

COMPTES RENDUS DES SÉANCES
DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE VARSOVIE.

Classe III.

XXIII Année 1930

Fascicule 7—9

SPRAWOZDANIA
z posiedzeń
TOWARZYSTWA NAUKOWEGO
WARSZAWSKIEGO

Wydział III
nauk matematyczno-fizycznych

Rok XXIII 1930

Zeszyt 7—9



WARSZAWA
NAKŁADEM TOWARZYSTWA NAUKOWEGO WARSZAWSKIEGO
Z ZASIŁKU MINISTERSTWA WYZNAŃ RELIGIJNYCH I OŚWIECENIA PUBLICZNEGO

1930

DRUK
ZAKŁADÓW
GRAFICZNO-
INTROLIGATORSKICH
J. DZIEWULSKI s. z o. o.
WARSZAWA, ZŁOTA 29

TREŚĆ ZESZYTU 7—9.

(Table des matières).

	Str.
L. Szperl. O działaniu siarkowodoru na chlorek o-ftalilu	155
L. Szperl i H. Morawski. O działaniu siarkowodoru na chlorek naftalilu	156
W. Lampe, M. Trenknerówna i S. Lipski. Nowa synteza pochodnych dwucynamioilometanu	156
Stefan Mazurkiewicz. O obrazach ciągłych zbiorów linjowych . .	158
V. W. Adkisson. O krzywych cyklicznie spójnych, których obszary uzupełnienia mają brzegi homeomorficzne z zachowaniem punktów rozgałęzienia	164
Antoni Morawiecki. Fosforyty północne Polski	194
Michał Kamiński. Ruch komety Wolfa I pod wpływem planety Urana w okresie 1884—1919.	199
—	
L. Szperl. Sur l'action de l'hydrogène sulfuré sur le chlorure d'o-phtalyle	155
L. Szperl, H. Morawski. Sur l'action de l'hydrogène sulfuré sur le chlorure de naphtalyle	156
W. Lampe, M. Trenknerówna, S. Lipski. Nouvelle synthèse des dérivées du dicinnamoyleméthane	156
Stefan Mazurkiewicz. Sur les images continues des ensembles linéaires	159
V. W. Adkisson. Cyclicly connected continuous curves whose complementary domain boundaries are homeomorphic, preserving branch points	164
Antoni Morawiecki. Les phosphorites de la partie septentrionale de la Pologne	198
M. Kamiński. Über den Einfluss des Planeten Uranus auf die Bewegung des Kometen Wolf I in dem Zeitraume 1884—1919. 199	199

SPRAWOZDANIA Z POSIEDZEŃ
TOWARZYSTWA NAUKOWEGO WARSZAWSKIEGO

Wydział III nauk matematyczno-fizycznych.

Posiedzenie

z dnia 11 października 1930 r.

L. Szperl.

o działaniu siarkowodoru na chlorek o-ftalilu.

Zgłoszono dnia 11 października 1930 r.

**Sur l'action de l'hydrogène sulfuré sur le chlorure
d'o-phtalyle.**

Mémoire présenté dans la séance de 11 Octobre 1930.

Streszczenie.

Chlorek o-ftalilu, podobnie, jak po raz pierwszy poddany przez autora współdziałaniu z siarkowodorem chlorek benzoilu (R. Ch. 10, 510, 1930), reaguje z tym gazem w temperaturze 195 — 200°. Z otrzymanych produktów, oprócz chlorowodoru i kwasu o-ftalowego, zostały wyodrębnione dwa związki, a mianowicie: siarczek o-ftalilu i dwusiarczek dwu-o-ftalilu o temp. topnienia 330 — 331,5°.

Praca wyjdzie in extenso w „Rocznikach Chemji”.

L. Szperl i H. Morawski.

O działaniu siarkowodoru na chlorek naftalilu.

Zgłoszono dnia 11 października 1930 r.

Sur l'action de l'hydrogène sulfuré sur le chlorure de naphthalyle.

Mémoire présenté dans la séance de 11 Octobre 1930.

Streszczenie.

Ze studjów nad działaniem siarkowodoru na chlorek naftalilu (chlorobezwodnik kwasu naftaleno-1, 8-dwukarboksyłowego) zostały otrzymane rezultaty następujące: chlorek naftalilu, poddany działaniu siarkowodoru w temp. około 100° , zmienia się na masę smolistą. Gdy tenże chlorek w roztworze ksylenu ogrzewać w temp. $80 - 86^{\circ}$, lub w roztworze benzenu do wrzenia, to wytwarza się krystaliczny tiobezwodnik kwasu naftalowego barwy pomarańczowej, nie notowany dotychczas w literaturze. Związek ten w temp. około 200° poczyna się rozkładać, szybko ogrzany, topnieje w temp. $205 - 206^{\circ}$. Łatwo ulega hydrolizie pod wpływem gorących roztworów sody, ługów, amonjaku.

Praca wyjdzie in extenso w „Rocznikach Chemji”.

W. Lampe, M. Trenknerówna i S. Lipski.

Nowa synteza pochodnych dwucynamoilometanu.

Komunikat przedstawiony dn. 11 października 1930 r.

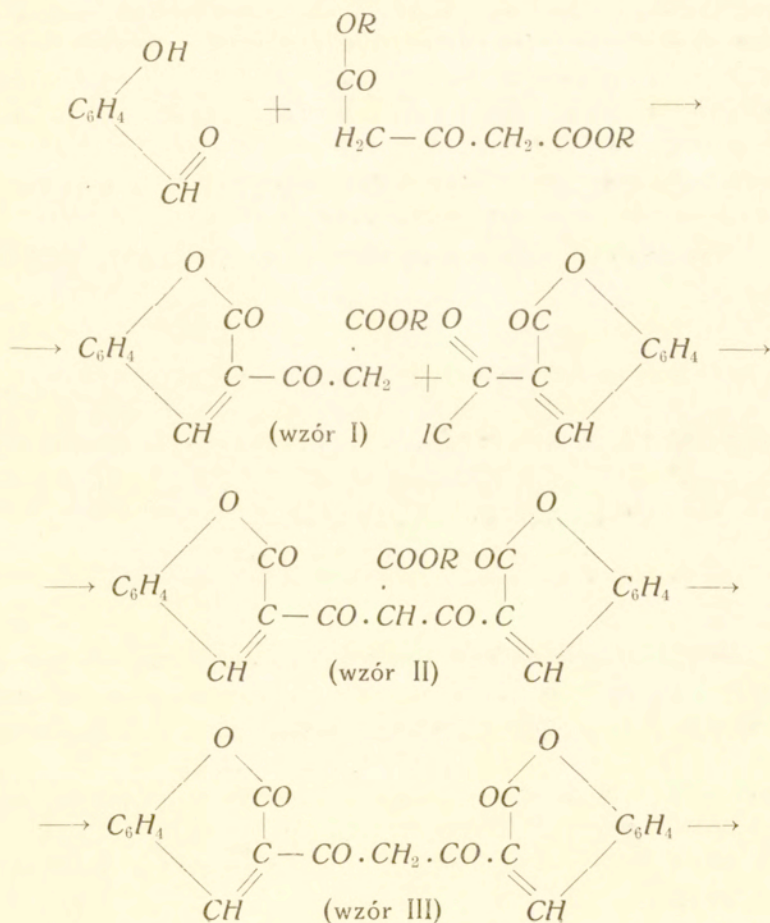
Nouvelle synthèse des dérivées du dicinnamoilméthane.

Mémoire présenté dans la séance de 11 Octobre 1930.

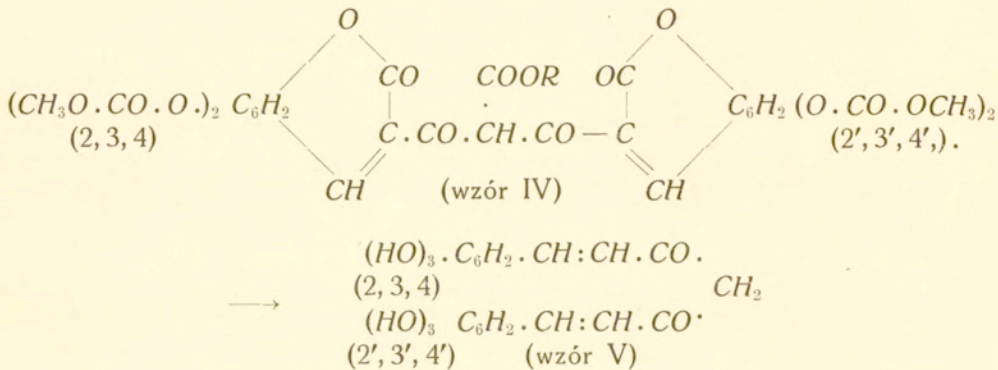
Streszczenie.

Dotychczasowa metoda otrzymywania dwucynamoilometanu, a także jego pochodnych, polegała: a) na wprowadzaniu do estru acetylooctowego najpierw jednego rodnika kw. cynamonowego i odszczepianiu od powstałego połączenia ugrupowania karboetoksyłowego, oraz b) na sprzęganiu wytworzonego w ten sposób dwuketonu z nową cząsteczką kw. cynamonowego i usuwaniu — za pomocą hydrolizy — rodnika kw. octowego w produkcji kondensacji.

W nowej metodzie produktami wyjściowymi są: oksyaldehyd aromatyczny i ester kw. acetonodwukarbonowego. Powstający podczas kondensacji składników związek — pochodna kumaryny (wzór I) — po przeprowadzeniu w sól miedziową poddawany zostaje kondensacji z chlorkiem kw. kumarynokarbonowego; w wytworzonym produkcie (wzór II) usuwana zostaje gr. karboetoksylova podczas ogrzewania z wodą w autoklawie. Rezultatem tych reakcji jest związek o dwóch pierścieniach kumarynowych (wzór III). Próby rozszczepienia tych heterocyklowych rdzeni, mające na celu otrzymanie dwuoksypochodnej dwucynamioilometanu dotychczas nie doprowadziły do rezultatu pożądanego.



O wiele łatwiejszem natomiast okazało się osiągnięcie syntezy sześciowodorotlenowej pochodnej. Zbudowany za pomocą nowej metody związek o wzorze IV ulega podczas ogrzewania z wodą w autoklawie rozkładowi, dzięki czemu powstaje sześćio-
 oksy-dwucynamoidometan (wzór V).



Rozpoczęte zostały badania doświadczalne, celem rozstrzygnięcia zagadnienia, czy opisana wyżej metoda będzie posiadała cechy ogólności.

Praca ogłoszona zostanie w „Rocznikach Chemji”.

Stefan Mazurkiewicz.

O obrazach ciągłych zbiorów linjowych.

Komunikat przedstawiony na posiedzeniu dnia 11 października 1930 r.

Streszczenie.

W pracy niniejszej dowodzę następującego twierdzenia.

Jeżeli spełniona jest hipoteza kontynuuum $2^{\aleph_0} = \aleph_1$, wówczas

- 1) istnieje \aleph_1 typów c linjowych nieporównywalnych między sobą,
- 2) istnieje nieskończony ciąg typów c linjowych malejących.

Wynik powyższy stanowi, przy założeniu hipotezy kontynuuum rozwiązanie zagadnień pp. Kuratowskiego i Sierpińskiego.

Stefan Mazurkiewicz.

Sur les images continues des ensembles linéaires.

Note présentée à la séance de 11 Octobre 1930.

Le but de cette Note est la démonstration du théorème suivant.

Théorème: Si $2^{\aleph_0} = \aleph_1$ alors: 1) il existe \aleph_1 types *c* linéaires qui ne sont pas comparables deux à deux; 2) il existe une suite infinie de types *c* linéaires décroissants ¹⁾.

Je ne considère que des ensembles linéaires.

J'appelle *ensemble de Lusin* tout ensemble non dénombrable E tel que tout ensemble parfait non dense contient au plus un ensemble dénombrable de points de E . L'existence de ces ensembles a été démontré par M. Lusin avec l'aide de l'hypothèse du continu ²⁾.

Soit X un ensemble, $f(x)$ une fonction définie pour $x \in X$, z un nombre réel. Nous désignerons par $f(X)$ l'ensemble des valeurs de $f(x)$ sur l'ensemble X , par $X(f(x) = z)$ l'ensemble de tous les points $x \in X$ tels que $f(x) = z$.

Lemme I. Y étant un ensemble de Lusin, $f(x)$ une fonction continue sur Y , l'ensemble $Y(f(x) = z)$ est au plus dénombrable sauf pour un ensemble de valeurs de z qui est au plus dénombrable.

$Y(f(x) = z) = Y \times \overline{[Y(f(x) = z)]}$, donc $Y(f(x) = z)$ est dénombrable si l'ensemble fermé $\overline{Y(f(x) = z)}$ est non dense. Or pour $z_1 \neq z_2$ on a $Y(f(x) = z_1) \times \overline{Y(f(x) = z_2)} = 0$, donc

¹⁾ La notion de type *c* a été introduite par M. Sierpiński; le type *c* d'un ensemble P sera désigné par cP ; si P est linéaire cP est un type *c* linéaire. P, Q étant deux ensembles on pose: $cP = cQ$ si P est une image continue de Q et Q une image continue de P ; on pose: $cP < cQ$ (ou $cQ > cP$) si P est une image continue de Q mais Q n'est pas une image continue de P , enfin on dit que cP et cQ ne sont pas comparables si P n'est pas une image continue de Q et Q n'est pas une image continue de P . Comp. Sierpiński: Fund. Math. XIV p. 234-236, p. 345-349 et Comptes rendus du 1^{er} Congrès des Mathématiciens des Pays Slaves p. 52-56.

La première partie de notre théorème donne la solution (conditionnée par l'admission de l'hypothèse du continu) d'un problème de M. Kuratowski, la seconde — d'un problème de M. Sierpiński.

²⁾ C. R. 158 p. 1258.

$\overline{Y(f(x)=z_1)} \times \overline{Y(f(x)=z_2)}$ ne contient aucun intervalle. Il en résulte que $\overline{Y(f(x)=z)}$ est non dense sauf pour un ensemble au plus dénombrable de valeurs de z .

Lemme II. *Y étant un ensemble de Lusin, Z_1, Z_2, \dots une suite d'ensembles non dénombrables, il existe un ensemble de Lusin $A \subset Y$, tel que pour tout m naturel, tout ensemble $A_1 \subset A$ et toute fonction $f(x)$ continue sur A_1 on a :*

$$(1) \quad Z_m - f(A_1) \neq 0$$

Rangeons en des suites de type Ω : α_1) les points de Y :

$$(2) \quad \{y_\mu\} \quad \mu < \Omega$$

α_2) les points de Z_n , pour $n=1, 2, \dots$:

$$(3) \quad \{z_\mu^{(n)}\} \quad \mu < \Omega$$

α_3) les couples (α, n) où n est un nombre naturel, α un nombre ordinal $< \Omega$:

$$(4) \quad \{(\alpha_\mu, n_\mu)\} \quad \mu < \Omega$$

enfin α_4) les fonctions continues sur des G_δ qui ont en commun avec Y un ensemble non dénombrable de points :

$$(5) \quad \{f_\mu(x)\} \quad \mu < \Omega.$$

Soit U_μ le G_δ sur lequel $f_\mu(x)$ est continue. Posons :

$$(6) \quad V_\mu = Y \times U_\mu.$$

V_μ est non dénombrable.

Définissons pour $\lambda < \Omega$ un point α_λ et un ensemble D_λ au plus dénombrable de manière suivante.

(I) α_1 est le premier point de la suite (2) contenu dans V_{α_1} .

(II) Soit $z_{\beta_1}^{(n_1)}$ le premier point de $\{z_\beta^{(n)}\}$, $\beta < \Omega$, différent de $f_{\alpha_1}(\alpha_1)$ et tel que $V_{\alpha_1}(f_{\alpha_1}(x) = z_{\beta_1}^{(n_1)})$ est au plus dénombrable; un tel point existe d'après le lemme I. Nous posons :

$$(7) \quad D_1 = V_{\alpha_1}(f_{\alpha_1}(x) = z_{\beta_1}^{(n_1)}).$$

Supposons les α_μ, D_μ déterminés pour $\mu < \lambda < \Omega$.

(III) α_λ est le premier point de (2) contenu dans $V_{\alpha_\lambda} - \sum_{\mu < \lambda} (D_\mu + \alpha_\mu)$ un tel point existe car $V_{\alpha_\lambda} \subset Y$ est non dénombrable et l'ensemble $\sum_{\mu < \lambda} (D_\mu + \alpha_\mu)$ est dénombrable.

(IV) Soit $z_{\beta_\lambda}^{(n_\lambda)}$ le premier point de la suite $\{z_{\beta}^{(n_\lambda)}\}$ qui n'est pas contenu dans l'ensemble dénombrable $\sum_{\mu \leq \lambda} f_{\alpha_\lambda}(\alpha_\mu)$ et qui est tel que $V_{\alpha_\lambda}(f_{\alpha_\lambda}(x) = z_{\beta_\lambda}^{(n_\lambda)})$ est au plus dénombrable. Un tel point existe d'après le lemme I. Nous posons:

$$(8) \quad D_\lambda = V_{\alpha_\lambda}(f_{\alpha_\lambda}(x) = z_{\beta_\lambda}^{(n_\lambda)})$$

Soit

$$(9) \quad A = \sum_{\lambda < \Omega} (\alpha_\lambda)$$

le dis que c'est l'ensemble cherché. D'abord $\lambda_1 \neq \lambda_2$ entraîne $\alpha_{\lambda_1} \neq \alpha_{\lambda_2}$, donc A est non dénombrable donc c'est un ensemble de Lusin.

D'après (II), (III), (IV) on a:

$$(10) \quad A \times \sum_{\lambda < \Omega} D_\lambda = 0.$$

Il résulte de (7), (8), (10) et de la définition de $V_{\alpha_\lambda}(f_{\alpha_\lambda}(x) = z_{\beta_\lambda}^{(n_\lambda)})$ que pour tout $\lambda < \Omega$, $z_{\beta_\lambda}^{(n_\lambda)}$ n'est pas contenu dans $f_{\alpha_\lambda}(V_{\alpha_\lambda} \times A)$. Donc:

$$(11) \quad Z_{n_\lambda} - f_{\alpha_\lambda}(V_{\alpha_\lambda} \times A) \neq 0 \quad \lambda < \Omega.$$

Soit $A_1 \subset A$, $f(x)$ continue sur A_1 , m un nombre naturel. Si A_1 est dénombrable on a certainement (1). Si A_1 est non dénombrable on peut étendre la définition de $f(x)$ en conservant la continuité à un certain ensemble G_β , que nous désignerons par U et qui contient A_1 .

On obtient une fonction $f^*(x)$ continue sur U et identique à $f(x)$ pour $x \in A_1$. On a: $Y \times U \supset A_1$, donc $Y \times U$ est non dénombrable. Donc $f^*(x)$ est contenue dans la suite (5); soit ρ son indice dans cette suite.

On a les relations:

$$(12) \quad f_\rho(x) = f(x) \quad x \in A_1$$

$$(13) \quad V_\rho \times A \supset A_1.$$

Le couple (ρ, m) est contenu dans la suite (4); soit σ son indice; on a alors: $\rho = \alpha_\sigma$, $m = n_\sigma$. Posons dans (11) $\lambda = \sigma$, on obtient:

$$(14) \quad Z_m - f_\rho(V_\rho \times A) \neq 0$$

(12), (13), (14) entraînent (1). Le lemme est donc démontré.

Lemme III. *Y étant un ensemble de Lusin, $Z_1, Z_2 \dots Z_n \dots$ une suite d'ensembles non dénombrables, il existe un ensemble de Lusin $B \subset Y$ tel que pour tout n naturel cB et cZ_n ne sont pas comparables.*

Considérons l'ensemble A du lemme précédent. Soit Γ l'ensemble de tous les sous-ensembles non dénombrables de A , Δ_n — l'ensemble de toutes les images continues de Z_n , enfin $\Delta = \sum_{n=1}^{\infty} \Delta_n$. La puissance de Γ est $2^{\aleph_1} > \aleph_1$, celle de Δ_n est \aleph_1 , donc celle de Δ de même \aleph_1 . Il en résulte que $\Gamma - \Delta \neq 0$. Soit $B \in \Gamma - \Delta$, on voit facilement que B possède les propriétés requises.

La première partie de notre théorème est une conséquence presque immédiate du lemme III.

X étant un ensemble linéaire, z un nombre réel désignons par $X^{(1)}(z)$ l'ensemble de points $x \in X$, $x < z$, par $X^{(2)}(z)$ l'ensemble de points: $x \in X$, $x > z$. Formons une suite d'ensembles de Lusin $\{L_n\}$ et une suite de points $\{z_n\}$ de manière suivante.

(I) L_1 est un ensemble de Lusin arbitraire, z_1 est tel que $L_1^{(1)}(z_1)$ et $L_1^{(2)}(z_1)$ sont non dénombrables, donc des ensembles de Lusin.

(II) L_{n+1} est un ensemble de Lusin contenu dans $L_n^{(2)}(z_n)$ et tel que $L_{n+1}^{(1)}(z_n)$ n'est pas une image continue d'aucun sous-ensemble de L_{n+1} ; un tel ensemble existe d'après lemme II; z_{n+1} est tel que $L_{n+1}^{(1)}(z_{n+1})$ et $L_{n+1}^{(2)}(z_{n+1})$ sont non dénombrables, donc des ensembles de Lusin.

Posons:

$$(15) \quad M_n = L_n^{(1)}(z_n).$$

$$(16) \quad N_n = \sum_{k=n}^{\infty} M_k \quad n = 1, 2, \dots$$

Je dis que:

$$(17) \quad cN_{n+1} < cN_n \quad n = 1, 2, \dots$$

On a:

$$(18) \quad N_n = M_n + N_{n+1}$$

$$(19) \quad N_{n+1} \subset L_{n+1} \subset L_n^{(2)}(z_n)$$

(15) et (19) entraînent:

$$(20) \quad (M_n \times \bar{N}_{n+1}) + (\bar{M}_n \times N_{n+1}) = 0.$$

Soit w_n un point arbitraire de N_n . Posons:

$$(21) \quad g_n(x) = w_{n+1} \quad \text{pour } x \in M_n$$

$$(22) \quad g_n(x) = X \quad \text{pour } x \in N_{n+1}$$

$g_n(x)$ est continue sur N_n d'après (18), (20). On a de plus:

$$(23) \quad g_n(N_n) = N_{n+1}(w_{n+1}) = N_{n+1}$$

c. à d. N_{n+1} est une image continue de N_n .

Soit $h(x)$ une fonction continue sur N_{n+1} . Supposons que:

$$(24) \quad h(N_{n+1}) = N_n.$$

Désignons par N_{n+1}^* l'ensemble des points $x \in N_{n+1}$ tels que $h(x) \in M_n$.

On aura d'après (19):

$$(25) \quad N_{n+1}^* \subset L_{n+1}$$

$$(26) \quad h(N_{n+1}^*) = M_n$$

$h(x)$ continue sur N_{n+1}^* . Mais ces relations sont en contradiction avec la définition de L_{n+1} . Donc (24) est impossible, c. à d. N_n n'est pas une image continue de N_{n+1} . (17) est ainsi démontrée.

V. W. Adkisson.

**O krzywych cyklicznie spójnych, których obszary
uzupełnienia mają brzegi homeomorficzne
z zachowaniem punktów rozgałęzienia.**

Komunikat przedstawiony przez K. Kuratowskiego dn. 11 października 1930 r.

Przedmiotem badań jest klasyfikacja i własności krzywych, wymienionych w tytule.

V. W. Adkisson.

**Cyclicly connected continuous curves whose
complementary domain boundaries are
homeomorphic, preserving branch points.**

Mémoire présenté par M. C. Kuratowski dans la séance de 11 Octobre 1930.

Introduction. It is the purpose of this paper to classify certain cyclicly connected continuous curves¹⁾ in Euclidian space of two dimensions.

The curves M considered²⁾, among others, are those that contain only a finite number of simple closed curves³⁾, and are such that any two complementary domain boundaries are homeomorphic, preserving branch points⁴⁾. Such curves are classified according to the number and orders of the branch points⁵⁾ on each complementary domain boundary.

¹⁾ Cf. G. T. Whyburn, *Cyclicly Connected Continuous Curves*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31—38.

²⁾ See *Notation and Definitions* in Part II.

³⁾ For a characterization of such curves see, P. Alexandroff, *Über endlich-hoch zusammenhängende stetige Kurven*, Fundamenta Mathematicae, vol. 13 (1929), pp. 34—41.

⁴⁾ Two complementary domain boundaries, J and J' , will be said to be homeomorphic if a continuous (1—1) correspondence, π , can be set up between them such that if A is a branch point on J of order α , then $\pi(A) = A'$ is a branch point on J' of order α .

⁵⁾ Cf. Menger's definition of order of a point on a curve, K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1927), pp. 279—280. A branch point, as here used, will refer to a point of Menger order greater than two.

In Part I is established a necessary and sufficient condition that a continuous (1—1) correspondence¹⁾ of M (on a sphere S) into itself be extendable to S ²⁾. Part II is devoted to the classification of the curves M indicated above. In Part III it is proved that any continuous (1—1) correspondence between any two complementary domain boundaries of M (on a sphere S) can be extended to M , and also to S , for the curves in which each complementary domain boundary contains exactly three branch points. Furthermore, it is shown that for any one of the curves M any continuous (1—1) correspondence sending the curve into itself can be extended to S .

Credit is due Professor J. R. Kline whose suggestions led to the present paper, and whose constant advice and criticisms have aided materially in its completion.

PART I.

*Definition*³⁾. If M is a cyclicly connected continuous curve lying on a sphere S , π a continuous (1—1) correspondence such that $\pi(M) = M$, C and C' any two simple closed curves of M such that $\pi(C) = C'$, $S - C = R_1 + R_2$, and $N_i = M \cdot R_i$, we shall say that regions are preserved under π provided that it is possible to so assign subscripts to the regions of $S - C'$, R_1' and R_2' , so that $\pi(N_i) = N_i'$, where $N_i' = M \cdot R_i'$.

Theorem 1. *If M is a cyclicly connected continuous curve lying on a sphere S , and π is a continuous (1—1) correspondence such that $\pi(M) = M$, a necessary and sufficient condition that π be extendable to S is that regions be preserved under π .*

Proof: The condition is necessary. For suppose there were a simple closed curve, C of M , enclosing N_1 of M in one region R_1 , and N_2 of M in the other region R_2 , such that $\pi(C) = C'$

1) The (1—1) correspondences considered in this paper are all (1—1) reciprocal.

2) If π is a continuous (1—1) correspondence such that $\pi(M) = M$, then π is extendable to S provided there exists a continuous (1—1) correspondence T such that $T(S) = S$, and such that for points of M , $T = \pi$. See H. M. Gehman, *On Extending a Continuous (1—1) Correspondence of Two Plane Continuous Curves to a Correspondence of Their Planes*, Transactions of the American Mathematical Society, vol. 28 (1926), pp. 252—265.

3) Cf. H. M. Gehman, loc. cit. Definition, p. 260.

does not enclose $\pi(N_1) = N_1'$ in one region R_1' , and $\pi(N_2) = N_2$, in the other region R_2' . (Either N_1 or N_2 may be vacuous. If both vacuous, the necessity of the theorem is trivial).

There are two cases to consider.

Case 1. If N_1 vacuous, then under our assumption N_2' must be divided into two sets, N_{21}' and N_{22}' , such that N_{21}' lies in R_1' and N_{22}' lies in R_2' . Let t be a simple arc joining a point of $\pi^{-1}(N_{21}') = N_{21}$ to a point of N_{22} and lying entirely in R_2 . Then $\pi(t) = t'$ must join a point of N_{21}' to a point of N_{22}' if π is to be extendable. Hence, t' must have a point P' in common with C' . But $\pi^{-1}(C') = C$, hence P must lie on C . But since P' lies on t' , P must lie on t , and since t has no points in common with C , we have a contradiction.

Case 2. If neither N_1 nor N_2 is vacuous, and regions not preserved under π , there must be points of N_1' and N_2' lying together in one of the regions R_i' ($i=1, 2$). Let P_1' be a point of N_1' and P_2' a point of N_2' lying together in the region R_1' . Let t' be a simple arc joining P_1' to P_2' , and lying entirely within R_1' . Then $\pi^{-1}(t') = t$ is an arc joining $\pi^{-1}(P_1') = P_1$ to $\pi^{-1}(P_2') = P_2$. But P_1 must lie in N_1 , and P_2 in N_2 . Hence, any arc joining P_1 and P_2 must have points on C . But this would mean that the point P on C corresponds under π^{-1} to two points, which is impossible if π is (1—1).

The condition is sufficient. Let C be any simple closed curve of M . Now consider one of the regions R_i ($i=1, 2$) bounded by C and the points N_i of M lying in R_i . Since regions are preserved under π , $\pi(N_1) = N_1'$ lies in the region R_1' bounded by C' , and no other points of M lie in this region. By making use of a theorem due to Gehman¹⁾, we see that the correspondence can be extended to the regions R_1 and R_1' . In order to do this we let R_2 and R_2' correspond to the unbounded domains in Gehman's Theorem. Thus any simple closed curve lying in R_1 (or in R_1') will have an interior and exterior such that N_2 (or N_2') lies in its exterior. Now since regions are preserved under π , any simple closed curve, J of M , lying in $R_1 + C$, and enclosing the points N_{11} of N_1 in its interior, must go into $\pi(J) = J'$ enclosing the points N_{11}' of N_1' in its interior.

¹⁾ Cf. H. M. Gehman, loc. cit., Theorem II, p. 261.

For if this were not the case there would be points of N_{11}' lying outside J' , that is, lying in the exterior of J' with N_2' and C' . But this would mean that regions are not preserved under π . Hence, for the regions R_1 and R_1' , „interiors are preserved” and we can apply Gehman's Theorem showing that the correspondence can be extended to these two regions. But precisely the same argument holds for R_2 and R_2' , and thus for the whole sphere S .

Theorem 2. *If M is a cyclicly connected continuous curve lying on a sphere S , and π is a continuous (1—1) correspondence such that $\pi(M) = M$, a necessary and sufficient condition that π be extendable to S is that for every boundary, J , of a complementary domain of M , $\pi(J) = J'$ be also the boundary of a complementary domain of M .*

Proof: The condition is necessary, for if J is the boundary of a complementary domain and J' fails to be the boundary of a complementary domain then obviously regions are not preserved under π . (Cf. Th. 1).

The condition is sufficient. For suppose that under the hypothesis of this theorem regions are not preserved. Then there is a simple closed curve, C of M , enclosing the points N_1 of M in a region R_1 , and the points N_2 of M in a region R_2 , such that C' does not enclose N_1' in a region R_1' and N_2' in a region R_2' . Then either N_1 is vacuous while N_2' lies in both R_1' and R_2' , or points of N_1' and N_2' lie together in R_1' or R_2' . Suppose N_1 vacuous. This means that C is the boundary of a complementary domain, and under our hypothesis C' must be the boundary of a complementary domain. But this contradicts the assumption on C and C' .

Now suppose neither N_1 nor N_2 vacuous, and points of N_1' lie in one of the regions R_i' with points of N_2' . There are two cases to consider.

Case 1. Suppose all of N_1' and N_2' lie together in one of the regions R_i' , say R_2' . Then C' must be the boundary of the complementary domain R_1' . But this implies that C is also the boundary of a complementary domain, contrary to assumption.

Case 2. Suppose a part of N_1' , say N_{11}' , lies in R_2' with points of N_2' . This means that N_1' , and hence N_1 , must be disconnected. For if N_1 were connected, there would be at least

one arc of N_1' joining points of N_{11}' in R_2' to points of N_{12}' in R_1' (where $N_1' = N_{11}' + N_{12}'$). This arc would necessarily have at least one point, P' , in common with C' , which means that for some point P_1 on C , $\pi(P_1) = P'$, and for some point P_2 in N_1 , $\pi(P_2) = P'$, and thus π would not be (1-1).

Now consider the complementary domain in R_1 bounded by points of N_{11} , C and N_{12} . That such a domain exists may be shown as follows. Let P_1 be any point in N_{11} , and P_2 any point in N_{12} . Let d be an arc joining P_1 and P_2 , but lying entirely within C . On the arc d from P_1 to P_2 , let Q_1 be the last point d has in common with N_{11} . Such a point exists since $N_{11} \cdot C$ is closed. On the arc from P_1 to P_2 let Q_2 , after Q_1 , be the first point d has on N_{12} . Since $N_{11} \cdot N_{12} = 0$, $Q_1 \neq Q_2$. Therefore, the subarc $d' = \overline{Q_1 Q_2}$ of d must lie in a complementary domain of M . Suppose the boundary J of this domain contained no points of C . Then $J = J_1 + J_2$ (where $J_1 \subset N_{11}$ and $J_2 \subset N_{12}$) would necessarily be disconnected, which is impossible since J must be a simple closed curve. Hence, there are points of C , N_{11} and N_{12} on J . But any simple closed curve J composed of points of C , N_{11} and N_{12} must correspond under π to a simple closed curve J' composed of points of C' , N_{11}' and N_{12}' . But such a simple closed curve cannot bound a complementary domain, for N_{11}' lies in R_2' and N_{12}' lies in R_1' , and hence, J' would divide S into two regions, each containing points of C' . Therefore, we have J the boundary of a complementary domain while J' is not the boundary of a complementary domain, contradicting our hypothesis in the theorem.

Theorem 3. *If M is a cyclicly connected continuous curve lying on a sphere S , such that no two complementary domain boundaries of M have a disconnected set in common, then every continuous (1-1) correspondence π such that $\pi(M) = M$ can be extended to S .*

Proof: Assume that under this hypothesis π is not extendable. Then from Theorem 2, we see there must be at least one simple closed curve J which is the boundary of a complementary domain, R_1 of M , such that $\pi(J) = J'$ is not the boundary of a complementary domain. If $M = J$, or J plus a simple arc t having only two points in common with J , it is easy to see, by making use of Theorem 2, that Theorem 3 is true. Therefore,

let us assume that M consists of points other than J and t . Let $M = J + N$, and since J is the boundary of a complementary domain R_1 , N must lie entirely in the region R_2 bounded by J . Since J' is not the boundary of a complementary domain, $\pi(N) = N'$ must lie a part in R_1' and a part in R_2' . Let $N' = N_1' + N_2'$ such that N_1' lies in R_1' , and N_2' lies in R_2' . Then N must be disconnected in such a manner that neither $N_1 = \pi^{-1}(N_1')$ nor $N_2 = \pi^{-1}(N_2')$ contains a limit point of the other. For suppose N_1 contained a limit point P of N_2 . Then $\pi(P) = P'$ must lie in R_1' , while N_2' lies entirely in R_2' . Hence, P' cannot be a limit point of N_2' . But since P is a limit point of N_2 , if the correspondence were continuous P' would be a limit point of N_2' . Thus we have a contradiction, and N must be disconnected in the manner indicated.

Now let P_i be the limit points of N_1 on J , and Q_i the limit points of N_2 on J . Since $M = J + N_1 + N_2$ is connected, J must contain at least one limit point, P_1 , of N_1 and one limit point, Q_1 , of N_2 . For obviously N_1 and N_2 cannot contain limit points of the closed set J . (It is possible that $P_1 = Q_1$). Now P_1 and Q_1 are not the only limit points of N lying on J . For if they were M would not be cyclicly connected¹⁾. Since P_1 is a limit point of N_1 , there must be at least one simple arc from P_1 to some other point P_2 on J such that this arc is a subset of $N_1 + P_1 + P_2$. In precisely the same manner we see there must be an arc $Q_1 T Q_2$ from Q_1 to some other point Q_2 on J such that this arc is a subset of $N_2 + Q_1 + Q_2$.

Let M_i be a maximal connected subset of N_i , and let $P_1 Z P_2$ be that particular arc of $M_1 + P_1 + P_2$ which can be taken with one of the arcs of J from P_1 to P_2 , say $P_1 X_1 P_2$, to form the boundary of the region of R_2 containing $M_1 - \underline{P_1 Z P_2}$. Now M_2 lies either in the region of R_2 bounded by $P_1 X_2 P_2 Z P_1$, or $P_1 X_1 P_2 Z P_1$, where $J = P_1 X_1 P_2 X_2 P_1$. There are two cases to consider.

Case 1. Suppose M_2 lies in the region of R_2 bounded by $P_1 X_2 P_2 Z P_1$. On the arc $P_1 X_2 P_2$ there must be at least two points of Q_i since M_2 lies inside the region of R_2 bounded by $P_1 X_2 P_2 Z P_1$. From the point P_1 in the order of $P_1 X_2 P_2$ on J

1) Cf. G. T. Whyburn, loc. cit., Theorem I, p. 31.

there must be a first limit point, T_1 , of M_2 . (It may be that $T_1 = P_1$). From the point P_2 in the order $P_2 X_2 P_1$ on J there must be a first limit point, T_2 , of M_2 . (It may be that $T_2 = P_2$ but $T_2 \neq T_1$, for this would mean only one point of Q_i on the arc $P_1 X_2 P_2$). Now there must exist a complementary domain, D of M , lying entirely in R_2 whose boundary contains as a proper subset the arc $P_1 Z P_2$, the arc $P_1 T_1$ of J taken in the order $P_1 X_2 P_2$, and the arc $P_2 T_2$ of J taken in the order $P_2 X_2 P_1$. It is easy to see that this boundary cannot contain the whole arc $P_1 X_2 P_2$ of J . For if it did the simple closed curve $P_1 X_2 P_2 Z P_1$ would enclose points of M_2 in a region of R_2 , and hence would not be the boundary of a domain D . Therefore, the points on the boundary of D that are also on J must form a disconnected set. But J is itself the boundary of a complementary domain R_1 . Hence, we have two complementary domains, R_1 and D , with a disconnected set in common contrary to our hypothesis, and the assumption that π be not extendable cannot hold in this case.

Case 2. Now suppose M_2 lies in the region of R_2 bounded by $P_1 X_1 P_2 Z P_1$, i. e., in the same region with M_1 . From P_1 in the order $P_1 X_1 P_2$ on J there must be, just as in case 1, a first limit point T_1 of M_2 . From P_2 on J in the order $P_2 X_1 P_1$ there must be a first limit point T_2 of M_2 . ($T_1 \neq T_2$). Now from T_1 to T_2 there must be an arc $T_1 V T_2$ which is a subset of $M_2 + T_1 + T_2$ such that if $T_2 X_3 T_1$ is an arc on J , $T_1 V T_2 X_3 T_1$ encloses a region of R_2 containing $M_2 - T_1 V T_2$. Now from T_1 on J in the order $T_1 X_2 T_2$ there must be a first limit point, U_1 , of M_1 , where U_1 may be T_1 . From T_2 on J in the order $T_2 X_2 T_1$ there must be a first limit point, U_2 , of M_1 . $U_2 \neq U_1$. The arc $U_2 T_2 V T_1 U_1$ must be a proper subset of the boundary of a complementary domain D lying entirely in R_2 . But this leads to the same contradiction as in case 1. Therefore, the theorem must be true.

PART II.

Notation and definitions. The symbol M will designate a cyclicly connected continuous curve containing a finite number of simple closed curves¹⁾, and hence, a finite number of comple-

¹⁾ If M contains only a finite number of simple closed curves, it follows that M contains no domain.

mentary domains D_i with boundaries J_i , such that for any i and j , J_i is homeomorphic with J_j . The curve M may lie either on the sphere or in the plane. It is convenient to consider M in the plane when speaking of interiors and exteriors.

The orders of the branch points A, B, C, D, E will be denoted by $\alpha, \beta, \gamma, \delta, \varepsilon$ respectively.

The number of branch points on J_i will be denoted by k .

If F_1 is any complementary domain boundary of M , let F_2 be a complementary domain boundary that has an arc $A_1 X_1 B_1$ on F_1 . If P_1 is the first branch point on F_2 from B_1 in the order $A_1 X_1 B_1 Y_1 P_1$, let F_3 be the boundary¹⁾ that contains the arc $B_1 Y_1 P_1$ of F_2 . Let B_2 be the last point that F_3 has in common with F_1 such that on F_1 we have the order $A_1 X_1 B_1 B_2$. (It may happen that $B_2 = B_1$). Let P_2 be the first branch point on F_3 from B_2 in the order $P_1 Y_1 B_1 B_2 Y_2 P_2$. Let F_4 be the boundary of M that contains the arc $B_2 Y_2 P_2$. In this manner we define F_i for all boundaries that have points on F_1 . Let C_i be the outer boundary of $F_1 + F_2 + \dots + F_i$, and R the number of boundaries with points on F_1 . Then if $F_i (i=2, 3 \dots R)$ does not have a disconnected set on C_{i-1} , C_R is a simple closed curve, and we can define $F_i (i=R+1, \dots)$ for those boundaries that have points on C_R in the same manner as $F_i (i=2, 3 \dots R)$ was defined. Let $(S-R)$ be the number of boundaries on C_R . If $F_i (i=R+1, \dots S)$ does not have a disconnected set on C_{i-1} , C_S will be a simple closed curve, and we can define C_T as C_R and C_S were defined. This process can be continued so long as the curve does not end or F_i does not have a disconnected set on C_{i-1} .

Theorem 4. *If each complementary domain boundary of M contains only two branch points, then M consists of h distinct arcs ($h > 2$) connecting the branch points A and B .*

Proof: Let J be the boundary of any complementary domain of M , A and B the two branch points of J , and M_1 the points A and B plus all simple arcs of M that join A to B . These arcs can have no branch points other than A and B , otherwise there would exist a boundary with at least three branch points.

¹⁾ Since M contains no domain, the term *boundary* when used with reference to M will mean complementary domain boundary.

Now suppose M consists of points other than M_1 . Let P be such a point. The points P and A must lie on a simple closed curve of M ; also the points P and B . Hence, there are two independent arcs from P to A , PXA and PYA . One of these arcs may pass through B . If it does we then have a simple arc from A to B passing through P , and P must belong to M_1 , contrary to assumption. Let PZB be an arc from P to B lying on the simple closed curve that contains P and B such that, Q , the last point this arc has in common with $PXAYA$ is different from A . We now have an arc from A to B passing through the branch point Q . But Q is different from A and B , and there cannot be an arc from A to B having branch points other than A and B . Therefore, $M = M_1$.

Theorem 5. *This theorem will consist of several parts, 5.1, 5.2, 5.3 and 5.4.*

5.1. *If M is such that $k > 5$, then F_i does not have a disconnected set on C_{i-1} . (If $i = 1$, C_0 is to be taken as the boundary of any complementary domain with points on F_1).*

Proof. Suppose M lies in the plane and two boundaries, J and J' , have a disconnected set in common such that no two boundaries lying within the outer boundary C of $J + J'$ have a disconnected set in common. If M contains boundaries that have disconnected sets in common this choice of J and J' is always possible. Otherwise M would necessarily contain an infinite number of complementary domains. Let $K_n = AXBYA$ be the simple closed curve lying within C such that $AXB \subset J$, and $AYB \subset J'$. Let A_1 be the first branch point from A on AYA_1B . There must be at least one such branch point different from B , otherwise there would lie inside K_n a boundary with a disconnected set on J . Let D_1 be the complementary domain lying inside K_n and having the arc AYA_1 on its boundary. Let D_2 be the complementary domain within K_n that lies adjacent to D_1 and has at least one boundary point on AXB , D_3 the complementary domain that lies adjacent to D_2 and has at least one boundary point on AXB , etc. Let J_i be the boundary of D_i . Then $J_1 \neq K_n$, for otherwise we would have a boundary with only two branch points, A and B . Let K_i be the outer boundary of $J_1 + J_2 + \dots + J_i$. Now J_2 cannot have a disconnected set on J_1 because of the choice of J and J' . However, J_i may have

a disconnected set on $K_{i-1} + K_n$, or a disconnected set on K_{i-1} if $i = 3, 4 \dots$. Let J_R be the last boundary that has points on AXB , and such that its domain lies inside K_n .

First, suppose that J_i ($i = 2, 3 \dots R$) does not have a disconnected set on $K_{i-1} + K_n$, or on K_{i-1} . Then $K_R \neq K_n$. For consider any boundary J_i . At most J_i has two branch points on AXB and therefore, has at least four branch points on $A_1 X_1 B_1$; where $A_1 X_1 B_1 = K_R - \underbrace{A_1 AXBB_1}$. Since only two of these four

branch points can be points of complementary domains lying inside K_R and adjacent to J_i , two of them (different from A and B) must have arcs extending into the exterior of K_R . Therefore, $K_R \neq K_n$.

Now if all branch points on K_n have not been exhausted, we can define J_i ($i = R + 1, R + 2, \dots$) for boundaries with points on $A_1 X_1 B_1$ as J_i ($i = 1, 2, \dots R$) was defined for boundaries with points on AXB . Let the number of such boundaries be $(S - R)$. Then $K_S \neq K_n$. For at most, J_i ($i = R + 1, \dots S$) can have only three branch points on K_R . If it had four, only two of these could lie on boundaries that lie inside K_S and adjacent to J_i , while the other two must lie on a boundary lying between J and K_R , having only these two points on K_R . But such a boundary would then have four branch points on AXB which is impossible. Hence, J_i must have at least three branch points on $A_2 X_2 B_2$; where $A_2 X_2 B_2 = K_S - \underbrace{A_2 AXBB_2}$. Only two

of these three can belong to complementary domains inside of K_S and adjacent to J_i , and therefore, at least one has arcs going into the exterior of K_S . Hence, $K_S \neq K_n$.

We can form K_T similar to K_R and K_S , and the corresponding $A_3 X_3 B_3$, etc., until all of the branch points on K_n have been exhausted, or until some J_i is encountered which has a disconnected set on K_{i-1} (or on $K_{i-1} + K_n$). We propose to show that in either case the boundaries continue indefinitely, which means that M must necessarily contain an infinite number of complementary domains.

If J_i does not have a disconnected set on either K_{i-1} or $K_{i-1} + K_n$, suppose the branch points on K_n have been exhausted at A_n, B_n . There are two cases, (1) $A_n \neq B_n$, (2) $A_n = B_n$.

Case (1). Since the branch points of K_n have been exhausted at A_n, B_n , there are no branch points on $\underbrace{A_n Y_n B_n}_{= K_n - A_n A X B B_n}$.

Let D_n be the complementary domain that has the arc $A_n Y_n B_n$ as a subset of its boundary, and lies within K_n . Let $\underbrace{A_n Y_n B_n Z_n A_n}_{= \text{boundary of } D_n}$ be the boundary of D_n . Now since there are no branch points on $\underbrace{A_n Y_n B_n}$, there must be a branch point on $\underbrace{A_n Z_n B_n}$. Moreover,

there are branch points on $A_n X_n B_n$ with arcs extending into the exterior of $A_n X_n B_n B X A A_n$ for the same reason that there were branch points on $A_2 X_2 B_2$ with arcs extending into the exterior of K_S . (It is to be noted that there must exist more than one such branch point on $A_n X_n B_n Z_n A_n$, otherwise M would not be cyclicly connected). We can now use the interior of $A_n X_n B_n Z_n A_n$ and the arc $A_n X_n B_n$ in the same manner as the interior of K_n and the arc $A X B$ were used originally to show that the complementary domains of M must continue to extend into the interior of $A_n X_n B_n Z_n A_n$.

Case 2. In this case $A_n X_n B_n$ is a simple closed curve. Let \bar{A} and \bar{B} be any two adjacent branch points of $\underbrace{A_n X_n B_n}$.

Let $\bar{A} X_n \bar{B}$ be the arc of the simple closed curve $A_n X_n B_n$ that does not contain A_n . We can now use the interior of $A_n X_n B_n$ and the arc $\bar{A} X_n \bar{B}$ as the interior of K_n and the arc $A X B$ were used originally, and show, as in case (1), that the complementary domains of M must continue to extend into the interior of $A_n X_n B_n$.

We have now shown that (provided J_i does not have a disconnected set on K_{i-1}) as soon as the branch points on K_n are exhausted, K_n can be replaced by another simple closed curve to which the original argument, used for the interior of K_n , can be applied and so continue indefinitely.

In the second place, suppose J_c has a disconnected set on K_{c-1} (or on $K_{c-1} + K_n$). Let $A_c X_c B_c Y_c A_c$ be the simple closed curve that lies inside the outer boundary of $J_c + K_{c-1}$ such that $A_c X_c B_c \subset K_{c-1}$ and $A_c Y_c B_c \subset J_c$. (If J_c has a disconnected set on $K_{c-1} + K_n$ we take $A_c X_c B_c Y_c A_c$ such that $A_c X_c B_c \subset K_{c-1} + K_n$, and $A_c Y_c B_c \subset J_c$. The argument is practically the same in either case).

Now repeat the process inside $A_c X_c B_c Y_c A_c$ that was used for the interior of K_n , etc., until we come to the last time that J_i has a disconnected set on K_{i-1} . Suppose this occurs at $i=b$. By adding primes to the notation originally employed, we have the simple closed curve $K'_n = A' X' B' Y' A'$, such that $A' X' B' \subset K_{b-1}$, $A' Y' B' \subset J_b$, $K'_R =$ the outer boundary of complementary domains that lie inside K'_n and have points on $A' X' B'$, ect.

Any boundary J'_i that lies inside K'_n and has points on $A' X' B'$ can have at most three branch points on $A' X' B'$. For suppose J'_i has four branch points, P_1, P_2, P_3 , and P_4 on $A' X' B'$, and situated in the order $A' P_1 P_2 P_3 P_4 B'$. The two branch points P_2 and P_3 cannot have arcs extending into the exterior of K_{b-1} , for such arcs would have to extend into the interior of J'_i . Therefore, P_2 and P_3 have arcs extending into the interior of K_{b-1} , and since they are adjacent they must lie on the same boundary, J'_H , which lies interior to K_{b-1} . Furthermore, J'_H can have only these two branch points on $A' X' B'$, for otherwise it would have a disconnected set on $A' X' B'$. If K'_T is the last simple closed curve of the K' 's encountered before reaching K_{b-1} , J'_H must have four branch points on K'_T which, in turn, means a complementary domain with four branch points on K'_S , etc., down to K'_R . But this is impossible since no boundary within K'_R can have more than two branch points on $A' X' B'$.

Since J'_i can have at most three branch points on $A' X' B'$, it has at least three on $A'_1 X'_1 B'_1$, one of which must have arcs extending into the exterior of K'_R . Therefore, $K'_R \neq K'_n$. Now since for no i does J_i , inside K'_n , have a disconnected set on K'_{i-1} (or on $K'_{i-1} + K'_n$) we can repeat the argument indefinitely, each time obtaining a K' with branch points extending into its exterior. Therefore, the assumption that two boundaries of M have a disconnected set in common leads to a contradiction of the hypothesis that M have only a finite number of complementary domains.

Now by assuming that F_i has a disconnected set on C_{i-1} a similar contradiction is reached, which requires a repetition of practically the same argument just completed. Therefore, the theorem is true.

5.2. If $k=3$, and each branch point of M is at least of order 6, then F_i does not have a disconnected set on C_{i-1} .

Proof: We use the same notation as in 5.1, and assume that no two boundaries, other than J and J' , within the outer boundary of $J+J'$ have a disconnected set in common. Suppose J and J' have the disconnected set H in common. Since $k=3$, H consists of just two points, or an arc and one point. Suppose the latter. There can be no branch point on \overline{AXB} or \overline{AYB} for this would require more than three branch points on either J or J' . Therefore, K_n is either the boundary of a complementary domain with only two branch points, A and B , which is impossible, or there lie within K_n two boundaries with a disconnected set $(A+B)$ in common, contrary to assumption.

Assume that H consists of just two points, A and B . Since $k=3$, there is only one branch point, A_1 , on \overline{AYB} , and one branch point, P , on \overline{AXB} . The points A_1 and P must exist, otherwise there would exist within the outer boundary of $J+J'$ two boundaries with a disconnected set in common. If J_i ($i=1, 2, \dots, R$) does not have a disconnected set on K_{i-1} , then $A_1 X_1 B_1$ ($B_1 = A_1$) is a simple closed curve. Consider one of the complementary domains that has just one boundary point, P , on \overline{AXB} . Such a domain must have two branch points on $A_1 X_1 A_1$, and each such branch point must have at least two arcs extending into the interior of $A_1 X_1 A_1$. But if there are two such branch points on $A_1 X_1 A_1$ there must be a third. For if not, there would exist within $A_1 X_1 A_1$ two boundaries with a disconnected set in common. Hence, in this case we can also obtain $A_2 X_2 B_2$.

Let S_1, S_2, S_3, \dots be the simple closed curves used in the succeeding steps of the above argument, i. e., $K_n = S_1$, $A_1 X_1 A_1 = S_2$, $A_i X_i B_i Y A_i = S_i$. On S_i ($i=1, 2, \dots$) there must always occur at least one branch point with arcs going into the interior of S_i . For suppose there were no such branch point on S_i . Let P be a branch point on S_i . From P there must extend at least three arcs, each arc ending on $A_{i-1} X_{i-1} B_{i-1}$. Let the three corresponding branch points on $A_{i-1} X_{i-1} B_{i-1}$ be P_{11}, P_{12} and P_{13} , such that the order on the arc is $A_{i-1} X_{i-1} P_{11} P_{12} P_{13} B_{i-1}$. Now since $k=3$, there must extend from P_{12} three arcs into the exterior of S_{i-1} . Let the three corresponding branch points on $A_{i-2} X_{i-2} B_{i-2}$ be P_{21}, P_{22} and P_{23} such that the order on the arc is $A_{i-2} X_{i-2} P_{21} P_{22} P_{23} B_{i-2}$.

There must extend from P_{22} at least three arcs into the exterior of S_{i-2} . Evidently we can continue in this manner until AXB is encountered. But there cannot exist on AXB three branch points such that the one lying between A and B has arcs extending into the exterior of S_1 (interior of J). Therefore, on S_i there must exist at least one branch point with arcs going into the interior of S_i . (This argument also shows that the number of such arcs going into S_i from any branch point must be at least two). But it is shown above that if there is one such branch point on S_i there must be at least three. Therefore, by selecting two adjacent branch points, A_i and B_i , on S_i that have arcs extending into the interior of S_i , and using the method already indicated for obtaining the S 's it is possible to get S_{i+1} , then S_{i+2} , etc., indefinitely unless J_i has a disconnected set on K_{i-1} .

The case where J_i has a disconnected set on K_{i-1} is treated in a manner similar to that in 5.1. Therefore, if F_i has a disconnected set on C_{i-1} , M cannot contain a finite number of complementary domains.

5.3. *If $k \geq 4$, and each branch point is at least of order 4, then F_i cannot have a disconnected set on C_{i-1} .*

Proof: Practically the same argument can be used here as in 5.1 and 5.2, the essential feature being (as above) that each S_i must have branch points with arcs extending into its interior. If there were an S_i with no such branch point, it is shown, as in 5.2, that this would require a branch point on AXB with arcs extending into the interior of J which is impossible.

$$\begin{array}{ll}
 5.4. & \text{If (1) } k=3, \quad p=3, 4, 5, \\
 & \text{or (2) } k=4, \quad p=3, \\
 & \text{or (3) } k=5, \quad p=3,
 \end{array}$$

where p is the order of the branch points of M , then F_i cannot have a disconnected set on C_{i-1} .

Proof: Case (1a). The theorem is trivial here.

Case (1b). $k=3, p=4$. Suppose two boundaries, J and J' , have a disconnected set H in common. Let J, J', K_n , etc., be chosen as in 5.1. Since $k=3$, H must consist of two points, A and B , or an arc and one point. The latter is impossible (Cf. 5.2). Suppose then, that $H=A+B$. The following possibilities occur.

- (a) \overbrace{AXB} with no branch point, and
 \overbrace{AYB} " " " " .
- (b) \overbrace{AXB} with one branch point, and
 \overbrace{AYB} " no " " .
- (c) \overbrace{AXB} with one branch point, and
 \overbrace{AYB} " " " " .

(a) Since A and B are of order 4, K_n will be a complementary domain with only two branch points, which is impossible.

(b) Since \overbrace{AYB} contains no branch point, there must be a boundary lying within K_n that has a disconnected set on J . But this contradicts the choice of J and J' .

(c) Let P_1 be the branch point on \overbrace{AXB} , and P_2 the branch point on \overbrace{AYB} . In this case there must exist two boundaries within K_n with the disconnected set $P_1 + P_2$ in common, contrary to assumption.

Now suppose F_i ($i=3, 4 \dots 7$) has a disconnected set on C_{i-1} . If this does not happen before F_1 , the curve is complete and the case proved. If H is the disconnected set that F_i has on C_{i-1} then H must consist of just two points, A and B . Now \overbrace{AXB} cannot have more than one branch point. This is easily seen by construction. Furthermore, \overbrace{AYB} cannot have more than one branch point since F_i already has two, i. e., A and B . We now have a repetition of cases (a), (b) and (c).

Case (1c). This case can be treated in very much the same manner as (1b).

Case (2). If J and J' have a disconnected set H in common, then H must consist of exactly two arcs. But this means there can be no branch point on either \overbrace{AXB} or \overbrace{AYB} since J and J' already have four branch points each. Therefore, K_n must be a boundary with only two branch points, which is impossible. By construction it is easy to see that no F_i ($i=3, 4, 5$) can have a disconnected set on C_{i-1} .

Case (3). As in case (2), H must consist of two arcs. Therefore, \overbrace{AXB} and \overbrace{AYB} can have at most one branch point

each, and again, we have a repetition of cases (a), (b), and (c). Suppose $F_i (i=3, 4 \dots 11)$ has a disconnected set on C_{i-1} . By construction we see that, at most, \overline{AYB} can have one branch point, and when one branch point does occur on \overline{AYB} there can be only one on \overline{AXB} . Hence, cases (a), (b) and (c) again.

Theorem 6. *If M' is a cyclicly connected continuous curve containing no domain, and such that the boundary of each complementary domain contains exactly three branch points of orders α' , β' and γ' , and if $\alpha' > \beta' > \gamma' > 4$, then M' must contain an infinite number of complementary domains.*

Proof: In the first place, A' , B' and C' must all be of even order. For consider the arcs extending from a branch point of order α' . Such arcs must alternate in going to branch points of orders β' and γ' , otherwise two branch points of the same order would lie on the same complementary domain boundary. Hence, A' must be of even order. For the same reason, B' and C' must be of even order. Let $\alpha' = 2\alpha$, $\beta' = 2\beta$, and $\gamma' = 2\gamma$.

Now delete each arc of M' that extends from a branch point of order γ' to a branch point of order α' or β' . This gives a curve \overline{M} for which $k = 2\gamma \geq 6$, and the branch points on each complementary domain boundary of \overline{M} alternate with orders α and β . Let F_1 be the boundary of a complementary domain of \overline{M} . Let \overline{C}_1 be the outer boundary of all domains that have points on F_1 . In general, let \overline{C}_i be the outer boundary of domain boundaries with points on \overline{C}_{i-1} . Then \overline{C}_i will be a simple closed curve since F_i cannot have a disconnected set on C_{i-1} , otherwise the theorem would be true by virtue of 5.1. Let t_i be the total number of branch points on \overline{C}_i . Let m_i be the number of branch points of \overline{C}_i that have arcs extending into the interior of \overline{C}_i , and $n_i = t_i - m_i$. Each point of m_i will have only one arc extending into the interior of \overline{C}_i since each boundary of \overline{M} has more than three branch points. It will be convenient to call the branch points of order α the A -points, and likewise for the B 's. Let a_i be the number of A -points of m_i and b_i the number of B -points of m_i , i. e., $m_i = a_i + b_i$.

Since the A -points and B -points alternate on the domain boundaries, each B -point on F_1 ends in an A -point on \bar{C}_1 , and each A -point ends in a B -point; it not, the theorem would be true by virtue of 5.1. Since there are γ B -points and γ A -points on F_1 , and as there are $(\alpha - 2)$ arcs extending from A into the exterior of F_1 (likewise for B) we have,

$$a_1 = \gamma(\beta - 2), \quad b_1 = \gamma(\alpha - 2).$$

Now each A -point on \bar{C}_{i-1} ends in a B -point on \bar{C}_i , and each B -point ends in an A -point. But each A -point of m_{i-1} has $(\alpha - 3)$ arcs extending into the exterior of \bar{C}_{i-1} while each A -point of n_{i-1} has $(\alpha - 2)$ such arcs. The same is true of the B 's. There are a_{i-1} A -points in m_{i-1} , and $\left(\frac{t_{i-1}}{2} - a_{i-1}\right)$ A -points in n_{i-1} . The expression $\left(\frac{t_{i-1}}{2} - a_{i-1}\right)$ is obtained by noting that the A 's and B 's alternate on any \bar{C}_i and hence, $\frac{t_{i-1}}{2}$ gives the total number of each in \bar{C}_{i-1} .

As each arc extending from \bar{C}_{i-1} to \bar{C}_i gives rise to a point of m , on \bar{C}_i , we can write

$$m_i = a_{i-1}(\alpha - 3) + \left(\frac{t_{i-1}}{2} - a_{i-1}\right)(\alpha - 2) + \\ b_{i-1}(\beta - 3) + \left(\frac{t_{i-1}}{2} - b_{i-1}\right)(\beta - 2).$$

In order to compute n_i we need to distinguish between the domain boundaries that have an arc in common with \bar{C}_{i-1} , and those that have only one point. We designate them by D -boundaries and R -boundaries respectively. Each D boundary gives rise to $(2\gamma - 4)$ points in n_i , and each R -boundary $(2\gamma - 3)$ points in n_i . Now for each A -point in m_{i-1} we have $(\alpha - 4)$ R -boundaries, or $(\alpha - 4)(2\gamma - 3)$ points in n_i , and since there are a_{i-1} such points, the total contribution to n_i from this source is $a_{i-1}(\alpha - 4)(2\gamma - 3)$. For each A -point in n_{i-1} we have $(\alpha - 3)$ R -boundaries, or $(\alpha - 3)(2\gamma - 3)$ points in n_i , and since

there are $\left(\frac{t_{i-1}}{2} - a_{i-1}\right)$ such points, the total contribution to n from this source is $\left(\frac{t_{i-1}}{2} - a_{i-1}\right) (\alpha - 3) (2\gamma - 3)$.

The B -points in m_{i-1} and n_{i-1} give similar expressions by changing the α 's and a 's to β 's and b 's.

As there are t_{i-1} branch points on \overline{C}_{i-1} , there are t_{i-1} D -boundaries. The number of points in n_i arising from this source is $t_{i-1}(2\gamma - 4)$.

We now have for the final total of n_i ,

$$n_i = a_{i-1}(\alpha - 4)(2\gamma - 3) + \left(\frac{t_{i-1}}{2} - a_{i-1}\right)(\alpha - 3)(2\gamma - 3) + \\ b_{i-1}(\beta - 4)(2\gamma - 3) + \left(\frac{t_{i-1}}{2} - b_{i-1}\right)(\beta - 3)(2\gamma - 3) + \\ t_{i-1}(2\gamma - 4).$$

Let $d_i = n_i - m_i$.

Then $d_i = Gm_{i-1}Hn_{i-1}$, where

$$G = (\gamma - 2)(\alpha + \beta) - 6\gamma + 11, \text{ and}$$

$$H = (\gamma - 2)(\alpha + \beta) - 4\gamma + 7.$$

If d_i is positive, the branch points of \overline{M} continue to increase as i increases, and \overline{M} will necessarily contain an infinite number of complementary domains. Since $H - G = 2\gamma - 4 > 0$, we see that d_i is certainly positive when $G > 0$, i. e., when

$$\gamma > 2 + \frac{1}{\alpha + \beta - 6}.$$

But in our hypothesis we have $2\alpha > 2\beta > 2\gamma > 4$. Therefore, the above condition is always satisfied, and the theorem is true.

Classification of the curves M . If any one of the curves M is such that each branch point is of order p , Theorem 5 shows that F_i cannot have a disconnected set on C_{i-1} . Therefore, we can use an argument similar to that used by Benton¹⁾ to show that M in this case exists with a finite number of complementary domains only when

¹⁾ Cf. T. C. Benton, *On Continuous Curves Which are Homogeneous Except for a Finite Number of Points*, *Fundamenta Mathematicae*, vol. 13 (1929), pp. 172–174, section 7.

- [A] (1) $k=3, p=3, 4, 5$
 (2) $k=4, p=3$
 (3) $k=5, p=3.$

We now consider all curves M for which $k=3$, and in which the branch points are not necessarily all of the same order. In the proof of Theorem 8 it will be shown that these curves are all unique.

(1) $\alpha=\beta=\gamma=3$, where α, β, γ represent the orders of the branch points on each complementary domain boundary. In light of Theorem 5.4, a curve of this type exists, i. e., the curve which is homeomorphic with the edges and vertices of a regular tetrahedron.

- (2) $\alpha=\beta=3, \gamma>3.$

This curve does not exist. Let the three arcs extending from a point, A , of order 3 be AB_1, AB_2 and AB_3 ; where B_1, B_2 and B_3 are the first branch points occurring on the arcs from A . Suppose $\beta_1=\beta$. Then $\beta_2=\gamma$, for B_1, A and B_2 all lie on the boundary of the same complementary domain. But for the same reason $\beta_3=\gamma$, and since B_3, A and B_2 all lie on the same boundary, $\beta_3=\beta$. This is impossible since $\beta \neq \gamma$.

- (3) $\alpha=3, \beta=\gamma=2n \ (n=2, 3, \dots)$

Suppose this curve M exists. Then remove every arc that extends from a point of order 3 to a point of order $2n$. For example, take a branch point A of order 3. The three arcs from A extend to points of order $2n$. The arcs going from a point B of order $2n$ alternate in going first to a point of order 3, then to one of order $2n$. Otherwise we would have two branch points of order 3 on the same boundary. Therefore, when the indicated arcs have been removed we have a curve \overline{M} , each boundary of which contains exactly 3 branch points of order n . But \overline{M} , and hence M , can exist only when $n < 6$ (Cf. [A]).

- (3.1) $\alpha=3, \beta=\gamma=4.$

This curve is obtained as follows. Construct on the sphere a curve M which is homeomorphic with the edges of a regular

tetrahedron. Then triangulate¹⁾ one complementary domain of M , that is, select a point P in a complementary domain and draw from this point three mutually exclusive arcs to the branch points lying on the domain boundary.

$$(3.2) \quad \alpha = 3, \quad \beta = \gamma = 6.$$

For this curve triangulate each complementary domain of the curve homeomorphic with the regular tetrahedron.

$$(3.3) \quad \alpha = 3, \quad \beta = \gamma = 8.$$

Triangulate the regular octahedron.

$$(3.4) \quad \alpha = 3, \quad \beta = \gamma = 10.$$

Triangulate the regular icosahedron.

$$(3.5) \quad \alpha = 3, \quad \beta = \gamma = 2n + 1.$$

This curve does not exist. For the arcs extending from a point, B , of order $(2n + 1)$ must alternate in going to points of order α and γ ; otherwise there would exist a boundary with two branch points of order α , or three of order γ .

$$(3.6) \quad \alpha = 3, \quad 3 < \beta \neq \gamma > 3.$$

This curve does not exist for the same reason that the curve in (2) does not exist.

Now consider those curves ($k = 3$) in which at least one branch point of every boundary is of order 4.

$$(4) \quad \alpha = \beta = \gamma = 4.$$

This curve is homeomorphic with the regular octahedron.

$$(5) \quad \alpha = \beta = 4, \quad \gamma > 4.$$

Construct a regular pyramid with a base of γ vertices. Then triangulate the base.

$$(6) \quad \alpha = 4, \quad \beta = \gamma = 2n.$$

In this curve delete all arcs extending from points of order 4. This gives a curve \bar{M} for which $k = 4$, and $p = n$. But \bar{M} , and hence M , can exist only when $n < 4$, (Cf. [A]).

$$(6.1) \quad \alpha = 4, \quad \beta = \gamma = 6.$$

¹⁾ This is a term adopted from. O. Veblen, *American Mathematical Society Colloquium Lectures*, vol. 5, Part II, p. 41.

Triangulate the cube.

$$(6.2) \quad \alpha = 4, \quad \beta = \gamma = 2n + 1.$$

This curve does not exist for the same reason that the curve in (2) does not exist.

$$(7) \quad \alpha = 4, \quad \beta > \gamma > 4.$$

In this case B and C must each be of even order, (Cf. 2). Therefore, let $\beta = 2m$ and $\gamma = 2n$. Delete all arcs extending from points of order 4. This gives a curve \bar{M} for which $k = 4$, and the branch points on each boundary are alternately of orders m and n . Now subdivide each complementary domain of \bar{M} by an arc extending between the two points of order m . We then have a curve M' for which $k = 3$, $\alpha' = \beta' = 2m$, and $\gamma' = n$. In M' delete each arc which extends from a point of order n , obtaining a curve \bar{M}' for which $k = n$, and $p = m$. Therefore, \bar{M}' , and hence M , exists only when $n = 3, m = 3, 4, 5; n = 4, m = 3; n = 5, m = 3$. (Cf. [A]).

$$(7.1) \quad \alpha = 4, \quad \beta = 6, \quad \gamma = 8.$$

This curve may be constructed as follows. Subdivide each complementary domain of the curve (3.3) by extending an arc from the branch point of order 3 to any point, P , lying between the two points of order 8. The arc joining these two points of order 8 is common to two domains, and hence, P is used in the subdivision of each of these domains. Therefore, P becomes a branch point of order 4, while each branch point of order 3 is doubled.

$$(7.2) \quad \alpha = 4, \quad \beta = 6, \quad \gamma = 10.$$

This curve is obtained from the curve of (3.4) as (7.1) was obtained from (3.3).

(8) Now consider the cases that have not already been disposed of for which $\alpha = \beta \neq \gamma$. Both A and B must be of even order, Cf. (2). Let $\alpha = \beta = 2n$, and $\gamma = m$. Delete the arcs from the points of order γ , and obtain a curve for which $k = \gamma$, and each branch point is of order n . (Cf. [A]).

$$(8.1) \quad \alpha = \beta = 6, \quad \gamma = 5.$$

Triangulate a regular dodecahedron. Theorem 6 shows that all curves M for which $k=3$, and $\alpha > \beta > \gamma > 4$ cannot exist with a finite number of complementary domains. This completes the classification of the curves when $k=3$.

Before considering the curves for which $k=4$, we prove the following theorem.

Theorem 7. *If M' is a cyclicly connected continuous curve, containing no domain, such that, (1) every complementary domain boundary has only a finite number of branch points, (2) any two boundaries are homeomorphic, (3) F_i does not have a disconnected set on C_{i-1} , (4) M' has an infinite number of complementary domains; and if \bar{M} is a cyclicly connected continuous curve such that, (1) any two complementary domain boundaries of \bar{M} are homeomorphic, (2) any complementary domain boundary of \bar{M} is homeomorphic with any complementary domain boundary of M' except for one branch point, which is of higher order in \bar{M} than in M' , then \bar{M} also contains an infinite number of complementary domains.*

Proof. Let F_i , C_i and \bar{F}_i , \bar{C}_i be defined for M' and \bar{M} respectively, as F_i and C_i were originally defined for M . (Cf. *Notation and Definitions*).

Since M' contains an infinite number of complementary domains, there must be arcs extending into the exterior of C_R . Hence, there must be arcs extending into the exterior of \bar{C}_R . For since all of the branch points on \bar{F}_1 are of the same order as those on F_1 , except one which is of higher order, there must lie between \bar{F}_1 and \bar{C}_R at least one more complementary domain than lies between F_1 and C_R of M' . Furthermore, since no F_i can have a disconnected set on C_{i-1} , and no \bar{F}_i a disconnected set on \bar{C}_{i-1} (otherwise the theorem would be true by virtue of Th. 5) this extra domain requires that the number of arcs extending into the exterior of \bar{C}_R be greater than the number extending into the exterior of C_R . But this, in turn, means a greater number of complementary domains lying between \bar{C}_R and \bar{C}_5 than lie between C_R and C_5 of M' . And since there are arcs of M' extending into the exterior of C_5 , there must be

arcs of \overline{M} extending into the exterior of \overline{C}_S . But this continues indefinitely for M' , and therefore for \overline{M} . Hence, \overline{M} must contain an infinite number of complementary domains. This completes the proof.

We now investigate the cases for which $k=4$.

$$(9) \quad \alpha = \beta = \gamma = \delta = 3.$$

This curve exists, and is homeomorphic with the edges of a cube.

$$(10) \quad \alpha = \beta = \gamma = 3, \quad \delta > 3.$$

Such a curve may be obtained by subdividing the two complementary domains, D_1 and D_2 , of a simple closed curve as follows. Select a point, P_1 , in D_1 . From P_1 draw δ mutually exclusive arcs to δ distinct points, $Q_1, Q_2, \dots, Q_\delta$, on C . Now select δ points, $R_1, R_2, \dots, R_\delta$, on C so that the first branch point on either side of an R -point is a Q -point. Then from a point, P_2 , in D_2 draw δ arcs to these δ R -points. This gives the required curve.

$$(11) \quad \alpha = \beta = 3, \quad \gamma = \delta > 3.$$

Such a curve is impossible if A and B lie adjacent. For consider a branch point, A , of order 3 that lies adjacent to D . There are three arcs from A , and they must alternate in going to points of order 3 and order δ . But since A is of odd order, this is impossible. Hence, assume that A and B separate C and D on the complementary domain boundary. Divide each complementary domain by an arc joining the two points of order $\gamma = \delta$. This gives a curve \overline{M} for which $k=3, \overline{\alpha}=3, \overline{\gamma}=\overline{\delta}=2\gamma$. But it has been shown that \overline{M} exists only when $\overline{\gamma}=\overline{\delta}=8, 10$. Cf. (3). Therefore, M exists only when $\gamma = \delta = 4, 5$.

$$(12) \quad \alpha = 3, \quad \beta = \gamma = \delta = 4.$$

Subdivide the complementary domains of the curve \overline{M} for which $k=3, \overline{\alpha}=\overline{\beta}=\overline{\gamma}=4$. Let D_1 be any complementary domain of \overline{M} . Let P be any point in D_1 . From P draw three arcs to three distinct points on the boundary of D_1 in such a manner that any two are separated by two of the three branch points $\overline{A}, \overline{B}$ and \overline{C} . Every other complementary domain of \overline{M} is

subdivided in the same manner, inserting only one branch point between any two branch points of \bar{M} .

$$(13) \quad \alpha = 3, \quad \beta = \gamma = \delta \geq 5.$$

Delete all arcs of this curve that extend from points of order 3. This gives a curve \bar{M} for which $k=6$, and each branch point is of order ≥ 3 . But since we have shown that the curve for which $k=6$ and $p=3$ does not exist with a finite number of complementary domains (Cf. [A]), we can make use of Theorem 7 to show that the curve in this case does not exist.

$$(14) \quad \alpha = 3, \quad \beta = \gamma \neq \alpha, \quad \alpha \neq \delta \neq \beta.$$

A and D must separate B and C since A is of odd order. Divide each complementary domain by an arc joining the two branch points A and D . This gives a curve \bar{M} for which $k=3$, $\bar{\alpha}=2\alpha$, $\bar{\delta}=2\delta$ and $\bar{\beta}=\beta$. The point B must be of even order. If $\bar{\delta} \leq 10$ and $\bar{\beta}=4$, \bar{M} exists, and therefore M exists.

(15) We have shown that for the special case $k=4$, and all branch points are of order 4, M must have an infinite number of complementary domains. Therefore, from this fact, and Theorem 7, we conclude that no curve M exists when $k=4$, and the branch points of lowest order are ≥ 4 .

This completes the classification of the curves M when $k=4$. Now we consider the curves for which $k=5$.

$$(16) \quad \alpha = \beta = \gamma = \delta = \varepsilon = 3.$$

This curve exists, and is homeomorphic with the regular dodecahedron.

$$(17) \quad \alpha = \beta = \gamma = \delta = 3, \quad \varepsilon > 3.$$

If we delete all arcs extending from points of order ε , we obtain a curve \bar{M} for which $k=\varepsilon$, and $p=3$. But \bar{M} , and hence M , exists only when $\varepsilon \leq 5$.

$$(18) \quad \alpha = \beta = \gamma = 3, \quad \delta > 3, \quad \varepsilon > 3.$$

Suppose D and E do not lie adjacent on the domain boundary. Delete all arcs going from points of order δ . This gives a curve \bar{M} for which $k=\delta$ and $p=\varepsilon$. But such a curve cannot exist with a finite number of complementary domains (Cf. [A]). Suppose D

and E lie adjacent. In this case, both D and E must be of even order. Subdivide each complementary domain of M by an arc extending from a point, P , between D and E to the point of order 3 that lies opposite to P , that is, the branch point that is not adjacent to either D or E . This gives a curve \bar{M} for which $k=4$ and each branch point is of order ≥ 4 . Therefore, \bar{M} , and hence M , cannot exist. (Cf. Th. 5.3 and [A]).

$$(19) \quad \alpha = \beta = 3, \quad \gamma, \delta, \varepsilon > 3.$$

Since the curve for which $\alpha = \beta = \gamma = 3$, $\delta, \varepsilon > 3$ does not exist, Theorem 7 shows that M under the above conditions cannot exist.

$$(20) \quad \alpha = 3, \quad \beta, \gamma, \delta, \varepsilon > 3.$$

Since the curve in (19) does not exist, Theorem 7 shows that this curve cannot exist.

(21) Since the curve for which $k=5$ and $p=4$ does not exist, Theorem 7 shows that the curves M for which $k=5$, and the branch points of lowest order are ≥ 4 cannot exist.

This completes the case for $k=5$.

Now consider all curves M for which $k \geq 6$. Since we have shown that these curves do not exist when $p \geq 3$, Theorem 7 shows that no curve for which $k \geq 6$ can exist with a finite number of complementary domains. This completes the classification of the curves M .

PART III.

Theorem 8. *The curves M for which $k=3$ are unique; that is, a curve in which the orders of the branch points on each complementary domain boundary are α, β, γ is homeomorphic with every other curve satisfying the same conditions.*

Proof: Case 1. From Theorem 5.4 it follows that if M lies on a sphere, and $k=3$, $p=3, 4, 5$, then M is homeomorphic with the edges of a regular tetrahedron, octahedron or icosahedron respectively. Therefore, for curves of this type the theorem must be true.

$$\text{Case 2. } \alpha = 3, \quad \beta = \gamma = 2n \geq 4.$$

Consider the arcs of M that go from a point of order 3. Each such arc will end in a point of order $2n$. These three points of order $2n$ must all lie on a simple closed curve, C , that does not pass through the branch point A . For there must be an arc between any two of these branch points that does not pass through A or the other point of order $2n$; otherwise there would exist a boundary with 3 points of order $2n$ and one of order 3. Delete all arcs extending from points of order 3, giving a curve \bar{M} . Then C becomes the boundary of a complementary domain of \bar{M} . The arcs extending from a point of order $2n$ alternate in going to points of order 3 and $2n$. Therefore, every point of order $2n$ will become a point of order n in \bar{M} . Hence, for \bar{M} we have $k=3$, $\bar{\alpha}=\bar{\beta}=\bar{\gamma}=n$.

If $n=2$, \bar{M} is a simple closed curve. For certainly C of \bar{M} exists. Suppose there were a point P of \bar{M} not on C . Since P is in M , there must exist an arc in M that joins P to C . As the 3 branch points on C are of order 4, there is only one arc from each extending into the exterior of C . These three arcs all go to the same branch point of order 3. For each one of these arcs is common to the boundary of two complementary domains, and each domain can have only one branch point of order 3. Now since A is of order 3, no other arcs can extend from A , and since M is connected, P must lie on one of the 3 arcs going to C from A . Therefore, P cannot lie in \bar{M} , and \bar{M} must be a simple closed curve.

For $n=3, 4, 5$ we use the same argument as for $n=2$, and obtain curves \bar{M} for which $k=3$ and $\bar{\alpha}=\bar{\beta}=\bar{\gamma}=3, 4, 5$ respectively.

This shows that the curves considered here must be homeomorphic with the curves obtained by triangulating each complementary domain of the following curves: $k=3$, $p=3, 4, 5$, or by properly subdividing the two complementary domains of a simple closed curve.

Case 3. $\alpha=4, 5$, $\beta=\gamma=6$.

Consider the arcs that go from a point of order 4 (or 5). Each such arc will end in a point of order 6. These 4 (or 5) points of order 6 must lie on a simple closed curve (Cf. Case 1). If

we delete all arcs of M that extend from a point of order 4 (or 5) to one of order 6, we obtain a curve \overline{M} for which $k=4$ (or 5) and $p=3$. This shows that the curves in the case are homeomorphic with the curves obtained by triangulating the curves for which $k=4$ (or 5) and $p=3$.

Case 4. $\alpha \geq 5, \beta = \gamma = 4$.

Delete all arcs extending from points of order α . This gives a simple closed curve as in case 1.

Case 5. $\alpha = 4, \beta = 6, \gamma = 8, 10$.

Delete all arcs extending from points of order 4. This gives a curve \overline{M} for which $k=4$, and the branch points on each boundary are alternately of orders 3 and 4 (or 5). Now if we subdivide each complementary domain of \overline{M} by an arc joining the two points of order 3, we obtain a curve M' for which $k=3$, and $\alpha'=4$ (or 5), $\beta'=\gamma'=6$. But we have shown that the theorem holds for this curve (Cf. case 3). It follows that the theorem must be true for the curve M . This completes the proof for the curves in which $k=3$.

We now consider the uniqueness of the curves for which $k=4$.

$$(9') \quad \alpha = \beta = \gamma = \delta = 3.$$

The uniqueness of this curve follows from Th. 5. 4.

$$(10') \quad \alpha = \beta = \gamma = 3, \quad \delta > 3.$$

Delete all arcs extending from a point of order δ . This gives a simple closed curve showing that M in this case is unique.

$$(11') \quad \alpha = \beta = 3, \quad \gamma = \delta = 4, 5.$$

In each complementary domain of M join the two boundary points of order $\gamma = \delta$. This gives a curve \overline{M} for which $k=3$, $\overline{\alpha} = \overline{\beta} = 8, 10$ and $\overline{\gamma} = 3$. But \overline{M} , and hence M , is unique. (Cf. case 2, Th. 8).

$$(12') \quad \alpha = \beta = \gamma = 4, \quad \delta = 3.$$

This curve is not unique. For if all the arcs extending from points of order 3 be deleted, in one case we obtain a curve \overline{M} for which $k=3$ and $p=4$, while in the other case a curve M' is obtained for which $k=4$, $\alpha'=\beta'=\gamma'=3$ and $\delta'=4$.

$$(14') \quad \alpha = \beta = 4, \quad \gamma = 3, \quad \delta = 5.$$

In each complementary domain join the two boundary points of order γ and δ . This gives a curve for which $\bar{\alpha} = 4$, $\bar{\beta} = 6$, $\bar{\gamma} = 10$. Hence M is unique.

The three curves for the case $k = 5$ are each unique.

$$(16') \quad \alpha = \beta = \gamma = \delta = \varepsilon = 3.$$

The uniqueness follows from Th. 5.4.

$$(17') \quad \alpha = \beta = \gamma = \delta = 3, \quad \varepsilon = 4, 5.$$

Delete all arcs extending from points of order ε . This gives a curve for which $k = 4, 5$ and $p = 3$. Hence, M must be unique.

Theorem 9. *If, (1) M lies on a sphere S , and $k = 3$, (2) F_1 and F_1' are any two complementary domain boundaries of M , and π a continuous (1—1) correspondence such that $\pi(F_1) = F_1'^1$, then there exists a continuous (1—1) correspondence T such that $T(M) = M$, $T(S) = S$ and for F_1 and F_1' , $T = \pi$.*

Proof: Let F_2 be any complementary domain boundary of M with an arc G_1 on F_1 . Let F_2' be the boundary with points $G_1' = \pi(G_1)$ on F_1' . For F_2 let T be any continuous (1—1) correspondence such that $T(F_2) = F_2'$, and for G_1 and G_1' , $T = \pi$. Let F_3 be a boundary of M that has the arc G_2 on F_2 , and has points on F_1 . Let F_3' be the boundary of M that has the points $G_2' = \pi(G_2)$ on C_2' . (C_i = the outer boundary of $F_1 + F_2 + \dots + F_i$). Then for F_3 , T is any correspondence such that $T(F_3) = F_3'$ and at the same time corresponds to the correspondence already defined for G_2 and G_2' . Since no F_i can have a disconnected set on C_{i-1} (Cf. Theorems 5 and 8) we can continue and define T for any F_i , and therefore, put all of M into correspondence with itself. Hence, $T(M) = M$.

¹⁾ The correspondence is such that branch points on F_1 go into branch points on F_1' , i. e., if A is a branch point on F_1 of order α , then $\pi(A) = A'$ is a point on F_1' , of order α .

Since no two complementary domain boundaries of M can have a disconnected set in common, T can be extended to S (Cf. Th. 3). Therefore, $T(S) = S$.

Theorem 9 holds for the curves M when $k=4$, excepting case (12') where $\alpha=\beta=\gamma=4$, $\delta=3$. If case (12') be omitted, practically the same proof as that in Theorem 9 can be given here. However, it can be shown that the Theorem holds for one of the curves in case (12'), i. e., the curve obtained by properly subdividing the complementary domains of the regular octahedron, Cf. (12). This may be shown as follows. Let M be the regular octahedron and M' the derived curve. Let D and D' be any two complementary domains of M' . Then D will lie in a complementary domain E of M and D' in a complementary domain E' of M . (It may happen that $E=E'$). Any continuous (1—1) correspondence of the boundary J of D into the boundary J' of D' (Cf. footnote to Th. 9) throws an arc t of the boundary F of E into an arc t' of the boundary F' of E' . The arc t (also t') contains one branch point of M of order 4. Let the three points on F that are branch points of M' but not of M be P_1, P_2, P_3 , and the corresponding points on F' be P_1', P_2', P_3' . The correspondence can be extended to F and F' such that $\pi(P_i) = P_i'$. ($i=1, 2, 3$). Furthermore, the correspondence can be extended to E and E' of M ¹⁾. But we have shown that such a correspondence can be extended to M and also to S . Therefore, the correspondence can be extended to M' and S .

Theorem 9 does not hold for the second curve under case (12'), i. e., the curve derived from the curve in which $\alpha=\beta=\gamma=3$, $\delta=4$. Even the extra condition that the correspondence preserve sense (or alter sense)²⁾ on each complementary domain boundary of the curve does not suffice for this case.

¹⁾ Cf. A. Schoenflies, *Beiträge zur Theorie der Punktmengen*, Mathematische Annalen, vol. 62 (1906), pp. 286—328. See also J. R. Kline, *A New Proof of a Theorem Due to Schoenflies*, Proceedings of the National Academy of Sciences, vol. 6 (1920), pp. 529—531.

²⁾ See, J. R. Kline, *A Definition of Sense on Closed Curves in Non-Metrical Plane Analysis Situs*, Annals of Mathematics, vol. 19 (1918), p. 185. In an unpublished paper I have shown that a continuous (1—1) correspondence π such that $\pi(M) = M$, and $\pi(S) = S$ must preserve sense or alter sense, on every complementary domain boundary of M .

In order for Theorem 9 to hold in the case where $k=5$, we need to add the further condition that the correspondence preserve sense on the complementary domain boundaries of the curve.

Finally, from Theorem 3, we see that any continuous (1—1) correspondence of any one of the curves M ($k=3, 4, 5$) into itself can be extended to the sphere.

University of Pennsylvania,
Philadelphia, Penna.

Posiedzenie

z dnia 4 grudnia 1930 r.

Antoni Morawiecki.

Fosforyty północne Polski.

Przedstawił St. J. Thugutt dn. 4 grudnia 1930 r.

Streszczenie.

W północnych częściach Polski fosforyty występują w utworach kredowych (senon), trzeciorzędowych (oligocen), dyluwialnych i aluwialnych.

Fosforyty kredowe występują w okolicach Grodna, w Mielniku nad Bugiem, w niektórych odsłonięciach w woj. Białostockiem, w woj. Poleskiem i w woj. Wołyńskim (np. okolice Ludwipola). Wiek ich nasuwać może pewne wątpliwości, gdyż są one pokryte (niejednokrotnie nawet przemieszane) utworami trzeciorzędowymi zawierającymi również fosforyty. Dostrzegamy je w odsłonięciach naturalnych w Grodnie i okolicach (Miały, Pyszki, Sopoćkinie i t. d.), w Tartaku nad Czarną Hańczą, w Piaskach i Strubnicy koło Wołkowyska, w Rosi, w Nowojelni, w Sawiczach, w Mielniku nad Bugiem, w Tuczynie na Wołyniu i t. d. Stwierdzono ich obecność wierceniami w Wilnie (Pohulanka), w Szubkowie na Wołyniu, w Warszawie, w Toruniu, w Grudziądzu, w Tucholi, w okolicach Tczewa i Gdańska, w Nowym Porcie, w środkowej części Helu i t. d.

W przeciwieństwie do fosforytów kredowych, występujących w szarym marglu lub kredzie, fosforyty trzeciorzędowe napotymano przeważnie w zielonych piaskach kwarcowych, zawierających domieszkę glaukonitu. Tu tworzą one rodzaj soczewek o zmien-

nej miąższości, nie przekraczającej wszakże 50 cm. Brak szczegółowych badań nie pozwala na ustalenie rozpiętości poszczególnych soczewek.

Fosforyty dyluwialne są na Pomorzu i na obszarze Wolnego Miasta Gdańska zjawiskiem pospolitem. Spotykamy je w morenach lodowcowych w postaci różnej wielkości otoczków wraz z otoczkami skał ogniowych i osadowych. Do ciekawszych należą odsłonięcia margli w stromem zboczu Kamiennej Góry pod Gdynią i w okolicach Sopot.

Pozatem tworzą fosforyty mniejsze lub większe skupienia w płatach starszych utworów kredowych lub trzeciorzędowych wtłoczonych w utwory dyluwialne. Na Pomorzu płyty powyższe (drobne kry lodowcowe) skupiają się przeważnie na zachód od linii kolejowej, prowadzącej z Tczewa do Gdańska. Do największych zdaje się należeć płat położony koło miejscowości Ulkowy. Powierzchnia jego wynosi około 1,5 km². Skądinąd znane są płyty pod Kłępinami, Kłodawą, Kleszczewem, Rozemberkiem, Łęgowem, (Langenau), Jasieniem i t. d.

Na Wołyniu zauważono fosforyty w dyluwiach kredowych osady wojskowej Bystrzyce nad Horyniem.

Fosforyty aluwialne spotykamy w żwirach i piaskach nadbrzeżnych morza Bałtyckiego w zatoce Puckiej (Gdynia, okolice Sopot). Są to prawie wyłącznie fosforyty trzeciorzędowe. W okresie dyluwialnym dostały się one do utworów lodowcowych, skąd wypłukały je fale morskie.

W utworach aluwialnych okolic Smordwy i Pełczy na Wołyniu spotykamy również fosforyty, które zostały wypłukane z nieodnalezionych pierwotnych pokładów cenomańskich.

Fosforyty dyluwialne i aluwialne leżą niewątpliwie na złożu wtórnym. Pochodzą one z utworów starszych bądź trzeciorzędowych, bądź kredowych, a może nawet jurajskich lub kambryjskich, rozpostartych na północ od opisywanego obszaru. Głównym czynnikiem powodującym ich wędrówkę było przesuwanie się skorupy lodowej w okresie dyluwialnym. W niektórych przypadkach poważne znaczenie miała również erozja i denudacja.

Fosforyty występują zazwyczaj w postaci buł rozmaitych wymiarów, dochodzących niekiedy do 15 cm. w średnicy. Fosforyty kredowe są przeważnie mniejsze, powierzchnię mają szorstką i kształt owalny. Fosforyty trzeciorzędowe przyjmują częściej

postać buł większych nieregularnych, o powierzchni bądź wygładzonej, bądź szorstkiej chropowatej. Fosforyty dyluwialne i aluwialne mają zazwyczaj powierzchnię wygładzoną wskutek działania czynników mechanicznych ¹⁾.

Rozmaitość struktury i tekstury poszczególnych konkrecyj jest duża. Trafiają się konkrecje owalne, złożone z substancyj fosforytowych, zawierających nieznaczne ilości ciał obcych. Spotykamy także odlewy muszli z niewielką zawartością tychże ciał. Skądinąd bywają konkrecje bądź owalne, bądź nieokreślonego kształtu, w których substancja fosforytowa schodzi do roli lepiszcza spajającego ziarna minerałów obcych.

Badania mikroskopowe fosforytów północnych doprowadziły do wyodrębnienia kilku substancyj fosforytowych, różniących się bądź składem bądź ustrojem, jak dalitu, stafelitu, frankolitu, koloфанitu i grodnolitu. W ich rzędzie odmiany bezpostaciowe mają znaczną przewagę nad odmianami krystalicznymi, zauważonymi w ilościach podrzędnych.

Pośród minerałów obcych wyróżniono nader pospolicity kwarc o wymiarach dochodzących do 3 mm., kalcyt, glaukonit, piryt, skalenie i muskowitz. Do rzadszych należał granat, cyrkon, rutyl, turmalin, biotyt i kilka innych.

W zależności od tekstury poszczególnych konkrecyj i ich składu mineralogicznego zmienia się w nich zawartość poszczególnych składników chemicznych. Fosforyty kredowe zawierają około 28—29% P_2O_5 , 48—51% CaO , 2—3% $Fe_2O_3 + Al_2O_3$, 5—6% CO_2 , do 18% cz. nierozp., niewielkie ilości fluoru, kwasu siarczanego, magnezu i t. d. W fosforytach trzeciorzędowych znaleziono średnio: 20,52% P_2O_5 , 31,46% CaO , 3,17% CO_2 , 1,00% SO_3 , 41,00% cz. nierozp., drobne ilości *F*, *Fe*, *Al*, *Mg*, *S*, i t. d. Fosforyty dyluwialne i aluwialne, pochodzące w przeważnej części z utwo-

¹⁾ Z pomiędzy prac, dotyczących rozpostarcia fosforytów w północnych częściach Polski, do najważniejszych należą: M. Hoyer: Über das Vorkommen von Phosphorit — und Grünsand — Geschiebe in Westpreussen, Zeit. d. deutsch. geolog. Gesellschaft, 1880; O. Helm: Über die in Westpreussen und den westlichen Provinzen Russlands vorkommenden Phosphoritknollen. Schriften der Naturforschenden Gesellschaft in Danzig, 1881—1885; A. Jentzsch: Über Phosphoritvorkommen in Westpreussen, Jahrbuch d. geolog. Landesanstalt, Berlin **39**, (1918), 96—132; W. Kaunhoben: Über russische Phosphorite, Zeits. f. praktisch. Geolog., Halle **27**, (1919), 71—76 i 89—93.

rów kredowych lub trzeciorzędowych, nie wykazują znaczniejszych odchyień. Trudno jest mówić o wzorze substancji fosforytowej, skoro mamy tu do czynienia z kilkoma jej odmianami, których rozdzielenie jest trudne, a niekiedy wręcz niemożliwe ¹⁾).

Z Grodzieńskich buł fosforytowych wyodrębnił dyr. J. Morozewicz nowy koloidalny gatunek substancji fosforytowej, zbliżony pod względem składu chemicznego do dalitu lub podolitu, nazwany przezeń grodnolitem ²⁾).

Przemysłowe znaczenie złóż wzmiankowanych, zdaje się, jest niewielkie. Niemcy, pozbawione w czasie wielkiej wojny dowozu surowca, zarządziły gorączkowe poszukiwania pokładów fosforytowych zarówno u siebie jak i w krajach okupowanych. Poszukiwania te nie dały wszakże poważniejszych wyników. Napotkane złoża były zbyt nikłe i nakładu pracy górniczej nie opłacały. A. Jentzsch, opierając się na mylnych przesłankach, określił wprawdzie zasoby warstw trzeciorzędowych Gdańska, Tczewa i okolic na 70.000.000 ton, jednak warunki geologiczne nie sprzyjały ich odbudowie. Do podobnego wniosku doszedł również W. Kaunhowen odnośnie do fosforytów, występujących w północno-wschodniej części Polski ³⁾), oraz J. Samsonowicz w odniesieniu do fosforytów Wołyńskich ⁴⁾).

¹⁾ Porów. J. Tokarski: Przyczynek do znajomości polskich fosforytów, *Przemysł Chemicz.*, Warszawa **11**, (1927), str. 63—66.

Z. Sujkowski: Przyczynek do znajomości fosforytów dorzecza górnego Niemna i Szczary, *Archiwum Mineral.*, Warszawa, **4**, (1928), 108—123, jak również prace poprzednio cytowane.

²⁾ J. Morozewicz: O dwóch nowych minerałach bardolicie i grodnolicie, *Pos. Nauk. P. I. G.*, zes. 7, Warszawa (1923), str. 2—3; Grodnolit koloidalny fosforan wapniowy, *Spraw. P. I. G.*, Warszawa, **2**, (1923—24), 223—224.

³⁾ l. c. str. 3.

⁴⁾ J. Samsonowicz: O dewonie i cenomanie w okolicach Pełczy, *Posiedz. P. I. G.*, Warszawa (1926), zes. 15 str. 43—44 i inne.

Antoni Morawiecki.

Les phosphorites de la partie septentrionale de la Pologne.

Mémoire présenté par M. S. J. Thugutt dans la séance du 4 Décembre 1930.

Résumé.

Dans la partie septentrionale de la Pologne on trouve des phosphorites dans les couches crétacées (Grodno, Mielnik sur Bug, Ludwipol en Volynie etc.), dans les couches des sables verts d'oligocène (Gdańsk, Tczew, Wołkowysk, Grodno, Tuczyn en Volynie etc.) et dans les terrains diluviens largement espacés en Poméranie. En outre on les trouve aussi dans les alluvions de Pelcza et Smordwa en Volynie ainsi que sur les bords de la mer Baltique (Gdynia, Sopoty).

La majeure partie de phosphorites diluviennes et alluviennes sont originaires du tertiaire et du crétacé.

On trouve les phosphorites dans des concrétions les plus variées à surface polie ou rugueuse. Ces concrétions se composent de la dahllite, de la staffélite, de la francolite, de la colophanite et de la grodnolite. Elles comprennent habituellement une plus ou moins grande quantité de grains d'autres minéraux, notamment du quartz, de la glauconie, de la calcite, des feldspaths, des zircons, des rutilés, des tourmalines, des amphiboles, de la moscovite, de la biotite, etc.

La composition chimique des concrétions est très variée. Les phosphorites crétacées comprennent le plus souvent 28—29% P_2O_5 , 48—51% CaO , 5—6% CO_2 , 2—3% $Al_2O_3 + Fe_2O_3$, 18% de résidus insolubles, une petite quantité Mg , F , etc.

Dans les phosphorites tertiaires nous rencontrons en moyenne 20,52% P_2O_5 , 31,46% CaO , 3,17% CO_2 , 1,00% SO_3 , 41,00% de résidus insolubles et une petite quantité Mg , F , Fe , Al , etc.

La majeure partie de phosphorites diluviennes et aluviennes sont originaires des couches crétacées et tertiaires tout en correspondant quant à leur composition aux précédentes.

En général les phosphorites tertiaires contiennent relativement plus de substances insolubles dans les acides et une moindre quantité de phosphate de calcium que les phosphorites crétacées.

Michał Kamiński.

**Ruch komety Wolfa I pod wpływem planety Urana
w okresie 1884—1919.**

Referat przedstawiony dn. 4 grudnia 1930 r.

W swym referacie autor przedstawia wyniki badań nad wpływem zakłóceń, pochodzących od Urana a oddziałujących na bieg komety Wolfa I w okresie 1884—1919. Z tych badań wynika, że zakłócenia te są dość znaczne (np. w anomalji średniej sięgają do $-10''$) — a więc powinny być uwzględniane przy zestawieniu różnych powrotów komety do słońca.

M. Kamiński.

**Über den Einfluss des Planeten Uranus
auf die Bewegung des Kometen Wolf I in dem
Zeitraume 1884—1919.**

Vorgelegt am 4 December 1930 r.

In der Mitteilung berichtet der Schriftsteller die Resultate seiner Untersuchungen über den Einfluss der Uranusstörungen auf die Bewegung des Kometen Wolf I in den Jahren 1884—1919. Aus den obigen Untersuchungen geht hervor, dass diese Störungen ziemlich bedeutend sind (z. B. in Mittlerer Anomalie erreichen sie bis $-10''$) — so dass sie, bei Zusammenstellung der verschiedenen Wiederkommen des Kometen zur Sonne, berücksichtigt sein müssen.

