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Damage identification method based on analysis of perturbation of elastic waves propagation

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New approach to the damage identification problem based on analysis of perturbation of elastic wave propagation is presented. The proposition is based on the use of pre-computed time dependent, dynamic influence matrix describing structural response to locally generated unit impulses. The global structural dynamic response can be decomposed on parts caused by external excitation in undamaged structure and perturbations caused by the structural defects. Assuming possible locations of all potential defects in advance, an optimisation technique with analytically calculated gradients can be applied to solve the problem of the most probable defects' location. Theoretical background as well as numerical results are presented.

Key words: *damage identification, Virtual Distortion Method, inverse dynamic analysis.*

1. Introduction

The damage detection systems based on array of piezoelectric transducers (Fig. 1) sending and receiving strain waves are intensively discussed by researchers recently [1, 2]. The signal-processing problem is the crucial point in this concept and the neural network method is one proposition to develop a numerically efficient solver for this problem [3].

The purpose of this paper is to propose an alternative approach to the inverse dynamic analysis problem. Generalising so called VDM approach (VDM – Virtual Distortion Method [4]) on dynamic problems, a dynamic influence matrix concept will be introduced. Pre-computing of the time dependent influence matrix D allows decomposition of the dynamic structural response on components caused by external excitation in undamaged structure and components describing perturbations caused by the internal defects. In the consequence, an analytical formulae for calculation of these perturbations and the corresponding gradients can be derived (cf. VDM based gradient calculations for non-linear static problem [5]). First formulation of this approach has been presented in [6].

The physical meaning of so-called *virtual distortions* used in this paper are externally induced strains (non-compatible in general, e.g. caused by piezoelectric transducers, similarly to the effect of non-homogeneous heating). The compatible strains and self-equilibrated stresses are structural responses for these distortions.

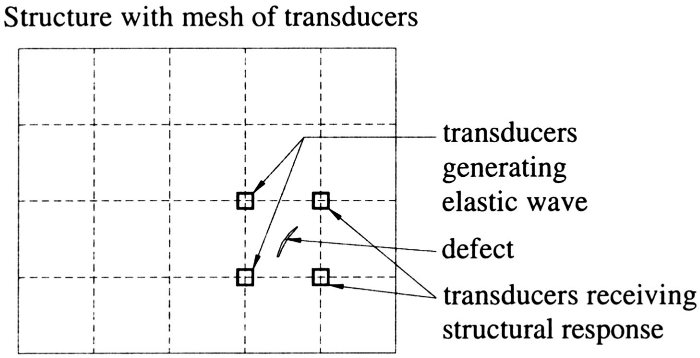


FIGURE 1. Array of piezo-transducers.

2. Virtual Distortion Method (VDM)

All our further proceedings are based on so-called Virtual Distortion Method (VDM). In this section the basic notions of the method as well as the method itself will be explained.

For simplicity of the explanation, let us assume a very simple structure (see Fig.2) consisting only of two bars welded together so that strains induced in one of them cause strains in another. Young's modulus and cross-section area for the left bar are: E_1, A_1^*, and for the right one: E_2, A_2^*. Let us assume that the element number 1 has been initially expanded. Strain ε_1^0 which corresponds to the expansion is called *virtual distortion* and its influence for the structure can be considered as a result similar to the effect of non-homogenous heating or as a lack-of-fit result.

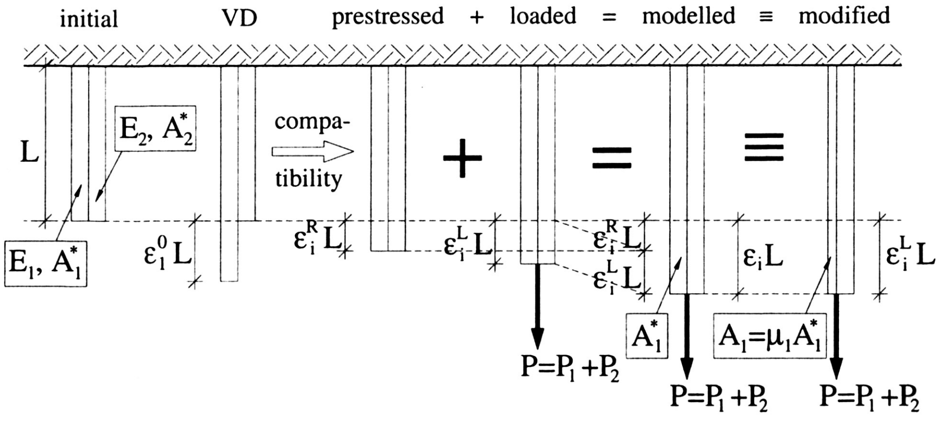


FIGURE 2. Explaining diagram for the Virtual Distortion Method.

Because of compatibility of the displacements we will get a *prestressed structure* with residual strains ε_i^R and associated self-equilibrated stresses σ_i^R $(i = 1, 2)$. These strains and stresses are described by the following equations:

$$\varepsilon_i^R = \sum_j D_{ij}\varepsilon_j^0, \qquad \sigma_i^R = E_i \sum_j (D_{ij} - \delta_{ij})\varepsilon_j^0. \tag{1}$$

Matrix D, used in the above formulae, is so-called *influence matrix*. It is fundamental for all numerical computations in the VDM. The notion of the influence matrix will be explained more thoroughly further, as well as its dynamic generalisation will be introduced. Here suffice it to say that D_{ij} element of the matrix determines strain in member i of the structure caused by unit distortion $\varepsilon_j^0 = 1$ applied in member j.

Now let us have the initial structure subjected to a load (Fig. 2). We consider two states of the structure: *prestressed* and *loaded*. Superposing those two states we obtain so-called structure *modelled by distortion*. Strains and stresses which are present in that structure are defined by the following formulae, where the influence matrix is used:

$$\varepsilon_i = \varepsilon_i^L + \varepsilon_i^R = \varepsilon_i^L + \sum_j D_{ij}\varepsilon_j^0, \tag{2}$$

$$\sigma_i = E_i\left(\varepsilon_i - \varepsilon_i^0\right) = E_i\left(\varepsilon_i^L + \varepsilon_i^R - \varepsilon_i^0\right) = \overbrace{E_i\varepsilon_i^L}^{\sigma_i^L} + \overbrace{E_i \sum_j \left(D_{ij} - \delta_{ij}\right)\varepsilon_j^0}^{\sigma_i^R} = \sigma_i^L + \sigma_i^R. \tag{3}$$

Here ε_i^L, σ_i^L are components of strains and stresses caused by the load, while ε_i^R, σ_i^R are components caused by the distortion.

Independently, we introduce a modification in the left bar of the loaded structure. This modification consists in a change of one of its structural parameters – for example cross-section area or Young's modulus. Now we demand that the *modelled* structure and the *modified* structure are *identical* in the sense of equality of their fields of strains and stresses. This means that introducing the virtual distortion in the left element is equal to the modification of the area of its cross-section. To explain it completely let us consider a particular case when $\varepsilon_1^0 = \varepsilon_1$, which means that marked in Fig. 2 displacement $\varepsilon_1^0 L$ should be equal to displacement $\varepsilon_1 L$ (apparently, Fig. 2 presents the general case, where the displacements are different). Assumed equality means that there are no stresses in the left element (from Eq. (3) we obtain $\sigma_1 = 0$), so it can be considered as non-existent. We may say now that the left bar has vanished and in the modified structure its cross-section area equals zero $(A_1 = 0)$.

We have assumed notation that A_i^* are initial values of the cross-section areas, while A_i are their modified values. Member axial forces adequate for (respectively) modelled and modified structure are as follows:

$$P_i = E_i A_i^*\left(\varepsilon_i - \varepsilon_i^0\right), \qquad P_i = E_i A_i \varepsilon_i. \tag{4}$$

Comparing these two equations we derive a formula for cross-section modification expressed by the virtual distortion:

$$\mu_i \equiv \frac{A_i}{A_i^*} = \frac{\varepsilon_i - \varepsilon_i^0}{\varepsilon_i}. \tag{5}$$

Defined above is so-called *modification parameter* μ_i, which describes here a change of the cross-section area of element i. We may define the parameter differently, e.g. as a modification of Young' modulus: $\mu_i \equiv E_i/E_i^*$, or a modification of member stiffness: $\mu_i \equiv (EA)_i/(EA)_i^*$. In every case the relationship between the modification parameter and the virtual distortion is the same as one presented in Eq. (5). We invert this equation so that now, knowing the modification of a structural parameter, we may calculate the virtual distortion which models its influence in the initial (i.e. unmodified) structure:

$$\varepsilon_i^0 = (1 - \mu_i)\varepsilon_i. \tag{6}$$

Thus in the presented method we have the virtual distortions which we may use to model a change in the structure (among other things: some modifications of its structural parameters).

3. Influence matrix

The influence matrix introduced in the previous section forms numerical basis for the VDM. We have mentioned that component D_{ij} of the matrix determines strain in structural element i caused by unit distortion applied in element j. Thus, the influence matrix is (generally) non-symmetric. For truss structures with n elements, its dimension is $n \times n$.

To compute one column of the influence matrix we must calculate responses in all structural elements caused by a proper local load applied to one of them. And the load (pair of forces in case of truss structures) must correspond to the unit-expansion of the unbounded element (see Fig. 3, assuming that $\varepsilon_3^0 = 1$).

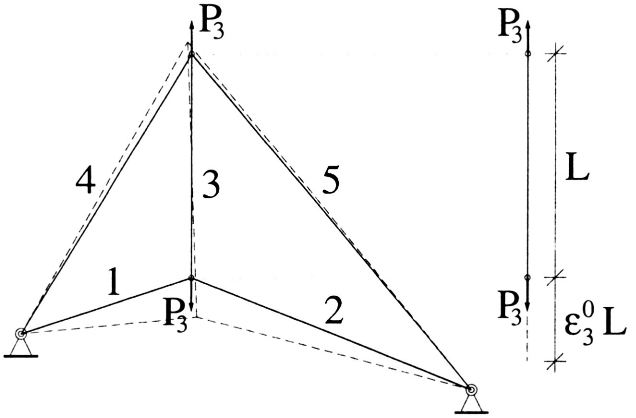


FIGURE 3. Influence of the unit distortion applied in one element.

Below two simple examples of influence matrices for two- and three-element trusses are presented. The examples provide analytical formulae for the influence matrices – obviously for real problems all calculations are to be performed numerically.

3.1. Example of two-bar truss influence matrix

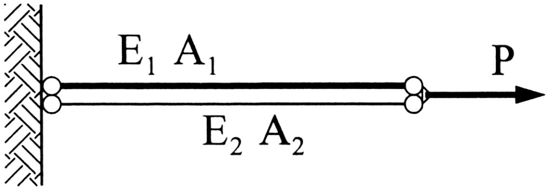


FIGURE 4. Simple two-bar truss.

Figure 4 presents the already-considered truss composed from two elastic bars. Assuming cross-section areas and Young's moduli for the bars as (respectively): A_1, A_2, and E_1, E_2, we may easily determine member axial forces and then strains for the structure loaded with force P:

$$N_1 = \frac{E_1A_1}{E_1A_1 + E_2A_2}P, \qquad N_2 = \frac{E_2A_2}{E_1A_1 + E_2A_2}P, \tag{7}$$

$$\varepsilon_1 = \varepsilon_2 = \frac{P}{E_1A_1 + E_2A_2}. \tag{8}$$

In this simple example applying pair of forces to a member is equivalent with the case, where the load P has got the same value. And the force from the self-equilibrated pair which causes unit expansion of free (i.e. unbounded) element i should be equal E_iA_i. Thus, substituting $P = E_iA_i$, from Eq. (8) we obtain formula for the influence matrix element:

$$D_{ij} = \frac{E_iA_i}{E_1A_1 + E_2A_2}. \tag{9}$$

The whole matrix is as follows:

$$D = \begin{bmatrix} \frac{E_1A_1}{E_1A_1 + E_2A_2} & \frac{E_2A_2}{E_1A_1 + E_2A_2} \\ \frac{E_1A_1}{E_1A_1 + E_2A_2} & \frac{E_2A_2}{E_1A_1 + E_2A_2} \end{bmatrix}. \tag{10}$$

In particular case when $E_1 = E_2$, the influence matrix for that simple two-bar truss equals:

$$D = \begin{bmatrix} \frac{A_1}{A_1 + A_2} & \frac{A_2}{A_1 + A_2} \\ \frac{A_1}{A_1 + A_2} & \frac{A_2}{A_1 + A_2} \end{bmatrix}. \tag{11}$$

3.2. Example of three-bar truss influence matrix

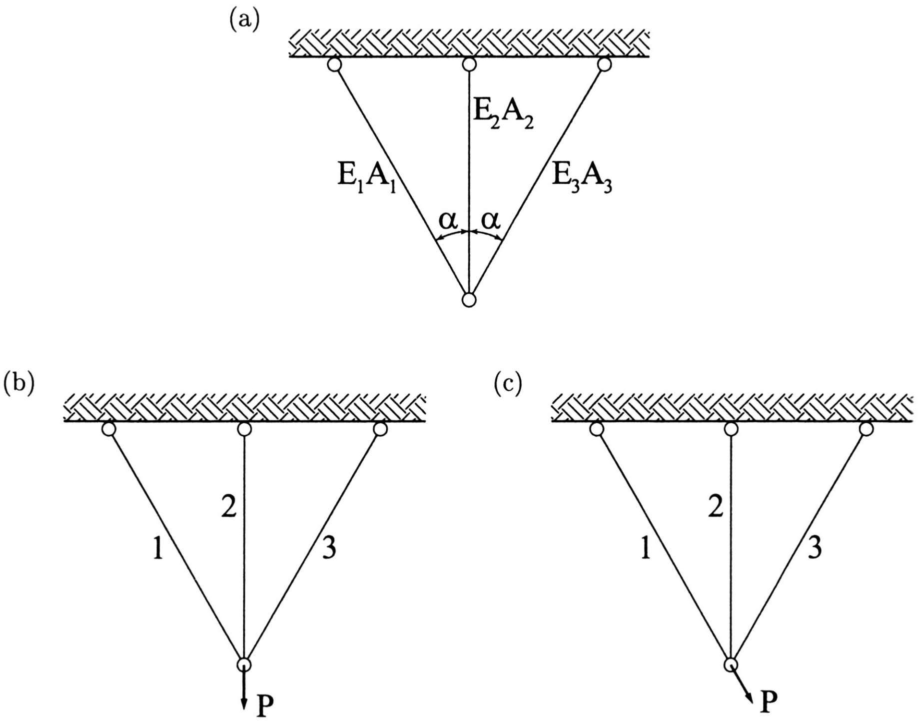


FIGURE 5. Elastic three-bar truss.

Figure 5a presents a truss composed from three elastic bars. Let us assume that structural stiffnesses of all the bars are equal, i.e.:

$$E_1 A_1 = E_2 A_2 = E_3 A_3. \tag{12}$$

For the load P applied as presented in Fig. 5b we achieve the following equations for axial member forces:

$$N_1 = N_3 = \frac{\cos^2(\alpha)}{1 + 2\cos^3(\alpha)} P, \qquad N_2 = \frac{1}{1 + 2\cos^3(\alpha)} P. \tag{13}$$

These formulae will be used to calculate elastic response (i.e. elastic strains) in the structure for the unit distortion applied in element 2.

For the load applied like in Fig. 5c the axial member forces are as follows:

$$N_1 = \frac{1 + 4\cos^3(\alpha)}{2 + 4\cos^3(\alpha)} P, \qquad N_2 = \frac{\cos(\alpha)}{1 + 2\cos^3(\alpha)} P, \qquad N_3 = -\frac{1}{2 + 4\cos^3(\alpha)} P. \tag{14}$$

The analogous equations we achieve for the load P applied in the same node but along the element 3.

A unit distortion for the central bar we realize by applying to its ends pair of axial forces E_2A_2 (which would stretch the free element twice). And this is equivalent with the case when $P = E_2A_2$ in Fig. 5b. Analogously a unit distortion for the left element is realized in the case when $P = E_1A_1$ in Fig. 5c. But because of the structural stiffnesses equality (14), we may assume in every case $P = EA$. The elastic response we achieve calculating elastic strains in every element: $\varepsilon_i = N_i/(E_iA_i) = N_i/(EA)$, $(i = 1, 2, 3)$.

First column components of the structure influence matrix are $D_{1i} = \varepsilon_i$, where ε_i are strains calculated for the unit distortion applied to the element 1, i.e. for the load $P = EA$ applied as presented in Fig. 5c. The central column components are $D_{2i} = \varepsilon_i$, where ε_i are calculated for the unit distortion applied to the element 2, i.e. for the load $P = EA$ applied as presented in Fig. 5b. Thus, using Eqs. (13) and (14) the complete influence matrix for the structure is determined:

$$D = \frac{1}{2 + 4\cos^3(\alpha)} \begin{bmatrix} 1 + 4\cos^3(\alpha) & 2\cos^2(\alpha) & -1 \\ 2\cos(\alpha) & 2 & 2\cos(\alpha) \\ -1 & 2\cos^2(\alpha) & 1 + 4\cos^3(\alpha) \end{bmatrix}. \tag{15}$$

4. Construction of time-dependent influence matrices

From now on, we will be dealing with dynamic analysis, so we need to introduce a time factor into the VDM. Thus we assume that the virtual distortion depends on time (as well as for example the corresponding load which may be used to realize the distortion). This means that the corresponding influence matrix also will be time-dependent, so we can say that it will be three dimensional matrix.

Any form of relationship can be composed from series of short impulses (see Fig. 6). So we calculate influence matrix as a 3-dimensional matrix of dynamic responses obtained for unit impulse excitations applied in time instant $t = 0$ (Fig. 7). It is important to notice that this impulse load we can simulate in initial velocity conditions of Newmark's integration. This means that in dynamic version of the VDM we do not need to use a (time-dependent) external load to realize the effect of a time-dependent distorsion. So to calculate a column of the time-dependent influence matrix we only need to solve a dynamic problem of the structure without any external load but with some proper initial conditions imposed on the adequate nodes. Having calculated all the columns of the influence matrix $D_{ij}(t)$ which gathers the dynamic response for impulse excitations imposed in time instant $t = 0$, it appears that we should compute the following matrices as well for the impulse excitations applied in the successive time instants $t = \tau$ (Fig. 8), but fortunately we do not need to do that, thanks to this obvious relationship:

$$D^{\tau}_{ij}(t) = \begin{cases} 0 & \text{for } t < \tau, \\ D_{ij}(t - \tau) & \text{for } t \geq \tau, \end{cases} \tag{16}$$

where $D_{ij}(t)$ is the already calculated dynamic influence matrix. So for all our further purposes we have only one time-dependent influence matrix computed for unit impulse excitation applied at the beginning of the assumed time period.

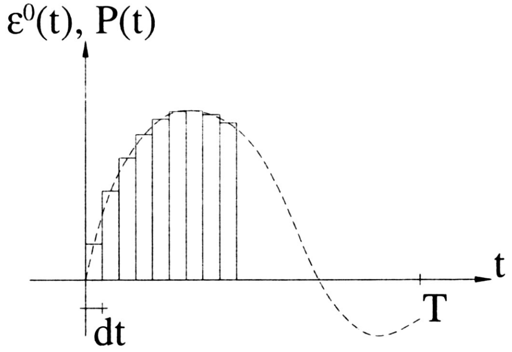


FIGURE 6. Short impulses composing a relationship.

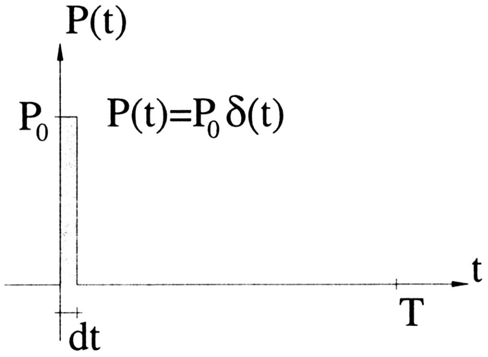


FIGURE 7. Unit impulse excitation applied at the beginning of the assumed time period.

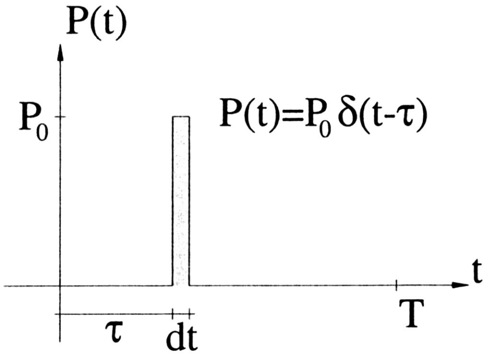


FIGURE 8. Unit impulse excitation applied at time instant τ.

5. Influence matrix based description of wave propagation

Let us describe the dynamic response of the strain increment $\Delta\varepsilon_A(t)$ in the location A and the time instance t as the superimposed response caused by impulses of *virtual distortions increments* $\Delta\varepsilon^0_\alpha(\tau)$ generated in the locations α and the time instances τ (see for example Fig. 9):

$$\Delta\varepsilon_A(t) = \sum_{\tau\leqslant t}\sum_{\alpha} D_{A\alpha}(t-\tau)\,\Delta\varepsilon^0_\alpha(\tau), \tag{17}$$

where the dynamic, time dependent, influence matrix $D_{A\alpha}(t-\tau)$ describes the corresponding dynamic response of the strain in location A and the time instance t, caused by the unit impulse virtual distortions forced in the locations α and time instances $\tau \leqslant t$. Note again that it is sufficient to compute only the matrix $D_{A\alpha}(t)$ which stores the response for the appropriate unit impulse distortion forced in the initial time instant $\tau = 0$. The virtual distortion increments $\Delta\varepsilon^0_\alpha(\tau)$ model excitations caused in locations α by the piezoelectric transducers (activated by an applied current increment). In the paper, we assume that small Greek subscripts (α) runs through all locations of wave-generators while the capital ones (A) runs through locations of wave-receivers.

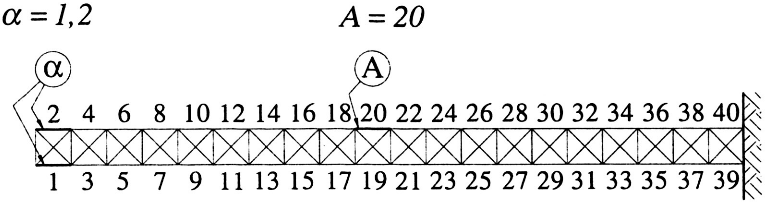


FIGURE 9. Truss-cantilever structure.

The elements of the influence matrix $D_{A\alpha}(t)$ can be determined through the integration of the motion equations (e.g. using the Newmark's method) computed for the unit impulse excitation generated sequentially in the structural elements α. The unit impulse excitation can be supplied in form of initial velocity conditions: $v(0) = P\,\Delta t/m$, where P denotes, so called, compensative force corresponding to locally generated unit virtual distortion impulse $\varepsilon^0 = 1$, Δt is the integration time step, and m is the mass concentrated in the charged node of the loaded structural element α. Assuming (for simplicity of presentation) a discrete truss structure model (Fig. 9), we can describe the transient function for the wave propagation generated in members $\alpha = 1,2$ and received in member $A = 20$. To this end it is necessary to determine, in advance, the time dependent dynamic influence matrix $D_{A\alpha}(t)$, where t runs through all time steps of the dynamic analysis: $t = \langle 0,T\rangle$. Generally, it is a three-dimensional matrix: $D_{A\times\alpha\times t}$, although, in case of only one receiver we can consider it as the two-dimensional $(A\times\alpha\times t \to 1\times 2\times(T+1) \equiv 2\times(T+1))$. Having the influence matrix computed, we can calculate the superposition (17), where $\Delta\varepsilon^0_\alpha(\tau)$ describes (for the sequence of τ instances) the shape of the excited signal. Then, we can achieve the form of the strain in location A and the time period $\langle 0,T\rangle$ by summing the strain increments for all successive time instances $t \in \langle 0,T\rangle$:

$$\varepsilon_A(t) = \sum_{\tau \leqslant t} \Delta\varepsilon_A(\tau) = \varepsilon_A(t-1) + \Delta\varepsilon_A(t). \tag{18}$$

In this way, the storage of the influence matrix $D_{A\alpha}(t)$, allows us to determine the transient function (between locations α and A) for any shape of the excited signal.

6. VDM based damage influence description

Let us follow the influence matrix based approach described above to the damage influence description. Three new, time dependent, influence matrices ($D_{Ai}(t)$, $D_{i\alpha}(t)$, $D_{ij}(t)$) will be introduced. The method of computation of the matrices is similar to the one described in the previous section.

In the case of any perturbation on elastic wave propagation caused by defects in structural members i, between the locations α of the wave generator and the location A of the wave observation, it is necessary to generalise the formula (17) adding the component $\Delta\varepsilon_A^R(t)$ related to the perturbation caused by these defects:

$$\begin{aligned}\Delta\varepsilon_A(t) &= \Delta\varepsilon_A^L(t) + \Delta\varepsilon_A^R(t)\\ &= \sum_{\tau \leqslant t}\left[\sum_{\alpha} D_{A\alpha}(t-\tau)\,\Delta\varepsilon_\alpha^0(\tau) + \sum_i D_{Ai}(t-\tau)\,\Delta\varepsilon_i^0(\tau)\right],\end{aligned} \tag{19}$$

where $\Delta\varepsilon_A^L(t)$ is the part of the strain increment caused by virtual distortion increments $\Delta\varepsilon_\alpha^0(t)$ modelling piezoelectric excitations, whereas $\Delta\varepsilon_A^R(t)$ is the component caused by virtual distortion increments $\Delta\varepsilon_i^0(t)$ simulating defects. From now on, we assume that small Latin indices (i, j, k, l) runs through all presumed locations of possible defects.

The defect-simulating virtual distortion increment can be expressed by the following formula which is dynamic generalization of the static Eq. (5):

$$\Delta\varepsilon_i^0(t) = (1-\mu_i)\,\Delta\varepsilon_i(t). \tag{20}$$

Here $\varepsilon_i(t)$ denotes the strain in member i and the time instance t, while $\mu_i = E_i/E_i'$ denotes the ratio of the damaged member Young's modulus to the initial one. Therefore, the parameter $\mu_i \in \langle 0, 1\rangle$ specifies the size of the defect in location i (actually $\mu_i = 1$ means that there is no damage, while $\mu_i = 0$ means that the member i is completely damaged so that it can sustain no stresses). If we assume several possible-defect locations i (eventually, all structural elements of the structure), we can agree that vector μ_i specifies also the distribution of these defects.

The above relation (20) comes from the more general formula (cf. Eq. (6), see [4]):

$$\mu_i = \frac{E_i}{E_i'} = \frac{\varepsilon_i(t) - \varepsilon_i^0(t)}{\varepsilon_i(t)} = \frac{\Delta\varepsilon_i(t) - \Delta\varepsilon_i^0(t)}{\Delta\varepsilon_i(t)}, \tag{21}$$

which applies virtual distortions to simulate material parameter modifications (or material redistribution $\mu_i = A_i/A_i'$ etc.).

Now, let us substitute the strains $\Delta\varepsilon_i(t)$ in the formula (20) through the formula analogous to equation (19):

$$\begin{aligned}\Delta\varepsilon_i(t) &= \Delta\varepsilon_i^L(t) + \Delta\varepsilon_i^R(t)\\ &= \sum_{\tau\leqslant t}\left[\sum_{\alpha} D_{i\alpha}(t-\tau)\,\Delta\varepsilon_\alpha^0(\tau) + \sum_j D_{ij}(t-\tau)\,\Delta\varepsilon_j^0(\tau)\right]. \qquad (22)\end{aligned}$$

Here, similarly, increment $\Delta\varepsilon_i^L(t)$ is caused by the virtual distortion $\Delta\varepsilon_\alpha^0(t)$, modelling piezoelectric excitations, while increment $\Delta\varepsilon_i^R(t)$ is caused by the defect-simulating virtual distortions $\Delta\varepsilon_j^0(t)$. Now, the following relation between the defect parameters μ_i and the simulating this defect (in the time instance t) virtual distortion increment $\Delta\varepsilon_i^0(t)$ can be reached:

$$\begin{aligned}&\sum_j [\delta_{ij} - (1-\mu_i)\,D_{ij}(0)]\,\Delta\varepsilon_j^0(t)\\ &\qquad = (1-\mu_i)\left[\sum_{\tau\leqslant t}\sum_{\alpha} D_{i\alpha}(t-\tau)\,\Delta\varepsilon_\alpha^0(\tau) + \sum_{\tau<t}\sum_j D_{ij}(t-\tau)\,\Delta\varepsilon_j^0(\tau)\right]. \qquad (23)\end{aligned}$$

Note that to achieve the above expression the following relation has been used:

$$\begin{aligned}\Delta\varepsilon_i^R(t) &= \sum_{\tau\leqslant t}\sum_j D_{ij}(t-\tau)\,\Delta\varepsilon_j^0(\tau)\\ &= \sum_j D_{ij}(0)\,\Delta\varepsilon_j^0(t) + \sum_{\tau<t}\sum_j D_{ij}(t-\tau)\,\Delta\varepsilon_j^0(\tau). \qquad (24)\end{aligned}$$

For the distinguished time instant t, the formula (23) represents a set of i equations with $j = i$ unknowns $\Delta\varepsilon_j^0(t)$. To obtain $\Delta\varepsilon_j^0(t)$ for the entire time period $\langle 0, T\rangle$, we have to solve (step by step) the set (23) for all successive time instances $t \in \langle 0, T\rangle$. However, it is highly important (for the computation cost) to notice that in our algorithm $D_{ij}(0) = 0$. Considering this, the system of equations (23) should be given in the simple diagonal form:

$$\Delta\varepsilon_i^0(t) = (1-\mu_i)\left[\sum_{\tau\leqslant t}\sum_{\alpha} D_{i\alpha}(t-\tau)\,\Delta\varepsilon_\alpha^0(\tau) + \sum_{\tau<t}\sum_j D_{ij}(t-\tau)\,\Delta\varepsilon_j^0(\tau)\right], \qquad (25)$$

which needs only computation of the right hand side expressions.

Knowing the defect parameters μ_i, the step by step (for the sequence of time instances t) determination of the increments $\Delta\varepsilon_i^0(t)$ can be performed making use of the formula (25). Then, knowing $\Delta\varepsilon_i^0(\tau)$ for $\tau \in \langle 0, t\rangle$, the strain increments in the observed location $\Delta\varepsilon_A(t)$ can be calculated making use of the equation (19). Summing these increments, like in the expression (18), we can determine the function of the strains $\varepsilon_A(t)$ in location A and the time period $\langle 0, T\rangle$.

7. Damage identification technique

The inverse problem of damage identification requires determination of the defect size and location (which are specified by the defect vector μ_i), knowing (from measurements) the functions of the strain response $\Delta\varepsilon_A^M(t)$ in locations A to the known excitation $\Delta\varepsilon_\alpha^0(\tau)$ generated in locations α. Therefore, the problem leads actually to the determination of the vector μ_i, where that assumed in advance locations i should allow every significant possibility of defect distribution.

Let us assume for the objective function f the sum of the following measures f_A of the distance between the observed response $\varepsilon_A^M(t)$ in location A and the appropriate possible response $\varepsilon_A(t)$, which depends on the defect-simulating virtual distortions $\Delta\varepsilon_j^0(t,\mu_i)$:

$$f = \sum_A f_A = \sum_A \sum_t \left[d_A(t)\right]^2, \tag{26}$$

where:

$$\begin{aligned} d_A(t) &= \varepsilon_A^M(t) - \varepsilon_A(t) = \varepsilon_A^M(t) - \left[\varepsilon_A^L(t) + \varepsilon_A^R(t)\right] \\ &= \varepsilon_A^M(t) - \sum_t \left[\Delta\varepsilon_A^L(t) + \Delta\varepsilon_A^R(t)\right] = \varepsilon_A^M(t) - \sum_{\tau\leqslant t} \sum_{\tau'\leqslant\tau} \left[\sum_\alpha D_{A\alpha}(\tau-\tau')\,\Delta\varepsilon_\alpha^0(\tau') \right. \\ &\qquad\qquad \left. + \sum_j D_{Aj}(\tau-\tau')\,\Delta\varepsilon_j^0(\tau',\mu_i) \right]. \end{aligned} \tag{27}$$

The most probable defect identification leads to the minimisation problem $\min f$, with respect to the control parameters μ_i. To this end, the gradient approach can be applied, with the following analytical gradient calculated from the formulae (26, 27):

$$\frac{\partial f}{\partial \mu_k} = \sum_A \frac{\partial f_A}{\partial \mu_k} = -2 \sum_A \sum_t d_A(t) \left[\sum_{\tau\leqslant t} \sum_{\tau'\leqslant\tau} \sum_j D_{Aj}(\tau-\tau') \frac{\partial \Delta\varepsilon_j^0(\tau',\mu_i)}{\partial \mu_k} \right], \tag{28}$$

where the partial derivatives $\partial\Delta\varepsilon_j^0/\partial\mu_k$ can be determined from the following systems of equations obtained through differentiation of the formula (23):

$$\begin{aligned} &\sum_j \left[\delta_{ij} - (1-\mu_i)\,D_{ij}(0)\right] \frac{\partial \Delta\varepsilon_j^0(t,\mu_l)}{\partial \mu_k} \\ &\qquad = -\delta_{ik}\,\Delta\varepsilon_i(t) + (1-\mu_i) \sum_{\tau<t} \sum_j D_{ij}(t-\tau) \frac{\partial \Delta\varepsilon_j^0(\tau,\mu_l)}{\partial \mu_k}. \end{aligned} \tag{29}$$

Actually, for the distinguished time instant t, we have got here k sets of equations, where every set consists of i linear equations with j unknowns (and of course $i = j = k$). Finally, taking advantage of $D_{ij}(0) = 0$ (see comment to Eq. (23)) we can simplify the system (29) to the following diagonal form:

$$\frac{\partial \Delta\varepsilon_i^0(t,\mu_l)}{\partial \mu_k} = -\delta_{ik}\,\Delta\varepsilon_i(t) + (1-\mu_i) \sum_{\tau<t} \sum_j D_{ij}(t-\tau) \frac{\partial \Delta\varepsilon_j^0(\tau,\mu_l)}{\partial \mu_k}. \tag{30}$$

The iterative algorithm for the multi-defect identification (cf. Table 1) requires calculation from (25) and (30) the defect-simulating distortion increments $\Delta\varepsilon_i^0(t)$ and their gradients $\partial\Delta\varepsilon_i^0/\partial\mu_j$, for each time step of the dynamic analysis. Making use of these components, the objective function (26, 27) and its gradient (28) can be calculated. Heaving and the gradient of the objective function determined, a modification of the material redistribution can be proposed:

$$\mu_i = \mu_i - \frac{\partial f}{\partial \mu_i}\Delta, \tag{31}$$

where the step length Δ can be adjusted e.g. due to the steepest descent optimisation strategy. Then, the calculation of the objective function and its gradient for the modified structure response can be performed in the next iteration. The cost of the initial computation is related to the determination of the structural dynamic responses for the unit impulses generated in all possible locations of the potential defects (the dynamic, time dependent influence matrix).

8. Numerical algorithm

The presented analysis allowed us to construct a damage identification algorithm based on the VDM. For the numerical efficiency purpose, several modifications to the way of calculation of the formulae described above have been made. The whole algorithm is presented in Table 1. Initial data for the algorithm are:

- distortion function which describes the excitation that causes the vibrations,
- transient function which should be obtained experimentally.

In the starting part of the algorithm influence matrices must be calculated and some initial values for the defect vector μ_i must be assumed. Then, at every step of the algorithm loop, we compute the gradient of the objective function and we use it to modify the defect vector.

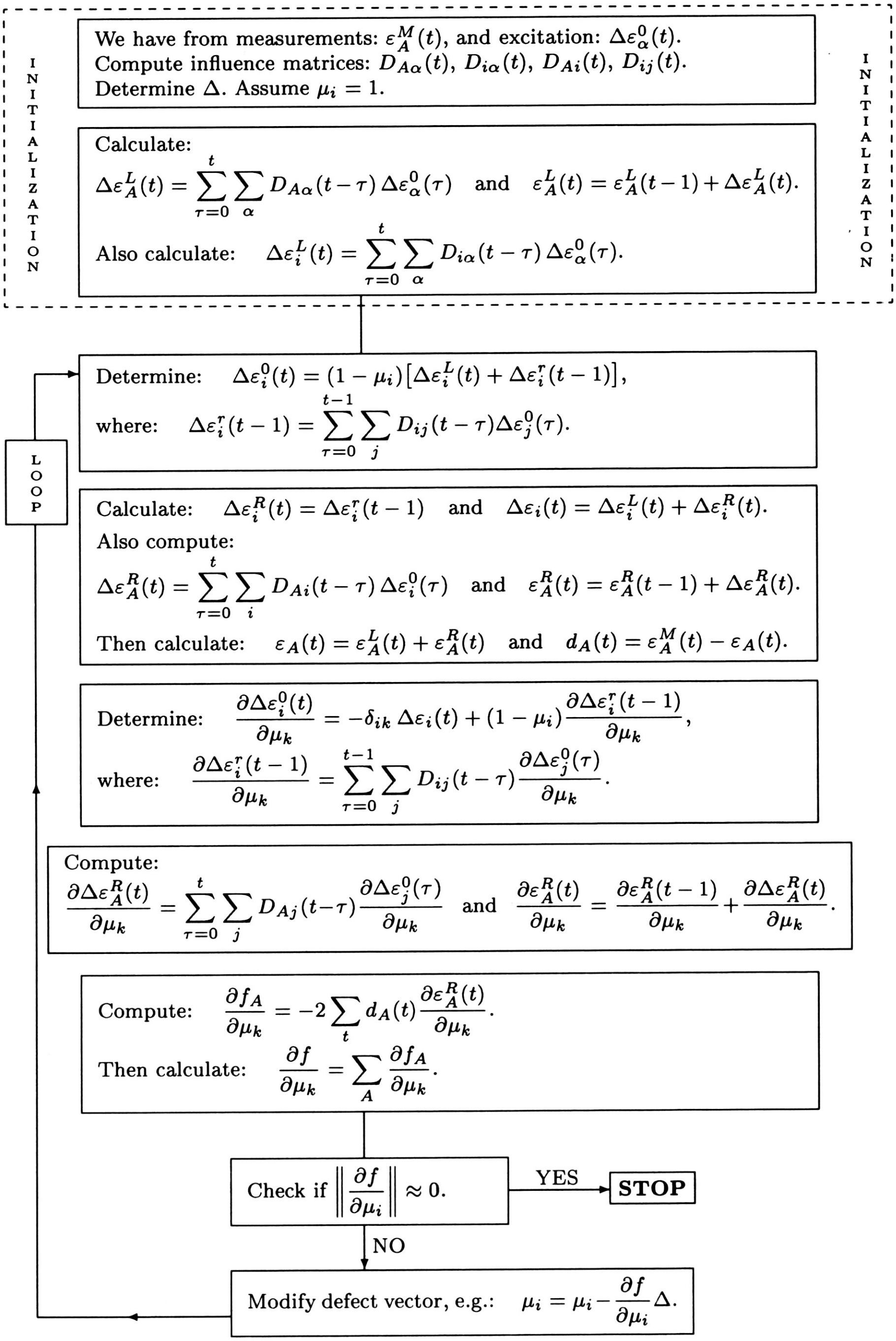
Apart from the initial computations: two Newmark's integration processes for excitations generated in locations α ($D_{A\alpha}(t)$, $D_{i\alpha}(t)$) and j ($D_{Aj}(t)$, $D_{ij}(t)$), the main numerical cost of the proposed damage identification approach is related to the computation of the sensitivity components for each iteration step. It can be checked that this cost is due to accumulation of the $(N+1)^2\, T^2/2$ number of multiplications (calculation of the second components in formulas for the strain increments and derivatives of these increments with respect to the μ_i modifications), where N denotes the number of assumed possible defect locations and T denotes the number of assumed time steps in dynamic analysis.

Estimations of the minimal numbers N and T necessary to reach the desired identification accuracy should be made before further applications of the proposed method.

9. Testing examples

Several simple testing examples have been performed to verify numerical operation of the proposed technique.

TABLE 1. The algorithm for the defect identification.



We have from measurements: $\varepsilon^M_A(t)$, and excitation: $\Delta\varepsilon^0_\alpha(t)$.
Compute influence matrices: $D_{A\alpha}(t)$, $D_{i\alpha}(t)$, $D_{Ai}(t)$, $D_{ij}(t)$.
Determine Δ. Assume $\mu_i = 1$.

Calculate:

$$\Delta\varepsilon^L_A(t) = \sum_{\tau=0}^{t}\sum_{\alpha} D_{A\alpha}(t-\tau)\,\Delta\varepsilon^0_\alpha(\tau) \quad\text{and}\quad \varepsilon^L_A(t) = \varepsilon^L_A(t-1) + \Delta\varepsilon^L_A(t).$$

Also calculate: $$\Delta\varepsilon^L_i(t) = \sum_{\tau=0}^{t}\sum_{\alpha} D_{i\alpha}(t-\tau)\,\Delta\varepsilon^0_\alpha(\tau).$$

Determine: $$\Delta\varepsilon^0_i(t) = (1-\mu_i)\big[\Delta\varepsilon^L_i(t) + \Delta\varepsilon^r_i(t-1)\big],$$

where: $$\Delta\varepsilon^r_i(t-1) = \sum_{\tau=0}^{t-1}\sum_{j} D_{ij}(t-\tau)\Delta\varepsilon^0_j(\tau).$$

Calculate: $\Delta\varepsilon^R_i(t) = \Delta\varepsilon^r_i(t-1)$ and $\Delta\varepsilon_i(t) = \Delta\varepsilon^L_i(t) + \Delta\varepsilon^R_i(t)$.
Also compute:

$$\Delta\varepsilon^R_A(t) = \sum_{\tau=0}^{t}\sum_{i} D_{Ai}(t-\tau)\,\Delta\varepsilon^0_i(\tau) \quad\text{and}\quad \varepsilon^R_A(t) = \varepsilon^R_A(t-1) + \Delta\varepsilon^R_A(t).$$

Then calculate: $\varepsilon_A(t) = \varepsilon^L_A(t) + \varepsilon^R_A(t)$ and $d_A(t) = \varepsilon^M_A(t) - \varepsilon_A(t)$.

Determine: $$\frac{\partial\Delta\varepsilon^0_i(t)}{\partial\mu_k} = -\delta_{ik}\,\Delta\varepsilon_i(t) + (1-\mu_i)\frac{\partial\Delta\varepsilon^r_i(t-1)}{\partial\mu_k},$$

where: $$\frac{\partial\Delta\varepsilon^r_i(t-1)}{\partial\mu_k} = \sum_{\tau=0}^{t-1}\sum_{j} D_{ij}(t-\tau)\frac{\partial\Delta\varepsilon^0_j(\tau)}{\partial\mu_k}.$$

Compute:

$$\frac{\partial\Delta\varepsilon^R_A(t)}{\partial\mu_k} = \sum_{\tau=0}^{t}\sum_{j} D_{Aj}(t-\tau)\frac{\partial\Delta\varepsilon^0_j(\tau)}{\partial\mu_k} \quad\text{and}\quad \frac{\partial\varepsilon^R_A(t)}{\partial\mu_k} = \frac{\partial\varepsilon^R_A(t-1)}{\partial\mu_k} + \frac{\partial\Delta\varepsilon^R_A(t)}{\partial\mu_k}.$$

Compute: $$\frac{\partial f_A}{\partial\mu_k} = -2\sum_{t} d_A(t)\frac{\partial\varepsilon^R_A(t)}{\partial\mu_k}.$$

Then calculate: $$\frac{\partial f}{\partial\mu_k} = \sum_{A}\frac{\partial f_A}{\partial\mu_k}.$$

Check if $\left\|\dfrac{\partial f}{\partial\mu_i}\right\| \approx 0.$

Modify defect vector, e.g.: $\mu_i = \mu_i - \dfrac{\partial f}{\partial\mu_i}\Delta.$

9.1. Basic numerical test

First the proposed approach was tested on a classical problem of forced axial vibrations of an elastic bar with a mass fixed to its end (Fig. 10a). It wasn't supposed to be a damage identification problem and the aim of the test was to compare three results:

- analytical solution,
- solution achieved from Newmark's integration,
- solution obtained using the influence matrix (VDM approach).

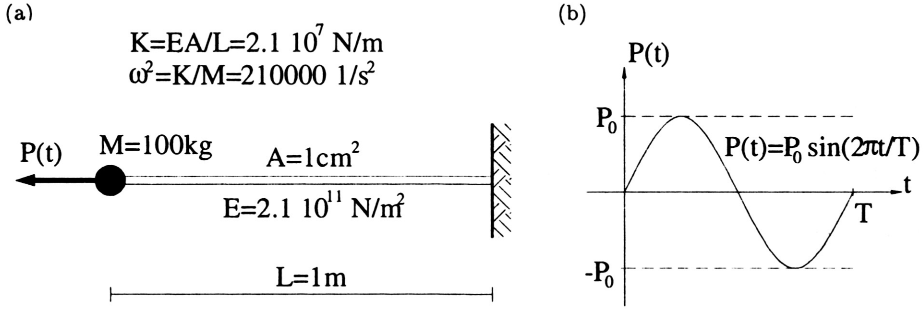


FIGURE 10. Basic test: elastic bar subjected to a sinusoidally time-dependent external load.

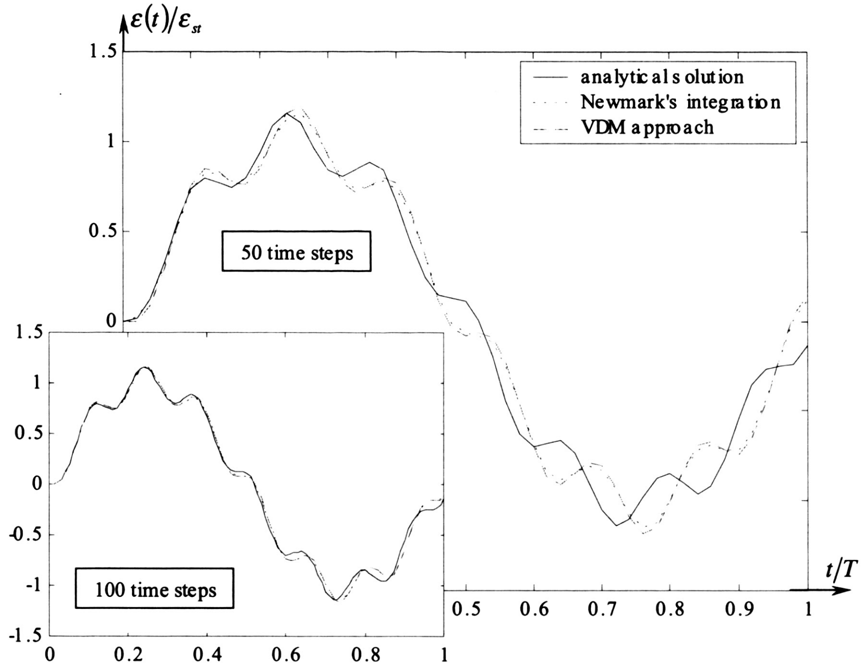


FIGURE 11. Analytical and two numerical solutions of the problem of axial vibrations.

For the simple bar loaded with sinusoidally time-dependent force $P(t)$ (Fig. 10b) the solutions reached through the three mentioned above paths are very close to each other for the case of 100 time steps applied in numerical integrations (Fig. 11). Note that even for 50 time steps the difference between the Newmark's and the VDM approaches is very small and it is remarquable that two numerical solutions are almost identical in comparison with the analytical result.

9.2. Further numerical examples

Finally, two simple numerical examples were performed to test the algorithm. Both the examples are sort of numerical experiments, where the transient function was determined numerically.

A truss cantilever model presented in Fig. 12 was used to verify operation of the proposed VDM based sensitivity analysis and the damage identification technique. It was assumed that the piezoelectric transducers generating sinusoidal shape of excitation had been located in members 1 and 2 (simultaneous extension of the same intensity but the opposite sign are generated, see Fig. 12). The piezoelectric sensor observing wave propagation was located in member 20. Then, two simulation processes of wave propagation were performed. The transient function was determined numerically for two following cases:

first case: there is a defect in member 10. The size of the defect is 40%, which means that the member's stiffness is reduced to 60% of the initial undamaged value.

second case: there are defects of different size in the following elements:

element no.: $i =$	8	9	10	11	12
defect size:	40%	20%	30%	20%	10%
defect parameter: $\mu_i =$	0.6	0.8	0.7	0.8	0.9

In the first example, the initial value for the defect vector was 1. Graph presented in Fig. 13 shows the gradient of the objective function in relationship to the defect

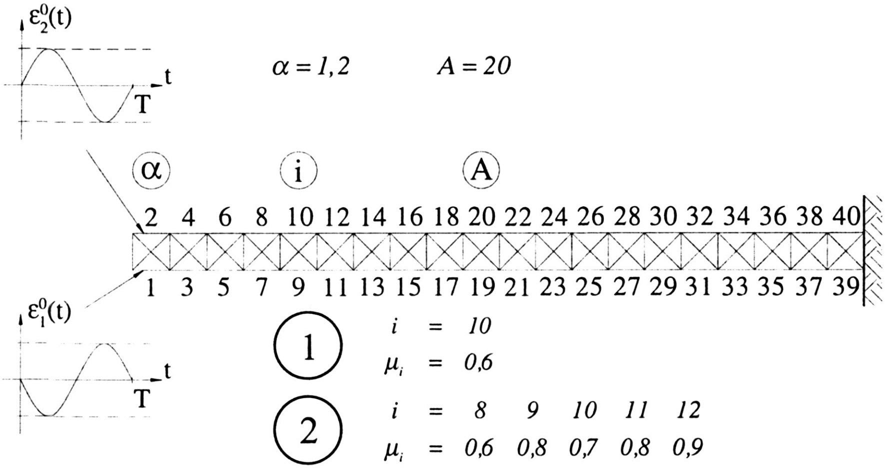


FIGURE 12. Truss-cantilever structure – two cases of defects.

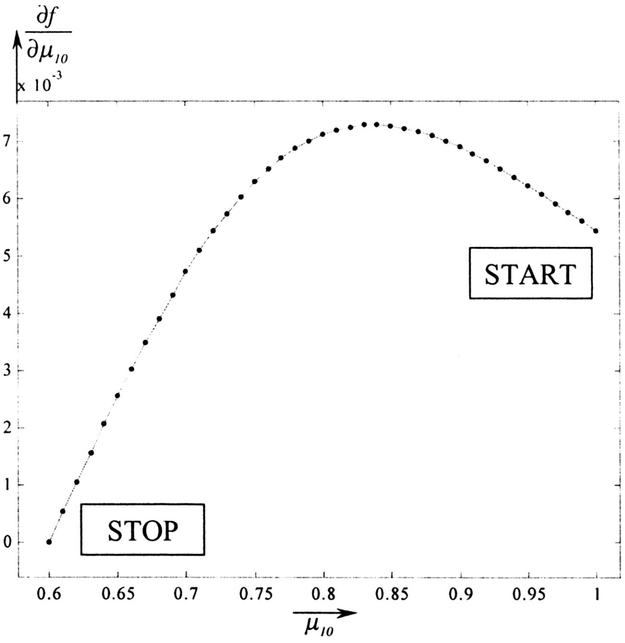


FIGURE 13. Damage identification process – gradients of the objective function.

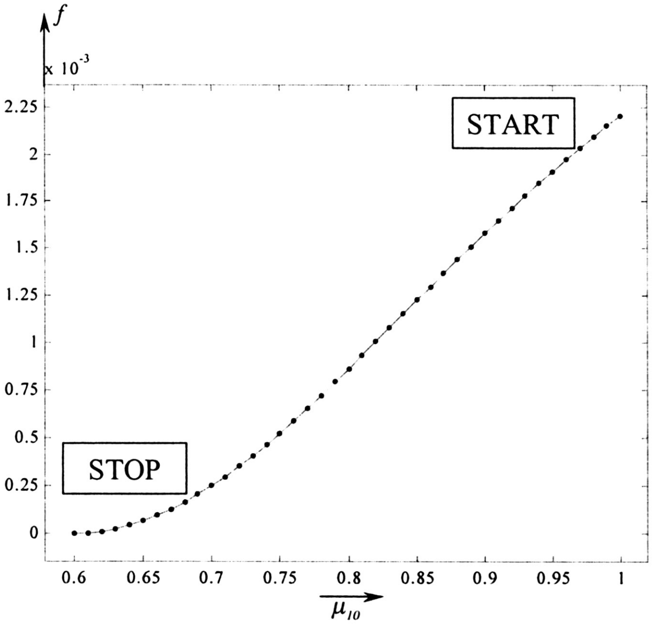


FIGURE 14. Damage identification process – the objective function.

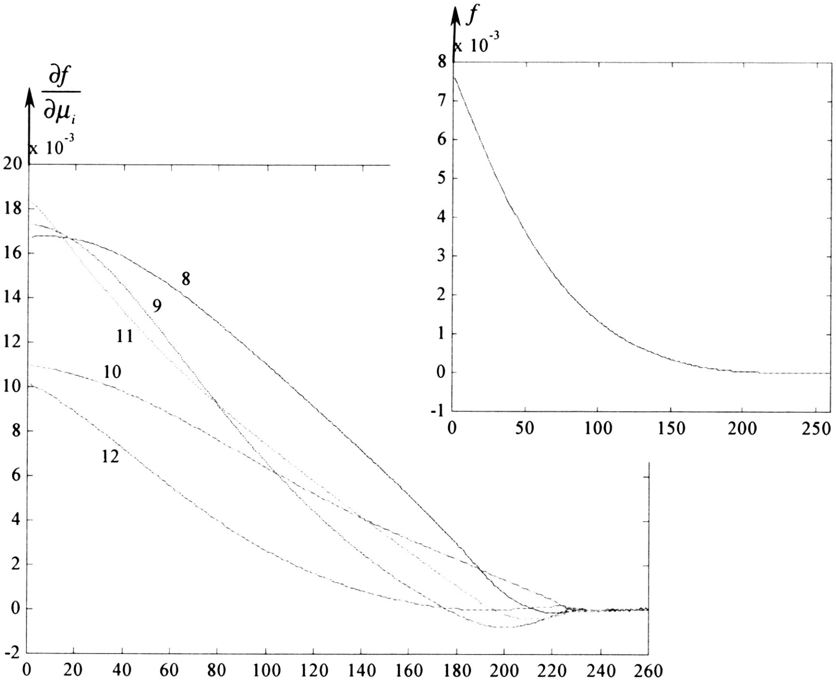


FIGURE 15. Damage identification process – the objective function and its gradients.

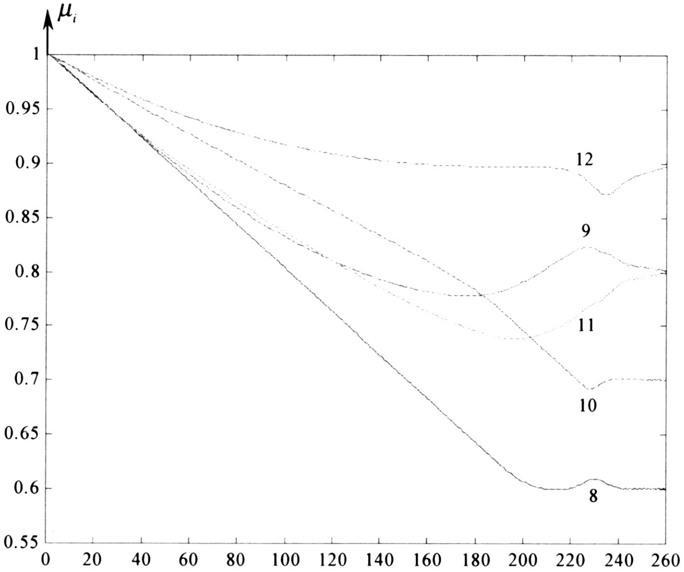


FIGURE 16. Damage identification process – the defect parameters.

parameter. As we can see the gradient is zero, when the defect parameter equals 0,6. Figure 14 shows the objective function – again, its value approaches zero for the valid value $\mu = 0,6$.

In the second example, the initial values for all the components of the defect vector were assumed as 1. Figures 15 and 16 present: the objective function, the components of its gradient and the defect vector components. Again the objective function and its gradient approach zero, while the defect intensities reach the proper values consistent with the values assumed in the numerical experiment. Additionally we present below Fig. 17 showing dynamic responses (transient functions) of the undamaged structure and structure with defects.

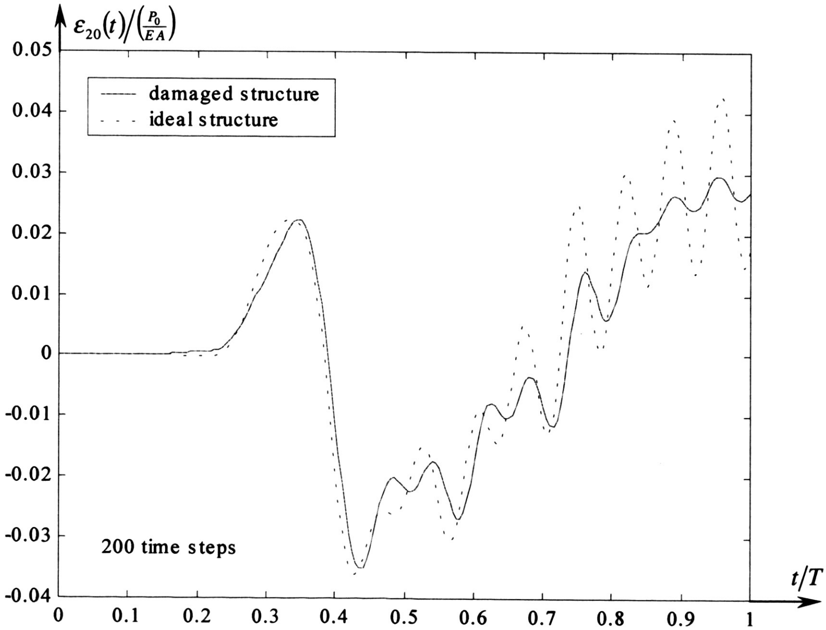


FIGURE 17. Responses of an ideal and damaged structure.

10. Final remarks, conclusions and further objectives

- New approach to the damage identification problem based on analysis of perturbation of elastic wave propagation has been presented.
- The proposition is based on the use of pre-computed time dependent, dynamic influence matrix describing structural response to locally generated unit impulses. The global structural dynamic response can be decomposed on the following two parts: the first one, caused by external excitation in undamaged structure and the second (perturbing) one, caused by the structural defects (modelled through so-called virtual distortions multiplied by the influence matrix).

- This VDM (Virtual Distortion Method) based formulation allows numerically efficient, analytical gradient calculation (with respect to local defect/virtual distortion intensity).
- Assuming possible locations of all potential defects in advance, an optimisation technique with analytically calculated gradients has been applied to solve the problem of the most probable defects' location.
- The proposed numerical tool for the inverse non-linear, dynamic problem analysis has been tested on the truss beam structure, identifying accurately five simultaneous defects with different intensities.

We think the VDM approach has a potential which allows to invent some new techniques for solving some of the classical problems. We are going to develop and improve the proposed method. Our further objectives are:

- Perform more tests using different multidimensional constrained nonlinear minimization procedures (cases with several wave-receivers and assuming more and more structural elements to be damageable).
- Apply the method to plates (rectangular plate elements with in-plane forces).
- Solve the influence matrix size problem.
- Try to modify the proposed method for use in the modal approach (using Virtual Distortions to deal with structure natural frequencies).

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