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THE PROBLEM OF AXIALLY SYMMETRIC FLOW OF IDEAL RIGID-PLASTIC HUBER-MISES MEDIUM WITHOUT ANY SPECIFIC HYPOTHESES

1. Introduction

The title problem is usually simplified by assuming Tresca yield condition and then reduced to a statically determined one by means of the Haar-Kármán hypothesis / Hencky [3] , Shield [4] / or to a kinematically determined one /Lippmann [5] /. Note that for most metals the Huber-Hencky-Mises yield condition "fits the data more closely than Tresca's" /[1], p. 21 /; this holds, especially if the associated flow rules are applied / [6] / .

In the present paper, in contradistinction to the known solutions, not only Huber-Hencky-Mises' yield condition not Tresca's is used, but also the Haar-Kármán hypothesis, or its substitute, are fully avoided. Instead of this, two new functions are introduced / cf. /4// as unknowns which allow to reduce the system of seven equations to that of two ones. Each of the two new quantities is both statical and kinematical. This feature enables us to find directly the proper solution, and not, as usually, its lower and /or upper estimations, as given by statical and/or kinematical approach respectively.

This idea of solving the problem seems to be new. It is true, that elimination of kinematical quantities and the scalar parameter λ presumably leads to similar results. However, this elimination has been suggested to be aimless and, to the author's knowledge, it has not led till now to any positive results. The objection against the latter method is that "the boundary values of stresses, and not their derivatives, are specified in a physical problem" / [1] p.263, [2] p. 13 /. In the present paper it will be shown that the said difficulty may be avoided since the necessary boundary values may be found by means of velocity equations. It will be shown, moreover, that the title problem may be led to the hyperbolic one. Let us recall that in the Shield solution hyperbolicity of the problem has been achieved only at the price of introduction of "an error of unknown magnitude" /by Tresca's condition and the Haar-Kármán hypothesis /cf. [1], p. 281/.

In spite of a rather complex form of the equations obtained, in view of the enormous progress of computing means in the last three decades the effective solution can surely be found.

2. The system of two equations resolving the problem

Let us consider the steady plastic flow as specified in the title through a curvilinear in its axial section perfectly smooth die of revolution. All body forces are neglected. Non-vanishing plastic flow is assumed within the whole medium volume under consideration. In the cylindrical coordinate system r, θ, z / z being the symmetry axis/ the basic relations have then the following form:

- geometrical relations after substitution of the associated flow rule (cf. /3/)

$$\varepsilon_r = u_{,r} = \lambda s_r \,, \quad \varepsilon_\theta = \frac{u}{r} = \lambda s_\theta \,, \quad \varepsilon_z = w_{,z} = \lambda s_z$$

$$2\gamma = u_{,z} + w_{,r} = 6\lambda\tau \qquad \lambda > 0 \qquad /1/$$

- two equilibrium equations

$$\sigma_{,r} + s_{r,r} + \tau_{,z} + \frac{s_r - s_\theta}{r} = 0, \quad \tau_{,r} + \sigma_{,z} + s_{z,z} + \frac{\tau}{r} = 0 \qquad /2/$$

- the Huber-Mises yield condition

$$(s_r - s_\theta)^2 + (s_\theta - s_z)^2 + (s_z - s_r)^2 + 6\tau^2 = 6k^2. \qquad /3/$$

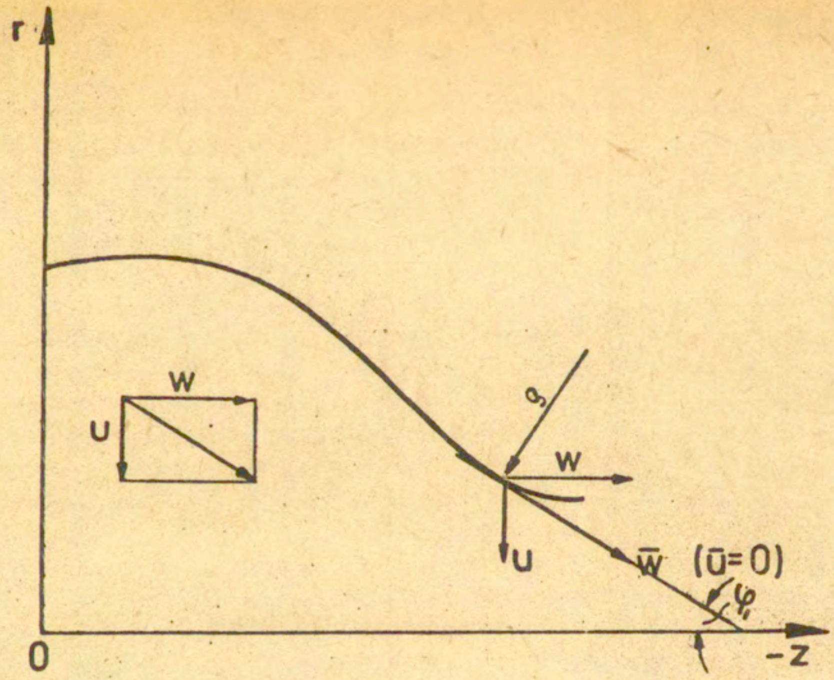

Fig. 1

Here u and w are the radial and axial velocity components respectively; ε_r, ε_θ, ε_z, γ, s_r, s_θ, s_z, τ are normal and shear components of the strain rate and stress deviator tensors respectively, referred to the said coordinate frame r, θ, z /subscripts $_{rz}$ of γ and τ are omitted/, σ denoting the mean normal stress and k being a material constant; λ /see /1// stands for a scalar parameter. Comma in subscript denotes differentiation with respect to the indicated variable.

No simplification of the problem will be made except the usual - also in analogical plane problems - assumptions that body forces are negligible, that all the functions considered, including that describing the axial section of the die, are of the class C^2 and that $u \neq 0$ for $r \neq 0$.

Let us introduce the following two unknown functions intended to be the clue of the solution:

$$\alpha = s_z / s_\theta \qquad \beta = \tau / s_\theta \qquad /4/$$

and denote further:

$$A = -\int \frac{\alpha_{,z}}{r}\, dr\ , \quad b \sin 2\varphi = 2\tau,\ d = s_z - s_r \qquad /5/$$

$$2m = 2\alpha + 1\ ,\ n = \beta^2 + \alpha^2 + \alpha + 1\ ,\ N = \alpha^2 + \alpha + 1\ ,$$

where φ is the inclination angle of the principal line of maximum stress with respect to the z axis, $-\frac{\pi}{2} < \varphi \leqslant \frac{\pi}{2}$ /cf. Fig. 1/.

By eliminating λ from /1/ we have for $r \neq 0$:

$$\frac{du}{u} = (s_r / s_\theta)\ \frac{dr}{r}\ . \qquad /6/$$

Then, integrating and taking into account that $s_r \to s_\theta$ and $u \to 0$ for $r \to 0$ and that $s_r = -s_z - s_\theta$, one obtains from $\gamma_{,rz} = \gamma_{,zr}$ /cf. /1/ / after cumbersome calculations

$$r\alpha_{,rr} - r\alpha_{,zz} - 2(\alpha + 1)\alpha_{,r} + (2\beta - 2Ar)\alpha_{,z} + \frac{2\alpha}{r}(\alpha + 2) = 2r\beta_{,zr} - 2Ar\beta_{,r} - 2\beta_{,z} - 2A\beta\ . \qquad /7/$$

Another equation in α and β can be obtained from / 2 /. Differentiation of the first one of these equations with respect to z and of the second one with respect to r and, then, subtraction from one another eliminate σ as to give

$$(s_r - s_z)_{,rz} + \tau_{,zz} - \tau_{,rr} + \frac{1}{r}\left[(s_r - s_z)_{,z} - \tau_{,r} + \frac{\tau}{r}\right] = 0 \qquad /8/$$

To obtain from /8/ the relation in terms of α and β, presenting with /7/ the system of equations, we should use /3/ taking into account that α and β impose, from their definition / cf. /4//, the orientation of s_z, s_r, s_θ with respect to the coordinate frame, whereas otherwise these components may be interchanged /cf. / 3/ /.

From the Mohr's circle we find /cf. / 5/ /

$$d^2 = b^2 - 4\tau^2 \qquad /9/$$

By elimination of s_r equation /3/ may be reduced to

$$\tau^2 = k^2 - \frac{E}{3} \qquad /10/$$

where $E = C^2 - CD + D^2$, $C = 2s_\theta + s_z$, $D = 2s_z + s_\theta$ or / cf. /4/, /5/ / $E = 3N s^2_\theta$ and $s^2_\theta = \frac{1}{3}(4k^2 - d^2 - 4\tau^2)$.

Substitution of the preceding relations into the last one and then into /10/ gives

$$\tau^2 = k^2 - \frac{1}{3} N(4k^2 - d^2 - 4\tau^2) \qquad /11/$$

where τ is determined by the relation

$$n\tau^2 = k^2\beta^2 \qquad /12/$$

From /11/ we have

$$d = \pm 2\, mn^{-\frac{1}{2}} k \qquad /13/$$

These relations and the well-known transformation formulae enable us to express the Eqs. / 2/ and /3/ as a relation between α and β . Its principal part is simpler after subtraction of Eq / 7/ multiplied by $\tau_{,\alpha}/r$ and then takes the form

$$-d_{,\alpha}\alpha_{,rz} + \tau_{,\beta}\beta_{,rr} - (d_{,\beta} + 2\tau_{,\alpha})\beta_{,rz} - \tau_{,\beta}\beta_{,zz} +$$

$$+\left[\tau_{,\alpha r} - d_{,\alpha z} + \frac{1}{r}\tau_{,\alpha}(2\alpha + 1)\right]\alpha_{,r} - \left[\tau_{,\alpha z} + \frac{1}{r}d_{,\alpha} + \right.$$

$$\left. + \left(\frac{\beta}{r} - A\right) 2\tau_{,\alpha}\right]\alpha_{,z} - \left(4A\,\tau_{,\alpha} - \tau_{,\beta r} + d_{,\beta z} + \frac{1}{r}\tau_{,\beta}\right)\beta_{,r} + \qquad /14/$$

$$- \left(\tau_{,\beta z} + \frac{2}{r}\tau_{,\alpha} + \frac{1}{r}d_{,\beta}\right)\beta_{,z} + \left[(\alpha^2 + \alpha - Ar)/r^2\right] 2\tau_{,\alpha} + \tau/r^2 = 0$$

Taking into account that $f_{q} = f_{,\alpha}\alpha_{,q} + f_{,\beta}\beta_{,q}$, where f is an arbitrary function of the class C^1 and q stands for r and/or z, on multiplying by $n^{7/2}k^{-1}$, we have

$$\pm(2n - m)n^2\alpha_{,rz} + (n^2 \pm \beta^2)n\,\beta_{,rr} \mp (1 + 2\beta)n^2 m\,\beta_{,rz} +$$

$$+ (-n^2 \pm \beta^2)\,n\,\beta_{,zz} + (\mp 6m^2 \pm n)\,n\beta\alpha_{,r}^2 + (\mp nm \pm n + 3)n\,\alpha_{,r}\alpha_{,z} +$$

$$+ (\mp 6\beta \pm n)nm\,\alpha_{,r}\beta_{,r} + \left(2n\beta \mp \frac{3}{2}m\beta\right)n\,\alpha_{,r}\beta_{,z} + (\mp n \pm 3m^2)n\,\alpha_{,z}\beta_{,r} +$$

$$+ (+n^2 \pm 5\beta^2 \pm 6n\beta \mp n^2)m\,\alpha_{,z}\beta_{,z} \pm (6m^2 - n)\,n\beta\alpha_{,z}^2 + \qquad /15/$$

$$+ (-n^2 \pm 5\beta^2 \mp 2n)\beta\beta_{,r}^2 \pm 3nm\,\beta\beta_{,r}\beta_{,z} +$$

$$+ (n^2 \pm 5\beta^2 \mp 2n)\beta\beta_{,z}^2 \pm \frac{2}{r}n^2 m^2\beta\alpha_{,r} - \left[\mp 2n \pm m + (\beta - Ar)\cdot\right.$$

$$\left.(\pm 2\beta m)\right] n^2 r^{-1}\alpha_{,z} - (\pm 4nm\beta A + n^2 r^{-1} \pm \beta^2 r^{-1})\,n\,\beta_{,r} +$$

$$+ (\mp 2\beta \mp 1)\,n^2 m r^{-1}\,\beta_{,z} \pm \left[2m(\alpha^2 + \alpha - Ar) - n\right]\beta n^2 r^{-2} = 0 \quad 1/$$

/cf. 4 and 5 /.

In such a way the system of two partial integro-differential equations / A being the integral , cf. / 5// in α and β , namely / 7/ and /15/ or , equivalent ones , / 7/and /14/, more convenient for further analysis, has been derived. Each of these equations is quasi-linear. Since the integrals A may be represented by functions without second-order derivatives and in any case do not enter principal parts of the equations, the said parts are rather simple. However, no second order derivative

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1/ Lower signs apply to hardly possible case $\alpha > -\frac{1}{2}$ only. /It has been assumed $s_z > 0 > s_r$, whence $d > 0$. /

can be eliminated from any of these equations. In view of the rather unusual treatment of the problem in the present paper, some general remarks referring to the final form of the solution we are looking for should be made in advance.

After having solved the system 7 and 14 /its boundary conditions are given in Sec. 3. and its hyperbolicity is analyzed in Sec. 4. /the functions $\alpha(r,z)$, $\beta(r,z)$ being given explicitely, one may determine the stress deviator components at each point consecutively from the relations: /4/,/5/,/11/ with the preceding one., /12/ and /13/.

The mean stress σ is to be evaluated by integrating the equilibrium equations /2/ for the prescribed boundary conditions /cf. Sec. 3./.

The velocity components u and w can be found, in the simplest way, by solving the following partial differential equations system /with appropriate boundary conditions /:

$$u_{,z} + w_{,r} = \tan 2\varphi(u_{,r} - w_{,z})$$

$$u_{,r} + w_{,z} + \frac{u}{r} = 0, \qquad /16/$$

where $\tan 2\varphi$ is known /see the last line of Sec. 3./. The both equations follow from the associated flow rule /cf./1//, the latter determining simply the material incompressibility.

3. Boundary conditions

The boundary conditions of the system /7/ and /14/ clearly should be formulated in terms of our new unknowns /α and β /, that is by the values of α , β , $\alpha_{,r}$, $\alpha_{,z}$, $\beta_{,r}$, $\beta_{,z}$ on the boundary. However, the analysis, necessary due to the "lack" of cinditions, cf. Sec. 1., is much simpler for the velocity components $\bar{u}$ and $\bar{w}$ in direction normal and tangential to the side of the die /channel / respectively.

The original boundary conditions of the title problem are of the mixed type, namely, for a perfectly smooth and rigid die,

$$\bar{u} = 0 \qquad \tau_{\bar{r}\bar{z}} = 0 \qquad /17/$$

that is, we have both the kinematical /17.1/ and statical condition /17.2/ [2/].

Here and below $\tau_{\bar{r}\bar{z}}$ is the shear stress in directions $\bar{r}$ and $\bar{z}$, the reference frame $\bar{r}$, $\bar{z}$ is the plane system r , z rotated by the angle φ_1 of the slope of the side with respect to the axis z, In view of /17.2/ and /1/ the directions of $\bar{u}$ and $\bar{w}$ are those of principal lines, that is, $\varphi_1 = \varphi$ /cf. /5/ , /16/ /.

Thus from the last but one equation of /1/ we have

$$\bar{u}_{,\bar{z}} + \bar{w}_{,\bar{r}} = 0 \qquad /18/$$

The two expressions of the angular velocity, namely

$\omega = \frac{1}{2}(\bar{u}_{,\bar{z}} - \bar{w}_{,\bar{r}})$ and $\omega = \frac{\bar{w}}{\rho}$ give, in view of /18/

$$\bar{u}_{,\bar{z}} = \frac{\bar{w}}{\rho} \quad \text{and} \quad \bar{w}_{,\bar{r}} = -\frac{\bar{w}}{\rho} ,$$

where ρ is the meridional curvature radius of the die at the given point. The consideration of the centripetal acceleration confirms only the statement, giving no other conditions.

The boundary conditions /17/ , as the analysis given shows, do not determine fully the problem since one quantity, say $s_{\bar{r}} / s_{\theta}$, may be at this stage chosen arbitrarily. However, by taking in consideration the principle of least work and the flow along the streampline , [12] , we obtain the necessary missing cindition on the boundary, what is quite sufficient for the full formulation of the boundary conditions. The said condition states that if the directions of the tangents of the stream line and of

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2/ The conditions /17/hold also for a smooth deformable die if the flow process is steady. However, in this case the deformed die contour should be taken into considerations from their beginning. In principle, friction could be also accounted for.

the principal line coincide at a given point, two principal stresses at this point are equal to each other, or in other words, in such a case the Haar-Kármán hypothesis is strictly satisfied [12] . Alternatively, we may find the same condition in a less rigorous way, without referring to [12] , by using the well-known case of the radial flow through the apex of the cone, for which the principle of the least work is satisfied for any included angle of the cone. In other words, in the latter case we use the same principle, applied in the whole region to give the deduction for the boundary by the inverse method. The latter deduction should, however, be extended to curvilinear boundary. It is done by means of cones of various included angles, our curvilinear boundary being the envelope of the generatrices of those cones. This deduction is fully consistent with the proof given in [12] .

Thus we have $s_{\bar r} = s_\theta$ whence it follows from /1/

$$\bar u_{,\bar r} = \frac{u}{r} \qquad /19/$$

On the other hand it follows from /1/ also the medium incompressibility condition /16.2/ . Notice that $\bar u_{,\bar r} + \bar w_{,\bar z} = u_{,r} + w_{,z}$, since either of these members presents the trace of the strain rate tensor in the meridional plane at the given point, that is, the first invariant of the tensor.

By substituting the last equation into /16.2/ and then subtracting /19/ we obtain

$$\bar w_{,\bar z} = - \frac{2u}{r} \qquad /20/$$

Since $u = \bar w \sin\varphi$ / cf. /17.1/ , see also fig. 1/, it follows $\bar w_{,\bar z} = -2(\bar w/r)\sin\varphi$. Thus $\bar w$ is determined by its boundary value $\bar w_0$ for $z = 0$ and the ordinary differential equation with separated variables $d\bar w/\bar w = -2\sin\varphi\, r^{-1}\, dz$.

It follows $d\bar w/\bar w = -2r'(1 + r'^2)^{-\frac{1}{2}}\, dz$, where $r(z)$ and the derivative $r'(z)$ are given . Thus $\bar w$ may be found along the boundary. It is needed for solving the boundary problem determined by the system /16/ .

Let us show that, in general, the six quantities α, β, $\alpha_{,r}$, $\alpha_{,z}$, $\beta_{,r}$, $\beta_{,z}$ constituting boundary values may be found directly as the functions of r, $\varphi(r,z)$, $\rho(r,z)$. From /4/, /5/, /19/, /20/ we have

$$\alpha = 3\sin^2\varphi - 2 \qquad /21/$$

Moreover, from /19/ using formula $\varphi_{,r} = \varphi_{,\bar{r}}\,\bar{r}_{,r} + \varphi_{,\bar{z}}\,\bar{z}_{,r}$ and analogous one for $\varphi_{,z}$ we have $\varphi_{,r} = (\sin\varphi)/r$ and $\varphi_{,\bar{z}} = 1/\rho$ /cf. fig. 1/. It follows that

$$\alpha_{,r} = 3\sin 2\varphi \sin\varphi \left[(\cos\varphi)/r + 1/\rho\right] \qquad /22/$$

and analogously

$$\alpha_{,z} = 3\sin 2\varphi\left[(\cos\varphi)/\rho + \sin^2\varphi/r\right] \qquad /23/$$

Similarly, since $\beta = \alpha\tau/s_z$ and $\tau = (s_r - s_z)\sin 2\varphi$, on substituting /21/ we have

$$\beta = 3(1 - 2\sin^2\varphi)\sin 2\varphi \qquad /24/$$

$$\beta_{,r} = 3(2 - 3\sin^2 2\varphi)\sin\varphi\left[(\cos\varphi)/r + 1/\rho\right] \qquad /25/$$

$$\beta_{,z} = 3\left[2\cos 2\varphi(1 - 2\sin^2\varphi) - 2\sin^2 2\varphi\right]\left[(\cos\varphi)/\rho + (\sin^2\varphi)/r\right] \qquad /26/$$

Thus all boundary conditions may be found from /21/ - /26/, since purely geometrical quantities φ, r and ρ are supposed to be known functions of z.

All considered quantities have been assumed to be represented by functions of the class C^2. In the cases, when the region under consideration contains some subregions with discontinuities on their boundaries, it is necessary to consider individually these subregions. To this end their boundaries, in general, should be assumed on the basis of respective known solutions including those for the plane strain. It should be noted that some boundaries of the subregions are the slip lines, that is, are inclined with respect to principal lines at the angle $\frac{\pi}{4}$ (cf. /39/).

For $\gamma \neq 0$, where γ denotes the angle made by principal line with a stream line (cf. [12], p. 705, where this angle is denoted by α) at any point respective principal stresses are not equal to each other. In such a case, however, similarly as in the case of friction on the boundary previously mentioned, values of the principal deviatoric stresses (s_i i = 1,2,3) may be found by means of /5/ ,/5'/ ,/5''/ ,(ibidem) whence we have

$$s_1/s_2 = 1/\ (\ 2 - 3\cos^2\gamma\ ) - 1 \ . \qquad /27/$$

The slopes of the streamlines may be found by means of the hodograph (cf. [6], p. 146-7) , whereas the inclination angle φ of the principal line with respect to z axis is determined at each point, in view of /12/ and /13/ , by $\tan 2\varphi = \pm 2\beta(mn)^{-1}$ /cf. /5// .

4. Determination and control of the type of the equation system.

The general problem in its original form, according to [1] and [10], is not hyperbolic. It will be shown, however, that this statement may not hold in the case of system of equations /7/ and /14/ in our unknowns α and β which are defined as ratios of the original unknown functions.

The type of a system of partial differential equations of the second order in any case depends only on the coefficients in the principal parts of the equations. Thus, let us apply the procedure, as presented in [8] for a system of linear equations of the second order, to our quasi-linear system /7/ and /14/. Accordingly, we find the characteristic determinant D of our system.

The system is hyperbolic when the equation D = 0 (cf. /28/) has two real roots. After dividing the equation /7/ by r we obtain this determinant in the form

$$D = \begin{vmatrix} \xi^2 - \eta^2 & 2\xi\eta \\ d_{,\alpha}\xi\eta & \tau_{,\beta}\xi^2 - (d_{,\beta} + 2\tau_{,\alpha})\xi\eta - \tau_{,\beta}\eta^2 \end{vmatrix} \qquad /28/$$

where $\xi = \omega_{,r}$, $\eta = \omega_{,z}$ and $\omega(r,z) = 0$ determines the characteristic manifold.

By dividing D by η^4 (for $\eta \neq 0$) and introducing $\chi = \dfrac{\xi}{\eta}$

we have

$$\tau_{,\beta}\chi^4 - (d_{,\beta} + 2\tau_{,\alpha})\chi^3 + 2(d_{,\alpha} - \tau_{,\beta})\chi^2 + (d_{,\beta} + 2\tau_{,\alpha})\chi + \tau_{,\beta} = 0 \qquad /29/$$

This is a symmetric quartic equation, but signs of the odd power terms are distinct. Such form of the symmetry is omitted in many mathematical handbooks, it has been found, however, in [14].

To solve the equation /29/ or , as in our case, to determine the number of its real solutions we introduce $y = \xi - \frac{1}{\xi}$, whence we have $\chi^2 - y\chi - 1 = 0$ with positively determined discriminant $y^2 + 4$. The equation /29/ is then, as usually, reduced to a quadratic one in y with the discriminant

$$\Delta = \left[(d_{,\beta} + 2\tau_{,\alpha}) \,/2\tau_{,\beta}\right]^2 - (2d_{,\alpha}/\tau_{,\beta}) \qquad /30/$$

From /11/and following equations we obtain $\tau_{,\beta} = -N n^{-3/2} k$,
$\tau_{,\alpha} = \frac{1}{2}(2\alpha+1)\, n^{-3/2}\beta k$; $2d_{,\alpha}/\tau_{,\beta} = -8 + \frac{1}{N}\left[6 + 2/(\beta^2 + N)\right]$
$d_{,\beta} = \pm (2\alpha + 1)\, n^{-3/2}\beta k$ whence

$$\Delta = 8 - \left[24 N + (2\alpha + 1)^2\beta^2\right]/4N^2 - 2/(N\beta^2 + N^2) \qquad /31/$$

In view of the statements given above the condition of the hyperbolicity[3/] of our system is $\Delta > 0$. Equating the right member of /31/ to zero we get a quartic equation in α . This equation, however, may be split into two quadratic ones, similarly as symmetric quartic equation, by means of N (cf. /5/) . Thus we have $(2\alpha + 1)^2 = 4N - 3$, whence

$$4N^2 - 6N - 3\beta^2 = 0 \qquad /32/$$

with the roots N_1 and N_2 of different signs since $N_1 N_2 = -\frac{3}{4}\beta^2$. Thus,(cf. /5/) , the relation

$$\alpha^2 + \alpha + 1 = N_1 \qquad /33/$$

has two real roots α_1 and α_2 , where $\alpha_1 \alpha_2 < 0$ while its

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3/ The importance of the hyperbolicity will be pointed out in the Sec. 5.

variant with N_2, not written, has no real roots since $N_1 > 0$ and $N_2 < 0$, while $\alpha^2 + \alpha + 1 > 0$.
The equation /32/ gives a very simple criterion for N to ensure $\Delta > 0$ (cf. /31/). It states that the value of N should be outside the closed interval bounded by the two roots N_2, N_1. This interval in /32/ is clearly $\frac{1}{2}\sqrt{9 + 12\beta^2}$. Thus, substitution, instead of β, of its maximum value β_{max} extends (or does not change) the said interval and condition $\Delta > 0$ is ensured for $N > N_1$, since N_2 gives no real roots in /33/. By substituting this condition for N in /33/ we convert the latter into inequality and by applying the same criterion of the sign of the quadratic trinomial, as above used for N, this time for α (after substitution of β_{max} in /32/), we obtain, similarly as above for N, the conditions $\alpha > \alpha_{1m}$ or alternatively $\alpha < \alpha_{2m}$, where α_{1m} and α_{2m} are respective roots α_1 and α_2 calculated for $\beta = \beta_{max}$.

The conditions found, without further analysis, enable us to state merely that the problem, in general, may be hyperbolic, as one of the alternatives, in the part only of the considered region. However, the fact that - after substituting β^2 by its upper bound - these conditions do not depend explicitely on the values of β and that they are - in either alternatives - one-sided suggests that translation of α or, in other words, substituting it by

$$\bar{\alpha} = \alpha + a \qquad /34/$$

where a is a constant to be determined, may satisfy one of the conditions $\bar{\alpha} > \alpha_1$ or $\bar{\alpha} < \alpha_2$, in contrast to α, in the whole region. Thus, by substituting /34/ into /7/ and /14/ a new system of equations in $\bar{\alpha}, \beta$ is obtained, where the derivatives of $\bar{\alpha}$ and β, the characteristic determinant and the conditions of the hyperbolicity are the same as previously, that is, $\bar{\alpha} > \alpha_1$ or $\bar{\alpha} < \alpha_2$ (cf. /33/) α_1 and α_2 having the same values as those obtained for α.

On the other side the system for $\bar{\alpha}$ is clearly distinct from that for α in the following four aspects:

1. The suitable choice of a ensures the hyperbolicity of the problem in $\bar{\alpha}$ in the whole region of the interest.[4/]

2. The boundary values for $\bar{\alpha}$ are obviously different than for α and / 20/ is to be replaced by

$$\bar{\alpha} = 3\sin^2\varphi - 2 + a \quad . \qquad /35/$$

3. In / 7/ the last term of the left-hand side should be $\frac{2}{r}(\bar{\alpha} \mp a)(\bar{\alpha} - a + 2)$ instead of $\frac{2}{r}\alpha(\alpha + 2)$.

4. In /14/ the last but one term will have the form $(\bar{\alpha} - a)^2 + \bar{\alpha} - a$ instead of $\alpha^2 + \alpha$.

To analyze the question and to determine the minimum absolute value of a in /34/ we should find the upper bound of β^2 and compare conditions we have got from /33/ with the upper and/or lower bounds of actually possible values of α.

To the estimation of α the Haar-Kármán hypothesis might be used. However, since such estimate would ensure neither upper nor lower bound which are needed, it is much better to use other method without resort to the Haar-Kármán hypothesis. We shall use instead the relations deduced by minimizing the energy dissipated along an arbitrary stream line in the flow process. These relations ([12] p. 706 (5), (5'), (5")) after replacing the notation α by γ can be rewritten in the form

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4/ Such a transformation, in contrast to that of coordinate frame only, may, in general, change the type of differential equations and in our case it may be seen directly from [11], p. 33. Let us, namely, compare our transformation with the conditions of conservation of the type and characteristics of the systems of quasi-linear partial differential equations of the <u>first</u> order of the hyperbolic type in transformations of unknowns as stated in [11], pp. 31-33. These conditions are in our case <u>not</u> fulfilled, since the determinant of the transformation $\Delta_{\alpha} = \partial[\bar{\alpha},\beta]/\partial[\alpha,\beta] = 0$.

$$f = s_1 - s_2 = Q \cos 2\gamma; \quad g = s_2 - s_3 = \frac{Q}{2}(1 - \cos 2\gamma); \qquad /36/$$
$$h = s_3 - s_1 = -\frac{Q}{2}(1 + \cos 2\gamma),$$

where s_1, s_2, s_3 are principal deviatoric stresses, $Q = (2 - \frac{3}{2}\sin 2\gamma)^{-\frac{1}{2}}$ and γ is the angle made by the principal line with the stream line at the considered point.

From /36/ it follows $s_1 = (f - h)/3$, $s_2 = (g - f)/3$, $s_3 = (h - g)/3$ and thus ratios s_i/s_j and τ/s_j, where s_i, s_j denote any of s_1, s_2, s_3 are independent of Q, since the latter appears in both terms of each ratio. Observe that s_i/s_j determine bounds of α and β since $s_z < s_i$ for $s_i > s_j$. Thus, it is sufficient to analyze

$$s_1/Q = (f - h)/3Q = (3\cos 2\gamma + 1)/6; \quad s_2/Q = (1 - 3\cos 2\gamma)/6$$
$$s_3/Q = (h - g)/3Q = -\frac{1}{3} \qquad /37/$$

The relations 36 have been deduced for γ, s_1 and s_2 in meridional plane (cf. [12], p.705) and clearly due to this fact γ does not appear in /37/ in the expression for s_3/Q in contrast to those for s_1/Q and s_2/Q.

Thus, $s_3 = s_\theta$ and taking into account that s_1/Q and s_2/Q are monotone functions for $0 \leqslant 2\gamma \leqslant \frac{\pi}{2}$, being cosinusoids, by using $|s_z| \leqslant |s_1|$ if $|s_1| \geqslant |s_2|$ we find that in any case $\alpha_{max} \leqslant (s_1/s_3)_{max}$ and $\alpha_{max} \leqslant (s_2/s_3)_{max} = 1$ while $\alpha_{min} \geqslant (s_1/s_3)_{min} = -2$ and $\alpha_{min} \geqslant (s_2/s_3)_{min}$.

To allow for the whole possible range of values of α we choose two suitable conditions and we get $-2 \leqslant \alpha \leqslant +1$. Observe that in the case of extruding signs of both s_1 and s_3, in general, are changed and therefore their ratio has the sign unaltered.

Further we have $\tau_{max} = (f/2)_{max} = \frac{Q}{2}$ (from /34/ for $\gamma = 0$ similarly as in preceding result), whence $\beta_{max} = \frac{f}{2s_3} = -\frac{3}{2}$. Substituting this value in /34/ we obtain

$$4 N^2 - 6 N - \frac{27}{4} = 0 \qquad /38/$$

with the roots $N_1 = 9/4$ and $N_2 = -\frac{3}{4}$.
Thus, $\alpha^2 + \alpha + 1 = 9/4$, whence $\alpha_1 = + 0.7247$ and $\alpha_2 = - 1.7247$.
From the limit values of α we _ nd that the needed translation a decreasing α ($a < 0$, cf. /34/) is determined by the condition $a \leqslant -[1 - (-1,7247)] = - 2,7247$ or, in integers, $a \leqslant - 3$, whereas for increasing α ($a > 0$) the condition is $a \geqslant -(-2) + 0.7247 = + 2.7247$ or, in integers, $a \geqslant + 3$, that is, a has the same absolute value as for α_2.

Let us note that our characteristics, in contrast to those of the plain strain, in general, are not the slip lines and the latter have the direct physical meaning, being occasionally discontinuity lines for tangential velocity components.

On denoting the slope of the slip lines with respect to z axis at any point by ψ we have $\psi = \varphi \pm \frac{\pi}{4}$ and in view of the relation given in the last line of the Sec. 3., we get (cf. /5/)

$$\cot 2\psi = \mp 2\beta\, mn^{-1} \qquad /39/$$

The existence of the solution follows, in our case, indirectly, since it has been proved for a system of first-order quasi-linear hyperbolic partial differential equations ([11], pp.65-70), to which our system may be led, and the corresponding boundary problem is the same for the both systems (cf. [8]). To this conclusion clearly leads the remark that our system may be considered as an intermediate stage of the transformation of one equation of the fourth-order into the system of four equations of the first-order.

The uniqueness of the solution presumably may be shown by means of the generalization of the theorem of Szmydt (of, [13]) formulated for canonical system, at least with the extension of the assumptions onto characteristics and their slopes. This is, however, beyond the scope of the present paper.

5. Final conclusions and notes.

In the present paper three main points may be emphasized. They have been considered in Sections 2, 3 and 4 respectively.

1. Reduction of the general system of the seven equations of the title problem to that of two ones by means of, among other things, introduction of new unknowns α and β.

2. Determination of necessary boundary values for posing boundary value problem in the case of hyperbolic system. The said determination is effected, among other things, by using results obtained by the author in his previous paper [12].

3. Reduction of the type of integro-differential equation system to a hyperbolic one in the whole region under consideration by means of suitable transformation of α, β into $\bar{\alpha}, \bar{\beta}$.

Let us recall that all the difficulties specified above have been avoided by Shield, his results, however, refer to Tresca's yield condition and assumption of the Haar-Kármán hypothesis. In the present paper no simplification of the problem by replacing Huber-Hencky-Mises' yield condition by Tresca's and by assumption of Haar-Kármán hypothesis or any its substitute is used.

Moreover, even if errors introduced by Tresca condition and the Haar-Kármán hypothesis were not occurring, Shield's method would give lower and/or upper estimation only, not the proper solution which may be obtained by the method shown in the present paper. In connection with this, the solution in velocities, as addition to static solution, is needed only, if one is interested in velocities and/or for verification.

With respect to the hyperbolicity of the problem it should be noted that it is highly desirable from following reasons:

a/ It enables the method of characteristics to be used (cf. [1], [3]).

b/ In the case of the ellipticity, that is, in the case of the main alternative of the hyperbolicity, we would have, in general, no uniqueness of the solution and some instability would occur, because our domain is not bounded ([15], [9]).

c/ Moreover, the elliptic problem is inconsisting with both function and its derivative being given on the boundary, as in our case ([9] , p. 142).

d/ In general case, we have hyperbolicity in a part only of the region under consideration and this fact may involve more cumbersome calculations than for any single solution.

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