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Review of control algorithms for active suspension systems

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The purpose of this paper is to provide a review of recent state of the art in the area of active control in the suspension system of vehicles. Included in the discussion are vertical vibration models of vehicles travelling with constant velocity on a randomly profiled road. The analysis is presented for three groups of control approaches, namely active controls for models without and with preview information and adaptive control. In all discussed approaches several modifications of linear quadratic control theory are used.

1. Introduction

During the last two decades in design and production of a new generation of cars the most important problem is to improve the safety and the passenger's comfort. Research and practical applications have been concentrated in four area: anti-lock brake and anti-slip control, four wheel steering systems and advanced suspension systems. The purpose of this paper is to provide a review and an assessment of the recent state of the art in the area of active control in the advanced suspension systems. We limit our consideration to vertical vibrations of ground vehicles. We note that advanced suspensions are basically required to improve the compromise between many conflicting ride and handling measured of vehicle performance, for instance between the passenger's comfort and safety and economics of producing advanced suspensions. Since the competition between producers of cars is very high research on advanced suspensions is growing significantly. This fact has been observed in the open literature and has been reported, for instance in recent survey papers (see [7, 11, 60, 25, 30]). To order systematically all types of suspension systems one required to take into account the following six main groups, each with three properties of considered model.

1. Deterministic, stochastic with complete observations, stochastic with incomplete observations.
2. Linear, bi-linear, nonlinear.
3. Active, semi-active continuous, semi-active switchable.
4. Without preview information, with information about the road profile ahead of the vehicle wheels, with information from the dynamics variables of preceding axles.

5. Unadaptive control, adaptive passive control, adaptive control with the reference model.
6. Slow control, fast control, slow-fast control.

As a separate very recent group one can treat smart and other advanced suspensions with application, for instance, fuzzy-logic or neural-network approaches. Although from the total number of possible groups of models i.e. $3^6 = 729$ only about 100 were considered in the open literature nevertheless this number is still to large for a survey paper with limited volume. These 100 groups of models covers over 1000 papers and therefore we have restricted our reviewing process to the three main groups of control approaches, namely active controls for models without and with preview information and adaptive control, omitting, for instance a wide group of semi-active suspension systems.

2. Active suspensions

The standard passive suspensions contain only springs and dampers, especially shock absorbers. They are simple, reliable and inexpensive. The characteristics of the shock absorbers are generally nonlinear unsymmetric in the jounce and rebound.

Active suspensions contain instead of springs and dampers actuators which act as force producers according to some control law. This fact is very often treated as a criterion of "activity of suspensions". The control law may contain information of any kind obtained from anywhere in the system. According to Elbeheiry et al. [11] usually two main groups of active suspensions are considered in the literature, namely "the fully active suspensions" or "high frequency active suspensions" and "slow active suspensions" or "low frequency active suspensions". The border between low and high frequency was suggested by Millikan [35] as 8 Hz. Taking into account the properties of road irregularities and their measurement one can may find three main group of models: deterministic and stochastic with and without complete observation of road irregularities. We consider below these groups separately.

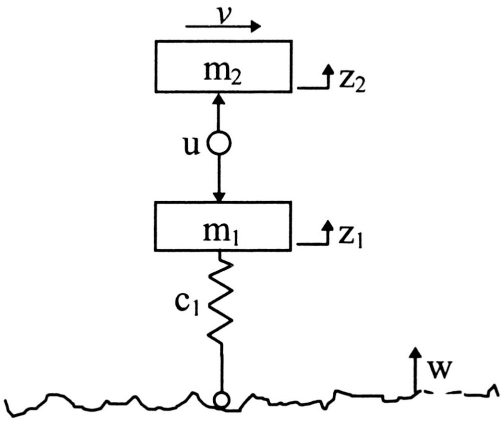

FIGURE 1. Model of suspension system (Eq. (1)).

2.1. Linear deterministic models

The linear quadratic (LQ) optimal control theory was first applied to the determination of active suspension by Thompson [61]. He considered 2-degree-of-freedom (2-DOF) quarter car model (see Fig. 1) described by the following state differential equations:

$$\begin{aligned} \frac{d\mathbf{x}(t)}{dt} &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{G}w(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \end{aligned} \tag{1}$$

where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$, u, w and $\mathbf{y}$ are the state vector, scalar active control, scalar deterministic disturbance and the observation vector, respectively, and

$$\mathbf{x} = \begin{bmatrix} z_1 - y \\ z_2 - y \\ \dot{z}_1 - \dot{y} \\ \dot{z}_2 - \dot{y} \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c_1/m_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ -1/m_1 \\ 1/m_2 \end{bmatrix},$$
$$\mathbf{G} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{C}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2}$$

The control strategy is designed to minimize the criterion

$$I = \frac{1}{2}\int_0^\infty [\rho u^2 + q_1(y - z_1)^2 + q_2(z_1 - z_2)^2]dt = \frac{1}{2}\int_0^\infty [\rho u^2 + \mathbf{x}^T\mathbf{Q}\mathbf{x}]dt, \tag{3}$$

where

$$\mathbf{Q} = \begin{bmatrix} q_1 + q_2 & -q_2 & 0 & 0 \\ -q_2 & q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4}$$

q_1, q_2 and ρ are weight coefficients.

The optimal control has the form

$$u_{\text{opt}}(t) = \mathbf{K}^T\mathbf{x}(t) = -\frac{1}{\rho}\mathbf{B}^T\mathbf{P}\mathbf{x}, \tag{5}$$

where $\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}^T$ and $\mathbf{P}$ is a positive definite solution of the algebraic Riccati equation

$$\mathbf{PA} + \mathbf{A}^T\mathbf{P} - \frac{1}{\rho}\mathbf{PBB}^T\mathbf{P} = \mathbf{0}. \tag{6}$$

The optimal control can be rewritten in the form

$$\begin{aligned} u_{\text{opt}}(t) &= k_1(x_1 - x_2) + k_3(x_3 - x_4) + (k_1 + k_2)x_2 + (k_3 + k_4)x_4 \\ &= k_1(z_1 - z_2) + k_3(\dot{z}_1 - \dot{z}_2) + (k_1 + k_2)(z_2 - y) + (k_3 + k_4)\dot{z}_2. \end{aligned} \tag{7}$$

From equation (7) it follows that the optimal control may be realized by an actuator producing a force

$$F_{\mathrm{opt}}(t) = (k_1 + k_2)(z_2 - y) + (k_3 + k_4)\dot{z}_2 \tag{8}$$

in parallel with a passive spring of rate k_1 and a damper with a damping rate k_3. This model was developed by Thompson and other authors, for instance in [61] Thopson proposed to replace in optimal control law (Eq. 7) the body displacement $x_2 = z_2 - y$ relative to the road by the corresponding displacement $z_2 - z_1$ relative to the wheel. Then a suboptimal control has the form

$$u_{\mathrm{subopt}}(t) = -k_2(z_1 - z_2) + k_3(\dot{z}_1 - \dot{z}_2) + (k_3 + k_4)\dot{z}_2. \tag{9}$$

Comparing coefficients in equations (7) and (9) it is seen that the suboptimal system will be optimal if $k_2 = -k_1$.

In furhter extensions of Thompson model Davis and Thompson [6, 63, 62] considered derivative and integral constraints, respectively. By suitable choice of new variables the authors have to extended linear models with a new quadratic criterion. LQ control theory was also applied to the determination of optimal control for multi-degree-of-freedom (MDOF) models of vehicles, for instance Li and Nagai [32] considered 4-DOF, 6-DOF and 8-DOF models.

We note that also nonlinear deterministic models were studied in the literature [1], [18]. To design nonlinear control Alleyne et al.[1] used feedback linearization approach while Fialho and Balas [18] proposed a novel approach based on linear parameter-varying control techniques.

2.2. Stochastic models with complete observations

2.2.1. Linear models. The linear quadratic Gaussian (LQG) optimal control theory was first applied to the determination of active suspension by Hac [21].

The standard LQG procedure in stationary case has the following form [31].

The dynamic system is described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{G}\boldsymbol{\xi}(t), \tag{10}$$

where $\mathbf{x}(t)$ is the state vector, $\mathbf{u}(t)$ the control vector, $\boldsymbol{\xi}(t)$ the disturbances vector which is assumed to be a zero mean white noise with covariance matrix $\mathbf{Q}_\xi$, $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{u} \in \mathbf{R}^m$, $\boldsymbol{\xi} \in \mathbf{R}^p$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{B}$ and $\mathbf{Q}_\xi$ are constant matrices of appropriate dimensions.

The mean-square criterion is defined by

$$I = \lim_{T\to\infty} E\left\{\frac{1}{T}\int_{t_0}^{T} \left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + 2\mathbf{x}^T(t)\mathbf{N}\mathbf{u}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right] dt\right\}, \tag{11}$$

where $\mathbf{Q}$, $\mathbf{N}$ and $\mathbf{R}$ are matrices of appropriate dimensions, $\mathbf{Q} \geq \mathbf{0}$ and $\mathbf{R} > \mathbf{0}$ are symmetric. It is assumed that the state vector $\mathbf{x}$ complete measurable. Using results of LQG theory the optimal control is determined by

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t), \tag{12}$$

where the control gain matrix $\mathbf{K}$ is constant and given by

$$\mathbf{K} = \mathbf{R}^{-1}(\mathbf{N}^T + \mathbf{B}^T\mathbf{P}), \tag{13}$$

where $\mathbf{P}$ is a symmetric, positive-definite solution of the algebraic Riccati equation

$$\mathbf{P}\mathbf{A}_N + \mathbf{A}_N^T\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q}_N = \mathbf{0}, \tag{14}$$

where

$$\mathbf{A}_N = \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{N}^T \geq \mathbf{0}, \qquad \mathbf{Q}_N = \mathbf{Q} - \mathbf{N}\mathbf{R}^{-1}\mathbf{N}^T \geq \mathbf{0}. \tag{15}$$

The performance index for optimal control is determined by

$$I = \mathrm{tr}(\mathbf{P}\mathbf{G}\mathbf{Q}_\xi\mathbf{G}^T) \tag{16}$$

or alternatively can be calculated from the algebraic Lyapunov equation

$$(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{V} + \mathbf{V}(\mathbf{A} - \mathbf{B}\mathbf{K}) + \mathbf{G}\mathbf{Q}_\xi\mathbf{G}^T = \mathbf{0}, \tag{17}$$

where $\mathbf{V}$ is the covariance matrix of vector state $\mathbf{x}$ i.e. $\mathbf{V}(t) = E[\mathbf{x}(t)\mathbf{x}^T(t)]$. Hac [21] has used this approach to determine an optimal active control in 2-DOF quarter car model (see Fig. 2).

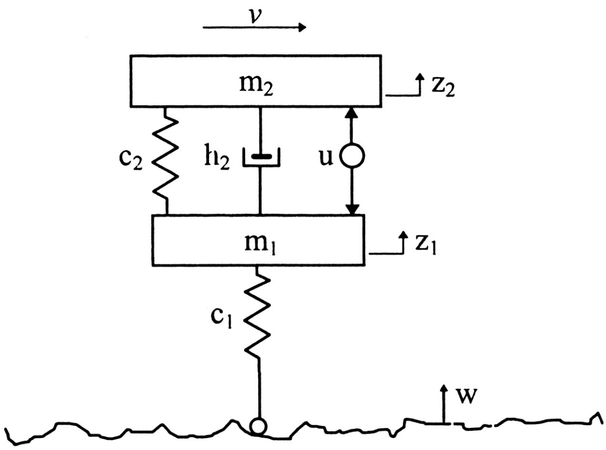

FIGURE 2. Model of linear suspension system (Eq. (18)).

The vector equation of motion is

$$\frac{d\mathbf{z}(t)}{dt} = \mathbf{A}_z\mathbf{z}(t) + \mathbf{B}_z u(t) + \mathbf{G}_z w(t), \tag{18}$$

where $\mathbf{z} = \begin{bmatrix} z_1 & \dot{z}_1 & z_2 & \dot{z}_2 \end{bmatrix}^T$, u and w are the state vector, scalar active control and scalar stochastic disturbance, respectively, whereas

$$
\mathbf{A}_z = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(c_2+c_1)/m_1 & -h_2/m_1 & c_2/m_1 & h_2/m_1 \\ 0 & 0 & 0 & 1 \\ c_2/m_2 & h_2/m_2 & -c_2/m_2 & -h_2/m_2 \end{bmatrix},
$$

$$
\mathbf{B}_z = \begin{bmatrix} 0 \\ -1/m_1 \\ 0 \\ 1/m_2 \end{bmatrix}, \qquad \mathbf{G}_z = \begin{bmatrix} 0 \\ c_1/m_1 \\ 0 \\ 0 \end{bmatrix}, \tag{19}
$$

where c_1, c_2 and h_1, h_2 are constant spring and damper parameters. The stochastic disturbance which described the road irregularities is a stationary coloured noise modelled as an output of first order linear filter with a white noise input process described by

$$
\dot{w}(t) = -avw(t) + \xi(t), \tag{20}
$$

where σ^2, a, v and ξ denote the variance of the road irregularities, a constant parameter describing the road surface, the vehicle velocity and the standard white noise process, respectively.

By introducing the notation $A_w = -av$, $G_w = \sigma\sqrt{2av}$, $Q_\xi = 2\sigma^2 av$ and $G_w = 1$ equations (18) and (20) can be rewritten in a joint vector form (10) where

$$
\mathbf{x} = \begin{bmatrix} \mathbf{z} \\ \eta \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \mathbf{A}_z & G_w \\ \mathbf{0} & A_w \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{B}_z \\ \mathbf{0} \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} \mathbf{0} \\ G_w \end{bmatrix}. \tag{21}
$$

The performance index I is defined by stationary response characteristics of considered suspension system

$$
I = I_1 + \rho_1 I_2 + \rho_2 I_3 + \rho_3 I_4. \tag{22}
$$

where $I_1 = E[\ddot{z}_2]$ represents a measure of ride comfort, $I_2 = E[(z_2 - z_1)^2]$ limits the space required for the suspension, $I_3 = E[(z_1 - w)^2]$ avoids loosing contact between the wheel and the road, $I_4 = E[u^2]$ limits the control force; ρ_i $(i = 1, \ldots, 3)$ are weight coefficients.

This criterion is an extended version of a criterion given for linear deterministic model by Thompson [61]. In new state variables the performance index I has the form

$$
\begin{aligned}
I = E\Bigg[&\left(-\frac{c_2}{m_2}x_1 - \frac{h_2}{m_2}x_2 + \frac{c_2}{m_2}x_3 + \frac{h_2}{m_2}x_4 + \frac{1}{m_2}u \right)^2 \\
&+ \rho_1(x_1 - x_3)^2 + \rho_2(x_1 - x_5)^2 + \rho_3 u^2 \Bigg] = E\big[\mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{x}^T\mathbf{N}\mathbf{u} + \mathbf{u}^T\mathbf{R}\mathbf{u}\big].
\end{aligned} \tag{23}
$$

Equation (23) defines matrices $\mathbf{Q}$, $\mathbf{N}$ and $\mathbf{R}$.

Another 2-DOF linear vehicle model with random speed v was considered by Elbeheiry [13]. To determine the quasi-optimal control he used a perturbation approach.

2.2.2. Nonlinear models. An application of LQG theory to the determination of quasi-optimal control for nonlinear models of active suspension systems was presented, for instance, by Gordon, Marsh and Milsted [20], Narayanan and Raju [41, 45, 40], Socha [55, 56]. Mainly the authors proposed iterative procedures where LQG technique and a linearization method were used. To present this approach we follow the results presented by Socha [55, 56]. We consider the linear 2-DOF vehicle model described in previous section with one nonlinear suspension spring between masses m_1 and m_2 (see Fig. 3).

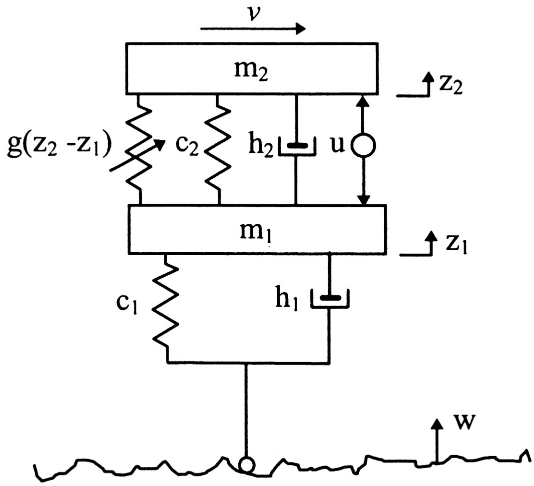

FIGURE 3. Model of nonlinear suspension system (Eq. (24)).

The equation of motion for new state variables are described by the following Ito equations

$$
\begin{aligned}
dx_1 &= x_3 dt,\\
dx_2 &= x_4 dt,\\
dx_3 &= \left[-\frac{c_1}{m_1}x_1 - \frac{h_1}{m_1}x_3 + \frac{c_2}{m_1}x_2 + \frac{h_2}{m_1}x_4 + \frac{1}{m_1}\phi(x_2)\right.\\
&\qquad \left. - \frac{1}{m_1}u + a_1 a_2 x_5 + (a_1 + a_2)x_6\right] dt - q d\xi,\\
dx_4 &= \left[\frac{c_1}{m_1}x_1 - \frac{m_1 + m_2}{m_1 m_2}c_2 x_2 + \frac{h_1}{m_1}x_3 - \frac{m_1 + m_2}{m_1 m_2}h_2 x_4\right.\\
&\qquad \left. - \frac{m_1 + m_2}{m_1 m_2}\phi(x_2) + \frac{m_1 + m_2}{m_1 m_2}u\right] dt,\\
dx_5 &= x_6 dt,\\
dx_6 &= [-a_1 a_2 x_5 - (a_1 + a_2)x_6]dt + q d\xi,
\end{aligned}
\tag{24}
$$

where the new state variables are defined by

$$\begin{aligned} x_1 &= z_1 - w, & x_2 &= z_2 - z_1, & x_3 &= \dot{z}_1 - \dot{w}, \\ x_4 &= \dot{z}_2 - \dot{z}_1, & x_5 &= w, & x_6 &= \dot{w}, \end{aligned} \tag{25}$$

c_1 and c_2 are stiffness constant parameters, h_1 and h_2 are damping constant parameters, ξ is a standard Wiener process, a_1, a_2 and q are constant parameters of linear filter defined by

$$a_1 = a_1^* v, \qquad a_2 = a_2^* v, \qquad q = q^* \sqrt{a_1 a_2 v}, \tag{26}$$

where a_1^*, a_2^* and q^* are constant parameters of random road profile, v is the constant speed of vehicle.

The objective of the use of the active control u is to minimize the modified performance index I defined by (23), namely

$$I = \frac{1}{m_2} E\left[\left(\frac{c_2}{m_2}x_2 + \frac{h_2}{m_2}x_4 + \frac{1}{m_2}\phi(x_2) - \frac{1}{m_2}u\right)^2 + \rho_1(x_2)^2 + \rho_2(x_3)^2 + \rho_3 u^2\right]. \tag{27}$$

Here the stationary moments are considered. If the nonlinear stiffness

$$Y = \phi(x_2) \tag{28}$$

can substitute by the following linearized form

$$Y = \alpha k x_2, \tag{29}$$

where α is the constant parameter and k is the linearization coefficient. Then using equations (24), (28) and (29) the optimal control problem can be transformed to the standard one

$$d\mathbf{x} = [\mathbf{A}(k)\mathbf{x} + \mathbf{B}u]dt + \mathbf{G}d\xi, \tag{30}$$

$$I = E[\mathbf{x}^T\mathbf{Q}(k)\mathbf{x} + 2\mathbf{x}^T\mathbf{N}(k)u + ru^2], \tag{31}$$

where matrices $\mathbf{A}$, $\mathbf{Q}$ and vectors $\mathbf{B}$, $\mathbf{G}$ $\mathbf{N}$ are defined by equations (24) and (29); $\mathbf{A}(k)$, $\mathbf{Q}(k)$ and $\mathbf{N}(k)$ for a given linearization coefficient k are constant matrices and vector, respectively.

Socha [55] has compared three methods of statistical linearization for Gaussian excitations corresponding to the following moment criteria:

Criterion 1. Equality of second order moments [29]:

$$E\left[(k_a x_2)^2\right] = E\left[(\phi(x_2))^2\right]. \tag{32}$$

Criterion 2. Mean-square error of displacements [29]:

$$E\left[(k_b x_2 - \phi(x_2))^2\right] \to \min. \tag{33}$$

Criterion 3. Mean-square error of potential energies [14]:

$$E\left[\left(\int_0^{x_2}[k_c v - \phi(v)]dv\right)^2\right] \to \min, \tag{34}$$

where k_a, k_b and k_c are linearization coefficients, and two criteria in the probability density functions space:

Criterion 4. Square probability metric [56]:

$$\int_{-\infty}^{+\infty} (g_N(y) - g_L(y, k_d))^2 dy. \tag{35}$$

Criterion 5. Pseudo-moment metric [56]:

$$\int_{-\infty}^{+\infty} |y|^{2l} |g_N(y) - g_L(y, k_e)| dy, \tag{36}$$

where $l = 1, 2, \ldots$, $g_N(y)$ and $g_L(y)$ are probability density functions of variables defined by

$$Y = \psi(x_2) = c_2 x_2 + \phi(x_2) \tag{37}$$

and

$$Y = k^* x_2, \tag{38}$$

respectively, where k^* is the linearization coefficient. Then the optimal control problem can be transformed to the modified version of (30) and (31). As an example it was considered a nonlinear function

$$\phi(x_2) = \alpha x_2^3. \tag{39}$$

In the case of moment criteria it was shown [29, 14] that the corresponding linearization coefficients have the form

$$k_a = \sqrt{15} E[x_2^2], \qquad k_b = 3E[x_2^2], \qquad k_b = 2.5E[x_2^2], \tag{40}$$

and in the case of criteria 4 and 5 the nonlinear function $\psi(x_2)$ and the corresponding probability density function have the form

$$Y = \psi(x_2) = c_2 x_2 + \alpha x_2^3, \tag{41}$$

$$g_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_L} \exp\left[-\frac{(v_1 + v_2)^2}{2\sigma_L^2}\right] \frac{1}{6a\varepsilon} \left(\frac{a+y}{v_1^2} + \frac{a-y}{v_2^2}\right), \tag{42}$$

where

$$v_1 = \left(\frac{y}{2\alpha} + \sqrt{\frac{y^2}{4\alpha^2} + \frac{c_2^3}{27\alpha^3}}\right)^{1/3}, \qquad v_2 = \left(\frac{y}{2\alpha} - \sqrt{\frac{y^2}{4\alpha^2} + \frac{c_2^3}{27\alpha^3}}\right)^{1/3},$$
$$a = \sqrt{y^2 + \frac{4c_2^3}{27\alpha}}. \tag{43}$$

The probability density of the linearized variable (38) has the form

$$g_L(y, k^*) = \frac{1}{\sqrt{2\pi} k^* \sigma_L} \exp\left\{-\frac{y^2}{2(k^*)^2 \sigma_L^2}\right\}, \tag{44}$$

where $\sigma_L^2 = E[x^2]$ is the variance of the input Gaussian variable, k^* is equal to k_d or k_e in criterion 4 or 5, respectively.

To determine the quasi-optimal control for a nonlinear system with nonlinear criterion the idea proposed in the literature (see for instance [69, 4]) consisting in application of the statistical linearization and LQG method was used. The following two iterative procedures were proposed by Socha [56].

Iterative procedures

Procedure A (for criteria 1-3):

Step 1. Assume that one of the linearization coefficients is equal to zero, for instance, $k = k_a = 0$.

Step 2. Calculate $\mathbf{A} = \mathbf{A}(k)$, $\mathbf{Q} = \mathbf{Q}(k)$ and $\mathbf{N} = \mathbf{N}(k)$ in (30)-(31) and then solve the algebraic Riccati equation

$$\mathbf{PA}_N + \mathbf{A}_N^T\mathbf{P} - \frac{1}{r}\mathbf{PBB}^T\mathbf{P} + \mathbf{Q}_N = \mathbf{0}, \tag{45}$$

where $\mathbf{A}_N = \mathbf{A} - \frac{1}{r}\mathbf{BN}^T \geq \mathbf{0}$, $\mathbf{Q}_N = \mathbf{Q} - \frac{1}{r}\mathbf{NN}^T \geq \mathbf{0}$. The solution is a symmetric positive definite matrix $\mathbf{P}$.

Step 3. Find the optimal control and the matrix $\mathbf{K}$.

$$\mathbf{u}(t) = -\mathbf{Kx}(t) = -\frac{1}{r}(\mathbf{N}^T + \mathbf{B}^T\mathbf{P})\mathbf{x}(t). \tag{46}$$

Next, substitute $\mathbf{K}$, $\mathbf{A}(k)$ and $\mathbf{N}(k)$ into the covariance equation

$$(\mathbf{A}(k) - \mathbf{BK})\mathbf{V_L} + \mathbf{V_L}(\mathbf{A}(k) - \mathbf{BK}) + \mathbf{GG}^T = \mathbf{0} \tag{47}$$

and solve the equation. The solution of equation (47) is $\mathbf{V}_L$.

Step 4. Substitute the element of covariance matrix $E[x_2^2] = V_{L_{22}}$ obtained in step 3 into the linearization coefficient k_a defined by (40).

Step 5. Calculate $\mathbf{P}$, $\mathbf{u}$ and $\mathbf{V_L}$ using equations (45)-(47) and the linearization coefficient k_a obtained in the last step.

Step 6. Iterate steps 2-5 until $\mathbf{V_L}$ and $\mathbf{P}$ converge.

Step 7. Calculate the optimal value of criterion I_{opt} using the solution of the Riccati equation obtained in step 5,

$$I = tr(\mathbf{PGG}^T). \tag{48}$$

Procedure B (for criteria 4-5):

Step 1. Assume $k^* = c_2$.

Step 2. Calculate modified matrices $\mathbf{A} = \mathbf{A}(k^*)$, $\mathbf{Q} = \mathbf{Q}(k^*)$ and $\mathbf{N} = \mathbf{N}(k^*)$ in (30)-(31) and then solve the algebraic Riccati equation (45) The solution is a symmetric positive definite matrix $\mathbf{P}$.

Step 3. Substitute $\mathbf{P}$ obtained in step 2 into equation (46) and find the matrix $\mathbf{K}$. Next, substitute $\mathbf{K}$, $\mathbf{A}(k)$ and $\mathbf{N}(k)$ into the covariance equation (47) and solve the equation. The solution of equation (47) is $\mathbf{V}_L$.

Step 4. For $\mathbf{V}_L$ obtained in previous step calculate for linearized element the variance $\sigma_{x_2}^2 = E[x_2^2]$ of the input Gaussian variable and next the corresponding probability density functions given by (42) and (44), respectively.

Step 5. For nonlinear element find the linearization coefficient k^* which minimize, for instance, Criterion 4.

Step 6. Substitute the linearization coefficient k^* obtained in step 5 into equation (47) and then solve the equation.

Step 7. If the error of accuracy is greater then a given parameter ε_1 then repeat steps 4-6 until $\mathbf{V}_L$ converges.

Step 8. Calculate new $\mathbf{A} = \mathbf{A}(k^*), \mathbf{Q} = \mathbf{Q}(k^*)$ and $\mathbf{N} = \mathbf{N}(k^*)$ and next using these matrices calculate $\mathbf{P}, \mathbf{K}$ and $\mathbf{V}_L$ using equations (45)-(47).

Step 9. Iterate steps 4-8 until $\mathbf{V}_L$ and $\mathbf{P}$ converge.

Step 10. Calculate the optimal value of criterion I_{opt} substituting the solution of the Riccati equation obtained in step 8 into relation (48).

An illustration of the obtained results in the form of a comparison of the criterion $I_{\rm opt}$ defined by (48) versus parameter alpha was presented in [56]. In this comparison three moment criteria and two criteria in probability density function space of linearization techniques, namely equality of second order moments of nonlinear and linearized elements, mean-square error of the displacement, mean-square error of the potential energies, square probability metric and pseudomoment probability metric are considered. The numerical results denoted by lines with circles, stars, squares, triangels and

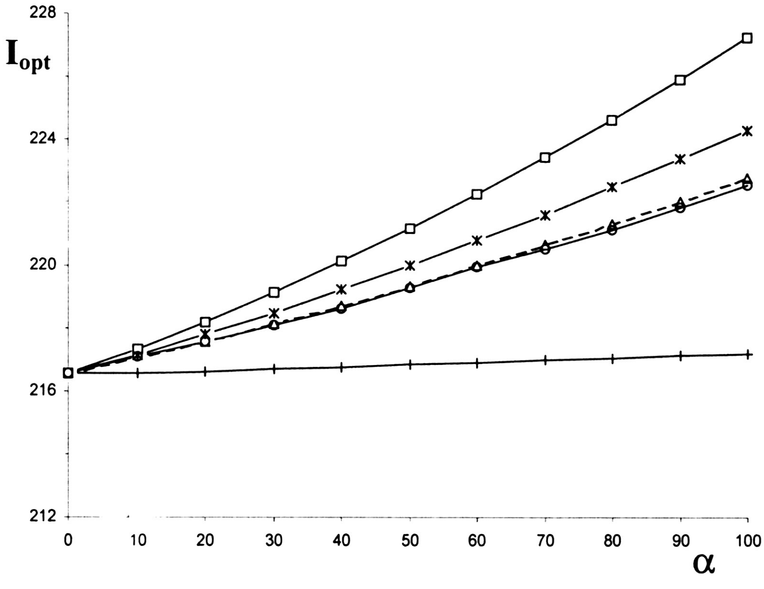

FIGURE 4. Comparison between optimal criteria obtained by application of different statistical linearization techniques versus parameter α.

crosses, respectively are presented in Fig. 4. The parameters selected for calculations and further simulations are $m_1 = 100$, $m_2 = 500$, $c_1 = 100$, $c_2 = 50$, $h_1 = 1$, $h_2 = 5$, $a_1^* = 0.025$, $a_2^* = 0.075$, $q^* = \sqrt{0.0067}$, $v = 20$, $\rho_1 = 1$, $\rho_2 = 1000$, $\rho_3 = 10000$, $\rho_4 = 1$.

Figure 5 shows the dependence of the criterion I_{opt} upon the speed of vehicle as it changes from 10^0 to 10^2. The other parameters are the same except $\rho_2 = 100$, $\rho_3 = 1000$ and $\alpha = 20$.

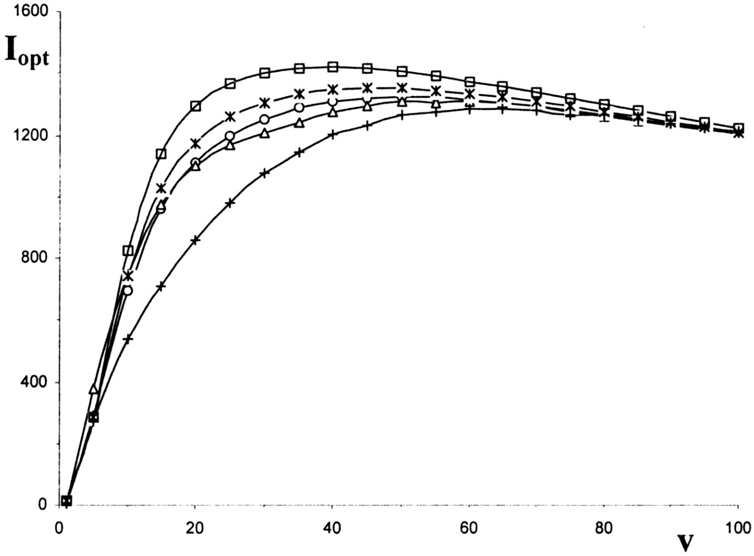

FIGURE 5. Comparison between optimal criteria obtained by application of different statistical linearization techniques versus parameter v [56].

From the numerical results it follows that for given sets of parameters there are no significant differences between mean-square criteria obtained by application considered linearization methods.

The modified approach of an application of equivalent linearization and LQG method was used to the determination a quasi-optimal control by Narayanan and Raju [41] for 1-DOF model and Raju and Narayanan [45] for 2-DOF model of vehicle with hysteresis.

The equations of motions of 2-DOF quarter car model with Bouc-Wen model of hysteresis can be written as

$$m_2\ddot{z}_2 + h_2(\dot{z}_2 - \dot{z}_1) + h|\dot{z}_2 - \dot{z}_1|(\dot{z}_2 - \dot{z}_1) + \alpha c_2(z_2 - z_1) + (1-\alpha)c_2 z_0 - u = 0 \quad (49)$$

$$m_1\ddot{z}_1 + h_2(\dot{z}_1 - \dot{z}_2) + h|\dot{z}_1 - \dot{z}_2|(\dot{z}_1 - \dot{z}_2) + \alpha c_2(z_1 - z_2) + (1-\alpha)c_2 z_0 + c_1(y_1 - w) + u = 0. \quad (50)$$

The hysteresis displacement is given by the nonlinear differential equation

$$\dot{z}_0 + \gamma|\dot{z}_2 - \dot{z}_1|z_0|z_0|^{p-1} + \eta\dot{z}_2 - \dot{z}_1||z_0|^p + A_h|\dot{z}_2 - \dot{z}_1| = 0, \quad (51)$$

where γ, η, A_h and p are constant parameters determined hysteresis. The road excitations and performance index have the form (20) and (23), respectively. Equations (49)-(51) and (20) can be rewritten in the form

$$d\mathbf{x} = \big[\mathbf{\Phi}(\mathbf{x}) + \mathbf{B}u\big]dt + \mathbf{G}d\xi, \tag{52}$$

where $\mathbf{x} = \begin{bmatrix} z_1 & \dot{z}_1 & z_2 & \dot{z}_2 & z_0 & w \end{bmatrix}^T$ and the criterion in new state variables has the modified form of (23)

$$\begin{aligned} I = E\Bigg[\Bigg(& -\frac{\alpha c_2}{m_2}x_1 - \frac{h_2}{m_2}x_2 + \frac{\alpha c_2}{m_2}x_3 + \frac{h_2}{m_2}x_4 \\ & - \frac{h|x_4 - x_2|(x_4 - x_2)}{m_2} - \frac{(1-\alpha)c_1}{m_2}x_5 + \frac{1}{m_2}u\Bigg)^2 \\ & + \rho_1(x_1 - x_3)^2 + \rho_2(x_1 - x_6)^2 + \rho_3 u^2\Bigg] = E[\mathbf{\Psi}(\mathbf{x}, u)]. \end{aligned} \tag{53}$$

Similar to the previous case, using equivalent linearization technique one can obtain the equations of motion and performance index in linearized form

$$m_2\ddot{z}_2 + (h_2 + h_{eq})(\dot{z}_2 - \dot{z}_1) + \alpha c_2(z_2 - z_1) + (1-\alpha)c_2 z_0 - u = 0, \tag{54}$$

$$m_1\ddot{z}_1 + (h_2 + h_{eq})(\dot{z}_1 - \dot{z}_2) + \alpha c_2(z_1 - z_2) + (1-\alpha)c_2 z_0 + c_1(z_1 - w) + u = 0, \tag{55}$$

$$\dot{z}_0 + h_h(\dot{z}_2 - \dot{z}_1) + k_h z_0 = 0, \tag{56}$$

$$\begin{aligned} I = E\Bigg[\Bigg(& -\frac{\alpha c_2}{m_2}x_1 - \frac{(h_2 + h_{eq})}{m_2}x_2 + \frac{\alpha c_2}{m_2}x_3 + \frac{(h_2 + h_{eq})}{m_2}x_4 \\ & - \frac{(1-\alpha)c_1}{m_2}x_5 + \frac{1}{m_2}u\Bigg)^2 + \rho_1(x_1 - x_3)^2 + \rho_2(x_1 - x_6)^2 + \rho_3 u^2\Bigg] \\ & = E\big[\mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{x}^T\mathbf{N}\mathbf{u} + \mathbf{u}^T\mathbf{R}\mathbf{u}\big], \end{aligned} \tag{57}$$

where h_{eq}, h_h and k_h are the linearization coefficients. Then the optimal control problem can be transformed to the standard one (30)-(31) i.e.

$$d\mathbf{x} = [\mathbf{A}(\mathbf{k})\mathbf{x} + \mathbf{B}u]dt + \mathbf{G}d\xi, \tag{58}$$

$$I = E[\mathbf{x}^T\mathbf{Q}(\mathbf{k})\mathbf{x} + 2\mathbf{x}^T\mathbf{N}(\mathbf{k})u + ru^2], \tag{59}$$

where $\mathbf{A}(\mathbf{k})$, $\mathbf{Q}(\mathbf{k})$ and $\mathbf{N}(\mathbf{k})$ are depending on vector of linearization coefficients $\mathbf{k} = \begin{bmatrix} h_{eq} & h_h & k_h \end{bmatrix}^T$ constant matrices and vector, respectively.

To determine the quasi-optimal control one may use a modified iterative procedure A.

2.3. Linear stochastic models with incomplete observations

The natural generalization of linear models with complete observations are linear stochastic models with incomplete observations described very well in control literature. An application of this approach to the determination of optimal control for suspension systems was shown, for instance by Yoshimura et al. [70], Raju and Narajanan [44], Ray [47], Elmadany and Abduljabbar [15] and Yu and Crolla [75]. The standard LQG procedure for linear stochastic models with incomplete observations has the following form [31, 57]:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{G}\boldsymbol{\xi}(t), \tag{60}$$

$$\mathbf{y}(t) = \mathbf{H}\mathbf{x}(t) + \boldsymbol{\eta}(t), \tag{61}$$

where $\mathbf{x}(t)$ is the state vector, $\mathbf{u}(t)$ the control vector, $\mathbf{y}(t)$ the observation vector; $\boldsymbol{\xi}(t)$ and $\boldsymbol{\eta}(t)$ are the disturbances vectors; $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{u} \in \mathbf{R}^m$, $\mathbf{y} \in \mathbf{R}^l$, $\boldsymbol{\xi} \in \mathbf{R}^p$, $\boldsymbol{\eta} \in \mathbf{R}^q$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{G}$, $\mathbf{H}$, $\mathbf{B}$ and are constant matrices of appropriate dimensions; $\mathbf{H}$ is the state-to-measurement transformation matrix $\boldsymbol{\xi}(t)$ and $\boldsymbol{\eta}(t)$ are assumed to be mutually independent a zero mean white noises with covariance matrices $\mathbf{Q}_\xi$ and $\mathbf{Q}_\eta$ defined by

$$E\big[\boldsymbol{\xi}(t_2)\boldsymbol{\xi}(t_1)^T\big] = \mathbf{Q}_\xi\delta(t_2 - t_1), \qquad E\big[\boldsymbol{\eta}(t_2)\boldsymbol{\eta}(t_1)^T\big] = \mathbf{Q}_\eta\delta(t_2 - t_1). \tag{62}$$

The mean-square criterion is defined by

$$I = E\left\{ \mathbf{x}^T(t_f)\mathbf{P}(t_f)\mathbf{x}(t_f) + \int_{t_0}^{t_f} \big[\mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{x}^T\mathbf{N}\mathbf{u} + \mathbf{u}^T\mathbf{R}\mathbf{u}\big]dt \right\}. \tag{63}$$

Using results of LQG theory and Kalman filtering approach the optimal control is determined by

$$\mathbf{u}(t) = -\mathbf{K}\hat{\mathbf{x}}(\mathbf{t}), \tag{64}$$

where the control gain matrix $\mathbf{K}(t)$ is time depending and given by

$$\mathbf{K}(t) = \mathbf{R}^{-1}(\mathbf{N}^T + \mathbf{B}^T\mathbf{P}(t)), \tag{65}$$

where $\mathbf{P}(t)$ is a symmetric, positive-definite solution of the differential Riccati equation

$$\dot{\mathbf{P}} + \mathbf{P}\mathbf{A}_N + \mathbf{A}_N^T\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q}_N = \mathbf{0} \tag{66}$$

with terminal condition $\mathbf{P}(t_f)$ and

$$\mathbf{A}_N = \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{N}^T \geq \mathbf{0}, \qquad \mathbf{Q}_N = \mathbf{Q} - \mathbf{N}\mathbf{R}^{-1}\mathbf{N}^T \geq \mathbf{0}, \tag{67}$$

$\hat{\mathbf{x}}(t)$ is the optimal estimates vector of system (60), governed by the Kalman filter

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}[\mathbf{H}(\mathbf{x} - \hat{\mathbf{x}}) + \boldsymbol{\eta}(t)]. \tag{68}$$

with time depending Kalman gain vector

$$\mathbf{L}(t) = \mathbf{S}\mathbf{H}^T\mathbf{Q}_\eta^{-1} \tag{69}$$

The covariance matrix $\mathbf{S}(t)$ of the estimation error $\mathbf{e}(t) = \mathbf{x} - \hat{\mathbf{x}}$ is the solution of another Riccati matrix differential equation for $t \geq t_0$

$$\dot{\mathbf{S}} - \mathbf{S}\mathbf{A}^T - \mathbf{A}\mathbf{S} + \mathbf{S}\mathbf{H}^T\mathbf{R_1}^{-1}\mathbf{H}\mathbf{S} - \mathbf{G}\mathbf{Q}_\xi\mathbf{G}^T = \mathbf{0}, \qquad t \geq t_0, \qquad \mathbf{S}(t_0) = \mathbf{S}_0. \tag{70}$$

The performance index for optimal control can be determined from the mean-square value of the state

$$\mathbf{V}(t) = E[\mathbf{x}(t)\mathbf{x}^T(t)] = \hat{\mathbf{V}}(t) + \mathbf{S}(t), \tag{71}$$

where $\hat{\mathbf{V}}(t)$ can be calculated from the matrix differential Lyapunov equation

$$\dot{\hat{\mathbf{V}}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\hat{\mathbf{V}} + \hat{\mathbf{V}}(\mathbf{A} - \mathbf{B}\mathbf{K}) + \mathbf{S}\mathbf{H}^\mathbf{T}\mathbf{Q}_\eta^{-1}\mathbf{H}\mathbf{S}, \qquad \hat{\mathbf{V}}(\mathbf{t_0}) = \mathbf{E}\mathbf{x}(\mathbf{0})\mathbf{E}\mathbf{x}^\mathbf{T}(\mathbf{0}), \tag{72}$$

Raju and Narajanan [44] have used this approach to determine an optimal active control in a 2-DOF quarter car model (see Fig. 2).

The equations of motions can be written as

$$m_2\ddot{z}_2 + h_2(\dot{z}_2 - \dot{z}_1) + c_2(z_2 - z_1) - u = 0, \tag{73}$$

$$m_1\ddot{z}_1 + h_2(\dot{z}_1 - \dot{z}_2) + c_2(z_1 - z_2) + c_1(z_1 - y) + u = 0. \tag{74}$$

The dynamics of the road input model is described by the differential equation

$$y(s)'' + (\alpha + \beta)y(s)' + \alpha\beta y(s) = \Gamma\xi(s), \tag{75}$$

where the prime denote the order of differentiation with respect to space variable s representing the traverse along the rough road; α and β are constant parameters of the road profile filter, $\Gamma = \beta\sigma_r\sqrt{2\alpha}$, σ_r^2 is the road profile line space variance and $\xi(s)$ is the spatial standard white noise. Since $s = s(t)$ is a function of time one can show that the differentiation with respect to variable s can be replaced by differentiation with respect to t according to formula $d(.)/ds = (dt/ds)(d(.)/dt)$. According to Harrison and Hammond [24], if $\xi_1(t)$ is a zero-mean stationary Gaussian white noise process then $\xi_1(t)/\sqrt{\dot{s}(t)}$ is equivalent in covariance sense, to the parametrized white noise process $\xi(s(t))$. Then equation (75) can be rewritten as a system of two equations in time domain

$$\begin{aligned} \dot{\bar{y}}_1 &= \dot{s}y_2, \\ \dot{\bar{y}}_2 &= -(\alpha + \beta)\dot{s}\bar{y}_2 - \alpha\beta\dot{s}\bar{y}_2 + \dot{s}\Gamma\xi\big(s(t)\big). \end{aligned} \tag{76}$$

If we denote $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T$ where $x_1 = z_1$, $x_2 = \dot{z}_1$, $x_3 = z_2$, $x_4 = \dot{z}_2$, $x_5 = \bar{y}_1$, $x_6 = \bar{y}_2$ and $\mathbf{y} = \begin{bmatrix} \bar{y}_1 & \bar{y}_2 \end{bmatrix}^T$ then equations (73), (74) and (76) can be rewritten in the vector forms (60) and (61).

The criterion is defined by integral version of (16), namely

$$I = E\left\{\mathbf{x}^T(t_f)\mathbf{P}(t_f)\mathbf{x}(t_f) + \int_{t_0}^{t_f}\left[\left(-\frac{c_2}{m_2}x_1 - \frac{h_2}{m_2}x_2 + \frac{c_2}{m_2}x_3 + \frac{h_2}{m_2}x_4 + \frac{1}{m_2}u\right)^2 \right.\right.$$
$$\left.\left. + \rho_1(x_1 - x_3)^2 + \rho_2(x_1 - \bar{y_1})^2 + \rho_3 u^2\right]dt\right\}$$
$$= E\left\{\mathbf{x}^T(t_f)\mathbf{P}(t_f)\mathbf{x}(t_f) + \int_{t_0}^{t_f}\left[\mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{x}^T\mathbf{N}\mathbf{u} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right]dt\right\}. \quad (77)$$

Equation (77) defines matrices $\mathbf{Q}$, $\mathbf{N}$ and $\mathbf{R}$.

3. Advanced suspension systems

As was mentioned in the Introduction very recent suspension systems one can treat as adaptive control systems and smart suspensions with application, for instance, fuzzy-logic or neural-network approaches.

The application of fuzzy reasoning to the active suspension systes has been proposed by Lin et al. [33], Roukieh and Titli [50] and developed with satisfactory performance by Yeh and Tsai [66], Yoshimura et al. [73, 74], Huang and Chao [26]. An application of neural networks for the identification of nonlinear vehicle model and the design of optimal active control was shown by Moran and Nagai [37, 38, 39] and Yoon and Kim [68].

We present bellow a short review of basic adaptive control algorithms.

3.1. Adaptive active suspensions

In the determination procedure of optimal or quasi-optimal control it is assumed that the considered model of the physical system is known. This assumption is very strong and not always satisfied. Therefore an adaptation process of considered model is required. It should be the process of changing the parameter structure and possibly the controls of system on the basis of information obtained during the control period. This idea was developed in control theory and to our knowledge it was first time implemented for vehicle suspensions by Sachs [51]. There are three groups of methods of adaptation of vehicle suspension models. The first one called "adaptation of passive systems" is an adaptation of both stiffness and damping for passive systems governed by a proper control program (see, for instance [28, 54, 48, 12]). Some information about technical implementations of "adaptive passive suspensions" have been reported by Mizuguchi et al. [36], Yokoya et al. [67] and Poyser [43]. To this group of adaptation methods one can include the adaptation of model with respect to input signals, for instance in Hac paper [22] the adaptation is oriented towards the compensation in excitaion. The control adopted to the changes of the velocity and the characteristics of the surface.

In the second group of adaptation mehods the proposed algorithms are self–tuning regulators, where for instance time–series analysis is applied [64, 46] or the controller incorporating a weighting contoller, state observer and parameter estimator is designed by LQG approach [76].

The third one called "model reference adaptive control" makes the output of an unknown suspension system asymptotically approach the output of a user-defined reference model that represents a desired suspension characteristics. In what follows the third group of control algorithms is reviewed.

3.1.1. Linear models. Model Reference Adaptive Control" (MRAC) approach for active suspension was presented by Sunwoo et al. [58]. The authors considered 1-DOF quarter car suspension model (see Fig. 6) described by

Active plant suspension model

$$\dot{\mathbf{x}}_p = \mathbf{A}_p\mathbf{X}_p + \mathbf{B}_p u_c + \mathbf{G}_p \ddot{w}, \tag{78}$$

where

$$\mathbf{X}_p = \begin{bmatrix} z_p - w \\ \dot{z}_p - \dot{w} \end{bmatrix}, \qquad \mathbf{A}_p = \begin{bmatrix} 0 & 1 \\ -c_p/m_p & -h_p/m_p \end{bmatrix},$$
$$\mathbf{B}_p = \begin{bmatrix} 0 \\ -1/m_p \end{bmatrix}, \qquad \mathbf{G}_p = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \tag{79}$$

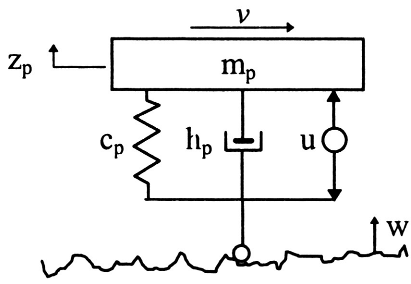

FIGURE 6. Active suspension model (Eq. (78)).

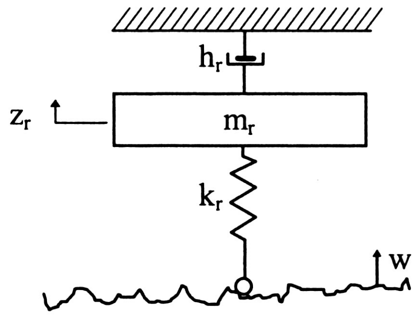

FIGURE 7. Sky-hook damping reference model (Eq. (81)).

c_p and h_p are the coefficients of linearized spring and damping rates and m_p is the mass of the sprig load. It is assumed that the exact values of the parameters c_p, h_p and m_p are unknown. u_c is the active suspension control input. It is assumed to be the adjustable part of MRAC in the form

$$u_c = K_r \mathbf{K}_p \mathbf{X}_p + K_r \dot{w}, \tag{80}$$

where K_r and $\mathbf{K}_p = \begin{bmatrix} k_{p1} & k_{p2} \end{bmatrix}^T$ are adjustable gain scalar and matrix, respectively.

The model chosen for MRAC was a 1-DOF quarter-car model with the sky-hook damper (see Fig. 7).

Reference suspension model

$$\dot{\mathbf{x}}_r = \mathbf{A}_r \mathbf{X}_r + \mathbf{B}_r \dot{w} + \mathbf{G}_r \ddot{w}, \tag{81}$$

where

$$\mathbf{X}_r = \begin{bmatrix} z_r - w \\ \dot{z}_r - \dot{w} \end{bmatrix}, \qquad \mathbf{A}_r = \begin{bmatrix} 0 & 1 \\ -c_r/m_r & -h_r/m_r \end{bmatrix},$$
$$\mathbf{B}_p = \begin{bmatrix} 0 \\ -h_r/m_r \end{bmatrix}, \qquad \mathbf{G}_r = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \tag{82}$$

c_r, h_r and m_r are the hypothetical spring and damping rates and the mass of the spring load, respectively.

To determine the adaptive control law Sunwoo et al. [58] have used Lyapunov stability approach. They have shown that the gain scalar K_r and gain matrix $\mathbf{K}_p$ satisfy the following differential equations

$$\dot{\mathbf{K}}_p = -\mathbf{B}_r^T \mathbf{P} \mathbf{e} \mathbf{X}_p \mathbf{M}^{-1}, \tag{83}$$

$$\dot{K}_r = -K_r \mathbf{B}_r^T \mathbf{P} \mathbf{e} u_c \mathbf{N}^{-1} K_r, \tag{84}$$

where $\mathbf{e} = \mathbf{X}_p - \mathbf{X}_r$, $\mathbf{P}$ is chosen to as a unique positive- definite solution $\mathbf{P} = \mathbf{P}^T \geq 0$ of the Lyapunov equation

$$\mathbf{A}_r \mathbf{P} + \mathbf{P} \mathbf{A}_r^T + \mathbf{Q} = \mathbf{0}, \tag{85}$$

$\mathbf{M} = \mathbf{M}^T \geq 0$, $\mathbf{N} = \mathbf{N}^T \geq 0$ and $\mathbf{Q} = \mathbf{Q}^T \geq 0$ are weighting matrices to be specified by the designer.

The results obtained by Sunwoo et al. [58] were developed by Esmailzadeh and Fahimi [17] for 7-DOF system (full-car model). The authors used the optimal active suspension LQ in stationary case (see section 2.1.1) as a reference model. MRAC approach was also applied by Bakhtiari-Nejad and Karami-Mohammadi [2] to the determination of active control for vehicles body modelled by an elastic beam. The reference model is a conceptual active discrete-continuous vibrating system with structural damping for body and sky-hook dampers on the suspensions and with an optimal suspension controller.

3.1.2. Nonlinear models. The application of (MRAC) approach to the study of the 1-DOF quarter car linear suspension model proposed by Sunwoo et al. [58] was extended by Vallurupalli et al. [65] for 1-DOF quarter nonlinear suspension model. The authors have used discrete time approach and a harmonic linearization. Dukkipati et al. [8] studied 4-DOF system (half-car model) of nonlinear plant and linear reference suspension models. The Taylor linearization and discrete time approach was used.

The MRAC approach was developed by Dukkipati and Vallurupalli [9] for nonlinear time depending 2-DOF systems (see Fig. 8).

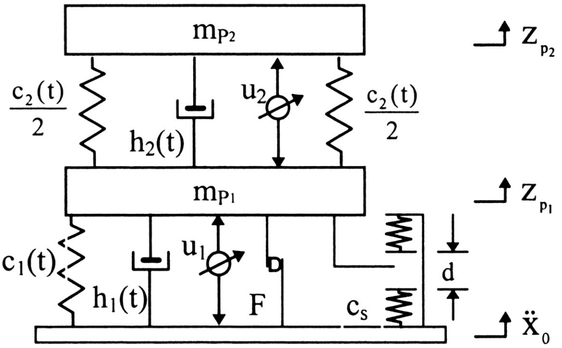

FIGURE 8. Adaptive active suspension model (Eq. (86)).

The equation of motion are

$$\begin{bmatrix} m_{p1}(t) & 0 \\ m_{p2}(t) & m_{p2}(t) \end{bmatrix} \begin{bmatrix} \ddot{x}_{p1} \\ \ddot{x}_{p2} \end{bmatrix} + \begin{bmatrix} h_1(t)|\dot{x}_{p1}| & -h_2(t) \\ 0 & h_2(t) \end{bmatrix} \begin{bmatrix} \dot{x}_{p1} \\ \dot{x}_{p2} \end{bmatrix}$$
$$+ \begin{bmatrix} c_1(t) + c_s s^* & -c_2(t) \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} x_{p1} \\ x_{p2} \end{bmatrix} + \begin{bmatrix} F\mathrm{sgn}(\dot{x}_{p1}) \\ -0 \end{bmatrix} + \begin{bmatrix} -c_s s^* \frac{d}{2}\mathrm{sgn}(x_{p1}) \\ -0 \end{bmatrix}$$
$$= \begin{bmatrix} m_{p1}(t) \\ m_{p2}(t) \end{bmatrix} \ddot{w} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{86}$$

where $x_{p1} = z_{p1} - w$, $x_{p2} = z_{p2} - z_{p1}$ are relative displacements of the plant (suspension system), $\ddot{w}$ denotes input acceleration, w represents road irregularities; $m_{p1}(t)$, $m_{p2}(t)$, $h_1(t)$, $h_2(t)$, $c_1(t)$ and $c_2(t)$ denotes the time depending mass, damping and stiffness coefficients, respectively. The forces $F\mathrm{sgn}(\dot{x}_{p1})$, $h_1(t)|\dot{x}_{p1}|$ and $c_s s^*(x_{p1} - \frac{d}{2}\mathrm{sgn}(x_{p1}))$ represent Coulomb damping, velocity squared viscous damping and elastic limit stop forces, respectively, where

$$s^* = \begin{cases} 0 & \text{if } \ |x_{p1}| \le \frac{d}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

Here, the actual values of coefficients F, c_s and h_2 are not known.

As a reference model Dukkipati and Vallurupalli [9] proposed a sky-hook linear time invariant model (see Fig. 9).

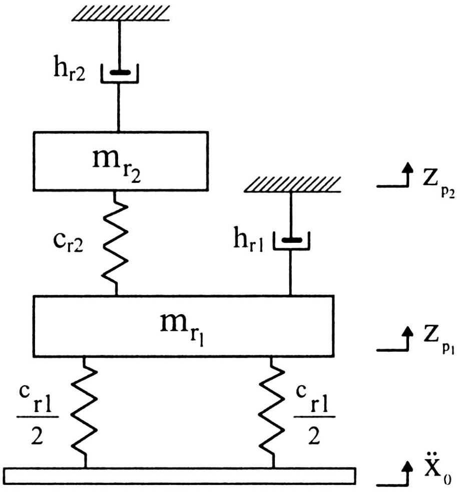

FIGURE 9. 2-DOF reference model (Eq. (87)).

The equation of motion are

$$\begin{bmatrix} m_{r1} & 0 \\ m_{r2} & m_{r2} \end{bmatrix} \begin{bmatrix} \ddot{x}_{r1} \\ \ddot{x}_{r2} \end{bmatrix} + \begin{bmatrix} h_{r1} & 0 \\ h_{r2} & h_{r2} \end{bmatrix} \begin{bmatrix} \dot{x}_{r1} \\ \dot{x}_{r2} \end{bmatrix} + \begin{bmatrix} c_{r1} & -c_{r2} \\ 0 & c_{r2} \end{bmatrix} \begin{bmatrix} x_{r1} \\ x_{r2} \end{bmatrix} = \begin{bmatrix} m_{r1} \\ m_{r2} \end{bmatrix} \ddot{w} + \begin{bmatrix} h_{r1} \\ h_{r2} \end{bmatrix} \dot{w}. \quad (87)$$

To determine adaptive control Dukkipati and Vallurupalli [9] first linearized suspension model by Taylor linearization approach and next divided the controller structure into three parts:

- feedforward controller,
- feedback controller,
- auxiliary signal.

The block diagram in Fig. 10 [9] illustrates the functions of various controllers and the flow of control signals. For detailed analysis and notations the reader is refered to [9].

The further development of MRAC approach for nonlinear multi-degree-of-suspension models was also proposed by Dukkipati and Vallurupalli [9]. They considered the system described by

$$\mathbf{M}(t)\ddot{\mathbf{X}} + \mathbf{C}(t, \mathbf{X}, \dot{\mathbf{x}})\dot{\mathbf{x}} + \mathbf{K}(t, \mathbf{X}) + \mathbf{L}(t, \dot{\mathbf{x}}) = \mathbf{D}(t)\ddot{X}_0 + \mathbf{PU}, \quad (88)$$

where $\mathbf{X}$ and $\mathbf{U}$ are n-dimensional vectors of relative displacemens and active control, respectively; $\mathbf{M}(t)$ and $\mathbf{C}(t, \mathbf{Z}, \dot{\mathbf{Z}})$ denotes the mass and damping $n \times n$ matrices; $\mathbf{K}(t, \mathbf{X})$ and $\mathbf{L}(t, \dot{\mathbf{x}})$ denotes n-dimensional vectors of the nonlinear stiffness and damping time depending coefficients; $\mathbf{D}(t)$ is n-dimensional time depending vector, $\mathbf{P}$ is $n \times n$ constant matrix, X_0 is a scalar input representing road irregularities.

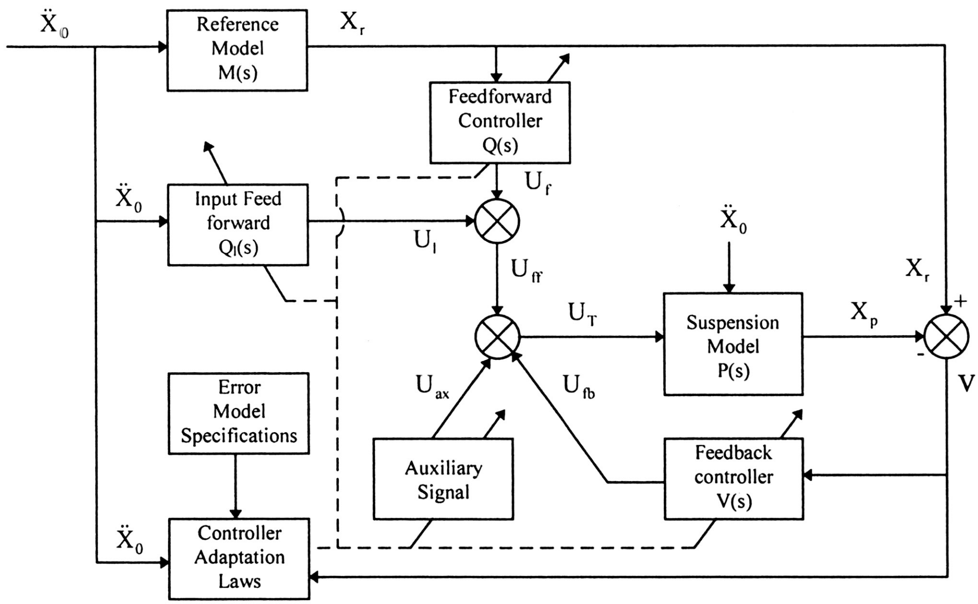

Figure 10. Block diagram representation of adaptive active control system [9].

4. Active suspensions with preview information

To design the optimal control for dynamic system we need information about disturbances acting on the system. Since the vehicle and control system has own dynamics the reaction of control system is delayed and therefore not optimal. This is the typical situation in the case of active suspension system. To improve the control process it would be fine to have in advance an information about disturbances acting on vehicle, for instance by measurements of road irregularities in front of the vehicle. This idea was first proposed by Bender [5], who showed that the use of preview information may improve suspension performance. He used spectral approach and Wiener filter theory to find the optimal preview control law. This idea was developed by Tomizuka [59] who considered the problem in discrete time domain and obtained the solution by application dynamic programming. Both authors considered 1-DOF vehicle models to illustrate the proposed control strategy. Further applications to 2-DOF models were given, for instance by Hac [23], Huisman et al. [27]. There are two basic preview concept for active vehicle suspension systems in the literature. In the first concept an information about the road profile ahead of the vehicle wheels is utilised while in the second concept an information from the dynamic variables of preceding axles is after suitable transformations used in the determination of optimal control. In both cases it is assumed that the displacement input to the rear wheels is a time delayed version of that to the front wheels.

4.1. Linear deterministic models

Following results in [23] and [27] we shortly discuss an application of preview control for to 2-DOF suspension models (see Fig. 11).

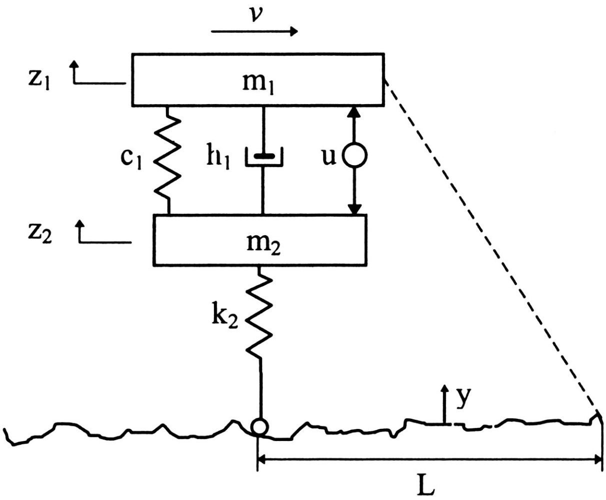

FIGURE 11. 2-DOF vehicle model with preview.

Consider the system

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{G}\mathbf{w}(t), \tag{89}$$

where $\mathbf{x} \in R^n$, $\mathbf{x_0} \in R^n$, $\mathbf{u} \in R^m$ and $\mathbf{w} \in R^q$ are the state vector, the vector of initial conditions, the control vector and the road disturbance deterministic vector, respectively. $\mathbf{A}$ and $\mathbf{B}$ and $\mathbf{G}$ are time invariant matrices of appropriate dimensions, $\mathbf{w}(t)$ is measured exactly up to T_d ahead of time t, $\mathbf{w}(\tau)$, $\tau \in [t, t+T_d]$ is assumed to be known.

The deterministic linear optimal preview control problem is a problem of finding the functional

$$\mathbf{u}(t) = \mathbf{f}\big[\mathbf{x}(\tau), \mathbf{w}(\sigma), t_0 \le \tau \le t, t_0 \le \sigma \le t + T_d\big], \tag{90}$$

which minimizes the criterion

$$I = \frac{1}{2}\mathbf{x}^T(T)\mathbf{P}_T\mathbf{x}(T) + \frac{1}{2}\int_{t_0}^{T} \big[\mathbf{x}^T\mathbf{Q}_1\mathbf{x} + 2\mathbf{x}^T\mathbf{N}_1\mathbf{u} + \mathbf{u}^T\mathbf{R}\mathbf{u} + 2\mathbf{x}^T\mathbf{Q}_{12}\mathbf{w} + \mathbf{w}^T\mathbf{Q}_2\mathbf{w}\big]dt, \tag{91}$$

where $\mathbf{Q}_1$, $\mathbf{R}$, $\mathbf{P}_T$ and $\mathbf{Q}_2$ are symmetric, time invariant matrices of appropriate dimensions, $\mathbf{R} > \mathbf{0}$, $\mathbf{Q}_N = \mathbf{Q}_1 - \mathbf{N}\mathbf{R}^{-1}\mathbf{N}^T \ge \mathbf{0}$.

The solution is

$$\mathbf{u}_{\text{opt}}(t) = -\mathbf{R}^{-1}\Big\{\big[\mathbf{N}^T + \mathbf{B}^T\mathbf{P}(t)\big]\mathbf{x}(t) + \mathbf{B}^T\mathbf{r}(t)\Big\}, \tag{92}$$

where $\mathbf{P}(t)$ is a positive definite solution of the Riccati equation

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}_N + \mathbf{A}_N^T\mathbf{P}(t) - \mathbf{P}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t) + \mathbf{Q}_N = \mathbf{0}, \qquad \mathbf{P}(T) = \mathbf{P}_T, \tag{93}$$

where $\mathbf{A}_N = \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{N}^T$ and the vector $\mathbf{r}(t) \in \mathbf{R}^n$ is given by

$$\dot{\mathbf{r}}(t) + \left[\mathbf{A}_N^T - \mathbf{P}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\right]\mathbf{r}(t) + \left[\mathbf{P}(t)\mathbf{G} + \mathbf{Q}_{12}\right]\mathbf{w}(t) = \mathbf{0}, \qquad \mathbf{r}(t_r) = \mathbf{0}, \tag{94}$$

where $t_r = \min(t + T_d, T)$.

An application of knowledge of the front wheel states to the determination an active control for both front and rear part of suspension system was presented, for instance by Pilbeam and Sharp [42] and Yu and Crolla [77].

4.2. Linear stochastic models with complete observations

An application of LQG optimal control theory to the study of preview control in linear stochastic models with complete observations of avtive suspension systems was given by several authors, for instance by Fruhauf, Kasper and Luckel [19], Louam, Wilson and Sharp [34], Senthil and Narayanan [52].

The standard LQG procedure for linear stochastic models with preview information and complete observations in stationary case has the following form (Senthil and Narayanan [52]).

The dynamic of vehicle model is described by

$$\dot{\mathbf{z}}(t) = \mathbf{A}_z\mathbf{z}(t) + \mathbf{B}_z\mathbf{u}(t) + \mathbf{G}_z\mathbf{w}(t), \tag{95}$$

where $\mathbf{z}(t)$ is the state vector, $\mathbf{u}(t)$ the control vector, $\mathbf{w}(t)$ the disturbance vector, $\mathbf{z} \in \mathbf{R}^n$, $\mathbf{u} \in \mathbf{R}^m$, $\mathbf{w} \in \mathbf{R}^p$, $\mathbf{A}_z$, $\mathbf{B}_z$, and $\mathbf{G}_z$ and are constant matrices of appropriate dimensions. The disturbance vector $\mathbf{w}(t)$ is modelled as the output of the first order shaping filter with white noise excitation

$$\dot{\mathbf{w}}(t) = \mathbf{A}_w\mathbf{w}(t) + \mathbf{G}_w\boldsymbol{\xi}(t), \tag{96}$$

where $\boldsymbol{\xi}(t)$ is assumed to be a zero mean white noise vector with covariance matrix $\mathbf{Q}_\xi$, $\boldsymbol{\xi} \in \mathbf{R}^q$, $\mathbf{A}_w$, $\mathbf{G}_w$ and $\mathbf{Q}_\xi$ are constant matrices with appropriate dimensions.

Equations (95) and (96) can be rewritten in a joint vector form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{G}\xi(t), \tag{97}$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{z} \\ \mathbf{w} \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \mathbf{A}_z & G_z \\ \mathbf{0} & A_w \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{B}_z \\ \mathbf{0} \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} \mathbf{0} \\ G_w \end{bmatrix}. \tag{98}$$

The mean-square criterion is defined by

$$I = \lim_{T\to\infty} E\left\{\frac{1}{T}\int_{t_0}^{T} \left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + 2\mathbf{x}^T(t)\mathbf{N}\mathbf{u}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right] dt\right\}, \tag{99}$$

where $\mathbf{Q}$, $\mathbf{N}$ and $\mathbf{R}$ are matrices of appropriate dimensions, $\mathbf{Q} \geq \mathbf{0}$ and $\mathbf{R} > \mathbf{0}$ are symmetric. It is assumed that the state vector $\mathbf{x}$ complete measurable and that the road uneveness $\mathbf{w}(\tau)$ for $\tau \in [t, t+T_d]$ is the preview information about $\mathbf{w}(t)$ up to T_d time units ahead of t is available.

Using results of LQG theory the optimal control is determined by

$$\mathbf{u}(t) = -\mathbf{K_1}\mathbf{x}(t) + \mathbf{K_2}\mathbf{r}(t), \tag{100}$$

where the control gain matrices $\mathbf{K_1}$ and $\mathbf{K_1}$ are constant and given by

$$\mathbf{K_1} = \mathbf{R}^{-1}(\mathbf{N}^T + \mathbf{B}^T\mathbf{P}), \qquad \mathbf{K_2} = \mathbf{R}^{-1}\mathbf{B}^T. \tag{101}$$

$\mathbf{P}$ is a symmetric, positive-definite solution of the differential Riccati equation

$$\dot{\mathbf{P}} + \mathbf{P}\mathbf{A}_N + \mathbf{A}_N^T\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q}_N = \mathbf{0} \tag{102}$$

where

$$\mathbf{A}_N = \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{N}^T \geq \mathbf{0}, \qquad \mathbf{Q}_N = \mathbf{Q} - \mathbf{N}\mathbf{R}^{-1}\mathbf{N}^T \geq \mathbf{0}, \tag{103}$$

and $\mathbf{r}$ is a vector satisfying

$$\mathbf{r}(t) = \int_0^{T_d} \exp\left\{\mathbf{A}_C^T\sigma\right\}\mathbf{P}(t+\sigma)\mathbf{G}\mathbf{w}(t+\sigma)d\sigma, \tag{104}$$

where $\mathbf{A}_C = \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}(\mathbf{N}^T + \mathbf{B}^T\mathbf{P})$ is stable matrix. The performance index for the optimal control is determined from the covariance equation

$$\begin{aligned}\dot{\mathbf{V}} = (\mathbf{A} - \mathbf{B}\mathbf{K_1})\mathbf{V} + \mathbf{V}(\mathbf{A} - \mathbf{B}\mathbf{K_1})^T \\ + G_w\mathbf{G}\mathbf{G}^T + \mathbf{V}_1\mathbf{G}^T + \mathbf{G}\mathbf{V}_1^T + \mathbf{B}\mathbf{K}_2\mathbf{V}_2^T + \mathbf{V}_2(\mathbf{B}\mathbf{K}_2)^T,\end{aligned} \tag{105}$$

where $\mathbf{V}$ is the covariance matrix of vector state $\mathbf{x}$, i.e. $\mathbf{V}(t) = E[\mathbf{x}(t)\mathbf{x}^T(t)]$ and

$$\mathbf{V}_1(t) = \frac{1}{2}\int_0^{T_d} \phi(\tau+\sigma, \tau)\mathbf{B}\mathbf{K}_2 \exp\left\{\mathbf{A}_C^T\sigma\right\}\mathbf{P}\mathbf{G}G_w d\sigma, \tag{106}$$

$$\begin{aligned}\mathbf{V}_2(t) = \frac{1}{2}\int_0^{T_d} \phi(t, t+\sigma)\mathbf{G}\mathbf{G}^T G_w\mathbf{P}\Big[\exp\left\{\mathbf{A}_C^T\sigma\right\}\Big]^T d\sigma \\ + \frac{1}{2}\int_0^{T_d}\int_0^{T_d} \mathbf{B}\mathbf{K}_2 \exp\left\{\mathbf{A}_C^T\sigma\right\}\mathbf{P}\mathbf{G}G_w\mathbf{G}^T\mathbf{P}\Big[\exp\left\{\mathbf{A}_C^T\sigma_1\right\}\Big]^T d\sigma d\sigma_1.\end{aligned} \tag{107}$$

From equation (100) it follows that the optimal control u is composed of a feedback and a feedforward term. The feedback control is exactly the same as for the system without preview information (see section 2.2.1). The feedforward control is constructed on the basis of preview information with respect to road input from the present time t up to T_d time units beyond t. An illustration of control structure is given in Fig. 12 [52].

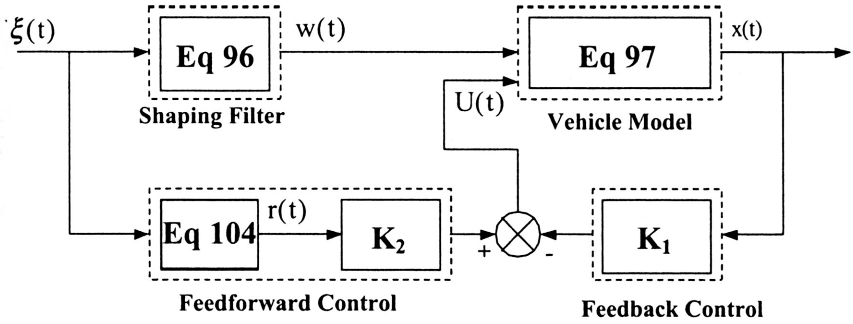

FIGURE 12. Block diagram representation of feedforward and feedback terms of active control in linear model with preview information [52].

Senthil and Narayanan [52] applied this approach for a 2-DOF quarter-car model (see Fig. 11).

The equation of motion and road irregularities are similar to already considered in section 2.2.1. In this case the matrices and vectors in equations (95) and (96) are defined by

$$\mathbf{z} = \begin{bmatrix} z_1 \\ \dot{z}_1 \\ z_2 \\ \dot{z}_2 \end{bmatrix}, \qquad \mathbf{A}_z = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(c_2+c_1)/m_1 & -h_2/m_1 & c_2/m_1 & h_2/m_1 \\ 0 & 0 & 0 & 1 \\ c_2/m_2 & h_2/m_2 & -c_2/m_2 & -h_2/m_2 \end{bmatrix},$$

$$\mathbf{B}_z = \begin{bmatrix} 0 \\ -1/m_1 \\ 0 \\ 1/m_2 \end{bmatrix}, \qquad \mathbf{G}_z = \begin{bmatrix} 0 \\ c_1/m_1 \\ 0 \\ 0 \end{bmatrix}, \tag{108}$$

$$\mathbf{A}_w = -av, \qquad \mathbf{G}_w = 1, \qquad \mathbf{Q}_\xi = 2\sigma^2 v,$$

where m_1, m_2, c_1, c_2 and h_1, h_2 are constant mass, spring and damper parameters, respectively. Matrices $\mathbf{Q}$, $\mathbf{N}$ and $\mathbf{R}$ are defined by the following equality

$$I = E\Bigg[\Big(-\frac{c_2}{m_2}x_1 - \frac{h_2}{m_2}x_2 + \frac{c_2}{m_2}x_3 + \frac{h_2}{m_2}x_4 + \frac{1}{m_2}u\Big)^2 + \rho_1(x_1-x_3)^2 + \rho_2(x_1-x_5)^2 + \rho_3 u^2\Bigg] = E\big[\mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{x}^T\mathbf{N}\mathbf{u} + \mathbf{u}^T\mathbf{R}\mathbf{u}\big]. \tag{109}$$

Similar approach was presented by Fruhauf, Kasper and Luckel [19], Louam, Wilson and Sharp [34] for the case when an information of a road profile from the response of the front wheel was used to the determination of optimal control.

4.3. Linear stochastic models with incomplete observations

The idea of using preview information to the determination of optimal control linear stochastic models with complete observations was also applied to the case of linear stochastic models with observations disturbed by a noise. see for instance Yoshimura and Ananthanarayana [71], Hac [23], Yoshimura et al. [72], Huisman et al. [27] and Roh and Park [49].

The standard LQG procedure for linear stochastic models with preview information and complete observations in stationary case has the following form [23].

The dynamic system is described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{G}w(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{110}$$

where $\mathbf{x}(t)$ is the state vector, $\mathbf{u}(t)$ the control vector, $\mathbf{w}(t)$ the disturbance vector , $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{u} \in \mathbf{R}^m$, $\mathbf{w} \in \mathbf{R}^p$, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{G}$ and are constant matrices with appropriate dimensions. $\mathbf{x}_0$ is a stochastic process with the mean value $\bar{\mathbf{x}}_0$ and variance $\mathbf{V}_0$. The disturbance vector $\mathbf{w}(t)$ is a vector zero mean stationary process.

The stochastic observation which described the road irregularities are given by

$$\begin{aligned} \mathbf{y}_1(t) &= \mathbf{C}_1^T\mathbf{x}(t) + \boldsymbol{\xi}(t), \\ \mathbf{y}_2(t) &= \mathbf{w}(t + T_d) + \boldsymbol{\eta}(t), \end{aligned} \tag{111}$$

where $\mathbf{C}_1$ is a constant $q \times n$ matrix, $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are observation variables, $\mathbf{y}_1(t)$, $\boldsymbol{\xi} \in \mathbf{R}^q$, $\mathbf{y}_2(t)$, $\boldsymbol{\eta} \in \mathbf{R}^r$, $\boldsymbol{\xi}(t)$ and $\boldsymbol{\eta}(t)$ are independent zero mean stationary white noises with covariance matrices $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$, respectively. Furthermore it is assumed that the initial state $\mathbf{x}_0$ is independent of $\boldsymbol{\xi}(t)$ and $\boldsymbol{\eta}(t)$.

The stochastic linear optimal preview control problem is a problem of finding the functional

$$\mathbf{u}(t) = \mathbf{f}\big[\mathbf{y}_1(\tau), \mathbf{y}_2(\tau), t_0 \le \tau \le t\big], \qquad t_0 \le t \le T, \tag{112}$$

which minimizes the criterion

$$I = \frac{1}{2}E\Bigg\{\mathbf{x}^T(T)\mathbf{P}_T\mathbf{x}(T) + \frac{1}{2}\int_{t_0}^{T}\big[\mathbf{x}^T\mathbf{Q}_1\mathbf{x} + 2\mathbf{x}^T\mathbf{N}\mathbf{u} + \mathbf{u}^T\mathbf{R}\mathbf{u} + 2\mathbf{x}^T\mathbf{Q}_{12}\mathbf{w} + \mathbf{w}^T\mathbf{Q}_2\mathbf{w}\big]dt\Bigg\} \tag{113}$$

where $\mathbf{Q}_1$, $\mathbf{R}$,$\mathbf{P}_T$ and $\mathbf{Q}_2$ are symmetric, time invariant matrices of appropriate dimensions; $\mathbf{R} > \mathbf{0}$, $\mathbf{Q}_N = \mathbf{Q}_1 - \mathbf{N}\mathbf{R}^{-1}\mathbf{N}^T \ge \mathbf{0}$.

The solution is

$$\mathbf{u}_{\text{opt}}(t) = -\mathbf{R}^{-1}\Big\{\big[\mathbf{N}^T + \mathbf{B}^T\mathbf{P}(t)\big]\hat{\mathbf{x}}(t) + \mathbf{B}^T\hat{\mathbf{r}}(t)\Big\}, \tag{114}$$

where $\mathbf{P}(t)$ is a positive definite solution of the following Riccati equation

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}_N + \mathbf{A}_N^T\mathbf{P}(t) - \mathbf{P}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t) + \mathbf{Q}_N = \mathbf{0}, \qquad \mathbf{P}(T) = \mathbf{P}_T, \tag{115}$$

for $\mathbf{A}_N = \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{N}^T$.

The estimation vectors $\hat{\mathbf{r}}(t) \in \mathbf{R}^n$ and $\hat{\mathbf{x}}(t) \in \mathbf{R}^n(t)$ satisfy the following differential equations

$$\dot{\hat{\mathbf{r}}}(t) + \left[\mathbf{A}_N^T - \mathbf{P}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\right]\hat{\mathbf{r}}(t) + \left[\mathbf{P}(t)\mathbf{G} + \mathbf{Q}_{12}\right]\mathbf{y}_2(t - T_d) = \mathbf{0}, \quad \hat{\mathbf{r}}(t_r) = \mathbf{0}, \tag{116}$$

where $t_r = \min(t + T_d, T)$ and

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{K}(t)\left[\mathbf{y}_1(t) - \mathbf{C}_1\hat{\mathbf{x}}(t)\right] + \mathbf{G}\mathbf{y}_2(t - T_d), \qquad \hat{\mathbf{x}}(t_0) = \bar{\mathbf{x}}_0, \tag{117}$$

where

$$\mathbf{K}(t) = \mathbf{S}(t)\mathbf{C}_1^T \nu_2^{-1}. \tag{118}$$

$\mathbf{S}(t)$ is a positive definite solution of another matrix Riccati equation

$$\dot{\mathbf{S}}(t) - \mathbf{S}(t)\mathbf{A} - \mathbf{A}^T\mathbf{S}(t) + \mathbf{S}(t)\mathbf{C}_1^T\nu_2^{-1}\mathbf{C}_1\mathbf{S}(t) - \mathbf{G}\nu_1\mathbf{G}^T = \mathbf{0}, \qquad \mathbf{S}(t_0) = \mathbf{S}_0. \tag{119}$$

Hac [23] used this approach to the 2-DOF quarter car model described in section 4.1 under white noise road irregularities with observations disturbed by an independent white noise i.e. the state equation of motion and observations are defined by equations (110) and (111) for

$$\mathbf{x} = \begin{bmatrix} z_1 \\ \dot{z}_1 \\ z_2 \\ \dot{z}_2 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(c_2 + c_1)/m_1 & -h_2/m_1 & c_2/m_1 & h_2/m_1 \\ 0 & 0 & 0 & 1 \\ c_2/m_2 & h_2/m_2 & -c_2/m_2 & -h_2/m_2 \end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} 0 \\ -1/m_1 \\ 0 \\ 1/m_2 \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} 0 \\ c_1/m_1 \\ 0 \\ 0 \end{bmatrix}, \tag{120}$$

where m_1, m_2, c_1, c_2 and h_1, h_2 are constant mass, spring and damper parameters. u and $\mathbf{w}$ are the scalar active control and scalar stochastic disturbance, respectively. $y_1(t)$ and $y_2(t)$ are scalar observation variables; $\mathbf{C}_1$ is a constant 4-dimensional vector. $\xi(t)$ and $\eta(t)$ are independent zero mean stationary white noises with intensities ν_1 and ν_2, respectively. Furhermore it is assumed that the initial state $\mathbf{x}_0$ is independent of $\xi(t)$ and $\eta(t)$.

The optimization criterion reduces to the following one

$$I = \frac{1}{2}\mathbf{x}^T(T)\mathbf{P}_T\mathbf{x}(T) + \frac{1}{2}\int_{t_0}^{T} \left[\mathbf{x}^T\mathbf{Q}_1\mathbf{x} + 2\mathbf{x}^T\mathbf{N}\mathbf{u} + \mathbf{u}^T R\mathbf{u} + 2\mathbf{x}^T\mathbf{Q}_{12}\mathbf{w} + \mathbf{w}^T Q_2\mathbf{w}\right] dt, \tag{121}$$

where the quantities $\mathbf{Q}_1$, $\mathbf{N}$, R, $\mathbf{Q}_{12}$ and Q_2 are defined by

$$\mathbf{Q}_1 = \frac{1}{m_2^2}\begin{bmatrix} c_2^2+(\rho_1+\rho_2)m_2^2 & c_2h_2 & -c_2^2-\rho_1 m_2^2 & -c_2h_2 \\ c_2h_2 & h_2^2 & -c_2h_2 & -h_2^2 \\ -(c_2^2+\rho_1 m_2^2) & -c_2h_2 & c_2^2+\rho_1 m_2^2 & c_2h_2 \\ -c_2h_2 & -h_2^2 & c_2h_2 & h_2^2 \end{bmatrix},$$

$$\mathbf{N} = \frac{1}{m_2^2}\begin{bmatrix} c_2 \\ h_2 \\ -c_2 \\ -h_2 \end{bmatrix}, \qquad \mathbf{Q}_{12} = \begin{bmatrix} -\rho_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{122}$$

$$R = 1/m_2^2 + \rho_3, \qquad Q_2 = \rho_2.$$

This approach was also applied to the determination of an active control for rail vehicle modelled by linear 3-DOF system with mean-square criterion by Yoshimura et al.(1993)

The equations for vehicle model, linear filter with white noise excitation and observation variables have the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{G}\mathbf{w}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{123}$$

where $\mathbf{x}(t)$ is the state vector, $\mathbf{u}(t)$ the control vector, $\mathbf{w}(t)$ the disturbance vector, $\mathbf{x} \in \mathbf{R}^6$, $\mathbf{u} \in \mathbf{R}^4$, $\mathbf{w} \in \mathbf{R}^6$, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{G}$ are constant matrices with appropriate dimensions. $\mathbf{x}_0$ is a stochastic process with the mean value $\bar{\mathbf{x}}_0$ and variance $\mathbf{V}_0$. The disturbance vector $\mathbf{w}(t)$ is a vector zero mean stationary coloured noise modelled by

$$\dot{\mathbf{w}}(t) = \mathbf{A}_w\mathbf{w}(t) + \boldsymbol{\xi}(t), \tag{124}$$

where $\boldsymbol{\xi}(t)$ is assumed to be a zero mean white noise vector with covariance matrix $\mathbf{Q}_\xi$, $\boldsymbol{\xi} \in \mathbf{R}^6$, $\mathbf{A}_w$ and $\mathbf{Q}_\xi$ are constant matrices with appropriate dimensions.

The stochastic observations which described the road irregularities are given by

$$\mathbf{y}(t) = \mathbf{H}\mathbf{x}(t) + \mathbf{C}\mathbf{w}(t) + \boldsymbol{\eta}(t), \tag{125}$$

where $\mathbf{C}$ and $\mathbf{C}$ are constant 6×6 matrices, $\mathbf{y}(t)$ is the observation variables vector, $\mathbf{y}(t)$, $\boldsymbol{\eta} \in \mathbf{R}^6$, $\boldsymbol{\xi}(t)$ and $\boldsymbol{\eta}(t)$ are independent zero mean stationary white noises with covariance matrices $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$, respectively. Furhermore it is assumed that the initial state $\mathbf{x}_0$ is independent of $\boldsymbol{\xi}(t)$ and $\boldsymbol{\eta}(t)$.

An extension of the discussed approach was proposed by Roh and Park [49] who considered the vehicle model in the form (110) with mean suare criterion (113) where the disturbances are modelled by a scalar white noise $w(t)$ with time depending intensity and one of observation variables depends on the acceleration of one state variable i.e.

$$y_1(t) = z_2(t) - z_1(t) + \eta_1(t), \qquad y_2(t) = \ddot{z}_2(t) + \eta_2(t), \tag{126}$$

where $\eta_1(t)$, $\eta_2(t)$ and $w(t)$ are mutually independent white noises with time depending intensities. The substitution of state variables to equations (126) leads to vector observation equation

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \boldsymbol{\eta}(t), \tag{127}$$

where $\mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$, $\boldsymbol{\eta} = \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix}^T$, $\mathbf{C}$ and $\mathbf{D}$ are constant matrices with appropriate dimensions.

To determine the optimal control Roh and Park [49] transformed the continuous time problem to the discrete time one.

At the end of this section we note that the recent deterministic and stochastic continuous time models considered in the open literature in the field of "Vehicle Dynamics" are particular cases of models presented by Balzer in 1981 [3], for instance in the case of linear stochastic models with incomplete observations the state and observation equations have the form

The dynamic system is described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{F}(t)z(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{128}$$

where $\mathbf{x}(t)$ is the state vector, $\mathbf{u}(t)$ the control vector, $\mathbf{w}(t)$ the disturbance vector , $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{u} \in \mathbf{R}^m$, $\mathbf{w} \in \mathbf{R}^p$, $\mathbf{A}(t)$, $\mathbf{B}(t)$, and $\mathbf{F}(t)$ are time depending matrices with appropriate dimensions. $\mathbf{x}_0$ is a stochastic process with the mean value $\bar{\mathbf{x}}_0$ and variance $\mathbf{P}_0$. The total disturbance vector $\mathbf{z}(t)$ consists of two parts: a previewable component $\mathbf{d}(t)$ and a vector zero mean stationary process (coloured noise) $\mathbf{w}(t)$. ie.

$$\mathbf{z}(t) = \mathbf{d}(t) + \mathbf{w}(t), \tag{129}$$

The coloured noise vector $\mathbf{w}(t)$ is modelled as the output of the shaping filter with white noise excitation

$$\dot{\mathbf{w}}(t) = \mathbf{A}_w\mathbf{w}(t) + \mathbf{G}_w\boldsymbol{\xi}(t), \tag{130}$$

where $\boldsymbol{\xi}(t)$ is assumed to be a zero mean Gaussian white noise vector with covariance matrix $\mathbf{Q}_\xi$, $\boldsymbol{\xi} \in \mathbf{R}^q$, $\mathbf{A}_w$, $\mathbf{G}_w$ and $\mathbf{Q}_\xi$ are constant matrices with appropriate dimensions.

The stochastic observation which described the road irregularities are given by

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) + \mathbf{E}(t)\mathbf{d}(t) + \boldsymbol{\eta}(t), \tag{131}$$

where $\mathbf{C}$, $\mathbf{D}(t)$, and $\mathbf{E}(t)$ are time depending matrices with appropriate dimensions, $\boldsymbol{\eta}(t)$ is assumed to be a zero mean Gaussian white noise vector with covariance matrix $\mathbf{Q}_\eta$ independent of $\boldsymbol{\xi}(t)$

the mean–square criterion has the form

$$I = \frac{1}{2}E\left\{\mathbf{x}^T(T)\mathbf{P}_T\mathbf{x}(T) + \frac{1}{2}\int\limits_{t_0}^{T} \left[\dot{\mathbf{x}}^T(t)\mathbf{Q}_1(t)\dot{\mathbf{x}}(t) + \mathbf{x}^T\mathbf{Q}_2(t)\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)\right]dt\right\} \tag{132}$$

where $\mathbf{Q}_1(t)$, $\mathbf{Q}_2(t)$ and $\mathbf{R}(t)$, are symmetric, time depending matrices of appropriate dimensions, $\mathbf{R}(t) > \mathbf{0}$, $\mathbf{Q}_1(t), \mathbf{Q}_2(t) \geq \mathbf{0}$.

An application of adaptive control to suspension system with preview information was considered very briefly by Esmailzadeh and Bateni [16] and the detailed analysis for nonlinear dynamic vehicle model with linear time invariant reference model during acceleration and decelleration maneuvers of the vehicle was given by Dukkipati and Vallurupalli [10]. The authors called their suspension system by "Smart Active Suspension".

5. Conclusions and future research

Based on the two or three decades of analytical developments it can be concluded that control algorithms developed in control and system theory have been systematically implemented to active suspension systems. However, the research was given for simple systems (a few-DOF models). A comparison study given in section 2.2.2 shows that the difference between performance indeeces obtained by control algorithms with different linearization techniques are small for considered sets of parameters. A significant improvement of performance indeeces obtained as well by control algorithms with preview information as by adaptive control algorithms has been observed in the literature. Slow-active suspensions are capable of a good performance over wide ranges of operating conditions. However, their behaviours following extreme event excitations is not fully understood. Based on already developed algorithms and implemented for active suspension systems one may expect that the future theoretical research can be focused on the following theoretical models:

- Optimal and adaptive control of linear models with parametric and external excitations;
- Optimal control of linear and nonlinear models with non-Gaussian (continuous or impulse) external excitations;
- Optimal control of linear and nonlinear models with multi-objective performance criteria of safety and passengers comfort;
- Adaptive control of suspension models with Gaussian and non-Gaussian external excitations.

Also other approaches are expected to be developed in the context of active suspension systems, namely:

- Information from multi-preview sensors;
- Neural Network identification and control;
- Combination of different types of suspension systems;
- Smart identification and choice of different types of suspension systems (algorithms and programs) including different area of interest, for instance, anti-lock brake and anti-slip control, four wheel steering systems.

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