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EVALUATION OF MECHANICAL RATCHETING USING ENDOCHRONIC THEORY

1. Motivation to study the problem

When a material element is under steady stress with superposed cyclic high stress then permanent deformations in the direction of steady stress accumulate and result in large deformations. This phenomenon will here be called ratcheting. For instance, in a nuclear reactor in operation, a situation may occur when a pressure vessel subjected simultaneously to an internal pressure and a cyclic heat flux expands gradually in each cycle provided that stresses are large enough. Other technological examples concern: turbin blades and disks subjected to a large temperature gradient in presence of centrifugal forces, pressurized piping under high bending or torsional loading; high-speed aircraft wings subjected to combined thermal and mechanical load, and alike.

In 1960 L.F. Coffin [23] noticed that after a number of cycles of plastic straining the head of a dial gauge slightly loaded had made a visible identation into material. This is another example of ratcheting which he designated as "the cyclic strain induced creep".

In relation with the ratcheting, an industrially observable phenomenon, we can distinguish several interesting and important questions realizing an appropriate analysis within the theory of plasticity:

a/ Under a cyclic loading other phenomena may appear which do not occur under monotous loading. The problem of the metallic material behaviour under cyclic plastic straining, has not yet been fully studied.

b/ The material behaviour under multiaxial cyclic agencies has not yet been given in terms of invariant relations employing all the basic invariants of the tensor arguments involved in constitutive relations.

c/ Plastic second order effects and highly complicated behaviour of dissipative materials have not yet been presented within the nonlinear continuum mechanics. The classical linear and infinitesimal constitutive relations are not sufficient for that purpose. It is important to establish constitutive equations which are appropriate and effective.

2. Actual state of knowledge in the domain

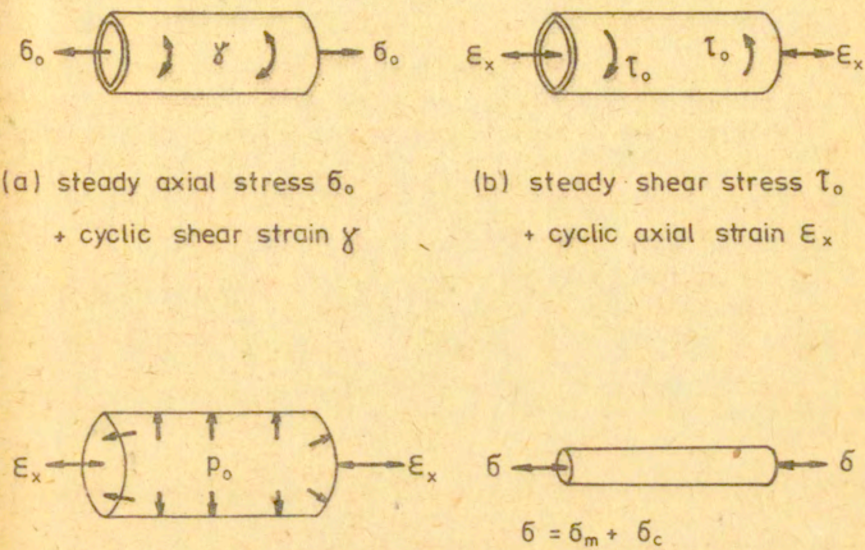
Almost all theoretical studies on the mechanical ratcheting done hitherto under a steady stress and a superposed cyclic loading were based upon the incremental theory of plasticity and the plastic potential flow law. To explain ratcheting due to a steady normal stress and a superposed cyclic shear strain of constant amplitude, /See Fig. 1.1/a/, Freudenthal and Ronay [19] introduced the third invariant of the deviatoric stress tensor into the plastic potential and concluded that the ratcheting is a second order effect due to the third invariant.

Introducing the notion of a field of workhardening moduli, Mróz [16] extended the theories of kinematic and isotropic hardening in an attempt to describe the material behaviour under cyclic loads. In the stress space he postulated a number of closed surfaces which do not intersect but consecutively contact and push each other, and can translate, expand or contract during plastic deformations. The surfaces $f_0, f_1, f_2 \ldots$ define regions of constant workhardening moduli; here f_0 denotes the yield surface and f_1, f_2 are the surface defining the regions of constant workhardening moduli. The equations of two neighboring surfaces f_l and f_{l+1} are

$$f(\sigma_{ij} - \alpha_{ij}^{(l)}) - (\sigma_0^{(l)})^n = 0, \qquad f(\sigma_{ij} - \alpha_{ij}^{(l+1)}) - (\sigma_0^{(l+1)})^n = 0, \qquad /2.1/$$

while the instantaneous translation of surface f_l is given by

$$d\alpha_{ij} = \frac{d\mu}{\sigma_o^{(l)}}\left[\left\{\sigma_o^{(l+1)} - \sigma_o^{(l)}\right\}\sigma_{ij}^{(l)} - \alpha_{ij}^{(l)}\,\sigma_o^{(l+1)} - \alpha_{ij}^{(l+1)}\,\sigma_o^{(l)}\right] \qquad /2.2/$$



(a) steady axial stress σ_o + cyclic shear strain γ

(b) steady shear stress τ_o + cyclic axial strain ε_x

(c) steady internal pressure p_o + cyclic axial strain ε_x

(d) mean stress σ_m + cyclic stress σ_c

Fig. 1.1 Examples of mechanical ratcheting where

$$d\mu = \frac{(\partial f/\partial \sigma_{ij})d\sigma_{ij}}{(\partial f/\partial \sigma_{kl})(\sigma_{kl}^{(l+1)} - \sigma_{kl}^{(l)})}$$

and the stress point defined by $\sigma_{ij}^{(l)}$ on f_l has not yet reached the surface f_{l+1}.

When analysing, within the proposed model, the ratcheting under steady normal stress and cyclic shear strain, he approximated the workhardening curve by a bilinear curve that does not change throughout cyclic straining process. Such on approach was not sufficient to describe a strong dependence of the ratchet strain on both the steady stress and the cyclic strain amplitude.

Valanis [1] proposed the endochronic theory of plasticity /for brevity we shall write further simply "ET"/. The ET /endochronic theory/ does not make use of the notion of yield surface but introduces an intrinsic time as a property of the material. For the sake of completness and further references the Valanis prototype ET will now be briefly. From the outset we confine our attention to rate independent materials, the temperature effects being disregarded.

The central concept of the ET of plasticity is the intrinsic time which is independent of the natural /Newtonian/ time but is intrinsically related to the deformation of the material. The intrinsic time should be a monotonically increasing function of deformation. First the time measure is defined by the relation

$$d\zeta^2 = P_{ijkl}\, dC_{ij}\, dC_{kl} \quad , \qquad /2.3/$$

where C_{ij} is the right Cauchy-Green strain tensor and P_{ijkl} is a fourth order tensor, the componants of which mass depend on C_{ij}. The positive definite nature of $d\zeta$ requires that P_{ijkl} be positive definite. In the case of small deformation /2.3/ reduces to

$$d\zeta^2 = p_{ijkl}\, d\varepsilon_{ij}\, d\varepsilon_{kl}, \qquad /2.4/$$

where ε_{ij} is the small deformation strain tensor and p_{ijkl} may depend on ε_{ij}. For isotropic materials

$$p_{ijkl} = k_1 \sigma_{ij} \sigma_{kl} + k_2 \sigma_{ik} \sigma_{jl}, \qquad /2.5/$$

where k_1 and k_2 are material constants such that $k_1 + \frac{1}{3}k_2 \geq 0$, $k_2 > 0$. Once a time measure is introduced an intrinsic time scale is needed. To this end the intrinsic time scale z is employed such that

$$z = z(\zeta) \quad ; \quad \frac{dz}{d\zeta} = f(\zeta) > 0 \qquad (0 < \zeta < \infty) \qquad /2.6/$$

thus it is a positive monotonously increasing function of ζ

The particular case

$$\zeta = \int_0^{\zeta} \{ d\varepsilon_{ij}\, d\varepsilon_{ij} \}^{1/2}, \qquad z = \zeta + \zeta_0 \qquad /2.7/$$

does not depend on the material constants k_1, k_2. It means that z in this case is independent of the material considered. It measures simply the lentgth of deformation history in the strain space. Pipkin and Rivlin [13] called it "the arc length". In this case 'z' is not material property.

A simple linear form of $f(\zeta)$ is obtained putting

$$\frac{dz}{d\zeta} = \frac{1}{(1+\beta\zeta)}. \qquad /2.8/$$

This form was found to give a good agreement in some significant experiments [1,2,3] .

The constitutive equation for an isotropic material is

$$\sigma_{ij}(z) = \delta_{ij} \int_0^z \lambda(z-z') \frac{\partial \varepsilon_{kk}}{\partial z'} dz' + 2\int_0^z \mu(z-z') \frac{\partial \varepsilon_{ij}}{\partial z'} dz' \quad ; \qquad /2.9/$$

$$\lambda(z) = \lambda_\infty + \sum_{r=1}^{n} \lambda_r e^{-\varrho_r z} \quad ; \quad \mu(z) = \mu_\infty + \sum_{r=1}^{n} \mu_r e^{-\alpha_r z},$$

where $\lambda_{\infty}, \lambda_r, \rho_r, \mu_{\infty}, \mu_r$ and α_r are positive constants.

For the analysis of ratcheting due to a steady normal stress and a superposed strain - controlled cyclic shear strain Valanis and Wu [3] introduced and intrinsic time measure such that

$$d\zeta = \bar{k}_2(\eta_{max})|d\eta| , \qquad /2.10/$$

where $d\eta$ is the shear strain increment and k_2 is a certain function of η_{max}, whereas η_{max} is the amplitude of the cyclic shear strain. The intrinsic time measure is here eventually an empirical function of the strain amplitude. Since they introduced the strain amplitude and the number of cycles, which are not state variables, the proposed equation is rather empirical. The relation is not in principle applicable to cyclic strain histories with varying strain amplitudes and complicated shapes of the cycles as well as to multidimensional situations. Also it cannot be used for stress - controlled ratcheting in which the strain amplitude changes in general.

Thereafter, Murakami and Nashiro [11] introduced the invariants of strain instead of the strain amplitude, which is not a state variable, into the intrinsic time measure in the following way

$$d\zeta^2 = k_1^2 d\varepsilon_{ii}\, d\varepsilon_{kk} + k_2^2 d\varepsilon_{ij}\, d\varepsilon_{ij} , \qquad /2.11/$$

where

$$k_1 = K_{10} + K_{11}|I_\varepsilon| + K_{12}|II_\varepsilon| + K_{13}|III_\varepsilon| ,$$

$$k_2 = K_{20} + K_{21}|I_\varepsilon| + K_{22}|II_\varepsilon| + K_{23}|III_\varepsilon| ,$$

and K_{ij} are material constants; I_ε , II_ε , III_ε are invariants of strain. It is an explicity generalized form of intrinsic time measure. To explain the influences of the strain amplitude upon a ratchet strain and the time measure they used the time scale

$$z = \dot{p}^{\alpha} \; , \qquad /2.12/$$

where α is constant such that $0 < \alpha < 1$.

The intrinsic time measure eq. (2.11) is appropriate to describe the nonlinear dependence of ratchet strain on the strain amplitude. Sawczuk and Nashiro [12] the pusblished data on the ratcheting behaviour interpreted in terms of the ratchet strain and the invariants of the strain amplitude.

Fig. 1.2 /a/ shows the axial ratchet strain-the second invariant of shear strain amplitude relation for 0.45 per cent carbon steel S45C by Udoguchi et al [30] under steady axial stress $\sigma_0 = 6.7$ kg/mm^2 and superposed axial strain amplitude γ^a = 1.05, 1.30, 2.15, 2.55 and 5.20% /engineering strain/ using cycle number n as a parameter. We can conclude from the figure that for S45C steel the axial ratchet strain is linear with respect to the second invariant of the axial strain amplitude. However it is also shown that the ratchet strain is not linear neither in the third invariant of the strain amplitude III_{ε^a} nor in the root of the sum of squares of the second and third invariants of strain amplitude$(\overline{II}^2_{\varepsilon^a} + \overline{III}^2_{\varepsilon^a})^{1/2}$.
Figs. 1,2/b/ give the axial ratchet strain and the second invariant of the strain amplitude for a superpure aluminum reported by Freudenthal and Ronay [19] . These figure show also that for the aluminum the ratchet strain is linear in the second invariant of strain amplitude.

3. Limitation of the former studies

Theories for ratcheting should based on the results of tremendous amount of recent experimental studies on the material behaviour under cyclic straining. The characteristic features of the behaviour under cyclic straining is given by the hysteresis loop at large cycle numbers. Observing the hysteresis loop /see Fig. 5.1/ we may point out the following two properties:

a/ Yielding is very unclear. It does not show sharp transition /as mild steel in simple tension test does/. Hence,

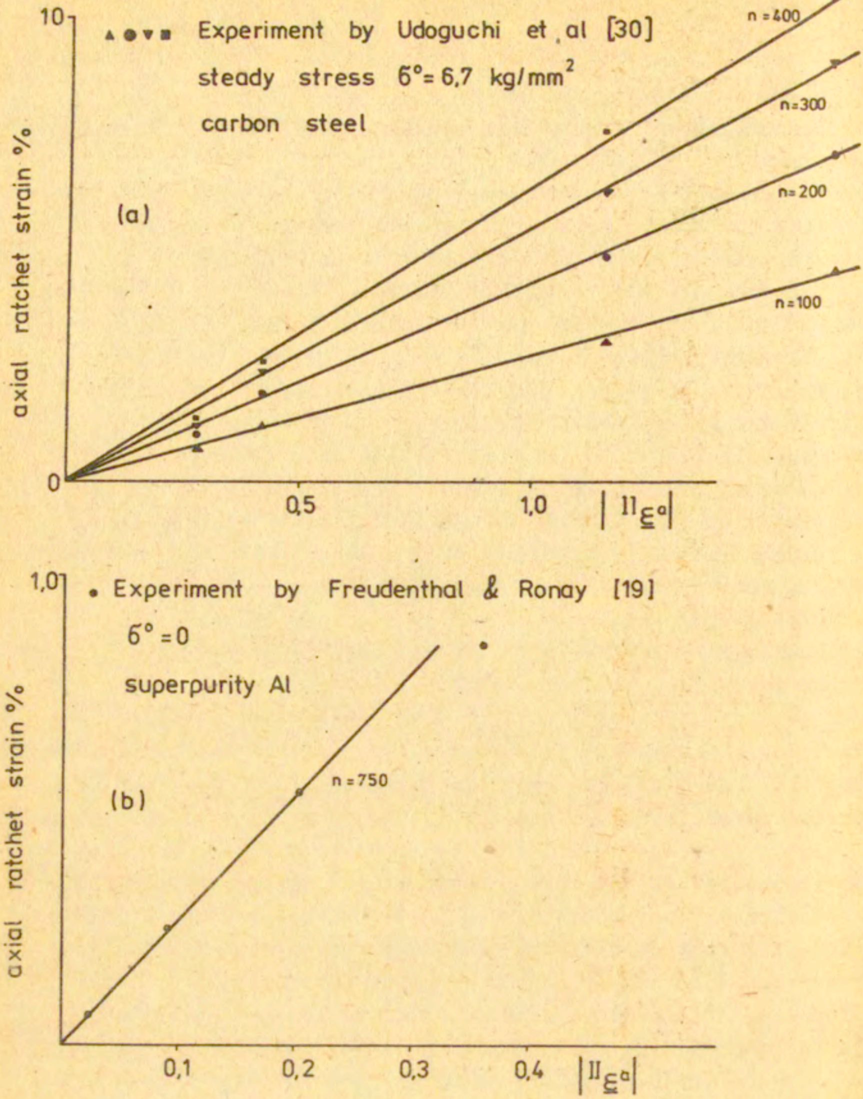


Fig. 1.2 Axial ratchet strain vs. the second invariant of strain amplitude

different definitions of yielding /e.g. deviation from linearity, 0,2% offset/ change markedly the values of yield stress and plastic modulus.

b/ In the loop it is not almost seen a linear part. It means that the elastic range is not well defined. Moreover the unloading part is not a straight line. It means that the inelastic strain rate is not zero there.

The notion of a yield surface is not appropriate at least to describe /a/ and /b/.

Related aspects of ratcheting phenomenon in the control strain and stress experiments are:

c/ Depending on its initial internal structure the material shows cyclic hardening or cyclic softening in the sense that both the peak stress and the shape of hysteresis loop change /see Fig. 1.3/.

d/ The Bauschinger effect, the cyclic stress relaxation.

Theories of ratcheting should have incorporated experimentally established properties of materials under cyclic straining.

Analyzing the ratchet mechanism, early works by Miller [20], Burgreen [21,22] assumed that a elasto-perfectly plastic or a bilinear stress - strain curves fully describe the plastic properties. Moreover, the employed relations could have not be influenced by a cyclic loading at all. The assumption of elasto-perfectly plastic relation has lead to the case in which the accumulation rate of ratchet strain is constant [20,21,22]. The bilinear approximation has predicted the ratchet strain tends to the asymptotic value [20] . These two approximations however, cannot describe the experimental results on a reduction of accumulation rate with the ratchet strain which has no asymptotic value.

For the description of ratcheting the following three criteria are conceivable:

1/ Nonlinear dependence of ratchet strain on amplitude of cyclic strain.

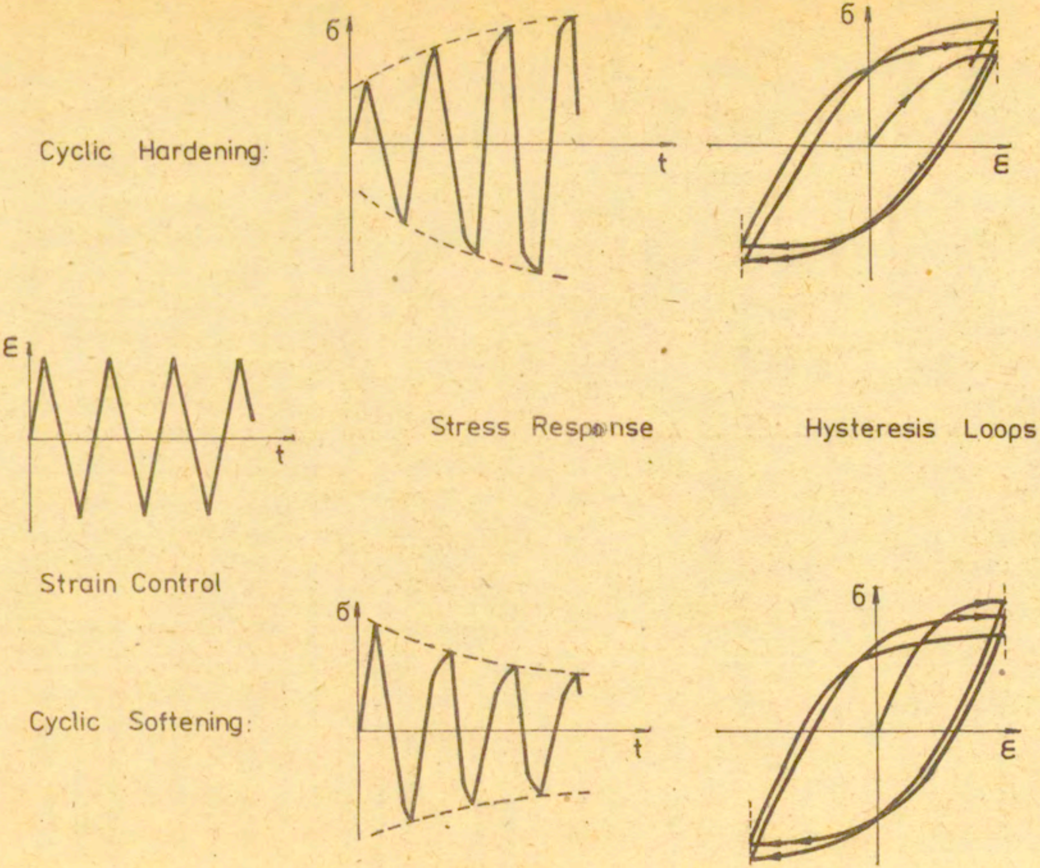


Fig.1.3 Cyclic hardening and cyclic softening (schematic)

ii/ Nonlinear dependence of ratchet strain on steady stress.
iii/ Consistent description of the other related aspects of the phenomenon especially the above mentioned /c/.

Valanis and Wu, Murakami and Nashiro could explain /i/ quantitatively well, and they succeded in description of ratchet strain versus cyclic number at a certain steady stress. But at large cycle number their hysteresis loops tend to coincide with elastic straight line. Their theories could not describe /c/ consistently. The influence of steady stress, /ii/, to the author's knowledge has not yet been described.

Chapter II Problem formulation

4. Subject of the thesis

Our first attempt is to construct a nonlinear finite ET of plasticity which consists of three constitutive relations. The relations are: a nonlinear finite constitutive equation, a nonlinear evolution equation and an intrinsic time equation, The theory proposed here is more general than the original due to Valanis. It means that the constitutive and evolution equations are finite and nonlinear, includes the Valanis theory as a special case and our intrinsic time measure and time scale are more appropriate and broader than the original ones [1] . We make a physically reasonable assumption that the internal variable satisfies incompressibility condition.

Secondly, this study aims at explaining technologically important phenomena in the domain of cyclin plasticity, i.e, /i/ mechanical ratcheting under various loading conditions, and /ii/ cyclic hardening and softening.

The influence of cyclic strain on stress amplitude was studied by Valanis and Wu [3] Murakami and Nashiro [11] and they could get a quantitatively good agreement. The other important factor in ratcheting is the steady stress /mean stress/. Experimentally it is known that the influence of steady stress on a ratchet strain is nonlinear [12, 30]. Most important object for analysis is to explain the nonlinear dependence of ratchet strain on steady stress by using proposed ET. We aim to predict not only the ratchet strain versus cyclic number relation as Valanis and Wu did, but also to describe real stress-strain curve at each cycle together with nonlinear dependence of ratchet strain on steady stress. Next important aim is a discussion of cyclic hardening and softening. Valanis explained this phenomenon but he gave mainly the prediction of the change of peak stress with respect to cyclic number. We shall try to describe the hysteresis curve at each cyclic because one needs not only the change of peak stress but also the shape of the hysteresis curve.

Fourthly, it was criticized [34,35] that though in uniaxial loading condition ET had shown its effectiveness in many examples [1,2,3] in multiaxial loading condition it had not yet fully verified its applicability. Since ratcheting analyzed herein, i.e., ratcheting at steady axial stress with cyclic shear as well as that of steady shear stress and cyclic axial strain superposed, concerns a multiaxial loading conditions, an aim of this study is to verify the effectiveness of ET under multiaxial loading condition.

5. Motivation of the used approach. Endochronic Theory

The ratcheting we are going to analyze herein is sometimes called "cyclic creep". It was well known among experimental researchers [30] that many properties of ratcheting are systematically expressible in terms of the cyclic number n and the cyclic number n can be treated formally like natural time 't'. Simultaneously reserchers who studied cyclic hardening and softening, cyclic stress-relaxation, empirically knew that the cyclic number 'n' formally plays the same role as the natural time does in viscoelasticity.

There are several similarities of ratcheting and creep. For instance, (1) the curve of ratchet strain accumulated versus the cyclic number 'n' of certain strain or stress amplitude is very similar to the curve of creep strain versus natural time 't'. (2) The ratchet strain accumulated versus cyclic number 'n' curve has three periods like creep curve /See Fig. 2.1/. (3) Ratchet strain depends on the steady stress nonlinearly as creep strain does. (4) In ratcheting, if the steady stress is increased in the midst of the process, the accumulation rate of ratchet strain increases rapidly at first and tends to be constant later; and if the steady stress is decreased, the rate decreases; and if the steady stress is removed completely the ratchet strain recovers similarly as we observe in creep /See Fig. 2.2/.

We meet frequently in the ET many terms of similarities with the classical viscoelasticity, for instance the Maxwell model. It seems that Valanis, founder of the ET, Bažant, Wemper and Anderson [10] who modified and developed the ET, noticed the analogy and similarity of cyclic number 'n' and natural 't', and recognized intuitively at least the important role of 'n'. In fact, one can see easily the formal identity between of the Valanis constitutive equation of ET and the equation of visco-elasticity, where the intrinsic time z is put instead of the natural time 't' /See eq. 2.9./.

Advantages and features of ET are given in the following list:

a/ First, the ET does not use the concept of yield surface. Abandoning the notion of yield surface is phenomenologically quite natural in cyclic plasticity, because the observation of stress-strain curve hysteresis curve under cyclic loading will tell us that yielding is rather gradual process, not a clear outset. Z.P. Bažant and his coworkers showed in their series of papers that the ET is very effective in the description of problems of concrete, for concrete does not show clear yielding [6-9].

b/ Second, a theoretical structure of the ET is very simple. It does not introduce yield surface which divides domain into two parts and so, it is not needed to use two different constitutive equations and not necessary to define the rule

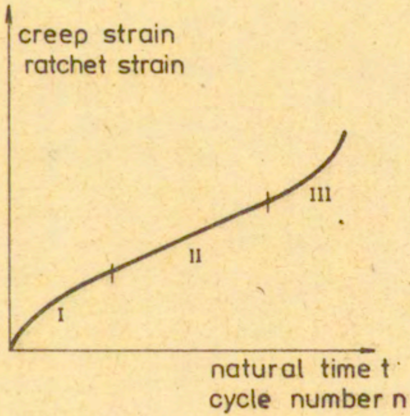


Fig. 2.1 Three stages of creep strain and ratchet strain

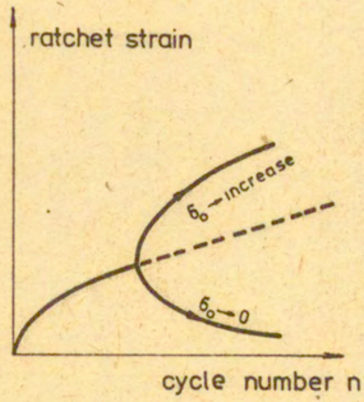


Fig.2.2 Influence of the change of steady stress

of moving of yield surface. The theory can model loading and unloading without the loading and unloading criteria [8]. Since mathematically a yield surface is singular part in the stress space, the elimination of it gives mathematical simplicity. The total strain is used directly and is not divided into elastic and plastic strains.

c/ Third, ET is developed in the field of the internal variable theory and has sound and solid thermodynamical foundations. The constitutive equation can be derived from a free energy density function, the evolution equation satisfies the dissipation inequality.

d/ Fourth, intrinsic time, the central concept of ET, is a material function. Intrinsic time measure is a locus in the strain stress space weighted by a certain value; it even includes the ideas of Iluyshin, Pipkin and Rivlin [13]. Intrinsic time scale is a function of the time measure.

The theory can incorporate many material properties, and can be applied to many domains. In fact it has been shown the effectiveness of ET in a broad domain and in many problems; for examples: loading-unloading loops of the stress-strain curve, cross hardening effect, ratcheting, cyclic stress relaxation, cyclic hardening and softening. ET is especially effective for the description of highly complicated mechanical behaviour of materials in plastic range. The fact that all of these explanation were done using only linear infinitesimal constitutive and evolution equations and a very simple form of intrinsic time, suggests its possibilities.

6. Framework of the theory. Assumptions and definitions

6.1. Free energy function

We assume that the elasto-plastic material is isotropic in isothermal conditions. Our study will be made in the framework of the internal variable theory presented by Coleman and Gurtin [16], Valanis [18], Perzyna and Wojno [17]. We restrict our investigation to the case that the internal variable

$\underline{q}$ is a symmetric tensor of the second order. For definiteness, we assume here that the internal state variable remains invariant upon change of the spatial frame.

We suppose that the internal variable $\underline{q}$ and the right Cauchy-Green tensor $\underline{C}$ are independent variables. For convenience we mainly use the following variables

$$\underline{E} = \frac{1}{2}(\underline{C} - \underline{1}) \quad , \quad \underline{\hat{q}} = \frac{1}{2}(\underline{q} - \underline{1}), \qquad /6.1/$$

where $\underline{C} = \underline{F}^T\underline{F}$ and $\underline{F}$ is the deformation gradient, and $\underline{1}$ is the unit tensor.

We choose $\underline{E}$ and $\underline{\hat{q}}$ as state variables σ , namely

$$\sigma = (\underline{E}, \underline{\hat{q}}). \qquad /6.2/$$

We assume the existence of free energy function Ψ which is a function of $\underline{E}$ and $\underline{\hat{q}}$ i.e.,

$$\Psi = \hat{\Psi}(\underline{E}, \underline{\hat{q}}) = \hat{\Psi}(\sigma). \qquad /6.3/$$

Representation theorem for an isotropic function of two tensors $\underline{C}$ and $\underline{q}$ /or $\underline{E}$ and $\underline{\hat{q}}$/ tells us that only 10 independant invariants exist in this case, and therefore the free energy function Ψ may be expressed as follows

$$\Psi = \hat{\Psi}(J_1, J_2, J_3, J_4, J_5, J_6, J_7, J_8, J_9, J_{10}), \qquad /6.4/$$

where

$$J_1 = 2I_{\underline{E}} = I_{\underline{C}} - 3$$

$$J_2 = 4II_{\underline{E}} = II_{\underline{C}} - 2(I_{\underline{C}} - 3) - 3$$

$$J_3 = 8III_{\underline{E}} = (III_{\underline{C}} - 1) - (II_{\underline{C}} - 3) + (I_{\underline{C}} - 3)$$

$$J_4 = 2I_{\underline{\hat{q}}} \qquad /6.5/$$

$$J_5 = 4II_{\underline{\hat{q}}}$$

$$J_6 = 8III_{\underline{\hat{q}}}$$

$$J_7 = 4\,\underline{E}\cdot\underline{\hat{q}}$$

$$J_8 = 8\,\underline{E}^2\cdot\underline{\hat{q}}$$

$$J_9 = 8\,\underline{E}\cdot\underline{\hat{q}}^2$$

$$J_{10} = 16\,\underline{E}^2\cdot\underline{\hat{q}}^2, \qquad /6.5/$$

where I_C, II_C, III_C, I_E, II_E, III_E are the first, second and third invariants of $\underline{C}$ and respectively $\underline{E}$; I_q, II_q, III_q, $I_{\hat{q}}$, $II_{\hat{q}}$, $III_{\hat{q}}$ are the first, second and third invariant of $\underline{q}$ and $\underline{\hat{q}}$ respectively, and '.' denotes the inner product.

We assume that the free energy function Ψ is a polynomial function of 10 invariants of $\underline{C}$ and $\underline{q}$ /or $\underline{E}$ and $\underline{\hat{q}}$/; then we can write without loss of generality

$$\Psi = \sum_{abc\ldots j} A_{abc\ldots j}\, J_1^a J_2^b J_3^c \ldots\ldots J_{10}^j\,, \qquad /6.6/$$

where A's are physical material constants and J's are invariants. Note that invariants J_7, J_8, J_9, J_{10} show elasto-plastic coupling. It means that our free energy function Ψ contains coupling terms, and predicts the existence of elasto-plastic coupling effects.

6.2. Evolution equation

To describe inelastic properties of the material in the framework of the internal variable theory we need an evolution equation. Valanis' ET assumes that an evolution equation is derivable from the free energy function. But we take different approach. Our evolution equation is postulated independently from the free energy.

In the internal variable theory, the evolution equation $\underline{\dot{q}}$ is a function of state variables $\sigma = /\underline{E}, \underline{\hat{q}}/$.

We assume two types of the evolution equations

Type /i/ $$\frac{d\underline{q}}{dt} = \hat{G}(\underline{c},\underline{q})$$ /6.7/

Type /ii/ $$\frac{d\underline{q}}{dz} = \hat{A}(\underline{c},\underline{q})$$

In type /i/, 't' is the natural time, and in type /ii/ 'z' is the intrinsic time in the sense of Valanis. These evolutions equations also satisfy the principle of material objectivity.

We restrict our analysis to type /ii/. Mathematically, the problem of evolution equation is an initial value problem of the differential equation

$$\frac{d\underline{q}}{dz} = \hat{A}(\underline{C},\underline{q}) \qquad \underline{q}(z)_{z=0} = \underline{q}_o \ , \quad z \in [0,\infty) \ . \qquad /6.8/$$

Now we are going to give the more precise definition of $\underline{q}$. We assume the tensor internal variable $\underline{q}$ to be a measure of inelastic deformation defined in the material coordinate system. Now let us perform a thought-experiment namely that we have a closed loop in the stress space starting and ending at zero stress $\underline{\tilde{T}} = \underline{0}$, but the closed cycle in stress results in some non-vanishing final deformation $\underline{B}_f$ /$\underline{B} = \underline{F}\underline{F}^T$/. At the end of such an experiment the material point exhibits plastic deformation and no elastic deformation. It means that $\underline{B}$ /= $\underline{B}_f$/, which measures the total deformations, is a pure permanent deformation. Since $\underline{q}$ has to measure the inelastic plastic deformation, at the end of the experiment $\underline{q}$ is equal to $\underline{C}_f$/for $\underline{B} = \underline{F}\underline{F}^T$, $\underline{C} = \underline{F}^T\underline{F}$, so $\underline{C}_f$ corresponds to $\underline{B}_f$/. Mathematically we define q through the following two relations

$$\begin{aligned} &\underline{\tilde{T}}(\underline{C},\underline{q})\big|_{\underline{C}=\underline{q}} = 0 \\ &\frac{d\underline{q}}{dz} = \hat{A}(\underline{C},\underline{q})\big|_{\underline{C}=\underline{q}} = 0 \end{aligned} \qquad /6.9/$$

Furthermore, eq. $/6.9/_2$ means that at the unloaded state $\underline{\tilde{T}} = \underline{0}$ and $\underline{C} = \underline{q}$, the rate of $\underline{q}$ with respect to intrinsic time is zero, $\dot{\underline{q}} = \underline{0}$.

We make at this stage an important assumption regarding $\underline{q}$. We assume that

$$\det(\underline{q}) = 1, \qquad /6.10/$$

i.e., in all the deformation process the inelastic deformation tensor is unimodular. This condition express the plastic /inelastic/ incompressibility requirement.

6.3. The second Piola-Kirchhoff stress tensor

We assume that there exists the second Piola-Kirchhoff stress tensor $\underline{\tilde{T}}$ such that

$$\underline{\tilde{T}} = \underline{\hat{T}}(\underline{c}, \underline{q}), \qquad /6.11/$$

It is well known that in the classical theory of material with internal variables, the second law of thermodynamics gives us /cf. Coleman and Gurtin [16],Valanis [18] /

$$\underline{\tilde{T}} = 2\rho_0 \frac{\partial \Psi}{\partial \underline{C}}$$

$$\underline{T} = 2\rho \underline{F} \frac{\partial \Psi}{\partial \underline{C}} \underline{F}^T \qquad /6.12/$$

where ρ_0 is the reference mass density, ρ is the present mass density, $\underline{T}$ is the Cauchy stress, $\underline{F}$ is deformation gradient, $\partial/\partial \underline{C}$ denotes derivative of scalar function with respect to $\underline{C}$.

6.4. Intrinsic time

Essential for the ET is the concept of intrinsic time which consists of intrinsic time measure and intrinsic time scale z. For brevity in the following analysis we simply call time measure and time scale. The time measure and the time scale are material functions.

We propose the following form of the intrinsic time measure

$$\left(\frac{d\zeta}{dt}\right)^2 = \tilde{M}\left(I_{\dot{\varepsilon}}, II_{\dot{\varepsilon}}, III_{\dot{\varepsilon}}\right) \quad ; \quad \dot{\varepsilon} = \frac{d}{dt}\left(\varepsilon(t)\right), \qquad /6.13/$$

where d/dt is natural time derivative.

Valanis [1] used time scale of the form

$$\frac{dz}{d\zeta} = \frac{1}{1+\beta\zeta} \quad ; \quad \beta = \text{const.} \qquad /6.14/$$

In the problems of cyclic plasticity in which the cyclic number is very big, the time scale, which reflects the deformation history, must cover the limit case $\zeta \to \infty$. In this limit case the Valanis time scale does not make any sense, for $dz/d\zeta$ tends to zero. Since at infinite ζ, $\dot{\underline{q}} = 0$, i.e., the inelastic strain rate is 0, then the material behaves as perfect elastic body. This is why Valanis' time scale eq. /2.8/ above mentioned cannot be used for the analysis of the material behavior under cyclic loading.

If we put $dz/d\zeta$ = const., it can merely describe elasto-perfectly plastic material.

The experimental techniques for cyclic loading, periodic input of either strain /strain-control/ or stress /stress-control/ is used. In steady-state, the output, stress /for strain-control test/ or strain /for stress-control/ are also periodic. Hence the constitutive relations proposed should have cyclic and periodic properties. We introduce for this purpose a time scale that has these properties required. Two possibilities are conceivable. The first one is to introduce a periodic function /e.g. sin x/ into the time scale. But this form restricts considerations to special loading conditions and is not applicable to different non-periodic loading conditions. Another possibility is to introduce state variables in the time scale as Bažant and his co-workers did [6-9]. To this end we propose the following time scale

$$\frac{dz}{d\zeta} = f(\underline{E}, \hat{\underline{q}}, \zeta); \qquad /6.15/$$

it is a function not only of the time measures but of the state variables $\underline{E}$ and $\hat{\underline{q}}$ as well. This form is different from Bažant's one for it includes the internal variable. When cyclic number is small, the material behavior is in general in transient-stage. In this stage, for instance, hystereris curve and the accumulation rate of ratchet strain are not constant. To describe the behavior in long deformation history or in wide range of cycle number, let the time scale be an explicit function of time measure. Therefore f $/\underline{E}, \hat{\underline{q}}, \zeta/$ is writen as

$$\frac{dz}{d\zeta} = f_1(\underline{E}, \hat{\underline{q}}) f_2(\zeta), \qquad /6.16/$$

where f_1 $/\underline{E}, \hat{\underline{q}}/$ expresses the periodic property of material behavior; $f_2(\zeta)$ expresses the influence of long deformation history on $dz/d\zeta$. In a steady or in a state of infinite values of ζ, the product $f_1 f_2$ has to express the periodicity.

6.5. Dual constitutive relations

For some boundary value problems, it is more suitable to formulate constitutive relations in terms of stress and the internal variable $\underline{q}$. Let us call it the stress-type ET.

In this dual formulation it is natural to claim that stress tensor $\tilde{\underline{T}}$ and $\hat{\underline{q}}$ are state variables

$$\omega = (\tilde{\underline{T}}, \hat{\underline{q}}). \qquad /6.17/$$

Now ω is a set of state variables $\tilde{\underline{T}}$ and $\hat{\underline{q}}$.

Using the same approach we can formulate the stress-type ET as follows.

There exists the complementary energy function

$$\varphi = tr(\tilde{\underline{T}}\,\underline{E}) - \varrho_0 \Psi, \qquad /6.18/$$

where $\tilde{\underline{T}}$ is given by eq. /6.12/ and $\underline{E}$ is the Green strain tensor.

We can get the Green strain tensor by the relation

$$\underline{E} = \frac{\partial \varphi}{\partial \tilde{\underline{T}}} \qquad /6.19/$$

Similarly, we formulate the evolution equation, intrinsic time measure and time scale in terms of the set of state variables $\omega = /\tilde{\underline{T}}, \hat{\underline{q}}/$.

$$\frac{d\hat{q}}{dz} = \hat{B}(\tilde{\underline{T}}, \hat{\underline{q}}) = \hat{B}(\omega)$$

$$\left(\frac{d\zeta}{dt}\right)^2 = \hat{L}(I_{\dot{\underline{T}}}, II_{\dot{\underline{T}}}, III_{\dot{\underline{T}}}) \qquad \dot{\underline{T}} = \frac{d}{dt}(\tilde{\underline{T}}(t)) \qquad /6.20/$$

$$\frac{dz}{d\zeta} = \tilde{g}(\underline{T}, \hat{\underline{q}}, \zeta) = \tilde{g}(\omega, \zeta)$$

These equations complete the framework of the proposed ET.

Chapter III Derivation of the constitutive, evolution and intrinsic time equations

7. Constitutive equation

7.1. Introduction. Order of invariants

When a function f is a polynomial function in a small quantity 'a' then the order of f is defined as the exponent of the lowest degree term.

We can choose a set of appropriate bases in which the right Cauchy-Green tensor $\underline{C}$ and the tensor internal variable $\underline{q}$ have only diagonal parts such that

$$\underline{c} = \begin{bmatrix} (1+\varepsilon_1)^2 & 0 & 0 \\ 0 & (1+\varepsilon_2)^2 & 0 \\ 0 & 0 & (1+\varepsilon_3)^2 \end{bmatrix} \qquad \underline{q} = \begin{bmatrix} (1+\beta_1)^2 & 0 & 0 \\ 0 & (1+\beta_2)^2 & 0 \\ 0 & 0 & (1+\beta_3)^2 \end{bmatrix}$$

We assume that ε_i and β_i are first order in a certain small parameter.

It should be noted that the incompressibility for $\underline{q}$ /eq. /6.10// changes the orders of invariants of $\underline{q}$. In fact, let us rewrite the condition in the following way

$$\det(\underline{q}) = (1+\beta_1)^2(1+\beta_2)^2(1+\beta_3)^2 = 1$$

$$\beta_1+\beta_2+\beta_3 = -(\beta_1\beta_2+\beta_2\beta_3+\beta_3\beta_1) - \beta_1\beta_2\beta_3 . \qquad /7.1/$$

With this relation the order of J_4 increases to two because of the equality

$$J_4 = \mathrm{tr}(\underline{q}) - 3 = 2(\beta_1+\beta_2+\beta_3) + (\beta_1^2+\beta_2^2+\beta_3^2) =$$
$$= -2(\beta_1\beta_2+\beta_2\beta_3+\beta_3\beta_1)+\beta_1^2+\beta_2^2+\beta_3^2-2\beta_1\beta_2\beta_3 . \qquad /7.2/$$

Order of J_5 is two also:

$$J_5 = II\underline{q} - 2(I\underline{q}-3) - 3 = 4(\beta_1\beta_2+\beta_2\beta_3+\beta_3\beta_1) +$$
$$+2(\beta_1\beta_2^2+\beta_1^2\beta_2+\beta_2\beta_3^2+\beta_2^2\beta_3+\beta_3\beta_1^2+\beta_3^2\beta_1)+(\beta_1^2\beta_2^2+\beta_2^2\beta_3^2+\beta_3^2\beta_1^2) . \qquad /7.3/$$

Because of incompressibility condition, i.e., det $/q/ = III_q = 1$, order of J_6 is equal to three, which is seen from the following relation

$$J_6 = (III\underline{q}-1)-(II\underline{q}-3)+(I\underline{q}-3) = -II\underline{q}+I\underline{q}$$
$$= 2\beta_1\beta_2\beta_3 - 2(\beta_1\beta_2^2+\beta_2\beta_3^2+\beta_3\beta_1^2) - 2(\beta_1^2\beta_2+\beta_2^2\beta_3+\beta_3^2\beta_1) +$$
$$+(\beta_1^2\beta_2^2+\beta_2^2\beta_3^2+\beta_3^2\beta_1^2) - (\beta_1\beta_2+\beta_2\beta_3+\beta_3\beta_1)^2 . \qquad /7.4/$$

From this reason we introduce a new invariant J_5' of order three using the standard method /see Rivlin [31] /

$$J_5' = -J_6 = II\underline{q} - I\underline{q} . \qquad /7.5/$$

Both J_4 and J_5 are order two. There are many perfect second order free energy functions. We choose J_4 and J_5' to make a perfect second order free energy. Note that there is no order one invariant of $\underline{q}$.

We express the free energy Ψ in the following way

$$\Psi = \hat{\Psi}(J_1\ J_2\ J_3\ J_4\ J_5\ J_6\ J_7\ J_8\ J_9\ J_{10}), \qquad /7.6/_1$$

where orders of these invariants are

$$\begin{array}{lll} \mathrm{ord}(J_1)=1 & \mathrm{ord}(J_4)=2 & \mathrm{ord}(J_8)=3 \\ \mathrm{ord}(J_2)=2 & \mathrm{ord}(J_5)=3 & \mathrm{ord}(J_9)=3 \\ \mathrm{ord}(J_3)=3 & \mathrm{ord}(J_7)=2 & \mathrm{ord}(J_{10})=4 \end{array} \qquad /7.6/_2$$

7.2. The second order approximation

Now we make a fundamental assumption:

We retain in the free energy function Ψ all terms up to and including order 3 in ε_i and β_i, i.e.,

$$\varepsilon_i\varepsilon_j\varepsilon_k\ ,\ \varepsilon_i\varepsilon_j\beta_k\ ,\ \varepsilon_i\beta_j\beta_k\ ,\ \beta_i\beta_j\beta_k$$

We have assumed in /6.6/ that the free energy function Ψ is a polynomial function of the invariants. With this assumption the complete form of Ψ, the second order approximation, is

$$\Psi = a_0 + a_1 J_1 + a_2 J_2 + a_3 J_3 + a_4 J_4 + a_5' J_5' + a_7 J_7 + a_8 J_8 +$$

$$+ a_9 J_9 + a_{10} J_1^2 + a_{11} J_1 J_2 + a_{12} J_1^3 + a_{13} J_1\ J_4 + a_{17} J_1\ J_7, \qquad /7.7/$$

where a's are material constants. Invariants J_7, J_8 and J_9 and terms $a_{13}J_1\ J_4$ and $a_{17}J_1J_7$ express the elasto-plastic coupling.

We assume that in the natural state $\Psi = 0$, then $a_0 = 0$. Since $\tilde{\underline{T}}$ is given by eq. /6.12/ and if we assume in the natural state $\tilde{\underline{T}} = \underline{0}$, then $a_1 = 0$.

The free energy function Ψ has 12 material constants. It is easy to show that without the assumption of incompressibility for $\underline{g}$ the free energy Ψ would be a function of 10 invariants of $\underline{C}$ and $\underline{g}$ with 19 material constants.

The second Piola-Kirchhoff stress $\tilde{\underline{T}}$ in this case is, using eq. /6.12/

$$\frac{1}{2\varrho_0}\tilde{\underline{T}} = \frac{\partial\Psi}{\partial J_1}\frac{\partial J_1}{\partial \underline{C}} + \frac{\partial\Psi}{\partial J_2}\frac{\partial J_2}{\partial \underline{C}} + \frac{\partial\Psi}{\partial J_3}\frac{\partial J_3}{\partial \underline{C}} + \frac{\partial\Psi}{\partial J_7}\frac{\partial J_7}{\partial \underline{C}} + \frac{\partial\Psi}{\partial J_8}\frac{\partial J_8}{\partial \underline{C}} + \frac{\partial\Psi}{\partial J_9}\frac{\partial J_9}{\partial \underline{C}}, \tag{7.8}$$

where

$$\frac{\partial\Psi}{\partial J_1} = 2a_{10} + a_{11}J_2 + 3a_{12}J_1^2 + a_{13}J_4 + a_{17}J_7 \qquad \frac{\partial\Psi}{\partial J_2} = a_2 + a_{11}J_1$$

$$\frac{\partial\Psi}{\partial J_3} = a_3 \qquad \frac{\partial\Psi}{\partial J_7} = a_7 + a_{17}J_1 \qquad \frac{\partial\Psi}{\partial J_8} = a_8 \qquad \frac{\partial\Psi}{\partial J_9} = a_9 \tag{7.9/1}$$

and

$$\frac{\partial J_1}{\partial \underline{c}} = 1 \qquad \frac{\partial J_2}{\partial \underline{c}} = (I_{\underline{c}} - 2)\underline{1} - \underline{c} \qquad \frac{\partial J_3}{\partial c} = (3 - I_{\underline{c}} + II_{\underline{c}})\underline{1} + (1 + I_{\underline{c}})\underline{c} + \underline{c}^2$$

$$\frac{\partial J_7}{\partial \underline{c}} = \underline{q} - \underline{1} \qquad \frac{\partial J_8}{\partial \underline{c}} = (\underline{c} - \underline{1})(\underline{q} - \underline{1}) + (\underline{q} - \underline{1})(\underline{c} - \underline{1}) \qquad \frac{\partial J_9}{\partial \underline{c}} = (\underline{q} - \underline{1})^2 \tag{7.9/2}$$

In terms of $\underline{E}$ and $\hat{\underline{q}}$

$$\begin{aligned}\frac{1}{2\varrho_0}\tilde{\underline{T}} = {} & [2(a_2 + 2a_{10})I_{\underline{E}} + 4(a_{11} + 3a_{12})I_{\underline{E}}^2 + 4(a_3 + a_{11})II_{\underline{E}}]\underline{1} + [2a_{13}I_{\hat{q}} + 4a_{17}\underline{E}\cdot\underline{q}]\underline{1} + \\ & - [a_2 + 2(a_3 + a_{11})I_{\underline{E}}]\underline{E} + 4a_3\underline{E}^2 + 2a_7\hat{\underline{q}} + 2a_{17}I_{\underline{E}}\hat{\underline{q}} + \\ & + 4a_8(\underline{E}\hat{\underline{q}} + \hat{\underline{q}}\underline{E}) + 4a_9\hat{\underline{q}}^2\end{aligned} \tag{7.10/2}$$

Now we should determine material constants.

7.3. Derivation of material constants.

7.3.1. Initial response from natural state.

The derivative of $\tilde{\underline{T}}$ with respect to $\underline{C}$ is

$$\frac{1}{2\varrho_0}\frac{\partial\tilde{T}_{ij}}{\partial C_{km}} = \begin{bmatrix}(a_2 + 2a_{10})\delta_{km} + (a_3 + a_{11})((1 + J_1)\delta_{km} - C_{km}) + (a_{11} + 3a_{12})2J_1\delta_{km} \\ + a_{17}(q_{km} - \delta_{km}) + \end{bmatrix}\delta_{ij}$$

$$-(a_3+a_{11})o_{km}(C_{ij}-\delta_{ij})-[a_2+(a_3+a_{11})J_1]\delta_{ik}\delta_{jm}$$

$$+2a_3\delta_{ik}(C_{jm}-\delta_{jm})+2a_8\delta_{ik}\delta_{sm}(q_{sj}-\delta_{sj}). \qquad /7.11/$$

The condition that both $\underline{C}$ and $\underline{q}$ are equal to $\underline{1}$ and $\underline{\tilde{T}} = \underline{0}$ defines the natural state. We assume that if both $\underline{C}$ and $\underline{q}$ tend to $\underline{1}$, the linear part of initial response is given by the classical Lame constants. For shear i=k=1, $_{j=m=2}$, for example,

$$\frac{1}{2\rho_o}\frac{\partial \tilde{T}_{12}}{\partial C_{12}} = -a_2 \qquad \text{when } \underline{C} = \underline{1}, \qquad /7.12/$$

and we may identify

$$-2\rho_o a_2 = \mu \qquad /7.13/$$

where μ is the shear modulus.

Similarly, for simple tension, i = j = k = m = 1 say,

$$\frac{1}{2\rho_o}\frac{\partial \tilde{T}_{12}}{\partial C_{11}} = 2a_{10} \qquad /7.14/$$

we may identify

$$4\rho_o a_{10} = \lambda + 2\mu, \qquad /7.15/$$

where λ is one of the Lame constants.

7.3.2. Response from arbitary unloaded /stress-free/ state.

To determine material constants we can use the definition of internal variable $\underline{q}$ /eq. 6.9 /, i.e., at unloaded $\underline{C} = \underline{q}$ /$\underline{E} = \hat{\underline{q}}$/ and $\underline{\tilde{T}} = \underline{0}$.

The following relations at an unloaded state hold

$$J_1 = J_4 , \; J_2 = J_5 , \; J_7 = J_4^2 - 2J_5 . \qquad /7.16/$$

It means that at an unloaded state total strain $\underline{C}$/or $\underline{E}$/ coincides with $\underline{q}$ /or $\hat{\underline{q}}$/ and invariants J_1, J_2, J_7 are expressible in terms of invariants of $\underline{q}$. Substitute eqs. /7.13/, /7.15/, /7.16/ into /7.10/, then we have a simple equation

$$\underline{Q} = \left(\frac{\lambda}{4\rho_o} + a_{13}\right) J_4 \underline{1} + \left(\frac{\mu}{2\rho_o} + 2a_7\right)\hat{\underline{q}} + (a_{11} + 3a_{12} + a_{17}) J_4^2 \underline{1} + (a_3 + a_{11} - 2a_{17}) J_5 \underline{1} +$$

$$+ 2(a_{17} + a_3 - a_{11}) J_4 \hat{\underline{q}} + 4(a_3 + 2a_8 + a_9)\hat{\underline{q}}^2 . \qquad /7.17/$$

It is possible to choose a basis in which $\hat{\underline{q}}$ has only the diagonal form such that

$$\hat{\underline{q}} = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} .$$

For convenience of algebraic manipulation $\hat{q}_{11}$, $\hat{q}_{22}$, $\hat{q}_{33}$ are written as x,y,z. Express eq./7.17/ in terms of $\hat{\underline{q}}$, then we have

$$\underline{Q} = a(x+y+z)\underline{1} + b\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} + c(x^2+y^2+z^2+2xy+2yz+2zx)\underline{1} +$$

$$+ d(xy+yz+zx)\underline{1} + e(x+y+z)\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} + f\begin{bmatrix} x^2 & 0 & 0 \\ 0 & y^2 & 0 \\ 0 & 0 & z^2 \end{bmatrix} , \qquad /7.18/$$

where we put

$$a = 2(\lambda/4\rho_o + a_{13}) \qquad d = 4(a_3 + a_{11} - 2a_{17})$$

$$b = 2(a_7 + \mu/2\rho_o) \qquad e = 4(-a_3 - a_{11} + a_{17}) \qquad /7.19/$$

$$c = 4(a_{11} + 3a_{12} + a_{17}) \qquad f = 4(a_3 + 2a_8 + a_9) .$$

From eq. /7.18/ we have a system of equations

$$(a+b)x+ay+az+(c+e+f)x^2+cy^2+cz^2+(2c+d+e)xy+(2c+d)yz+(2c+d+e)zx=0$$
$$ax+(a+b)y+az+cx^2+(c+e+f)y^2+cz^2+(2c+d+e)xy+(2c+d+e)yz+(2c+d)zx=0 \qquad /7.20/$$
$$ax+ay+(a+b)z+cx^2+cy^2+(c+e+f)z^2+(2c+d)xy+(2c+d+e)yz+(2c+d+e)zx=0 .$$

This system of equations is valid for all values of x,y,z. For simplicity we choose a very simple condition, that is the plain strain condition, then z = 0. The incompressibility condition /6.10/ gives the following relation

$$\det(\underline{q}) = (1+2x)(1+2y)(1+2z) = 1, \qquad /7.21/$$

since $z \equiv 0$,

$$y = -\frac{x}{1+2x} . \qquad /7.22/$$

If we substitute eq. /7.22/ into $/7.20/_1$ we get

$$(2a-d)+(4a-2d)x+4cx^2=0 . \qquad /7.23/$$

Since this equation is valid for all x,

$$2a=d , \quad c=0 . \qquad /7.24/$$

Substituting of /7.24/ into $/7.20/_2$ results in

$$b+(2b+f)x+2(e+f)x^2=0 , \qquad /7.25/$$

which gives us

$$b=f=e=0 . \qquad /7.26/$$

With eq. /7.19/ and eqs. /7.24/, /7.26/ we get :

$$-4\rho_o a_{13}=4\rho_o(a_2+2a_{10})=\lambda \qquad a_3+a_{11}=a_3+a_{12}=a_{17}=0$$
$$-2\rho_o a_7=-2\rho_o a_2=\mu \qquad a_3+2a_8+a_9=0 . \qquad /7.27/$$

Substituting relations /7.27/ into /7.10/ gives the constitutive equation:

$$\tilde{T} = \lambda \operatorname{tr}(\underline{E}-\hat{\underline{q}})\underline{1} + 2\mu(\underline{E}-\hat{\underline{q}}) + 8\rho_o[a_3\underline{E}^2 - \frac{1}{2}(a_3+a_9)(\underline{E}\hat{\underline{q}}+\hat{\underline{q}}\underline{E}) + a_9\underline{q}^2], \qquad /7.28/$$

where λ , μ are Lame constants; a_3 and a_9 are unknown material constants.

7.4. Determination of the third order constant a_3

Let us make the following assumption: The free energy function Ψ can be decomposed into the purely elastic strain energy Ψ_e and the energy of the elasto-plastic coupling Ψ_i i.e.,

$$\Psi(\underline{E},\hat{\underline{q}}) = \Psi_e(\underline{E}) + \Psi_i(\underline{E},\hat{\underline{q}}), \qquad /7.29/$$

where

$$\Psi_e = a_{10}J_1^2 + a_2J_2 + a_{12}J_1^3 + a_{11}J_1J_2 + a_3J_3$$

$$= 4a_{10}I_{\underline{E}}^2 + 4a_2II_{\underline{E}} + 8a_{12}I_{\underline{E}}^3 + 8a_{11}I_{\underline{E}}II_{\underline{E}} + 8a_3III_{\underline{E}}. \qquad /7.30/$$

$$\Psi_i = a_4J_4 + a_5'J_5' + a_7J_7 + a_8J_8 + a_9J_9 + a_{13}J_1J_4 + a_{17}J_1J_7$$

In determining the material constants in the expression for Ψ_e we will use the acoustic method.

In the acoustic-elasticity the strain energy function for the isotropic material

$$\Psi = \frac{1}{2}(\lambda+2\mu)I_{\underline{E}}^2 - 2\mu II_{\underline{E}} + \frac{1}{6}(\vartheta_1+6\vartheta_2+8\vartheta_3)I_{\underline{E}}^3 - 2(\vartheta_2+2\vartheta_3)I_{\underline{E}}II_{\underline{E}} + 4\vartheta_3III_{\underline{E}} \qquad /7.31/$$

is used. The material constant a_3, which we are going to determine is related to the term $4\vartheta_3III_{\underline{E}}$ in eq. /7.31/. Comparing /7.30/$_1$ with /7.31/ we identify

$$2\rho_o a_3 = \vartheta_3. \qquad /7.32/$$

This second order elastic constant ϑ_3 can be determined by experimental acoustic method.

7.5. Final forms of constitutive equation

The final form is

$$\tilde{\underline{T}} = \lambda \operatorname{tr}(\underline{E} - \hat{\underline{q}})\underline{1} + 2\mu(\underline{E} - \hat{\underline{q}}) + 4\vartheta_3 \underline{E}^2 - 2(\vartheta_3 + 2\rho_0 a_9)(\underline{E}\hat{\underline{q}} + \hat{\underline{q}}\underline{E}) + 8\rho_0 a_9 \hat{\underline{q}}^2, \qquad /7.33/$$

where λ , μ are Lame constants; ϑ_3 is "the third order elastic constant" which can be measured by the acoustic method.

The first group is the linear part of the constitutive equation. It includes Valanis' theory as a special case.

The second group contains the higher order terms with only one unknown material constant a_9.

Under the condition $2\rho_0 a_9 = \vartheta_3$ we get a special case of the constitutive equation, i.e.,

$$\tilde{\underline{T}} = \lambda \operatorname{tr}(\underline{E} - \hat{\underline{q}})\underline{1} + 2\mu(\underline{E} - \hat{\underline{q}}) + 4\vartheta_3(\underline{E} - \hat{\underline{q}})^2. \qquad /7.34/$$

It consists of linear and nonlinear parts. In both the cases the linear part of the constitutive equation is:

$$\tilde{\underline{T}} = \operatorname{tr}(\underline{E} - \hat{\underline{q}})\underline{1} - 2\mu(\underline{E} - \hat{\underline{q}}). \qquad /7.35/$$

8. Evolution equation

In the following way we can specify the evolution equation using two restrictions /6.9/, /6.10/. The restriction det /$\underline{q}$/ = 1 will be satisfied if

$$\frac{d}{dz}(\det(\underline{q})) = \operatorname{tr}(\underline{q}^{-1}\frac{d\underline{q}}{dz}) = \operatorname{tr}(\frac{d\underline{q}}{dz}\underline{q}^{-1}) = 0 \qquad /8.1/$$

and

$$\det(\underline{q}(z))\big|_{z=0} = 1$$

in all deformation processes. It means that the product $\underline{q}^{-1}\underline{q}'$ and $\underline{q}'\underline{q}^{-1}$ should be the deviatoric part of some tensor $\underline{s}$

treated as a function of $\underline{E}$ and $\hat{\underline{q}}$ i.e.,

$$\acute{\underline{q}}\,\underline{q}^{-1} = \text{dev}\,\underline{s} \quad \text{and} \quad \underline{q}^{-1}\,\acute{\underline{q}} = \text{dev}\,\underline{s}, \qquad /8.2/$$

where dev $\underline{s}$ means the deviatoric part of $\underline{s}$.
Since

$$\frac{d\hat{\underline{q}}}{dz} = \frac{1}{2}[\underline{q}(\text{dev}\,\underline{s}) + (\text{dev}\,\underline{s})\underline{q}] \qquad /8.3/$$

the function $\underline{s} = \tilde{\underline{s}}\,/\underline{E},\hat{\underline{q}}/$ should be linear in $\underline{E}$ and $\hat{\underline{q}}$ in order to be consistent with the assumption of the second order approximation:

$$\tilde{\underline{s}}(\underline{E},\hat{\underline{q}}) = b_o\underline{E} + b_1\hat{\underline{q}} + b_2\,\text{tr}(\underline{E})\underline{1}, \qquad /8.4/$$

where b_o, b_1 and b_2 are material constants.

The condition /6.9/ results in $b_1 = b_o$, $b_2 = 0$. Hence the final form of the evolution equation is following

$$\frac{1}{b_o}\frac{d\hat{\underline{q}}}{dz} = (\underline{E}-\hat{\underline{q}}) - \frac{1}{3}\text{tr}(\underline{E}-\hat{\underline{q}})\underline{1} + (\underline{E}-\hat{\underline{q}})\hat{\underline{q}} + \hat{\underline{q}}(\underline{E}-\hat{\underline{q}}) - \frac{2}{3}(\text{tr}(\underline{E}-\hat{\underline{q}}))\hat{\underline{q}}, \qquad /8.5/_1$$

or

$$\frac{1}{b_o}\frac{d\hat{q}_{11}}{dz} = (E_1-\hat{q}_1) - \frac{1}{3}\text{tr}(\underline{E}-\hat{\underline{q}})(1+2\hat{q}_1) + 2(E_1-\hat{q}_1)\hat{q}_1 + 2(E_{12}-\hat{q}_{12})\hat{q}_{12},$$

$$\frac{1}{b_o}\frac{d\hat{q}_{22}}{dz} = (E_2-\hat{q}_2) - \frac{1}{3}\text{tr}(\underline{E}-\hat{\underline{q}})(1+2\hat{q}_2) + 2(E_2-\hat{q}_2)\hat{q}_2 + 2(E_{12}-\hat{q}_{12})\hat{q}_{12},$$

$$\frac{1}{b_o}\frac{d\hat{q}_{33}}{dz} = (E_3-\hat{q}_3) - \frac{1}{3}\text{tr}(\underline{E}-\hat{\underline{q}})(1+2\hat{q}_3) + 2(E_3-\hat{q}_3)\hat{q}_3, \qquad /8.5/_2$$

$$\frac{1}{b_o}\frac{d\hat{q}_{12}}{dz} = (E_{12}-\hat{q}_{12}) - \frac{2}{3}\text{tr}(\underline{E}-\hat{\underline{q}})\hat{q}_{12} + (E_1-\hat{q}_1+E_2-\hat{q}_2)\hat{q}_{12} + (E_{12}-\hat{q}_{12})(\hat{q}_1+\hat{q}_2), \ldots$$

The first two terms are linear. The nonlinear terms manifest elasto-plastic coupling. For small deformations, the incompressibility condition reduces to tr $/d\hat{\underline{q}}/dz/ = 0$, and two linear terms are sufficient. At finite deformations the incompressibility condition is no longer tr $/d\underline{q}/dz/ = 0$ but $d/dz /\det /\underline{q}/ / = 0$ so, the second group must be included to satisfy it. Note that there is no linear evolution equation that satisfies the condition $\det /\underline{q}/ = 1$. Obtained above evolution equation is deformation history dependent and it has only material constart b_0. Similarly we can derive higher order evolution equations satisfing two restrictions /6.9/ and /6.10/.

9. Intrinsic time measure and time scale

Within the framework formulated in Chapter II we define explicitely the forms of intrinsic time measure and time scale.

We propose the following form of intrinsic time measure.

$$\left(\frac{d\zeta}{dt}\right)^2 = k_1^2 \left| I_{\dot{\underline{E}}} \right|^2 + k_2^2 \left| II_{\dot{\underline{E}}} \right| + k_3^2 \left| III_{\dot{\underline{E}}} \right|^{2/3} \qquad /9.1/$$

here 't' is real time not an intrinsic time; $I_{\dot{\underline{E}}}$, $II_{\dot{\underline{E}}}$ and $III_{\dot{\underline{E}}}$ are the first, second and third invariants of $\dot{\underline{E}}$, respectively: k_1, k_2 and k_3 are material constants, but in general they may depend on invariants of $\underline{E}$ and $\dot{\underline{E}}$.

We have found out that the following form is suitable for many problems of cyclic plasticity:

$$f_1(\underline{E}, \hat{\underline{q}}) = \left| II(\underline{E} - \hat{\underline{q}}) \right|^{c_2}, \qquad /9.2/$$

that is, f_1 is a function of the second invariant of the difference $\underline{E} - \hat{\underline{q}}$ and c_2 is material constant.

For f_2 the following relation is used in this thesis:

$$f_2(\zeta) = (c_3 + c_4 e^{-c_5 \zeta}), \qquad /9.3/$$

where c_3, c_4 and c_5 are material constants.

The following form is also applicable to certain problems:

$$f(\zeta) = (1 + \beta\zeta)^{\alpha} \qquad /9.4/$$

where β , α are constants. Valanis used only the case

$\alpha = -1$ because of analytical reason. The author has tried cases $\alpha = -0.1, -0.6$, but then it is not possible to find closed form of the intrinsic time and numerical methods should be used. The case /9.4/ can describe cyclic hardening behaviour of certain materials; for example, annealed copper with almost linearly hardens up to small cycle number, say $n = 6$.

10. Dual constitutive relations

In this section we derive dual constitutive relations. We call they the stress-type ET. It means that the stress is an independent variable, whereas the strain is expressed in terms of the stress and the internal variable $\underset{\sim}{q}$.

10.1. The complementary energy function

We assume that each component of $\underset{\sim}{\tilde{T}}$ and $\underset{\sim}{\hat{q}}$ is order one in some small parameter.
We retain in the complementary energy function φ all terms up to and including order three. The complete form of φ is

$$\varphi = b_0 + b_1 I_{\tilde{T}} + b_2 II_{\tilde{T}} + b_3 III_{\tilde{T}} + b_4 I_{\hat{q}} + b_5' J_5' + b_7 \underset{\sim}{\tilde{T}} \cdot \underset{\sim}{\hat{q}} + b_8 \underset{\sim}{\tilde{T}} \cdot \underset{\sim}{\hat{q}}^2 +$$

$$b_9 \underset{\sim}{\tilde{T}}^2 \cdot \underset{\sim}{q} + b_{10} I_{\tilde{T}}^2 + b_{11} I_{\tilde{T}} II_{\tilde{T}} + b_{12} I_{\tilde{T}}^3 + b_{13} I_{\tilde{T}} I_{\hat{q}} + b_{17} I_{\tilde{T}} \underset{\sim}{\tilde{T}} \cdot \underset{\sim}{\hat{q}}, \qquad /10.1/$$

where b's are physical constants; $I_{\tilde{T}}$, $II_{\tilde{T}}$ and $III_{\tilde{T}}$ are the first, second and third invariants of $\underset{\sim}{\tilde{T}}$, respectively; $I_{\hat{q}}$ is the first invariant of $\underset{\sim}{\hat{q}}$, and J_5' is defined in eq. /7.5/; and '. ' denotes inner product. This complementary energy function also shows the existence of elasto-plastic coupling. We assume in the natural state, $\varphi = 0$, then $b_0 = 0$. Since $\underset{\sim}{E}$ given through eq. /6.19/ and if we assume the natural state $\underset{\sim}{E} = \underset{\sim}{0}$, then $b_1 = 0$. So, the complementary energy function contains 12 material constants. It is easy to show that in the general case without the assumption of incompressibility the function φ has 19 material constants in complete form.

10.2. The constitutive equation

The strain $\underline{E}$ is given by the relation

$$\underline{E} = \frac{\partial\varphi}{\partial I_I}\frac{\partial I_I}{\partial \underline{T}} + \frac{\partial\varphi}{\partial II_I}\frac{\partial II_I}{\partial \underline{T}} + \frac{\partial\varphi}{\partial III_I}\frac{\partial III_I}{\partial \underline{T}} + \frac{\partial\varphi}{\partial(\tilde{T}\cdot\hat{q})}\frac{\partial(\tilde{T}\cdot\hat{q})}{\partial \underline{T}} + \frac{\partial\varphi}{\partial(\tilde{T}\cdot\hat{q}^2)}\frac{\partial(\tilde{T}\cdot\hat{q}^2)}{\partial \underline{T}} + \frac{\partial\varphi}{\partial(\tilde{T}^2\cdot\hat{q})}\frac{\partial(\tilde{T}^2\cdot\hat{q})}{\partial \underline{T}} \qquad /10.2/$$

The following relations are easy to obtain

$$\frac{\partial\varphi}{\partial I_I} = 2b_{10}I_I + b_{11}II_I + 3b_{12}I_I^2 + b_{13}I_{\hat{q}} + b_{17}\tilde{\underline{T}}\hat{\underline{q}}\ , \qquad \frac{\partial\varphi}{\partial II_I} = b_2 + b_{11}II_I\ ,$$

$$\frac{\partial\varphi}{\partial III_I} = b_3\ , \quad \frac{\partial\varphi}{\partial(\tilde{\underline{T}}\hat{\underline{q}})} = b_7\ , \quad \frac{\partial\varphi}{\partial(\tilde{\underline{T}}\hat{\underline{q}}^2)} = b_8\ , \quad \frac{\partial\varphi}{\partial(\tilde{\underline{T}}^2\hat{\underline{q}})} = b_9\ ,$$

$$\frac{\partial I_I}{\partial \underline{I}} = \underline{1}\ , \quad \frac{\partial II_I}{\partial \underline{I}} = I_I\underline{1} - \underline{I}\ , \quad \frac{\partial III_I}{\partial \underline{I}} = II_I\underline{1} - I_I\tilde{\underline{T}} + \tilde{\underline{T}}^2, \qquad /10.3/$$

$$\frac{\partial(\tilde{T}\cdot\hat{q})}{\partial \underline{T}} = \hat{\underline{q}}\ , \quad \frac{\partial(\tilde{T}\cdot\hat{q}^2)}{\partial \underline{T}} = \underline{q}^2\ , \quad \frac{\partial(\tilde{T}^2\hat{q})}{\partial \underline{T}} = (\tilde{\underline{T}}\hat{\underline{q}} + \hat{\underline{q}}\tilde{\underline{T}})\ .$$

The eq. /10.2/ together with eqs /10.3/ result in

$$E = [2b_{10}I_I + b_{11}II_I + 3b_{12}I_I^2 + b_{13}I_{\hat{q}} + b_{17}\tilde{\underline{T}}\hat{\underline{q}}]\underline{1} + [b_2 + b_{11}I_I](I_I\underline{1} - \tilde{\underline{T}}) +$$

$$+ b_3(II_I\underline{1} - I_I\tilde{\underline{T}} + \tilde{\underline{T}}^2) + b_7\hat{\underline{q}} + b_8\hat{\underline{q}}^2 + b_9(\tilde{\underline{T}}\hat{\underline{q}} + \hat{\underline{q}}\tilde{\underline{T}})\ . \qquad /10.4/$$

Eq/10.4/ must satisfy the definition of the internal variable $\underline{q}$ given by eq. /6.9/ i.e., at unloaded state, $\tilde{\underline{T}} = \underline{0}$, $\underline{E} = \hat{\underline{q}}$, that is

$$E\big|_{\tilde{\underline{T}}=\underline{0}} = b_{13}I_{\hat{q}}\underline{1} + b_7\hat{\underline{q}} + b_8\hat{\underline{q}}^2 = \hat{\underline{q}}\ . \qquad /10.5/$$

To satisfy the condition /10.5/ for all $\underline{q}$, the following relations must exist:

$$b_8 = b_{13} = 0 \qquad b_7 = 1\ . \qquad /10.6/$$

Eq. /10.4/ now reduces to

$$\underline{E} = [(b_2 + 2b_{10}) I_{\underline{T}} + (b_{11} + 3b_{12}) I_{\underline{T}}^2 + (b_3 + b_{11}) II_{\underline{T}}] \underline{1} - [b_2 (b_3 + b_{11}) I_{\underline{T}}] \tilde{\underline{T}} + b_3 \tilde{\underline{T}}^2 + \hat{\underline{q}} + b_{17} (\tilde{\underline{T}} \cdot \hat{\underline{q}}) \underline{1} + b_9 (\tilde{\underline{T}} \hat{\underline{q}} + \hat{\underline{q}} \tilde{\underline{T}}). \qquad /10.7/$$

This equation has seven constants; two constants b_{17} and b_9 are the so-called plastic constants.

The initial response from the natural state $/\tilde{\underline{T}}, \hat{\underline{q}} = \underline{0}/$ can be described by the following linear equation /compare with the point 7.3/

$$\text{if } \tilde{\underline{T}} \to 0 \quad \text{then} \quad \underline{E} = (b_2 + 2b_{10}) I_{\underline{T}} \underline{1} - b_2 \tilde{\underline{T}} + O(\tilde{T}^2, \tilde{T}\hat{q}). \qquad /10.8/$$

For simple shear this equation reduces to

$$E_{12} = -b_2 \tilde{T}_{12} \qquad /10.9/$$

hence the materials constant b_2 is given by one of Lame constants μ as follows,

$$-b_2 = \frac{1}{2\mu} \qquad /10.10/$$

In the same way, for simple tension

$$E_{11} = 2 b_{10} \tilde{T}_{11}, \qquad /10.11/$$

then b_{10} is given by Young's modulus E_0

$$2 b_{10} = \frac{1}{E_0}. \qquad /10.12/$$

The inversed constitutive equation is

$$E = -\frac{\nu}{E_0} I_{\underline{T}} \underline{1} + \frac{1+\nu}{E_0} \tilde{\underline{T}} + [(b_3 + 3b_{12}) I_{\underline{T}}^2 + (b_3 + b_{11}) II_{\underline{T}}] \underline{1} - (b_3 + b_{11}) I_{\underline{T}} \tilde{\underline{T}} + b_3 \tilde{\underline{T}}^2 + \hat{\underline{q}} + b_{17} (\tilde{\underline{T}} \cdot \hat{\underline{q}}) + b_9 (\tilde{\underline{T}} \hat{\underline{q}} + \hat{\underline{q}} \tilde{\underline{T}}), \qquad /10.13/$$

where $2\mu = E_0 / (1 + \nu)$, and ν is Poisson's ratio.

10.3. Determination of the second order constants b_3, b_{11} and b_{12}

Similarly, using the idea presented in 7.4, the complementary energy function φ is decomposed into the purely elastic complementary energy φ_e and the elasto-plastic coupling complementary energy φ_i i.e.,

$$\varphi(\tilde{\underline{T}},\hat{\underline{q}}) = \varphi_e(\tilde{\underline{T}}) + \varphi_i(\tilde{\underline{T}}\hat{\underline{q}}), \qquad /10.14/$$

where

$$\varphi_e = b_{10} I_T^2 + b_2 II_T + b_{12} I_T^3 + b_{11} I_T II_T + b_3 III_T$$

$$\varphi_i = b_4 I_{\hat{q}} + b_5' J_5' + b_7 \tilde{\underline{T}}\cdot\hat{\underline{q}} + b_8 \tilde{\underline{T}}\cdot\hat{\underline{q}}^2 + b_9 \tilde{\underline{T}}^2\cdot\hat{\underline{q}} + b_{13} I_T I_{\hat{q}} + b_{17} I_T \tilde{\underline{T}}\cdot\hat{\underline{q}}. \qquad /10.15/$$

To determine the material constants in the expression for φ_e we can apply the acoustic method. To this end we can use eq. /6.18/ i.e.,

$$\varphi_e = \mathrm{tr}(\tilde{\underline{T}}_e \underline{E}) - \rho_0 \Psi_e, \qquad /10.16/$$

where

$$\tilde{\underline{T}}_e = \rho_0 \frac{\partial \Psi_e(\underline{E})}{\partial \underline{E}}. \qquad /10.17/$$

Hence, the strain $\underline{E}$ is derived from

$$\underline{E} = \frac{\partial \varphi_e}{\partial \tilde{\underline{T}}}$$

$$= \frac{-\nu}{E_0} I_T \underline{1} + \frac{1+\nu}{E_0}\tilde{\underline{T}} + [(b_3+3b_{12}) I_T^2 + (b_3+b_{11}) II_T]\underline{1} - (b_3+b_{11}) I_T \tilde{\underline{T}} + b_3 \tilde{\underline{T}}^2 \qquad /10.18/$$

Our method has the following procedure:

1/ Using eq. /10.18/ we can express the I_E, II_E, III_E in terms of I_T, II_T, III_T and b_3, b_{11}, b_{12}. . In this operation we use the consistent second order approximation.

2/ The above φ in eq. /10.16/ is expressed in terms of I_E, II_E, III_E and λ, μ, ν_3 by I_T, II_T, III_T and λ ,

μ , ϑ_3 , b_3 , b_{11} , b_{12}.

3/ We identify the equation obtained above with eq. /10.1/ in which besides terms with $I_{\underline{T}}$, $II_{\underline{T}}$, $III_{\underline{T}}$ the unknown constants b_3, b_{11}, b_{12} take place.

4/ Then, we can express unknown material constants b_3, b_{11}, b_{12} in terms of known constants λ , μ , ϑ_3.

For convenience of algebraic manipulation we rewrite eq. /10.18/ as

$$\underline{E} = c I_{\underline{T}} \underline{1} + d\underline{\tilde{T}} + (k I_{\underline{T}}^2 + l\, II_{\underline{T}})\underline{1} - l I_{\underline{T}} \underline{\tilde{T}} + n\underline{\tilde{T}}^2, \qquad /10.19/$$

where

$$c = -\frac{\vartheta}{E_0},\ d = \frac{1+\vartheta}{E_0},\ k = (b_3 + 3b_{12}),\ l = (b_3 + b_{11}),\ n = b_3 . \qquad /10.20/$$

We list up here the following relations for clarity

$$\begin{aligned}
&\mathrm{tr}(\underline{\tilde{T}}\,\underline{E}) = (c+d)I_{\underline{T}}^2 - 2d\, II_{\underline{T}} + (k-l+n)I_{\underline{T}}^3 + (3l-3n)I_{\underline{T}} II_{\underline{T}} + 3n\, III_{\underline{T}} ,\\
&I_{\underline{E}} = (3c+d)I_{\underline{T}} + (3k-l+n)I_{\underline{T}} + (3l-2n)\, II_{\underline{T}},\\
&I_{\underline{E}}^2 = (3c+d)^2 I_{\underline{T}}^2 + 2(3c+d)(3k-l+n)I_{\underline{T}}^3 + 2(3c+d)(3l-2n)I_{\underline{T}} II_{\underline{T}},\\
&I_{\underline{E}}^3 = (3c+d)^3 I_{\underline{T}}^3 ,\\
&II_{\underline{E}} = \frac{1}{2}\Big\{(6c^2+4cd)I_{\underline{T}}^2 + 2d^2 II_{\underline{T}} + [5c(3k-l+n) + d(5k-l+n)]I_{\underline{T}}^3 +\\
&\qquad + [5c(3l-2n) + d(3l-n)]I_{\underline{T}} II_{\underline{T}} - 3dn\, III_{\underline{T}}\Big\},\\
&III_{\underline{E}} = \frac{1}{6}[(3c+d)^3 - 3(3c+d)(3c^2+2cd+d^2) + 2(3c^3+3c^2d+3cd^2+d^3)]I_{\underline{T}}^3 +\\
&\qquad + [(3c+d)d^2 - (2cd^2+3d^3)]I_{\underline{T}} II_{\underline{T}} + d^3 III_{\underline{T}},
\end{aligned} \qquad /10.21/$$

Put the relations /10.21/ into eq. /10.16/ and identify the result with eq. /10.1/ /using $\underline{Q} = \underline{0}$/, then we can express b_3, b_{11} and b_{12} in terms of known constants λ , μ and ϑ_3

$$b_3 = \vartheta_3 \left(\frac{1}{\mu}\right)^3 ,$$

$$b_{11} = \frac{b_3[(\lambda+2\mu)(3c+d) - \mu(5c+2d)] + 4\vartheta_3(cd^2-3cd-d^2)}{2-3(\lambda+2\mu)(3c+d) - 3\mu(5c+d)},$$

$$b_{12} = \frac{2(1+\vartheta)}{12\vartheta-1}\left[b_3\frac{1+10\vartheta}{2(1+\vartheta)} - b_{11}\left\{1 - \frac{(1-\vartheta)}{(1+\vartheta)} + \mu(5c+d)\right\} + 4\vartheta_3(11c^3+17c^2d+5cd^2+\frac{1}{3}d^3)\right]. \qquad /10.22/$$

Our constitutive eq. /10.13/ has now only two unknown material constants b_9 and b_{17}.

10.4. Constitutive equation in terms of the Cauchy stress

Let us express eq. /10.13/ in terms of the Cauchy stress. Using known relation the second Piola-Kirchhoff stress $\tilde{\underline{T}}$ is given by

$$\tilde{\underline{T}} = (\det \underline{C})^{1/2}\underline{F}^{-1}\underline{T}(\underline{F}^{-1})^T. \qquad /10.23/$$

For the analysis of further examples we restrict our attention to the Cauchy stress of the following form /in some Cartesian coordinate system/

$$\underline{T} = \begin{bmatrix} \overset{c}{T}_1 & \overset{c}{T}_{12} & 0 \\ \overset{c}{T}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad /10.24/$$

This form of the stress $\tilde{\underline{T}}$ results in the deformation gradient $\underline{F}$

$$\underline{F} = \begin{bmatrix} a & d & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \qquad /10.25/$$

The second Piola-Kirchhoff stress is

$$\tilde{T} = |abc| \begin{bmatrix} \frac{1}{a} & \frac{-d}{ab} & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix} \begin{bmatrix} \overset{c}{T}_1 & \overset{c}{T}_{12} & 0 \\ \overset{c}{T}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ \frac{-d}{ab} & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix} \begin{bmatrix} \frac{c}{a}(b\overset{c}{T}_1 - 2d\overset{c}{T}_{12}) & c\overset{c}{T}_{12} & 0 \\ c\overset{c}{T}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{/10.26/}$$

The constitutive eq. /10.13/ can be written using eq. /10.26/ as follows

$$E_{11} = \frac{1}{E_o}\tilde{T}_1 + (b_3 - b_{11} + 3b_{12})\tilde{T}_1^2 - b_{11}\tilde{T}_{12}^2 + \hat{q}_{11} + (2b_9 + b_{17})\tilde{T}_1\hat{q}_1 + 2(b_9 + b_{17})\tilde{T}_{12}\hat{q}_{12}, \tag{/10.27/}$$

$$E_{22} = \frac{-\nu}{E_o}\tilde{T}_1 + (b_3 + 3b_{12})\tilde{T}_1^2 - b_{11}\tilde{T}_{12}^2 + \hat{q}_{22} + b_{17}\tilde{T}_1\hat{q}_1 + 2(b_9 + b_{17})\tilde{T}_{12}\hat{q}_{12},$$

$$E_{33} = \frac{-\nu}{E_o}\tilde{T}_1 + (b_3 + 3b_{12})\tilde{T}_1^2 - (b_3 + b_{11})\tilde{T}_{12}^2 + \hat{q}_{33} + b_{17}(\tilde{T}_1\hat{q}_1 + \tilde{T}_{12}\hat{q}_{12}),$$

$$E_{12} = \frac{1+\nu}{E} T_{12} - b_{11}\tilde{T}_1\tilde{T}_{12} + \hat{q}_{12} + b_9(\tilde{T}_1\hat{q}_{12} + \tilde{T}_{12}(\hat{q}_1 + \hat{q}_2)),$$

where

$$\tilde{T}_1 = \left(\frac{bc}{a}\right)\overset{c}{T}_1 - \left(\frac{2cd}{a}\right)\overset{c}{T}_{12}, \tag{/10.28/}$$

$$\tilde{T}_{12} = c\,\overset{c}{T}_{12}.$$

10.5. Evolution equation

To obtain the dual form of the evolution equation we can use two restrictions /6.9/ and /6.10/. In parallel with the approach adopted in section 8 we can specify the stress-type evolution equation. Because of the incompressibility condition /8.1/ the product $\underline{g}^{-1}\,\dot{\underline{g}}$ and $\dot{\underline{g}}\,\underline{g}^{-1}$ should be the deviatoric part of a tensor $\underline{t}$ treated as a function of $\tilde{\underline{T}}$ and $\hat{\underline{q}}$ i.e.,

$$\underline{q}^{-1}\,\underline{\acute{q}} = \mathrm{dev}\,\underline{t} \quad \text{and} \quad \underline{\acute{q}}\,\underline{q}^{-1} = \mathrm{dev}\,\underline{t}\,, \qquad /10.29/$$

where dev $\underline{t}$ means the deviatoric part of $\underline{t}$. Since

$$\frac{d\underline{\hat{q}}}{dz} = \frac{1}{2}\left[\,\underline{q}\,(\mathrm{dev}\,\underline{t}) + (\mathrm{dev}\,\underline{t}\,)\,\underline{q}\,\right] \qquad /10.30/$$

the function $\underline{t} = \underline{\tilde{t}}\ /\underline{\tilde{T}},\underline{\hat{q}}/$ should be linear in $\underline{\tilde{T}}$ and $\underline{\hat{q}}$ in order to be consistent with the assumption of the second order approximation:

$$\tilde{t}(\underline{\tilde{T}},\underline{\hat{q}}) = d_o\underline{\tilde{T}} + d_1\underline{\hat{q}} + d_2 \mathrm{tr}(\underline{\tilde{T}})\underline{1}, \qquad /10.31/$$

where d_o, d_1 and d_2 are material constants. The condition /6.9/ results in $d_1 = 0$. Hence the final form of the stress-type evolution equation is

$$\frac{1}{d_o}\frac{d\underline{\hat{q}}}{dz} = \underline{\tilde{T}}' + \underline{\tilde{T}}'\underline{\hat{q}} + \underline{\hat{q}}\,\underline{\tilde{T}}' \qquad /10.32/$$

where $\underline{\tilde{T}}'$ is the deviatoric part of the stress $\underline{\tilde{T}}$. Eq. /10.32/ has linear term and nonlinear coupled terms. The equation obtained above depends on the stress history and involves one material constant d_o.

If we interpret q as a plastic strain, the first term can be rewritten as follows

$$d\underline{\hat{q}} = d_o\underline{\tilde{T}}'dz \quad ; \quad d\hat{q}_{ij} = d_o\tilde{T}'_{ij}\,dz\,. \qquad /10.33/$$

That is, the increment of plastic strain $d\hat{q}_{ij}$ depends on the deviatoric stress $\tilde{T}'_{ij}$ multiplied by a scalar function dz. It is similar to the Prandtl-Reuss equation. In other words we may say that our stress-type evolution equation includes the Prandtl-Reuss equation.

10.6 Intrinsic time

We formulate the stress-based intrinsic time measure

$$\left(\frac{d\zeta_T}{dt}\right)^2 = R_1^2 \left|I_{\dot{\underline{T}}}\right|^2 + R_2^2 \left|II_{\dot{\underline{T}}}\right| + R_3^2 \left|III_{\dot{\underline{T}}}\right|^{2/3} \qquad /10.34/$$

with

$$\dot{\underline{T}} = \frac{d}{dt}\left(\tilde{\underline{T}}(t)\right),$$

where ʹtʹ is natural time; Rʹs are material constants. In general they may depend on invariants of $\dot{\underline{T}}$ and $\tilde{\underline{T}}$; $I_{\dot{T}}$, $II_{\dot{T}}$ and $III_{\dot{T}}$ are the first second and third invariants of $\dot{\underline{T}}$, respectively.

For a time scale we propose /cf. /6.16/

$$\frac{dz_T}{d\zeta_T} = g(\tilde{\underline{T}},\hat{\underline{q}},\zeta_T) = g_1(\tilde{\underline{T}},\hat{\underline{q}})\,g_2(\zeta_T), \qquad /10.35/$$

where g_1 is a scalar-valued function of two tensors $\tilde{\underline{T}}$ and $\hat{\underline{q}}$, and g_2 is a scalar-valued function of the scalar ζ_T.

10.7. Equality of linearized constitutive relations

It is possible to invert the linear equation /7.35/ to the following form:

$$\underline{E} = -\frac{\nu}{E_o}\,\mathrm{tr}(\tilde{\underline{T}})\underline{1} + \frac{1+\nu}{E_o}\tilde{\underline{T}} + \hat{\underline{q}}\,. \qquad /10.36/$$

It is exactly the same with eq. /10.13/ which we have obtained from the complementary energy function.

It is easy to show the equality of the strain-type evolution eq. /8.5/ and the stress-type eq. /10.32/ under the assumption of the existence of the linear constitutive equation /7.35/. Substitution of /7.35/ into /8.5/ furnishes

$$\frac{1}{d_o}\,\frac{d\hat{\underline{q}}}{dz} = \tilde{\underline{T}} - \frac{1}{3}\mathrm{tr}(\tilde{\underline{T}})\underline{1} + \tilde{\underline{T}}\,\hat{\underline{q}} + \hat{\underline{q}}\,\tilde{\underline{T}} - \frac{2}{3}\mathrm{tr}(\tilde{\underline{T}})\hat{\underline{q}}$$

$$d_o = \frac{b_o}{2\mu} \qquad /10.37/$$

Note that two material constants d_o and b_o are connected by eq. $/10.37/_2$.

11. Linear elastic response from unloaded state
/Elasto-plastic coupling. Existence of elasticity in ET/

In this section we shall show important results which are mainly due to the definition of the evolution equation /6.9/ at unloaded state. We write

$$\left.\frac{d\underline{q}}{dz(\wp)}\right|_{\underline{c}=\underline{q}} = \underline{0} \qquad \left.\tilde{\underline{T}}(\underline{c},\underline{q})\right|_{\underline{c}=\underline{q}} = \underline{0} \qquad /11.1/$$

From any unloaded state $\wp = \wp^*$, we make small increment $\Delta\wp$ then Taylor's series expansion of stress $\tilde{\underline{T}}(\wp^* + \Delta\wp)$ is

$$\tilde{\underline{T}}(\wp^* + \Delta\wp) = \tilde{\underline{T}}(\wp^*) + \tilde{\underline{T}}'(\wp^*)\Delta\wp + \frac{1}{2}\tilde{\underline{T}}''(\wp^*)\Delta\wp^2 + \ldots$$
$$+ \tilde{\underline{T}}'(\wp^*)\Delta\wp + \underline{0}(\Delta\wp^2), \qquad /11.2/$$

since at unloaded state stress $\tilde{\underline{T}}(\wp^*) = \underline{0}$ by $/11.1/_2$. The first term of the right-hand side of the above equation is

$$\tilde{\underline{T}}'(\rho^*) = \tilde{\underline{T}}'(\underline{C}(\rho^*), \underline{q}(\rho^*))$$
$$= \left.\frac{\partial\tilde{\underline{T}}}{\partial\underline{c}}\frac{\partial\underline{c}}{\partial\wp}\right|_{\wp=\wp^*} + \left.\frac{\partial\tilde{\underline{T}}}{\partial\underline{q}}\frac{\partial\underline{q}}{\partial\wp}\right|_{\wp=\wp^*}$$
$$= \left.\frac{\partial\tilde{\underline{T}}}{\partial\underline{c}}\frac{\partial\underline{c}}{\partial\wp}\right|_{\wp=\wp^*}. \qquad /11.3/$$

Note that the increment of $\tilde{\underline{T}}$ from any unloaded state is given by the increment of $\underline{C}$ only, not by $\underline{q}$ because the linear increment of $\underline{q}$ from any unloaded state is $\underline{0}$ by /11.1/. Equations /11.1/. Equations /11.1/ and /11.3/ give us

$$\tilde{T}(\zeta^{*}+\Delta\zeta)_{ij} = \frac{\partial \tilde{T}_{ij}}{\partial C_{km}} \frac{\partial C_{km}}{\partial \zeta}\Bigg|_{\substack{q=c \\ \zeta=\zeta^{*}}} + 0(\Delta\zeta^{2})$$

/11.4/

$$= \lambda(tr(\Delta \underline{E})\delta_{ij} + 2\mu \Delta E_{ij} + (4\rho_0 a_9 - 2\vartheta_3)(\hat{q}_{im}\Delta E_{mj} + \Delta E_{im}\hat{q}_{mj}) + 0(\Delta\zeta^{2}),$$

where

$$2\Delta E_{km} = \frac{\partial C_{km}(\zeta^{*})}{\partial \zeta}\Delta\zeta .$$

/11.5/

At initial natural state $\underline{q}^{*} = \underline{0}$, hence eq. /11.4/ together with eq. /11.5/ reduce to

$$T(0+\Delta\zeta)_{ij} = \lambda\, tr(\Delta \underline{E})\delta_{ij} + 2\mu\Delta E_{ij} + 0(\Delta\zeta^{2}) \qquad /11.6/$$

Eq. /11.4/ together with /11.5/ shows that the material response from any unloaded state as well as from the natural state is linear elastic within the order $\Delta\zeta$ /and neglecting higher order terms/. This equation shows the existence of elasticity without introducing any yield criterion.

In his very recent paper [5] Valanis showed the existence of elasticity in ET using a new definition of the intrinsic time measure. However, our result on the existence of elasticity is obtained mainly owing to the evolution equation /11.1/.

The response from an unloaded state is linear elastic but the response depends on deformation history through the plastic /permanent/ strain $\underline{q}$. The second part of the eq. /11.4/ shows that the $\hat{q}_{im}\Delta E_{mj} + \Delta E_{im}\hat{q}_{mj}$ renders material anisotropy /tensorial property/ in the elastic response from unloaded state. At virgin state $\underline{q}\big|_{\zeta=0} = \underline{0}$ and the material response from virgin state is isotropic. After some deformation history $\underset{\sim}{\varepsilon}^{r} \neq \underset{\sim}{0}$ in general, the material acquires an anisotropy in linear elastic response. It means that elastic properties from unloaded state depend on the direction of previous deformations. The terms $a_8 J_8$ and $a_9 J_9$ of the free energy Ψ in eq. /7.7/ introduce the anisotropy in the initial linear response from unloaded states.

Chapter IV Mechanical Ratcheting

In this Chapter we analyze two strain-controlled ratchetings. The ratchet strain can result in large strain, say 15%, so we have to use large deformation theory and the Cauchy stress. Valanis and Wu, for instance, analyzed ratcheting, however, used the infinitesimal theory. When the ratchet plastic strain attains large strain, the incompressibility condition cannot be approximated by tr /d$\mathfrak{q}$/ = 0, but det /$\mathfrak{q}$/ = 1 should be used. The evolution equation /8.5/ with the nonlinear coupling term must be applied to the large ratchet plastic strain condition. For this complex phenomenon under multiaxial loading condition we use the equations /10.27/ with /10.28/, /8.5/, /9.1/, /9.2/ and /9.3/.

Interpreting the available experimental data in the invariants of strain amplitude space, we show the features of the influence of the steady stress in next section.

12. Experimental evidence of influences of steady stress on ratchet strain

In this section the influence of steady stress is surveyed using available experimental data. The ratchet strain-steady stress curves of certain constant strain amplitude were drawn using cycle number as a parameter. All figures, except Fig. 4.5, were drawn rewriting the published data.

Fig. 4.1 shows a set of ratchet strain-steady stress curves having rewritten the data by Mayor and Sinclair [28] for OFHC copper under the steady axial stress σ_o changing from 1.4 to 7.5 kg/mm^2 superposed on the cyclic shear strain amplitude γ^o = 0.275% /engineering strain/ at cyclic numbers n = 1000, 5000 and 10000. Ordinate denotes the axial ratchet strain, abscissa denotes the steady stress σ_o and the nondimensional stress $\sigma_o / \sigma_{y\,o\cdot2}$ where $\sigma_{yo\cdot2}$ is the 0.2 per cent of the yield stress. The marks (• • •) denote experimental results and solid lines denote calculated values using an empirical power relation:

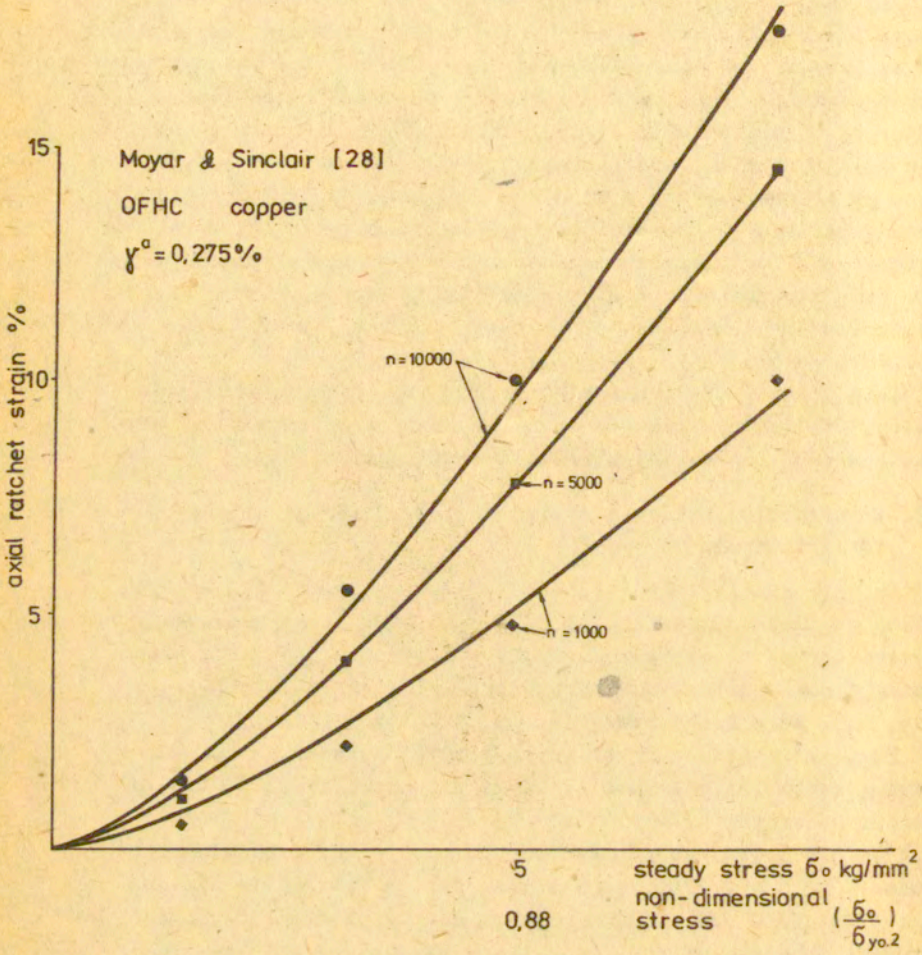


Fig. 4.1 Ratchet strain vs. steady stress σ_o and nondimensional stress $(\sigma_o/\sigma_{y0.2})$

$$\varepsilon^r = A(n)\left(\frac{\sigma_o}{\sigma_{Yoz}}\right)^k \qquad /12.1/$$

where ε^r is the ratchet strain; k is a material constant; A /n/ is an empirical function of cycle number n. The used values for calculation are

$$k = 1.43;\ A(1000) = 6.17,\ A(5000) = 9.44,\ A(10000) = 5.22, \qquad /12.2/$$

One can see from the Figure that the power relation /12.1/ can well describe the influence of steady stress.

Fig. 4.2 gives a set of ratcheting strain curves having been rewritten the data by Wood and Cousland [26] for copper under steady axial stresses σ_o = 1.6, 3.9 and 7.5 kg/mm^2 superposed cyclic shear strain γ^a = 0,45% at cycle numbers n = 100, 1000, 10000. Similarly, the marks '• ▲ ■' denote experimental data, and solid lines were drawn using an empirical equation /12.1/. We used values

$$k = 1.35 \quad A(100) = 0.501 \quad A(1000) = 0.538 \quad A(10000) = 0.573. \qquad /12.3/$$

One can see in the Figure that the power equation /12.1/ with values /12.3/ can well describe the real behaviour.

Fig. 4.3 shows a set of ratchet strain curves by Udoguchi et al [30] for 6:4 Brass tubes under the steady axial stress σ_o changing from -3.05 to +6.1 kg/mm^2 superposed on cyclic shear strain amplitude γ^a = 2.56%. Solid lines denote calculated values using empirical relation /12.1/ or lines which connect experimental values simply. It can be seen that even at zero steady stress σ_o = 0 the axial strain accumulates by cyclic shear straining. This is the so-called "second order strain" and is similar to the Poynting effect. Dashed lines are extension of straight lines from σ_o = 0 to σ_o = 1.53 kg/mm^2 and drawn to show the discrepancy of the ratchet strain from the linear relation. From σ_o = 0 to around σ_o = 2.8 kg/mm^2 it is approximated by lines. In the right-hand side of the Fi-

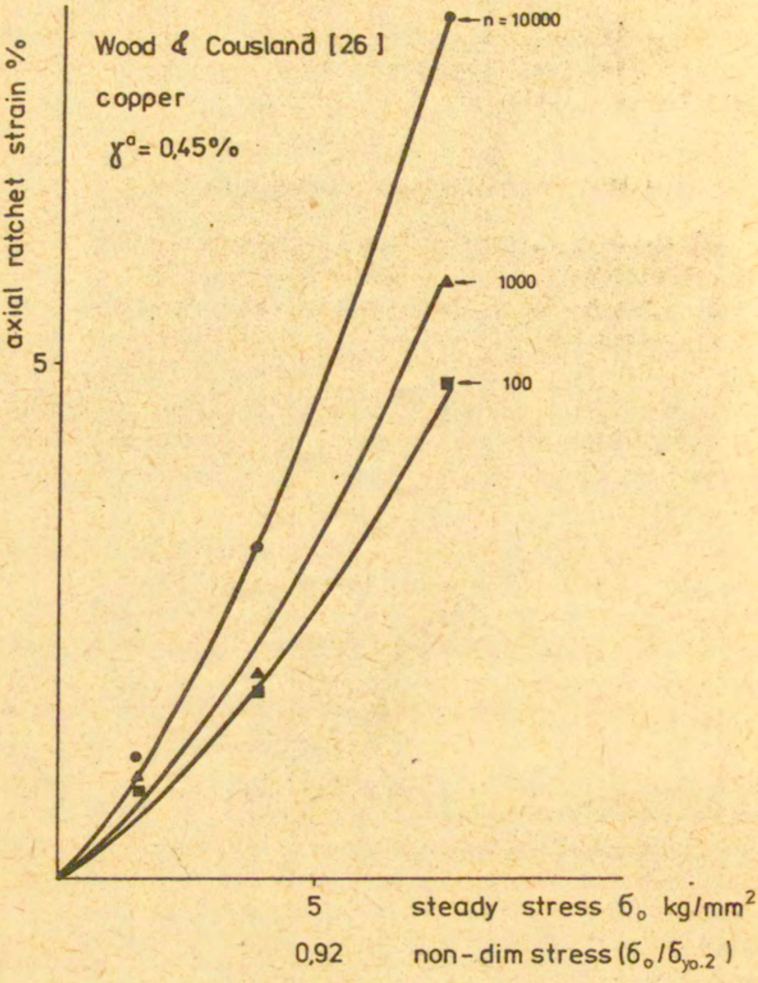


Fig.4.2 Axial ratchet strain vs. steady stress and non-dimensional stress

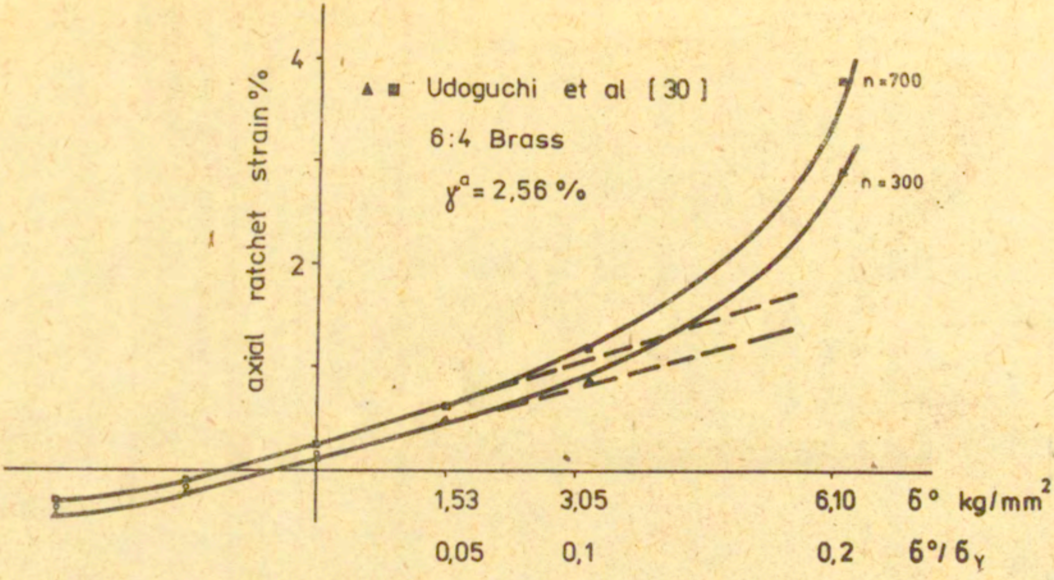


Fig. 4.3 Axial ratchet strain vs. steady axial stress and non-dimensional stress

gure the curves are drawn in the domain from σ_o = 2.4 to 6.2 kg/mm^2 using the empirical equation /12.1/ together with values

$$k=1.79 \quad A(300)=51.7 \quad A(700)=67.8 \,. \qquad /12.4/$$

The behaviour can be expressed by the power in this domain of steady stress. In the domain of compression /negative/ stresses the compressive /negative/ ratchet strain accumulates. But ratchet strains for negative and positive σ_o are not the same. It may be attributed to the existence of a correlation between the positive axial strain accumulation by the cyclic shearing and the negative axial strain by the superposition of negative σ_o . Since there are not many studies on the influence of negative σ_o it is difficult describe this quantitatively but we can point out that curves are not symmetry with respect to the origin.

Fig. 4.4 shows the ratchet strain versus square of the steady stress using the same data by Mayar and Sinclair at n = 1000, 3000, 5000, 7000, 10000. All above curves Figs. 4.1-4.3 are not linear. It means that ratchet strain is generally not linear in the steady stress. It limits the value k of equation /12.5/ to 1 < k. Moreover, Fig. 4.4 shows that the ratchet strain is not linear in $(\sigma_o)^2$. It limits the value of k to k < 2. Hence the value k must be 1 < k < 2.

According to the author's knowledge, the most systematic experiments on ratcheting were carried out by Udoguchi et al [30] so it is worthwhile to study in detail their results. They found both the steady axial stress-cyclic shear strain ratcheting and the steady shear stress-cyclic axial strain ratcheting can be expressed by similar empirical equations. They devided the strains into elastic and accumulation strains as follows:

$$\varepsilon^r = \varepsilon_e + \varepsilon_{pII}(\gamma_{pa}, n) + \varepsilon_{p\sigma}(\sigma_o/\sigma_y, \gamma_{pa}, n) \qquad /12.5/$$

$$\gamma^r = \gamma_e + \gamma_{p\tau}(\tau_o/\tau_y, \gamma_{pa}, n). \qquad /12.5/$$

The cyclic shear strain without axial stress is denoted as $\varepsilon_{p\Pi}$. The $\varepsilon_{p\Pi}$ is a functions of plastic strain amplitude and cycle number n. The ε_{pa} and γ_{pa} are functions of the nondimensional stresses σ_o/σ_y, τ_o/τ_y and the plastic strain amplitudes γ_{pa} and ε_{pa}, and cycle number n. For 0.45 per cent carbon steel S45C the empirical functions $\varepsilon_{p\sigma}$ and $\gamma_{p\tau}$ are

$$\varepsilon_{po} = \left[k_1(\gamma_{pa})^{k2}(2n)^{B_\sigma}\right](\sigma_o/\sigma_y)^{k3},$$
$$\gamma_{p\tau} = \left[2k_1(2\varepsilon_{pa})(2n)^{B_\tau}\right](\tau_o/\tau_y)^{k_3}, \qquad /12.6/$$

where $k_1 = 0.77$, $k_2 = 0.9$, $k_3 = 1.1$ are constants; B_σ and B_τ are certain functions of plastic strain amplitude. The influence of steady stress on ratchet strain is separately expressed through $(\sigma_o/\sigma_y)^{1.1}$ and $(\tau_o/\tau_y)^{1.1}$. Eq. /12.6/ shows a weak nonlinear dependence on steady stresses. For 6:4 Brass, the influence is written by the following equations

$$\varepsilon_{po} = C_\sigma(\sigma_o/\sigma_y)\, F_o(\gamma_{pa}, n),$$
$$\gamma_{pa} = C_\tau(\tau_o/\tau_y)\, F_\tau(\varepsilon_{pa}/n), \qquad /12.7/$$

with the products of the functions of cycle number n and strain amplitude, F_σ and F_τ and the functions of steady stresses, C_σ and C_τ. The both functions C_σ and C_τ are shown in Fig. 4.5. They are slightly discrepant from linear functions in the region $0 < (\sigma_o/\sigma_y\ \tau_o/\tau_y) < 0.1$ but around $\sigma_o/\sigma_y = \tau_o/\tau_y = 0.2$ the discrepancy from linearity is about 50%. In other words in the small steady stress region, the influence is almost linear, but in large steady stress region the nonlinear dependenee is significant. For stainless-steel SUS 304 the influence is expressible by the empirical relations

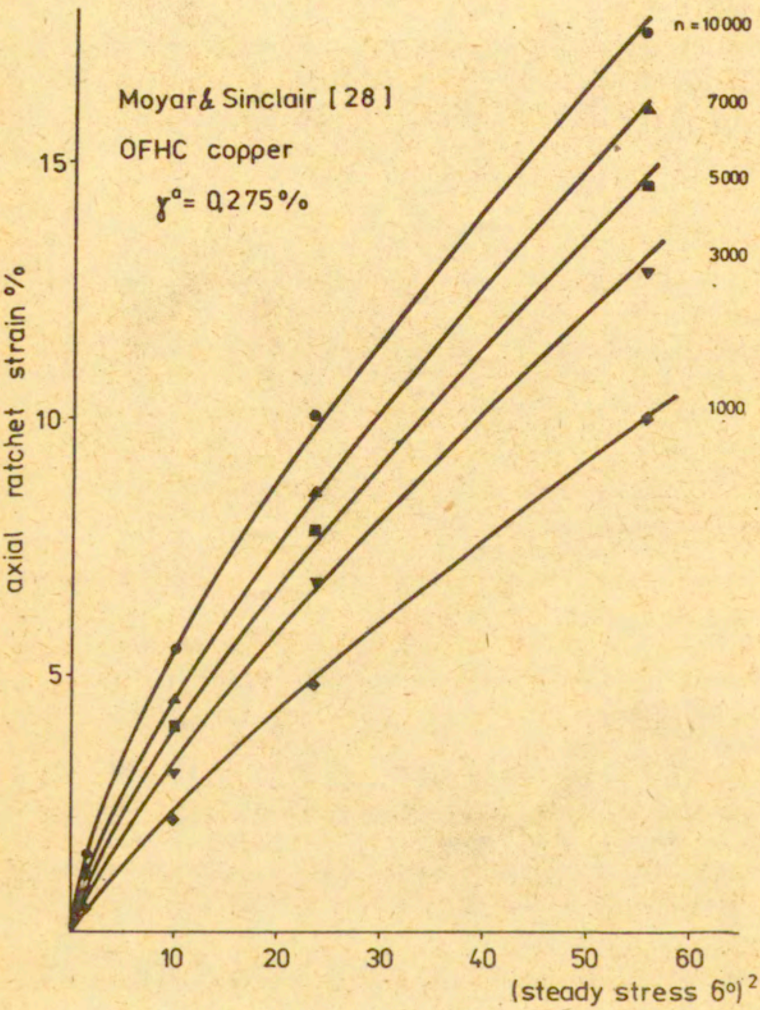


Fig. 4.4 Ratchet strain vs. (steady stress)^2

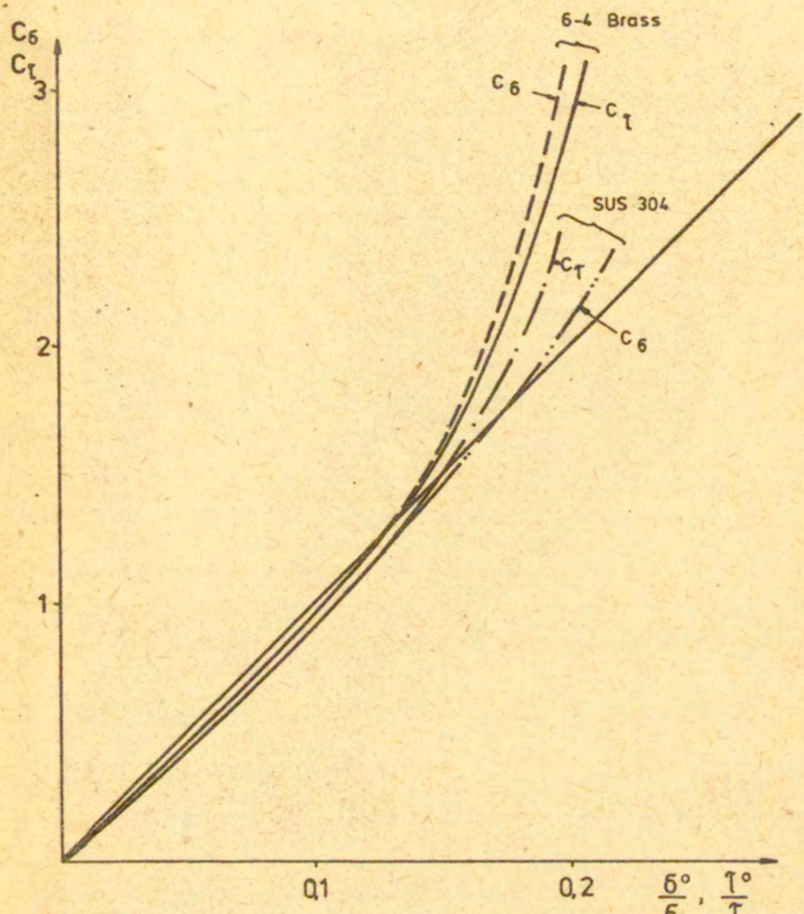


Fig.4.5 Emprical functions of steady stress by Udoguchi et al. [30]

$$\varepsilon_{p\sigma} = C_\sigma (\sigma_o / \sigma_y) F (\gamma_{pa} / n)$$

$$\gamma_{pa} = C_\tau (\tau_o / \tau_y) 0.75 F (3.4 \varepsilon_{pa}, n), \qquad /12.8/$$

where F is a function of the plastic strain amplitude and cycle number n .

Summary of this section in which we has surveyed the influence of steady stress on ratchet strain is:

i/ Generally the influence of the steady stress on the ratchet strain can be expressed by a power relation

$$\text{ratchet strain} \sim \left(\frac{\text{steady stress}}{\text{yield stress}} \right)^k \quad 1 < k < 2, \qquad /12.9/$$

ii/ For small steady stresses, the relation can be approximated by the linear one, but for large steady stresses the relation shows obvious and striking nonlinearity. /In many theories [3,15,19] which analyzed hitherto the ratcheting, the ratchet strain is linear in steady stress so their prediction ability is limited/.

iii/ The form above written is similar to Norton's law in creep. In ratcheting and creep the influence of steady stresses on strains are formally the same.
In the following sections, the influence of steady stress will be described within the proposed theory.

13. Steady axial stress and cyclic shear strain ratcheting

13.1. Explicit constitutive relations

We first address ourselves to the representative ratcheting i.e. to the case of a steady normal stress superposed cyclic shear strain /see Fig. 1.1/a/ /. Under this simple loading condition many properties have been studied and various interesting phenomena have been reported. So it is worthwhile to study in detail the ratcheting under this loading condition.

Then the Cauchy stress $\underline{T}$ in the constitutive equation must have the form /cf. /16.24/ /.

$$\underline{T} = \begin{bmatrix} \overset{c}{T}{}^0_1 & \overset{c}{T}(t)_{12} & 0 \\ \overset{c}{T}(t)_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad /13.1/$$

in the Cartesian coordinate system /we use Cartesian coordinate system throughout/. That is, $\overset{c}{T}{}^0_1$ is constant throughout the experiment, and there exists time dependent shear stress $\overset{c}{T}_{12}$ /t/. This stress $\overset{c}{T}_{12}$ is not a control quantity but should be expressed by strain $\underline{E}$ and internal variable $\underline{\hat{q}}$. All remaining components of $\underline{T}$ are zero.

The strain $\underline{E}$ and the internal state variable $\underline{\hat{q}}$ related to $\underline{T}$, are

$$\underline{E} = \begin{bmatrix} E_1 & E_{12} & 0 \\ E_{12} & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix} \qquad \underline{\hat{q}} = \begin{bmatrix} \hat{q}_1 & \hat{q}_{12} & 0 \\ \hat{q}_{12} & \hat{q}_2 & 0 \\ 0 & 0 & \hat{q}_3 \end{bmatrix}. \qquad /13.2/$$

The constitutive equation is given by eq. /10.27/ together with eq. /13.3/

$$\tilde{T}_1 = \left(\frac{bc}{a}\right)\overset{c}{T}{}^0_1 - \left(\frac{2cd}{a}\right)\overset{c}{T}_{12}(t) \qquad \tilde{T}_{12} = c\,\overset{c}{T}_{12}(t). \qquad /13.3/$$

To distinguish the Cauchy stress components and the second Piola-Kirchhoff stress components, the 'C' is put for the former and the '~' is written for the later stress components. Component of $\underline{F}$, using /10.25/, are

$$a = \sqrt{1 + 2E_{11}} \quad , \quad d = 2E_{12} / \sqrt{1 + 2E_{11}} \; ,$$

$$b = \sqrt{1 + 2E_{22} \; d^2} \quad , \quad c \; \sqrt{1 + 2E_{33}} \; . \qquad /13.4/$$

The evolution equation /8.5/ is used in the further investigation. For strain-control ratcheting it is better to use strain-based intrinsic time measure. Under the present loading condition /13.2/ three invariants of $\underline{E}'$ are

$$I_{\dot{\underline{E}}} = \dot{E}_1 + \dot{E}_2 + \dot{E}_3 \quad II_{\dot{\underline{E}}} = \dot{E}_1\dot{E}_2 + \dot{E}_2\dot{E}_3 + \dot{E}_3\dot{E}_1 - (\dot{E}_{12})^2 \quad III_{\dot{\underline{E}}} = \dot{E}_1\dot{E}_2\dot{E}_3 - (\dot{E}_{12})^2\dot{E}_3 \qquad /13.5/$$

To analyze actual problems we make an approximation for time being, it means that we use linearized part of equation /10.13/

$$\dot{\underline{E}} = -\frac{\nu}{E_o} I_{\tilde{T}} \underline{1} + \frac{1+\nu}{E_o} \tilde{\underline{T}} + \hat{\underline{q}} \qquad /13.6/$$

Then each component of $\underline{E}$ is written as:

$$E_1' = \frac{1}{E_o}\tilde{T}_1 + \hat{q}_1 \qquad E_3 = -\frac{\nu}{E_o}\tilde{T}_1 + \hat{q}_3$$

$$E_2 = -\frac{\nu}{E_o}\tilde{T}_1 + \hat{q}_2 \qquad E_{12} = \frac{1+\nu}{E_o}\tilde{T}_{12} + \hat{q}_{12} \qquad /13.7/$$

The above relation can be simplified further, by making use of the relation $/13.7/_1$ i.e., $T_1 = E_0 \, /E_1' - \hat{q}_1/$

$$E_2 - \hat{q}_2 = E_3 - \hat{q}_3 = -\frac{\nu}{E}\tilde{T}_1 = -\nu \, (E_1 - \hat{q}_1) \qquad /13.8/$$

where $/E_2 - \hat{q}_2/$ and $/E_3 - \hat{q}_3/$ are expressed $E_1 - q_1$. Note that above relation does not mean $E_2 = E_3$ or $\hat{q}_2 = \hat{q}_3$ but it means that the differences of $E_2 - \hat{q}_2$ and $E_3 - \hat{q}_3$ are the same.

Time derivative of the control quantity E_{12} ... also be written above to express $\dot{E}_1$ by known variable $\dot{E}_{12}$. The time derivative of a component of the second Piola-Kirchhoff stress, $\dot{T}_1$ is expressible in terms of the Cauchy stress and deformation gradient components

$$\tilde{T}_1 = \left(\frac{\dot{a}bc - a\dot{b}c - ab\dot{c}}{a^2}\right)\overset{c}{T}{}^{\circ}_1 + 2\left(\frac{\dot{a}cd - a\dot{c}d - ac\dot{d}}{a^2}\right)\overset{c}{T}_{12} - \left(\frac{2cd}{a}\right)\dot{\overset{c}{T}}_{12} . \qquad /13.9/$$

Since we control the shear strain E_{12} not the shear stress $\overset{c}{T}_{12}$ we have to express the unknown quantity $\overset{c}{T}'_{12}$ by the independent quantity E_{12}, the same with T_{12} using /10.28/, we have

$$\tilde{T}_{12} = \dot{c}\,\overset{c}{T}_{12} + c\,\dot{\overset{c}{T}}_{12} . \qquad /13.10/$$

Substituting /13.9/ into /13.10/ leads to

$$\dot{\overset{c}{T}}_{12} = \frac{1}{c}\,\frac{E_o}{1+\nu}\left(\dot{E}_{12} - \dot{q}_{12}\right) - \left(\frac{\dot{c}}{c}\right)\overset{c}{T}_{12} . \qquad /13.11/$$

Hence, $\dot{\tilde{T}}'_1$ in eq. /13.7/ is written by the Cauchy stress and deformation gradient as:

$$\dot{\tilde{T}}_1 = \left(\frac{a\dot{b}c + ab\dot{c} - abc}{a^2}\right)\overset{c}{T}{}^{\circ}_1 - 2\left(\frac{a\dot{c}d + ac\dot{d} - \dot{a}cd}{a^2}\right)\overset{c}{T}_{12}$$

$$- \left(\left(\frac{2d}{a}\right)\frac{E_o}{1+\nu}\left(\dot{E}_{12} - \dot{q}_{12}\right) - \left(\frac{2d\dot{c}}{a}\right)\overset{c}{T}_{12}\right), \qquad /13.12/$$

where each component of the time derivative of deformation gradient is

$$\dot{a} = \dot{E}_{11}\Big/\sqrt{1 + 2E_{11}} \qquad \dot{d} = 2\left(\dot{a}E_{12} - a\dot{E}_{12}\right)\Big/a^2$$

$$\dot{b} = \left(\dot{E}_{22} - d\dot{d}\right)\Big/\sqrt{1 + 2E_{22} + d^2} \qquad \dot{c} = \dot{E}_{33}\Big/\sqrt{1 + 2E_{33}} \qquad /13.13/$$

Then the time measure ζ is

$$\dot{\zeta} = k_1 \left| \dot{E}_1 + \dot{E}_2 + \dot{E}_3 \right| + k_2 \left| \dot{E}_1\dot{E}_2 + \dot{E}_2\dot{E}_3 + \dot{E}_3\dot{E}_1 - (\dot{E}_{12})^2 \right|^{1/2} +$$

$$+ k_3 \left| \dot{E}_1\dot{E}_2\dot{E}_3 - (\dot{E}_{12})^2 \dot{E}_3 \right|^{1/3} \qquad /13.14/$$

together with eqs. /13.7/, /13.8/, /13.12/ and /13.13/.
Though a study of the experimental data shows that the increase in $\dot{E}_1$, $\dot{E}_2$, and $\dot{E}_3$ per cycle are small in the sense that

$$|\dot{E}_1|, \ |\dot{E}_2|, \ |\dot{E}_3| \ll |\dot{E}_{12}|$$

we do not make the following approximation

$$d\zeta = k_2 \left| dE_{12} \right|$$

as Valanis and [3] did. We maintains that three invariants of $\dot{\underline{E}}$ in eq. /13.5/ have different properties. For instance, we assume $\frac{d}{dt}$ /det /$\underline{q}$/ / = 0, but do not assume $\frac{d}{dt}$ /det /$\underline{E}$/ / = 0, so cannot drop $III_{\dot{\underline{E}}}$. Time scale under this loading condition is

$$\frac{dz}{d\zeta} = \left| \frac{(\vartheta^2 - 2\vartheta)}{E_o^2} \tilde{T}_1^2 - (E_{12} - \hat{q}_{12})^2 \right|^{c_2} (c_3 + c_4 e^{-c_5\zeta}), \qquad /13.15/$$

where

$$\tilde{T}_1 = \left(\frac{bc}{a}\right) T_1^o - \left(\frac{2cd}{a}\right) T_{12} .$$

We have completed explicit forms of all constitutive relations for this ratcheting problem.

Valanis and Wu's theory for axial ratcheting strain ε_1 is

$$\varepsilon_1 = \frac{\alpha \sigma_o}{\beta E_o} \log (1 + \beta\zeta)$$

where α , β , E_0 are material constants and σ_o is axial steady stress. In their theory the axial ratchet strain ε_1 linearly depends on the steady stress σ_0 so it is impossible to describe nonlinear dependence on the steady stress. In our theory the axial strain E_1 is written omitting reversible "elastic strain" by $/10.27/_1$, /8.5/, /13.3/ i.e.

$$E_1 = \dots + \hat{q}_1 + (b_{17} + 2b_9)\left(\frac{bc}{a}\right)\overset{c}{T}{}^{o}_{1}\hat{q}_1 - (b_{17} + 2b_9)\left(\frac{2cd}{a}\right)\overset{c}{T}_{12}\hat{q}_1 +$$

$$+ (2b_{17} + 2b_9)c\,\overset{c}{T}_{12}\hat{q}_{12},$$

$$\frac{1}{b_0}\frac{dq_1}{dz} = \frac{2}{3E_0}\left(\frac{bc}{a}\right)\overset{c}{T}{}^{o}_{1} - \frac{2}{3E_0}\left(\frac{2cd}{a}\right)\overset{c}{T}_{12} + \left(\frac{1-\nu}{E_0}q_{12} - \frac{2}{E_0}q_1\right) \quad /13.16/$$

$$\left(\left(\frac{bc}{a}\right)\overset{c}{T}{}^{o}_{1} - \left(\frac{2cd}{a}\right)\overset{c}{T}_{12}\right)$$

One can see in eq. $/13.16/_2$ that $\acute{q}_1$ has term$\left(\frac{2}{3E_0}\left(\frac{bc}{a}\right)\overset{c}{T}{}^{o}_{1}\right.$ which is linear in $\overset{c}{T}{}^{o}_{1}$. Moreover, in the second term of eq. $/13.16/_1$, $\hat{q}_1$ /that is linear in $\overset{c}{T}{}^{o}_{1}$ / is multiplied by T_1. Hence E_1 depends nonlinearly on steady stress $\overset{c}{T}{}^{o}_{1}$. Numerically, we will demonstrate this nonlinear dependence immediately. If we put

$$b_{17} = b_9 = 0 \quad /13.17/$$

eq. $/13.16/_1$ reduces to

$$E_1 = \dots + q_1 \quad /13.18/$$

This relation together with the quasi-linear evolution equation $/13.16/_1$, is linear in steady stress. Under the condition /13.17/ eq. /13.16/ reduces essentially to Valanis and Wu's linear relation.

13.2. Comparison with experimental results

We shall compare our theoretical description with experimental data reported by Ikegami et al [29] based on observations of thin-walled tube of annealed 6:4 Brass, under steady axial stress and strain-controlled cyclic shear strain.

material	6:4 Brass	
Young's modulus E_0	10960 kg/mm^2	/13.19/
shear modulus $G = \mu$	4248 -	
Poissons ratio	0.29.	

For numerical calculation the used material constants are:

$$b_0 = 1300 \qquad c_2 = 0.2 \qquad c_3 = 0.6 \qquad c_4 = 0.4 \qquad c_5 = 2.0$$

$$b_9 = 0.0001\frac{mm^2}{kg},\ b_{17} = 0.0001\frac{mm^2}{kg},\ b_3 = b_{11} = b_{12} = 0. \qquad /13.20/$$

Especially for brass, it is impossible to measure "the third order elastic constants" [32] so we put $b_3 = b_{11} = b_{12} = 0$.

Fig. 4.6 shows two steady axial stresses $T_1^o = 6.88$, 9.63 kg/ /mm^2 superposed shear strain amplitude $\gamma^o = 1.73\%$ /engineering strain/ ratcheting curves. Ordinate denotes axial ratchet strain accumulated, abscissa denotes cycle number. The marks ' ▲ ' and ' • ' denote experimental data and the solid lines corresponds to calculated values. The material constants are choosen to fit data at n = 20. Though in small cycle number agreement is not good, but as a whole it can describe the ratchet strain accumulating behaviour in cyclic process.

Fig. 4.7 shows the influence of the steady axial stress on the axial ratchet strain accumulated. Ordinate denotes ratchet strain, abscissa denotes steady stress. At representative cycle numbers n = 10,20 the ratchet strain-steady stress curves were drawn. As one can see in Figure, it can describe the nonlinear influence of the steady stress on the ratchet strain. To the author's knowledge, this is the first theory that has succeeded in explaining this nonlinear dependence.

Fig. 4.8 is the cyclic shear stress-strain curve. The material hardens but tends to draw the steady hysteresis loop. The hysteresis loop does not behave in large cycle number like pure elastic response as Valanis original theory predicted.

14. Prediction of steady shear stress and cyclic axial strain ratcheting

14.1. Explicit constitutive relations

We shall analyze in this section another representative ratcheting problem in biaxial stress condition, i.e., the steady shear stress and superposed cyclic axial strain ratcheting

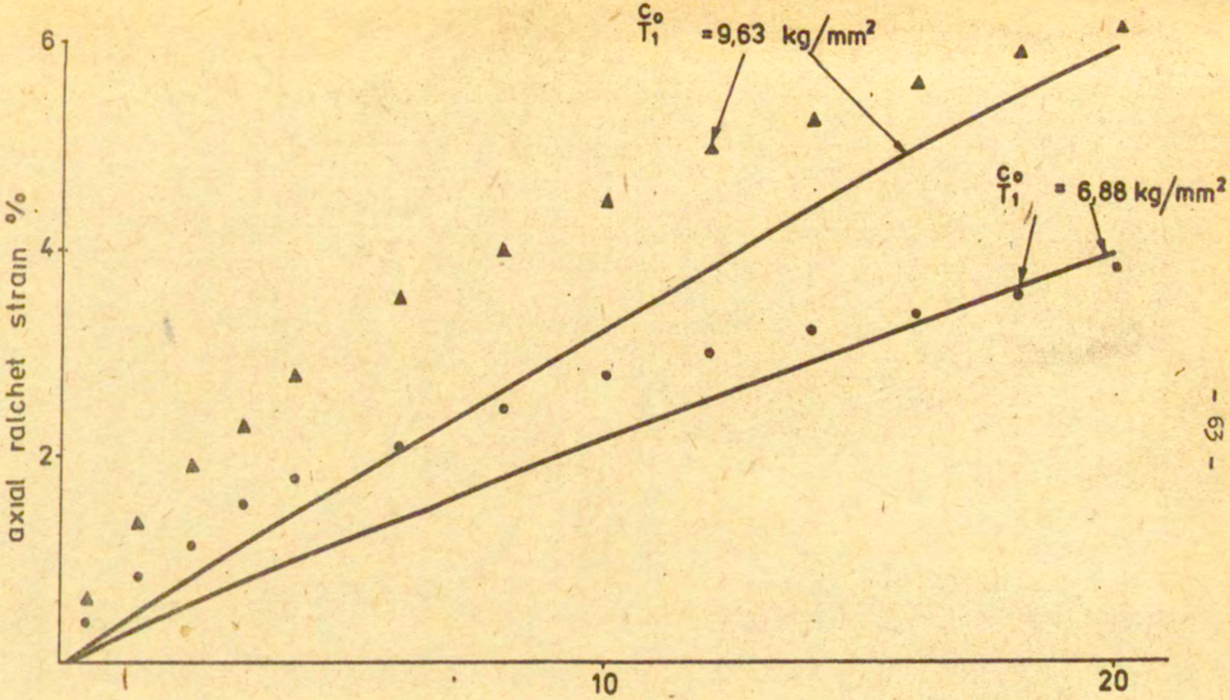


Fig.4.6 Axial ratchet stain accumulation

▲ • Experiment (Ikegami et al [29])

— Theory

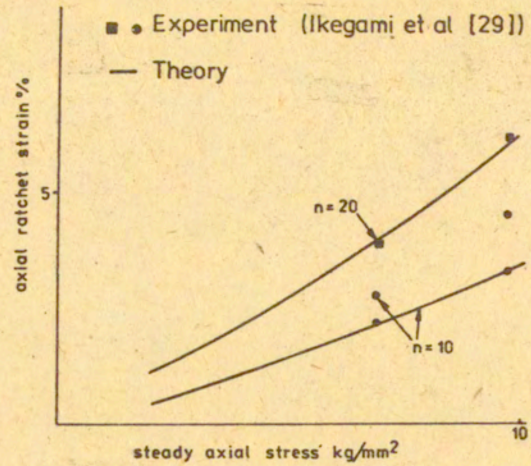


Fig. 4.7 Influence of steady stress

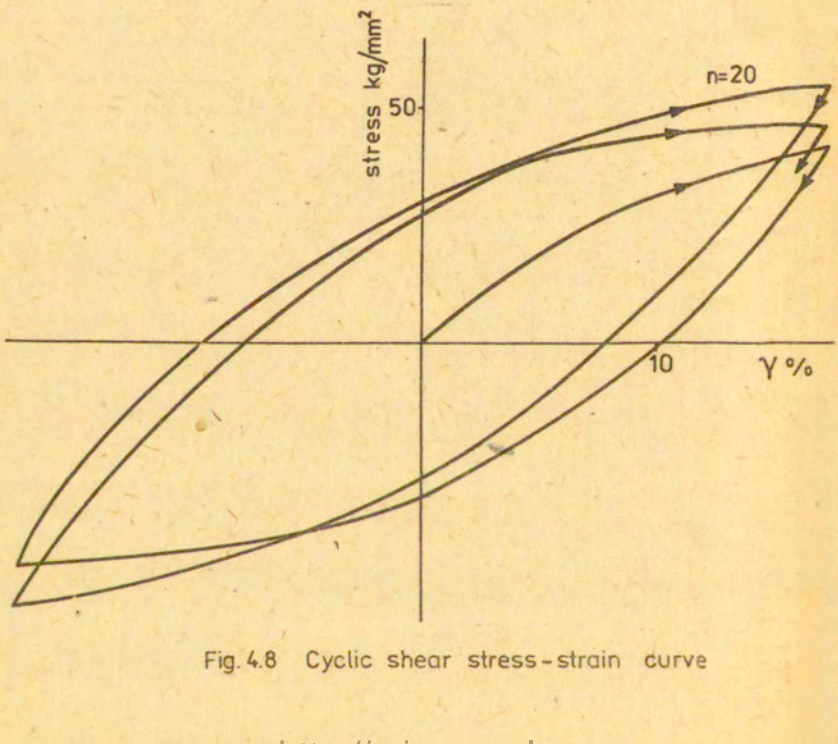


Fig. 4.8 Cyclic shear stress-strain curve

/see Fig. 1.1 /b/ /.

Using the same method shown in sect. 13 ratcheting under this loading condition will be analyzed. The Cauchy stress $\underline{T}$ has the non-zero time independent $\overset{c}{T}{}^{o}_{12}$ component and the time dependent $\overset{c}{T}_1$ component only:

$$\underline{T} = \begin{bmatrix} \overset{c}{T}_1(t) & \overset{c}{T}{}^{o}_{12} & 0 \\ \overset{c}{T}{}^{o}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \qquad /14.1/$$

The strain $\underline{E}$ and the internal variable $\underline{q}$ relating to the above $\underline{T}$ are given by eq. /13.2/. The constitutive equation is eq. /10.27/. But under this loading condition the non-zero components of the second Piola-Kirchhoff tensor $\underline{\tilde{T}}$ are:

$$\tilde{T}_1 = \left(\frac{bc}{a}\right) \overset{c}{T}_1 - \left(\frac{2cd}{a}\right) \overset{c}{T}{}^{o}_{12}$$

$$\tilde{T}_{12} = c\, \overset{c}{T}{}^{o}_{12} , \qquad /14.2/$$

where $\overset{c}{T}{}^{o}_{12}$ is the steady shear stress, $\overset{c}{T}_1$ is the axial stress of Cauchy stress; a,b,c and d are given by eq. /13.3/.

The evolution equation is given by /8.5/. For time being the same approximation, eqs. /13.5/, /13.6/, /13.7/ and /13.8/ are used, similarly.

The two non-zero components of $\dot{\underline{\tilde{T}}}$ are given as

$$\dot{\tilde{T}}_1 = \left(\frac{a\dot{b}c + ab\dot{c} - \dot{a}bc}{a^2}\right) \overset{c}{T}_1 + \left(\frac{bc}{a}\right) \dot{\overset{c}{T}}_1 - 2\left(\frac{a\dot{c}d + ac\dot{d} - \dot{a}cd}{a^2}\right) \overset{c}{T}{}^{o}_{12} ,$$

$$\dot{\tilde{T}}_{12} = \overset{c}{T}{}^{o}_{12}\, \dot{c} . \qquad /14.3/$$

The dependent quantity $\dot{\overset{c}{T}}_1$ in eq. $/14.3/_1$ is expressed with the help of $/13.7/_1$ by

$$\dot{\overset{c}{T}}_1 = \left(\frac{a}{bc}\right)\left[E_o(\dot{E}_1 - \dot{q}_1) - \left(\frac{a\dot{b}c - ab\dot{c} - \dot{a}bc}{a^2}\right) \overset{c}{T}_1\right] . \qquad /14.4/$$

$$+2\left(\frac{a\dot{c}d + ac\dot{d} - \dot{a}cd}{a^2}\right)\overset{c}{T}{}^{o}_{12}\Big]. \qquad /14.4/$$

Thus, $\overset{c}{T}_1$ has been expressed in terms of the Cauchy stress and the deformation gradient. Each $\dot{a}$, $\dot{b}$, $\dot{c}$ and $\dot{d}$, and a,b,c, and d are given by eqs. /13.13/ and /13.4/.
Then the time measure is given by /13.14/ together with eqs. /13.7/ and /14.3/. Time scale under this loading condition is

$$\frac{dz}{d\varphi} = (E_{11}-\hat{q}_{11})(E_{22}-\hat{q}_{22}) + (E_{22}-\hat{q}_{22}) + (E_{33}-\hat{q}_{33})(E_{11}-\hat{q}_{11}) - \left(\frac{1+\nu}{E_o}\tilde{T}_{12}\right)^2\Big| \times \qquad /14.5/$$

$$\times (c_3 + c_4 e^{c_5\varphi}),$$

where

$$\tilde{T}_{12} = c\,\overset{c}{T}{}^{o}_{12}, \qquad /14.6/$$

and eq. /13.8/ should be used along. We have completed the derivation of explicit forms of all constitutive relations for this ratcheting.

14.2. Prediction and comparison with experimental result

We shall predict the ratcheting of steady shear stress and cyclic axial strain using material constants /13.19/ and /13.20/ and compare the results with experimental results by Ikegami et al [29] .

Fig. 4.9 shows the ratcheting at the steady shear stress $\overset{c}{T}{}^{o}_{12}$ = 5.57 and 3.97 kg/mm^2 superposed axial strain amplitude E_1^a = 0.01. Ordinate denotes shear ratchet strain accumulated, abscissa denotes cycle number. The marks '▲' and '•' denote the experimental results and solid lines concern the calculated values.

As a whole one can say that the predicted results agree satisfactorily with experimental data. It is noteworthy that our

ET can describe consistently two ratcheting phenomena under multiaxial loading condition. It is another advantage of our ET.

Fig. 4.10 shows the influence of the steady shear stress on the axial ratchet strain accumulating. At representative cycle numbers n = 10,20 the ratchet strain-steady stress curves were drawn. As one can see from the Figure it can describe the influence of the steady stress on ratchet strain.

Fig. 4.11 is the cyclic axial strain-stress curve. The material hardens but tends to draw the steady hysteresis loop. At large cycle numbers the hysteresis loop does not coincides with linear elastic response.

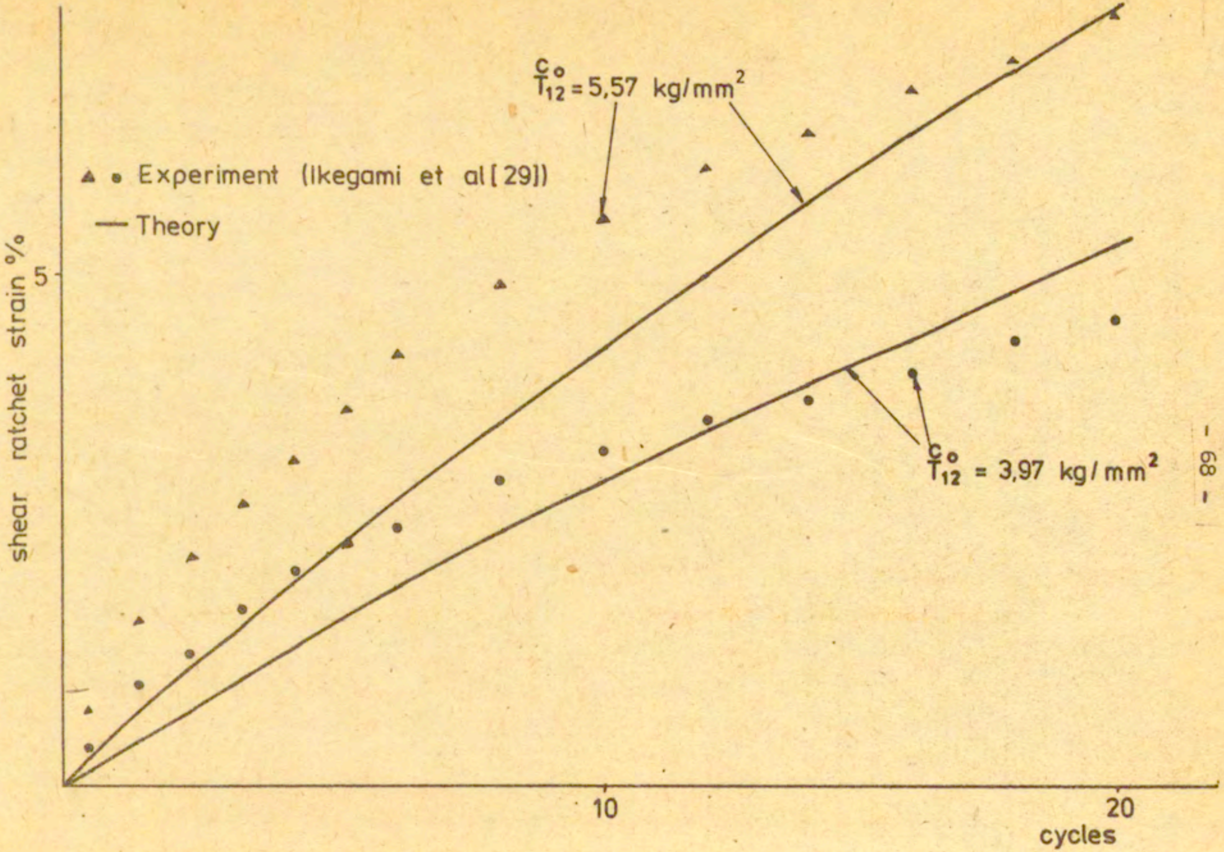


Fig. 4.9 Prediction of shear ratchet strain

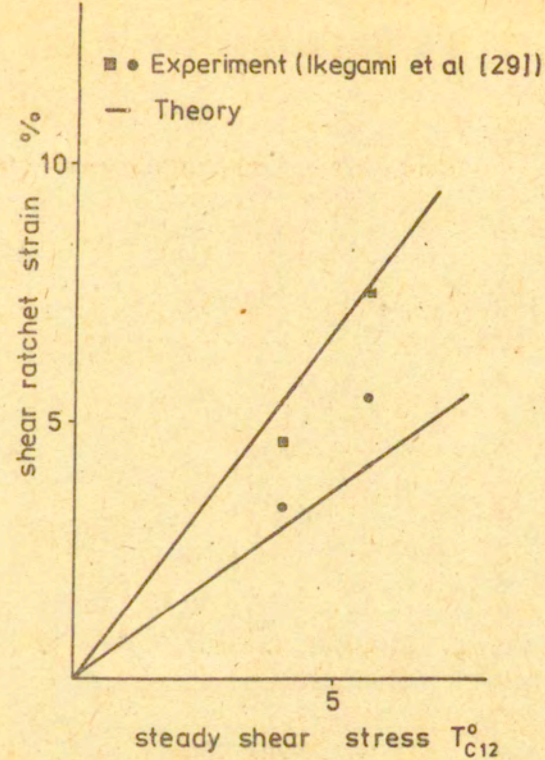


Fig. 4.10 Influence of steady stress

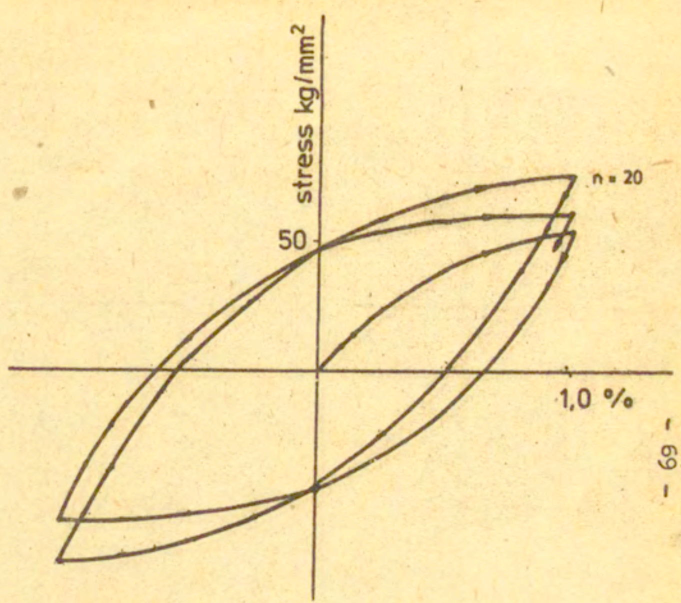


Fig. 4.11 Cyclic axial stress-strain curve

Chapter V Cyclic hardening and cyclic softening

In the following three sections, we describe cyclic hardening and softening /See Fig. 1.3/. In examples described in this chapter, strain is limited to less than 0.007 = 0.7%. So we may equivalently use the second Piola-Kirchhoff stress and the Cauchy stress, and the constitutive equation /7.35/. Moreover the cyclic hardening and softening, as features of stress-strain relations in cyclic processes, are not "second order effects". The evolution equation /8.5/, the time measure /9.1/, and the time scale /9.2/ are used in this Chapter.

15. Description of torsional cyclic hardening

15.1. Explicit constitutive relations

For torsional cyclic straining we apply only shear stress then the stress $\underline{T}$ has the following form:

$$\underline{\tilde{T}} = \begin{bmatrix} 0 & \tilde{T}_{12} & 0 \\ \tilde{T}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

When we apply only the shear stress $\tilde{T}_{12}$ component, we can assume the strain $\underline{E}$ has the following components

$$\underline{E} = \begin{bmatrix} E_1 & E_{12} & 0 \\ E_{12} & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix}, \qquad /15.1/$$

where E_{12} is the shear strain component to be controlled, E_1, E_2 and E_3 are normal strain components.

Corresponding to this $\underline{E}$ we may put $\underline{\hat{q}}$ as

$$\underline{\hat{q}} = \begin{bmatrix} \hat{q}_1 & \hat{q}_{12} & 0 \\ \hat{q}_{12} & \hat{q}_2 & 0 \\ 0 & 0 & \hat{q}_3 \end{bmatrix}. \qquad /15.2/$$

Constitutive equation /7.35/ under this condition may be written in the component form

$$\begin{aligned}
T_{12} &= 2\mu(E_{12} - \hat{q}_{12}) \\
0 &= (\lambda + 2\mu)(E_1 - \hat{q}_1) + \lambda(E_2 - \hat{q}_2) + \lambda(E_3 - \hat{q}_3) \\
0 &= \lambda(E_1 - \hat{q}_1) + (\lambda + 2\mu)(E_2 - \hat{q}_2) + \lambda(E_3 - \hat{q}_3) \\
0 &= \lambda(E_1 - \hat{q}_1) + \lambda(E_2 - \hat{q}_2) + (\lambda + 2\mu)(E_3 - \hat{q}_3).
\end{aligned} \qquad /15.3/$$

The fact that three normal stresses are simultaneously zero leads to a system of equations which results in:

$$E_1 - q_1 = E_2 - q_2 = E_3 - q_3 = 0$$

or $$E_1 = \hat{q}_2 \,, \; E_2 = \hat{q}_2 \,, \; E_3 = \hat{q}_3 . \qquad /15.4/$$

Note that it does not mean that $E_1 = E_2 = E_3 = 0$. But it means that each normal strain component coincides with corresponding each normal component of $\hat{q}$. These relations simplify the constitutive equation as follows

$$\tilde{T}_{12} = 2\mu(E_{12} - \hat{q}_{12}) . \qquad /15.5/$$

This equation should be used together /15.4/.

The evolution equation under cyclic torsion is

$$\begin{aligned}
\frac{d\hat{q}_1}{dz} &= \frac{d\hat{q}_2}{dz} = 2b_o(E_{12} - \hat{q}_{12})\hat{q}_{12} \\
\frac{d\hat{q}_{12}}{dz} &= b_o(E_{12} - \hat{q}_{12})(1 + 2\hat{q}_{12})
\end{aligned} \qquad /15.6/$$

The derivatives $d\hat{q}_{11}/dz, d\hat{q}_{22}/dz$ do not possess linear parts. It means that in the infinitesimal theory $d\hat{q}_{11}/dz = d\hat{q}_{22}/dz = 0$ i.e. $\hat{q}_1 = \hat{q}_2 = 0$. The relations $d\hat{q}_1/dz = d\hat{q}_2/dz$, $d\hat{q}_3/dz = 0$ mean that the tensor $d\hat{q}/dz$ can be expressed by two components in this case. The derivative $d\hat{q}_1/dz$ is given by $/E_{12} - \hat{q}_{12}/\hat{q}_{12}$, it means that $d\hat{q}_1/dz$ is smaller than $d\hat{q}_{12}/dz$ in deformation process, since we apply small cyclic strain /say $E_{12} = 0.005$/ $|q_{12}| < 0.005$. Hence we may, say: the increment of permanent strain in the direction of axis is very small in comparison with $\hat{q}_{12}$. But $/E_{12}-\hat{q}_{12}/\ \hat{q}_{12}$ will result in large deformation after long cyclic deformation process as reported [19, 30].

Let us write down the intrinsic time. The second invariant of $/\underline{E}-\underline{\hat{q}}/$ can be given simply by one component only:

$$\left| II(\underline{E}-\underline{\hat{q}}) \right|^{c_2} = \left| E_{12} - \hat{q}_{12} \right|^{2c_2} . \qquad /15.7/$$

Then,

$$\frac{dz}{d\zeta} = \left| E_{12} - q_{12} \right|^{2c_2} (c_3 + c_4 e^{-c_5 \zeta}) . \qquad /15.8/$$

Noting that $\hat{q}_1 = \hat{q}_2$, $\hat{q}_3 = 0$ and $E_1 = \hat{q}_1$, $E_2 = \hat{q}_2$, and $\underline{E}$ is expressed in terms of E_{12} and $\hat{q}_1$ in the following way

$$\underline{E} = \begin{bmatrix} E_1 & E_{12} & 0 \\ E_{12} & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix} = \begin{bmatrix} q_1 & E_{12} & 0 \\ E_{12} & q_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \qquad /15.9/$$

To obtain the time measure we need the invariants $I_{\dot{\underline{E}}}$, $II_{\dot{\underline{E}}}$, and $III_{\dot{\underline{E}}}$. They are represented as follows:

$$I_{\dot{\underline{E}}} = 2\dot{\hat{q}}_1 \qquad II_{\dot{\underline{E}}} = (\dot{\hat{q}}_1)^2 - (\dot{E}_{12})^2 \qquad III_{\dot{\underline{E}}} = 0 . \qquad /15.10/$$

Then, the time measure is

$$d\zeta = k_1 \left| 2d\hat{q}_1 \right| + k_2 \left| (d\hat{q}_1)^2 - (dE_{12})^2 \right|^{1/2} . \qquad /15.11/$$

15.2. Comparison with experimental data

Using above equations numerical calculations will be executed and compared with experimental data. We compare with Lamba's [33] hysteresis curve of annealed OFHC copper under cyclic shear at engineering strain amplitude $\gamma_a = 2E^a_{12} =$ $= 0.011$.

Material constants are the following:

material	annealed OFHC copper	
Young's modulus E_0	16700 ksi	/15.12/
shear modulus G	5600 ksi	
Poisson's ratio	0.33 .	

Material constants used for numerical calculations are

$$b_o = 14270 \qquad k_1 = k_2 = k_3 = 1 \qquad /15.13/$$

$$c_2 = 0.125 \qquad c_3 = 0.4 \qquad c_4 = 0.6 \qquad c_5 = 20.0 \ .$$

It seems impossible to get any analytical solution thus an incremental method is used for numerical calculation. At first one put $\hat{q}_1 = \hat{q}_2 = \hat{q}_3 = E_{12} = 0$. We apply small increment $dE_{12} = 0.0001$ and put $d\hat{q}_1 = d\hat{q}_2 = d\hat{q}_3 = d\hat{q}_{12} = 0$ to calculate the first $d\zeta$. Next, in eqs. /15.8/ , /15.6/ we put $\hat{q}_1 = \hat{q}_2 = \hat{q}_3 = \hat{q}_{12} = 0$, $E_{12} = dE_{12}$ to calculate the increment dz and the components of increments $d\hat{q}$. The stress $\tilde{T}_{12}$ for dE_{12} is given by /15.5/ when we put $\hat{q} = 0$, $E_{12} = dE_{12}$. For first strain increment dE_{12}, $T_{12} = 2\mu dE_{12}$. The components $d\hat{q}_1$, $d\hat{q}_2$, $d\hat{q}_3$, $d\hat{q}_{12}$ are the initial values for the next step strain increment dE_{12}. For the next increment, we put $E_{12} =$ $2dE_{12}$, $\hat{q}_1 = d\hat{q}_1$, $\hat{q}_2 = d\hat{q}_2$, $\hat{q}_3 = d\hat{q}_3$, $\hat{q}_{12} = d\hat{q}_{12}$, $\zeta = \Delta\zeta$ $z = dz$ and calculated new values of ζ , z and $\tilde{T}_{12}$ in the same way.

Fig. 5.1 shows the experimental results by Lamba but the next Fig. 5.2 shows our theoretical prediction using the

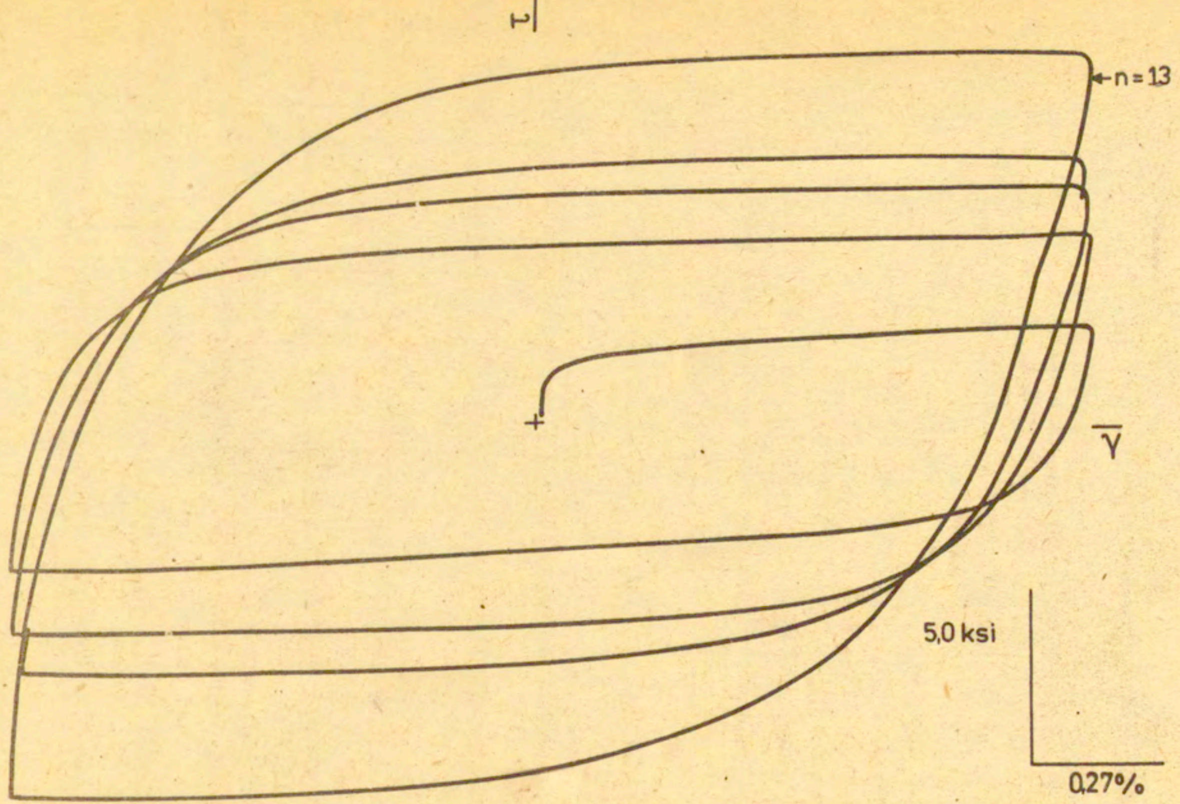


Fig. 5.1 Reording of torsional cyclic hardening by Lamba [33]

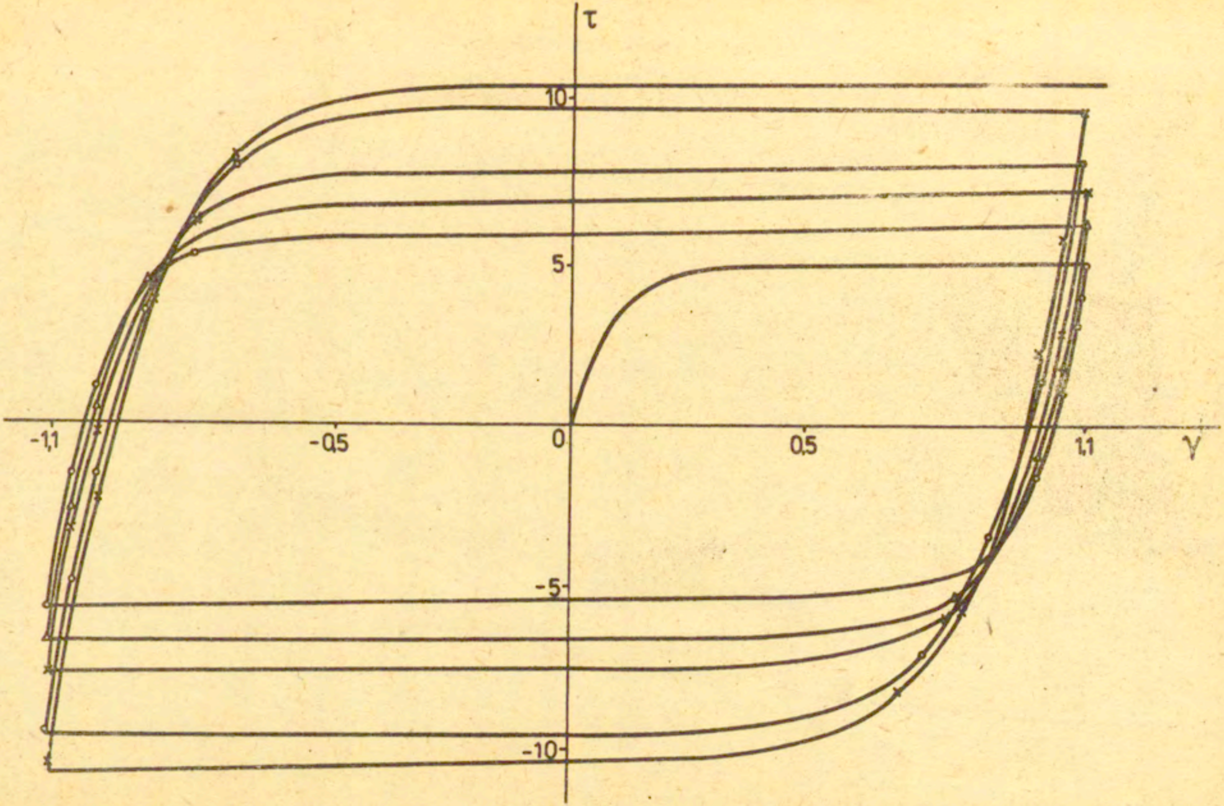


Fig. 5.2 Description of torsional cyclic hardening

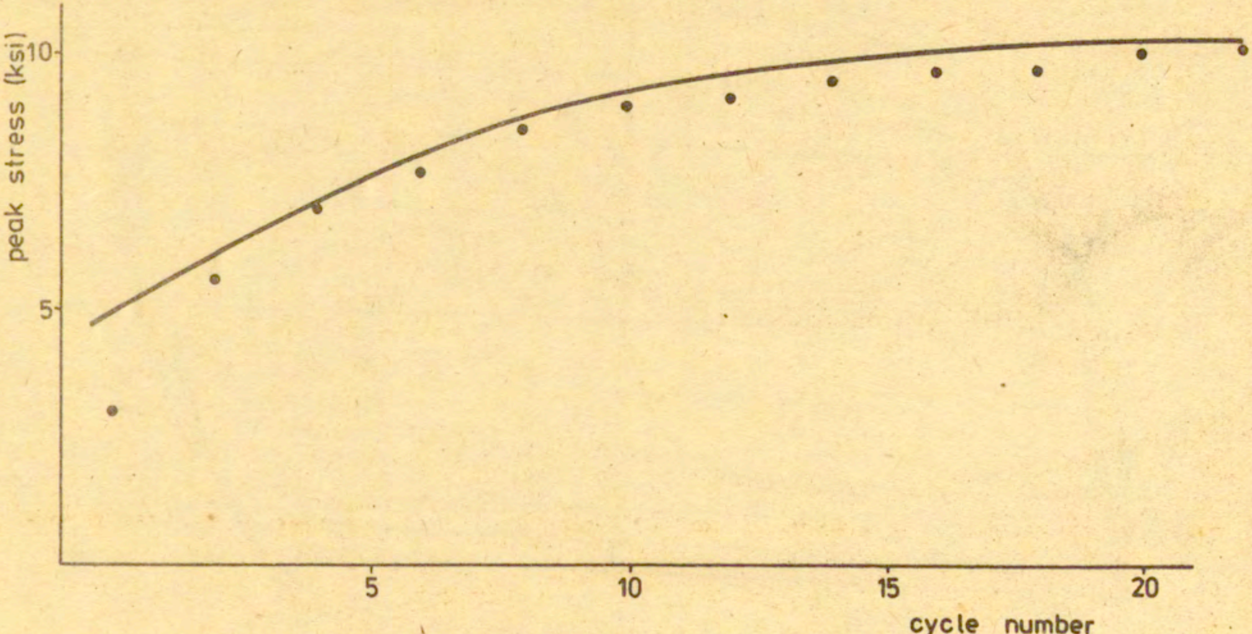


Fig. 5.3 Change of peak stress

material constants /15.12/ and /15.13/. As one can see the numerical calculation gives good agreement with experimental data.

Fig. 5.3 shows the changes of peak stress at shear strain $2E^a_{12} = 0.011$ and comparison with Lamba's data. It can describe the behaviour of cyclic hardening. The experiment reports that the peak stress in a steady state is 10.5 ksi, this calculation shows the same result; around n = 30 the peak stress reaches 10.5 ksi and keeps the same value thereafter.

We may affirm that the ET proposed herein can describe well not only each hysteresis curve at each cycle number in transient stage, but also hysteresis loop in a steady state.

16. Prediction of axial cyclic hardening

16.1. Explicit constitutive relations

Let us predict material behaviour under axial cyclic straining using the same material constants that were used for torsional hardening.

When we apply only axial stress $\tilde{T}_{11}$, the strain $\underline{E}$ has non-zero diagonal components only. Corresponding to $\underline{E}$, the tensor $\hat{\underline{q}}$ has also non-zero diagonal components only

$$\tilde{\underline{T}} = \begin{bmatrix} \tilde{T}_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{E} = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix} \qquad \hat{\underline{q}} = \begin{bmatrix} \hat{q}_1 & 0 & 0 \\ 0 & \hat{q}_2 & 0 \\ 0 & 0 & \hat{q}_3 \end{bmatrix} . \qquad /16.1/$$

Using /16.1/ the constitutive equation written in the component form is

$$\begin{aligned} \tilde{T}_1 &= (\lambda+2\mu)(E_1-\hat{q}_1)+\lambda(E_2-\hat{q}_2)+\lambda(E_3-\hat{q}_3) \\ 0 &= \lambda(E_1-\hat{q}_1)+(\lambda+2\mu)(E_2-\hat{q}_2)+\lambda(E_3-\hat{q}_3) \\ 0 &= \lambda(E_1-\hat{q}_1)+\lambda(E_2-\hat{q}_2)+(\lambda+2\mu)(E_3-\hat{q}_3) . \end{aligned} \qquad /16.2/$$

These relations /16.2/ will result in further considerations. The following relation can be derived

$$E_2-\hat{q}_2=E_3-\hat{q}_3=-\nu(E_1-\hat{q}_1), \tag{16.3}$$

where $\nu=\lambda/2(\lambda+\mu)$ is Poisson's ratio. Hence,

$$\tilde{T}_1=E_o(E_1-\hat{q}_1), \tag{16.4}$$

where $E_o = 2\mu(1+\nu)$ is Young's modulus.

The evolution equation takes the following components form

$$\frac{1}{b_o}\frac{d\hat{q}_1}{dz}=\frac{2(1+\nu)}{3}(E_1-\hat{q}_1)(1+2\hat{q}_1),$$
$$\frac{1}{b_o}\frac{d\hat{q}_2}{dz}=\frac{-(1+\nu)}{3}(E_1-\hat{q}_1)(1+2\hat{q}_2),$$
$$\frac{1}{b_o}\frac{d\hat{q}_3}{dz}=\frac{(1+\nu)}{3}(E_1-\hat{q}_1)(1+2\hat{q}_3). \tag{16.5}$$

From the last two equations and /16.3/ we can conclude that:

$$\hat{q}_2=\hat{q}_3\quad,\quad E_2=E_3=\hat{q}_2-\nu(E_1-\hat{q}_1). \tag{16.6}$$

Together with the relation /16.6/ the equation /16.5/ can be further simplified to:

$$\frac{1}{b_o}\frac{d\hat{q}_1}{dz}=\frac{2(1+\nu)}{3}(E_1-\hat{q}_1)(1+2\hat{q}_1)$$
$$\frac{1}{b}\frac{d\hat{q}_2}{dz}=\frac{-(1+\nu)}{3}(E_1-\hat{q}_1)(1+2\hat{q}_2)\;,\;\hat{q}_2=\hat{q}_3. \tag{16.7}$$

The relation $\hat{q}_2 = \hat{q}_3$ obtained here is quite natural. But we should remember that if a shear stress is superposed on tension-compression, the equation $\hat{q}_2 = \hat{q}_3$ does not hold any more.

We can not say that $d\hat{q}_{22}/dz = d\hat{q}_3/dz = -\frac{1}{2}d\hat{q}_1/dz$ like in the small deformation theory because the evolution equation has nonlinear terms. This is the difference between the nonlinear theory and linear one. Furthermore since

$$tr\left(\frac{d\hat{q}}{dz}\right)=\left(\frac{2(1+\vartheta)}{3}-2\,\frac{(1+\vartheta)}{3}\right)(E_1-\hat{q}_1)+\frac{4(1+\vartheta)}{3}(\hat{q}_1-\hat{q}\;)(E_1-\hat{q}_1)\neq 0 \qquad /16.8/$$

we require $\frac{d}{dz}$ /det/1+2$\underline{q}$/ = 0. The first term of the right-hand side of the equation /16.8/ is a linear infinitesimal term and vanishes. We are sure now that the linear part of our evolution equation satisfies classical infinitesimal plastic incompressibility condition.

For the intrinsic time for this loading condition we can use the same method as in 15.1. The time scale and measure are given in the following forms

$$\left|II_{(\underline{E}-\hat{\underline{q}})}\right|^{c_2}=\left|\vartheta(2-\vartheta)(E_1-\hat{q}_1)^2\right|^{c_2}, \qquad /16.9/$$

$$\frac{dz}{d\zeta}=\left|\vartheta(2-\vartheta)(E-q)^2\right|^{c_2}(c_3+c_4e^{-c_5\zeta}).$$

Using the relations

$$I_{\dot{\underline{E}}}=\dot{E}_1+2\dot{E}_2 \qquad II_{\dot{\underline{E}}}=2\dot{E}_1\dot{E}_2+(\dot{E}_2)^2 \qquad III_{\dot{\underline{E}}}=\dot{E}_1(\dot{E}_2)^2, \qquad /16.10/$$

the time measure is given

$$d\zeta=k_1\left|dE_1+2dE_2\right|+k_2\left|2dE_1\,dE_2+(d\,E_2)^2\right|^{1/2}+k_3\left|dE_1(dE_2)^2\right|^{1/3}. \qquad /16.11/$$

Note that we can express $d\zeta$ by only one total strain increment dE_1, the axial strain which we control /cf./16.6/ /.

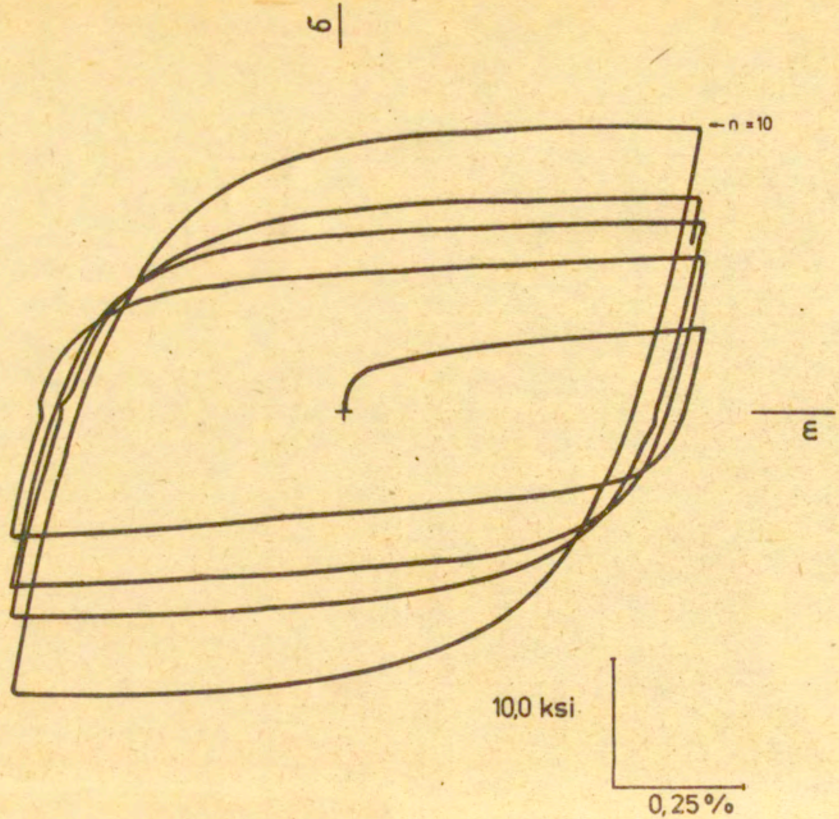


Fig.5.4 Reording of axial cyclic hardening by Lamba [33]

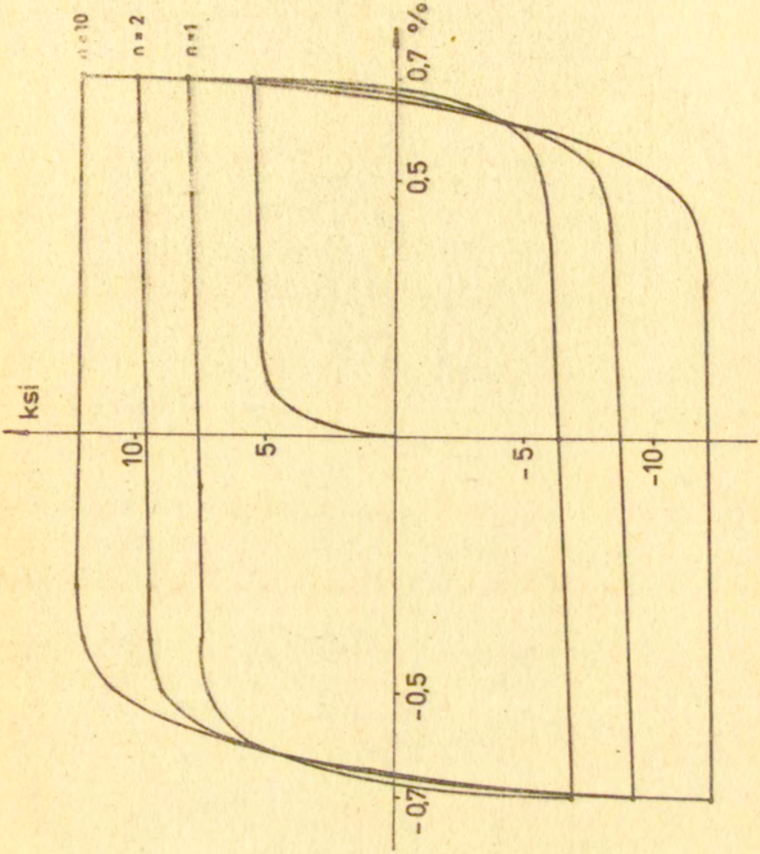


Fig.5.5 Axial cyclic hardening

16.2. Comparison with experimental data

Fig. 5.5 shows the prediction of axial cyclic hardening of annealed copper using material constants /15.13/.

The stress-strain curve were drawn for the same material under cyclic axial strain of amplitude E_1^a = 0.0007. The experimental data for the same condition is shown in Fig. 5.4. Peak stress was calculated up to n = 100. In calculation peak stress does not increase from around n = 6 and a steady state continues up to n = 100. One can see here the basic features of the material behaviour: at first the stress-strain curve is in transient state, i.e. the peak stress and the shape of the curve change quickly, but later they arrive at a steady state in which the peak stress and the loop do not change. In the steady state unfortunately predicted peak stress is much less than the experimental one.

17. Simulation of cyclic softening

In this section we propose some numerical description of cyclic softening behaviour. We do not have at hand any systematic experimental data of cyclic hardening and softening under various loading condition for the same material but in different initial structure /or state/. So, we can make only numerical simulation of cyclic softening of material in imaginary initially fully coldworked state.

For this simulation the constitutive relations, previously used for analysis of cyclic hardening will be applied. For the numerical calculation the same values which appear in /15.12/ and /15.13/ are used with the exception of

$$c_3 = 1.2 \qquad c_4 = 0.2 \qquad c_5 = 20.0 \,. \qquad /17.1/$$

Fig. 5.6 shows the simulation of cyclic softening under cyclic simple shear deformation of strain amplitude of $2E_{12}^a$ = 0.011. One can see from the figure that the theory can describe the changes of peak stress and hysteresis loop in both transient and steady states.

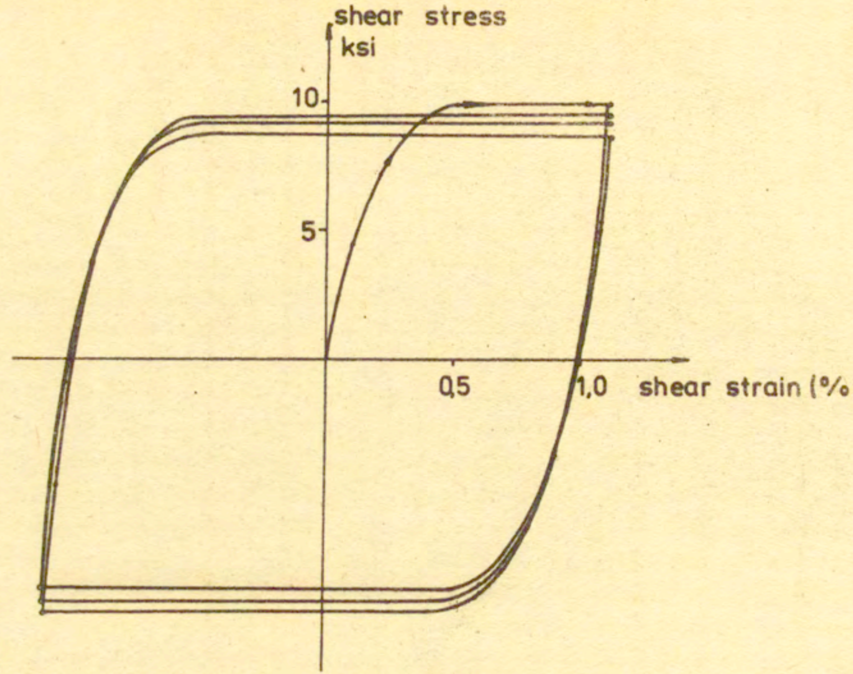


Fig. 5.6 Simulation of torsional cyclic softening

Chapter VI Conclusion

The main objective of the thesis was to construct a nonlinear endochronic theory that can describe the nonlinear dependence of ratchet strain on the steady stress.

To construct a ET consisted of general constitutive relations, fundamental assumptions concerning one internal state variable tensor $\underline{q}$ and the form of the free energy function were made at the beginning. We claim the evolution equation for $\underline{q}$ to satisfy simultaneously two conditions, i.e., at unloaded state the internal variable $\underline{q}$ coincides with the strain $\underline{C}$ and the derivative of $\underline{q}$ with respect to intrinsic time is zero. Moreover, the $\underline{q}$ is unimodular in deformation processes. In Valanis theory, physical meaning of the internal variable $\underline{q}$ is not clearly defined.

For the free energy function, the complete second order approximation was made. The condition of the unloaded state renders strong restriction on the relation between material constants and helps to reduce the number of constants. The constitutive equation finally obtained has the linear part and the nonlinear part as well as coupling terms. Since the linear part is equal to the original Valanis one, we could assert that the present constitutive equation is a generalization of Valanis one.

In the derivation of the evolution equation the incompressibility condition and two restrictions of unloaded states lead to the equation consisted of a linear part and quadratic coupling term with only one material constant.

State variables were introduced in the time scale and a suitable time scale for the problems of cyclic plasticity was proposed. The Valanis time scale has its shortcomings, that is, at large cycle numbers the material response becomes elastic. Our time scale gets rid of it.

Using the same idea and method the dual constitutive relations were constructed for ratcheting analysis, because the resulting strain is important in this analysis. The dual evolution equation has a term which increases linearly depending on the steady stress. The dual constitutive equation has a term which is linear in $\underline{q}$ and has the coupling of the stress and $\underline{q}$. Hence the whole constitutive relations lead to the equation in which the ratchet strain depends on the linear term as in Valanis theory and moreover on nonlinear term in the steady stress.

To show clearly the nonlinear dependence of ratchet strain on the steady stress, two examples were chosen. To describe the steady stress-cyclic shear strain ratcheting behaviour, the values of material constants were found. The nonlinear dependence of ratchet strain on the steady stress was explained for the first time. The explanation by present ET is consistent in the sense that it can describe the related aspects of the phenomenon simultaneously, i.e., /a/ the ratchet strain accumulation process, /b/ the nonlinear dependence on the steady stress, and /c/ the stress-strain relation. The ratcheting of steady shear stress-cyclic axial strain was predicted. The prediction has succeeded to explain the shear ratchet strain accumulation process and the influence of steady shear stress. We may conclude that the proposed ET can consistently explain two different ratcheting problems and related aspects of the phenomenon. We conclude further that the present ET has been verified under multiaxial loading condition. Since ratcheting can easily result in large deformation, finite strain, the Cauchy stress, and the incompressibility condition of type det $/\underline{g}/ = 1$ were used.

The next result obtained is that even the linear part of the constitutive equation can describe cyclic hardening and softening. The description of above cyclic hardening and softening is better than Valanis' one in the sense that it can describe not only the change of the peak stress but also the change of the shape of the hysteresis curve, consistently. The proposed ET has solved two problems i.e. the ratcheting, cyclic hardening and softening.

Owing mainly to the definition of evolution equation it has been proved the existence of elasticity in ET; the response from any unloaded state is linear elastic neglecting the higher order terms. Because of the nonlinear constitutive equation the linear elastic response from any unloaded state depends on the deformation history of the material. It means that the existence of elasto-plastic coupling has been shown. Furthermore even in the linear part of the response from unloaded states, elasto-plastic anisotropy has been shown.

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