality (5.33) takes the form.

$$(5.40) \quad \|X^{-1}(t)f[t, X(t)a]\| \leq \alpha(t)e^{t^2/3}(2+t)|a_1|^{1/3} \leq \alpha(t)e^{t^2/3} t\, 3(|a_1|+|a_2|)^{1/3}$$

for all $t \geq 1$ and $|a_1| + |a_2| < \infty$. Therefore the functions v and g can be taken in the form.

$$(5.41) \qquad v(t) = \alpha(t)\, t\, e^{t^2/3} \quad , \qquad g(\mu) = 3\mu^{1/3}$$

If the function $\alpha(t)$ satisfies the following inequality

$$(5.42) \qquad \int_1^\infty t\,\alpha(t)e^{t^2/3} dt < \infty$$

then the assumptions of theorem 3 (part 4,p.4.2,$w = 0$) are satisfied and therefore all conditions under which the asymptotical representation (5.36) holds are also satisfied.

On the basis of the properties of the fundamental matrix $X(t)$ we conclude, that all solutions of the system of the differential equations (5.37) are limited for $t \geq 1$ and tend to zero for $t \to \infty$.

Example 2. Let us consider a system of differential equations in the form

$$(5.43) \qquad \dot{y}_1 = -ty_1 + \alpha(t)y_1^3 \quad , \qquad \dot{y}_2 = y_1 - ty_2 + \alpha(t)y_1^3$$

where $\alpha(t) \geq 0$ is a continuous function. This system can be written in the following form

$$(5.44) \qquad \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -t, & 0 \\ 1, & -t \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \alpha(t)y_1^3 \\ \alpha(t)y_1^3 \end{bmatrix}$$

The system of comparison differential equations has in the considered case the same form as in the example 1, that is the form (5.39). Therefore the fundamental matrix $X(t)$ is limited and tends to zero for $t \to \infty$ if $t \geqslant 1$.

Performing the analogous calculationc as in the example 1 we get

$$X^{-1}(t)f[t, X(t)a] = \begin{bmatrix} \alpha(t)e^{-t^2}a_1^3 \\ (1-t)\alpha(t)e^{-t^2}a_1^3 \end{bmatrix}$$

$$\| X^{-1}(t)f[t, X(t)a] \| \leqslant \alpha(t)\,t\,e^{-t^2}3\left(|a_1|+|a_2|\right)^3 \tag{5.45}$$

for all $t \geqslant 1$ and $|a_1| + |a_2| < \infty$. Therefore the functions v and g can be taken in the form

$$v(t) = \alpha(t)\,t\,e^{-t^2} \quad , \quad g(\mu) = 3\mu^3$$

If the function $\alpha(t)$ satisfies the following inequality

$$\int_1^{\infty} t\,\alpha(t)e^{-t^2}dt < \infty \tag{5.46}$$

then the assumptions of theorem 4 (part 4, p.4.2, $w = 0$) are satisfied and therefore all conditions under which the asymptotical representation (5.36) holds are also satisfied.

On the basis of the properties of the fundamental matrix $X(t)$ we conclude that all solutions of the system of the differential equations (5.43) are limited for $t \geqslant 1$ and tend to zero for $t \to \infty$.

Example 3. Let us consider a system of differential equations in the form

$$\dot{y}_1 = -t y_1 + \alpha(t) y_1 \quad , \quad \dot{y}_2 = y_1 - t y_2 + \alpha(t) y_1 \tag{5.47}$$

where $\alpha(t) \geqslant 0$ is a continuous function. This system can be written in the following form

$$(5.48)\qquad \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -t, & 0 \\ 1, & -t \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \alpha(t)y_1 \\ \alpha(t)y_1 \end{bmatrix}$$

The system of comparison differential equations has in the considered case the same form as in the examples 1 and 2, that is the form (5.39). Therefore the fundamental matrix $X(t)$ is limited and tends to zero for $t \to \infty$, if $t \geqslant 1$. Performing the analogous calculations as in the example 1 we get

$$X^{-1}(t) f[t, X(t)a] = \begin{bmatrix} \alpha(t)a_1 \\ (1-t)\alpha(t)a_1 \end{bmatrix}$$

$$(5.49)\qquad \| X^{-1}(t) f[t, X(t)a] \| \leqslant \alpha(t)\, t\, 3\,(|a_1| + |a_2|)$$

for all $t \geqslant 1$ and $|a_1| + |a_2| < \infty$. Therefore the functions v and g can be taken in the form

$$v(t) = t\,\alpha(t) \qquad , \qquad g(\mu) = 3\mu$$

If the function $\alpha(t)$ satisfies the following inequality

$$(.5.50)\qquad \int_1^\infty t\,\alpha(t)\,dt < \infty$$

then the assumptions of theorem 5 (part 4, p.4.2, $w = 0$) are satisfied and therefore all conditions under which the asymptotical representation (.5.36) holds are also satisfied.

On the basis of the properties of the fundamental matrix $X(t)$ we conclude, that all solutions of the system of the differential equations (5.47) are limited for $t \geqslant 1$ and tend to zero for $t \to \infty$.

5.4. The influence of non-linear couplings on the behaviour of the solutions of the equations of motion of a mechanical system.

Let us consider a mechanical system described by N differential equations of the first order. Let us assume that this system can be separated into two sub-systems having l and m (l + m = N) differential equations respectively, and coupled in a non-linear manner.

Therefore let us consider a system of differential equations having the form

$$\dot{y} = A(t)y + f(t,x,y) + p(t) \tag{5.51}$$

$$\dot{x} = B(t)x + \varphi(t,x,y) + \pi(t) \tag{5.52}$$

$$x(t_0) = x_0 \quad , \quad y(t_0) = y_0$$

where

$$x = col\,[x_1,\dots,x_l] \quad , \quad y = col\,[y_1,\dots,y_m] \quad , \quad l+m = N$$

$A(t)$ is a square matrix m x m and $B(t)$ a square matrix $l \times l$. These matrices are continuous for $t \in [t_0, \infty)$. The column matrices

$$f = col\,[f_1,\dots,f_m] \quad , \quad \varphi = col\,[\varphi_1,\dots,\varphi_l]$$

are assumed to satisfy the conditions of existence and uniqueness of the solutions of the equations (5.51) in the interval $[t_0, \infty)$. The matrices

$$p = col\,[p_1,\dots,p_m] \quad , \quad \pi = col\,[\pi_1,\dots,\pi_l]$$

are also assumed to be continuous for $t \in [t_0, \infty)$.

Together with the equations (5.51) and (5.52) let us consider the comparison linear differential equations in the form

$$\dot{\eta} = A(t)\eta + p(t) \tag{5.53}$$

(5.54) $\dot{r} = A(t)\, r$

(5.55) $\dot{\xi} = B(t)\, \xi + \pi(t)$

(5.56) $\dot{q} = B(t)\, q$

where

$$\eta = \mathrm{col}\,[\eta_1, \dots, \eta_m] \quad , \quad r = \mathrm{col}\,[r_1, \dots, r_m]$$

$$\xi = \mathrm{col}\,[\xi_1, \dots, \xi_l] \quad , \quad q = \mathrm{col}\,[q_1, \dots, q_l]$$

Let the initial values of these functions satisfy the conditions

$$y(t_0) = \eta(t_0) = r(t_0) = y_0 \quad , \quad x(t_0) = \xi(t_0) = q(t_0) = x_0$$

Let us denote by $R(t)$ the fundamental matrix of the solutions of the equation (5.54) and by $Q(t)$ the fundamental matrix of the solutions of the equation (5.56).

Let us introduce the following assumptions

1) $\| R(t) R^{-1}(s) \| \leqslant \sigma_1 e^{-n_1(t-s)}$, $\| Q(t) Q^{-1}(s) \| \leqslant \sigma_2 e^{-n_2(t-s)}$

where σ_1 , n_1 , σ_2 , n_2 are positive constants

2) $\| \eta(t) \| \leqslant c_1 < \infty$, $\| \xi(t) \| \leqslant c_2 < \infty$ for $t \in [t_0, \infty)$

where c_1 , c_2 are positive constants

3) $\| f(t, x, y) \| \leqslant k_1 g_1(\|x\| + \|y\|)$, $\| \varphi(t, x, y) \| \leqslant k_2 g_2(\|x\| + \|y\|)$

for $t \in [t_0, \infty)$ and $\|x\| + \|y\| < \infty$,where k_1, k_2 are non-negative constants,and $g_1(u)$, $g_2(u)$ continuous, non-negative,non-decreasing functions for $u \geqslant 0$ and $g_1(0) = g_2(0) = 0$.

Let us transform the system of differential equations into a equivalent system of the integral equations having theform.

$$(5.57)\quad y = R(t)R^{-1}(t_0)y_0 + \int_{t_0}^{t} R(t)R^{-1}(s)f[s,x(s),y(s)]\,ds + \int_{t_0}^{t} R(t)R^{-1}(s)p(s)\,ds$$

$$(5.58)\quad x = Q(t)Q^{-1}(t_0)x_0 + \int_{t_0}^{t} Q(t)Q^{-1}(s)\varphi[s,x(s),y(s)]\,ds + \int_{t_0}^{t} Q(t)Q^{-1}(s)\pi(s)\,ds$$

The solutions of the differential equations (5.53) and (5.55) have the form

$$(5.59)\quad \eta = R(t)R^{-1}(t_0)\eta_0 + \int_{t_0}^{t} R(t)R^{-1}(s)p(s)\,ds$$

$$(5.60)\quad \xi = Q(t)Q^{-1}(t_0)\xi_0 + \int_{t_0}^{t} Q(t)Q^{-1}(s)\pi(s)\,ds$$

Since it has been assumed that $\eta(t_0) = \eta_0 = y_0$ and $\xi(t_0) = \xi_0 = x_0$ therefore the system of integral equations (5.57) and (5.58) takes the form

$$(5.61)\quad y = \eta + \int_{t_0}^{t} R(t)R^{-1}(s)f[s,x(s),y(s)]\,ds$$

$$(5.62)\quad x = \xi + \int_{t_0}^{t} Q(t)Q^{-1}(s)\varphi[s,x(s),y(s)]\,ds$$

Taking the norm of both sides of these equations, we obtain on the basis of the assumptions (1), (2), (3)

$$(5.63)\quad \|y\| \le c_1 + \int_{t_0}^{t} k_1\sigma_1 e^{-n_1(t-s)} g_1(\|x\|+\|y\|)\,ds$$

$$(5.64)\quad \|x\| \le c_2 + \int_{t_0}^{t} k_2\sigma_2 e^{-n_2(t-s)} g_2(\|x\|+\|y\|)\,ds$$

Let us denote

$$\|x\| + \|y\| = u \quad , \qquad \sigma k = \max(\sigma_1 k_1 , \sigma_2 k_2)$$

$$n = \min(n_1, n_2) \quad , \qquad c = c_1 + c_2$$

$g(u) \geqslant \max\limits_{u \geqslant 0}(g_1, g_2)$, $g(0) = 0$ and g is a continuous, non-negative, non-decreasing function for $u \geqslant 0$.

Adding the inequalities (5.63) and (5.64) we obtain

$$u \leq c + \sigma k \int_{t_0}^{t} e^{-n(t-s)} g(u(s))\, ds \tag{5.65}$$

that is the integral inequality of the 1-st kind (see (4.1))

On the basis of the formula (4.9) we have

$$u(t) = \|x\| + \|y\| \leq (1+\lambda_0)c \tag{5.66}$$

where λ_0 is given by the relation (4.10). Therefore the following inequalities are also satisfied

$$\|x\| \leq (1+\lambda_0)c \quad , \quad \|y\| \leq (1+\lambda_0)c \tag{5.67}$$

On the basis of the formula (4.10)

$$\lambda_0 = \frac{\sigma k}{n} \frac{1}{c} g[(1+\lambda_0)c] \tag{5.68}$$

we conclude that if $c = \text{const} > 0$, then for $\frac{\sigma k}{n} \to 0$ we have $\lambda_0 \to 0$. In other words if $\frac{\sigma k}{n}$ is sufficiently small, which depends either on the matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ (the ratio $\frac{\sigma}{n}$) or on the functions f and φ being small (constant k), then λ_0 can be made arbitrarily small for $c = \text{const} > 0$.

Let us assume additionally, that the function g satisfies the condition (see (4.11))

$$\lim_{\vartheta \to 0} \frac{g(\vartheta)}{\vartheta} = \Omega \tag{5.69}$$

we obtain (see (4.12))

$$\lambda_0 = \frac{\frac{\sigma k}{n}}{1 - \frac{\sigma k}{n}\Omega} \quad \text{for} \quad \frac{\sigma k}{n}\Omega < 1 \ \text{and} \ c \to 0 \tag{5.70}$$

If therefore $\frac{\sigma k}{n} \to 0$, we have also $\lambda_0 \to 0$ for $c \to 0$ and $\frac{\sigma k}{n}\Omega < 1$. From (5.67) it follows that $\|x\|$ and $\|y\|$ can be made arbotrarily small if $\frac{\sigma k}{n}$ and c are sufficiently small and $\frac{\sigma k}{n}\Omega < 1$.

Now we shall determine the vicinities in which the solutions of the considered sub-systems, coupled in a non-linear manner, are contained. On the basis of (5.61), (5.62) and the assumptions (1) and (3) we have

$$\|y-\eta\| \leq \sigma_1 k_1 \int_{t_0}^{t} e^{-n_1(t-s)} g_1(\|x\|+\|y\|)\,ds$$

$$\|x-\xi\| \leq \sigma_2 k_2 \int_{t_0}^{t} e^{-n_2(t-s)} g_2(\|x\|+\|y\|)\,ds$$

In view of (5.66) these inequalities take the form

$$\|y-\eta\| \leq \sigma_1 k_1\, g_1[(1+\lambda_0)c] \int_{t_0}^{t} e^{-n_1(t-s)} ds = \frac{\sigma_1 k_1}{n_1} g_1[(1+\lambda_0)c]\left(1-e^{-n_1(t-t_0)}\right)$$

$$\|x-\xi\| \leq \sigma_2 k_2\, g_2[(1+\lambda_0)c] \int_{t_0}^{t} e^{-n_2(t-s)} ds = \frac{\sigma_2 k_2}{n_2} g_2[(1+\lambda_0)c]\left(1-e^{-n_2(t-t_0)}\right)$$

Hence

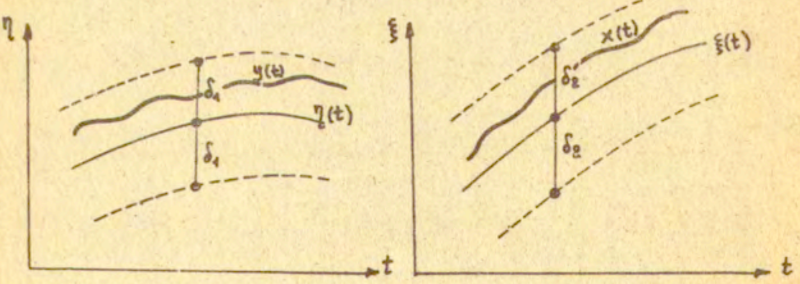
$$\|y-\eta\| \leq \frac{\sigma_1 k_1}{n_1} g_1[(1+\lambda_0)c] = \delta_1$$

$$\|x-\xi\| \leq \frac{\sigma_2 k_2}{n_2} g_2[(1+\lambda_0)c] = \delta_2$$

If, for instance, the norm is assumed to be a sum of absolute values of the elements of the matrices, then we get

(5.71) $$\eta_i(t)-\delta_1 \leq y_i(t) \leq \eta_i(t)+\delta_1 \quad , \quad i=1,\dots,m$$

(5.72) $$\xi_j(t)-\delta_2 \leq x_j(t) \leq \xi_j(t)+\delta_2 \quad , \quad j=1,\dots,l$$



From the above considerations it follows, that we can determine the vicinities (5.71) and (5.72) in which the solutions $x(t)$, $y(t)$ are contained if we have information given in the assumptions (1),(2),(3). From the point of view of synthesis of a system these results show, how the matrices A and B , the coupling functions f and φ and the functions p and π should be selected, in order that the vicinities (5.71) and (5.72) may be arbitrarily narrow. In other words the above results provide the sufficient conditions for determining the manner, in which the synthesis is to be performed for a system - so that the non-linear couplings may be neglected in practice.

5.5. The investigation of the behaviour of solutions of some non-linear ordinary differential equation of the second order.

Let us consider the following scalar ordinary differential equation of the second order (see part 2, p.2.2, Eq.(2.42))

$$(5.73) \qquad \ddot{x} + 2h\dot{x} + kx = \beta(t)x^r , \quad x(0) = x_0 , \quad \dot{x}(0) = \dot{x}_0$$

where h , k , r are positive constants and $\beta(t)$ is a continuous function.

We shall investigate the behaviour of solutions $x = x(t)$ of the equation (5.73) corresponding to non-zero initial values, with respect to zero solution $x = 0$. To this end we assume, that the distance between solution $x = x(t)$ and zero solution $x = 0$ has the form

$$(5.74) \qquad \rho = \dot{x}^2 + 2hx\dot{x} + kx^2 = (k-h^2)x^2 + (hx+\dot{x})^2$$

Let us determine the time derivative of distance (5.74) for solutions of Eq.(5.73), assuming that

$$(5.75) \qquad k - h^2 > 0$$

$$\dot{\rho} = 2[(k-h^2)x\dot{x} + (hx+\dot{x})(h\dot{x}+\ddot{x}) =$$

$$= 2[(k-h^2)x\dot{x} + (hx+\dot{x})(-h\dot{x} - kx + \beta(t)x^r)]$$

$$\dot{\varrho} = 2\left(kx\dot{x} - h^2 x\dot{x} - h^2 x\dot{x} - hkx^2 - h\dot{x}^2 - kx\dot{x}\right) + 2(hx+\dot{x})\beta(t)x^{\tau}$$

$$\dot{\varrho} = 2\left(-h\dot{x}^2 - 2h^2 x\dot{x} - hkx^2\right) + 2(hx+\dot{x})\beta(t)x^{\tau}$$

$$\dot{\varrho} = -2h(\dot{x}^2 + 2hx\dot{x} + kx^2) + 2(hx+\dot{x})\beta(t)x^{\tau}$$

In view of (5.74) we have

$$\dot{\varrho} = -2h\varrho + 2(hx+\dot{x})\beta(t)x^{\tau} \tag{5.76}$$

Let us assume,that (see (2.49))

$$2(hx+\dot{x})\beta(t)x^{\tau} \leq \gamma(t)\varrho + \omega(t,\varrho) \tag{5.77}$$

Let us denote

$$w(t) = \gamma(t) - 2h \tag{5.78}$$

From (5.76),(5.77),(5.78) we get the differential inequality (see (2.51))

$$\dot{\varrho} \leq w(t)\varrho + \omega(t,\varrho) \tag{5.79}$$

Integration of both sides of (5.79) with respect to time provides the following non-linear integral inequality (see (2.52))

$$\varrho \leq c + \int_0^t \left[w(s)\varrho(s) + \omega(s,\varrho(s))\right]ds \tag{5.80}$$

We shall first analyze the behaviour of distance $\varrho = \varrho(t)$ and construct its estimation,and next use the obtained results to estimate $x = x(t)$.

Let us find the estimation (5.77).

$$2(hx+\dot{x})\beta(t)x^{\tau} \leq (hx+\dot{x})^2 + \beta^2(t)x^{2\tau}$$

$$2(hx+\dot{x})\beta(t)x^{\tau} \leq \left[(k-h^2)x^2+(hx+\dot{x})^2\right] + \frac{\beta^2(t)}{(k-h^2)^{\tau}}\left[(k-h^2)x^2\right]^{\tau}$$

$$2(hx+\dot{x})\beta(t)x^{\tau} \leq \left[(k-h^2)x^2+(hx+\dot{x})^2\right] + \frac{\beta^2(t)}{(k-h^2)^{\tau}}\left[(k-h^2)x^2+(hx+\dot{x})^2\right]^{\tau}$$

On the basis of (5.74) we have

(5.81) $$2(hx+\dot{x})\beta(t)x^{\tau} \leq \varrho + \frac{\beta^2(t)}{(k-h^2)^{\tau}}\varrho^{\tau}$$

Let us denote

$$b(t) = \frac{\beta^2(t)}{(k-h^2)^{\tau}}$$

Hence the inequality (5.81) takes the form

(5.82) $$2(hx+\dot{x})\beta(t)x^{\tau} \leq \varrho + b(t)x^{\tau}$$

From (5.77) and (5.82) we have

(5.83) $$\gamma = 1 \quad , \quad \omega(t,\varrho) = b(t)x^{\tau}$$

Therefore we have

(5.84) $$\lambda = w(t) = \gamma - 2h = 1-2h = \mathrm{const} \neq 0$$

The differential and integral inequalities take respectively the forms

(5.85) $$\dot{\varrho} \leq (1-2h)\varrho + b(t)\varrho^{\tau}$$

(5.86) $$\varrho \leq c + \int_0^t \left[(1-2h)\varrho + b(s)\varrho^{\tau}\right]ds$$

For the further considerations we assume that the function $\beta(t)$ has the form

(5.87) $$\beta(t) = \left(\beta_0 e^{-[\nu+\lambda(\tau-1)]t}\right)^{1/2} \quad , \quad \lambda = 1-2h$$

where β_0 , ϑ are positive constants.

Now we shall apply theorem 1 of the part 4 (p.4.2). At first we write relation (4.14)

$$\omega\left[t, z\exp\left(\int_0^t w\,ds\right)\right]\exp\left(-\int_0^t w\,ds\right) \leq v(t)g(z)$$

$$b(t)z^{\tau}e^{\tau\lambda t}e^{-\lambda t} \leq v(t)g(z)$$

$$\frac{\beta^2(t)}{(k-h^2)^{\tau}}z^{\tau}e^{\lambda(\tau-1)t} \leq v(t)g(z)$$

In view of (5.87) we have

$$\frac{\beta_0}{(k-h^2)^{\tau}}z^{\tau}e^{-\vartheta t}e^{-\lambda(\tau-1)t}e^{\lambda(\tau-1)t} = \frac{\beta_0}{(k-h^2)^{\tau}}z^{\tau}e^{-\vartheta t} \leq v(t)g(z)$$

Therefore the functions $v(t)$ and $g(z)$ can be taken in the form

$$(5.88) \qquad v(t) = \frac{\beta_0}{(k-h^2)^{\tau}}e^{-\vartheta t} \quad , \qquad g(z) = z^{\tau}$$

For further discussion we denote

$$(5.89) \qquad \mu(t) = \int_0^t v(s)\,ds = \frac{\beta_0}{\vartheta(k-h^2)^{\tau}}\left(1-e^{-\vartheta t}\right)$$

Before to constructing an effective estimation of the distance ϱ in a closed form, let us use the theorems 2 - 5 (part 4, p.4.2) to find out, what results are obtainable in the case under consideration. It is evident, that all results obtained in the theorems 2 - 5 are excluded, because in the considered case it is $w(t) = \lambda = 1 - 2h = \text{const}$ and therefore the integral

$$\int_0^{\infty} w(t)\,dt = \infty$$

is divergent, which makes the theorems 2 - 5 useless. But in the considered case the integral inequality (5.86) was obtained from the differential inequality (5.85) and assuming that it is

(5.90) $\quad w(t) = -\alpha(t) = 1-2h < 0 \quad , \quad \alpha = 2h-1 > 0$

we obtain the same problem as for the differential and integral inequalities (4.45) and (4.46). Therefore in the considered case we can apply the theorems 3a - 5a , because in view of (5.89) we have

(5.91) $$\mu(\infty) = \int_0^\infty v(t)\,dt = \frac{\beta_0}{\nu(k-h^2)^r} < \infty$$

To determine the results contained in the theorems 3a-5a let us first calculate

(5.92) for $0 < r < 1$, $\int_0^A \frac{ds}{s^r} = \frac{A^{1-r}}{1-r} < \int_A^\infty \frac{ds}{s^r} = \infty$

(5.93) for $r > 1$, $\int_A^\infty \frac{ds}{s^r} = \frac{1}{(r-1)A^{r-1}} < \int_0^A \frac{ds}{s^r} = \infty$

(5.94) for $r = 1$ $\quad \int_0^A \frac{ds}{s} = \int_A^\infty \frac{ds}{s} = \infty$

Therefore the results contained in the theorems 3a , 4a, 5a are obtainable in the respective sases (5.92), (5.93) and (5.94). These results, that is the closed estimation (4.51) for the distance ρ have the forms:

a) The result based on the theorem 3a (the case $0 < r < 1$)

(5.95) $$\rho \leq \left[c^{1-r} + \frac{\beta_0(1-r)}{\nu(k-h^2)^r}\left(1-e^{-\nu t}\right)\right]^{\frac{1}{1-r}} e^{(1-2h)t} = \Phi(t)$$

It is evident that inequality (5.95) holds for $t \in [0, \infty)$ and for $c \in (0, \tilde{c})$ where $\tilde{c}$ can be arbitrarily large.

b) The result based on the theorem 4a (the case $r > 1$)

(5.96) $$\rho \leq \frac{e^{(1-2h)t}}{\left[\frac{1}{c^{r-1}} - \frac{\beta_0(r-1)}{\nu(k-h^2)^r}\left(1-e^{-\nu t}\right)\right]^{\frac{1}{r-1}}} = \Phi(t)$$

This inequality holds for $t \in [0, \infty)$ and for $c \in (0, \tilde{c})$ where $\tilde{c}$ is determined by the formula (see theorem 4a)

(5.97) $$\int_0^\infty v(t)dt = \int_{\tilde{c}}^\infty \frac{ds}{s^r} \quad , \quad r > 1$$

From (5.91),(5.93) and (5.97) we have

(5.98) $$\frac{\beta_0}{\nu(k-h^2)^r} = \frac{1}{(r-1)\tilde{c}^{r-1}} \quad , \quad \tilde{c} = \left(\frac{\nu(k-h^2)^r}{\beta_0(r-1)}\right)^{\frac{1}{r-1}}$$

It is evident, that if $0 < c < \tilde{c}$ then the denominator of the right-hand side of (5.96) is positive. Moreover $M(c) \to 0$ for $c \to 0$.

c) The result based on the theorem 5a (linear case $r = 1$).

In this case instead of (5.88) we have

(5.99) $$v(t) = \frac{\beta_0}{k-h^2} e^{-\nu t} \quad , \quad g(z) = z$$

Let us notice, that in the two preceding cases ($0 < r < 1$ and $r > 1$) $\beta = \beta(t)$ must be a function of time of the type

$$\beta(t) = O(e^{-[\nu + \lambda(r-1)]t})$$

because otherwise the first conditions of theorems 3a and 4a cannot be satisfied. In the linear case ($r = 1$), to obtain the result based on the theorem 5a, $\beta = \beta(t)$ can be an arbitrary continuous function, or a constant. For this reason we assume for further discussion, that in the linear case it is

(5.100) $$v(t) = \frac{\beta_0}{k-h^2}\varphi(t) \quad , \quad \mu(t) = \int_0^t v(s)ds = \frac{\beta_0}{k-h^2}\int_0^t \varphi(s)ds$$

where $\varphi(t)$ is an arbitrary continuous function.

In this case the inequality (4.51) takes the form

(5.101) $$\rho \leq c\, exp\left[-\left((2h-1)t - \frac{\beta_0}{k-h^2}\int_0^t \varphi(s)ds\right)\right] = \Phi(t)$$

where $2h - 1 > 0$

If $\varphi(t) \equiv 1$, we have

(5.102) $$\rho \leq c\, exp\left[-\left(2h-1 - \frac{\beta_0}{k-h^2}\right)t\right] = \Phi(t)$$

It is evident, that inequality (5.101) (and (5.102)) holds for $t \in [0, \infty)$ and for $c \in (0, \tilde{c})$ where $\tilde{c}$ can be arbitrarily large. Moreover $M(c) \to 0$ for $c \to 0$.

If the condition

$$(5.103) \qquad \lim_{t \to \infty} \frac{1}{t} \int_0^t \varphi(s)\,ds = \varphi_0 < \frac{(2h-1)(k-h^2)}{\beta_0}$$

is satisfied, and in the case $\varphi(t) \equiv 1$ the condition

$$\frac{(2h-1)(k-h^2)}{\beta_0} > 1$$

holds, then

$$(5.104) \qquad \lim_{t \to \infty} \Phi(t) = 0$$

In each three cases based on the theorems 3a, 4a, 5a considered above, having the function $\Phi(t)$, we can easily find the estimation of the solution $x(t)$ of the differential equation (5.73). Indeed, on the basis of (5.74) we have

$$\varrho = (k-h^2)x^2 + (hx+\dot{x})^2 \leq \Phi(t)$$

$$(k-h^2)x^2 \leq \Phi(t)$$

$$(5.105) \qquad x^2 \leq \frac{1}{k-h^2}\,\Phi(t)$$

5.6. The investigation of the behaviour of solutions of some non-linear partial differential equation of the second order.

Let us consider the following partial differential equation

$$(5.106) \qquad \frac{\partial^2 u}{\partial t^2} + 2h\frac{\partial u}{\partial t} + ku = a\frac{\partial^2 u}{\partial x^2} + f\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right)$$

where the function f has the form

$$(5.107) \qquad f = \beta_1(x,t)\,u^{r_1} + \beta_2(x,t)\left(\frac{\partial u}{\partial x}\right)^{r_2}$$

and h , k , a , r_1 , r_2 are positive constants. Let us assume, that the functions $\beta_1(x,t)$ and $\beta_2(x,t)$ have the form

$$(5.108)\quad \beta_1(x,t) = \left(\beta_{10}(x)e^{-[\nu_1+\lambda(r_1-1)]t}\right)^{1/2}$$

$$(5.109)\quad \beta_2(x,t) = \left(\beta_{20}(x)e^{-[\nu_2+\lambda(r_2-1)]t}\right)^{1/2}$$

where $\beta_{10}(x)$, $\beta_{20}(x)$ denote continuous positive functions, ν_1 , ν_2 are positive constants, r_1 , $r_2 \in (0,1)$ and $\lambda = 1 - 2h = \mathrm{const} \neq 0$.

Let the initial and boundary conditions have the form

$$(5.110)\quad \begin{cases} u(x,0) = \varphi_1(x) \quad , & \dfrac{\partial u(x,0)}{\partial t} = \varphi_2(x) \\ u(0,t) = 0 \quad , & u(l,t) = 0 \end{cases}$$

We shall investigate the behaviour of solutions $u = u(x,t)$ of the equation (5.106) corresponding to non-zero initial values, with respect to zero solution $u = 0$. To this end, we assume, that the distance between solution $u(x,t)$ and zero solution $u = 0$ has the form

$$(5.111)\quad \rho(u) = \int_0^l \left[\left(\frac{\partial u}{\partial t}\right)^2 + a\left(\frac{\partial u}{\partial x}\right)^2 + 2hu\frac{\partial u}{\partial t} + ku^2\right]dx =$$

$$= \int_0^l \left[(k-h^2)u^2 + a\left(\frac{\partial u}{\partial x}\right)^2 + \left(hu + \frac{\partial u}{\partial t}\right)^2\right]dx$$

where is assumed that

$$(5.112)\quad k - h^2 > 0$$

Determining the time derivative of the distance (5.111) for solutions of Eq. (5.106) we have obtained in the part 3 (see p.3.3.8 formula (3.179)) the following relation

$$(5.113)\quad \dot{\rho} = -2h\rho + 2\int_0^l \left(hu + \frac{\partial u}{\partial t}\right)f\,dx$$

Assuming that

(5.114) $$2\int_0^l \left(hu+\frac{\partial u}{\partial t}\right) f\,dx \le \gamma(t)\varrho + \omega(t,\varrho)$$

and denoting

(5.115) $$w(t) = \gamma(t) - 2h$$

we have obtained in the part 3 (see p.3.3.8 formulas (3.182) and (3.183)) the following differential and corresponding integral inequalities.

(5.116) $$\dot{\varrho} \le w(t)\varrho + \omega(t,\varrho)$$

(5.117) $$\varrho \le c + \int_0^t \left[w(s)\varrho(s) + \omega(s,\varrho(s))\right] ds$$

We shall first analyze the behaviour of distance $\varrho=\varrho(t)$ and construct its estimation, and next use the obtained results to estimate $u=u(x,t)$.

Let us find the estimation (5.114). We have

(5.118) $$2\int_0^l \left(hu+\frac{\partial u}{\partial t}\right) f\,dx \le 2\int_0^l \left|\left(hu+\frac{\partial u}{\partial t}\right) f\right| dx \le \int_0^l \left(hu+\frac{\partial u}{\partial t}\right)^2 dx + \int_0^l f^2 dx$$

The integral of the function f^2 at the right-hand side is estimated as follows

$$\int_0^l f^2 dx = \int_0^l \left[\beta_1^2 u^{2\tau_1} + \beta_2^2 \left(\frac{\partial u}{\partial x}\right)^{2\tau_2} + 2\beta_1\beta_2\, u^{\tau_1}\left(\frac{\partial u}{\partial x}\right)^{\tau_2}\right] dx \le$$

$$\le \int_0^l \left[\beta_1^2 u^{2\tau_1} + \beta_2^2 \left(\frac{\partial u}{\partial x}\right)^{2\tau_2} + \beta_1^2 u^{2\tau_1} + \beta_2^2\left(\frac{\partial u}{\partial x}\right)^{2\tau_2}\right] dx$$

$$\int_0^l f^2 dx \le 2\int_0^l \beta_1^2 u^{2\tau_1} dx + 2\int_0^l \beta_2^2 \left(\frac{\partial u}{\partial x}\right)^{2\tau_2} dx$$

Using the Hölder's inequality

$$\int_0^l PQ\,dx \le \left(\int_0^l |P|^p dx\right)^{1/p} \left(\int_0^l |Q|^q dx\right)^{1/q}, \quad p>1, \quad \frac{1}{p}+\frac{1}{q}=1$$

we have

$$\int_0^l f^2 dx \le 2\left(\int_0^l \beta_1^{\frac{2}{1-\tau_1}} dx\right)^{1-\tau_1}\left(\int_0^l u^2 dx\right)^{\tau_1} + 2\left(\int_0^l \beta_2^{\frac{2}{1-\tau_2}} dx\right)^{1-\tau_2}\left(\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx\right)^{\tau_2}$$

$$\int_0^l f^2 dx \le \frac{2}{(k-h^2)^{\tau_1}}\left(\int_0^l \beta_1^{\frac{2}{1-\tau_1}} dx\right)^{1-\tau_1}\left(\int_0^l (k-h^2)u^2 dx\right)^{\tau_1} +$$

$$+ \frac{2}{a^{\tau_2}}\left(\int_0^l \beta_2^{\frac{2}{1-\tau_2}} dx\right)^{1-\tau_2}\left(\int_0^l a\left(\frac{\partial u}{\partial x}\right)^2 dx\right)^{\tau_2}$$

Substituting into (5.118) and supplementing the respective integrands to obtain (5.111) we have

$$(5.119)\quad 2\int_0^l \left(hu + \frac{\partial u}{\partial t}\right) f\, dx \le \varrho + \frac{2}{(k-h^2)^{\tau_1}}\left(\int_0^l \beta_1^{\frac{2}{1-\tau_1}} dx\right)^{1-\tau_1} \varrho^{\tau_1} +$$

$$+ \frac{2}{a^{\tau_2}}\left(\int_0^l \beta_2^{\frac{2}{1-\tau_2}} dx\right)^{1-\tau_2} \varrho^{\tau_2}$$

From (5.108) and (5.109) it follows

$$\frac{2}{(k-h^2)^{\tau_1}}\left(\int_0^l \beta_1^{\frac{2}{1-\tau_1}} dx\right)^{1-\tau_1} = \frac{2}{(k-h^2)^{\tau_1}}\left(\int_0^l \beta_{10}^{\frac{1}{1-\tau_1}} dx\right)^{1-\tau_1} e^{-[\nu_1+\lambda(\tau_1-1)]t}$$

$$\frac{2}{a^{\tau_2}}\left(\int_0^l \beta_2^{\frac{2}{1-\tau_2}} dx\right)^{1-\tau_2} = \frac{2}{a^{\tau_2}}\left(\int_0^l \beta_{20}^{\frac{1}{1-\tau_2}} dx\right)^{1-\tau_2} e^{-[\nu_2+\lambda(\tau_2-1)]t}$$

Let us denote

$$\delta_{10} = \frac{2}{(k-h^2)^{\tau_1}}\left(\int_0^l \beta_{10}^{\frac{1}{1-\tau_1}} dx\right)^{1-\tau_1}$$

$$\delta_{20} = \frac{2}{a^{\tau_2}}\left(\int_0^l \beta_{20}^{\frac{1}{1-\tau_2}} dx\right)^{1-\tau_2}$$

If $\beta_{10} = \text{const}$ and $\beta_{20} = \text{const}$, then

$$\delta_{10} = \frac{2\beta_{10}}{(k-h^2)^{\tau_1}}\, l^{1-\tau_1}, \qquad \delta_{20} = \frac{2\beta_{20}}{a^{\tau_2}}\, l^{1-\tau_2}$$

In view of (5.108) and (5.109) this corresponds to the case in which the coefficients β_1, β_2 of the non-linear function (5.107) depend only on time.

Next let us denote

$$\delta = \max(\delta_{10}, \delta_{20}), \quad \nu = \min(\nu_1, \nu_2), \quad \tau = \max(\tau_1, \tau_2)$$

(5.120) $$b(t) = \delta e^{-[\nu + \lambda(\tau - 1)]t}$$

Then the inequality (5.119) takes the form

(5.121) $$2\int_0^l \left(hu + \frac{\partial u}{\partial t}\right) f\,dx \leq \varrho + b(t)\varrho^{\tau}$$

From (5.114) and (5.121) we have

(5.122) $$\gamma = 1 \quad , \qquad \omega(t,\varrho) = b(t)\varrho^{\tau}$$

Let us denote (see (5.115))

(5.123) $$\lambda = w(t) = \gamma - 2h = 1 - 2h = \text{const} \neq 0$$

In view of (5.123) the differential inequality (5.116) and the integral inequality (5.117) take the form

(5.124) $$\dot{\varrho} \leq (1-2h)\varrho + b(t)\varrho^{\tau}$$

(5.125) $$\varrho \leq c + \int_0^t \left[(1-2h)\varrho + b(s)\varrho^{\tau}\right] ds \quad , \quad c = \varrho(0)$$

These inequalities have the same form as in the point 5.5 (see (5.85) and (5.86)).

Now we shall apply theorem 1 of the part 4 (p.4.2). In the same way as in the point 5.5 we arrive to the conclusion, that the functions $v(t)$ and $g(z)$ can be taken in the form

(5.126) $$v(t) = \delta e^{-\nu t} \quad , \qquad g(z) = z^{\tau}$$

For further discussion we denote

(5.127) $$\mu(t) = \int_0^t v(s)\,ds = \frac{\delta}{\nu}\left(1 - e^{-\nu t}\right)$$

On the same basis as in the point 5.5 we conlude, that all results obtained in the theorems 2 - 5 (part 4, p.4.2) are excluded, because it is $w(t) = \lambda = 1 - 2h =$ const and therefore the integral

$$\int_0^\infty w(t)\,dt$$

is divergent, which makes the theorems 2 - 5 useless. But in the considered case the integral inequality (5.125) was obtained from the differential inequality (5.124), and assuming that it is

$$w(t) = -\alpha(t) = 1-2h < 0 \quad , \quad \alpha = 2h-1 > 0 \tag{5.128}$$

we obtain the same problem as for the differential and integral inequalities (4.45) and (4.46). Therefore in the considered case we can apply one of the theorems 3a - 5a, because in view of (5.127) we have

$$\mu(\infty) = \int_0^\infty v(t)\,dt = \frac{\delta}{\nu} < \infty \tag{5.129}$$

Since it is $0 < r < 1$ therefore we have

$$\int_0^A \frac{ds}{s^r} = \frac{A^{1-r}}{1-r} < \int_A^\infty \frac{ds}{s^r} = \infty \tag{5.130}$$

Thus we conclude that only the result based on the theorem 3a is obtainable in the case under consideration.

Let us now determine this result, that is construct effectively the estimation (4.51) in a closed form.

We shall first determine the function (4.16)

$$G(u) = \int_{u_0}^u \frac{ds}{s^r} = \frac{s^{1-r}}{1-r}\Big|_{u_0}^u = \frac{u^{1-r} - u_0^{1-r}}{1-r} = \lambda$$

where $0 < r < 1$.

The inverse function G^{-1} has the form

$$G^{-1}(\lambda) = \left[\lambda(1-r) + u_0^{1-r}\right]^{\frac{1}{1-r}} \tag{5.131}$$

The term within the square brackets in (4.51), taking into account (5.127) has the form

$$(5.132)\quad G(c)+\int_0^t v(s)ds=\int_{u_0}^{c}\frac{ds}{s^{\tau}}+\frac{\delta}{\nu}(1-e^{-\nu t})=\frac{c^{1-\tau}-u_0^{1-\tau}}{1-\tau}+\frac{\delta}{\nu}(1-e^{-\nu t})$$

Therefore in view of (5.131) and (5.132) the inequality (4.51) takes the form

$$\varsigma\leq\left[\left(\frac{c^{1-\tau}-u_0^{1-\tau}}{1-\tau}+\frac{\delta}{\nu}(1-e^{-\nu t})\right)(1-\tau)+u_0^{1-\tau}\right]^{\frac{1}{1-\tau}}e^{(1-2h)t}$$

$$(5.133)\quad \varsigma\leq\left[c^{1-\tau}+\frac{\delta}{\nu}(1-\tau)(1-e^{-\nu t})\right]^{\frac{1}{1-\tau}}e^{(1-2h)t}=\Phi(t)$$

where in view of (5.128) it is $1-2h<0$.

It is evident, that the inequality (5.133) holds for $t\in[0,\infty)$ and for $c\in(0,\tilde{c})$ where $\tilde{c}$ can be arbitrarily large.

For comparison, we shall consider a linear case in which $r_1=r_2=1$, in function (5.107). We assume in this case that the coefficients β_1, and β_2 are the functions of time only, because the dependence on this variable is here most important and interesting. Therefore in the considered case the function (5.107) has the form

$$(5.134)\qquad f=\beta_1(t)u+\beta_2(t)\frac{\partial u}{\partial x}$$

The functions (5.108) and (5.109) have the form

$$(5.135)\qquad \beta_1(t)=(\beta_{10}e^{-\nu_1 t})^{1/2},\qquad \beta_2(t)=(\beta_{20}e^{-\nu_2 t})^{1/2}$$

where β_{10} and β_{20} are constants.

On the basis of (5.120) we have for $r=1$

$$(5.136)\qquad b(t)=\delta e^{-\nu t}$$

Thus, the inequality (5.121) takes the form

$$(5.137)\quad 2\int_0^L\left(hu+\frac{\partial u}{\partial t}\right)f\,dx\leq\varsigma+b(t)\varsigma$$

From (5.122) it follows

(5.138) $\quad \gamma = 1 \quad , \quad \omega(t, \varrho) = b(t)\varrho$

Denoting

(5.139) $\quad \lambda = w(t) = \gamma - 2h = 1 - 2h = const \neq 0$

we get the inequalities (5.124) and (5.125) in the form

(5.140) $\quad \dot{\varrho} \leqslant (1-2h)\varrho + b(t)\varrho$

(5.141) $\quad \varrho \leqslant c + \int_0^t [(1-2h)\varrho + b(s)\varrho]ds \quad , \quad c = \varrho(0)$

Now we shall apply theorem 1 of the part 4 (p.4.2).We write the relation (4.14)

(5.142) $\quad b(t)\, z\, e^{\lambda t} e^{-\lambda t} \leqslant v(t)\, g(z)$

Thus,on the basis of (5.136) we can take the functions v(t) and g(z) in the form

(5.143) $\quad v(t) = b(t) = \delta e^{-\nu t} \quad , \quad g(z) = z$

Since it is $r = 1$ therefore we have

(5.144) $\quad \int_0^A \frac{ds}{s} = \int_A^\infty \frac{ds}{s} = \infty$

Because it is $w(t) = \lambda = 1 - 2h = const$,which provoke that the integral

$$\int_0^\infty w(t)\, dt$$

is divergent,therefore the theorems 2 - 5 (part 4,p.4.2) cannot be applied in the considered case.But the integral inequality (5.141) was obtained from the differential inequality (5.140),therefore assuming that it is

(5.145) $\quad w(t) = -\alpha(t) = 1 - 2h < 0 \quad , \quad \alpha = 2h - 1 > 0$

we obtain the same problem as for the differential and integral inequalities (4.45) and (4.46). As it is (see (5.127))

$$(5.146)\quad \mu(t)=\int_0^t v(s)ds=\frac{\delta}{\nu}(1-e^{-\nu t})\ ,\ \mu(\infty)=\int_0^\infty v(t)dt=\frac{\delta}{\nu}<\infty$$

therefore on the basis of (5.144) and (5.146) the result based on the theorem 5a is obtainable in the considered case. Let us find this result, that is construct effectively the estimation (4.51) in a closed form.

We shall first determine the function (4.16)

$$(5.147)\qquad G(u)=\int_{u_0}^{u}\frac{ds}{s}=\ln\frac{u}{u_0}=\lambda$$

The inverse function G^{-1} has the form

$$(5.148)\qquad G^{-1}(\lambda)=u_0e^{\lambda}$$

The term within the square brackets in (4.51), taking into account (5.146), has the form

$$(5.149)\quad G(c)+\int_0^t v(s)ds=\int_0^c\frac{ds}{s}+\frac{\delta}{\nu}(1-e^{-\nu t})=$$
$$=\ln\frac{c}{u_0}+\frac{\delta}{\nu}(1-e^{-\nu t})$$

Therefore in view of (5.148) and (5.149) the inequality (4.51) takes the form

$$\rho\leq u_0\exp\left[\ln\frac{c}{u_0}+\frac{\delta}{\nu}(1-e^{-\nu t})\right]\exp(1-2h)t$$

$$(5.150)\quad \rho\leq c\exp\left[-\left((2h-1)t-\frac{\delta}{\nu}(1-e^{-\nu t})\right)\right]=\Phi(t)$$

where in view of (5.145) it is $2h-1>0$.

It is evident, that the inequality (5.150) holds for $t\in[0,\infty)$ and for $c\in(0,\tilde{c})$ where $\tilde{c}$ can be arbitrarily large. Moreover $M(c)\to 0$ for $c\to 0$.

It should be stressed here, that in the preceding case ($0<r<1$) β_1 and β_2 had to be functions of time of the type

$$\beta_1 = O\left(e^{-[\vartheta_1+\lambda(\tau_1-1)]t}\right) \quad , \quad \beta_2 = O\left(e^{-[\vartheta_2+\lambda(\tau_2-1)]t}\right)$$

and they could not be, for example, functions of variable x only, or constants, since the conditions of theorem 3a could not be satisfied. But in the linear case considered most recently ($r=1$) the functions β_1 and β_2 can be arbitrary continuous functions of time, not necessarily of the form (5.135), and they can be also constants, to obtain the result based on the theorem 5a.

Indeed, let us assume

$$(5.151) \qquad \beta_1 = \left(\beta_{10}\,\varphi_1(t)\right)^{1/2} \quad , \quad \beta_2 = \left(\beta_{20}\,\varphi_2(t)\right)^{1/2}$$

where β_{10} , β_{20} are positive constants and φ_1 , φ_2 are positive continuous functions. Then, denoting

$$(5.152) \qquad \varphi(t) = \sup_t \left(\varphi_1(t), \varphi_2(t)\right)$$

we get first of the relations (5.143) in the form

$$b(t) = \delta\,\varphi(t)$$

Therefore the estimation (5.150) takes the form

$$(5.153) \qquad \varrho \leq c\exp\left[-\left((2h-1)t - \delta\int_0^t \varphi(s)ds\right)\right] = \Phi(t)$$

If the condition

$$(5.154) \qquad \lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi(s)ds = \varphi_0 < \frac{2h-1}{\delta}$$

and in the case $b(t) = b_0 = \text{const}$ the condition

$$(5.155) \qquad \frac{2h-1}{b_0} > 1$$

is satisfied, then

$$\lim_{t\to\infty} \Phi(t) = 0 \tag{5.156}$$

Having the function $\Phi(t)$ we can easily find the estimation of the solution $u(x,t)$ of the differential equation (5.106). Indeed, on the basis of (5.111) we have

$$\varrho = \int_0^l \left[(k-h^2)u^2 + a\left(\frac{\partial u}{\partial x}\right)^2 + \left(hu + \frac{\partial u}{\partial t}\right)^2 \right] dx \leqslant \Phi(t)$$

$$a\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx \leqslant \Phi(t)$$

Using the inequality (3.33) we get

$$u^2(x,t) \leqslant \frac{l}{a}\Phi(t) \tag{5.157}$$

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