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Marek Sokołowski

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Marek Sokołowski
Zakład Teorii Ośrodków Ciągłych IPPT PAN

Application of the Method of Equivalent Moments to Statical and Dynamical Problems of Elastic Media with Cylindrical Holes and Inclusions

Abstract

A method of approximate analysis of stress fields in elastic media containing structural defects was proposed in paper [1], where statical problems of this kind were discussed. In the present paper it is shown that the method may also be applied to dynamical problems involving, for instance, elastic wave propagation across a medium containing arbitrarily distributed holes, cracks and inclusions. Numerical examples demonstrate that fairly accurate information on multiple elastic wave reflections and stress concentration are obtained in a body containing two cylindrical cavities by applying the integral transforms technique, the entire problem being reduced to the solution of a simple set of four linear algebraic equations.

1. Introduction. A method of approximate analysis of stationary stress states in unbounded elastic media containing cylindrical inclusions, holes and cracks was presented by this author in paper (1). A possibility of extending this method of analysis to quasi-static and dynamic cases was suggested in (2,3). This suggestion will further be explored here by discussing a simple case of an unbounded body containing two cylindrical circular inclusions or cavities and subject to the action of a plane elastic wave. The consideration will be preceded by a presentation of fundamental features of the method as applied to stationary problems and by simple numerical examples; they will be confined to the antiplane strain case, i.e. to the state characterized by a single non-vanishing displacement component $w = w(x,y)$ directed along the z-axis of a Cartesian coordinate

system x, y, z, two components of the strain tensor ε_{xz} , ε_{yz} , and two stress components σ_{xz} , σ_{yz} ,

$$\sigma_{xz} = \mu \frac{\partial w}{\partial x} , \qquad \sigma_{yz} = \mu \frac{\partial w}{\partial y} .$$

Function $w(x,y)$ satisfies the equation

$$\mu\nabla^2 w(x,y) = - p(x,y)$$

$p(x,y)$ denoting here the body force intensity.

Let us shortly recall the fundamental notions used in the analysis (presented in more detail in [1]). A concentrated body force P distributed along the z-axis, i.e. $P\ \delta(x)\ \delta(y)$, is called a moment of order zero and denoted M^{oo}. Such load produces in an unbounded elastic body stresses

$$\sigma_{xz} = - \frac{M^{oo}}{2\pi} \frac{\cos\theta}{r} , \qquad \sigma_{yz} = - \frac{M^{oo}}{2\pi} \frac{\sin\theta}{r}$$

$$\sigma_{\theta z} = - \sigma_{xz} \sin\theta + \sigma_{yz} \cos\theta , \quad \sigma_{rz} = \sigma_{xz} \cos\theta + \sigma_{yz} \sin\theta ,$$

$$\sigma_{rz} = - \frac{M^{oo}}{2\pi} \frac{1}{r} , \quad \sigma_{\theta z} = 0 ,$$

where r, θ denote polar coordinates of point (x,y), $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, $r^2 = x^2 + y^2$.

Confining the considerations to moments of orders 1, 2, 3 and remembering that moments M^{mn} lead to stresses

$$\sigma_{iz}^{mn} = \frac{\partial^m}{\partial x^m} \frac{\partial^n}{\partial x^n} \sigma_{iz}^{oo} , \qquad i = x,y$$

explicit formulae can be written for the stresses.

$$\sigma_{xz} = + \frac{M^{10}}{2\pi}\frac{\cos 2\theta}{r^2} + \frac{M^{01}}{2\pi}\frac{\sin 2\theta}{r^2} - \frac{M^{20}}{2\pi}\frac{2\cos 3\theta}{r^3} - \frac{M^{11}}{2\pi}\frac{2\sin 3\theta}{r^3} + \frac{M^{30}}{2\pi}\frac{6\cos 4\theta}{r^4} + \frac{M^{21}}{2\pi}\frac{6\sin 4\theta}{r^4}$$

(1.1)

$$\sigma_{yz} = + \frac{M^{10}}{2\pi}\frac{\sin 2\theta}{r^2} - \frac{M^{01}}{2\pi}\frac{\cos 2\theta}{r^2} - \frac{M^{20}}{2\pi}\frac{2\sin 3\theta}{r^3} + \frac{M^{11}}{2\pi}\frac{2\cos 3\theta}{r^3} + \frac{M^{30}}{2\pi}\frac{6\sin 4\theta}{r^4} - \frac{M^{21}}{2\pi}\frac{6\cos 4\theta}{r^4} .$$

It was shown in paper (1) that the stress fields in a medium containing inclusions and holes centered at points (x_i, y_i) and loaded at infinity (or far enough from the defects) by forces σ_{xz}^{∞} σ_{yz}^{∞}, may be evaluated approximately in the following manner: the holes (inclusions) are replaced with concentrated moments M^{mn} of suitably selected order and intensity, which are then applied to points (x_i, y_i) of the solid body; such moments are called *equivalent*. *Primary* stress σ_{rz} at (x_i, y_i) is the stress which would exist there in absence of that particular defect. Such a stress, denoted σ^{pr}, is the result of action of both the external loads σ^{∞} and the remaining defects (replaced by the corresponding equivalent moments).

In case of moments of orders 1 and 2 , intensities of the equivalent moments are found from the formulae

$$M^{01} = -2\pi\varkappa a^2 p_o \quad , \quad M^{10} = -2\pi\varkappa a^2 q_o$$

(1.2)

$$M^{11} = \pi\varkappa a^3 p_1 \quad , \quad M^{20} = \pi\varkappa a^3 q_1$$

Here p_0 , q_0 , p_1 , q_1 are the primary stresses and stress gradients evaluated at the given point

$$p_0 = \sigma_{xz}^{pr} \quad , \quad q_0 = \sigma_{yz}^{pr}$$

$$p_1 = \frac{\partial \sigma_{yz}^{pr}}{\partial(x/a)} \quad , \quad q_1 = \frac{\partial \sigma_{yz}^{pr}}{\partial(y/a)} \quad .$$

If $x_i = y_i = 0$, the stresses may be expanded in the neighborhood of that point into power series with the initial terms

$$\sigma_{xz} = q_0 + q_1\, x/a + p_1\, y/a \; ,$$
(1.3)
$$\sigma_{yz} = p_0 - q_1\, y/a + p_1\, x/a \; .$$

Here a is the radius of the circular inclusion, $\varkappa$ - the inhomogeneity measure: $\varkappa = (\mu - \mu')/(\mu + \mu')$, μ and μ' - shear moduli of the medium and inclusion, respectively. In what follows it will be assumed that $\varkappa = 1$ what means that we are dealing with a hole.

Formulae (1.1) are valid outside the holes and were shown in (1) to lead to satisfactory results provided the hole distribution was not too dense. In practice it means that the distances between the edges of the holes should be of the order of 1.5 hole diameters. In case of two holes of radii a and distance between their centers l it will be assumed that $\lambda = l/a > 5$.

Case of a single hole.

In case of an infinite body containing a single cylindrical cavity at $x = y = 0$ and loaded at infinity by linearly variable distributed forces σ_{xz}^{∞}, σ_{yz}^{∞} according to (1.3), the primary stress distribution is uniform and given by the same formula; thus the equivalent moments follow directly from (1.1) (the first four terms in each formula). The final state of stress (outside the hole, $r > a$), i.e. the sum of stresses (1.1) and (1.3), yields *accurate* results, what follows from the simple external stress distribution assumed. For instance,

$$\sigma_{rz} = (1 - 1/\rho^2) \cos\theta + p_0 (1 - 1/\rho^2) \sin\theta$$

(1.4)

$$+ q_1(\rho - 1/\rho^3) \cos 2\theta + p_1(\rho - 1/\rho^3) \sin 2\theta .$$

In the case when the primary stress distribution is more complicated (e.g. disturbed by other holes), formula (1.4) will be only approximate, thus creating the necessity of introducing higher order moments.

2. Stationary case of two holes. Consider now in more detail the case of stationary, antiplane state of stress in an unbounded body containing two circular cylindrical holes of equal diameters $2a$, their axes lying at a distance of $l = \lambda a$ from each other (Fig.1). The body is loaded at infinity by uniformly distributed forces $\sigma^{\infty}_{xz} = q_0$ and $\sigma^{\infty}_{yz} = p_0$.

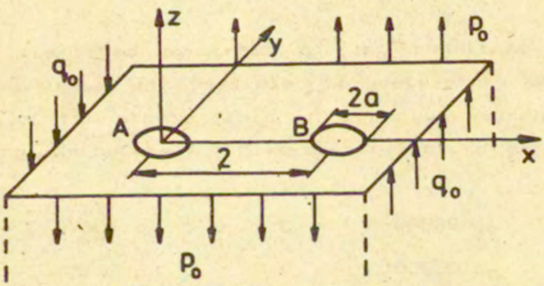

Fig. 1

First order approximation.

Replace holes A and B with 1st order moments M^{10}_A, M^{01}_A and M^{10}_B, M^{01}_B. According to Eqs.(1.2), the equivalent moment M^{10}_B is obtained by multiplying the primary stress value σ^{pr}_{xz} at B by the coefficient $-2\pi a^2$. Eqs. (1.2) may be rewritten in the form

$$M_B^{10} = -2\pi a^2 \ \sigma_{xz}^{pr}(B) \quad , \quad M_B^{01} = -2\pi a^2 \ \sigma_{yz}^{pr}(B) \quad .$$

Moment M_B^{10} which replaces (together with M_B^{01}) the action of hole B , depends on the primary stress value $\sigma_{xz}^{pr}(B)$ produced at the point (0,1) by the external load (q_0 in the present case, and by the moments of unknown intensities M_A^{10} and M_A^{01} applied to point (0,0), Fig.1. Using the corresponding formulae (1.1) (where only the terms involving M^{10} and M^{01} should be preserved), and substituting $\theta = \theta_{AB} = 0$, $r = r_{AB} = 1$, we obtain one equation with two unknowns

(2.1)
$$M_B^{10} = -2\pi a^2 \left[q_0 + \frac{M^{10}}{2\pi} \frac{1}{1^2} \right]$$

and a similar equation corresponding to σ_{yz} ,

(2.2)
$$M_B^{01} = -2\pi a^2 \left[p_0 - \frac{M^{01}}{2\pi} \frac{1}{1^2} \right]$$

Applying the analogous procedure to point A (here $\theta_{BA} = \pi$ and $r_{BA} = 1$), we obtain

(2.3)
$$M_A^{10} = -2\pi a^2 \left[q_0 + \frac{M^{10}}{2\pi} \frac{1}{1^2} \right] \quad ,$$

$$M_A^{01} = -2\pi a^2 \left[p_0 - \frac{M^{01}}{2\pi} \frac{1}{1^2} \right] \quad .$$

The four algebraic equations with four unknowns enable the determination of the first-order approximation of the solution. In view of the symmetry

(2.4)
$$M_A^{10} = M_B^{10} = M^{10} \quad , \quad M_A^{01} = M_B^{01} = M^{01} \quad , \quad \text{and}$$

$$M^{10} = \frac{-2\pi \ a^2}{1+1/\lambda^2} \, q_0 \quad , \quad M^{01} = \frac{-2\pi \ a^2}{1-1/\lambda^2} \, p_0 \quad .$$

By substituting these values into (1.1) and adding the external stresses q_o and p_o , we obtain the approximate formulae for stresses. For instance, along the x-axis (Fig1)

$$\sigma_{xz} = q_o \left[1 - \frac{1}{1+1/\lambda^2} \left(\frac{1}{\xi_A^2} + \frac{1}{\xi_B^2} \right) \right] ,$$

$$\sigma_{yz} = p_o \left[1 + \frac{1}{1-1/\lambda^2} \left(\frac{1}{\xi_A^2} + \frac{1}{\xi_B^2} \right) \right] ,$$

where additional notations for the dimensionless distances from the centers of holes A and B are introduced: $\xi_A = \frac{x}{a}$, $\xi_B = \frac{x-l}{a}$. Accuracy of the present approximation may be estimated by evaluating the σ_{xz} - stress variation along the x-axis in the case of $\lambda = 6$. The stresses should vanish at points $\xi_A = \pm 1$ and $\xi_B = \pm 1$ (at the hole edges). In our present solution the stresses are zero at $\xi_A = -0.9964$ and 1.0062, and at $\xi_B = -0.9938$ and 1.0036, the error being thus of the order of 0.3-0.6%.

Second order approximation. In order to determine the second order approximation of the problem shown in Fig.1 let us introduce new notations for the unknown equivalent moments M^{01} , M^{10} , M^{11}, M^{20} at points A and B. Using relations (1.2), (1.3) define four parameters at hole A,

$$q_0^A = \sigma_{xz}(A) \quad , \quad p_0^A = \sigma_{yz}(A) \quad ,$$

(2.5)

$$q_1^A = a \frac{\partial \sigma_{xz}}{\partial x} \quad , \quad p_1^A = a \frac{\partial \sigma_{yz}}{\partial y} \quad ,$$

and similar four parameters at the other hole B. Using these notations and Eqs. (1.1), and substituting for θ and r the corresponding values, we obtain the following set of eight linear algebraic equations with eight unknowns:

$$q_0^B = q_0 - q_0^A/\lambda^2 - q_1^A/\lambda^3 \quad ; \quad q_0^A = q_0 - q_0^B/\lambda^2 + q_1^B/\lambda^3 \quad ,$$

$$p_0^B = p_0 + p_0^A/\lambda^2 + p_1^A/\lambda^3 \quad ; \quad p_0^A = p_0 + p_0^B/\lambda^2 - p_1^B/\lambda^3 \quad ,$$

(2.6)

$$q_1^B = 0 + 2q_0^A/\lambda^3 + 3q_1^A/\lambda^4 \quad ; \quad q_1^A = 0 - 2q_0^B/\lambda^3 + 3q_1^B/\lambda^4 \quad ,$$

$$p_1^B = 0 - 2p_0^A/\lambda^3 - 3p_1^A/\lambda^4 \quad ; \quad p_1^A = 0 + 2p_0^B/\lambda^3 - 3p_1^B/\lambda^4 \quad .$$

From the symmetry of the problem and structure of these equations it follows that the number of unknowns may be reduced to four, since

$$q_0^B = q_0^A = Q_0 \ , \ p_0^B = p_0^A = P_0 \ , \ q_1^A = -q_1^B = Q_1 \ , \ p_1^A = -p_1^B = P_1$$

what reduces the set of Eqs.(2.6) to the following for ones

$$Q_0 \left[1 + \frac{1}{\lambda^2}\right] + Q_1 \frac{1}{\lambda^3} = q_0 \quad ,$$

$$Q_0 \frac{2}{\lambda^3} + Q_1 \left[1 + \frac{3}{\lambda^4}\right] = 0 \quad ,$$

$$P_0 \left[1 - \frac{1}{\lambda^2}\right] - P_1 \frac{1}{\lambda^3} = p_0 \quad ,$$

$$P_0 \frac{2}{\lambda^3} - P_1 \left[1 - \frac{3}{\lambda^4}\right] = 0 \quad ,$$

with the solutions

$$Q_0 = \frac{q_0}{1 + \frac{1}{\lambda^2} - \frac{2}{\lambda^6}\;\frac{1}{1+3/\lambda^4}} ,$$

$$Q_1 = \frac{-2q_0/\lambda^3}{1 + \frac{1}{\lambda^2} + \frac{3}{\lambda^4} + \frac{1}{\lambda^6}} ,$$

(2.7)

$$P_0 = \frac{p_0}{1 - \frac{1}{\lambda^2} - \frac{2}{\lambda^6}\;\frac{1}{1-3/\lambda^4}} ,$$

$$P_1 = \frac{2p_0/\lambda^3}{1 - \frac{1}{\lambda^2} - \frac{3}{\lambda^4} + \frac{1}{\lambda^6}} .$$

The results, completed by terms q_0, p_0 resulting from the external load, make it possible to determine the second approximation of the stress field in a medium weakened by two holes; assuming, as before $\lambda = 6$ we obtain from (2.7)

$$Q_0 = 0.97801\ q_0 \quad , \quad Q_1 = -\ 0.00899\ q_0 \quad ,$$
$$P_0 = 1.02862\ p_0 \quad , \quad P_1 = \ 0.00955\ p_0 \quad .$$

The σ_{xz} - stress variation along the x-axis is now given by the formula

(2.8) $$\sigma_{xz} = 1 - Q_0 \left[\frac{1}{\xi_A^2} + \frac{1}{\xi_B^2} \right] - Q_1 \left[\frac{1}{\xi_A^3} + \frac{1}{\xi_B^3} \right] ,$$

where attention should be paid to the sings of corresponding expressions $\frac{1}{\xi_A^3}$ and $\frac{1}{\xi_B^3}$ on both sides of points A and B.

Simple analysis of (2.8) shows that the accuracy of the second approximation (measured by positions of points at which

$\sigma_{xz} = 0$) is higher than before, since now

$$\sigma_{xz}(\xi_A) = 0 \quad \text{for} \quad \xi_A = 1.00153 \quad \text{and} \quad -0.99906 \; ,$$

and

$$\sigma_{xz}(\xi_B) = 0 \quad \text{for} \quad \xi_B = 1.00094 \quad \text{and} \quad -0.99847 \; ;$$

The error is thus of the order of 0.1%.

3. <u>Dynamic case.</u> Consider now the case of an unbounded body remaining stress-free for time $t < 0$. Assume the body to be loaded at time $t = 0$ by moments $M^{10}(t)$, $M^{01}(t)$ distributed uniformly along the z-axis. It was shown in (3) that by applying the Laplace transform

$$\bar{w}(p) = \int_0^\infty w(t)\, e^{-pt}\, dt \; ,$$

the wave equation

$$\nabla^2 w - \frac{1}{c^2} \frac{\partial^2 w}{\partial t^2} = - \frac{P(t)}{\mu}$$

(here $c = \sqrt{\mu/\rho}$ is the shear-wave speed) can be solved to yield the following solution

$$\bar{w} = - \frac{\bar{M}^{10} \cos\theta + \bar{M}^{01} \sin\theta}{2\pi\, \mu\, c}\, p\, K_1\left(\frac{pr}{c}\right) \; ,$$

(3.1)

$$\sigma_{rz} = \frac{\bar{M}^{10} \cos\theta + \bar{M}^{01} \sin\theta}{2\pi\, c}\, p \left[\frac{1}{r} K_1\left(\frac{pr}{c}\right) + \frac{p}{c} K_0\left(\frac{pr}{c}\right) \right].$$

Explicit inverse transform of (3.1) is available under the assumption that M^{10}, M^{01} are kept constant for $t > 0$, i.e.

$$M^{10}(t) = M^{10}\, \eta(t) \quad , \quad M^{01}(t) = M^{01}\, \eta(t) \; .$$

Then the cylindrical stress wave assumes the form

(3.2) $$w(r,\theta,t) = -\frac{M^{10}\cos\theta + M^{01}\sin\theta}{2\pi\mu}\frac{1}{r}\frac{1}{\sqrt{1-R^2}}$$

and, e.g.,

(3.3)

$$\sigma_{rz}(r,\theta,t) = \frac{M^{10}\cos\theta + M^{01}\sin\theta}{2\pi}\frac{1}{r^2}\left[\frac{1}{\sqrt{1-R^2}} - \frac{R^2}{\sqrt{1-R^2}^{\,3}}\right]$$

where $R = r/ct$; solutions (3.2), (3.3) are valid for $R < 1$; for $r > ct$ (ahead of the wave front) displacements and stresses vanish.

It is seen that for a fixed value of r and $t \to \infty$, R tends to 0 and Eqs. (3.2), (3.3) approach the statical form

$$w(r,\theta) = -\frac{M^{10}}{2\pi\mu}\frac{\cos\theta}{r} - \frac{M^{01}}{2\pi\mu}\frac{\sin\theta}{r} ,$$

(3.4)

$$\sigma_{rz}(r,\theta) = \frac{M^{10}}{2\pi}\frac{\cos\theta}{r^2} + \frac{M^{01}}{2\pi}\frac{\sin\theta}{r^2} .$$

Stress (3.3) exhibits two kinds of singularities: one is located at $r = 0$ (point of application of the load), as in the static case. The other singularity appears at the wave-front $R = 1$, i.e. at $r = ct$, and is propagated into the body. This makes it difficult to use Eq. (3.3) for simulating the action of a hole in the body in the dynamic case, and this is why a different approach was proposed in (3).

Consider a body containing a hole of diameter $2a$ centered at $x = y = r = 0$, and a plane stress wave $\sigma_{xz}(x,t) = \sigma_0\,\eta(t-x/c)$ travelling across the body. At time $t = 0$ the wave reaches the point $r = 0$. To simulate the action of the hole, at time $t = 0$ a moment M^{10} should be applied to the same point. Its intensity may be calculated from the formula analogous to (1.2), (3.5) $$M^{10} = -2\pi\, a^2 \sigma_{xz}^{pr}(0) \quad ,$$

rewritten in the form of L-transforms

(3.6) $$\bar{M}^{10}(p) = -2\pi\, a^2 \bar{\sigma}_{xz}^{pr}(0,p) \quad ,$$

$\bar{\sigma}_{xz}^{pr}(0,p)$ denoting the L-transform of the primary stress produced at $r = 0$ by the travelling plane wave. Since in the present case

$$\bar{\sigma}_{xz}^{pr}(x,p) = \int_0^\infty \sigma_0 \eta(t-x/c)\, e^{-pt}\, dt = \sigma_0 \frac{1}{p}\, e^{-px/c} \quad ,$$

the equivalent moment $\bar{M}^{10}$ is

(3.7) $$\bar{M}^{10}(p) = -2\pi\, a^2 \sigma_0 / p \quad .$$

Superposition of the stresses produced by the external load (plane stress wave) and by the moment (3.7) (cylindrical wave) leads to the solution (cf. Eq. (3.1))

(3.8)
$$\bar{\sigma}_{rz} = \sigma_0 \cos\theta \left[\frac{1}{p}\, e^{-pr\cos/\theta} - \frac{a^2}{c} \left(\frac{1}{r} K_1\left(\frac{pr}{c}\right) + \frac{p}{c} K_0\left(\frac{pr}{c}\right) \right) \right] .$$

Inverse transform of (3.6) yields (cf. Eq. (3.3))

(3.9) $$\sigma_{rz} = \sigma_0 \cos\theta \left[\eta(t-r\cos\theta/c) - \frac{a^2}{c^2} \left(\frac{1}{\sqrt{1-R^2}} - \frac{R^2}{\sqrt{1-R^2}^{\,3}} \right) \right] .$$

It is seen that solution (3.7) satisfies the condition of stress-free boundary of the hole (at $r = a$) only approximately: for large values of time $t \to \infty$ we have $R \to 0$ and then

$$\sigma_{rz}(a,t) \cong \sigma_0 \cos\theta \left[1 - \left(1 - \frac{R^2}{2} \right) \right] \longrightarrow 0$$

In addition, at the wave front $R = 1$ the unjustified stress singularity remains; that is why another approximate approach to the problem was proposed in (3).

In order to fulfil the condition $\sigma_{rz}(a,\theta) = 0$ and to avoid the wave-front stress singularity, the concentrated equivalent moment M^{10} should be replaced by a suitable distribution of stresses σ_{rz} applied (at time $t=0$) to the lateral surface of a cylindrical hole in the body; however, as it was shown in (3), such problem leads to Laplace transforms which cannot be inverted. A fairly good approximate solution was proposed

(3.10)
$$\sigma_{rz}(r,\theta,t) = \frac{M^{10}}{2\pi} \frac{\cos\theta}{r^2} \eta(t-r/c) \quad ,$$

$$\bar{\sigma}_{rz}(r,\theta,p) = \frac{M^{10}}{2\pi} \frac{\cos\theta}{r^2} \frac{1}{p} e^{-pr/c} \quad ,$$

which represents simply the statical solution cut off by the suitable wave factor. Assuming the value of equivalent moment M^{10} in the form (3.7) and adding the plane elastic wave, we obtain

(3.11)
$$\sigma_{rz}(r,\theta,t) = \sigma_0 \cos\theta \left[\eta(t-r\cos\theta/c) - \frac{a^2}{r^2} \eta(t-r/c) \right] .$$

In the transformed form

(3.12)
$$\bar{\sigma}_{rz}(r,\theta,p) = \frac{\sigma_0 \cos\theta}{p} \left[e^{-pr\cos\theta/c} - \frac{a^2}{r^2} e^{-pr/c} \right] .$$

It is seen that solution (3.11) satisfies the condition

$\sigma_{rz}(a,\theta,t) = 0$ for all times $t > a/c$, and that for $t \to \infty$ it approaches the static solution (1.4),

$$\sigma_{rz}(r,\theta) = \sigma_0 \cos\theta \left[1 - \frac{a^2}{r^2}\right].$$

Obviously, (3.11) does not satisfy the wave equation first of all, close to the wave front), but it may be assumed that the resulting errors will not be to high.

4. Another approach to problems involving several holes.

Let us now demonstrate how the method presented in Sec.2 could be applied to the analysis of an elastic antiplane stress wave propagating across a medium containing several holes. To simplify the discussion, assume there are two holes A and B of equal diameters 2a located at the x-axis at a distance of l from each other (Fig.2). Confine the considerations to the wave

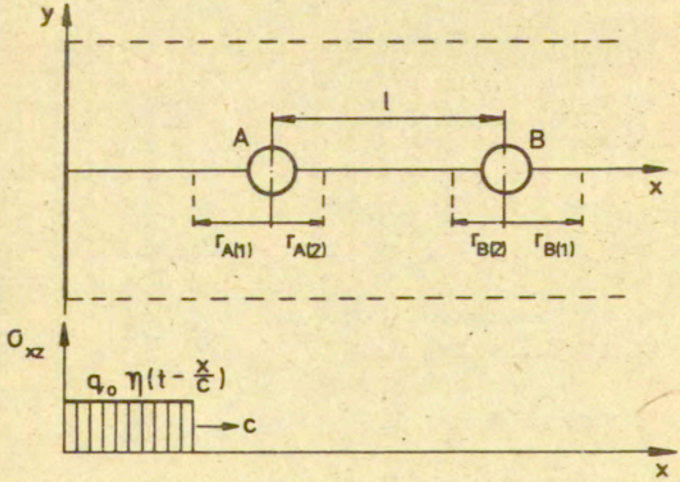

Fig. 2

propagation along the x-axis, and consider first the wave produced by one of the equivalent moments, for instance M_A^{10}, replacing the hole A. If the transformed primary stress produced at A by external load (incoming wave) is denoted by $\sigma_{xz}^{pr}(A)$ then, according to (3.6),

$$\bar{M}_A^{10}(p) = -2\pi\, a^2\, \bar{\sigma}_{xz}^{pr}(A) \quad . \tag{4.1}$$

Denote now the absolute value of distance from the center of A by r_A , $r_A = |x - x_A|$. If point A is reached at time $t_o = x_A/c$ by the stress wave $\sigma_{xz} = q_o\, \eta(t - x/c)$, then is follows from (4.1) that

$$\bar{M}_A^{10}(p) = -2\pi\, a^2\, (q_o/p)\, e^{-pt_o} \quad , \tag{4.2}$$

and the stress wave produced by $\bar{M}_A^{10}$ is given by Eq. (3.10)

$$\bar{\sigma}_{xz}(x,p) = -q_o\, \frac{a^2}{r_A^2}\, \frac{1}{p}\, e^{-p(t_o+r_A/c)} \tag{4.3}$$

since here $\sigma_{xz} = \sigma_{rz} \cos\theta$. Inverse transformation yields

$$\sigma_{xz}(x,t) = -q_o\, \frac{a^2}{\rho_A^2}\, \eta(t - t_o - r_A/c) \tag{4.4}$$

with the notation $\rho_A = r_A/a$. Ratio l/a is denoted by λ , and it is assumed again that $\lambda > 5$. The σ_{xz} - stress at a time instant t , $t_o < t < t_1$ due to the incoming wave $q_o\, \eta(t - x/c)$ and the wave produced by M_A^{10} is sketched in Fig. 3.

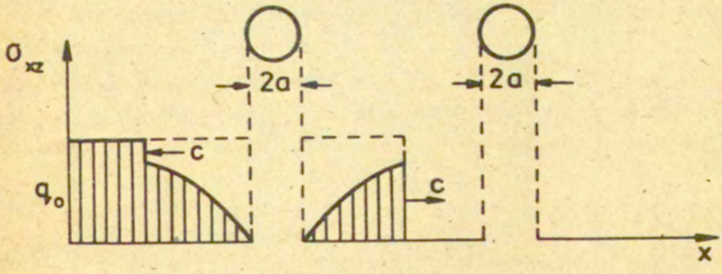

Fig. 3

Denote by t_1 the time at which the wave produced at A will reach point B ; its intensity (amplitude), according to (4.4), will be reduced by the factor $-1/\lambda^2$ to $-q_0/\lambda^2$. The wave produced at time t_1 at B returns to A at time t_2 , etc. In general, $t_n = (x_A + nl)/c$, $n = 0,1,2,\dots,\infty$.

Evaluate the intensities of moments M^{10} produced subsequently at A and B.

(1) At $t = t_0$ stress $\sigma^{pr}_{xz} = q_0$ arrives in A and produces moment $M^{1(}_{A(0)} = -2\pi a^2 q_0$.

(2) At $t = t_1$ stress $q_0 - q_0/\lambda^2$ arrives in B and produces $M^{1(}_{B(1)} = -2\pi a^2 q_0(1 -1/\lambda^2)$.

(3) At $t = t_2$ stress $-(q_0/\lambda^2)(1 -1/\lambda^2)$ arrives in A and produces $M^{1(}_{A(2)} = 2\pi a^2 q_0(1/\lambda^2 -1/\lambda^4)$, etc.

L-transforms for the expressions

$M^{1(}_{A(0)}\,\eta(t - t_0)$, $M^{1(}_{B(1)}\,\eta(t - t_1)$, $M^{1(}_{A(2)} = \eta(t - t_2)$ are

$$\bar{M}^{1(}_{A(0)} = -2\pi a^2 q_0 \frac{1}{p} e^{-pt_0} ,$$

$$\bar{M}^{1(}_{B(1)} = -2\pi a^2 q_0 (1 -1/\lambda^2) \frac{1}{p} e^{-pt_0} , \text{ etc.}$$

Adding together the subsequent results and remembering that $t_n = (x_A + nl)/c$, we obtain

$$\bar{M}^{10}_A = -2\pi a^2 q_0 \frac{1}{p} e^{-px_A/c} *$$

$$* \left[1 - (1/\lambda^2 -1/\lambda^4) e^{-2pl/c} - (1/\lambda^6 -1/\lambda^8) e^{-4pl/c} - \dots\right]$$

(4.5)

$$\bar{M}^{10}_B = -2\pi a^2 q_0 \frac{1}{p} e^{-px_A/c} *$$

$$* \left[(1 - 1/\lambda^2) e^{-pl/c} + (1/\lambda^4 -1/\lambda^6) e^{-3pl/c} - \dots\right] .$$

The geometric series (4.5) may easily be summed up to yield the final results

$$\bar{M}_A^{10} = -2\pi\ a^2 q_0\ \frac{e^{-px_A/c}}{p}\ \frac{1-(1/\lambda^2)\ e^{-2pl/c}}{1-(1/\lambda^4)\ e^{-2pl/c}}$$

(4.6)

$$\bar{M}_B^{10} = -2\pi\ a^2 q_0\ \frac{e^{-p(x_A+l)/c}}{p}\ \frac{1-1/\lambda^2}{1-(1/\lambda^4)\ e^{-2pl/c}}\ .$$

Consider two limiting cases of solution (4.6).

(1) If the holes are located very far apart, i.e. if $l \gg a$, then, with $\lambda \rightarrow 0$,

$$\bar{M}_A^{10} = -2\pi\ a^2 q_0\ \frac{1}{p}\ e^{-px_A/c}\ ,$$

$$\bar{M}_B^{10} = -2\pi\ a^2 q_0\ \frac{1}{p}\ e^{-p(x_A+l)/c}\ ,$$

what means that there is no interaction between the holes, and equal moments $\bar{M}_A^{10} = \bar{M}_B^{10} = -2\pi\ a^2 q_0$ are "generated" at time instants $t_0 = x_A/c$ and $t_1 = (x_A+l)/c$ at the respective points A and B.

(2) The stationary values of $\bar{M}_A^{10}$, $\bar{M}_B^{10}$ at $t \rightarrow \infty$ are found by assuming $p \rightarrow 0$ in Eqs. (4.6),

$$\bar{M}_A^{10} = -\frac{2\pi\ a^2}{1+1/\lambda^2}\ q_0\ \frac{1}{p}\ e^{-px_A/c}\ ,$$

$$\bar{M}_B^{10} = -\frac{2\pi\ a^2}{1+1/\lambda^2}\ q_0\ \frac{1}{p}\ e^{-p(x_A+l)/c}\ ;$$

Inverse transformations for large values of t yield

$$\bar{M}_A^{10} = \bar{M}_B^{10} = -\frac{2\pi\ a^2}{1+1/\lambda^2}\ q_0\ .$$

5. <u>Direct approach</u>. <u>Conclusions.</u> Solutions (4.6) were determined by a rather laborious procedure of summing up the results of constant generation of new moments M_A^{10} and M_B^{10}. The

same result may, however, be achieved directly by means of a procedure analogous to that applied in Section 2 to the statical problem of two holes under constant shear loading $\sigma_{xz} = q_0$. If we put $p_0 = 0$ in Eqs.(2.1), (2.2), the two unknown equivalent moments M_A^{10}, M_B^{10} are determined from the set of two equations

$$M_B^{10} = -2\pi\, a^2 \left[q_0 + \frac{M_A^{10}}{2\pi} \frac{1}{l^2} \right] ,$$

$$M_A^{10} = -2\pi\, a^2 \left[q_0 + \frac{M_B^{10}}{2\pi} \frac{1}{l^2} \right] ,$$

whence it follows that

$$M_A^{10} = M_B^{10} = - \frac{2\pi\, a^2}{1 + 1/\lambda^2}\, q_0 \quad .$$

In the dynamic case, by using an analogous approach to the transformed values $\bar{M}_A^{10}$, $\bar{M}_B^{10}$ (cf. Eqs. (4.1), (4.2)), we may write formally

$$\bar{M}_B^{10} = -2\pi\, a^2 \left[q_0 \frac{1}{p} e^{-p(x_A+l)/c} + \frac{M_A^{10}}{2\pi} \frac{1}{l^2} e^{-pl/c} \right] ,$$

(5.2)

$$\bar{M}_A^{10} = -2\pi\, a^2 \left[q_0 \frac{1}{p} e^{-px_A/c} + \frac{M_B^{10}}{2\pi} \frac{1}{l^2} e^{-pl/c} \right] .$$

Factors exp (-pl/c) multiplying the right-hand M^{10}- terms correspond to the time delay needed by the elastic wave to travel from A to B or from B to A.

Solution of the set of Eqs. (5.2) is identical with (4.6). This proves the possibility of applying the equivalent moment procedure described in (1) (after certain modifications) to media containing multiple structural defects. The whole problem is then reduced to a solution of a simple set of algebraic equations of the type of (5.2). Solution (4.6) may then be expanded back into a power series of $1/\lambda^2$, as in Eq. (5.5), to demonstrate the multiple wave reflections at the structural defects.

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Zastosowanie metody momentów zastępczych do statycznych i dynamicznych zagadnień ośrodków sprężystych z cylindrycznymi otworami i inkluzjami

W pracy [1], zaproponowano metodę przybliżonej analizy pól naprężeń w ośrodkach sprężystych zawierających defekty strukturalne, ograniczając się przy tym do przypadków statycznych. W niniejszej pracy pokazano, że metodę tę stosować można również do zagadnień dynamicznych, takich jak np. propagacja fali sprężystej w ośrodku zawierającym dowolne układy otworów, szczelin i inkluzji. Na przykładach liczbowych wykazano, że przy zastosowaniu techniki transformacji całkowych można uzyskać dość dokładne informacje o wielokrotnych odbiciach i koncentracjach naprężeń przy przechodzeniu fali płaskiej przez ośrodek zawierający dwa otwory cylindryczne, sprowadzając całe zagadnienie do rozwiązania prostego układu czterech liniowych równań algebraicznych.