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Reliability-based optimization of truss structures

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The paper studies two alternative formulations of Reliability-Based Optimization (RBO) problem for truss structures subject to various load cases. The considered RBO problem concerns minimization of structural volume under probabilistic reliability constraints. For the reliability analysis, FORM and Ditlevsen bounds methods are employed. The design variables are taken as cross-sectional areas of truss bars. The constraints imposed on reliability indices of failure elements and failure systems are considered. The failure due to various load cases is modeled as a set of limit states (elemental approach) or as a system of limit states (system approach). A numerical example of spatial truss is presented to illustrate two alternative formulations. Results of optimization are compared and conclusions are drawn.

1. Introduction

The assumptions made for design of structures, regarding material properties, loadings, geometry, or mathematical model, are characterized by some uncertainty. Information about the uncertainty can be introduced to the structural model by specifying reliability performance measures. Reliability-Based Optimization can be briefly described as the well-known problem of deterministic optimization, enhanced with reliability performance measures and formulated within probabilistic framework. The RBO problem can be formulated as minimization of the initial cost under constraints imposed on values of reliability indices (see eg. Madsen and Friss Hansen 1992, Kleiber, Siemaszko and Stocki 1999). Reliability indices are functions of failure probabilities for single limit states and/or for systems of limit states. Usually, during a design process structures are studied subject to various load cases. In the framework of reliability analysis, structure performance under various load cases can be assessed in various ways. For instance, several limit states can be specified for one type of failure, which model it subject to various straints can be imposed on the reliability indices corresponding to the same failure type. Alternatively, these limit states can be combined into a series system. Therefore to assess the influence of a few various load cases only one reliability constraint for each failure type must be specified. These two kinds of formulations are not mathematically equivalent, and it can be expected that they affect a solution of the optimization problem differently. The main purpose of this paper is to investigate the influence of these two alternative approaches on the results of the reliability-based optimization problem.

2. The RBO problem formulation considered

There are various formulations of reliability based optimization problem (Madsen and Friss Hansen 1992, Kuschel and Rackwitz 1997). In this paper the problem of minimization of the initial cost of the structure under reliability constraints is considered. It is assumed that elemental and system reliability constraints can be imposed. Deterministic constraints are allowed as well. The design variables are subject to simple bounds, or in case of discrete variables they are assumed to take values from finite sets. Thus, the considered optimization problem can be stated as:

$$\text{minimize}: \quad C_{\mathrm{I}}(\mathbf{x}^c, \mathbf{x}^d), \tag{2.1}$$

$$\text{subject to}: \quad \beta_i\left(\mathbf{x}^c, \mathbf{x}^d\right) \geqslant \beta_i^{\min}, \quad i = 1, \ldots, m_r, \tag{2.2}$$

$$\beta_{\mathrm{sys}_i}\left(\mathbf{x}^c, \mathbf{x}^d\right) \geqslant \beta_{\mathrm{sys}_i}^{\min}, \quad i = 1, \ldots, m_s, \tag{2.3}$$

$$c_i\left(\mathbf{x}^c, \mathbf{x}^d\right) \geqslant 0, \quad i = 1, \ldots, m_d, \tag{2.4}$$

$${}^l x_k^c \leqslant x_k^c \leqslant {}^u x_k^c, \quad k = 1, \ldots, n, \tag{2.5}$$

$$x_k^d \in Z_k = \left\{z_{k,1}, z_{k,2}, \ldots, z_{k,J_k}\right\}, \quad k = 1, \ldots, N, \tag{2.6}$$

where C_I is the initial cost/weight of the structure, $\mathbf{x}^c$ and $\mathbf{x}^d$ are the continuous and discrete design variables, respectively, β_i $(i = 1, \ldots, m_r)$ are the componental reliability indices, β_{sys_i} $(i = 1, \ldots, m_s)$ are the system reliability indices, c_i $(i = 1, \ldots, m_d)$ are deterministic constraints and ${}^l x_k^c$, ${}^u x_k^c$ $(k = 1, \ldots, n)$ are the lower and upper bounds, respectively, imposed on the continuous variables. Each discrete variable, x_k^d $(k = 1, \ldots, N)$ belongs to a discrete set of real numbers Z_k with J_k elements (Stocki, Kolanek, Jendo and Kleiber 2001).

Since the reliability analysis is performed with techniques like FORM (Rackwitz and Fiessler 1978) or SORM (Hohenbichler and Rackwitz 1988), a formulation given above is a so-called two-level optimization problem (Kuschel and Rackwitz 1997). The top level corresponds to minimization of the objective function. The lower level is used for finding the β-point within FORM and SORM methods, which is an optimization problem as well.

3. Analysis of structure failure for various load cases

Most structures are designed with an assumption of being subject to various load cases. In the framework of reliability-based optimization, uncertainty due to various load cases can be taken into account in many ways.

A solution for this problem can be found by the following typical design practice. First, optimal designs for each considered load case should be found. Then, usually taking advantage of engineering experience, obtained results might be compined into a single design with the smallest cost function value, at the same time satisfying constraints for all considered loads. That formulation is rather unsatisfactory. However, compared to the formulations that follow, it allows for conducting several analyses with smaller number of constraints. That, in some cases, makes it the only numerically feasible approach to solve the problem.

The other approach is to perform the optimization process with several constraints specified for each limit state corresponding to all considered load cases. Assuming that there are m limit states and n load cases, $m \times n$ constraints should be defined. Thus the reliability constraints can be expressed by:

$$\beta_i^{(j)}(\mathbf{x}^c, \mathbf{x}^d) \geqslant \beta_i^{(j)^{\min}}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n, \tag{3.1}$$

where i denotes limit states and j denotes load case. Compared to the previous formulation, solution of the problem stated in this way would lead straightforwardly to the optimal structure feasible for all load cases. Moreover, no additional difficulty with codifying human judgment would be necessary. However, for this approach, due to possibly large number of constraints, numerical analysis would be difficult. This approach is called later on an *elemental approach*, because no explicit system analysis is employed in the formulation.

Another possibility to assess the problem of multi-loading is by defining constraints for failure systems. The reliability constraints corresponding to the limit states for the same failure type and different load cases can be combined in series systems. For this formulation, for m limit states and n load cases, the reliability constraints can be stated as follows:

$$\beta_{\mathrm{sys}_i}(\mathbf{x}^c, \mathbf{x}^d) \geqslant \beta_{\mathrm{sys}_i}^{\min}, \quad i = 1, \ldots, m, \tag{3.2}$$

where the system reliability index is defined by:

$$\beta_{\text{sys}_i} = -\Phi^{-1}\left(P\left(\bigcup_{j=1}^{n}\left\{g_i^{(j)}\left(\mathbf{Y},\mathbf{x}^c,\mathbf{x}^d\right)\leqslant 0\right\}\right)\right), \quad i=1,\ldots,m. \qquad (3.3)$$

Here Φ is the normal distribution function, P is the probability and $g_i^{(j)}$ are the limit state functions. For this approach the number of constraints for the top-level optimization remains the same as for single load case analysis. On the other hand, a rather difficult analysis of system reliability is introduced. This formulation is called later on a *system approach*, because system analysis is employed.

4. Numerical example

In the following, a numerical example is presented to illustrate how different approaches to model an uncertainty connected with load cases affect the solution of optimization problem. The analyzed problem means the minimization of initial cost of a structure, subject to various load cases, with reliability constraints corresponding to admissible displacement of a central node and a global loss of stability limit states. The structure is a spatial cylindrical truss. It consists of 474 steel elements connected at 167 nodes. The ends of the

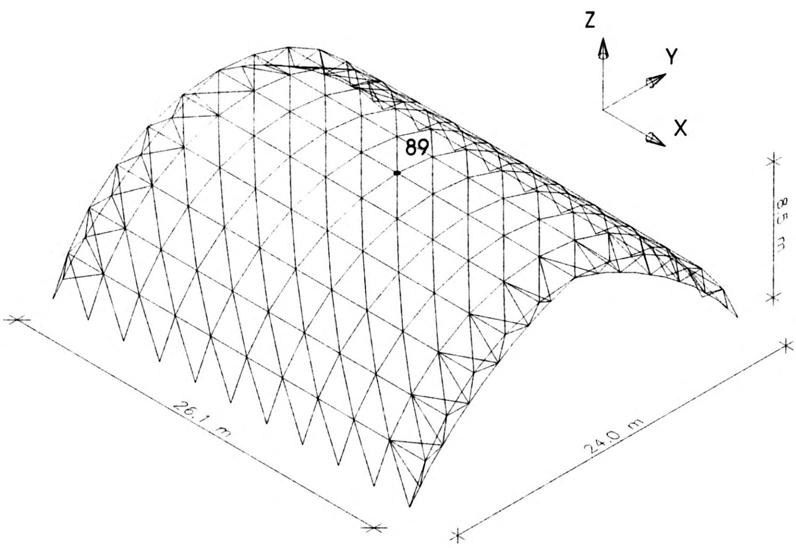

FIGURE 1. 474-element cylindrically shaped shell truss.

structure are reinforced with three hinge arches (Fig. 1). The nodes located at the ground level are fixed. The elements of the structure are divided into 7 groups (Figs. 2 and 3). Design variables are mean values of cross-sectional areas of the elements, one for each group. It is assumed that the considered truss is a support structure for a roof of a warehouse. The structure is analyzed subject to dead weight, snow and wind loadings. Schemes of loadings corresponding to each type of loading are shown in Fig. 4. The linear elastic theory is employed. The stochastic description consists of 447 independent

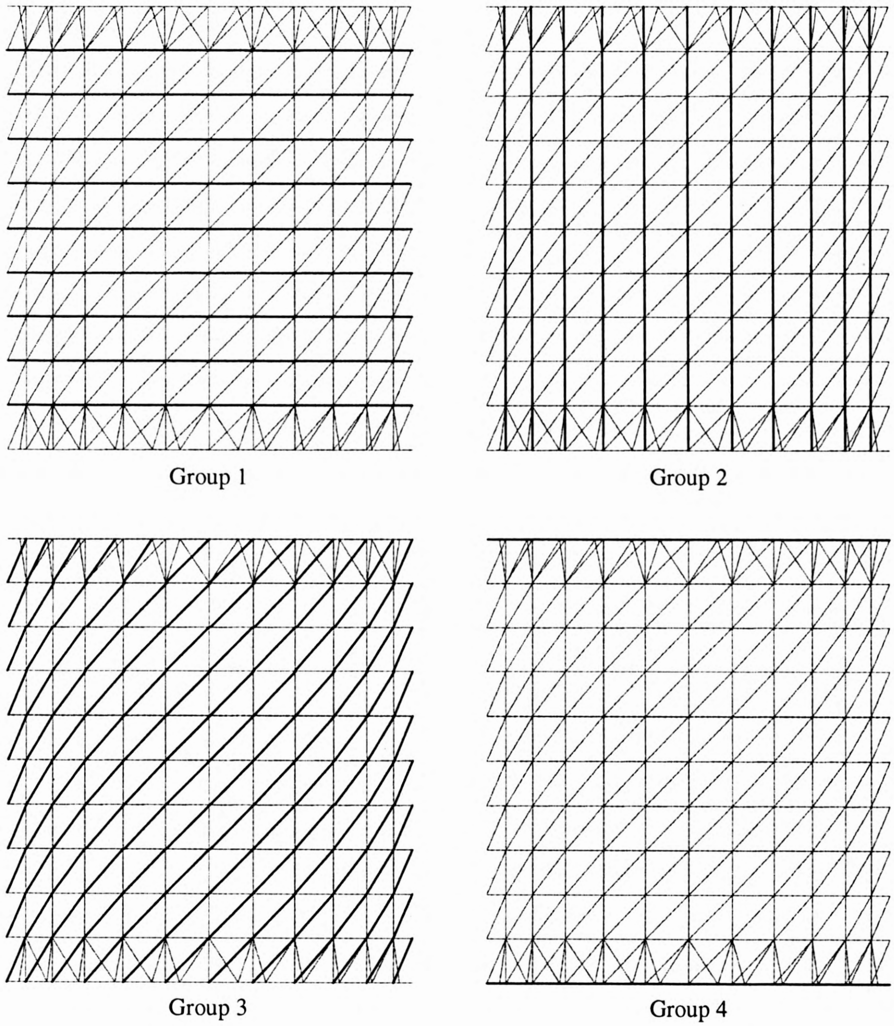

FIGURE 2. Groups of elements 1 – 4.

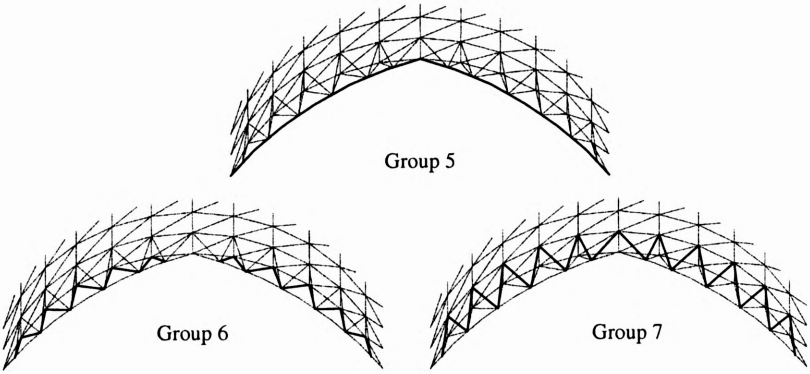

FIGURE 3. Groups of elements 5 – 7.

random variables. They are as follows:

- 7 random variables corresponding to cross-sectional areas of structural elements from each group. The variables have lognormal distribution with standard deviation of 5% of mean value,
- Young modulus of material, lognormally distributed with mean value $2.1 \cdot 10^{12}\,\mathrm{kN/cm^2}$ and standard deviation $5.0 \cdot 10^{10}\,\mathrm{kN/cm^2}$,
- 437 chosen nodal coordinates with normal distribution and the standard deviation of 1.0 cm,
- 3 multipliers of load cases with Gumbel distribution, the mean value 1.0, the standard deviation 0.18.

Two limit states for three load cases are considered. The first limit state is admissible displacement of the central node No. 89. The limit state function is as follows:

$$g_1^{(j)}\left(\mathbf{q}\left(\mathbf{Y},\mathbf{x}\right)\right) = 1 - \frac{\left|q_{89}\left(\mathbf{Y},\mathbf{x}\right)\right|}{q_a}, \quad j = 1,2,3, \tag{4.1}$$

where $\mathbf{Y}$ is the vector of random variables, $\mathbf{x}$ is the vector of design variables, $\mathbf{q}$ is the vector of nodal displacements, q_{89} is the vertical displacement of the central node No. 89 (Fig. 1), q_a is a prescribed allowable displacement taken here as 2.5 cm. The indices j ($j = 1,2,3$) correspond to three considered load cases.

The second limit state is a global loss of stability expressed by the function:

$$g_2^{(j)}\left(\lambda_{cr}\left(\mathbf{Y},\mathbf{x}\right)\right) = \lambda_{cr}\left(\mathbf{Y},\mathbf{x}\right) - 1, \quad j = 1,2,3, \tag{4.2}$$

where λ_{cr} is the critical load factor that leads to the failure due to snap-through effect.

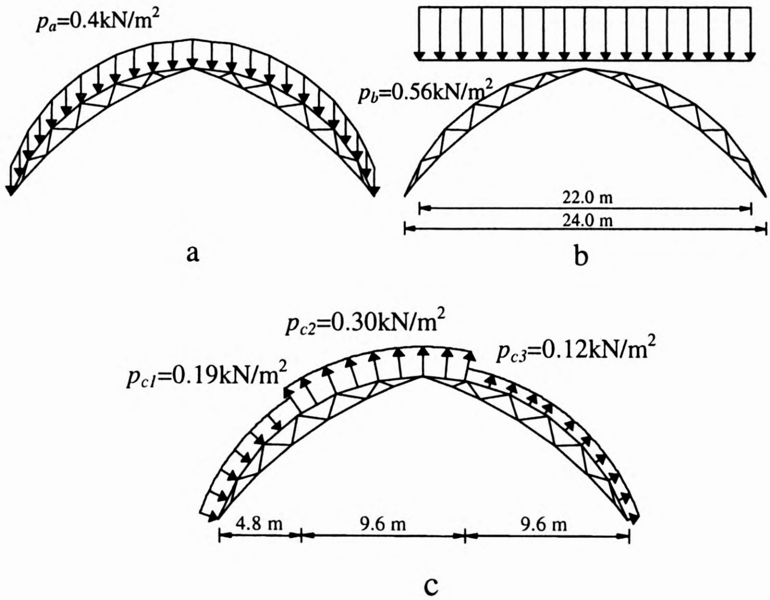

FIGURE 4. Schemes of loadings: a – dead load, b – snow load, c – wind load.

The structure is assumed to be subject to three load cases:

- load case 1 = dead weight + snow load,
- load case 2 = dead weight + wind load,
- load case 3 = dead weight + snow load + wind load.

The optimization problem was solved for two alternative formulations of reliability constraints. For the elemental approach six constraints were imposed on the values of β indices. For displacement constraints the admissible value of β was taken equal to 4.2. Similarly, for constraints corresponding to global loss of stability β was set equal to 3.2. The optimization problem is formulated as follows:

$$\text{minimize}: \quad C_{\mathrm{I}}(\mathbf{x}) = \sum_{i=1}^{7} x_i l_i, \tag{4.3}$$

$$\text{subject to}: \quad \beta_1^{(j)}(\mathbf{x}) \geqslant 4.2, \quad j = 1, 2, 3, \tag{4.4}$$

$$\beta_2^{(j)}(\mathbf{x}) \geqslant 3.2, \quad j = 1, 2, 3, \tag{4.5}$$

$$1.0\,\mathrm{cm}^2 \leqslant x_k \leqslant 300.0\mathrm{cm}^2, \quad k = 1, \ldots, 7, \tag{4.6}$$

where x_i are the design variables – the mean values of the cross-sectional area, l_i is the total length of elements in the group.

The system approach for this example implies an optimization problem with two reliability constraints imposed on the values of β for series systems. The systems consist of three considered limit states defined for the same type of failure and different load cases. Thus the reliability constraints (4.4) are replaced by:

$$\beta_{\text{sys}_1}(\mathbf{x}) \geqslant 4.2, \tag{4.7}$$

as well as constraints (4.5) by:

$$\beta_{\text{sys}_2}(\mathbf{x}) \geqslant 3.2, \tag{4.8}$$

where

$$\beta_{\text{sys}_i} = -\Phi^{-1}\left(P\left(\bigcup_{j=1}^{3}\left\{g_i^{(j)}(\mathbf{Y},\mathbf{x}) \leqslant 0\right\}\right)\right), \quad i = 1,2. \tag{4.9}$$

The top-level optimization, in the presented numerical experiment, was performed with the NLPQL (Schittkowski 1985) algorithm. Using the RBO system POLSAP-RBO (Kleiber, Siemaszko and Stocki 1999) for the elemental approach, the solution was obtained after 37 iterations while for the system approach after 19 iterations. For the reliability analysis the STRUREL – a structural reliability analysis system – was utilized (*STRUREL: Users manual* 1999). The FORM method was employed in both cases. The approximation of failure probability for the series system was made with Ditlevsen bounds (Ditlevsen 1979).

The same starting point for the optimization process was chosen for both formulations. The initial values for all the design variables were equal to $80.0\,\text{cm}^2$ and the structural volume was $1.064\cdot 10^7\,\text{cm}^3$. The solution with the elemental approach, giving the structural volume of $1.723\cdot 10^7$ was smaller by 4.87% compared to the solution with the system approach, giving the volume of $1.811\cdot 10^7$. The values of design variables for the optimal designs are shown in Table 1. For the elemental approach two reliability constraints were active. The first active constraint was the one corresponding to the displacement limit state defined for the first load case ($\beta_1^{(1)}$). The second constraint, which was active, corresponds to the global loss of stability of the structure subject to the third load case ($\beta_2^{(3)}$). For the solution with system approach, the constraint β_{sys_2}, corresponding to the global loss of stability limit state, was active. The other constraint β_{sys_1}, specified for admissible displacement limit state, was satisfied with the margin 0.979. For the solution with the elemental approach an analysis for the system approach was performed yielding the following values of the reliability indices: $\beta_{\text{sys}_1} = 4.2$ and $\beta_{\text{sys}_2} = 3.024$. Thus, the solution with the elemental approach turned out to be infeasible for the system formulation. The example proves that different mathematical models of the same problem may affect the solution of the optimization problem.

TABLE 1. Optimization results.

Design variable	**Imposed reliability constraints**	
	$\beta_{\text{sys}_1}(\mathbf{x}) \geqslant 4.2$ $\beta_{\text{sys}_2}(\mathbf{x}) \geqslant 3.2$	$\beta_1^{(j)}(\mathbf{x}) \geqslant 4.2$ $\beta_2^{(j)}(\mathbf{x}) \geqslant 3.2$ $j = 1, 2, 3$
x_1 [cm^2]	$8.423 \cdot 10^1$	$8.385 \cdot 10^1$
x_2 [cm^2]	$2.231 \cdot 10^2$	$2.109 \cdot 10^2$
x_3 [cm^2]	$1.667 \cdot 10^2$	$1.629 \cdot 10^2$
x_4 [cm^2]	$1.039 \cdot 10^2$	$9.909 \cdot 10^1$
x_5 [cm^2]	$5.660 \cdot 10^1$	$3.088 \cdot 10^1$
x_6 [cm^2]	$2.055 \cdot 10^1$	$1.688 \cdot 10^1$
x_7 [cm^2]	$6.123 \cdot 10^1$	$5.049 \cdot 10^1$
$C_{\text{I}}(\mathbf{x})$ [cm^3] – structural volume	$1.811 \cdot 10^7$	$1.723 \cdot 10^7$

5. Conclusions

In the paper two alternative approaches for modelling uncertainty due to multiple loads were formulated. The first one, the so called elemental approach, is based on idea of defining for one type of failure many reliability constraints corresponding to different load cases. The second one, the so called system approach, combines limit states for one type of failure and different load cases into one failure system with imposed reliability constraint. The Reliability-Based Optimization problem with constraints imposed on elemental and system indices values was formulated. The numerical example shows that proposed formulations for load modelling lead to different results of the optimization process.

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