

Reliability based optimization of truss structures with members' imperfections

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The paper deals with the effect of dimensional imperfections of truss elements on the optimum design of a structure. It is assumed that each imperfection can't exceed the assumed 'a priori' tolerances of design variables. The incorporation of these tolerances in optimum design is achieved by diminishing the limit values of state variables by the product of given tolerances and appropriate sensitivities. Therefore, the given method allows to introduce deviations of design variables into design in a relatively simple way and ensures safe results. The paper is illustrated with three examples of the truss optimum design, where structural imperfections are considered as variations of members cross-sections. The considered method proved to be an efficient tool for the reliable optimum design.

Key words: truss structures, reliability, optimization, manufacturing tolerances.

One of the most important limits of standard optimum design codes are their limited possibilities of practical implementations. As it was reported by several researches (e.g. by Bauer and Gutkowski in [1]) this fact is caused by a rapid growth of sensitivity of some state functions to variations of design variables while approaching to optimum solution. These variations, which correspond to differences between the real structure and intended optimum design, come from the accuracy of manufacture. In most design problems the mentioned factor can limit the safety of the optimal construction by the violation of the active constraints or a substantial change in structural performance. The mentioned circumstances require an incorporation of the structural imperfections directly into the analysis and reliable optimum search.

The importance of the reliability in design has attracted attention in a number of publications. One of the ideas to deal with the uncertainties in optimization is represented by stochastic approach. A detailed review of the books and papers devoted to this subject is given in [8].

An efficient engineering approach to the reliability design is given by Lee and Park in [6]. The authors define a new cost function, which is a weighted sum of the mean and the standard deviation of the original objective function. Moreover, the constraints are supplemented by adding a penalty term to the original constraints. This method is illustrated by three examples, including an optimization problem of an electric car body.

Another approach to the reliability design is considered in [10]. The idea consists of introducing two objective functions. The first one exhibits high imperfection tolerance and the second one reflects the high buckling load. Applying multiobjective optimization, an attempt is made to build imperfection tolerant structures. The paper is illustrated by the example of performance analysis of thin-walled structures endangered by buckling due to geometric imperfections.

In [4] a review of several further works devoted to the subject is presented. Afterwards, the authors propose their own methodology of searching for a near-optimal design that remains safe, even though the design variables occur in the problem vary due to manufacturing tolerances. This is done by changing the right hand side of inequality constraints and replacing the initial zero value in the original problem by a small positive number called a safety margin. Next the structure is re-optimized and checked, if new design parameters have specified earlier tolerances, while compared to the design variables in the original problem. If not, the safety margin is increased and the redesign procedure is run again. The process is repeated until the assumed tolerances of all design parameters are achieved. In the discussed paper the presented method is applied to the example of optimum design of a hat-stiffened composite panel.

In the present paper, a more formal engineering approach is proposed to optimize structures considering imperfections of the system. The presented method is a further development of an original concept worked out by Gutkowski and Bauer in [2]. In the discussed paper the authors incorporate dimensional imperfections (i.e. manufacturing tolerances) directly into optimum design search. The proposed method is illustrated by an example of truss optimum design with constraints imposed on stresses and displacements.

1. Statement of problem

Consider a truss structure with i_0 nodes and j_0 elements having nominal cross-sections $\mathbf{A} = \{A_1, A_2, \dots, A_j, \dots, A_{j_0}\}^T$ and members lengths $\mathbf{l} = \{l_1, l_2, \dots, l_j, \dots, l_{j_0}\}^T$ respectively. The truss members are manufactured from the same material each, with a given elasticity modulus E . It is

assumed that all actual cross-sections may deviate from their nominal values by an allowable summand t . These variations are represented by a tolerance vector $\mathbf{t} = \{t_1, t_2, \dots, t_j, \dots, t_{j_0}\}^T$. Therefore the actual value of j^{th} member's cross-section stays within a range $\langle A_j - t_j; A_j + t_j \rangle$.

The task of optimization problem is to find the minimum volume of a structure

$$f = \mathbf{A}^T \mathbf{1}, \quad (1.1)$$

with inequality constraints imposed on stresses, displacements and minimum cross-section

$$\sigma^0 - \sigma_j \geq 0 \quad \text{for } j = 1, \dots, j_0, \quad (1.2)$$

$$u^0 - u_i \geq 0 \quad \text{for } i = 1, \dots, i_0, \quad (1.3)$$

$$A_j - A^0 \geq 0 \quad \text{for } j = 1, \dots, j_0. \quad (1.4)$$

2. Structural imperfections in optimum design

The main idea to deal with the stated problem is based on incorporation of structural imperfections into optimization problem inequality constraints, while solving the equilibrium equations for nominal values of design variables.

Following this concept let's consider the variation of the j^{th} design variable A_j . The cross-section imperfection causes the variations of the structural performance (e.g. displacements and stresses). Considering the k^{th} component of the performance vector \mathbf{g} its new value is:

$$g_k := g_k + s_{kj} t_j \quad (2.1)$$

where the right-hand side g_k corresponds to the structure with ideal (nominal) values of design variables and $s_{kj} = \frac{dg_k}{dA_j}$ is the sensitivity of the k^{th} performance with the respect to design variable A_j .

In general case the allowable deviations of design variables are of unknown sign. Moreover, the derived sensitivity may be positive or negative. It means that their product may also be either positive or negative. In order to be sure to stay on a 'safe' side, the absolute value of the $s_{kj} \cdot t_j$ product must be taken into account in the above relation. Having this in mind, the optimization inequality constraints can be formulated as follows:

$$g_k^0 - [g_k + |s_{kj} t_j|] \geq 0 \quad (2.2)$$

or

$$[g_k^0 - |s_{kj} t_j|] - g_k \geq 0, \quad (2.3)$$

where the g_k^0 is a limit value of a constraint imposed on a k^{th} component of structural performance vector.

Extending considerations to the variations of all design variables the relation (2.3) is as follows

$$\left[g_k^0 - \sum_{j=1}^{j_0} |s_{kj} t_j| \right] - g_k \geq 0. \quad (2.4)$$

The above relation states that, while incorporating the elements imperfections into constraints, one has to decrease the limit value of a constraint by an absolute value of a product of allowed tolerances and appropriate sensitivities.

In further calculations, without lost of generality, a linear dependence of tolerances to nominal values of design variables is assumed - i.e. $t_j = \mu A_j$.

3. Solution method

The stated problem belongs to the class of nonlinear programming methods described in many monographs. It can be solved by one of the standard algorithms from the library of nonlinear programming methods, i.e. SLP or NLPQL. The author decided to develop and improve previously worked out algorithm, based on the Kuhn-Tucker necessary conditions. The given problem is solved by successive approximations solution of a set of equations and inequalities arising from the Kuhn-Tucker necessary conditions.

The sensitivities required for the calculations of modified constraints are determined according to the adjoint variable method

$$\mathbf{s}_n = -\Phi^T \frac{d\mathbf{K}}{d\mathbf{A}} \mathbf{u}_n, \quad (3.1)$$

where

$$\Phi = \left[\frac{\partial \mathbf{g}}{\partial \mathbf{u}} \mathbf{K}^{-1} \right]^T. \quad (3.2)$$

In the above relations \mathbf{K} corresponds to the stiffness matrix, Φ corresponds to the adjoint variable and the subscript n denotes the load case ($n = 1, \dots, n_0$).

Having this in mind one can define the Lagrangian for the discussed design problem as follows [5]:

$$\begin{aligned}
 L = & -\mathbf{A}^T \mathbf{l} + \sum_{n=1}^{n_0} \lambda_n^{eT} (\mathbf{K} \mathbf{u}_n - \mathbf{P}_n) + \sum_{n=1}^{n_0} \lambda_n^{tT} \left[\mathbf{s}_n + \Phi^T \left(\frac{\partial \mathbf{K}}{\partial \mathbf{A}} \mathbf{u}_n \right) \right] + \\
 & + \lambda^{aT} \left(\frac{\partial \mathbf{g}^T}{\partial \mathbf{u}} - \mathbf{K} \Phi \right) + \sum_{n=1}^{n_0} \lambda_n^{dT} (\mathbf{u}^0 - \mathbf{u}_n - \mu \cdot \mathbf{s}_n \mathbf{A}) + \\
 & + \sum_{n=1}^{n_0} \lambda_n^{sT} (\sigma^0 - E \mathbf{B} \mathbf{u}_n - \mu \cdot \mathbf{s}_n \mathbf{A}) + \lambda^{cT} (\mathbf{A} - \mathbf{A}^{\min}). \quad (3.3)
 \end{aligned}$$

Applying the Kuhn-Tucker theorem defining necessary conditions for an optimum problem containing equality and inequality constraints we arrive at a system of the following equations and inequalities which have to be fulfilled

$$\frac{\partial L}{\partial \lambda_n^e} = \mathbf{K} \cdot \mathbf{u}_n - \mathbf{P}_n = 0, \quad (3.4)$$

$$\frac{\partial L}{\partial \lambda_n^t} = \mathbf{s}_n + \Phi^T \frac{\partial \mathbf{K}}{\partial \mathbf{A}} \mathbf{u}_n = 0, \quad (3.5)$$

$$\frac{\partial L}{\partial \lambda^a} = \frac{\partial \mathbf{g}^T}{\partial \mathbf{u}} - \mathbf{K} \Phi = 0, \quad (3.6)$$

$$\frac{\partial L}{\partial \Phi} = \lambda_n^{tT} \frac{\partial \mathbf{K}}{\partial \mathbf{A}} \mathbf{u}_n + \lambda^{aT} \mathbf{K} = 0, \quad (3.7)$$

$$\frac{\partial L}{\partial \mathbf{u}_n} = \mathbf{K}(\mathbf{A}, \mathbf{l}) \lambda_n^e - \lambda_n^{dT} - \lambda^{sT} E \mathbf{B} - \lambda^{tT} \Phi_{(k)}^T \frac{\partial \mathbf{K}}{\partial \mathbf{A}} = 0, \quad (3.8)$$

$$\frac{\partial L}{\partial \mathbf{A}} = -\mathbf{l} + \sum_{n=1}^{n_0} \lambda_n^{eT} \frac{\partial \mathbf{K}(\mathbf{A}, \mathbf{l})}{\partial \mathbf{A}} \mathbf{u}_n - \lambda^{aT} \frac{\partial \mathbf{K}(\mathbf{A}, \mathbf{l})}{\partial \mathbf{A}} \Phi + \lambda^{cT} = 0 \quad (3.9)$$

and

$$\sum_{n=1}^{n_0} \lambda_n^{dT} \left[\mathbf{u}^0 - (\mathbf{u}_n + \mu \cdot \mathbf{s}^u \mathbf{A}) \right] = 0, \quad (3.10)$$

$$\sum_{n=1}^{n_0} \lambda_n^{sT} \left[\sigma^0 - (\mathbf{u}_n + \mu \cdot \mathbf{s}^u \mathbf{A}) \right] = 0, \quad (3.11)$$

$$\lambda^{cT} (\mathbf{A} - \mathbf{A}^{\min}) = 0, \quad (3.12)$$

$$\lambda_n^d, \lambda_n^s, \lambda^c \geq 0, \quad (3.13)$$

where the variables $\lambda_n^e, \lambda_n^t, \lambda_n^a, \lambda_n^d, \lambda_n^s, \lambda^c$ are the Lagrange multipliers associated with the equations of equilibrium (3.4), sensitivity (3.5), adjoint (3.6)

and inequalities of displacements (3.10), stresses (3.11) and minimum cross-section (3.12) respectively.

The n equations of equilibrium (3.4) and adjoint equation (3.6) are solved by FEM software (system Algor), while the remaining ones are solved by separate procedures. The reader is referred to the paper [3] for more details about the solution algorithm.

4. Numerical example

To illustrate the effectiveness of the proposed method for reliability based design three example problems dealing with truss structures are presented. These are the typical benchmark-problems, often used for numerical tests by a number of authors.

4.1. Example 1

The first example is a ten bar truss under a single point load presented in Fig. 1 subject to symmetric displacement constraint $u^0 = 2$ in imposed on all nodal displacements. Moreover, a constraint on minimum cross-section $A^{\min} = 0.5 \text{ in}^2$ is given. As it was proved by Svanberg [9] minimizing the weight of a truss subject to symmetric displacement constraints results in a convex optimization problem. Thus the obtained solution is a global optimum. The material properties ($E = 10^7$ psi and $\rho = 0.1 \text{ lbm/in}^3$) correspond to aluminium alloy; the load is set to be $Q_1 = 100000 \text{ lb}$.

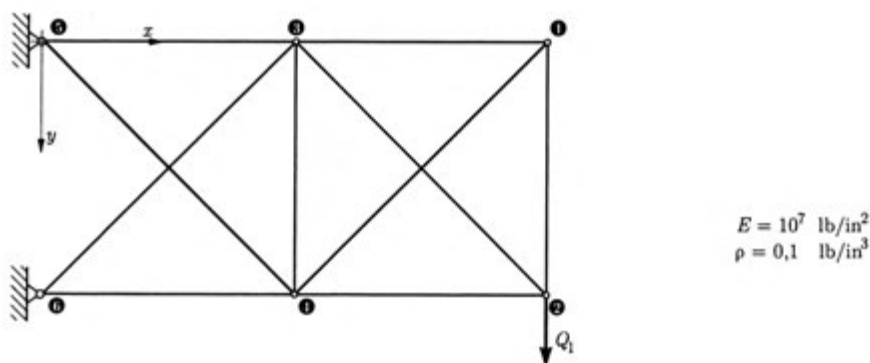


FIGURE 1. 10-bar truss benchmark test – convex optimum search problem.

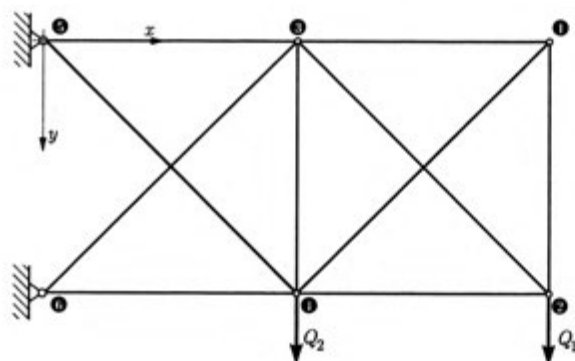
The detailed results for the problem are collected in Table 1.

TABLE 1. Results (A_j) for the example No 1: 10-bar truss convex optimum design.

Element #	μ				
	0.00	0.01	0.02	0.03	0.05
1	28.77	28.87	29.01	29.16	29.44
2	0.10	0.10	0.10	0.10	0.10
3	14.47	14.54	14.61	14.68	14.82
4	14.33	14.41	14.48	14.55	14.69
5	0.10	0.10	0.10	0.10	0.10
6	0.10	0.10	0.10	0.10	0.10
7	0.10	0.10	0.10	0.10	0.10
8	20.27	20.37	20.47	20.58	20.78
9	20.28	20.38	20.48	20.58	20.79
10	0.10	0.10	0.10	0.10	0.10
f [lb]	4156.38	4177.12	4197.86	4218.59	4260.06
Δf [%]	—	+0.50	+0.99	+1.50	+2.50

4.2. Example 2

In the second example the same structure is considered. As distinguished from the previous example the loads are given at two nodes, see Fig. 2. Constraints are imposed on the displacements $u^0 = 2$ in, stresses $\sigma^0 = 25000$ psi and minimum cross-section $A^{\min} = 0.1$ in². Material properties and loads' values stay the same as in the first example.



$$E = 10^7 \text{ lb/in}^2$$

$$\rho = 0.1 \text{ lb/in}^3$$

FIGURE 2. 10-bar truss benchmark test – non-convex optimum search problem.

The changes in the total weights for optimum structures are collected in Table 2.

TABLE 2. Results for the example No 2: 10-bar truss non-convex optimum design.

Total weight	μ				
	0.00	0.01	0.02	0.03	0.05
f [lb]	5060.84	5085.76	5110.68	5135.57	5249.73
Δf [%]	—	+0.49	+0.98	+1.48	+3.73

4.3. Example 3

In the third test a structure with two independent load cases is considered. The discussed cantilever consists of 36 elements, with the loads $Q_1 = Q_2 = 100000$ lb applied horizontally and vertically at the lower node #10 – see Fig. 3. Constraints are imposed on displacements $u^0 = 2$ in and on minimum cross-section area $A^{\min} = 5.0$ in². Material properties stay the same as in the first example. Details concerning coordinates of nodal points and their linking are given in [7].

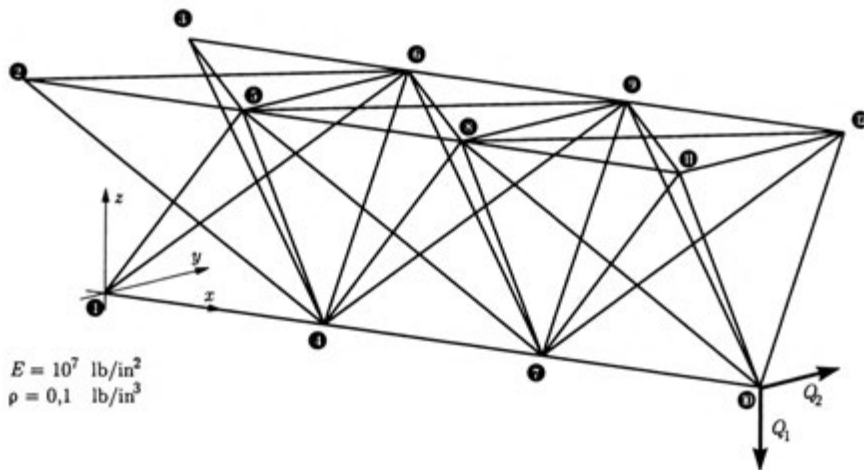


FIGURE 3. 36-bar cantilever with two load cases.

The final results of changes in weights of example 3 optimum structures are collected in Table 3.

In Fig. 4 an increase of structural weight for all the analysed examples is given. The weights of achieved optimum structures' are compared to the weights of systems with linearly increased elements' cross sections by the considered imperfections coefficient (black line).

TABLE 3. Results for the example No 3: 36-bar truss.

Total weight	μ				
	0.00	0.01	0.02	0.03	0.05
f [lb]	32532.97	32679.06	32824.99	32970.93	33263.64
Δf [%]	—	+0.45	+0.90	+1.35	+2.25

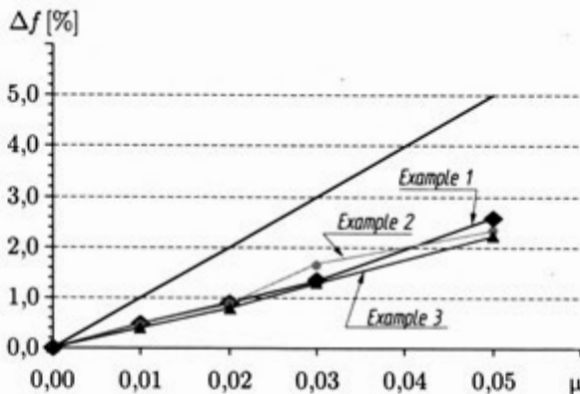


FIGURE 4. Changes in optimal structural weight while compared to linear weight increase due to their tolerances.

5. Conclusions

In the presented paper an efficient approach to reliability based design of structures is presented. The derived analysis and the results of given numerical examples allow for the following conclusions:

- The algorithm presented in this study offers an efficient approach to incorporate structural imperfections into safe optimum design. The systems reliability is achieved by considering the absolute value of the design variables tolerances and appropriate sensitivities. It corresponds to taking into account the 'worst' possible case of imperfections effect.
- As a result of incorporating tolerances in minimum weight design an expected increase of total weight of a structure is observed. The observed phenomenon is not a proportional one. The total weight of an optimal structure with incorporated tolerances is smaller than compared to the weight of a truss with linear increase of elements cross sections due to their tolerances. Moreover, some cross sections may stay unchanged for loosened tolerances.

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