Controllability and Stabilization

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Topics on deterministic and stochastic controllability and stabilization of distributed parameter systems: theory and numerical approximations

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The aim of these notes is to discuss selected mathematical problems, related to exact and approximate controllability as well as stabilization of systems with distributed parameters. The presentation covers wave equations, elastic solids and structures like plates and shells, Stokes, Navier–Stokes, Boussinesq and Burgers equations. Numerical methods and stochastic controllability are also discussed.

1. Introduction

One can distinguish three approaches to problems of controllability. The first approach is typical for engineering literature where one is interested in solving particular problems of controllability or stabilization by using specific sensors and/or actuators. The papers contained in this volume are mostly of this type, cf. also [71, 177, 178].

The second approach is highly theoretical, the so-called "control-theoretic approach". Here it suffices to mention the books by Avdonin and Ivanov [10, 11] Fattorini [49], Klamka [82], Lasiecka and Triggiani [127] and Zabczyk [193], cf. also [147, 174].

In these notes we follow the third approach, which combines mathematical rigour and elegance with applicability to solving controllability problems arising in physics and applied mechanics. Our study is focused on infinite-dimensional systems or distributed parameter systems. Typical for the third approach are the books by Banks [14], Banks *et al.* [15], Komornik [94], Lagnese [103], Lagnese and Lions [109], Lasiecka [118], Lions [144, 145], de Queiroz *et al.* [167].

It is not possible to consider all fundamental aspects of controllability of practically important systems in this paper of limited number of pages. However, we shall try to convey those ideas which are of interest to mechanical community.

Applied controllability pertains to:

1. Linear and nonlinear wave equations, cf. Sections 2 and 3,
2. Schrödinger equation, cf. [131],
3. Maxwell's equations [93, 94, 104],

4. Stokes and Navier–Stokes equations (control of turbulence), cf. Sec. 4,
5. Combustion, e.g. maximization of turbulence, cf. Sec. 4,
6. Korteweg–de Vries, Boussinesq and Burgers equations, cf. Sections 4, 9.3 and [172, 189],
7. Structural acoustic (fluid–structure interactions), cf. Lasiecka [118],
8. Dynamic elasticity and viscoelasticity , cf. Sec. 5 and [25, 139, 140, 141, 194],
9. Structures: beams, membranes, plates, shells and junctions, cf. Sec. 7, 8 and [41, 125, 142, 185],
10. Linear and nonlinear diffusion equations, cf. Glowinski and Lions [66, 67, 81, 133, 149, 194],
11. Thermoelasticity and thermoviscoelasticity including structures, cf. [119, 135, 136, 152, 190, 194],
12. Problems involving a small parameter and homogenization, cf. [13, 32, 33, 34, 54, 146, 183, 187],
13. Emerging theory: formation theory, cf. Renardy and Russell [170].

In all these a control may act on a part of the body (distributed control) or on a part of the boundary (boundary control). In the subsequent chapters both types of control will be discussed. In these lecture notes we shall only discuss some controllability problems related to points 1, 4, 5, 6, 8, 9. Controllability of stochastic systems will also be investigated. The book [180] will be devoted to a complete treatment of applied controllability problems.

2. One-dimensional wave equation (vibrating string)

Let us start with a simple example of one-dimensional wave equation where the material coefficient is equal to 1. Let $I = (a, b)$ be a bounded interval, $T > 0$. Consider the problem of small transversal vibrations of a string:

$$u_{tt}(x,t) - u_{xx}(x,t) = 0, \quad (x,t) \in I \times (0,T), \tag{1}$$

$$u(a,t) = v_a(t) \quad \text{and} \quad u(b,t) = v_b(t), \quad t \in [0,T], \tag{2}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x), \quad x \in I. \tag{3}$$

Here $u_{tt} = \ddot{u} = \frac{\partial^2 u}{\partial x^2}$. The problem (1)-(3) is said to be *exactly controllable* if for "arbitrarily" given initial state (u_0, u_1) there exist suitable control functions v_a and v_b such that the solution of (1)-(3) satisfies

$$u(x,T) = u_t(x,T) = 0, \quad x \in I, \tag{4}$$

where $u_t = \dfrac{\partial u}{\partial x}$. The solution of Eqs. (1)-(3) is a function

$$u \in C\big([0,T], H^1(I)\big) \cap C^1\big([0,T], L^2(I)\big) \tag{5}$$

satisfying Eq. (1) in the distributional sense, the equalities (2) pointwise and the equalities (3) almost everywhere (a.e.). The definitions of the relevant function spaces are given in the books by Adams [4], Kufner *et al.* [101].

The exact controllability theorem is formulated as follows.

Theorem 1: *Let $T = b - a$ and let $(u_0, u_1) \in H^1(I) \times L^2(I)$ be such that*

$$u_0(a) + u_0(b) + \int_a^b u_1(s)\, ds = 0.$$

Then there is a unique choice of functions

$$v_a,\, v_b \in H^1(0,T)$$

such that the solution of Eqs. (1)-(3) satisfies (4). The functions v_a, and v_b are given by

$$2v_a(t) = u_0(a+t) + u_0(a) + \int_a^{a+t} u_1(s)\, ds,$$

$$2v_b(t) = u_0(b-t) + u_0(b) + \int_{b-t}^{b} u_1(s)\, ds.$$

Moreover, the solution u has the following property

$$u(a,t) + u(b,t) + \int_a^b u_t(x,t)\, dx = 0, \quad \forall\, t \in [0,T].$$

The proof of Komornik [94] of the above theorem exploits d'Alembert's formula for the solutions of Eq. (1).

Remark 1.

(i) The problem (1)-(3) is also exactly controllable if $T > b - a$, however not if $T < b - a$.

(ii) Similar exact controllability theorem holds for the general case:

$$u(x,T) = u_T^0(x), \qquad u_t(x,T) = u_T^1(x).$$

Now the final state is prescribed by the functions $(u_T^0, u_T^1) \in H^1(I) \times L^2(I)$.

(iii) The feedback law

$$(u_x - u_t)(a,t) = (u_x + u_t)(b,t) = 0, \quad t \in \mathbb{R}_+ \tag{6}$$

ensures exact controllability of the equation

$$u_{tt}(x,t) - u_{xx}(x,t) = 0, \quad (x,t) \in I \times \mathbb{R}_+ \tag{7}$$

satisfying (3). Indeed, it can be shown that the solution of Eqs. (1), (6) and (7) satisfies

$$u(x,t) = 0 \quad \forall (x,t) \in I \times [b-a, +\infty].$$

Krabs [99] studied L^p-controllability of a vibrating string in the case where $(2)_1$ has simpler form

$$u(a,t) = 0, \quad t \in [0,T].$$

This author assumes weaker assumption on the initial data:

$$(u_0, u_1) \in L^p(a,b) \times W^{-1,p}(0,1).$$

Krabs [98] (see also [180]) used *moment equations* to study the following problem of a vibrating string

$$u_{xx}(x,t) - u_{tt}(x,t) = v(x,t), \quad (x,t) \in (0,1) \times (0,T), \tag{8}$$

$$\begin{aligned} &u(0,t) = u(1,t) = 0, \quad t \in [0,T], \\ &u(x,0) = x(1-x), \qquad u_t(x,0) = 0, \quad x \in [0,1]. \end{aligned} \tag{9}$$

The *distributed control function* v can be chosen in $L^\infty\big([0,T], L^2(0,1)\big)$. The requirement of exact controllability (4) ($x \in [0,1]$) can now be shown to be equivalent to finding a sequence $\{v_j\}_{j\in\mathbb{N}}$ in $L^\infty([0,T], L^2)$ such that

$$v_{2i} = 0 \quad \text{a.e. for all } i \in \mathbb{N} \tag{10}$$

and

$$\begin{aligned} &\int_0^T v_{2i-1}(t)\cos(2i-1)\pi t\, dt = 0, \\ &\int_0^T v_{2i-1}(t)\sin(2i-1)\pi t\, dt = \frac{4\sqrt{2}}{[(2i-1)\pi]^2}, \end{aligned} \tag{11}$$

for all $i \in \mathbb{N}$. The control function is then given by

$$v(x,t) = \sqrt{2}\sum_{i=1}^{\infty} v_{2i-1}(t)\sin(2i-1)\pi x$$

and (4) is satisfied.

Krabs [98] elaborated a general method of finding a sequence $\{f_j\}_{j\in\mathbb{N}}$ functions $f_j \in L^\infty[0,T]$ satisfying an infinite sequence of moment equations of the form

$$\begin{aligned} &\int_0^T f_j(t)\cos\sqrt{\lambda_j}t\, dt = c_j^1, \\ &\int_0^T f_j(t)\sin\sqrt{\lambda}t\, dt = c_j^2, \end{aligned} \tag{12}$$

for all $j \in \mathbb{N}$, where the sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ of positive reals, is increasing, and satisfies $\lim_{j\to\infty} \lambda_j = \infty$. The sequences $\{c_j^\alpha\}_{j\in\mathbb{N}}$ ($\alpha = 1,2$) are in l^2. It can be shown that the problem of finding a sequence $f = \{f_j\}_{j\in\mathbb{N}}$ in $L^\infty[0,T]$ which satisfies (2.12) and which minimizes

$$\gamma_T(f) = \text{ess}\sup_{t\in[0,T]}\left(\sum_{j=1}^{\infty} f_j^2(t)\right)^{1/2} \tag{13}$$

is solvable. Let now $\gamma_T(f)$ be replaced by

$$\Delta_T(f) = \left[\sum_{j=1}^{\infty}(\text{ess}\sup_{t\in[0,T]}|f_j(t)|)^2\right]^{1/2}. \tag{14}$$

Necessary and sufficient conditions were given for the sequence f_j which minimizes $\Delta_T(f)$ also to minimize $\gamma_T(f)$ and to satisfy $\gamma_T(f) = \Delta_T(f)$. In the particular case of the vibrating string

$$\begin{aligned} c_j^1 &= 0 \quad \text{for all } j \in \mathbb{N}, \\ c_{2i}^2 &= 0 \quad \text{for all } i \in \mathbb{N}, (10) \text{ is satisfied}, \\ c_{2i-1}^2 &= \frac{4\sqrt{2}}{[(2i-1)\pi]^2} \quad \text{for all } i \in \mathbb{N}, \\ \lambda_j &= (j\pi)^2 \quad \text{for all } j \in \mathbb{N}. \end{aligned}$$

Numerical results of an approximation of Δ_T and γ_T were also given.

The same author Krabs [98] considered also boundary control of vibrations. An extension to boundary null-controllability of the motion of a one-dimensional medium satisfying a differential equation of the form

$$u_{tt}(x,t) + Lu(x,t) = 0 \tag{15}$$

was investigated by Krabs [98]. Here L is a linear differential operator of order $2n$ with respect to x whose coefficients are time-independent. The control is applied to one of the boundary conditions and the control functions is allowed to vary in

$$V_0^{1,p} = \{v : [0,T] \to \mathbb{R} \mid v_t \in L^p[0,T],\ v(0) = 0\} \quad \text{for } p \in [2,\infty].$$

It is shown that the problem of null-controllability is equivalent to a trigonometric moment problem in $L^p[0,T]$.

Avdonin *et al.* [12] considered several controllability/observability problems for the string equation and reduced the problem to question concerning Riesz bases of exponentials in Sobolev spaces, cf. also Avdonin and Ivanov [10, 11]. Avdonin *et al.* [12] studied: the regular string equation, a string with piecewise constant density and the system of two string equations with unit density.

Datko (see [180]) investigated the problem of stabilization of one-dimensional wave equation and Euler–Bernoulli beam equation provided that small time delays are introduced into velocity feedbacks, see also [39, 40]. Consider the system (the one-dimensional wave equation)

$$
\begin{aligned}
& u_{tt} - u_{xx} = 0, \qquad x \in (0,1), \quad t > 0, \\
& u(0,t) = 0, \\
& u_x(1,t) = -K u_t(1, t-h),
\end{aligned} \tag{16}
$$

where $K > 0$ and $h \geq 0$ are fixed. The constant K is called a gain. If $h = 0$, the system (15) is uniformly exponentially stable and the term $-Ku_t(1,t))$ is considered a velocity feedback, cf. Komornik [94], Lions [144, 147]. Datko (see [180]) proved the following theorem.

Theorem 2: *Let $K > 1$ in (16)$_3$. Then, given any $R_0 > 0$ and $h_0 > 0$, there exist $R_1 > R_0$ and $0 < h_1 < h_0$ such that when $h = h_1$ the system (16) has a nontrivial solution of the form*

$$u(x,t) = e^{\lambda t} u_0(x) \tag{17}$$

where $Re(\lambda) = R_1$.

Obviously $Re(\lambda)$ denotes the real part of λ. The proof exploits the fact that the function

$$u(x,t) = e^{-\lambda h} \sinh \lambda x$$

satisfies $(16)_{1,2}$. Satisfaction of $(16)_3$ leads to the following condition for λ:

$$-\frac{1}{K} = e^{-\lambda h} \tanh \lambda.$$

Using now some asymptotic properties of the hyperbolic function $\tanh \lambda$, after some calculation the result follows.

Datko (see [180]) argues that result such as the last theorem is paradoxical since if the gain near $K = 1$ varies slightly to the right or left of one and $h \to 0^+$ the system has radically different qualitative behaviour on both sides of one.

Lions [144] solved the problem of pointwise controllability of the wave equation (the space dimension $n = 1$) by using the Hilbert Uniqueness Method (HUM). The same problem was reexamined by Fabre and Puel [46], see also the references therein. However, the point of departure is different. These authors start with the solutions of exact controllability problems when the controls have their supports in an interval $(a, a+\varepsilon)$, where a is any fixed point of $(0,\pi)$, i.e. the domain is $\Omega = (0,\pi)$. Fabre and Puel [46] carefully studied the limit passage when ε tends to zero. The space if initial data which are exactly pointwise controllable was specified. The point a is not arbitrary, cf. Lions [144, p. 425].

3. Wave equations

From the viewpoint of applied controllability as we understand it here, linear and nonlinear wave equations have been the most frequently studied, cf. Telega [180]. In the present section we shall introduce the Hilbert Uniqueness Method (HUM), devised by Lions [144] and next briefly comment on other developments. A numerical algorithm will also be discussed.

3.1. Exact controllability of linear wave equation

Let $\Omega \subset \mathbb{R}^n$ be a bounded sufficiently regular domain and $\Gamma = \partial\Omega$ its boundary. Fix $T > 0$ and consider the problem

$$\begin{aligned} \ddot{u} - \Delta u + qu = 0 \quad & \text{in } \Omega \times (0,T), \\ u = v \quad & \text{on } \Gamma \times (0,T), \\ u(0) = u^0, \quad & \dot{u}(0) = u^1. \end{aligned} \tag{18}$$

Here $\ddot{u} = \frac{du}{dt}$; etc. We will often write $u(t)$ instead of $u(x,t)$ cf. Adams [4], Kufner *et al.* [101]. It means that we set $u(t) = \{u(x,t)|x \in \Omega\}$. Problem (18) is slightly more general than one originally studied by Lions [144], where $q = 0$. Here $q : \Omega \to \mathbb{R}$ is a *nonnegative* function, cf. Komornik [94]. More precisely, we assume that

$$q \in L^\infty(\Omega). \tag{19}$$

It can be shown that for any given $u^0 \in L^2(\Omega)$, $u^1 \in H^{-1}(\Omega)$ and $v \in L^2(0,T;L^2(\Gamma))$ problem (18) has a unique solution $(u,\dot{u}) \in L(0,T;L^2(\Omega) \times H^{-1}(\Omega))$.

Definition 1: *Problem (18) is* exactly controllable *if for any given* $(u^0,u^1),(u^0_T,u^1_T) \in L^2(\Omega) \times H^{-1}(\Omega)$ *there exists* $v \in L^2(0,T;L^2(\Gamma))$ *such that (18) satisfies*

$$u(T) = u^0_T \quad \textit{and} \quad \dot{u}(T) = u^1_T. \tag{20}$$

Fix now $\mathbf{x}^0 \in \mathbb{R}^n$ arbitrarily and set: $\tilde{q} = \sup_\Omega q$ and

$$q_1 = \begin{cases} 2R\dfrac{\tilde{q}}{\sqrt{\lambda_1}}, & \text{if } n \geq 2, \\ 2R\dfrac{\tilde{q}}{\sqrt{\lambda_1}} + \dfrac{\tilde{q}}{\lambda_1}, & \text{if } n = 1, \end{cases}$$

where

$$\mathbf{m}(\mathbf{x}) := \mathbf{x} - \mathbf{x}^0, \quad \mathbf{x} \in \mathbb{R}^n, \tag{21}$$

$$R = R(\mathbf{x}^0) := \sup\{|\mathbf{x} - \mathbf{x}^0| : \mathbf{x} \in \Omega\}, \tag{22}$$

$$\Gamma(\mathbf{x}^0) := \{\mathbf{x} \in \Gamma : \mathbf{m}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) > 0\}, \tag{23}$$

$$\Gamma_*(\mathbf{x}^0) = \Gamma \setminus \Gamma(x^0) := \{\mathbf{x} \in \Gamma : \mathbf{m}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \leq 0\}. \tag{24}$$

Here $\mathbf{n}(\mathbf{x}) = (n_i(\mathbf{x}))$ denotes the outward unit normal vector to Γ. The constant λ_1 is the biggest constant such that

$$\int_\Omega \left(|\nabla w|^2 + q|w|^2\right) dx \geq \lambda_1 \int_\Omega |v|^2 dx, \quad \forall v \in H^1_0(\Omega). \tag{25}$$

The energy of system (18) is given by

$$E(t) = \frac{1}{2}\int_{\Omega} \left(|\dot{u}|^2 + |\nabla u|^2 + q|u|^2\right) dx. \tag{26}$$

It can easily be shown that this energy does not depend on t, i.e. $dE/dt = 0$. Prior to the formulation of the exact controllability theorem we introduce indispensable lemmas.

Lemma 1: *Consider the problem*

$$\begin{aligned} \ddot{y} - \Delta y + qy = 0 \quad & \textit{in } \Omega \times \mathbb{R}, \\ y = 0 \quad & \textit{on } \Gamma \times \mathbb{R}, \\ y(0) = y^0 \quad \textit{and} \quad \dot{y}(0) = y^1. & \end{aligned} \tag{27}$$

Then the so-called direct inequality *holds true for every interval I:*

$$\int_I \int_{\Gamma} |\partial_{\mathbf{n}} y|^2 d\Gamma dt \le cE, \tag{28}$$

where c is a positive constant and

$$E(t) = \frac{1}{2}\int_{\Omega} \left(|\dot{y}|^2 + |\nabla y|^2 + q|y|^2\right) dy.$$

Proof. The proof relies on the following identity, cf. Komornik [94],

$$\begin{aligned} \int_S^T \int_{\Gamma} \left[2(\partial_{\mathbf{n}} y)\mathbf{h}\cdot\nabla y + (\mathbf{h}\cdot\mathbf{n})(\dot{y})^2 - (\mathbf{h}\cdot\mathbf{n})|\nabla y|^2\right] d\Gamma dt = \left(\int_{\Omega} 2\dot{y}\mathbf{h}\cdot\nabla y dx\right)\Bigg|_S^T \\ + \int_S^T \int_{\Omega} \left\{(\operatorname{div}\mathbf{h})\left[(\dot{y})^2 - |\nabla y|^2\right] + 2qy\mathbf{h}\cdot\nabla y + 2(\partial_i h_j)(\partial_i y)(\partial_j y)\right\} dx dt. \end{aligned} \tag{29}$$

Here $\mathbf{h} : \bar{\Omega} \to \mathbb{R}^n$ is a vector field of class C^1 and $-\infty < S < T < \infty$. Obviously, the dot denotes the scalar product in $\mathbb{R}^n$. To prove the last identity it suffices to use the *multiplier method*, cf. [94]. More precisely, Eq. (27) is multiplied by $2\mathbf{h}\cdot\nabla y = 2h_j\partial_j y$ and integrated by parts (we apply the summation convention).

Since the energy is conserved therefore

$$||y(t)||_V^2 + ||\dot{y}(t)||_H = ||y^0||_V^2 + ||y^1||_H^2 \tag{30}$$

where $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$.

Applying identity (29) with $S = -T$ and with the vector field $\mathbf{h}$ such that $\mathbf{h} = \mathbf{n}$ on Γ, the l.h.s. of (29) becomes

$$\int_{-T}^T \int_{\Gamma} |\partial_{\mathbf{n}} y|^2 d\Gamma dt.$$

Since $\mathbf{h}$ is of class C^1, there exists a constant $c_1 > 0$ such that

$$|\mathbf{h}(x)| \leq c_1 \quad \text{and} \quad |\partial_i h_j(x)| \leq c_1, \quad \forall x \in \bar{\Omega}.$$

Using (30), the last inequalities and recalling that $q \in L^\infty(\Omega)$ is a nonnegative function we readily arrive at

$$\int_{-T}^{T}\int_{\Gamma_0} |\partial_{\mathbf{n}} y|^2 d\Gamma dt \leq c_2 \left(||y^0||_V^2 + ||y^1||_H^2\right) \tag{31}$$

and consequently at (28) with a suitable posi tive constant c. Indeed, we have

$$\int_\Omega q|y|^2 dy \leq c_3 \int_\Omega |y|^2 dy = c_3||y||_H^2, \quad \forall y \in L^2(\Omega).$$

We recall that $\partial_{\mathbf{n}} y = \frac{\partial y}{\partial \mathbf{n}}$ denotes the normal derivative of a function y. □

Lemma 2: *Assume that*

$$q_1 < 1 \tag{32}$$

and let I be an interval of length

$$|I| > 2R(1 - q_1).$$

Then there is a constant $c' > 0$ such that

$$\int_I \int_\Gamma |\partial_{\mathbf{n}} y|^2 d\Gamma_m dt \geq c' E, \quad \forall (y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega) \tag{33}$$

where $d\Gamma_m = (\mathbf{m} \cdot \mathbf{n}) d\Gamma$.

Proof. First we observe that (33) implies the inverse inequality

$$\int_I \int_\Gamma |\partial_{\mathbf{n}} y|^2 d\Gamma dt \geq \tilde{c} E \tag{34}$$

with $\tilde{c} = c'/R$.

We pass to the proof of (33). Applying identity (29) with $\mathbf{h} = \mathbf{m}$ and knowing that $\partial_i m_j = \delta_{ij}$, $\operatorname{div} \mathbf{m} = \mathbf{n}$ we get

$$\int_S^T \int_\Gamma \left\{2(\partial_{\mathbf{n}} y)\mathbf{m} \cdot \nabla y + (\mathbf{m} \cdot \mathbf{n}) \left[(\dot{y})^2 - |\nabla y|^2\right]\right\} d\Gamma dt = \left(\int_\Omega 2\dot{y}\mathbf{m} \cdot \nabla y dx\right)\Bigg|_S^T$$

$$+ \int_S^T \int_\Omega \left[n(\dot{y})^2 + (2 - n)|\nabla y|^2 + 2qy\mathbf{m} \cdot \nabla y\right] dx dt. \tag{35}$$

Now we multiply Eq. $(27)_1$ by y and integrate by parts. We get

$$\int_S^T \int_\Gamma y \partial_{\mathbf{n}} y d\Gamma dt = \left(\int_\Omega y \dot{y} dx \right) \Big|_S^T + \int_S^T \int_\Omega \left[-(\dot{y})^2 + |\nabla y|^2 + q y^2 \right] dx dt. \tag{36}$$

Putting

$$My := 2\mathbf{m} \cdot \nabla y + (n-1) y$$

from (35) and (36) we conclude that

$$\int_S^T \int_\Gamma \left[(\partial_{\mathbf{n}} y) My + (\mathbf{m} \cdot \mathbf{n})((\dot{y})^2 - |\nabla y|^2) \right] d\Gamma dt = \left(\int_\Omega \dot{y} My dx dt \right) \Big|_S^T$$
$$+ \int_S^T \int_\Omega \left[(\dot{y})^2 + |\nabla y|^2 + (n-1) q y^2 + 2 q y \mathbf{m} \cdot \nabla y \right] dx dt. \tag{37}$$

Using the expression for E we write (37) in the form

$$\int_S^T \int_\Gamma \left[(\partial_{\mathbf{n}} y) My + (\mathbf{m} \cdot \mathbf{n})((\dot{y})^2 - |\nabla y|^2) \right] d\Gamma dt = \left(\int_\Omega \dot{y} My dx dt \right) \Big|_S^T$$
$$+ 2|I|E + \int_S^T \int_\Omega \left[(n-2) q y^2 + 2 q y \mathbf{m} \cdot \nabla y \right] dx dt. \tag{38}$$

Since $y = 0$ on Γ therefore $y = \dot{y} = 0$ and $\nabla y = (\partial_{\mathbf{n}} y)\mathbf{n}$ on Γ. This permits us to reduce the l.h.s. of (38) to

$$\int_I \int_\Gamma |\partial_{\mathbf{n}} y|^2 d\Gamma_m dt.$$

On the other hand, using (25) the last integral in (38) is bounded from below by $-2|I| q_1 E$. Indeed, the case $q \equiv 0$ is trivial. If $q \neq 0$ and $n \geq 2$, then we have

$$\int_\Omega \left[(n-2) q y^2 + 2 q y \mathbf{m} \cdot \nabla y \right] dx \geq -2 R q_1 ||y||_H ||\nabla y||_H$$
$$\geq -q_1 \int_\Omega \left(|\nabla y|^2 + q y^2 \right) dx \geq -2 q_1 E.$$

If $q \neq 0$ and $n = 1$, then we have

$$\int_\Omega \left[(n-2) q y^2 + 2 q y \mathbf{m} \cdot \nabla y \right] dx \geq \int_\Omega \left(-q_1 y^2 + 2 q y \mathbf{m} \cdot \nabla y \right) dx.$$

Repeating the above computation and using the definition of q_1 for $n = 1$, this integral is again bounded from below by $-2q_1 E$. Consequently, we get the following inequality

$$\int_I \int_\Gamma |\partial_{\mathbf{n}} y|^2 d\Gamma_m dt \geq 2|I|(1-q_1)E + \left(\int_\Omega \dot{y} M y dx \right) \Big|_S^T . \tag{39}$$

To bound from below the last term of (39) we shall use the next lemma.

Lemma 3: *The solution of (27) satisfies the estimate*

$$\left| \int_\Omega \dot{y} M y dx \right| \leq 2RE, \quad \forall\, t \in \mathbb{R}.$$

Proof. First we show that

$$||My||_{L^2(\Omega)} \leq ||2\mathbf{m} \cdot \nabla y||_{L^2(\Omega)}. \tag{40}$$

Indeed, the application of Green's formula yields

$$\begin{aligned}
||My||^2_{L^2(\Omega)} - ||2\mathbf{m} \cdot \nabla y||^2_{L^2(\Omega)} &= ||2\mathbf{m} \cdot \nabla y + (n-1)y||^2_{L^2(\Omega)} - ||2\mathbf{m} \cdot \nabla y||^2_{L^2(\Omega)} \\
&= \int_\Omega \left\{ |2\mathbf{m} \cdot \nabla y + (n-1)y|^2 - |2\mathbf{m} \cdot \nabla y|^2 \right\} dx \\
= \int_\Omega \left[(n-1)^2 y^2 + 4(n-1) y \mathbf{m} \cdot \nabla y \right] dx &= \int_\Omega \left[(n-1)^2 y^2 + 2(n-1)\mathbf{m} \cdot \nabla (y)^2 \right] dx \\
&= 2(n-1) \int_\Gamma (\mathbf{m} \cdot \mathbf{n}) y^2 d\Gamma + \int_\Omega \left[(n-1)^2 y^2 - 2(n-1)(\operatorname{div} \mathbf{m}) y^2 \right] dx \\
&= 2(n-1) \int_\Gamma (\mathbf{m} \cdot \mathbf{n}) y^2 d\Gamma + (1-n^2) \int_\Omega y^2 dx = (1-n^2) \int_\Omega y^2 dx \leq 0.
\end{aligned}$$

Here we used the fact that $\partial_i m_j = \delta_{ij}$, $\operatorname{div} \mathbf{m} = n$ and $y = 0$ on Γ. From (40) and the definition of energy it follows that

$$\begin{aligned}
\left| \int_\Omega y' M y dx \right| &\leq ||\dot{y}||_{L^2(\Omega)} ||My||_{L^2(\Omega)} \leq ||\dot{y}||_{L^2(\Omega)} ||2\mathbf{m} \cdot \nabla y||_{L^2(\Omega)} \\
&\leq R||\dot{y}||^2_{L^2(\Omega)} + \frac{1}{R} ||\mathbf{m} \cdot \nabla y||^2_{L^2(\Omega)} = \int_\Omega \left(R|\dot{y}|^2 + R^{-1} |\mathbf{m} \cdot \nabla y|^2 \right) dx \\
&\leq R \int_\Omega \left(|\dot{y}|^2 + |\nabla y|^2 \right) dx \leq 2RE.
\end{aligned}$$

Thus the proof is complete. □

Applying the last lemma with $t = S$ and $t = T$ from (39) we conclude that

$$\int_I \int_\Gamma |\partial_{\mathbf{n}} y|^2 d\Gamma_m dt \geq 2|I|(1-q_1)E - 4RE,$$

and inequality (33) follows with

$$c' = 2|I|(1-q_1) - 4R.$$

Remark 2. Inequality (39) implies the following uniqueness result: If the solution of (27) satisfies the condition $\partial_{\mathbf{n}} y = 0$ on $\Gamma(\mathbf{x}^0)$ with $|I| > 2R(1-q_1)$ then $y^0 = y^1 = 0$ and consequently $y \equiv 0$ on $\Gamma \times \mathbb{R}$.

The same inequality shows that the "observation" of $\partial_{\mathbf{n}} y = 0$ on $\Gamma(\mathbf{x}^0) \times I$ permits one to distinguish the initial data provided I is sufficiently large.

Now we are in a position to formulate the exact controllability theorem.

Theorem 3: *Assume that*

$$q_1 < 1 \tag{41}$$

and let

$$T > \frac{2R}{1-q_1}. \tag{42}$$

Then for any given $(u^0, u^1), (u^0_T, u^1_T) \in L^2(\Omega) \times H^{-1}(\Omega)$ *there exists* $v \in L^2(0,T; L^2(\Gamma))$ *such that*

$$v = 0 \quad a.e.\ on\ \Gamma_* \times (0,T) \tag{43}$$

and the solution of (18) satisfies

$$u(T) = u^0(T) \quad and \quad \dot{u}(T) = u^1_T. \tag{44}$$

Sketch of the proof. To prove this theorem we use the HUM. In this case the procedure runs as follows. Let us consider the solution of the problem

$$\begin{aligned} \ddot{u}_1 - \Delta u_1 + q u_1 = 0 \quad &\text{in} \quad \Omega \times (0,T), \\ u_1 = 0 \quad &\text{on} \quad \Gamma \times (0,T), \\ u_1(T) = u^0_T \quad \text{and} \quad \dot{u}_1(T) = u^1_T.& \end{aligned}$$

Assume that there exists a unique function $v \in L^2(0,T; L^2(\Gamma))$ satisfying $v = 0$ (a.e.) on $\Gamma_* \times (0,T)$ and such that the solution of the problem

$$\begin{aligned} \ddot{u}_2 - \Delta u_2 + q u_2 = 0 \quad &\text{in} \quad \Omega \times (0,T), \\ u_2 = v \quad &\text{on} \quad \Gamma \times (0,T), \\ u_2(0) = u^0 - u_1(0) \quad \text{and} \quad \dot{u}_2(T) = u^1 - \dot{u}_1(0),& \end{aligned}$$

satisfies $u_2(T) = 0$, $\dot{u}_2(T) = 0$. Then $u := u_1 + u_2$ is a solution of (18) and it satisfies the following condition:

$$u(T) = u^0(T) \quad \text{and} \quad \dot{u}(T) = u^1_T.$$

Consequently, it suffices to prove Theorem 3 in the special case where $u^0_T = u^1_T = 0$ (exact null controllability).

Now we can pass to the essence of the HUM. Consider the *homogeneous* problem:

$$\begin{aligned} \ddot{\phi} - \Delta\phi + q\phi = 0 \quad &\text{in} \quad \Omega \times (0,T), \\ \phi = 0 \quad &\text{on} \quad \Gamma \times (0,T), \\ \phi(0) = \phi^0 \quad \text{and} &\quad \dot{\phi}(0) = \phi^1, \end{aligned} \tag{45}$$

where $(\phi^0, \phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$. It is known that problem (45) has a unique solution. Moreover, $\partial_{\mathbf{n}}\phi \in L^2(0,T;L^2(\Gamma))$ and the linear map $(\phi^0,\phi^1) \to \partial_{\mathbf{n}}\phi$ is continuous from $H_0^1(\Omega) \times L^2(\Omega)$ into $L^2(0,T;L^2(\Gamma))$, cf. [94, 144].

Consider now the *backward* problem:

$$\begin{aligned} &\ddot{\Psi} - \Delta\Psi + q\Psi \quad \text{in} \quad \Omega \times (0,T), \\ &\Psi = \begin{cases} \dfrac{\partial\phi}{\partial\mathbf{n}} & \text{on} \quad \Gamma(\mathbf{x}^0) \times (0,T), \\ 0 & \text{on} \quad \Sigma_*; \end{cases} \\ &\Psi(T) = 0 \quad \text{and} \quad \dot{\Psi}(T) = 0. \end{aligned} \tag{46}$$

We recall that $\Sigma_* = (\Gamma \setminus \Gamma(\mathbf{x}_0)) \times (0,T)$. The backward problem has a unique solution satisfying $(\Psi(0), \dot{\Psi}(0)) \in L^2(\Omega) \times H^{-1}(\Omega)$ and the linear map $(\phi^0,\phi^1) \to (\Psi(0), \dot{\Psi}(0))$ is continuous from $H_0^1(\Omega) \times L^2(\Omega)$ into $L^2(\Omega) \times H^{-1}(\Omega)$.

Set now $u = \Psi$. Then since $\Psi(T) = 0$ and $\dot{\Psi}(T) = 0$ was imposed the control function

$$v = \begin{cases} \dfrac{\partial\phi}{\partial\mathbf{n}} & \text{on} \quad \Gamma(\mathbf{x}^0) \times (0,T), \\ 0 & \text{on} \quad \Sigma_*, \end{cases} \tag{47}$$

drives the system (18) to rest. We recall that we assumed $u_T^0 = 0$ and $u_T^1 = 0$. Hence we conclude that the controllability theorem will be proved if we show that the map

$$H_0^1(\Omega) \times L^2(\Omega) \ni (\phi^0,\phi^1) \to (u(0), \dot{u}(0)) \in L^2(\Omega) \times H^{-1}(\Omega)$$

is surjective. It is more convenient to investigate the surjectivity of the map

$$\Lambda : H_0^1(\Omega) \times L^2(\Omega) \to H^{-1}(\Omega) \times L^2(\Omega)$$

defined by

$$\Lambda(\phi^0,\phi^1) := (\dot{u}(0), -u(0)).$$

We have

Lemma 4: *Under the assumption (41) and (42) the operator Λ is an isomorphism of $H_0^1(\Omega) \times L^2(\Omega)$ onto $H^{-1}(\Omega) \times L^2(\Omega)$.*

Proof. It is evident that Λ is a bounded linear mapping. By Lax–Milgram lemma (see Yosida [188]), it is sufficient to show the existence of a constant $c > 0$ such that

$$\left\langle \Lambda(\phi^0,\phi^1), (\phi^0,\phi^1) \right\rangle_{F' \times F} \geq c\|(\phi^0,\phi^1)\|_F^2 \tag{48}$$

where we set $F = H_0^1(\Omega)\times L^2(\Omega)$. The last inequality is to be valid for every $(\phi^0,\phi^1)\in F$. Since $\Lambda : F \to F'$ is continuous, it is sufficient to prove this inequality for $\phi^0,\phi^1 \in Z$, where $Z\times Z$ is a dense subset in F.

Multiplying Eq. $(46)_1$ by ϕ where $\Psi = u$ and integrating by parts we get

$$0 = \int_0^T\int_\Omega \phi(\ddot{u} - \Delta u + qu)dxdt = \left[\int_\Omega(\phi\dot{u} - \dot{\phi}u)dx\right]\Bigg|_0^T + \int_0^T\int_\Omega(\ddot{\phi} - \Delta\phi + q\phi)udxdt$$

$$+\int_0^T\int_\Gamma(-\phi\partial_{\mathbf{n}}u + u\partial_{\mathbf{n}}\phi)d\Gamma dt = \int_\Omega(-\phi^0\dot{u}(0) + \phi^1 u(0))dx + \int_0^T\int_{\Gamma(\mathbf{x}^0)}|\partial_{\mathbf{n}}\phi|^2 d\Gamma dt.$$

Hence

$$\left\langle\Lambda(\phi^0,\phi^1),(\phi^0,\phi^1)\right\rangle_{F'\times F} = \int_0^T\int_{\Gamma(\mathbf{x}^0)}|\partial_{\mathbf{n}}\phi|^2 d\Gamma dt.$$

Due to (41) and (42) we may apply Lemma 2. We conclude that

$$\left\langle\Lambda(\phi^0,\phi^1),(\phi^0,\phi^1)\right\rangle_{F'\times F} \geq c'E$$

with a positive constant $c' = c'(T)$. Using the definition of the energy E we prove (48) with $c = c'/2$. □

Remark 3. We have studied the exact controllability of the linear wave equation with the Dirichlet control. The Hilbert Uniqueness Method can likewise be used to different control functions, for instance of Neumann or Robin type, cf. [94, 144].

This method was applied to many *linear* problems, including solids and structures, cf. [94, 144, 145, 180]. It is not clear whether this method can be generalized to nonlinear problems. In subsequent sections we shall also discuss some methods suitable for nonlinear problems of optimal control.

Remark 4. A nice feature of the Hilbert Uniqueness Method is provided by a theorem which follows. Introduce the set of admissible controls

$$U_{ad} := \{w \in L^2(\Sigma(\mathbf{x}^0)) \mid u(T;w) = \dot{u}(T;w) = 0 \text{ in } \Omega\} \tag{49}$$

where $\Sigma(\mathbf{x}^0) = \Gamma(\mathbf{x}^0)\times(0,T)$ and $u(t;w)$ is a solution to (18). The control function is here denoted by w. It can be shown that U_w contains infinitely many elements, cf. Lions [144].

Theorem 4: *For each pair of initial conditions $(u^0,u^1)\in L^2(\Omega)\times H^{-1}(\Omega)$ the control $v\in L^2(\Sigma(\mathbf{x}^0))$ given by the HUM is such that*

$$J(v) = \min\left\{\frac{1}{2}\int_{\Sigma(\mathbf{x}^0)}|w|^2 d\Gamma\, dt \;\middle|\; w \in U_{ad}\right\}.$$

The functional J defined by

$$J(w) = \frac{1}{2} \int\limits_{\Sigma(\mathbf{x}^0)} |w|^2 \, d\Gamma \, dt$$

is obviously the *cost functional.*

3.2. Overview of other results related to controllability of wave equations

As we have already mentioned, from the point of controllability linear and nonlinear wave equations have been most frequently studied. Now we provide an overview of important results, see also Komornik [94], Lasiecka [118], Lasiecka and Triggiani [127], Lions [144, 145].

As can be inferred from our presentation, one usually considers controllability provided that the underlying domain Ω is regular. Grisvard [68] used the Hilbert Uniqueness Method in the case where Ω has singular points like corners and fissures.

More general problem of the Dirichlet and Neumann boundary controllability was studied by Lasiecka *et al.* [130]. The control function acts on a part of the boundary. The basic equation of motion now involves also a linear, first-order differential operator in all variables $(t, x_1, \ldots, x_n)$, i.e.,

$$\ddot{u} - Au = F_1(u) \quad \text{in} \quad Q.$$

The regularity assumption on the coefficients a_{ij} is strong : $a_{ij} \in C^4(\Omega)$. This assumption is required by the adopted differential geometry methods combined with the Carleman differential multipliers for proving the inverse inequalities (the continuous observability inequalities). We observe that the Riemannian metric g is introduced by

$$\mathbf{G}(x) = [\mathbf{a}(x)]^{-1} = \big(g_{ij}(x)\big), \quad i, j = 1, \ldots, n; \quad x \in \mathbb{R}^n.$$

The books by Avdonin and Ivanov [10, 11] provide a systematic presentation of the theory of exponential families and some applications to controllability, particularly to orthogonal membranes and systems of strings, cf. also Avdonin *et al.* [12]. We also observe that exponential families are closely related to Riesz bases, cf. also Russell [174].

The boundary controllability of the wave equation with space- and time-dependent coefficients was studied by Miranda [158] and Liu and Williams [150]. The first author considered the following system

$$\begin{gathered}
Lu = \ddot{u} - \frac{\partial}{\partial x_i}\left[(\delta_{ij} - (\dot{k})^2 x_i x_j) k^{-2} \frac{\partial u}{\partial x_j}\right] - 2\dot{k}k^{-1} x_i \frac{\partial \dot{u}}{\partial x_i} \\
+ \left[(1-n)(\dot{k})^2 - \ddot{k}k\right] k^{-2} x_i \frac{\partial u}{\partial x_i} = 0 \qquad \text{in} \quad Q, \\
u = \begin{cases} v & \text{on} \quad \Sigma(x^0), \\ 0 & \text{on} \quad \Sigma \backslash \Sigma(x^0), \end{cases} \\
u(0) = u^0, \qquad \dot{u}(0) = u^1,
\end{gathered} \tag{50}$$

where $\Sigma(\mathbf{x}^0) = \Gamma \times (0, T)$. The function $k : [0, \infty) \mapsto (0, \infty)$ is assumed to be continuous and satisfies the following conditions:

(i) $k \in W^{3,\infty}_{\text{loc}}(0, \infty)$,

(ii) $0 < k_0 = \inf\limits_{t \geq 0} k(t), \ \sup\limits_{t \geq 0} k(t) = k_1 < \infty$,

(iii) $\sup\limits_{t \geq 0} \mid \dot{k}(t) \mid = \tau < \dfrac{1}{M}, \ \displaystyle\int\limits_0^\infty |\ddot{k}| \, dt < \infty$,

(iv) $\displaystyle\int\limits_0^\infty |\dot{k}| \, dt < \infty, \ \int\limits_0^\infty |\ddot{k}| \, dt < \infty.$

The domain $\Omega \subset \mathbb{R}^n$ contains the origin of $\mathbb{R}^n$ and the boundary Γ is of class C^2. The HUM was used to show the exact controllability. Primarily, the direct and inverse inequalities were proved.

From the physical point of view more interesting is the exact boundary controllability problem with Neumann boundary control considered by Liu and Williams [150]. These authors studied the following problem

$$\begin{aligned} \ddot{u} - \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_i}\right) &= 0 \quad \text{in} \quad \Omega \times (0,T), \\ \frac{\partial u}{\partial \mathbf{n}_A} &= v \quad \text{on} \quad \Sigma = \Gamma \times (0,T), \\ u(0) = u^0, \qquad \dot{u}(0) &= u^1. \end{aligned} \tag{51}$$

Here $\partial u/\partial \mathbf{n}_A$ denotes the co-normal derivative of A, where

$$A = \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right),$$

and

$$\frac{\partial u}{\partial \mathbf{n}_A} = a_{ij}(x,t)\frac{\partial u}{\partial x_j} n_i. \tag{52}$$

Under suitable assumptions on the coefficients $a_{ij}(x,t)$ and the domain Ω the problem of exact controllability was solved by using HUM. Similar approach has been used by the present author to solve the problem of Neumann controllability of linear elastic, anisotropic and inhomogeneous solid, cf. Sec. 5.

The assumptions imposed on the coefficients $a_{ij}(x,t)$ preclude bodies made of layered materials. In this case, for time-independent coefficients, $a_{ij} \in L^\infty(\Omega)$. Such problems lead to the study of transmission problems, cf. Sec. 3.4.

3.3. Stabilization of wave equations

In the present section we shall review recent developments concerning the decay of the solutions of the wave equation under suitable feedbacks. Well-known Russell's principle [174] states that stabilizability of a linear reversible system implies its exact controllability, cf. also Komornik [94]. The book by Komornik [94] may serve as a systematic introduction to linear and nonlinear stabilization problems, cf. also Komornik [86, 87, 88, 90], Lasiecka [111, 118], Lasiecka and Triggiani [124, 127]. Below we review some recent results not covered by these contributions.

Komornik (see [180]) considered the following problem

$$\begin{aligned} &\ddot{u} - \Delta y = 0 && \text{in} \quad \Omega \times (0, \infty), \\ &u = 0 && \text{on} \quad \Gamma_0 \times (0, \infty), \\ &u = v && \text{on} \quad \Gamma_1 \times (0, \infty), \\ &u(0) = u^0, \quad \dot{u}(0) = u^1 && \text{in} \quad \Omega, \end{aligned} \tag{53}$$

where Ω is a bounded open set in $\mathbb{R}^n$, Γ_1 is an open subset of $\Gamma = \partial\Omega$, and $\Gamma_0 = \Gamma \backslash \Gamma_1$. Komornik (see [180]) proved the following theorem.

Theorem 5: *Fix an arbitrarily large positive number ω. Then there exist two bounded linear maps*

$$P : H^{-1}(\Omega) \to H_0^1(\Omega), \qquad Q : L^2(\Omega) \to H_0^1(\Omega), \tag{54}$$

and a constant M such that, putting

$$v = \frac{\partial}{\partial n}(P\dot{u} + Qu), \tag{55}$$

problem (53), (54) is well posed in $\mathcal{H} := L^2(\Omega) \times H^{-1}(\Omega)$ and its solutions satisfy the estimates

$$\|(u, \dot{u})\|_{\mathcal{H}} \leq M \|(u^0, u^1)\|_{\mathcal{H}} e^{-\omega t}, \tag{56}$$

for all $t \geq 0$ and for all $(u^0, u^1) \in \mathcal{H}$.

The above theorem is valid under some condition on Γ. Two possibilities were discussed:

(i) $\Gamma = \partial\Omega$ is analytic and Γ_1 satisfies the geometrical conditions of Bardos *et al.* [16]: there exists a positive number T such that every ray of geometrical optics in $\bar{\Omega}$ hits Γ_1 at a nondiffractive point in some time $\leq T$.

(ii) Γ is only of class C^2 but there exists a point $x^0 \in \mathbb{R}^n$ such that Γ_1 contains all points x of Γ satisfying $(x - x^0) \cdot \boldsymbol{n}(x) > 0$.

Theorem 5 improves some earlier results due to Lions [147] and Lasiecka and Triggiani [123] mainly in that ω may be prescribed as large as we like. Komornik (see [180]) proved also a similar theorem for Neumann feedback control on Γ.

Lasiecka and Tataru [121] considered the following semilinear wave equation

$$\begin{aligned} &\ddot{u} = \Delta u - f_0(u) && \text{in} \quad \Omega \times (0, \infty), \\ &u = 0 && \text{on} \quad \Gamma_0 \times (0, \infty), \\ &\frac{\partial u}{\partial n} = -g(\dot{u}) - f_1(u) && \text{on} \quad \Gamma_1 \times (0, \infty), \\ &u(0) = u^0, \quad \dot{u}(0) = u^1 && \text{in} \quad \Omega, \end{aligned} \tag{57}$$

where Ω is a bounded and regular domain in $\mathbb{R}^n$, $n \geq 1$ and $u^0 \in H^1_{\Gamma_0}(\Omega)$, $u^1 \in L^2(\Omega)$. Here

$$H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0\}. \tag{58}$$

The sets Γ_0 and Γ_1 are closed and disjoint.

Under suitable assumptions on the nonlinear functions f_α, $\alpha = 0, 1$, and g it has been proved that solutions to (57) exist in

$$C\big(0,\infty; H^1_{\Gamma_0}(\Omega)\big) \cap C^1\big(0,\infty; L^2(\Omega)\big).$$

Moreover, they decay to zero with uniform rates when $t \to \infty$. We observe that uniqueness is not required. In essence, the approach employed is based on the following two steps.

(i) One first obtains certain integral estimates for the energy functional (in place of the usual differential estimates as in the Lyapunov approach). These integral estimates have the advantage of allowing application of certain nonlinear compactness-uniqueness arguments which in turn lead to a nonlinear functional (not differential) relation for the energy function.

(ii) Secondly, one proves comparison theorems which relate the asymptotic behaviour of the energy and of the solutions to an appropriate nonlinear ordinary differential equation.

The nonlinear damped wave system:

$$\begin{aligned} &\ddot{u} - \Delta u + g(\dot{u}) = 0 && \text{in} \quad \Omega \times (0,\infty), \\ &u = 0 && \text{on} \quad \Gamma \times (0,\infty), \\ &u(0) = u^0, \quad \dot{u}(0) = u^1 && \text{in} \quad \Omega, \end{aligned} \tag{59}$$

and its variants were studied by many authors, cf. Aassila [1], Komornik [94] and the references therein. Aassila (see [180]), weakened the assumptions usually imposed on the function g and the domain Ω. He assumes that g is not monotone and has not polynomial growth near the origin and the domain Ω is of finite measure. Consequently, LaSalle's invariance principle cannot be applied due to the lack of compactness. The main result states that the energy of every solution of (59) tends to 0 as $t \to \infty$. The proof is based on the following two lemmas.

Lemma 5: *We have*

$$\int_0^t \int_\Omega |ug(\dot{u})| \, dx d\tau = o(t), \ t \to +\infty.$$

Lemma 6: *We have*

$$\int_0^t \int_\Omega |\dot{u}|^2 \, dx d\tau = o(t), \ t \to +\infty.$$

Proof of the main theorem is then easy (by contradiction).

Possible extensions include a semilinear wave equation and a wave equation with higher-order damping. Martinez (see [180]) studied the problem of the wave equation damped by a nonlinear boundary feedback where

$$\frac{\partial u}{\partial n} + m \cdot n g_1(\dot{u}) = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+.$$

Here $\mathbf{m}(\mathbf{x}) = \mathbf{x} - \mathbf{x}^0$, $x^0 \in \mathbb{R}^n$ is fixed. It is assumed that there exist a strictly increasing and odd function γ of class C^1 on $[-1, 1]$ and two positive constants c_1 and c_2 such that

$$\forall\, z \in [-1,1], \quad |\gamma(z)| \le |g_1(z)| \le |\gamma^{-1}(z)|,$$

$$\forall\, |z| \ge 1, \qquad c_1|z| \le |g_1(z)| \le c_1|z|,$$

where γ^{-1} denotes the inverse function of γ. Under some geometrical assumptions the energy $E(t)$, defined by

$$E(t) = \frac{1}{2} \int\limits_{\Omega} \left(\dot{u}^2 + |\nabla u|^2\right) dx,$$

decays as

$$\forall\, t \ge 1, \quad E(t) \le c \left[\gamma^{-1}\left(\frac{1}{t}\right)\right]^2,$$

where c is a positive constant. Particularly, if

$$\gamma(z) = e^{-1/z^p} \le g_1(z) \le \gamma^{-1}(z) = \left(\frac{-1}{\ln z}\right)^{1/p} \quad \text{if} \quad 0 < z < 1,$$

for some $p > 0$, then

$$\forall\, t \ge 2, \quad E(t) \le \frac{c}{(\ln t)^{2/p}}.$$

Martinez [156] examined the decay property of the system (59) provided that

$$g(\dot{u}) = a(x)\dot{u}, \tag{60}$$

where Ω is a smooth bounded open set of $\mathbb{R}^n$ and $a : \bar{\Omega} \to \mathbb{R}^+$ a nonnegative smooth function. This author proved that the decay rate of energy depends on the degeneracy of $a(x)$, i.e., on the speed of convergence to zero at the boundary as well as the regularity of the solution itself. More precisely, the degeneracy of $a(x)$ has a more pronounced effect than the regularity on the decay rate; for instance, if a decays exponentially to zero, the energy seems to decay only in a logarithmic way whatever the regularity is. The proof of the main result is based on some new integral inequalities and an identity given by the multiplier method. We observe that the stabilization results of Martinez [154] are intermediate between those of strong asymptotic stability of Haraux (see [180]) and those of Nakao [159]. The last author studied the case of degenerate dissipation: set $m > n/2$ and $x^0 \in \mathbb{R}^n$, assume that $\mathfrak{O}$ is a neighborhood of $\Gamma(x^0)$. Take $(u^0, u^1) \in H^{m+1}(\Omega) \times H^m(\Omega)$ that satisfy the so-called *compatibility condition of the m-th order* associated to (59) with g specified by (60). Then if a belongs to $C^{m-1}(\bar{\Omega})$ and satisfies

$$\int\limits_{\mathfrak{O}} \frac{1}{a(x)^p}\, dx < \infty$$

for some $p \in (0,1)$, the energy decays polynomially:

$$E(t) \leq c\big(\|u^0\|_{H^{m+1}(\Omega)} + \|u^1\|_{H^m(\Omega)}\big)t^{-2mp/n}.$$

Aassila [2] and Komornik and Rao [96, 97] investigated the energy decay properties of the following coupled system

$$\begin{aligned}
\ddot{u}_1 - \Delta u_1 + \gamma(u_1 - u_2) = 0 &\quad \text{in} \quad \Omega \times (0,\infty),\\
\ddot{u}_2 - \Delta u_2 + \gamma(u_2 - u_1) = 0 &\quad \text{in} \quad \Omega \times (0,\infty),\\
u_\alpha = 0 &\quad \text{on} \quad \Gamma_0 \times (0,\infty),\\
\frac{\partial u_\alpha}{\partial \boldsymbol{n}} + a_\alpha u_\alpha + g_\alpha(\dot{u}_\alpha) = 0 &\quad \text{in} \quad \Omega,\\
u_\alpha(0) = u_\alpha^0, \quad \dot{u}_\alpha(0) = u_\alpha^1 &\quad \text{in} \quad \Omega.
\end{aligned} \tag{61}$$

where $\alpha = 1,2$, and Ω is a bounded open domain in $\mathbb{R}^n$. The summation convention does not apply to Eq. (59). The energy of the above system is given by

$$E(t) = \frac{1}{2}\int\limits_\Omega \left[\dot{u}_1^2 + |\nabla u_1|^2 + \dot{u}_2^2 + |\nabla u_2|^2 + \gamma(u_1 - u_2)^2\right] dx + \frac{1}{2}\int\limits_{\Gamma_1} \left(a_1 u_1^2 + a_2 u_2^2\right) d\Gamma. \tag{62}$$

Under appropriate assumptions, standard calculation shows that the energy is non-increasing. Proofs of theorems on the energy decay are given in Aassila [2] and Komornik and Rao [97].

Komornik and Loreti [95] studied the observability of coupled linear distributed system. Both the Dirichlet- and Neumann-type boundary conditions were considered. The investigated system couples the wave equation with Petrovsky system.

In Section 3.1 we have discussed pointwise controllability of a vibrating string. Jaffard *et al.* [78] studied the asymptotic behaviour of solutions of the following wave equation:

$$\begin{aligned}
&\ddot{u} - \Delta u + g(\dot{u})\delta_\gamma = 0 &&\text{in} \quad \Omega \times (0,\infty),\\
&u = 0 &&\text{on} \quad \Gamma \times (0,\infty),\\
&u(0) = u^0, \quad \dot{u}(0) = u^1 &&\text{in} \quad \Omega.
\end{aligned} \tag{63}$$

Here $\Omega \subset \mathbb{R}^n$, $n = 1,2$, is an open bounded set, $\gamma \subset \Omega$ is a closed simple curve (in two space dimensions) or $\gamma = a$, $a \in \Omega$ (in one space dimension) and δ_γ is the Dirac mass concentrated on γ. System (63) is suggested by various models of vibrating structures provided with piezoelectric actuators. Thus the damping term is concentrated on an interior curve or in an interior point. The system (63) is dissipative since

$$\frac{dE}{dt} = -\int\limits_\gamma g(\dot{u}(x,t))\dot{u}(x,t)\, d\gamma, \tag{64}$$

where

$$E(t) = \frac{1}{2} \int_{\Omega} (\dot{u}^2 + |\nabla u|^2) \, dx \tag{65}$$

is the energy.

In the one-dimensional case we have strong stabilization for a dense set of control points, called "strategic". By using Russell's "stabilizability-controllability" argument [174] we conclude that the energy decay is not exponential, see also the references cited in Jaffard *et al.* [78].

In the two-dimensional case Jaffard *et al.* [78] proved the existence of a dense set of "strategic curves" such that every solution of (62) decays to zero when t tends to infinity.

Conrad and Pierre (see [180]) studied strong asymptotic stability of problems described by an abstract evolution differential inclusion of the form

$$\ddot{u} + Au + \partial\psi(\dot{u}) \ni 0, \tag{66}$$

where A is a linear and self-adjoint operator with dense domain $D(A)$ in a Hilbert space H and $\partial\psi(\dot{u})$ describes a nonlinear dissipative mechanism characterized by the subdifferential $\partial\psi$ of a functional ψ. Introducing a suitably defined nonlinear operator B, the nonlinear semi-group of contractions $S_B(t)$ on $\overline{D(B)}$ is introduced. To study the asymptotic behaviour of the semi-group $S_B(t)$ the following closed convex set is introduced:

$$K_{\psi} = \{\psi \in V | \psi(v) = 0\}, \tag{67}$$

where $V = D(A^{1/2})$. The operator $A^{1/2}$ is well-defined since A is assumed to be coercive on H. Assume further that the resolvent of A is compact and denote by V_i the associated eigenspaces. They are obviously of finite dimension and

$$V = \bigoplus_{i \geq 1} V_i.$$

In what concerns the functional ψ we suppose that $0 \in \partial\psi(0)$, thus after a normalization

$$\min_{v \in V} \psi(0) = \psi(0) = 0. \tag{68}$$

Now we are in a position to formulate the basic result due to Conrad and Pierre (see [180]).

Theorem 6: *Assume that the operators A and B have compact resolvents and (67) holds. Then*

$$\forall (u_0, u_1) \in \overline{D(B)}, \quad \lim_{t \to \infty} S_B(t)(u_0, u_1) = 0, \quad \textit{in } V \times H$$

if and only if

$$\forall i \geq 1, \quad K_{\psi} \cap (-K_{\psi}) \cap F_i = \{0\}.$$

The main tool in the proof is the invariance principle of LaSalle.

We observe the differential inclusion (66) enables one to investigate the strong stabilization of a class of problems involving *unilateral conditions*. For instance, one can consider the following boundary conditions for the wave equation (the space dimension $N \geq 1$):

$$u = 0 \quad \text{on} \quad \Gamma_0,\ t > 0,$$

$$\frac{\partial u}{\partial \boldsymbol{n}} = -a(x)g(\dot{u}), \quad x \in \Gamma_1, \quad t > 0,$$

where $\Gamma = \partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, $g = \partial j$, $g(0) = 0$, $a > 0$. Thus for g one can take a maximal monotone graph.

Lasiecka [110] proved asymptotic stability theorems for the wave equation in the case of:

(i) Dirichlet boundary condition

$$u(x,t) + g\big(A_1\dot{u}(x,t)\big) \ni 0, \qquad x \in \Gamma_1, \quad t > 0. \tag{69}$$

(ii) Neumann boundary condition

$$\frac{\partial u}{\partial s}(x,t) + g\big(A_1\dot{u}(x,t)\big) \ni 0, \qquad x \in \Gamma_1, \quad t > 0. \tag{70}$$

Possible candidates for the linear operator A_1 have also been provided. We observe that the general framework of Lasiecka [110] is more general than the one developed by Conrad and Pierre (see [180]). Anyway, none of them incorporates Coulomb's friction and elasto-plastic solids. The last two classes seem to remain untouched by from the point of view of exact controllability and stabilization.

The problem of exponential decay of the energy of the wave equation in a bounded domain by the use of a boundary feedback requires special geometrical conditions. Martinez [156] weakened the usual assumption of star-shapedness of the domain using adapted multipliers.

As usual let $\{\Gamma_0, \Gamma_1\}$ be a partition of the boundary Γ of Ω, is an open bounded set of $\mathbb{R}^n$.

$(\Omega, \Gamma_0, \Gamma_1)$ is an *almost star-shaped* domain if there exist $\varphi \in C^2(\bar{\Omega})$ such that

$$\sup\{\Delta\varphi(x) - 2\lambda_1(x) \,|\, x \in \Omega\} < \inf\{\Delta\varphi(x) \,|\, x \in \Omega\},$$

$$\frac{\partial \varphi}{\partial \boldsymbol{n}} \leq 0 \quad \text{on} \quad \Gamma_0,$$

$$\frac{\partial \varphi}{\partial \boldsymbol{n}} \geq 0 \quad \text{on} \quad \Gamma_1,$$

where $\lambda_1(x)$ is the smallest eigenvalue of the real symmetric squared matrix $D^2\varphi(x)$. In particular, $(\Omega, \Gamma_0, \Gamma_1)$ is almost star-shaped if there exists $\varphi \in C^2(\bar{\Omega})$ such that

$$\Delta\varphi = 1 \quad \text{in} \quad \Omega,$$

$$\lambda_1(\varphi) = \inf\{\lambda_1(x) \,|\, x \in \Omega\} > 0,$$

$$\frac{\partial \varphi}{\partial \boldsymbol{n}} \leq 0 \quad \text{on} \quad \Gamma_0,$$

$$\frac{\partial \varphi}{\partial \boldsymbol{n}} \geq 0 \quad \text{on} \quad \Gamma_1.$$

Martinez [156] provided a non-trivial example of an almost star-shaped domain which is not star-shaped.

The boundary damping given by:

$$\frac{\partial u}{\partial n} + au + l\dot{u} = 0 \quad \text{on} \quad \Gamma_1$$

was studied by many authors, cf. Komornik [94], Martinez [156] and the references therein. The last author carefully studied the exponential energy decay for the wave equation in almost star-shaped domains assuming that

$$a = \frac{c}{2\|\nabla\varphi\|_\infty^2}\frac{\partial\varphi}{\partial n} \quad \text{and} \quad l = \frac{1}{\|\nabla\varphi\|_\infty}\frac{\partial\varphi}{\partial n},$$

where $c > 0$ is a computable constant.

Lebeau [132] studied a model problem of stabilization on a Riemannian manifold. Let us denote this manifold my (M, g). It is assumed that this manifold is of class C^∞, compact, and connected with the boundary ∂M of a class C^∞. By $\Delta = \Delta_g$ we denote the Laplace operator on M for the metric g; $a \in C^\infty(\overline{M}, \mathbb{R}^+)$. The evolution problem

$$\begin{aligned} \left(\partial_t^2 - \Delta + 2a(x)\partial_t\right)u = 0, &\quad \text{in} \quad M \times (0,\infty), \\ u = 0, &\quad \text{on} \quad \partial M \times [0,\infty) \\ u(0) = u^0 \in H_0^1(M), \qquad \dot{u}(0) = u^1 \in L^2(M), & \end{aligned} \tag{71}$$

possesses a unique solution $u \in C^0(\mathbb{R}_t, H_0^1) \cap C^1(\mathbb{R}_t, L^2)$, obtained for instance by application of the theorem of Hille-Yosida to the unbounded operator on the Hilbert space $H = H_0^1(M) \times L^2(M)$

$$A_a = \begin{bmatrix} 0 & Id \\ \Delta & -2a \end{bmatrix}, \qquad D(A_a) = (H_0^1 \cap H^2) \cap H_0^1.$$

Here $\partial_t u = \dot{u} = \frac{\partial u}{\partial t}$. For $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda \notin [-2\|a\|_{L^\infty}, 0]$, $(\lambda - A_a)$ is a bijection from $D(A_a)$ on H. The imbedding $D(A_a) \to H$ being compact, the spectrum of A_a, denoted by $\operatorname{sp}(A_a)$ consists of a sequence of complex numbers λ_j with $\operatorname{Re}\lambda_j \in [-2\|a\|_{L^\infty}, 0]$, $|\lambda_j| \to \infty$ as $j \to \infty$. For $u(t,x)$ solving the system (71) the energy is given by

$$E(t) = \frac{1}{2}\int_M (\dot{u}^2 + |\nabla u|^2)\,dx. \tag{72}$$

It can easily be verified that

$$E(t) - E(0) = \int_0^t \int_M 2a(x)|\frac{\partial u}{\partial \tau}(x,\tau)|^2\,dx d\tau. \tag{73}$$

Lebeau [132] proved two stabilization theorems. Let us formulate one of them.

Theorem 7: *Suppose that* $0 \le a(x) \not\equiv 0$.

(i) If $\partial M \neq \emptyset$, *one has* $Re\lambda < 0$ *for* $\lambda \in sp(A_a)$; *if* $\partial M = \emptyset$, *then* $\lambda = 0$ *is the only spectral value with zero real value associated to constant solutions of the system (71).*

(ii) For each $(u_0, u_1) \in H_0^1 \times L^2$, *the solution* u *of (71) verifies* $\lim_{t\to\infty} E(t) = 0$.

(iii) Suppose that the geodesics of $\overline{M}$ *do not have the contact of infinite order with* ∂M *and that there exists a time* $T > 0$ *such that for each generalized geodesics of* M *the length greater than* T_0 *meets the open set* $\{x, a(x) > 0\}$.

Then there exist c_0, $c_1 > 0$ *such that*

$$\forall (u^0, u^1) \in H, \qquad E(t) \le c_0 e^{-c_1 t} E(0), \qquad \forall\, t \ge 0. \tag{74}$$

(The generalized geodesics are projections on M *of* C^∞*-rays of Melrose-Sjöstrand (see the references cited in [132, 180]).*

The second theorems deals with the case possibility of spectrum accumulated on the imaginary axis.

Lebeau and Robbiano [134] considered stabilization of the wave equation on the Riemannian manifold (M, g) with $a = 0$ and the following Neumann condition

$$\frac{\partial u}{\partial n} + b(x)\dot{u} = 0 \quad \text{on} \quad \partial M \times \mathbb{R}^+. \tag{75}$$

Here $b \in C^\infty(\partial M, \mathbb{R}^+)$ and $\{x \in \partial M \mid b > 0\} \neq \emptyset$.

3.4. Transmission problems

The controllability and stabilizability problems considered until now were posed on rather simple domains, if viewed from the viewpoint of the mechanics and physics. To cope with a larger class of domains one has to consider the so-called transmission problems. In essence, one studies then problems posed on a domain consisting of linked subdomains. Therefore the expression "multi-link structures" or "junctions" is also used in the relevant literature, cf. Ciarlet [30], Lagnese [107], Lagnese and Leugering [108], Liu and Williams [151]. On the common boundary of adjacent subdomains *transmission conditions* have to be imposed, cf. also Lions [144, Chapt. VI]. Obviously, a synonym for the transmission condition is the *interface condition.* It is worth noting that the class of transmission problems includes layered materials, which were precluded from the previous considerations. Unfortunately, the general case of layered bodies still remains open, even in the simple case of bodies consisting of two materials.

Liu and Williams [151] examined the transmission of the wave equation with lower-order terms. Now the boundary $\Gamma = \partial\Omega$ of class C^2 consists of two parts, S_1 and S_2, cf. Fig. 1.

S_1 is assumed to be either empty or to have a nonempty interior and $S_2 \neq \emptyset$ and relatively open in Γ. Assume $\bar{S}_1 \cap \bar{S}_2 = \emptyset$. Let S_0 with $\bar{S}_0 \cap \bar{S}_1 = \bar{S}_0 \cap \bar{S}_2 = \emptyset$ be a regular hypersurface of class C^2, which separates Ω into two domains, Ω_1 and Ω_2, such that $S_1 \subset \Gamma_1 = \partial\Omega_1$ and $S_2 \subset \Gamma_2 = \partial\Omega_2$.

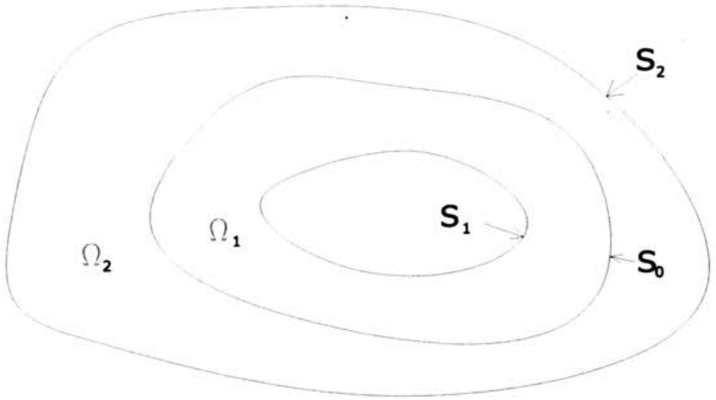

FIGURE 1.

Consider now the wave equation with lower–order terms and with dissipative boundary condition of Robin type:

$$\begin{aligned}
&\ddot{u}_\alpha - a_\alpha \Delta u_\alpha + q u_\alpha = 0 && \text{in} \quad \Omega_\alpha \times (0,\infty),\\
&u_\alpha(0) = u_\alpha^0, \quad \dot{u}_\alpha(0) = u_\alpha^1(0) && \text{in} \quad \Omega_\alpha,\\
&u_1 = 0 && \text{on} \quad S_1 \times (0,\infty),\\
&\frac{\partial u_2}{\partial \mathbf{n}} + \beta(\mathbf{x}) u_2 + \sigma(\mathbf{x}) \dot{u}_2 = 0 && \text{on} \quad S_2 \times (0,\infty),\\
&u_1 = u_2, \quad a_1 \frac{\partial u_1}{\partial \mathbf{n}} = a_2 \frac{\partial u_2}{\partial \mathbf{n}} && \text{on} \quad S_0 \times (0,\infty).
\end{aligned} \tag{76}$$

Here $\mathbf{n} = (n_i)$ denotes the unit normal on Γ and S_0 directed towards the exterior of Ω and Ω_1, a_1 and a_2 are positive constants; $\alpha = 1, 2$, and to $(76)_1$ the summation convention does not apply.

The functions $q : \Omega \to \mathbb{R}$, $\beta, \sigma : S_2 \to \mathbb{R}$ are nonnegative and satisfy

$$q \in L^\infty(\Omega), \qquad \beta, \sigma \in C^1(S_2). \tag{77}$$

Liu and Williams [151] studied the problem of rate of exponential decay of energy for the system (76). The energy of system (76) is given by

$$E(t) = \frac{1}{2} \int\limits_\Omega \left[|\dot{u}|^2 + a(x) |\nabla u|^2 + q(x) |u|^2 \right] dx + \frac{1}{2} \int\limits_{S_2} \beta(x) a(x) |u|^2 \, d\Gamma. \tag{78}$$

Here

$$u = \begin{cases} u_1, & x \in \Omega_1, \\ u_2, & x \in \Omega_2, \end{cases} \qquad a(x) = \begin{cases} a_1, & x \in \Omega_1, \\ a_2, & x \in \Omega_2. \end{cases} \tag{79}$$

The functions u^0 and u^1 are defined similarly. The main result is formulated as follows.

Theorem 8: *Let $l(x) = (l_1(x), \dots, l_n(x))$ be a vector field of class $C^2(\overline{\Omega})$ such that*

(i) $\boldsymbol{l} \cdot \boldsymbol{n} \le 0$ almost everywhere on S_1 with respect to the $(n-1)$-dimensional surface measure,

(ii) $\boldsymbol{l} \cdot \boldsymbol{n} \ge \eta > 0$ almost everywhere on S_2,

(iii) $(a_1 - a_2)\boldsymbol{l} \cdot \boldsymbol{n} \ge 0$ almost everywhere on S_0

(iv) the matrix $(\frac{\partial l_i}{\partial x_j} + \frac{\partial l_j}{\partial x_i})$ is uniformly positive definite on Ω,

(v) there exists a constants $\sigma_0 > 0$ such that

$$\sigma \ge \sigma_0 \quad on \quad S_2.$$

Then there are positive constants M, ω such that

$$E(t) \le M e^{-\omega t} E(0), \text{ for all } t \ge 0,$$

for all solutions of (76) with $(u^0, u^1) \in H^1_{S_1}(\Omega) \times L^2(\Omega, S_1)$.

The spaces $L^2(\Omega, S_1)$ and $H^1_{S_1}(\Omega)$ are defined in the following way:

$$L^2(\Omega, S_1) = \begin{cases} \{u \in L^2(\Omega) \mid \int\limits_\Omega u(x)\,dx = 0\}, & \text{if} \quad S_1 = \emptyset,\ q \equiv 0 \text{ and } \beta \equiv 0, \\ L^2(\Omega), & \text{otherwise;} \end{cases}$$

$$H^1_{S_1}(\Omega) = \begin{cases} H^1(\Omega), & \text{if} \quad S_1 = \emptyset, \\ \{u \in H^1(\Omega) \mid u = 0 \quad \text{on} \quad S_1\}, & \text{otherwise.} \end{cases}$$

We observe that the vector field $\boldsymbol{l}(x)$ is more general than the vector field $\boldsymbol{m}(x)$ used previously. However, Liu and Williams [151] did not provide a non trivial example of $\boldsymbol{l}(x)$. We observe that β is not required to be small.

The proof of Theorem 8 consists of several steps. First, one proves the well-posed ness of problem (76) by applying the theory of semigroups. Next, the case of $\beta_0 = \max_{x \in S_2} \beta(x)$ small enough is considered. Finally,the case of β_0 being arbitrary is studied. In the last case Russell's "controllability via stabilizability" principle is employed, cf. Russell [178].

A transmission problem for the wave equation in a more complex domain was investigated by Lagnese and Leugering [108]. More precisely, these authors applied domain decomposition methods (DDMs) for both the approximate and exact controllability problems. Each version of the Schwartz alternating algorithm and the introduction of skew-symmetric, Robin iterative transmission conditions between the subdomains Ω_i $(i = 1, \dots, m)$, that couple the direct and adjoint states in the optimality systems associated with the approximate, resp., the exact, controllability problem. For both the approximate and exact controllability problems, the corresponding DDM is a *sequence* of boundary value problems on the region $Q_i = \Omega_i \times (0, T)$.

Nicaise (see [180]) showed how to solve the problem of boundary exact controllability for a two-dimensional polygonal topological network ($2-d$ network for short) which is a subset of $\mathbb{R}^n$. We observe that Ω is not necessarily a plane polygon. This author carefully studied the exact controllability of the wave equation in such domain. The main idea consists of replacing the boundary control with its regular part and add to the space of controls the coefficients of the singularities. This leads to a classical boundary control but with an internal control which is a distribution with a support equal to the singular vertices. For other results on controllability and stabilization of the wave equations the reader is referred to [7, 24, 44, 45, 79, 84, 88, 89, 122, 173, 175].

3.5. Approximation methods

Papers on approximate solutions of problems treated in this contribution are not numerous. Glowinski and Lions [67] discussed the problem of numerical solution of exact and approximate controllability of the wave equation. It was proved that a direct discretization leads to an ill-posed discrete problem. Glowinski and Lions [67] summarized the attempts to overcome this drawback. A successful remedy was proposed by Glowinski [62], who used a multigrid filtering technique inspired from a similar problem which arises in the numerical solution of the Stokes problem. A finite element implementation was elaborated by Glowinski [62]. A finite difference implementation of this method was given by Asch and Lebeau [8]. The essential points of the last approach will now be presented. Primarily, however, we have to summarize the continuous (distributed) problem.

Consider the classical wave equation with Dirichlet control on a part of the boundary in the two-dimensional case

$$\begin{gathered} \Box u := \ddot{u} - \Delta u = 0 \qquad \text{in} \quad Q = \Omega \times (0,T), \\ u = \begin{cases} g & \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,T), \\ 0 & \text{on} \quad \Sigma \backslash \Sigma_0 = \Gamma \backslash \Gamma_0 \times (0,T). \end{cases} \\ u(x,0) = u^0(x), \qquad \dot{u}(x,0) = u^1(x) \qquad \text{in} \quad \Omega, \end{gathered} \tag{80}$$

where Ω is a bounded domain of $\mathbb{R}^2$ and $\Gamma_0 = \Sigma(\mathbf{x}^0)$.

Brief description of HUM

Let

$$V = H_0^1(\Omega) \times L^2(\Omega), \qquad V' = H^{-1}(\Omega) \times L^2(\Omega),$$

and define the operator $\Lambda : V \to V'$ as follows

(i) Take $\boldsymbol{e} = \{e^0, e^1\} \in V$ and solve from $t = 0$ to $t = T$

$$\begin{aligned} \ddot{\phi} - \Delta\phi &= 0 \qquad \text{in} \quad \Omega \times (0,T), \\ \phi(x,t) &= 0 \qquad \text{on} \quad \Sigma = \Gamma \times (0,T), \\ \phi(x,0) &= e^0, \quad \dot{\phi}(x,0) = e^1 \qquad \text{in} \quad \Omega. \end{aligned} \tag{81}$$

(ii) Then solve (backwards) from $t = T$ to $t = 0$

$$\begin{aligned} \ddot{\psi} - \Delta\psi &= 0 \qquad \text{in} \quad \Omega \times (0,T), \\ \psi(x,t) &= \begin{cases} \dfrac{\partial\phi}{\partial \boldsymbol{n}} & \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,T), \\ 0 & \text{on} \quad \Sigma \backslash \Sigma_0. \end{cases} \end{aligned} \tag{82}$$

(iii) Define the operator Λ as follows

$$\Lambda \boldsymbol{e} = \{\dot{\psi}(0), -\psi(0)\}.$$

Let us formulate the theorem due to Lions [145], cf. also Sec. 3.1.

Theorem 9: *The operator Λ is linear and continuous from V into V'. Λ is an isomorphism from V onto V' provided that T is sufficiently large ($> T_{min} = 2\|\mathbf{x} - \mathbf{x}^0\|_{L^\infty(\Omega)^2}$) and Γ_0 is of the following type*

$$\Gamma_0 = \Gamma(\mathbf{x}^0) = \{\mathbf{x} \in \Gamma \,|\, (\mathbf{x} - \mathbf{x}^0) \cdot \boldsymbol{n} > 0\},$$

where $\mathbf{x}^0 \in \mathbb{R}^2$ is an arbitrary point and $\boldsymbol{n}$ is the outward normal to Γ.

To apply Theorem 9 to the control of the wave equation (80) we assume that

$$u^0 \in L^2(\Omega), \quad u^1 \in H^{-1}(\Omega)$$

are prescribed. The procedure runs then as follows:

1. Take $\boldsymbol{f} = \{u^1, -u^0\}$, i.e. we identify u with ψ,
2. solve $\Lambda \boldsymbol{e} = \boldsymbol{f}$ to obtain e^0, e^1, the initial data for the ϕ wave equation,
3. solve the wave equation (81) forwards in time using e^0, e^1 as the initial data,
4. calculate the normal derivative of the solution of the ϕ-wave equation and set $g = \dfrac{\partial \phi}{\partial \boldsymbol{n}}$ on Σ_0,
5. solve the ψ-wave equation (82) backwards in time using g as the boundary data,
6. set $u = \psi$, then since $\psi(x,T) = 0$, $\dot{\psi}(x,T) = 0$ was imposed, g (the boundary control) gives the exact boundary controllability with

$$u(x,T) = \dot{u}(x,T) = 0, \quad \forall x \in \Omega.$$

Remark 5. The operator Λ is symmetric and V–elliptic, cf. Lions [145]. Consequently, the equation $\Lambda \boldsymbol{e} = \boldsymbol{f}$ can be solved by a conjugate gradient algorithm, cf. also Glowinski and Lions [67].

A numerical algorithm based on the HUM

The constructive nature of the HUM combined with advantageous properties of the operator Λ, enable us to elaborate a numerical algorithm based on a conjugate gradient method. After Asch and Lebeau [8], this algorithm consists of three steps:

(a) Presentation of a general conjugate gradient algorithm (Algorithm (G-0)),
(b) application of algorithm CG-0 to the boundary controllability problem for the wave equation based on the HUM (Algorithm CG-HUM),
(c) discretization of algorithm CG-HUM using a multigrid technique (Algorithm CG-h).

The conjugate gradient solution

The variational formulation of the equation $\Lambda \boldsymbol{e} = \boldsymbol{f}$, $\boldsymbol{f} = \{u^1, -u^0\}$, reads:
Find $\boldsymbol{e} \in V$ such that

$$\langle \Lambda \boldsymbol{e}, \check{e} \rangle = \langle \{u^1, -u^0\}, \check{e} \rangle, \quad \forall \check{e} \in V \tag{83}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V and V'. We have

$$\langle \Lambda \boldsymbol{e}, \check{\boldsymbol{e}} \rangle = \int_\Omega \left[\dot{\psi}(x,0)\check{\phi}(x,0) - \psi(x,0)\dot{\check{\phi}}(x,0) \right] dx = \int_\Sigma \frac{\partial \phi}{\partial \boldsymbol{n}} \frac{\partial \check{\phi}}{\partial \boldsymbol{n}} \, d\Gamma dt, \quad \forall \boldsymbol{e}, \check{\boldsymbol{e}} \in V. \tag{84}$$

We recall that $\dot{\check{\phi}} = \check{\phi}_t = \frac{\partial \check{\phi}}{\partial t}$. Consequently, for T large enough, problem (83) and hence the equation $\Lambda \boldsymbol{e} = \boldsymbol{f}$ can be solved by a conjugate gradient algorithm.

Problem (83) is a particular case of the well-known Lax–Milgram problem, cf. Yosida [188],

$$\left| \begin{array}{l} \text{Find } u \in V \text{ such that} \\ a(u,v) = L(v), \ v \in V, \end{array} \right. \tag{85}$$

where

- V is a real Hilbert space for the scalar product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$,
- $a : V \times V \to \mathbb{R}$ is a bilinear, continuous, symmetric and V-elliptic (or coercive) form,
- $L : V \to \mathbb{R}$ is linear and continuous.

Under these assumptions, problem (85) possesses a unique solution which can be computed by the following conjugate gradient algorithm:

Algorithm CG-0

Step 0. Initialization:

- $u^0 \in V$ is given;
- calculate the residual $g^0 \in V$ by solving

$$(g^0, v) = a(u^0, v) - L(v), \quad \forall v \in V;$$

- if $g^0 = 0$ is small, set $u = u^0$ and STOP; if not, set the first search direction $w^0 = g^0$ (steepest descent).

Then for $k = 0, 1, 2, \ldots$, assuming that u^k, g^k, w^k (solution, residual, search direction) are known, compute the next iterates $u^{k+1}, g^{k+1}, w^{k+1}$ as follows:

Step 1. Descent:

- minimize (85) in the search direction by calculating

$$\rho_k = \frac{\|g^k\|^2}{a(w^k, w^k)};$$

- update the solution

$$u^{k+1} = u^k - \rho_k w^k.$$

Step 2. Convergence test and new descent direction:

- calculate the residual $g^{k+1} \in V$ by solving

$$(g^{k+1}, v) = (g^k, v) - \rho_k a(w^k, v), \quad \forall v \in V;$$

- if $g^{k+1} = 0$ or is small, set $u = u^{k+1}$ and STOP; if not, calculate

$$\gamma_k = \frac{\|g^{k+1}\|^2}{\|g^k\|^2};$$

- define the new conjugate search direction as

$$w^{k+1} = g^{k+1} + \gamma_k w^k;$$

- set $k = k+1$ and go to step 1.

Obviously, there is no summation over repeated indices in steps 1 and 2.

Application of CG algorithm to the boundary control problem

Algorithm CG-0 is now applied to the solution of the wave equation (80) in the variational form (83). We recall that now

$$V = H_0^1(\Omega) \times L^2(\Omega), \qquad V' = H^{-1}(\Omega) \times L^2(\Omega).$$

We define the following inner product and norm on V:

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega (v^0_{,\alpha} w^0_{,\alpha} + v^1 w^1)\, dx, \tag{86}$$

$$\|\boldsymbol{e}\|_V^2 = (\boldsymbol{e}, \boldsymbol{e})_V. \tag{87}$$

The bilinear form a is

$$a(\cdot,\cdot) = \langle \Lambda\cdot, \cdot \rangle,$$

whilst the linear form L in Eq. (85) is specified by the r.h.s. of Eq. (83). The summation convention applies to Eq. (86), where $\alpha = 1, 2$.

We observe that the positive definite operator Λ is composed of the solution of *two* wave equations: in ϕ and ψ. Consequently, one never ends up with a simple positive definite matrix (after discretization) as would be the case in the solution of a large linear system by conjugate gradient methods.

Algorithm CG-HUM

Step 0. Initialization:

- $e_0^0 \in H_0^1(\Omega)$, $e_0^1 \in L^2(\Omega)$ are given;
- calculate the residual $\boldsymbol{g}_0 = \{g_0^0, g_0^1\} \in V$ by solving
  - forwards in time

$$\begin{gathered}\Box\phi_0 = 0 \quad \text{in} \quad Q = \Omega \times (0,T), \\ \phi_0(x,0) = e_0^0, \quad \frac{\partial \phi_0}{\partial t}(x,0) = e_0^1 \quad \text{in} \quad \Omega, \\ \phi_0(x,t) = 0 \quad \text{on} \quad \Gamma \times (0,T) = \Sigma;\end{gathered} \tag{88}$$

– backwards in time from $t = T$ to $t = 0$

$$\Box\psi_0 = 0 \quad \text{in} \quad Q = \Omega \times (0,T),$$

$$\psi_0(x,T) = 0, \quad \frac{\partial\psi_0}{\partial t}(x,T) = 0 \quad \text{in} \quad \Omega, \tag{89}$$

$$\psi_0(x,t) = \frac{\partial\phi_0}{\partial t} \quad \text{on} \quad \Sigma_0, \qquad \psi_0(x,t) = 0 \quad \text{on} \quad \Sigma\backslash\Sigma_0;$$

– finally

$$-\Delta g_0^0 = \frac{\partial\psi_0}{\partial t} - u^1 \quad \text{in} \quad \Omega,$$
$$g_0^0 = 0 \quad \text{on} \quad \Gamma, \tag{90}$$

with

$$g_0^1 = u^0 - \psi_0(x,0) \quad \text{in} \quad \Omega;$$

- if $\boldsymbol{g}_0 = \boldsymbol{0}$ or is small, set $\boldsymbol{e} = \boldsymbol{e}_0$ and STOP; if not, set the first search direction $\boldsymbol{w}_0 = \boldsymbol{g}_0$ (stepest descent).

Then for $k = 0, 1, 2 \ldots$, assuming that $\boldsymbol{e}_k, \boldsymbol{g}_k, \boldsymbol{w}_k$ are known, compute the next iterates $\boldsymbol{e}_{k+1}$, $\boldsymbol{g}_{k+1}$, $\boldsymbol{w}_{k+1}$ as follows.

Step 1. Descent:

- minimize (83) in the search direction by calculating

$$\rho_k = \frac{\|\boldsymbol{g}\|_V^2}{\langle \Lambda \boldsymbol{w}_k, \boldsymbol{w}_k \rangle} = \frac{\|\boldsymbol{g}_k\|_V^2}{(\overline{\boldsymbol{g}}_k, \boldsymbol{w}_k)_V},$$

  where $\overline{\boldsymbol{g}}_h = \{\overline{g}_k, \overline{g}_k^1\}$ is obtained by solving

  – forwards in time

$$\Box\overline{\phi}_k = 0 \quad \text{in} \quad \Omega \times (0,T),$$

$$\overline{\phi}_k(x,0) = w_0^0, \quad \frac{\partial\overline{\phi}_k}{\partial t}(x,0) = w_0^1 \quad \text{in} \quad \Omega, \tag{91}$$

$$\overline{\phi}_k(x,t) = 0 \quad \text{on} \quad \Sigma = \Gamma \times (0,T);$$

  – backwards in time from $t = T$ to $t = 0$

$$\Box\overline{\psi}_k = 0 \quad \text{in} \quad Q = \Omega \times (0,T),$$

$$\overline{\psi}_k(x,0) = 0, \quad \frac{\partial\overline{\psi}_k}{\partial t}(x,T) = 0 \quad \text{in} \quad \Omega, \tag{92}$$

$$\overline{\psi}_k(x,t) = \frac{\partial\overline{\phi}_k}{\partial t} \quad \text{on} \quad \Sigma_0, \qquad \overline{\psi}_k = 0 \quad \text{on} \quad \Sigma\backslash\Sigma_0;$$

  – finally

$$-\Delta\overline{g}_k^0 = \frac{\partial\overline{\psi}_k}{\partial t} \quad \text{in} \quad \Omega,$$
$$\overline{g}_k^0 = 0 \quad \text{on} \quad \Gamma, \tag{93}$$

with

$$\overline{g}_k^1 = -\overline{\psi}_k(x,0) \quad \text{in} \quad \Omega;$$

- update all quantities

$$\boldsymbol{e}_{k+1} = \boldsymbol{e}_k - \rho_k \boldsymbol{w}_k,$$

$$\phi_{k+1} = \phi_k - \rho_k \overline{\phi}_k,$$

$$\psi_{k+1} = \psi_k - \rho_k \overline{\psi}_k,$$

$$\boldsymbol{g}_{k+1} = \boldsymbol{g}_k - \rho_k \overline{\boldsymbol{g}}_k.$$

Step 2. Convergence test and new descent direction:

- if $\boldsymbol{g}_{k+1} = 0$ or is small, set $\boldsymbol{e} = \boldsymbol{e}_{k+1}$, $\phi = \phi_{k+1}$, $\psi = \psi_{k+1}$ and STOP;
- else
  - calculate

$$\gamma_k = \frac{\|\boldsymbol{g}_{k+1}\|_V^2}{\|\boldsymbol{g}_k\|_V^2};$$

  - define the new conjugate search direction as

$$\boldsymbol{w}_{k+1} = \boldsymbol{g}_{k+1} + \gamma_k \boldsymbol{w}_k;$$

- set $k = k+1$ and go to Step 1.

Remark 6. We observe that in the above conjugate gradient algorithm we seek (by minimization of the residual) the good initial conditions, e^0, e^1 of the ϕ-wave equation – not those of the original u wave equation. Once we have obtained these initial conditions, we can solve the ϕ-wave equation and calculate the boundary control $g = \frac{\partial \phi}{\partial \boldsymbol{n}}\big|_{\Sigma_0}$ for the ψ-wave equation. However, we imposed the condition

$$\psi(x,T) = \dot{\psi}(x,T) = 0.$$

Thus the solution of the ψ equation, using the converged value of g, will give us the exact controllability by simple identification with the u wave equation. The only role played by the initial conditions of u is in the calculation of the residue in the zeroth iteration of the conjugate gradient.

The discretization of the CG algorithm using a multigrid filtering technique

The direct discretization leads to an ill-posed discrete problem. The ill-posedness stems from the high frequency of the solution of the discrete problem:

$$\Lambda_{h,\Delta t} \boldsymbol{e}_h = \boldsymbol{f}_h.$$

According to Glowinski [65], the remedy is to eliminate the short wave-lengths components of the initial conditions of the ϕ-wave equation by defining them on a coarse finite difference grid of twice the step-size, $2h$.

Two operators are required for the passage from grid to grid:
- an interpolation operator,
- an injection operator.

The interpolation operator maps the coarse grid onto the finite grid:

$$I_{2h}^{h} : \Omega^{2h} \to \Omega^{h},$$

and is defined by

$$\begin{aligned}
\phi_{2i,2j}^{h} &= \phi_{ij}^{2h},\\
\phi_{2i+1,j}^{h} &= \frac{1}{2}(\phi_{ij}^{2h} + \phi_{i+1,j}^{2h}),\\
\phi_{i,2j+1}^{h} &= \frac{1}{2}(\phi_{ij}^{2h} + \phi_{i,j+1}^{2h}),\\
\phi_{2i+1,2j+1}^{h} &= \frac{1}{4}(\phi_{ij}^{2h} + \phi_{i+1,j}^{2h} + \phi_{i,j+1}^{2h} + \phi_{i+1,j+1}^{2h}),
\end{aligned}$$

for $0 \le i,j \le I/2-1$, where I is the number of elements in the fine grid. The injection operator maps the fine grid into the coarse:

$$I_{h}^{2h} : \Omega^{h} \to \Omega^{2h},$$

by simply assigning the fine grid values to the corresponding coarse grid points

$$\phi_{i,j}^{2h} = \phi_{2i-1,2j-1}^{h},$$

for $i,j = 1,\dots,I-1$.

Now we pass to the presentation of the conjugate gradient solution of the approximate problem. The discrete space, V_h, which approximates V is defined on the discrete domain Ω_{2h} at the points of the finite difference mesh. The $L^2(\Omega)$ inner product, $(\cdot,\cdot)_h$, is defined by a trapezoid integration over the discrete domain.

The fundamental equation $\Lambda \boldsymbol{e} = \boldsymbol{f}$ is approximated by the variational problem in V_h:

$$\left|\begin{array}{l} \text{Find} \quad \boldsymbol{e}_h \in V_h \text{ such that} \\ \langle \Lambda_{h,\Delta t} \boldsymbol{e}_h, \boldsymbol{v}_h \rangle = \langle u^1, v_h^0 \rangle - \int\limits_{\Omega} u^0 v_h^1 \, dx, \quad \forall \boldsymbol{v}_h = \{v_h^0, v_h^1\} \in V_h, \end{array}\right. \tag{94}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. The discrete operator $\langle \Lambda_{h,\Delta t} \cdot, \cdot \rangle$ is symmetric and positive definite for T large enough. Consequently, problem (94) can be solved by a conjugate gradient algorithm operating in V_h. Asch and Lebeau [8] described this algorithm in the case where $\Omega = (0,1) \times (0,1)$. As we have mentioned earlier, these authors used finite difference approximations. Various norms of the control g and the energetic cost of control have also been discussed. The norm of g under oscillatory initial data is also oscillatory.

Example 1. Asch and Lebeau [8] solved the problem of Dirichlet control of the unit square with a square cavity, cf. Fig. 2. The results of numerical calculations are depicted in Figs. 3-5.

The results obtained show that controlling on the interior boundary requires substantially more time and much larger control energy.

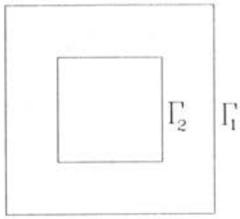

FIGURE 2. The square cavity

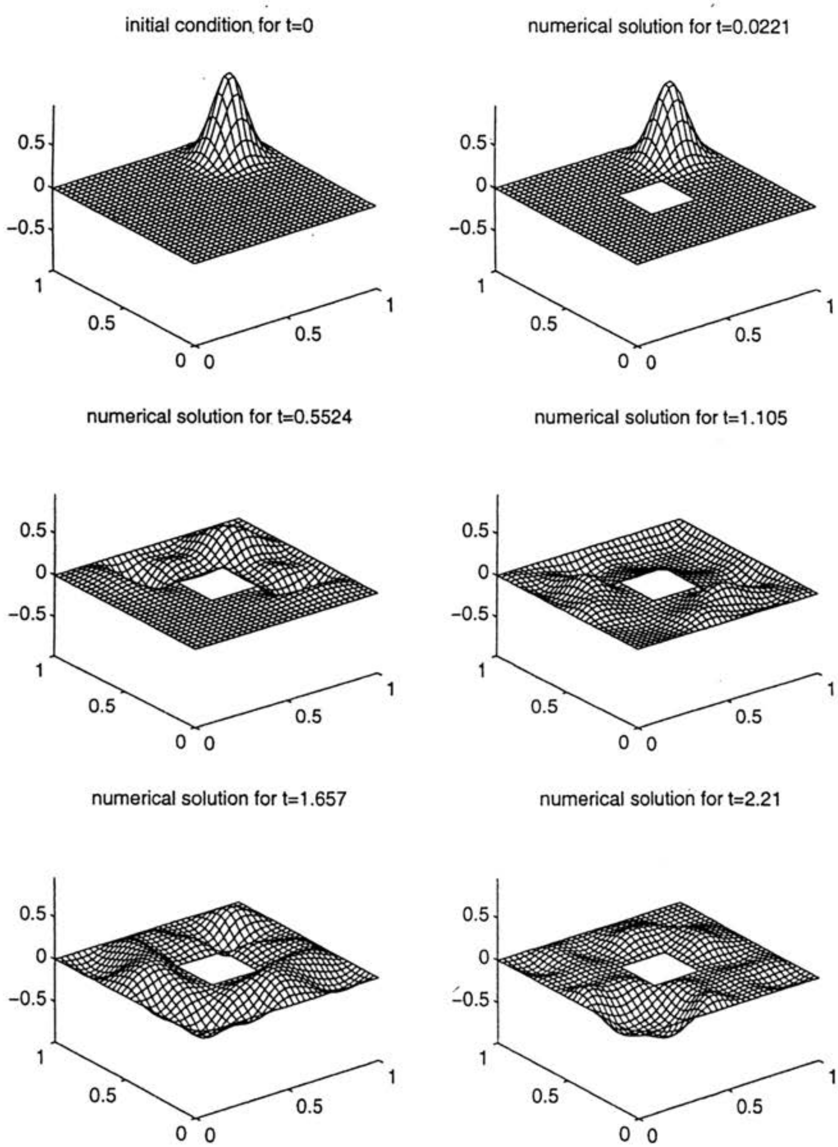

FIGURE 3. Solution of wave equation on square cavity – no control, after [8]

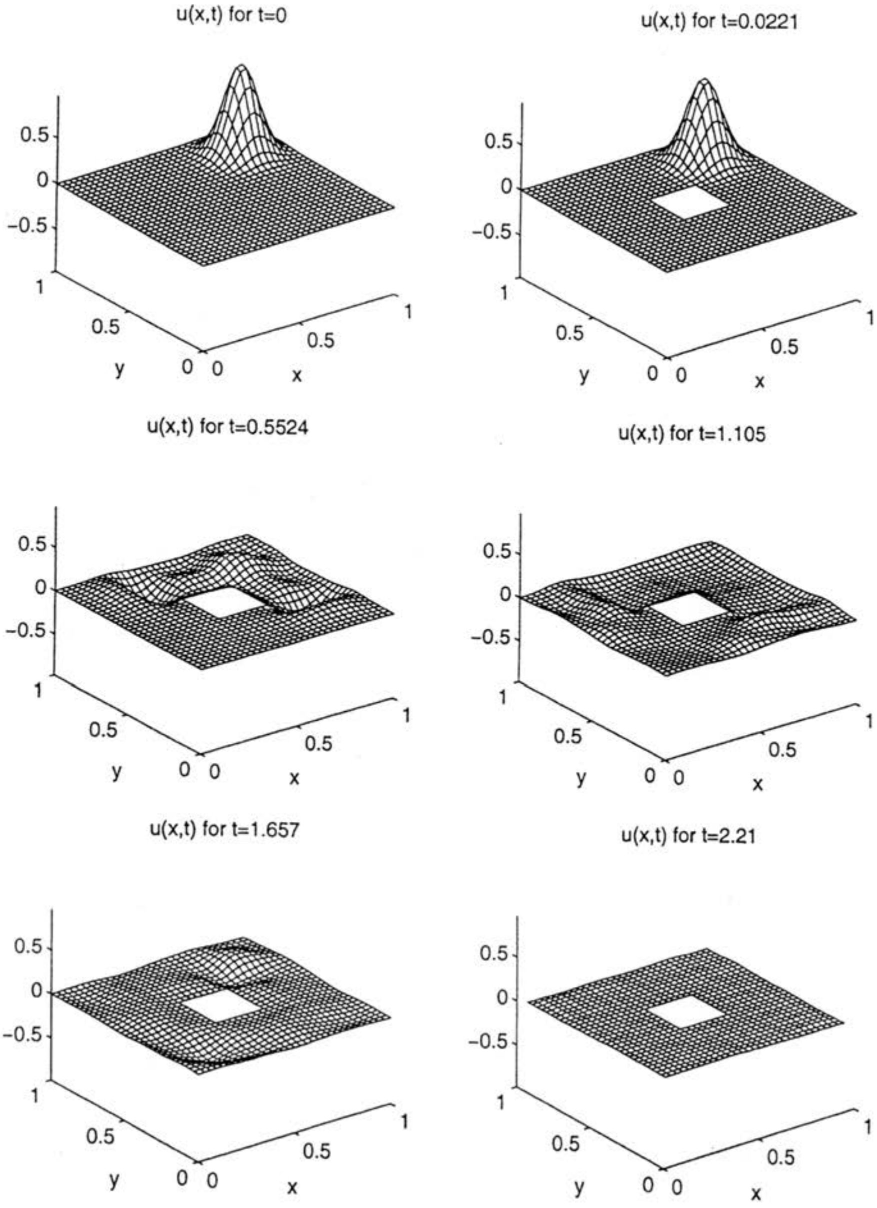

FIGURE 4. Solution of wave equation on square cavity – control on Γ_1, after [8]

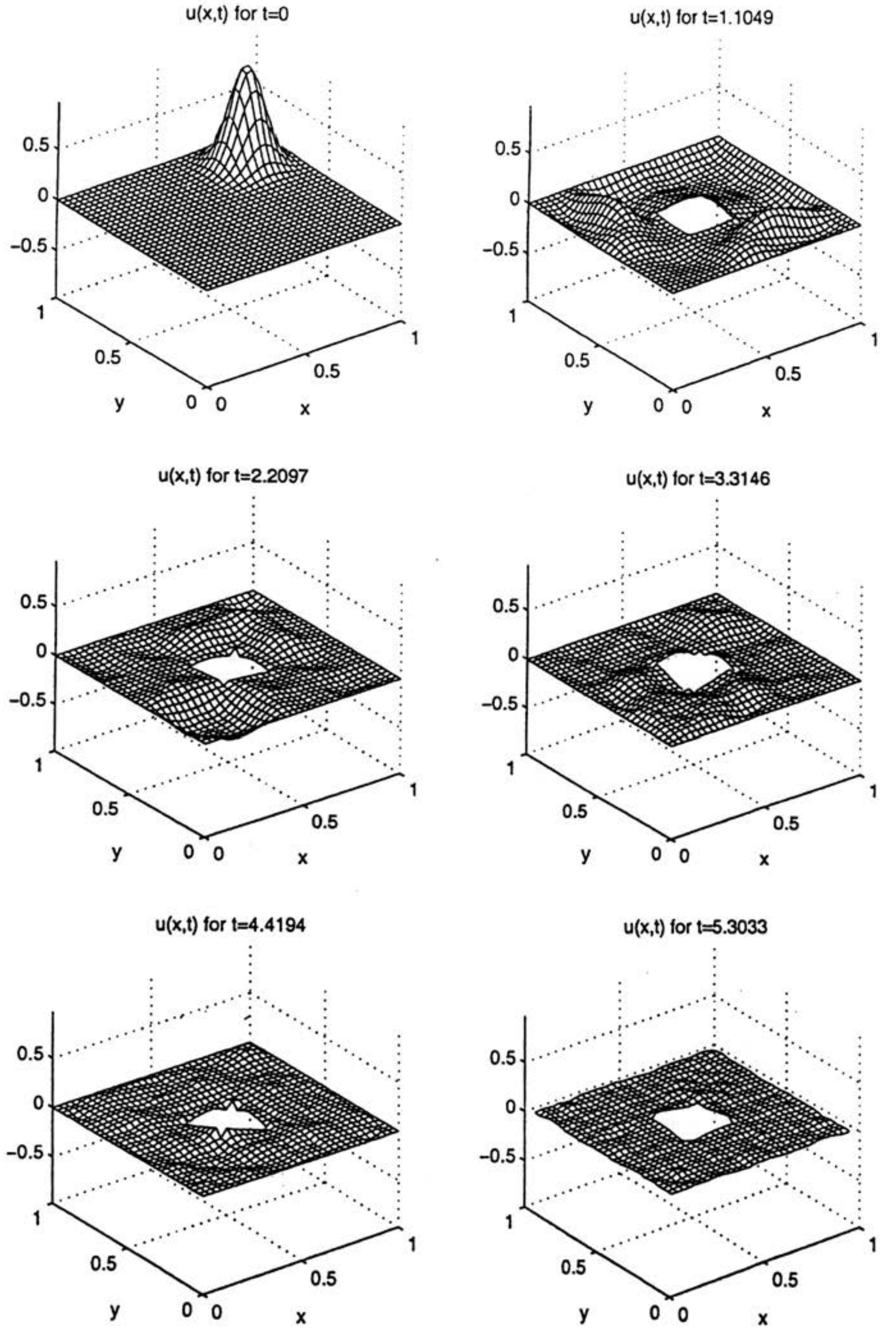

FIGURE 5. Solution of wave equation on square cavity – control on Γ_2, after [8]

Remark 7. Glowinski and Lions [67] investigated not only the exact controllability of the wave equation but also the approximate Dirichlet and Neumann boundary controllability. These authors discussed also the dual formulation, cf. also Glowinski and Lions [66].

FEM algorithms for partially observed system were developed in [79, 127], see also [180]. One can also consider distributed and pointwise control of the wave equation cf. [46, 171, 180].

4. The Stokes, Navier–Stokes, Boussinesq and combustion equations

The Stokes and Navier–Stokes equations are a fascinating subject of many papers. It is thus not surprising that they have also been studied from the point of view of controllability. In this section we shall review the papers by Coron [35], Fattorini and Sritharan [51], Fursikov and Imanuvilov [60], Glowinski and Lions [67], and Imanuvilov [76], where previous results have also been discussed, including the controllability of perfect fluids (Euler equations), cf. also [36–38, 50, 52, 55–59, 64]. Boussinesq equations describe flow of viscous fluids with thermal effects [61, 62]. Also, the combustion equations are a generalization of the Navier–Stokes equations [51].

4.1. The Navier–Stokes equations

Imanuvilov [76] studied the *local exact controllability* of the Navier–Stokes equations, defined on a bounded domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ with the boundary $\Gamma = \partial\Omega$ of class C^∞. More precisely, let us consider the nonstationary Navier–Stokes equations

$$\dot{\boldsymbol{v}}(x,t) - \Delta \boldsymbol{v}(x,t) + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}(x,t) + \nabla p(x,t) = \boldsymbol{f}(x) + \chi_\omega \boldsymbol{u}(x,t), \quad \text{in} \quad \Omega\times(0,T), \tag{95}$$

$$\operatorname{div}\boldsymbol{v} = 0, \quad \text{in} \quad \Omega\times(0,T), \tag{96}$$

$$\boldsymbol{v} = \boldsymbol{0}, \quad \text{on} \quad \Sigma = \Gamma\times(0,T), \tag{97}$$

$$\boldsymbol{v}(x,0) = \boldsymbol{v}^0(x), \quad \text{in} \quad \Omega. \tag{98}$$

Here $\boldsymbol{v}(x,t) = \big(v_1(x,t),\dots,v_n(x,t)\big)$ is the fluid velocity, p denotes the pressure, $\boldsymbol{f}(x) = \big(f_1(x),\dots,f_n(x)\big)$ is the density of body forces, and $\boldsymbol{u}(x,t)$ is a control distributed in an arbitrary fixed subdomain ω of Ω; χ_ω denotes the characteristic function of the set ω.

Let $(\hat{\boldsymbol{v}}, \hat{p}(x))$ be a steady-state solution of the Navier–Stokes equations

$$-\Delta\hat{\boldsymbol{v}} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla\hat{p} = \boldsymbol{f}(x), \quad \text{in} \quad \Omega, \tag{99}$$

$$\operatorname{div}\hat{\boldsymbol{v}} = 0, \quad \text{in} \quad \Omega, \tag{100}$$

$$\hat{\boldsymbol{v}} = \boldsymbol{0}, \quad \text{on} \quad \Gamma, \tag{101}$$

sufficiently close to the initial condition

$$\left\|\boldsymbol{v}^0 - \hat{\boldsymbol{v}}\right\|_{V^1(\Omega)} \le \varepsilon. \tag{102}$$

Here the parameter ε is sufficiently small and

$$V^1(\Omega) = \Big\{ \boldsymbol{v}(x) = \big(v_1(x), \ldots, v_n(x)\big) \quad | \quad v_i \in H^1(\Omega), \quad \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \quad \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma \Big\}.$$

Our aim is to find a control $\boldsymbol{u}$ such that for given $T > 0$, the following inequality holds

$$\boldsymbol{v}(x, T) = \hat{\boldsymbol{v}}(x). \tag{103}$$

We introduce also the space

$$\mathcal{V}(Q) = \Big\{ \boldsymbol{v}(x,t) \in H^1(Q)^n \quad | \quad \operatorname{div} \boldsymbol{v} = \boldsymbol{0} \text{ in } Q, \quad \boldsymbol{v} = \boldsymbol{0} \text{ on } \Sigma \Big\}.$$

We recall that $Q = \Omega \times (0,T)$. The main result due to Imanuvilov [76] can be stated as follows.

Theorem 10: *Let $\boldsymbol{v}^0 \in V^1(\Omega)$ and let $(\hat{\boldsymbol{v}}, \hat{p}) \in [(V^1(\Omega) \cap W^{1,\infty}(\Omega)^n] \times H^1(\Omega)$ be a given steady state solution of the Navier–Stokes equations (99)-(101) such that* $\operatorname{supp}\hat{\boldsymbol{v}} \subset\subset \Omega$. *Then for sufficiently small ε there exists a solution $(\boldsymbol{v}, p, \boldsymbol{u}) \in V^1(Q) \times L^2(0,T; H^1(\Omega) \times L^2(Q_\omega)^n$ of problem (95)-(98), (102), (103).*

Here Q_ω denotes the set $\omega \times (0,T)$. To prove the last theorem, Imanuvilov [76] used a variant of the implicit function theorem. The only nontrivial condition to be verified is to show that the derivative of the corresponding mapping at some point is an epimorphism. In the case discussed, this problem is equivalent to the (exact) null-controllability of the linearized Navier–Stokes equations at the point $\hat{\boldsymbol{v}}$. We observe that now the final velocity at $t = T$ cannot be arbitrarily prescribed, since it is a solution of Eqs. (99)-(101).

Remark 8. To clarify Theorem 10, let us assume that $\hat{\boldsymbol{v}} = \boldsymbol{0}$ on Γ and $\hat{\boldsymbol{v}}$ is an unstable singular point of the dynamical system generated by Eqs. (95), (96) in the phase space of solenoidal vector fields vanishing on Γ. Let $\boldsymbol{v}^0$ be an initial condition in a neighborhood of the function $\hat{\boldsymbol{v}}$. Theorem 10 shows that one can construct a locally distributed control such that the trajectory goes out of the point $\hat{\boldsymbol{v}}^0$ and reaches $\hat{\boldsymbol{v}}$ in finite time. In other words, by means of the locally distributed control, one can suppress the generation of turbulence.

Remark 9. Fursikov and Imanuvilov [60] studied the exact local boundary controllability problem for the Navier–Stokes equations. Then the control $\boldsymbol{u}$ is defined on $\Sigma = \Gamma \times (0,T)$; the body forces are allowed to depend on x and t.

Coron [35] examined the problem of approximate controllability of the 2D Navier–Stokes equations, provided that the Navier slip boundary condition occurs on a part of the boundary.

Let us pass to the presentation of the main result due to Coron [35]. Let Ω be a bounded nonempty connected open subset of $\mathbb{R}^2$ of class C^∞. Let ω be an open subset of Ω and let Γ_0 be an open subset of $\Gamma = \partial\Omega$. We assume that

$$\omega \cup \Gamma_0 \neq \emptyset. \tag{104}$$

On the part of the boundary $\Gamma \backslash \Gamma_0$, where there is no control, the fluid slips and the following condition is satisfied

$$\boldsymbol{v} \cdot \boldsymbol{n} = v_\alpha n_\alpha, \quad \text{on} \quad \Gamma \backslash \Gamma_0. \tag{105}$$

The Navier slip boundary condition is given by

$$\bar{\sigma}\boldsymbol{v}\cdot\boldsymbol{\tau} + 2(1-\bar{\sigma})e_{\alpha\beta}(\boldsymbol{v})n_\alpha\tau_\beta = 0, \quad \text{on} \quad \Gamma\backslash\Gamma_0, \tag{106}$$

where

$$e_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right), \tag{107}$$

and $\boldsymbol{\tau}$ denotes the unit tangent vector field on Γ such that $\{\boldsymbol{\tau},\boldsymbol{n}\}$ is a direct basis of $\mathbb{R}^2$; $\bar{\sigma}$ is a constant in $[0,1)$. We observe that the classical no-slip condition is expressed by

$$\boldsymbol{v} = \boldsymbol{0}, \quad \text{on} \quad \Gamma\backslash\Gamma_0,$$

and corresponds to $\bar{\sigma} = 1$. This case was not studied by Coron [35]. The next extreme case, $\bar{\sigma} = 0$, correspond to the case where the fluid slips (on $\Gamma\backslash\Gamma_0$) without friction. The case $\bar{\sigma} \in (0,1)$ corresponds to the case where the fluid slips on the wall with friction.

Let us pass to the problem of approximate controllability. Let $T > 0$ and $\boldsymbol{v}^0$ and $\boldsymbol{v}_T$ in $C^\infty(\bar{\Omega})^2$ be such that

$$\operatorname{div}\boldsymbol{v}^0 = 0, \quad \text{in} \quad \bar{\Omega}, \tag{108}$$

$$\operatorname{div}\boldsymbol{v}_T = 0, \quad \text{in} \quad \bar{\Omega}, \tag{109}$$

$$\boldsymbol{v}^0\cdot\boldsymbol{n} = 0, \quad \text{on} \quad \Gamma\backslash\Gamma_0, \tag{110}$$

$$\boldsymbol{v}_T\cdot\boldsymbol{n} = 0, \quad \text{on} \quad \Gamma\backslash\Gamma_0, \tag{111}$$

$$\sigma\boldsymbol{v}^0\cdot\boldsymbol{\tau} + \operatorname{curl}\boldsymbol{v}^0 = 0, \quad \text{on} \quad \Gamma\backslash\Gamma_0, \tag{112}$$

$$\sigma\boldsymbol{v}_T\cdot\boldsymbol{\tau} + \operatorname{curl}\boldsymbol{v}_T = 0, \quad \text{on} \quad \Gamma\backslash\Gamma_0, \tag{113}$$

where

$$\sigma(x) = \frac{2(1-\bar{\sigma}-\kappa(x)-\bar{\sigma})}{1-\bar{\sigma}}, \quad x \in \Gamma. \tag{114}$$

Here κ is the curvature of Γ defined by $\frac{\partial\boldsymbol{n}}{\partial\boldsymbol{\tau}} = \kappa\boldsymbol{\tau}$. We observe that (106) is equivalent to

$$\sigma\boldsymbol{v}\cdot\boldsymbol{\tau} + \operatorname{curl}\boldsymbol{v} = 0, \quad \text{on} \quad \Gamma\backslash\Gamma_0, \tag{115}$$

where σ is defined by (114). To obtain Eq. (115), the relation (105) have been taken into account.

Now the question is whether there exists $\boldsymbol{v} \in C^\infty(\Omega\times[0,T])^2$ and $p \in (\bar{\Omega}\times[0,T])$ such that

$$\dot{\boldsymbol{v}} - \Delta\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p = 0, \quad \text{in} \quad (\bar{\Omega}\backslash\omega)\times[0,T], \tag{116}$$

$$\operatorname{div}\boldsymbol{v} = 0, \quad \text{in} \quad \bar{\Omega}\times[0,T], \tag{117}$$

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0, \quad \text{on} \quad (\Gamma\backslash\Gamma_0)\times[0,T], \tag{118}$$

$$\sigma\boldsymbol{v}\cdot\boldsymbol{\tau} + \operatorname{curl}\boldsymbol{v} = 0, \quad \text{on} \quad (\Gamma\backslash\Gamma_0)\times[0,T], \tag{119}$$

$$\boldsymbol{v}(x,0) = \boldsymbol{v}^0(x), \quad \text{in} \quad \bar{\Omega}, \tag{120}$$

and, in a suitable topology,

$$v(x,T) \text{ is "close" to } v_T. \tag{121}$$

In other words, we ask whether starting with the initial data v^0 for the Navier–Stokes equations, there are solutions which, at a fixed time T, approach arbitrarily closely to the given velocity field v_T.

We recall that (116)-(120) have many solutions. In order to have uniqueness one needs to add additional conditions. Obviously, these extra conditions are the controls. A possible choice of the controls is

$$v \cdot n, \quad \text{on} \quad \Gamma_0 \times [0,T], \tag{122}$$

$$\sigma v \cdot \tau = \operatorname{curl} v, \quad \text{on} \quad \Gamma_0 \times [0,T], \tag{123}$$

$$\dot{v} - \Delta v + (v \cdot \nabla) v + \nabla p, \quad \text{in} \quad \omega \times [0,T]. \tag{124}$$

More precisely, let $v^s \in C^\infty(\bar{\Omega} \times [0,T])^2$ and $p^s \in C^\infty(\bar{\Omega} \times [0,T])$ be such that (116)-(120) hold for $(v,p) = (v^s, p^s)$. Let us consider the following Cauchy problem:

Find $v \in C^\infty(\bar{\Omega} \times [0,T])^2$ and $p \in C^\infty(\bar{\Omega} \times [0,T])$ such that (116)-(120) and

$$v \cdot n = v^s \cdot n, \quad \text{on} \quad \Gamma_0 \times [0,T],$$

$$\sigma v \cdot \tau + \operatorname{curl} v = \sigma v^s \cdot \tau + \operatorname{curl} v^s, \quad \text{on} \quad \Gamma_0 \times [0,T],$$

$$\dot{v} - \Delta v + (v \cdot \nabla) v + \nabla p = \dot{v}^s - \Delta v^s + (v^s \cdot \nabla) v^s + \nabla p^s, \quad \text{in} \quad \omega \times [0,T],$$

has, up to an arbitrary function depending only on time added to p, one and only one solution which is $(v,p) = (v^s, p^s)$.

One can also use for the control (122), (124), and $\operatorname{curl} v$ on $\Gamma_0 \times [0,T]$. Another possibility for the control is (124) and v on $\Gamma_0 \times [0,T]$.

Let us define the distance function $d \in C(\bar{\Omega})$ by

$$d(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \Gamma) = \min\{|\mathbf{x} - \mathbf{x}'|;\ \mathbf{x}' \in \Gamma\}.$$

The controllability result due to Coron [35] is formulated as follows.

Theorem 11: *Let $T > 0$, let v^0 and v_T in $C^\infty(\bar{\Omega})^2$ be such that (106)-(113) hold. Then, there exist a sequence $\{v^k\}_{k\in\mathbb{N}}$ in $C^\infty(\bar{\Omega} \times [0,T])^2$ and a sequence $\{p^k\}_{k\in\mathbb{N}}$ in $C^\infty(\bar{\Omega} \times [0,T])$ such that, for all $k \in \mathbb{N}$, (116)-(120) hold for $v = v^k$ and $p = p^k$ and such that, as $k \to \infty$,*

$$\int_\Omega d^\mu(x) |v^k(\mathbf{x},T) - v_T(\mathbf{x})| \, dx \to 0, \quad \forall\, \mu > 0, \tag{125}$$

$$\|v^k(\cdot,T) - v_T\|_{W^{-1,\infty}(\Omega)} \to 0, \tag{126}$$

and, for all compact K included in $\Omega \cup \Gamma_0$,

$$\|v^k(\cdot,T) - v_T\|_{L^\infty(K)} + \|\operatorname{curl} v^k(\cdot,T) - \operatorname{curl} v_T\|_{L^\infty(K)} \to 0. \tag{127}$$

Remark 10. In special circumstances stronger convergence in (126) occurs, see Remark 2.4 in Coron [35].

The proof of the Theorem 11 is divided into two cases. First, one considers the case where $\Gamma_0 = \emptyset$, hence $\omega \neq \emptyset$. Then one constructs solutions $(\boldsymbol{v}, p)$ of the controllability problem in a special manner by a decomposition technique. Next, the case $\Gamma_0 \neq \emptyset$ is considered and one applies the results of the previous case.

Controllability of the Boussinesq system

Fursikov and Imanuvilov [61, 62] solved the problem of exact, locally approximate and approximate controllability of the Boussinesq system. The procedure used is an extension of the approach applied to the Navier–Stokes equations . Therefore we shall only formulate the exact controllability problem.

Consider the Boussinesq system in $\Omega \times (0,T)$:

$$\begin{aligned}
\dot{\mathbf{v}}(\mathbf{x},t) - \nu\Delta\mathbf{v}(\mathbf{x},t) + (\mathbf{v}\cdot\nabla)\mathbf{v} + \theta(\mathbf{x},t)\mathbf{e} + \nabla p(\mathbf{x},t) &= \mathbf{u}(\mathbf{x},t),\\
\dot{\theta}(\mathbf{x},t) - \kappa\Delta\theta + (\mathbf{v}\cdot\nabla)\theta &= r(\mathbf{x},t),\\
\operatorname{div}\mathbf{v} &= 0,
\end{aligned} \tag{128}$$

$$\mathbf{v}(\mathbf{x},t) = \boldsymbol{\alpha}(\mathbf{x},t), \quad \theta(\mathbf{x},t) = \beta(\mathbf{x},t), \quad \text{on} \quad \Gamma \times (0,T),$$

$$\mathbf{v}(\mathbf{x},0) = \mathbf{v}^0(\mathbf{x}), \quad \theta(\mathbf{x},0) = \theta^0(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

Here $\theta(\mathbf{x},t)$ is the fluid temperature and $\mathbf{e}$ denotes the direction of the gravity forces.

The boundary exact controllability is now formulated as follows: let

$$(\hat{\mathbf{v}}, \nabla\hat{p}, \hat{\theta}) \in C^1\left(0,T; V^4(\Omega)\right) \times L^2(Q)^n \times C^1\left(0,T; H^4(\Omega)\right)$$

satisfies $(128)_{1,2}$; find control functions

$$(\boldsymbol{\alpha}, \beta) \in L^2\left(0,T; H^{3/2}(\Gamma)^n\right) \times L^2\left(0,t; H^{3/2}(\Gamma)\right)$$

such that the solution $(\mathbf{v}, p, \theta)$ of (128) satisfies

$$\mathbf{v}(\mathbf{x},T) = \hat{\mathbf{v}}(\mathbf{x},T), \qquad \theta(\mathbf{x},T) = \hat{\theta}(\mathbf{x},T). \tag{129}$$

Under physically plausible assumptions the Boussinesq system is exactly controllable.

4.2. Turbulent flow and its control

The most important conclusion which follows from the previous section is that one can minimize turbulence. In combustion the objective is to increase turbulence for a better mixing of the fuel and its oxidant.

4.2.1. State and control constraints via nonlinear programing in Banach spaces: the Navier–Stokes and combustion equations. Fattorini and Sritharan [51] used nonlinear programming theory in Banach spaces and derived a version of Pontryagin's maximum principle and applied it to distributed parameter systems under state and control constraints. General results were applied to the Navier–Stokes equations and combustion problem, cf. also [3, 49].

Consider the following Navier–Stokes equations

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p(\mathbf{x},t) + \nu\Delta\mathbf{v}(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t) \quad &\text{in} \quad \Omega\times(0,T),\\ \operatorname{div}\mathbf{v} = 0 \quad &\text{in} \quad \Omega\times(0,T),\\ \mathbf{v} = \mathbf{0} \quad &\text{on} \quad \Gamma\times(0,T),\\ \mathbf{v}(\mathbf{x},t) = \mathbf{v}^0(\mathbf{x}) \quad &\text{in} \quad \Omega. \end{aligned} \tag{130}$$

Here $\nu > 0$ is the coefficient of kinematic viscosity and, as usual, $\mathbf{u}(\mathbf{x},t)$ denotes an external force considered as a control. We observe that distributed force control of this type can be realized in practice by electromagnetic forcing. Realistic constraints on the control are either pointwise,

$$|\mathbf{u}(\mathbf{x},t)| \le C, \quad (\mathbf{x},t)\in\Omega\times[0,T], \tag{131}$$

or integral,

$$\int_\Omega |\mathbf{u}(\mathbf{x},t)|^p dx \le C, \quad t\in[0,T], \tag{132}$$

with $1 \le p < \infty$, where $|\,.\,|$ denotes a norm in $\mathbb{R}^n$. We recall that T is the time at which the control process terminates. Both constraints can be written in the form

$$\mathbf{u}(\cdot,t)\in\mathbb{U}, \quad t\in[0,T] \tag{133}$$

where $\mathbb{U}$ is a subset of $L^r(\Omega)^n$, $1\le r\le\infty$.

Natural state constraints are the velocity constraint

$$|\mathbf{v}(\mathbf{x},t)| \le C, \quad (\mathbf{x},t)\in\Omega\times(0,T), \tag{134}$$

the vorticity constraint

$$|\nabla\times\mathbf{v}(\mathbf{x},t)| \le C, \quad (\mathbf{x},t)\in\Omega\times(0,T), \tag{135}$$

and the strain rate constraint

$$|\mathbf{e}(\mathbf{v}(\mathbf{x},t))| \le C, \quad (\mathbf{x},t)\in\Omega\times(0,T), \tag{136}$$

where

$$e_{ij}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{137}$$

All of the above are particular cases of the constraint

$$\mathcal{S}\mathbf{v}(\mathbf{x},t)\in\mathcal{M}_S\subseteq\mathbb{R}^k, \quad (\mathbf{x},t)\in\Omega\times(0,T),$$

where $\mathcal{S}$ is a differential operator of the form

$$\mathcal{S}\mathbf{v}(\mathbf{x},t) = \chi_j(\mathbf{x})v_j(\mathbf{x},t) + \chi_{ij}(\mathbf{x})\frac{\partial y_i(\mathbf{x},t)}{\partial x_j}, \quad i,j=1,\dots,n. \tag{138}$$

Here $\chi_j(\mathbf{x})$ and $\chi_{ij}(\mathbf{x})$ are defined in $\bar{\Omega}$ and the summation convention applies to repeated indices.

Consider also a separate state constraint at the terminal time T, given by

$$\mathcal{T}\mathbf{v}(\mathbf{x},t) \in \mathcal{M}_\Upsilon \subseteq \mathbb{R}^l, \quad \mathbf{x} \in \Omega, \tag{139}$$

and called a *target condition*, with $\mathcal{T}$ a differential operator of the same form as $\mathcal{S}$

$$\mathcal{T}\mathbf{v}(\mathbf{x},t) = \eta_j(\mathbf{x})v_j(\mathbf{x},t) + \eta_{ij}(\mathbf{x})\frac{\partial v_i(\mathbf{x},t)}{\partial x_j}, \tag{140}$$

where η_j and η_{ij} are defined in $\bar{\Omega}$.

In certain situations, such as a turbulence suppression in selected regions, state constraints are only required in a subset $\omega \subset \Omega$. This can be handled by multiplying the coefficients of $\mathcal{S}, T$ by the characteristic function of ω. State conditions can also be of integral type, say

$$\mathcal{S}\mathbf{v}(\cdot,t) \in \mathcal{M}_S, \quad \mathbf{x} \in \Omega, \quad t \in [0,T], \tag{141}$$

where $\mathcal{M}_S$ is a subset of $L^r(\Omega)^k$ and the same applies to target conditions. State constraints and target conditions can also be expressed by nonlinear differential operators.

It is convenient to consider the Navier–Stokes equations as abstract differential equations [3, 49, 51]. It can be shown that the system (130) can be written as the abstract semilinear equation

$$\dot{\mathbf{v}}(t) = A_p\mathbf{v}(t) + N(\mathbf{v}(t)) + B\mathbf{u}(t), \quad \mathbf{v}(0) = \mathbf{v}^0, \tag{142}$$

for the velocity $\mathbf{v}(t)(\mathbf{x}) = \mathbf{v}(t,\mathbf{x})$ in the space $X^p(\Omega)$, where

$$N(\mathbf{v}) = -P_p(\mathbf{v}\cdot\nabla)\mathbf{v}, \quad B = P_pI_p. \tag{143}$$

Here I_p is the identity operator from $L^r(\Omega)^n$ into $L^p(\Omega)^n$. The *Stokes operator* A_p is defined by

$$A_p = \nu P_p\Delta_p, \quad D(A_p) = X^p(\Omega)^n \cap D(\Delta_p). \tag{144}$$

The space $X^p(\Omega)^n$ is the closure in $L^p(\Omega)^n$ of divergence–free function from $C_0^\infty(\Omega)^n$; we have

$$L^p(\Omega)^n = X^p(\Omega)^n \oplus G^p(\Omega)^n. \tag{145}$$

The direct sum is orthogonal if $p = 2$; if $p \neq 2$ the sum is norm–direct, that is, the projection

$$P_p : L^p(\Omega)^n \to X^p(\Omega)^n$$

is a bounded operator. The operator Δ_p is the (n-vector) Laplacian in $L^p(\Omega)^n$ with domain

$$D(A_p) = W_0^{2,p}(\Omega)^n = \left\{\mathbf{v} \in W^{2,p}(\Omega)^n \mid \mathbf{v} = \mathbf{0} \quad \text{on } \Gamma\right\}.$$

Fattorini and Sritharan [51] presented the solution of Eq. (142) in the form of a nonlinear integral equation. These authors claim that local solutions of this equation can be constructed by successive approximations or as a fixed points of a contraction map.

Optimal problem for (130) deals with the minimalization of the *cost functional*

$$J_0(t,\mathbf{u}) = \int_0^t f_0(\tau,\mathbf{v},\mathbf{u})d\tau$$

over a fixed or variable time interval $0 \le t \le T$, where the control $\mathbf{u}(\mathbf{x},t)$ satisfies the control constraint and is such that the solution $\mathbf{v}(\mathbf{x},t)$ satisfies the state constraint; the integrand $f_0(t,\mathbf{v},\mathbf{u})$ under consideration include, for instance,

$$f_0(t,\mathbf{v},\mathbf{u}) = \int_\Omega \left\{ |\nabla(\mathbf{v}(\mathbf{x},t) - \mathbf{v}_d(\mathbf{x}))|^2 + |\mathbf{u}(\mathbf{x},t)|^2 \right\} dx$$

which penalizes deviation from a desired state $\mathbf{v}_d(\mathbf{x})$ and control cost.

Fattorini and Sritharan [51] derived a version of Pontryagin's principle for the following general model

$$\dot{\mathbf{v}}(t) = A\mathbf{v}(t) + g(t,\mathbf{v}(t),\mathbf{u}(t)). \tag{146}$$

We see that Eq. (142) is a specific case of the last equation. Since even the formulation of the Pontryagin's principle for the Navier–Stokes system with state and control constraints is quite sophisticated, we refer the interested reader to [51].

Instead, we shall briefly comment on the combustion model. Similarly to the Navier–Stokes equations, $\mathbf{v}(\mathbf{x},t)$ and $p(\mathbf{x},t)$ denote the velocity and pressure [51] and the functions $\Psi_1(\mathbf{x},t)\ldots\Psi_m(\mathbf{x},t)$ are the scalar fields used in combustion: $\Psi_1(\mathbf{x},t)$ is the temperature field and $\Psi_2(\mathbf{x},t)\ldots\Psi_m(\mathbf{x},t)$ are the components of the reactant and burnt product. Introducing the vector function $\boldsymbol{\Psi}(\mathbf{x},t) = (\Psi_1(\mathbf{x},t)\ldots\Psi_m(\mathbf{x},t))$ the equations are

$$\begin{aligned}
\dot{\mathbf{v}} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p(\mathbf{x},t) + \nu\Delta\mathbf{v}(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t), &\quad \text{in} \quad \Omega\times(0,T),\\
\operatorname{div}\mathbf{v} = 0, &\quad \text{in} \quad \Omega\times(0,T),\\
\mathbf{v} = \mathbf{0} &\quad \text{in} \quad \Gamma\times(0,T),\\
\dot{\boldsymbol{\Psi}} + (\mathbf{v}\cdot\nabla)\boldsymbol{\Psi} = \mathbf{h}(\boldsymbol{\Psi}) + \Delta'\boldsymbol{\Psi}(\mathbf{x},t) + \mathbf{w}(\mathbf{x},t) &\quad \text{in} \quad \Omega\times(0,T),
\end{aligned} \tag{147}$$

where $\Delta'\boldsymbol{\Psi} = (\nu_1\Delta\Psi_1,\ldots,\nu_m\Delta\Psi_m)$, $\mathbf{w}(\mathbf{x},t)$ is an additional m-vector control, and the components $h_j(\Psi_1,\ldots,\Psi_m)$ of $\mathbf{h}$ come from Arrhenius' combustion law. The functions Ψ_j satisfy boundary conditions on Γ, either Dirichlet as $\mathbf{v}$,

$$\boldsymbol{\Psi} = \mathbf{0} \quad \text{on} \quad \Gamma\times(0,T),$$

or of variational type. Control and state constraints, optimal problems and cost functionals are formulated in the same manner as for the Navier–Stokes system.

4.2.2. Minimization of vorticity. In a seminal paper Abergel and Temam [3] studied three problems of control of turbulence. The first problem deals with the minimization of turbulence in a Navier–Stokes flow by *distributed controls* whilst the remaining two problems are specific cases of the Boussinesq equations. Here we shall investigate only the first problem of optimal control.

Consider a system of Navier–Stokes equations in an open set Ω of $\mathbb{R}^2$ of class C^2, cf. Sec. 4.2.1

$$
\begin{aligned}
r\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f} + \nu\Delta\mathbf{v} \quad &\text{in} \quad \Omega\times(0,T),\\
\operatorname{div}\mathbf{v} = 0 \quad &\text{in} \quad \Omega\times(0,T),\\
\mathbf{v} = \mathbf{0} \quad &\text{on} \quad \Gamma\times(0,T),\\
\mathbf{v}(\mathbf{x},0) = \mathbf{v}^0(\mathbf{x}) \quad &\text{in} \quad \Omega,
\end{aligned}
\tag{148}
$$

where the forcing term $\mathbf{f}$ is the control. We set

$$
X = \left\{\mathbf{v}\in C_0^\infty(\Omega)^2 \mid \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\right\}
$$

and denote by H (resp. V) the closure of X in $L^2(\Omega)^2$ (resp. $H_0^1(\Omega)^2$). As we already know, the Navier–Stokes equations in Ω can be written in the following form

$$
\begin{aligned}
\frac{d\mathbf{v}}{dt} + \nu A\mathbf{v} + N(\mathbf{v}) = \mathbf{f} \quad &\text{in} \quad Q = \Omega\times(0,T),\\
\mathbf{v}(\cdot,t)\in V \quad &\forall\, t\in(0,T),\\
\mathbf{v}(0) = \mathbf{v}^0 \quad &\text{in} \quad \Omega.
\end{aligned}
\tag{149}
$$

The forcing term $\mathbf{f}$ is in $L^2(0,T;H)$ and $\mathbf{v}^0$ in H. We see that the considerations performed in [3] are confined to $p=2$, i.e. to the Hilbert spaces. The operators A, N are defined as follows: $A\mathbf{v} = -P\Delta\mathbf{v}$, where P is the orthogonal projector from $L^2(\Omega)^2$ onto H and N is the nonlinear operator from V into its dual V', such that

$$
\langle N(\mathbf{v}),\mathbf{w}\rangle_{V'\times V} = \int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,dx \quad \forall\,\mathbf{w}\in V.
$$

The variational formulation is easily obtained by multiplying $(149)_1$ by $\mathbf{w}\in X$, integrating by parts and taking into account $(149)_{2,3}$. Then the pressure is excluded. The classical existence and uniqueness results can be found in the book by Temam, cf. [3].

Proposition 1.

(i) Let $\mathbf{f}$, $\mathbf{v}^0$ be given in $L^2(0,T;V')$ and H; there exists a unique weak solution $\mathbf{v}$ of (149), and $\mathbf{v}$ belongs to $C(0,T;H)\cap L^2(0,T;V)$.

(ii) If we assume furthermore that $\mathbf{v}^0\in V$ and $\mathbf{f}\in L^2(0,T;H)$, then $\mathbf{v}\in C(0,T;V)\cap L^2(0,T;H^2(\Omega)^2)$ and $d\mathbf{v}/dt\in L^2(0,T;H)$.

Define the bilinear mapping $\mathfrak{B}(\mathbf{v},\mathbf{w}) = (\mathbf{v}\cdot\nabla)\mathbf{w}$ where $\mathbf{v},\mathbf{w}\in V$. Then $N(\mathbf{v}) = \mathfrak{B}(\mathbf{v},\mathbf{v})$ in V'. We have

Lemma 7: *Let $b(\mathbf{u},\mathbf{v},\mathbf{z})$ be the trilinear form on $V\times V\times V$ defined by*

$$
b(\mathbf{u},\mathbf{v},\mathbf{z}) = \int_\Omega u_\alpha v_{\beta,\alpha} z_\beta\,dx
$$

where $v_{\beta,\alpha} = \partial v_\beta/\partial x_\alpha$. Then *(i) b has the following properties*

$$\begin{aligned} &\textit{orthogonality:} \quad b(\mathbf{u},\mathbf{v},\mathbf{z}) = 0,\\ &\forall(\mathbf{u},\mathbf{v},\mathbf{z}) \in V^3 \quad |b(\mathbf{u},\mathbf{v},\mathbf{z})| \le ||\mathbf{u}||\,||\mathbf{v}||\,||\mathbf{z}||,\\ &\forall(\mathbf{u},\mathbf{v},\mathbf{z}) \in V^3 \quad |b(\mathbf{u},\mathbf{v},\mathbf{z})| \le \sqrt{2}\,|\mathbf{u}|^{1/2}\,||\mathbf{u}||^{1/2}\,||\mathbf{v}||\,|\mathbf{z}|^{1/2}\,||\mathbf{z}||^{1/2}, \end{aligned}$$

(ii) $\mathbf{v} \to N(\mathbf{v})$ *is differentiable from* V *into* V'*, and we have*

$$\forall\ \mathbf{v} \in V, N'(\mathbf{v})\mathbf{w} = \mathfrak{B}(\mathbf{v},\mathbf{w}) + \mathfrak{B}(\mathbf{w},\mathbf{v}).$$

(iii) Let $N'(\mathbf{v})^*$ *denote the adjoint of* $N'(\mathbf{v})$ *for the duality between* V *and* V'*, i.e.,* $\langle N'(\mathbf{u},\mathbf{v},\mathbf{z})\rangle = \langle \mathbf{v}N'(\mathbf{u})^*\mathbf{w}\rangle$*; we have*

$$\langle N'(\mathbf{u})^*\mathbf{v},\mathbf{w}\rangle_{V'\times V} = \int\limits_\Omega w_\beta\Big(\frac{\partial u_\alpha}{\partial x_\beta}v_\alpha - a_\alpha\frac{\partial v_\beta}{\partial x_\alpha}\Big)dx.$$

Here $||\mathbf{v}|| = ||\mathbf{v}||_V$ *and* $|\mathbf{v}| = ||\mathbf{v}||_H$.

In the case of three–dimensional flows the situation is more complicated since then the state equation may not be well-posed. However, the control of turbulence in such a case was also studied, cf. [3] and Sec. 4.2.1.

The *optimal control* problem is formulated as follows:

$$(\mathcal{P})\qquad \left|\begin{array}{l} \text{Find a control } \mathbf{f} \text{ minimizing the cost functional}\\[4pt] J(\mathbf{f}) = \dfrac{1}{2}\displaystyle\int\limits_0^T\int\limits_\Omega |\mathbf{f}(\mathbf{x},t)|^2dxdt + \dfrac{1}{2}\int\limits_0^T\int\limits_\Omega |\nabla\times\mathbf{v}_\mathbf{f}(\mathbf{x},t)|^2dxdt\\[4pt] \text{where } \mathbf{v}_\mathbf{f} \text{ is the solution of (149) associated with } \mathbf{f}. \end{array}\right.$$

Here $\nabla\times\mathbf{v}_\mathbf{f}$ denotes the curl of $\mathbf{v}_\mathbf{f}$. The second term in the functional $J(\mathbf{f})$ measures the vorticity whilst the term

$$\int\limits_0^T\int\limits_\Omega |\mathbf{f}|^2dxdt$$

is a concession to the demands of mathematical rigour (the coercivity term).

For problem $(\mathcal{P})$, the following results were obtained:

- the existence of an optimal control, which may not be unique,
- the determination of the first–order necessary conditions of optimality, which are obtained in a straightforward manner upon differentiation of the functional $J(\mathbf{f})$, they involve the so-called adjoint state corresponding to the adjoint of a linearized version of system (148).

More precisely, we have

Theorem 12:

(i) Let $(\bar{\mathbf{f}},\bar{\mathbf{v}})$ *be an optimal pair for problem* $(\mathcal{P})$*; the following equality holds*

$$\bar{\mathbf{f}} + \tilde{\mathbf{w}}(\nabla\times(\nabla\times\bar{\mathbf{v}})) = \mathbf{0},$$

where $\tilde{\mathbf{w}}(\nabla \times (\nabla \times \bar{\mathbf{v}}))$ *is the adjoint state that is the solution of the linearized adjoint problem*

$$
\begin{aligned}
-\frac{\partial \tilde{\mathbf{w}}}{\partial t} - \mu \Delta \tilde{\mathbf{w}} + (\nabla \bar{\mathbf{v}})^T \cdot \tilde{\mathbf{w}} - (\bar{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{w}} + \nabla \tilde{q} = \nabla \times (\nabla \times \bar{\mathbf{v}}) \quad &in \quad Q,\\
div\, \tilde{\mathbf{w}} = 0 \quad &in \quad Q,\\
\tilde{\mathbf{w}} = \mathbf{0} \quad &on \quad \Gamma \times (0,T),\\
\tilde{\mathbf{w}}(\mathbf{x},T) = \mathbf{0} \quad &in \quad \Omega.
\end{aligned}
$$

(ii) $\bar{\mathbf{f}}$ *is in* $L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)^2)$.

Numerical algorithms

We pass to the presentation of two classical numerical algorithms for optimization problems, describing them in the situation considered. Due to the nonconvexity of $J(\mathbf{f})$, the convergence of such algorithms is conditional, depending on good 'initial guess'.

The *Simple Gradient Algorithm* proceeds as follows:

$$
(\mathcal{G}) \quad \left|\begin{array}{l} \text{Given } \mathbf{f} \text{ in } L^2(Q)^2, \rho > 0, \text{ define } \mathbf{f}_n \text{ recursively by}\\ \mathbf{f}_{n+1} = \mathbf{f}_n - \rho(\mathbf{f}_n + \tilde{\mathbf{w}}_{\mathbf{f}_n}). \end{array}\right.
$$

For a given $\mathbf{f}_n$, $\mathbf{v}_{\mathbf{f}_n}$ and therefore $\tilde{\mathbf{w}}_{\mathbf{f}_n}$ are uniquely determined, so that the sequence $\mathbf{f}_n$ is well defined.

The convergence of the algorithm $(\mathcal{G})$ to a minimum of J stems from the following two observations: First, if $\mathbf{f}_n$ is convergent, then its limit $\mathbf{f}$ must be a critical point of J, for we then have $\bar{\mathbf{f}} = \bar{\mathbf{f}} - \rho(\mathbf{f} + \tilde{\mathbf{w}}_{\bar{\mathbf{f}}})$, hence $\bar{\mathbf{f}} + \tilde{\mathbf{w}}_{\mathbf{f}} = \mathbf{0}$, precisely the critical–point condition of the last theorem. Secondly, if the second derivative of $\mathbf{f} \to \mathbf{v}_{\mathbf{f}}$ is sufficiently well behaved, the sequence $J(\mathbf{f}_n)$ is decreasing for small values of ρ, and the limit $\bar{\mathbf{f}}$ of $\mathbf{f}_n$ must be a local minimum of J.

A more refined algorithm is the *Conjugate Gradient Algorithm*:

$$
(\mathcal{CG}) \quad \left|\begin{array}{l} \text{Given } \mathbf{f}_0, \rho > 0, \text{ define } \mathbf{f}_n \text{ recursively as follows}\\ \mathbf{f}_{n+1} = \mathbf{f}_n - \rho \mathbf{k}_n,\\ \text{where}\\ \mathbf{k}_0 = \mathbf{f}_0 + \tilde{\mathbf{w}}_{\mathbf{f}_0},\\ \mathbf{k}_n = \mathbf{f}_n + \tilde{\mathbf{w}}_{\mathbf{f}_n} + \mathbf{k}_{n-1} \dfrac{\int_Q \left(\mathbf{f}_n - \mathbf{f}_{n-1} + \tilde{\mathbf{w}}_{\mathbf{f}_n} - \tilde{\mathbf{w}}_{\mathbf{f}_{n-1}}\right) \cdot (\mathbf{f}_n + \tilde{\mathbf{w}}_{\mathbf{f}_n})\, dx dt}{\int_Q \left|\mathbf{f}_{n-1} + \tilde{\mathbf{w}}_{\mathbf{f}_{n-1}}\right|^2 dx dt}. \end{array}\right.
$$

Example 2. Temam *et al.* [184] considered the flow in a three–dimensional channel as a simplified form of the flow in a wind tunnel. The channel occupies the region $\Omega = (0,l_1) \times (0,l_2) \times (0,l_3)$. The flow is maintained by an unspecified pressure in the x_1 (streamwise) direction. The flow is controlled by the normal velocity of the upper wall Γ_w, $\{x_2 = l_2\}$ (boundary control):

$$
\mathbf{v} = \phi \quad \text{on } \Gamma_w \times (0,T).
$$

The cost functional consists of two terms, $J = J_0 + J_1$. The first term, e.g.

$$J_0(\phi) = \frac{a^2}{2} \int_0^T \int_{\Gamma_w} |\phi|^2 dx_1 dx_3 dt,$$

accounts for the cost of control. The second term may be related to turbulence as previously. Other choices are:

$$J_{1a}(\mathbf{v}) = \int_0^T \int_{\Gamma_w} \frac{\partial v_1}{\partial x_2} dx_2 dx_3 dt,$$

$$J_{1b}(\mathbf{v}) = \frac{1}{2} \int_\Omega |\mathbf{v}(x,T)|^2 dx.$$

J_{1b} represents the time–averaged value of the drag whilst J_{1a} is the terminal value of the turbulent kinetic energy.

Figure 6 presents the coherent structures which appear near the wall in a turbulent channel flow and which we want to annihilate. The figure corresponds to the Reynolds number $Re_\tau = 180$, for which optimally controlled results seem not to be yet available. The results below corresponds to $Re_\tau = 100$. The results of the numerical simulations are depicted in Figs. 6, 7.

FIGURE 6. Coherent structures of a turbulent flow at $Re_\tau = 100$, after Temam *et al.*[184]

The algorithm consists in dividing the interval $(0, T)$ into intervals of length τ; then on each interval $(m\tau, (m+1)\tau)$ the control problem is solved. On each interval $(m\tau, (m+1)\tau)$ the Navier–Stokes equations are discretized with a time step $\Delta t \ll \tau$.

The results presented in Figs. 6, 7 show a tendency to relaminerization.

Remark 11. In the papers [19, 20] *robust* control theory applicable to fluid mechanics has been developed. In essence, when designing a robust controller one must plan on a finite component of the worst–case disturbance aggravating the system, and design a controller which is suited to handle even this extreme situation.

We shall not dwell upon this intriguing problem. To make this concept more palpable, consider only a linear robust regulation.

Consider the linear state equation with additional forcing due to an external disturbance $\boldsymbol{\chi}$:

$$\dot{\mathbf{v}} = A\mathbf{v} + B_1\mathbf{u} + B_2\boldsymbol{\chi}, \tag{150}$$

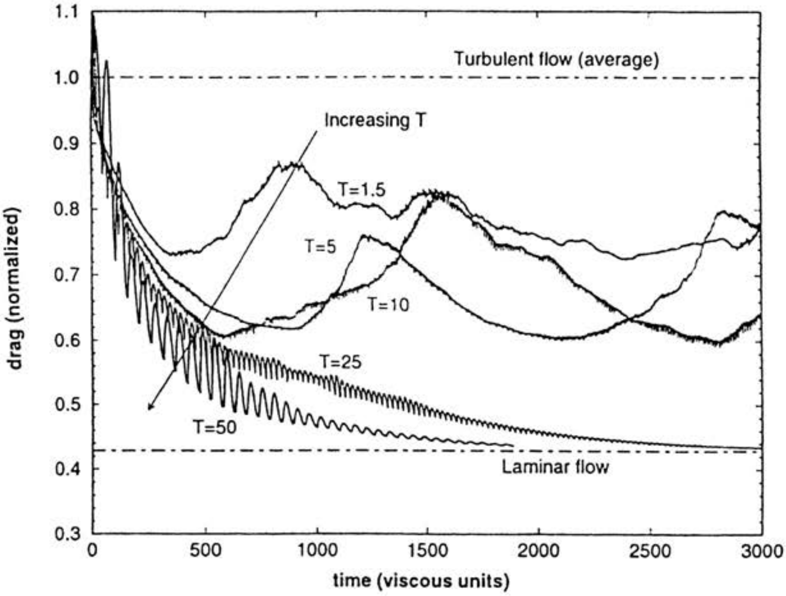

Figure 7. Time evolution of the drag for different values of τ (denoted T in the figure) after Temam *et al.* [184]

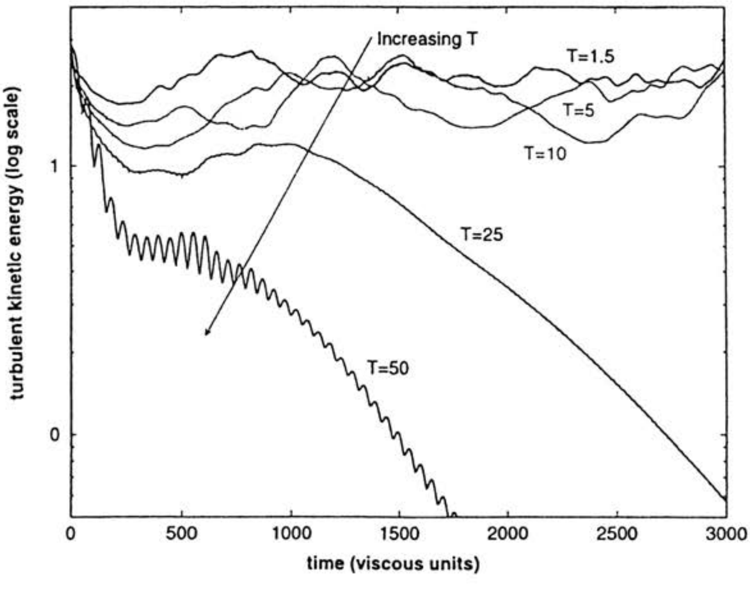

Figure 8. Time evolution of the turbulent kinetic energy for different values of τ (denoted T in the figure), after Temam *et al.* [184]

with the initial condition

$$\mathbf{v}(0) = \mathbf{v}^0. \tag{151}$$

Problem (150), (151) may be viewed as a general linear problem. The cost functional incorporates a term which accounts for the magnitude of the disturbance used to aggravate the system, for instance

$$J(\mathbf{u}, \boldsymbol{\chi}) = \frac{1}{2T} \int_0^T \left(\|\mathbf{v}\|^2 + a^2 \|\mathbf{u}\|^2 - \gamma^2 \|\boldsymbol{\chi}\|^2 \right) dt \tag{152}$$

with suitably defined norms. Now we *minimize* J with respect to the control $\mathbf{u}$ and simultaneously *maximize* with respect to the disturbance $\boldsymbol{\chi}$.

4.3. The Stokes system

Some controllability results can be deduced from Sec. 4.1 by deleting the nonlinear term in the Navier–Stokes equations, cf. Imanuvilov [76].

The aim of this section is different: we shall concisely present the approximate controllability results of the Stokes equations due to Glowinski and Lions [67]. Moreover, the conjugate gradient algorithm elaborated by Glowinski and his coworkers will also be discussed, cf. Glowinski and Lions [67] and the references cited therein.

4.3.1. Controllability results. In this case the control $\boldsymbol{u}$ is distributed over Ω with support in $\bar{\mathfrak{O}} \subset \Omega$, where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$. The *state equation* is given by

$$\begin{aligned} \dot{\boldsymbol{v}} - \Delta \boldsymbol{v} + \nabla p = \boldsymbol{u}\chi_{\mathfrak{O}} \quad &\text{in} \quad Q = \Omega \times [0,T], \\ \operatorname{div} \boldsymbol{v} = 0 \quad &\text{in} \quad Q. \end{aligned} \tag{153}$$

For the sake of simplicity, the following boundary and initial conditions are assumed

$$\boldsymbol{v} = \mathbf{0} \quad \text{on} \quad \Sigma, \qquad \boldsymbol{v}(0) = \mathbf{0} \quad \text{in} \quad \Omega. \tag{154}$$

In $(153)_1$ the control $\boldsymbol{u}$ is assumed to belong to

$$\mathcal{V} = \text{closed subspace of } L^2\big(\mathfrak{O} \times (0,T)\big)^n. \tag{155}$$

To fix ideas we shall take $n = 3$, and consider the following cases for $\mathcal{V}$:

$$\mathcal{V} = L^2\big(\mathfrak{O} \times (0,T)\big)^3, \tag{156}$$

$$\mathcal{V} = \Big\{ (u_1, u_2, 0) \mid (u_1, u_2) \in L^2\big(\mathfrak{O} \times (0,T)\big)^2 \Big\}, \tag{157}$$

$$\mathcal{V} = \Big\{ (u_1, 0, 0) \mid u_1 \in L^2\big(\mathfrak{O} \times (0,T)\big) \Big\}. \tag{158}$$

Problem (153), (154) possesses a unique solution, such that (in particular)

$$\begin{aligned} &\boldsymbol{v}(t; \boldsymbol{u}) \in L^2\big(0, T; H^1(\Omega)^3\big), \quad \operatorname{div} \boldsymbol{v} = 0, \\ &\dot{\boldsymbol{v}}(t, \boldsymbol{u}) \in L^2(0, T; V'), \end{aligned} \tag{159}$$

where V' is the dual space of V with

$$V = \{\boldsymbol{\varphi} \mid \boldsymbol{\varphi} \in H_0^1(\Omega)^3,\ \operatorname{div}\boldsymbol{\varphi} = 0\}. \tag{160}$$

It follows from (159) that

$$t \to \boldsymbol{v}(t;\boldsymbol{u}) \text{ belongs to } C\big([0,T];H\big), \tag{161}$$

where

$$H = \{\boldsymbol{\varphi} \in L^2(\Omega)^3 \mid \operatorname{div}\boldsymbol{\varphi} = 0 \text{ in } \ \Omega,\ \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0 \text{ on } \ \Gamma\}. \tag{162}$$

The scalar product is denoted by $(\cdot\,,\cdot)$.

The density result which follows implies (at least) approximate controllability.

Proposition 2. If $\mathcal{V}$ is defined by either (156) or (157), then the space spanned by $\boldsymbol{v}(T;\boldsymbol{u})$ is dense in H. □

Now we are in a position to formulate *two approximate controllability problems.* The *first problem* means evaluating

$$\min\left\{\frac{1}{2}\int_0^T\int_{\mathfrak{D}}|\boldsymbol{u}|^2\,dxdt \,\Big|\, \boldsymbol{u}\in\mathcal{U}\right\}, \tag{163}$$

where

$$\mathcal{U} = \big\{\boldsymbol{u}\in\mathcal{V} \;\big|\; \{\boldsymbol{u},\boldsymbol{v}\} \text{ verifies (153), (154) and } \boldsymbol{v}(T)\in\boldsymbol{v}_T+\beta B_H\big\}. \tag{164}$$

In (164), $\boldsymbol{v}_T$ is given in H, β is an arbitrary small positive number, B_H is the closed unit ball of H and – to fix ideas – $\mathcal{V}$ is defined by (157).

The *second problem* is obtained by *penalization* of the terminal condition $\boldsymbol{v}(T) = \boldsymbol{v}_T$:

$$\min\left\{\frac{1}{2}\int_0^T\int_{\mathfrak{D}}|\boldsymbol{u}|^2\,dxdt + \frac{1}{2}k\int_\Omega|\boldsymbol{v}(T)-\boldsymbol{v}_T|^2\,dx \;\Big|\; \boldsymbol{v}\in\mathcal{V}\right\}, \tag{165}$$

where k is an arbitrary large positive number whilst $\boldsymbol{v}$ is obtained from $\boldsymbol{u}$ via (153), (154). Proposition 2 implies unique solvability of both control problems (163) and (165).

Let us pass to optimality conditions. For the simpler problem (165), denoting by J_k the penalized cost functional, we get

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon\neq 0}}\frac{J_k(\boldsymbol{u}+\varepsilon\boldsymbol{w})-J_k(\boldsymbol{u})}{\varepsilon} = \big(J_k'(\boldsymbol{u}),\boldsymbol{w}\big) = \int_0^T\int_{\mathfrak{D}}(\boldsymbol{u}-\boldsymbol{p})\cdot\boldsymbol{w}\,dxdt. \tag{166}$$

Here the *adjoint velocity field* $\boldsymbol{p}$ is a solution to the following backward Stokes problem

$$\begin{aligned} -\dot{\boldsymbol{p}} - \Delta\boldsymbol{p} + \nabla\xi = \boldsymbol{0} \quad &\text{in} \quad Q,\\ \operatorname{div}\boldsymbol{p} = 0 \quad &\text{in} \quad Q; \end{aligned} \tag{167}$$

$$\boldsymbol{p} = \boldsymbol{0} \quad \text{on} \quad \Sigma, \qquad \boldsymbol{p}(T) = k\big(\boldsymbol{v}_T - \boldsymbol{v}(T)\big). \tag{168}$$

Suppose now that $\bar{\boldsymbol{u}}$ is the unique solution of problem (165). Obviously, $\bar{\boldsymbol{u}}$ is characterized by

$$\bar{\boldsymbol{u}} \in \mathcal{V}, \quad \big(J_k'(\bar{\boldsymbol{u}}), \boldsymbol{w}\big) = 0, \quad \forall\, \boldsymbol{w} \in \mathcal{V}. \tag{169}$$

Hence the *optimal triple* $\{\bar{\boldsymbol{u}}, \boldsymbol{v}, \boldsymbol{p}\}$ is characterized by (153), (154), (167), (168) and

$$\bar{u}_1 = p_1|_{\mathfrak{D}}, \qquad \bar{u}_2 = p_2|_{\mathfrak{D}}, \qquad \bar{u}_3 = 0. \tag{170}$$

To derive the *dual problem* of (165) from the above optimality conditions, we introduce an operator $\Lambda \in \mathcal{L}(H, H)$ defined as follows

$$\Lambda \hat{\boldsymbol{f}} = \hat{\boldsymbol{\varphi}}(T), \quad \forall\, \hat{\boldsymbol{f}} \in H, \tag{171}$$

and exploit the theory of duality [43]. To obtain $\hat{\boldsymbol{\varphi}}(T)$ we solve first

$$\begin{gathered} -\frac{\partial \hat{\boldsymbol{\psi}}}{\partial t} - \Delta \hat{\boldsymbol{\psi}} + \nabla \hat{\xi} = \boldsymbol{0} \quad \text{in} \quad Q, \\ \operatorname{div} \boldsymbol{v} = 0 \quad \text{in} \quad Q; \end{gathered} \tag{172}$$

$$\hat{\boldsymbol{\psi}}(T) = \hat{\boldsymbol{f}}, \quad \hat{\boldsymbol{\psi}} = \boldsymbol{0} \quad \text{on} \quad \Sigma, \tag{173}$$

and then

$$\begin{gathered} \frac{\partial \hat{\boldsymbol{\phi}}}{\partial t} - \Delta \hat{\boldsymbol{\phi}} + \nabla \hat{\xi} = \{\hat{\psi}_1, \hat{\psi}_2, 0\}\chi_{\mathfrak{D}} \quad \text{in} \quad Q, \\ \operatorname{div} \hat{\boldsymbol{\varphi}} = 0 \quad \text{in} \quad Q; \end{gathered} \tag{174}$$

$$\hat{\boldsymbol{\varphi}}(0) = \boldsymbol{0}, \quad \hat{\boldsymbol{\varphi}} = \boldsymbol{0} \quad \text{on} \quad \Sigma. \tag{175}$$

The two above Stokes problems are uniquely solvable.

Integrating by parts in time and using Green's formula it can be shown that

$$\int_{\Omega} (\Lambda \hat{\boldsymbol{f}}) \cdot \tilde{\boldsymbol{f}}\, dx = \int_0^T \!\! \int_{\mathfrak{D}} (\hat{\psi}_1 \tilde{\psi}_2 + \hat{\psi}_2 \tilde{\psi}_2)\, dx dt, \quad \forall\, \hat{\boldsymbol{f}}, \tilde{\boldsymbol{f}} \in H. \tag{176}$$

Hence we deduce that the operator Λ is *symmetric* and *positive semi-definite* over H; in fact it can be shown that this operator is *positive definite* over H.

Let us denote by $\boldsymbol{f}$ the function $\boldsymbol{p}(T)$ appearing in the *optimality conditions*. The definition of Λ and Eq. (176) yield

$$\Lambda \boldsymbol{f} + k^{-1} \boldsymbol{f} = \boldsymbol{v}_T, \tag{177}$$

which is precisely the dual problem of (165).

The properties of Λ suggest that Eq. (177) can be solved by a *conjugate gradient algorithm* operating in the space H.

To derive the *dual* problem of (163), one can apply the Fenchel-Rockafellar duality theory, presented in Ekeland and Temam [43]. It can be shown that the unique solution $\bar{\bar{\boldsymbol{u}}}$ of problem (163) can be obtained via

$$\bar{\bar{u}}_1 = p_1 \chi_{\mathfrak{D}}, \qquad \bar{\bar{u}}_2 = p_2 \chi_{\mathfrak{D}}, \qquad \bar{\bar{u}}_3 = 0, \tag{178}$$

where $\boldsymbol{p}$ is the solution of the *backward* Stokes problem

$$\begin{aligned} -\dot{\boldsymbol{p}} - \Delta \boldsymbol{p} + \nabla \xi = \mathbf{0} \quad &\text{in} \quad Q, \\ \operatorname{div} \boldsymbol{p} = 0 \quad &\text{in} \quad Q, \\ \boldsymbol{p}(T) = \boldsymbol{f}, \quad \boldsymbol{p} = \mathbf{0} \quad &\text{on} \quad \Sigma. \end{aligned} \tag{179}$$

Here $\boldsymbol{f}$ is the solution of the following *variational inequality*

$$\boldsymbol{f} \in H, \quad \int\limits_{\Omega} (\Lambda \boldsymbol{f}) \cdot (\hat{\boldsymbol{f}} - \boldsymbol{f})\, dx + \beta \|\hat{\boldsymbol{f}}\|_H - \beta \|\boldsymbol{f}\|_H \geq \int\limits_{\Omega} \boldsymbol{v}_T \cdot (\hat{\boldsymbol{f}} - \boldsymbol{f})\, dx, \quad \forall\, \hat{\boldsymbol{f}} \in H, \tag{180}$$

and

$$\|\boldsymbol{f}\|_H^2 = \int\limits_{\Omega} |\boldsymbol{f}|^2\, dx.$$

Inequality (180) represents the dual of problem (163).

4.3.2. Approximation and time discretization. Now we shall discuss the *direct solution* of the control problem (165) by a *conjugate gradient algorithm*. Applying this algorithm to problem (169) we get:

$$\bar{\boldsymbol{u}}^0 \text{ chosen in } \mathcal{V}; \tag{181}$$

solve

$$\begin{aligned} \frac{\partial \boldsymbol{v}^0}{\partial t} - \Delta \boldsymbol{v}^0 + \nabla p^0 = \bar{\boldsymbol{u}}^0 \chi_{\mathfrak{O}} \quad &\text{in} \quad Q, \\ \operatorname{div} \boldsymbol{v}^0 = 0 \quad &\text{in} \quad Q, \\ \boldsymbol{v}^0(0) = \mathbf{0}, \quad \boldsymbol{v}^0 = \mathbf{0} \quad &\text{on} \quad \Sigma; \end{aligned} \tag{182}$$

and then

$$\begin{aligned} -\frac{\partial \boldsymbol{p}^0}{\partial t} - \Delta \boldsymbol{p}^0 + \nabla \xi^0 = \mathbf{0} \quad &\text{in} \quad Q, \\ \operatorname{div} \boldsymbol{p}^0 = 0 \quad &\text{in} \quad Q, \\ \boldsymbol{p}^0 = \mathbf{0} \quad \text{on} \quad \Sigma, \quad \boldsymbol{p}^0(T) = k\big(\boldsymbol{v}_T - \boldsymbol{v}^0(T)\big)&. \end{aligned} \tag{183}$$

Solve now

$$\boldsymbol{g}^0 \in \mathcal{V}, \quad \int\limits_0^T\!\!\int\limits_{\mathfrak{O}} \boldsymbol{g}^0 \cdot \boldsymbol{z}\, dxdt = \int\limits_0^T\!\!\int\limits_{\mathfrak{O}} (\bar{\boldsymbol{u}}^0 - \boldsymbol{p}^0) \cdot \boldsymbol{z}\, dxdt, \quad \forall\, \boldsymbol{z} \in \mathcal{V}, \tag{184}$$

and set

$$\boldsymbol{w}^0 = \boldsymbol{g}^0. \tag{185}$$

Then, for $n \geq 0$, assuming $\bar{\boldsymbol{u}}^n$, $\boldsymbol{g}^n$, $\boldsymbol{w}^n$ are known, we obtain $\bar{\boldsymbol{u}}^{n+1}$, $\boldsymbol{g}^{n+1}$, $\boldsymbol{w}^{n+1}$ as follows.

Solve

$$
\begin{aligned}
\frac{\partial \bar{\boldsymbol{v}}^n}{\partial t} - \Delta \bar{\boldsymbol{v}}^n + \nabla \bar{\boldsymbol{p}}^n = \boldsymbol{w}^n \chi_{\mathfrak{O}} \quad & \text{in} \quad Q, \\
\operatorname{div} \bar{\boldsymbol{v}}^n = 0 \quad & \text{in} \quad Q, \\
\bar{\boldsymbol{v}}^n(0) = \mathbf{0}, \quad \bar{\boldsymbol{v}}^n = \mathbf{0} \quad & \text{on} \quad \Sigma;
\end{aligned}
\tag{186}
$$

and then

$$
\begin{aligned}
-\frac{\partial \bar{\boldsymbol{p}}^n}{\partial t} - \Delta \bar{\boldsymbol{p}}^n + \nabla \bar{\xi}^n = \mathbf{0} \quad & \text{in} \quad Q, \\
\operatorname{div} \bar{\boldsymbol{p}}^n = \mathbf{0} \quad & \text{in} \quad Q, \\
\bar{\boldsymbol{p}}^n = \mathbf{0} \quad \text{on} \quad \Sigma, \quad \bar{\boldsymbol{p}}^n(T) & = -k\bar{\boldsymbol{v}}^n(T).
\end{aligned}
\tag{187}
$$

Solve now

$$
\bar{\boldsymbol{g}}^n \in \mathcal{V}, \quad \int_0^T \int_{\mathfrak{O}} \bar{\boldsymbol{g}}^n \cdot \boldsymbol{z}\, dx dt = \int_0^T \int_{\mathfrak{O}} (\tilde{\boldsymbol{u}}^n - \bar{\boldsymbol{p}}^n) \cdot \boldsymbol{z}\, dx dt, \quad \forall\, \boldsymbol{z} \in \mathcal{V}, \tag{188}
$$

and compute

$$
\rho_n = \frac{\displaystyle\int_0^T \int_{\mathfrak{O}} |\boldsymbol{g}^n|^2 \, dx dt}{\displaystyle\int_0^T \int_{\mathfrak{O}} \bar{\boldsymbol{g}}^n \cdot \boldsymbol{w}^n \, dx dt}, \tag{189}
$$

$$
\bar{\boldsymbol{u}}^{n+1} = \bar{\boldsymbol{u}}^n - \rho_n \boldsymbol{w}^n, \tag{190}
$$

$$
\boldsymbol{g}^{n+1} = \boldsymbol{g}^n - \rho_n \bar{\boldsymbol{g}}^n. \tag{191}
$$

If

$$
\frac{\|\boldsymbol{g}^{n+1}\|_{L^2(\mathfrak{O}\times(0,T))}}{\|\boldsymbol{g}^0\|_{L^2(\mathfrak{O}\times(0,T))}} \le \varepsilon
$$

take $\bar{\boldsymbol{u}} = \bar{\boldsymbol{u}}^{n+1}$; else, compute

$$
\gamma_n = \frac{\displaystyle\int_0^T \int_{\mathfrak{O}} |\boldsymbol{g}^{n+1}|^2 \, dx dt}{\displaystyle\int_0^T \int_{\mathfrak{O}} |\boldsymbol{g}^n|^2 \, dx dt}, \tag{192}
$$

and update $\boldsymbol{w}^n$ by

$$
\boldsymbol{w}^{n+1} = \boldsymbol{g}^{n+1} + \gamma_n \boldsymbol{w}^n. \tag{193}
$$

Do $n := n + 1$ and go to (185).

We observe that for a given value of ε the number of iterations necessary to obtain the convergence of algorithm (181)-(193) varies like $k^{1/2}$.

In practice, the implementation of this algorithm requires space and time approximations of the control problem (165). Consider the time discretization. We introduce a time discretization step $\Delta t = T/N$ (with N a positive integer), denote by $\boldsymbol{u}$ the vector $\{\boldsymbol{u}^n\}_{1\le n\le N}$ and approximate problem (165) by

$$\min\left\{\frac{\Delta t}{2}\sum_{n=1}^{N}\int_{\mathfrak{O}}|\boldsymbol{u}^n|^2\,dx+\frac{k}{2}\int_{\Omega}|\boldsymbol{v}^N-\boldsymbol{v}_T|^2\,dx\;\middle|\;\boldsymbol{u}\in\mathcal{V}^{\Delta t}\right\}, \tag{194}$$

where, by analogy with (156)-(158), $\mathcal{V}^{\Delta t}$ is defined by either

$$\mathcal{V}^{\Delta t}=\Big\{\{\boldsymbol{u}^n\}_{1\le n\le N}\;\big|\;\boldsymbol{u}^n=\{u_1^n,u_2^n,u_3^n\}\in L^2(\mathcal{O})^3,\ \forall\, n=1,\dots,N\Big\}$$

$$\mathcal{V}^{\Delta t}=\Big\{\{\boldsymbol{u}^n\}_{1\le n\le N}\;\big|\;\boldsymbol{u}^n=\{u_1^n,u_2^n,0\},\ \{u_1^n,u_2^n\}\in L^2(\mathcal{O})^2,\ \forall\, n=1,\dots,N\Big\},$$

$$\mathcal{V}^{\Delta t}=\Big\{\{\boldsymbol{u}^n\}_{1\le n\le N}\;\big|\;\boldsymbol{u}^n=\{u_1^n,0,0\},\ u_1^n\in L^2(\mathfrak{O}),\ \forall\, n=1,\dots,N\Big\},$$

and where $\boldsymbol{v}^n$ is obtained from $\boldsymbol{u}$ via

$$\boldsymbol{v}^0=\boldsymbol{0}. \tag{195}$$

For $n = 1,2,\dots,N$, we obtain $\{\boldsymbol{v}^n, p^n\}$ from $\boldsymbol{v}^{n-1}$ by solving the following steady Stokes-type problem

$$\begin{aligned}\frac{\boldsymbol{v}^n-\boldsymbol{v}^{n-1}}{\Delta t}-\Delta\boldsymbol{v}^n+\nabla p^n=\boldsymbol{u}^n\chi_{\mathfrak{O}}\quad&\text{in}\quad\Omega,\\ \operatorname{div}\boldsymbol{v}^n=0\quad&\text{in}\quad\Omega,\\ \boldsymbol{v}^n=\boldsymbol{0}\quad&\text{on}\quad\Gamma.\end{aligned} \tag{196}$$

The above scheme is a *backward Euler* time discretization of problem (153), (154).

The results of numerical experiments were given in Glowinski and Lions [67], cf. also the references therein.

5. Linear elasticity

Lions [144, Chap. 4] applied the HUM to solve the problem of exact controllability of linear elastic bodies made of homogeneous and isotropic materials. Both Dirichlet and Neumann boundary control were examined. An extension to homogeneous and anisotropic materials was studied by Telega and Bielski [182]. The same problem was later considered by Alabau and Komornik [5] for the domain Ω in $\mathbb{R}^3$ being a ball of radius R whilst arbitrary bounded and regular domains were assumed in Alabau and Komornik [6].

In this section we shall present recent results on controllability and stabilization of linear elasticity systems. Numerical realization of the HUM follows along the lines sketched follows for the wave equation in Sec. 3.5, cf. Asch and Vai [9], Telega and Bielski [194]. Earlier results have obviously been discussed in the papers presented below.

We observe that the available results are confined to geometrically linear problems (the theory of small displacements). Geometrically nonlinear problems concern only plates and shells, cf. Sec. 7.4 and 8.3 below.

Recent ideas due to Renardy and Russell [170], though not directly related to the subject of this paper, are also worth of being mentioned. Russell (see [170, 180]) introduced the term *formation theory*, which refers to the controlled modification of the geometric configuration, or *shape* of an elastic body by means of attached or embedded actuators. According to Renardy and Russell [170], the subject material of formation theory concerns the relationships between the applied controls, the actuator distribution and the resulting deformation of the structure. In the above two papers only static, linear elastic problems were investigated. It seems that this new emerging theory may find applications in optimal design and biomechanics.

5.1. Exact Neumann boundary controllability for dynamic equations of anisotropic and inhomogeneous elasticity

Using HUM we are going to present our results concerning exact controllability of the equations of dynamic linear elasticity provided that the solid is anisotropic and inhomogeneous, cf. Telega [179, 180]. More difficult case of boundary Neumann controllability will be discussed.

Let $\Omega \in \mathbb{R}^n$ (n=1,2 or 3) be an open, bounded and sufficiently regular domain of class C^2. For the sake of simplicity we set $\rho = 1$ (the density). Consider the following system:

$$\begin{gathered}\ddot{\mathbf{u}} - \operatorname{div}(\mathbf{a}(\mathbf{x},t)\mathbf{e}(\mathbf{u})) = 0 \quad \text{in} \quad Q = \Omega \times (0,T),\\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}^0(\mathbf{x}), \quad \dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{u}^1(\mathbf{x}) \quad \text{in} \quad \Omega, \\ \sigma(\mathbf{u})\mathbf{n} = (a_{ijkl}e_{kl}(\mathbf{u})n_j) = \mathbf{g} \quad \text{on} \quad \Sigma = \Gamma \times (0,T).\end{gathered} \tag{197}$$

Here $\mathbf{u}$ denotes the displacement vector and $\sigma = (\sigma_{ij})$ is the stress tensor. The strain tensor $\mathbf{e}(\mathbf{u})$ is given by

$$e_{ij} = u_{(i,j)} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{198}$$

Exact boundary controllability problem reads: Given $T > 0$ for any initial state $(\mathbf{u}^0, \mathbf{u}^1)$ and any terminal state $(\mathbf{z}^0, \mathbf{z}^1)$ in a suitable Hilbert space $\mathcal{H}$, find a boundary control $\mathbf{g}$ such that the solution $\mathbf{u} = \mathbf{u}(\mathbf{x}, t\mathbf{g})$ of (197) satisfies

$$\mathbf{u}(\mathbf{x},T,\mathbf{g}) = \mathbf{z}^0, \quad \dot{\mathbf{u}}(\mathbf{x},T,\mathbf{g}) = \mathbf{z}^1 \quad \text{in} \quad \Omega. \tag{199}$$

Linearity of (197) implies that it suffices to look for controls driving this system to rest, i.e.,

$$\mathbf{u}(\mathbf{x},T,\mathbf{g}) = \mathbf{0}, \quad \dot{\mathbf{u}}(\mathbf{x},T,\mathbf{g}) = \mathbf{0} \quad \text{in} \quad \Omega. \tag{200}$$

The elastic moduli are assumed to depend also on time t; for instance, this is the case of adaptive elasticity.

As we remember, the HUM involves a point $\mathbf{x}^0 \in \mathbb{R}^n$, cf. Komornik [94], Lions [144] and Sec. 3. Now this point may be a function of t. More precisely, let $\mathbf{x}^0(t) \in C^1([0,\infty];\mathbb{R}^n)$ and set:

$$\begin{gathered}
\mathbf{m}(\mathbf{x},t) = \mathbf{x} - \mathbf{x}^0(t) = (x_1 - x_1^0(t), \ldots, x_n - x_n^0(t)) = (m_1(\mathbf{x},t), \ldots, m_n(\mathbf{x},t)), \\
\Sigma(\mathbf{x}^0) = \{(\mathbf{x},t) \in \Sigma | \mathbf{m}(\mathbf{x},t)\cdot \mathbf{n}(\mathbf{x}) > 0\}, \\
\Sigma_*(\mathbf{x}^0) = \Sigma \setminus \Sigma(\mathbf{x}^0), \\
\Gamma(\mathbf{x}^0(0)) = \{\mathbf{x} \in \Gamma | \mathbf{m}(\mathbf{x},0)\cdot \mathbf{n}(x) > 0\}, \\
\Sigma(\mathbf{x}^0(0)) = \Gamma(\mathbf{x}^0(0)) \times (0,T), \\
R(t) = \max_{\mathbf{x}\in\bar\Omega} |\mathbf{m}(\mathbf{x},t)| = \max_{\mathbf{x}\in\bar\Omega} \left|\sum_{k=1}^n (x_k - x_k^0(t))^2\right|^{1/2}, \\
R_1(t) = \max_{\mathbf{x}\in\bar\Omega} |\mathbf{m}'(\mathbf{x},t)| = \max_{\mathbf{x}\in\bar\Omega} \left|\sum_{k=1}^n ((x_k^0)'(t))^2\right|^{1/2}, \\
R_0 = \max_{0\le t\le\infty} R(t).
\end{gathered} \tag{201}$$

Figure 9 presents two examples of the set $\Gamma(x^0(0))$.

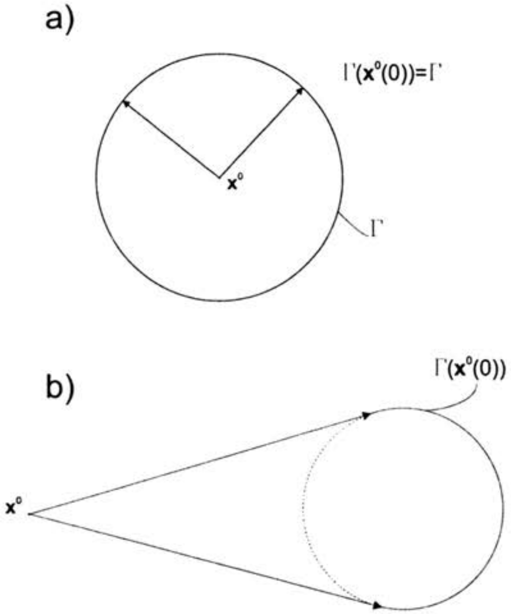

FIGURE 9.

Before the formulation of the exact controllability theorem we make some assumptions:

$$
\begin{aligned}
&a_{ijkl}(\mathbf{x},t), \dot{a}_{ijkl}(\mathbf{x},t), \ddot{a}_{ijkl}(\mathbf{x},t) \in C\big([0,\infty); L^\infty(\Omega)\big),\\
&\frac{\partial a_{ijkl}}{\partial x_m} \in L^\infty\big(\Omega \times (0,\infty)\big), \quad i,j,k,l,m = 1,\ldots,n.
\end{aligned}
\tag{202}
$$

There exists a constant $\alpha > 0$ such that

$$
a_{ijkl}(\mathbf{x},t) E_{ij} E_{kl} \geq \alpha |\mathbf{E}|^2, \quad \forall\, \mathbf{E} \in \mathbb{E}^n_s, \forall (\mathbf{x},t) \in Q. \tag{203}
$$

Here $\mathbb{E}^n_s$ denotes the space of symmetric $n \times n$ matrices and $|\mathbf{E}|^2 = E_{ij}E_{ij}$.

Remark 12. Assumption $(202)_2$ precludes the case of layered solids. □

Furthermore, we set

$$
\begin{aligned}
a(t) &= \frac{n}{\alpha} \max_{1 \leq i,j,k,l \leq n} \max_{\mathbf{x} \in \Omega} \big|\dot{a}_{ijkl}(\mathbf{x},t)\big|,\\
b(t) &= \frac{n}{\alpha} \max_{1 \leq i,j,k,l \leq n} \max_{\mathbf{x} \in \Omega} \left|\frac{\partial a_{ijkl}(\mathbf{x},t)}{\partial x_m}\right|,
\end{aligned}
\tag{204}
$$

If

$$
a(t), b(t), R_1(t) \in L^1(0,\infty), \tag{205}
$$

we set

$$
T_0 = \left[R_0 ||b||_{0,1} + \frac{R_0}{\sqrt{\alpha}} \left(1 + e^{-||a||_{0,1}}\right) + \frac{||R_1||_{0,1}}{\sqrt{\alpha}}\right] e^{2||a||_{0,1}}, \tag{206}
$$

where $||\cdot||_{0,1}$ denotes the norm of $L^1(0,\infty)$. If

$$
\dot{a}_{ijkl}(\mathbf{x},t) E_{ij} E_{kl} \leq 0, \quad \forall\ (\mathbf{x},t) \in \Omega \times [0,\infty),\ \forall\ \mathbf{E} \in \mathbb{E}^n_s. \tag{207}
$$

or

$$
\dot{a}_{ijkl}(\mathbf{x},t) E_{ij} E_{kl} \geq 0, \quad \forall\ (\mathbf{x},t) \in \Omega \times [0,\infty),\ \forall\ \mathbf{E} \in \mathbb{E}^n_s. \tag{208}
$$

then T_0 can be refined slightly to

$$
T_0 = \left(R_0 ||b||_{0,1} + \frac{2R_0}{\sqrt{\alpha}} + \frac{||R_1||_{0,1}}{\sqrt{\alpha}}\right) e^{||a||_{0,1}}. \tag{209}
$$

If

$$
a(t), b(t), R_1(t) \in L^\infty(0,\infty) \tag{210}
$$

we suppose

$$
3R_0 ||a||_{0,\infty} + R_0\sqrt{\alpha}||b||_{0,\infty} + ||R_1||_{0,\infty} < \sqrt{\alpha} \tag{211}
$$

where $||\cdot||_{0,\infty}$ denotes the norm of $L^\infty(0,+\infty)$.

Now we are in position to state the exact controllability result.

Theorem 13: *Let Ω be a bounded domain with the boundary Γ of class C^2. Suppose (202) and (203) hold and $\Sigma\big(\mathbf{x}^0(0)\big) \in \Sigma(\mathbf{x}^0)$. If either*

$$
a(t), b(t), R_1(t) \in L^1(0,\infty)
$$

holds and $T > T_0$ *or*

$$a(t), b(t), R_1(t) \in L^\infty(0,\infty)$$

and

$$3R_0||a||_{0,\infty} + R_0\sqrt{\alpha}||b||_{0,\infty} + ||R_1||_{0,\infty} < \sqrt{\alpha}$$

hold and T *is large enough so that*

$$3R_0||a||_{0,\infty} + R_0\sqrt{\alpha}||b||_{0,\infty} + ||R_1||_{0,\infty} < \frac{\sqrt{\alpha}T - 2R_0}{T}$$

then for all initial states

$$(\mathbf{u}^0, \mathbf{u}^1) \in L^2(\Omega)^3 \times \left[H^1(\Omega)'\right]^n$$

there exists a control

$$\mathbf{g} = \begin{cases} \mathbf{g}^0 & on \quad \Sigma(\mathbf{x}^0), \\ \mathbf{g}^1 & on \quad \Sigma_*(\mathbf{x}^0), \end{cases} \tag{212}$$

with $\mathbf{g}^0 \in \left[\left(H^1\left(\Sigma(\mathbf{x}^0)\right)\right)'\right]^3$ *and* $\mathbf{g}^0 \in \left[\left(H^1(\Sigma_*(\mathbf{x}^0))\right)'\right]^3$ *such that the solution* $\mathbf{u} = \mathbf{u}(\mathbf{x}, t, \mathbf{g})$ *of (197) satisfies (200).*

The proof of the last theorem will be given in [179]. Here we only provide some comments on the control function $\mathbf{g}$ and the basic inequality.

By applying HUM it can be shown that

$$\mathbf{g}^0 = -\mathbf{u} + \frac{\partial \dot{u}}{\partial t} \quad \text{on} \quad \Sigma(\mathbf{x}^0),$$

$$\mathbf{g}^1 = L_{\Sigma_*(\mathbf{x}_0)}(\mathbf{u}) \quad \text{on} \quad \Sigma_*(\mathbf{x}^0),$$

The linear operator is constructed as follows. Let

$$\frac{\partial \mathbf{u}}{\partial x_j} = \beta^j \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} + A^j\mathbf{u} \quad \text{on} \quad \Gamma.$$

Here $A\mathbf{u} = \{A^j\mathbf{u}\}_{j=1}^n$ defines the tangential gradient of $\mathbf{u}$ on Γ. For any subset Σ_1 of Σ the operators A^j are linear and continuous from $H^1(\Sigma_1)^3$ to $L^2(\Sigma_1)^3$. Then we set

$$-L_{\Sigma_1} = \sum_{j=1}^n (A^j)^* A^j,$$

where $(A^j)^*$ denotes the adjoint of A^j.

The crucial role in the application of HUM plays the *inverse* or *observability* inequality.

Proposition 3. Suppose $\Sigma\left(\mathbf{x}^0(0)\right) \subset \Sigma(\mathbf{x}^0)$ and let (202) and (203) hold. If either (205) holds and $T > T_0$ or (210) and (211) hold and T is large enough so that

$$3R_0||a||_{0,\infty} + R_0\sqrt{\alpha}||b||_{0,\infty} + ||R_1||_{0,\infty} < \frac{\sqrt{\alpha}T - 2R_0}{T},$$

then there exists a constant $c = c(T) > 0$ such that

$$\int\limits_{\Sigma(\mathbf{x}^0)} \left(|\dot{\mathbf{\Phi}}|^2 + |\mathbf{\Phi}|^2\right) d\Sigma + \int\limits_{\Sigma_*(\mathbf{x}^0)} |A\mathbf{\Phi}|^2 d\Sigma \geq c\left(||\mathbf{\Phi}^0||^2_{H^1} + ||\mathbf{\Phi}^1||^2_{L^2}\right)$$

where $A\mathbf{\Phi} = \{A^j\mathbf{\Phi}\}^n_{j=1}$ and $\mathbf{\Phi}$ is a strong solution to the following homogeneous Neumann boundary–initial value problem:

$$\begin{aligned} \ddot{\mathbf{\Phi}} - \operatorname{div}\left(\mathbf{a}(\mathbf{x},t)\mathbf{e}(\mathbf{\Phi})\right) = \mathbf{0} \quad &\text{in} \quad Q, \\ \boldsymbol{\sigma}(\mathbf{\Phi})\mathbf{n} = \left(a_{ijkl}e_{kl}(\mathbf{\Phi})n_j\right) = \mathbf{0} \quad &\text{on} \quad \Sigma, \\ \mathbf{\Phi}(\mathbf{x},0) = \mathbf{\Phi}^0, \quad \dot{\mathbf{\Phi}}(\mathbf{x},0) = \mathbf{\Phi}^1 \quad &\text{on} \quad \Sigma. \end{aligned} \tag{213}$$

5.2. Transmission problem

Lagnese [107] generalized a transmission problem considered by Lions [144, Chap. 6] for two wave equations, to the case of anisotropic elasticity, cf. also Sec. 3.4. More precisely, following Lagnese [107] we shall consider the exact Dirichlet boundary controllability in such a problem.

Let Ω, Ω_1 be bounded, open, connected sets in $\mathbb{R}^n$ (in practice $n = 1, 2$ or 3) with smooth boundaries Γ and Γ_1, respectively, such that $\bar{\Omega}_1 \subset \Omega$. We set $\Omega_2 = \Omega\backslash\bar{\Omega}_1$, $\Gamma_2 = \partial\Omega_2$; obviously $\Gamma_2 = \Gamma \cup \Gamma_1$. This assumption precludes the case of elastic body made of two layers. Two linear elastic bodies are identified with $\bar{\Omega}_1$ and $\bar{\Omega}_2$. Their elastic moduli a^{α}_{ijkl} ($\alpha = 1, 2$; $i, j, k, l = 1, \ldots, n$) satisfy the usual symmetry conditions

$$a^{(\alpha)}_{ijkl} = a^{(\alpha)}_{jikl} = a^{(\alpha)}_{klij}, \tag{214}$$

and the following ellipticity condition

$$\exists\ c_0 > 0,\ \forall\ \boldsymbol{E} \in \mathbb{E}^n_s, \quad a^{(\alpha)}_{ijkl}E_{ij}E_{kl} \geq c_0|\boldsymbol{E}|^2. \tag{215}$$

Here $\mathbb{E}^n_s$ denotes the space of all real symmetric $n \times n$ matrices.

Lagnese [107] considered also weaker assumptions on a^{α}_{ijkl}, where instead of (214) we only have

$$a^{(\alpha)}_{ijkl} = a^{(\alpha)}_{klij}, \quad \alpha = 1, 2;\ i, k = 1, \ldots, m;\ j, l = 1, \ldots, n.$$

However, this case is not interesting for the classical linear elasticity. The summation convention over repeated indices is used, unless otherwise stated.

Consider the following *problem of transmission*

$$\begin{cases} \ddot{u}_{1i} - a^{(1)}_{ijkl}u_{1k,lj} = 0, \quad \text{in} \quad Q_1 = \Omega_1 \times (0,T), \\ \ddot{u}_{2i} - a^{(2)}_{ijkl}u_{2k,lj} = 0, \quad \text{in} \quad Q_2 = \Omega_2 \times (0,T); \end{cases} \tag{216}$$

$$\boldsymbol{u}_2 = \boldsymbol{v}, \quad \text{on} \quad \Sigma = \Gamma \times (0,T), \tag{217}$$

$$\boldsymbol{u}_1 = \boldsymbol{u}_2, \quad a^{(1)}_{ijkl}e_{kl}(\boldsymbol{u}_1)n_j = a^{(2)}_{ijkl}e_{kl}(\boldsymbol{u}_2)n_j, \quad \text{on} \quad \Sigma_1 = \Gamma_1 \times (0,T), \tag{218}$$

$$\boldsymbol{u}_\alpha(x,0) = \dot{\boldsymbol{u}}_\alpha(x,0) = \boldsymbol{0}, \quad \text{in} \quad \Omega_\alpha. \tag{219}$$

Here $\boldsymbol{n} = (n_i)$ is the unit normal to Γ_1 pointing into Ω_1 and

$$e_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{220}$$

Obviously, the function $\boldsymbol{v}$ in (217) is a control function. The density ϱ_α $(\alpha = 1,2)$ does not appear. It is either incorporated into $\boldsymbol{a}^{(\alpha)}$ or one simply puts $\varrho_\alpha = 1$. Such an assumption is unessential.

Let us define

$$A_\alpha = \inf_{\boldsymbol{e}\in\mathbb{E}^n_s,\, \boldsymbol{e}\neq\boldsymbol{0}} \frac{a^{(2)}_{ijkl} E_{ij} E_{kl}}{E_{pq} E_{pq}}. \tag{221}$$

The main result of Lagnese is summarized as follows.

Theorem 14: *Assume that Γ_1 is star-shaped with respect to some point $\mathbf{x}^0 \in \Omega_1$ and let*

$$\Gamma(\mathbf{x}_0) = \{\mathbf{x} \in \Gamma \mid (\mathbf{x} - \mathbf{x}_0)\cdot \boldsymbol{n} > 0\}, \qquad \Sigma(\mathbf{x}^0) = \Gamma(\mathbf{x}^0) \times (0,T), \tag{222}$$

$$R(\mathbf{x}^0) = \max_{x\in\bar{\Omega}_2} |\mathbf{x} - \mathbf{x}^0|, \tag{223}$$

where $\boldsymbol{n}$ is the unit outer normal to Γ. Let

$$\mathcal{V}_T = \{(\boldsymbol{u}_1(\cdot,T), \boldsymbol{u}_2(\cdot,T), \dot{\boldsymbol{u}}_1(\cdot,T), \dot{\boldsymbol{u}}_2(\cdot,T) \mid \boldsymbol{v} \in L^2(\Sigma)^n,\ \boldsymbol{v} = \boldsymbol{0} \ on\ \Sigma\backslash\Sigma(x^0)\}.$$

If

$$\forall\, \boldsymbol{e} \in \mathbb{E}^n_s, \quad a^{(1)}_{ijkl} E_{ij} E_{kl} \geq a^{(2)}_{ijkl} E_{ij} E_{kl}, \tag{224}$$

and if

$$T > T(x^0) = \frac{2\sqrt{2}R(x^0)}{\sqrt{A_2}}, \tag{225}$$

then $\mathcal{V}_T = H \times V'$ where

$$H = L^2(\Omega_1)^n \times L^2(\Omega_2)^n,$$

$$V = \{(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \in H^1(\Omega_1)^n \times H^1(\Omega_2)^n \mid \boldsymbol{\varphi}_{2|\Gamma} = \boldsymbol{0},\ \boldsymbol{\varphi}_{1|\Gamma_1} = \boldsymbol{\varphi}_{2|\Gamma_1}\}.$$

The proof uses classical multipliers to derive a priori estimates and HUM.

Remark 13.

(i) Theorem 14 can be extended to the situation involving Ω and $p \geq 2$ nested open sets $\omega_1, \ldots, \omega_p$ with $\bar{\omega}_i \subset \omega_{i+1}$, $i = 1, \ldots, p-1$, $\omega_p = \Omega$. Set: $\Omega_1 = \omega_1$, $\Omega_i = \omega_i\backslash\bar{\omega}_{i-1}$, $i = 1, \ldots, p$; $\Gamma_i = \partial\omega_i$. The boundary Γ_i, $i = 1, \ldots, p-1$, is star-shaped with respect to a point $x^0 \in \Omega_1$.

(ii) It is not known whether the monotonicity condition (224) is necessary for exact controllability in dimension $n \geq 2$.

(iii) For isotropic materials the monotonicity condition (224) is satisfied provided that

$$\mu_1 \geq \mu_2 \quad \text{and} \quad \lambda_1 \geq \lambda_2,$$

where μ_α, λ_α, $(\alpha = 1,2)$ are the Lamé coefficients.

(iv) Nicaise [162] studied Dirichlet-Neumann boundary controllability of isotropic homogeneous elastic bodies identified with $\bar{\Omega}$, where Ω is a polygonal domain of the plane or a polyhedral domain of the space. Primarily, in Nicaise [160, 161], the regularity of solutions was examined for both $n = 2$ (corners) and $n = 3$ (vertex and edge singularities). The results of Nicaise extended those obtained by Grisvard [68] for the wave equation.

5.3. Approximate controllability by means of planar body forces

Consider the following system of linear elasticity, cf. Zuazua [191],

$$\begin{aligned} \ddot{\boldsymbol{u}} - \mu\Delta\boldsymbol{u} - (\lambda+\mu)\nabla\mathrm{div}\,\boldsymbol{u} = \boldsymbol{f}\chi_{\mathfrak{D}} &\quad \text{in} \quad \Omega\times(0,T), \\ \boldsymbol{u} = \boldsymbol{0} &\quad \text{on} \quad \Gamma\times(0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}^0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}^1 &\quad \text{in} \quad \Omega, \end{aligned} \tag{226}$$

where λ, μ denote Lamé's coefficients. Here $\mathfrak{D}$ is an open and nonempty subset of Ω. We assume that $\boldsymbol{f} \in L^2(Q)^n$ $(n = 2, 3)$ is of the form

$$\boldsymbol{f} = (f_1, \ldots, f_{n-1}, 0), \tag{227}$$

Prior to the formulation of the approximate controllability result we have to introduce indispensible notations. Let

$$T(\Omega) = \frac{2\delta_n(\Omega; \mathfrak{D})}{\sqrt{\mu}}, \tag{228}$$

the quantity δ_n being defined as follows. For any open subset Ω_1 of Ω

$$\delta_n(\Omega; \Omega_1) := \sup_{x\in\Omega\setminus\Omega_1} \inf_{\gamma\in\xi(x;\Omega_1)} l(\gamma), \tag{229}$$

where $\xi(x; \Omega_1)$ denotes the set of curves in Ω joining x and $\bar{\Omega}_1$ and $l(\cdot)$ stands for the length of the curve. We set $\delta_n(\Omega; \emptyset) = \infty$.

By $\Omega^{n-1} \subset \mathbb{R}^{n-1}$ and $\Omega^1 \subset \mathbb{R}$ we denote, respectively, the projections of Ω on the hyperplane $x_n = 0$ and on the axis $0x_n$. Furthermore, by $\mathfrak{U}^{n-1} \subset \mathbb{R}^{n-1}$ (resp. $\mathfrak{U}^1 \subset \mathbb{R}$) we denote the union of the projections on the hyperplane $x_n = 0$ (resp., on the axis $0x_n$) of all those components of the boundary Γ that can be written in the form $x_n = h(x_1, \ldots, x_{n-1})$ with h of class C^2 and such that

$$\left|\nabla' h(x_1, \ldots, x_{n-1})\right|^2 \neq \frac{\lambda + 2\mu}{\mu}$$

or

$$\Delta' h(x_1, \ldots, x_{n-1}) \neq 0.$$

By ∇' and Δ' we denote the gradient and Laplacian in the variables $(x_1, \ldots, x_{n-1})$.

The approximate controllability result proved by Zuazua [191] is formulated as follows.

Theorem 15: *Let Ω satisfies the following four conditions:*

(i) Ω is a piecewise C^2-bounded domain.

(ii) Some open and nonempty C^2 component of Γ can be written in the form: $x_n = h(x_1, \ldots, x_{n-1})$ with $|\nabla' h|^2 \neq (\lambda + 2\mu)/\mu$ or $\Delta' h \neq 0$ everywhere on that component.

(iii) There exists a point of a C^2 component of the boundary of Ω where the tangent hyperplane to Ω exists, and it is parallel to the axis Ox_n.

(iv) When $n = 3$, either

$(iv)_1$ an open subset of Γ is contained in a plane of the form $x_3 = c$

or

$(iv)_2$ Ω is not symmetric with respect to a plane of the form $x_3 = c$.

Then, if

$$T > 2\frac{\delta_n(\Omega; \mathfrak{O})}{\sqrt{\mu}} + T^*(\Omega),$$

system (226) is approximately controllable at time T under the constraint (227), where

$$T^*(\Omega) = 2\min\left(\frac{1}{\sqrt{\mu}}\delta_{n-1}\left(\Omega^{n-1}; \mathfrak{U}^{n-1}\right), \frac{1}{\sqrt{\lambda + 2\mu}}\delta_1\left(\Omega^1; \mathfrak{U}^1\right)\right).$$

More precisely, for all $(\boldsymbol{u}^0, \boldsymbol{u}^1)$ and $(\boldsymbol{u}_T^0, \boldsymbol{u}_T^1)$ in $H_0^1(\Omega)^n \times L^2(\Omega)^n$ and $\varepsilon > 0$ there exists $\boldsymbol{f} \in L^2(Q)^n$ obeying (227) such that the solution of (226) satisfies

$$\left[\|\boldsymbol{u}(T) - \boldsymbol{u}_T^0\|^2_{H_0^1(\Omega)^n} + \|\dot{\boldsymbol{u}}(T) - \boldsymbol{u}_T^1\|^2_{L^2(\Omega)^n}\right]^{1/2} \leq \varepsilon.$$

Remark 14.

(a) Without the constraint (227), exact controllability with $L^2(Q)^n$-controls holds for a certain class of $\mathfrak{O}$'s, cf. Lions [144]. For instance, if $\mathfrak{O}$ is a neighborhood of the boundary of Ω the exact controllability holds with $T(\Omega) = \text{diam}\,(\Omega \setminus \mathfrak{O})/\sqrt{\mu}$.

(b) Zuazua [191] constructed a two-dimensional domain for which $T^*(\Omega) > 0$. This author provided also two examples of noncontrollability.

(c) Under the constraint (227), the approximate controllability cannot be obtained directly from Holmgren's uniqueness theorem. Zuazua [191] solved the problem of uniqueness of the corresponding homogeneous (forward) system by reducing the proof to uniqueness result for scalar wave equations.

5.4. Stabilization of linear elastic bodies

Earlier result on boundary stabilization of three-dimensional linear elastodynamic system are due to Lagnese [102]. The same author studied also the case of plane strain (two-dimensional elasticity), cf. Lagnese [105]. In both cases the elastic bodies are made of homogeneous isotropic materials.

The aim of the present section is to present the results obtained afterwards by other authors. The papers by Lagnese [102, 105], however, largely influenced the developments which followed.

5.4.1. Asymptotic stability of isotropic bodies with internal damping. Aassila [1] extended the approach used for the damped wave equation to the case of homogeneous isotropic geometrically linear elastic bodies, cf. Sec. 3.3

Consider the following elasticity system with internal damping

$$
\begin{aligned}
\ddot{\boldsymbol{u}} - \mu\Delta\boldsymbol{u} - (\lambda+\mu)\nabla\boldsymbol{u} + \boldsymbol{G}(\dot{\boldsymbol{u}}) = \boldsymbol{0} &\quad \text{in} \quad \Omega\times\mathbb{R}^+, \\
\boldsymbol{u} = \boldsymbol{0} &\quad \text{in} \quad \Gamma_0\times\mathbb{R}^+, \\
\mu\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}} + (\lambda+\mu)(\operatorname{div},\boldsymbol{u})\boldsymbol{n} + g_1\boldsymbol{u} = \boldsymbol{0} &\quad \text{in} \quad \Gamma_1\times\mathbb{R}^+, \\
\boldsymbol{u}(0) = \boldsymbol{u}^0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}^1 &\quad \text{in} \quad \Omega.
\end{aligned}
\tag{230}
$$

Here Ω is a bounded open domain in $\mathbb{R}^n$ having a boundary Γ of class C^2, $\{\Gamma_0, \Gamma_1\}$ is a partition of Γ such that $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and $g_1 : \Gamma_1 \to \mathbb{R}^+$ is a continuously differentiable function.

Each component of the vector $\boldsymbol{G}(\dot{\boldsymbol{u}})$ is specified by $g(\dot{\boldsymbol{u}})$. The function g satisfies the following conditions:

(H_1) g is an increasing function of class C^1,

(H_2) $zg(z) > 0$ for all $z \neq 0$,

(H_3) there exists a number $q \geq 2$ satisfying $(n-2)q < 2n$ and two positive constants c_1, c_2 such that

$$
c_1|z| \leq |g(z)| \leq c_2|z|^{q-2} \quad \text{for all} \quad |z| \geq 1.
$$

We observe that no growth condition at the origin is imposed on g. Similarly to the case of the wave equation considered in Sec. 3.3, it suffices to assume that Ω is of finite measure (not necessarily bounded). The assumptions imposed on g preclude the possibility of construction of a standard Lyapunov function, which played an important role in the study performed by Lagnese [102, 105].

Using the standard nonlinear semigroup theory we conclude that for any given $(\boldsymbol{u}^0, \boldsymbol{u}^1) \in H^1_{\Gamma_0}(\Omega)^n \times L^2(\Omega)^n$, there exists a unique *mild (weak) solution* $\boldsymbol{u} \in C\big(\mathbb{R}^+, H^1(\Omega)^n\big) \cap C^1\big(\mathbb{R}^+, L^2(\Omega)^n\big)$ and the linear mapping $(\boldsymbol{u}^0, \boldsymbol{u}^1) \to \boldsymbol{u}$ is continuous with respect to these topologies. The space $H^1_{\Gamma_0}(\Omega)^n$ is defined by

$$
H^1_{\Gamma_0}(\Omega) = \big\{u \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma_0\big\}. \tag{231}
$$

If $\boldsymbol{u}^0 \in (H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega))^n$, $\boldsymbol{u}^1 \in H^1_{\Gamma_0}(\Omega)^n$ and

$$
\mu\frac{\partial\boldsymbol{u}^0}{\partial\boldsymbol{n}} + (\lambda+\mu)(\operatorname{div}\boldsymbol{u}^0)\boldsymbol{n} + g_1\boldsymbol{u}^0 = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \quad g(\boldsymbol{u}^1) \in L^2(\Omega),
$$

then we have the following regularity property

$$
\boldsymbol{u} \in C\big(\mathbb{R}^+, H^2(\Omega)^n\big) \cap C^1\big(\mathbb{R}^+, H^1(\Omega)^n\big) \cap C^2\big(\mathbb{R}^+, L^2(\Omega)^n\big).
$$

In this case we say that $\boldsymbol{u}$ is a *strong solution*.

The energy of the solution is defined by

$$
E(t) = \frac{1}{2}\int_{\Omega} \big(|\dot{\boldsymbol{u}}|^2 + \mu|\nabla\boldsymbol{u}|^2 + (\lambda+\mu)(\operatorname{div}\boldsymbol{u})^2\big)\, dx + \frac{1}{2}\int_{\Gamma_1} g_1(x)|\boldsymbol{u}|^2\, d\Gamma. \tag{232}
$$

If $\boldsymbol{u}$ is a strong solution, standard calculation yields, cf. Komornik [94]

$$E(S) - E(T) = \int_S^T \int_\Omega \dot{\boldsymbol{u}} \cdot \boldsymbol{G}(\dot{\boldsymbol{u}})\, dxdt \tag{233}$$

for all $0 \leq S < T < +\infty$. The last identity remains valid for all mild solutions by a density argument. By (H_2) we conclude that $E(t)$ is non-increasing. The strong asymptotic stability result is formulated as follows.

Theorem 16: *For every solution of system (230) we have*

$$E(t) \to 0 \quad as \quad t \to +\infty.$$

The proof of the last theorem is based on two lemmas similar to those formulated in Sec. 3.3.

Remark 15. Caution is needed when reading Aassila's paper [1] since in his Eq. (1.1) the damping term is a function with values in $\mathbb{R}^3$ and not in $\mathbb{R}^+$. Consequently, one can consider more general damping by assuming that each component of $\boldsymbol{G}$ is not necessarily the same.

5.4.2. Boundary stabilization of isotropic and anisotropic linear elastic bodies Komornik [92] and Alabau and Komornik [6] devised a constructive method to boundary stabilization problems primarily studied by Lagnese [102, 105]. More precisely, Komornik [92] studied *isotropic* linear elastic bodies whilst Alabau and Komornik [6] investigated *anisotropic* bodies. In both cases the bodies are made of homogeneous materials. It is thus sufficient to present the results contained in Alabau and Komornik [6]. These authors applied suitable dissipative boundary feedbacks. A nonlinear boundary feedback was applied by Martinez [155]. The last author, however, considered linear elastic bodies made of materials with only cubic symmetry.

Consider the following system, cf. Alabau and Komornik [6]

$$\begin{aligned} \ddot{\boldsymbol{u}} - \operatorname{div} \boldsymbol{\sigma} = \boldsymbol{0} \quad &\text{in} \quad \Omega \times \mathbb{R}^+, \\ \boldsymbol{u} = \boldsymbol{0} \quad &\text{on} \quad \Gamma_0 \times \mathbb{R}^+, \\ \boldsymbol{\sigma n} + A\boldsymbol{u} + B\dot{\boldsymbol{u}} = \boldsymbol{0} \quad &\text{on} \quad \Gamma_1 \times \mathbb{R}^+, \\ \boldsymbol{u}(0) = \boldsymbol{u}^0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}^1 \quad &\text{in} \quad \Omega, \end{aligned} \tag{234}$$

where $\boldsymbol{\sigma} = (\sigma_{ij})$, $i, j = 1, \ldots, n$, is the stress tensor defined by

$$\sigma_{ij} = a_{ijkl} e_{kl}(\boldsymbol{u}). \tag{235}$$

The strain tensor is given by (220) and the elastic moduli a_{ijkl} satisfy (214) and (215). In Eq. $(234)_3$, A, B are given nonnegative coefficients, for simplicity. One can, however, easily extend the result which follows to the case where A and B are nonnegative functions of class C^1 on Γ_1.

The energy of the solution of (234) is given by

$$E(t) = \frac{1}{2}\int_{\Omega} \left(|\dot{\boldsymbol{u}}|^2 + \sigma_{ij}e_{ij}(\boldsymbol{u})\right) dx + \frac{1}{2}\int_{\Gamma_1} A|\boldsymbol{u}|^2\, d\Gamma, \tag{236}$$

and is a nonincreasing function of $t \in \mathbb{R}^+$.

Geometric assumptions are rather restrictive: we assume that

$$\Omega = \Omega_1 \setminus \bar{\Omega}_0, \tag{237}$$

where Ω_1 is an open ball, say $\Omega_1 = B(x^0, R)$, Ω_0 is a star-shaped domain with respect to x^0 whose closure belongs to Ω_1, and

$$\Gamma_0 = \partial\Omega_0, \qquad \Gamma_1 = \partial\Omega_1. \tag{238}$$

The case $\Omega_0 = \emptyset$ is not excluded. The following theorem was proved by Alabau and Komornik [6].

Theorem 17: *Let the elasticity tensor (a_{ijkl}) satisfy (214), (215), and let Ω, Γ_0 and Γ_1 be defined by (237), (238). Given two positive constants A and B with $A < c_0/(4R)$, there exists a positive number ω such that all (weak) solutions of (234) satisfy the energy estimate*

$$E(t) \le E(0)e^{1-\omega t}, \tag{239}$$

for all $t \ge 0$.

If $\Gamma_0 \ne \emptyset$, then the result holds also for $A = 0$.

Remark 16.

(i) The proof is based on a Lyapunov-type method and a new identity which allows to estimate certain boundary integrals, cf. Sec. 6.4 below.

(ii) The proof can be adapted to domains such that Ω_1 is *close* to a ball.

(iii) The proof of Theorem 17 provides an explicit form of ω which involves a constant depending on A and B but not on the choice of the initial data.

(iv) Alabau and Komornik [6] formulated also a general theorem allowing to construct boundary feedback for observable systems which lead to arbitrarily large decay rates. The second theorem applies to all bounded domains of class C^2, choosing, for instance, $\Gamma_0 = \emptyset$ and $\Gamma_1 = \Gamma$.

(v) Liu (see [180]), Th.2.2 improved the result due to Alabau and Komornik [6] in the case of isotropic bodies: the domain Ω may be star-shaped and the assumption on the *function* $A(x)$ in $(234)_3$ can really be weakened, as conjectured in the second paper.

Remark 17. Martinez [155] investigated a class of nonlinear boundary feedback laws for bodies made of materials with cubic symmetry, cf. Chernykh [28]. Such materials are characterized by *three independent coefficients.* We recall that isotropic materials are described by only two coefficients (the Lamé constants). The boundary feedback law is given by

$$\boldsymbol{\sigma n} + a\boldsymbol{u} + bg(\dot{\boldsymbol{u}}) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \tag{240}$$

where $a, b : \Gamma_1 \to \mathbb{R}^+$ are two continuously differentiable functions whilst $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that

$$\forall\, z \in \mathbb{R}, \qquad |g(z)| \leq 1 + c|z|, \tag{241}$$

for some positive constant c. Martinez [155] proved a uniform stabilization theorem and derived rather precise decay estimates.

Remark 18. Horn [72] established an exponential (uniform) decay of solution for the elastodynamic system of isotropic elasticity. Only the velocity feedback is acting through the boundary:

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = -\dot{\boldsymbol{u}} \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+,$$

where $\partial\Omega = \Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. In our opinion, one should write:

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = -\alpha\dot{\boldsymbol{u}}, \quad \alpha > 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+.$$

The constants α is not dimensionless; particularly one can take $\alpha = 1$ (in appropriate units, depending on the units of the velocity $\dot{\boldsymbol{u}}$ and tractions $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}$.

The uniform stability theorem of Horn [72] does not require the usual strong geometric assumption on Γ_1. Under the usual assumption of smooth boundary of Ω, it suffices to impose the following standard condition:

$$\boldsymbol{m}(x) \cdot \boldsymbol{n} \leq 0 \quad \text{on} \quad \Gamma_0.$$

The uniform stability theorem is based on the multiplier method and sharp trace estimates for the tangential derivative of the displacements on the boundary as well as on the unique continuation results for the corresponding static system.

6. Mathematical and control–theoretic complements

This section starts with the dynamic programming equation called Bellman's equation in the finite–dimensional case. Then the linear regulator problems and Ricatti's equations are discussed. Both finite and infinite time horizon problems are studied. Prior to extension to infinite–dimensional spaces elements of the theory of semigroups are presented.

6.1. Bellman's equation

Consider the finite–dimensional control system, cf. Fattorini [49], Klamka [82], Zabczyk [193]

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{u}), \qquad \mathbf{y}(0) = \mathbf{y}^0. \tag{242}$$

Here $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m$. When $T < \infty$, then the *cost functional* is

$$J_T(\mathbf{y}^0, \mathbf{u}) = \int_0^T g\big(\mathbf{y}(t), \mathbf{u}(t)\big)dt + G\big(\mathbf{y}(T)\big). \tag{243}$$

If the control interval is $[0, +\infty]$, then the cost functional is given by

$$J(\mathbf{y}^0, \mathbf{u}) = \int\limits_0^\infty f\big(\mathbf{y}(t), \mathbf{u}(t)\big)\, dt. \tag{243a}$$

We want to find a control $\hat{\mathbf{u}}$ such that for all admissible controls $\mathbf{u}$

$$J_T(\mathbf{y}^0, \hat{\mathbf{u}}) \leq J_T(\mathbf{y}^0, \mathbf{u}), \tag{244}$$

or

$$J(\mathbf{y}^0, \hat{\mathbf{u}}) \leq J(\mathbf{y}^0, \mathbf{u}). \tag{245}$$

In essence, there are two methods for finding controls minimizing cost functionals (243) or (243a). One of them embeds a given minimization problem into a parametrized family of similar problems. The embedding should be such that the minimal value, as a function of the parameter, satisfies an analytic equation. If the selected parameter is the initial state and the length of the control interval, then the minimal value of the cost functional is called the *value function* and the analytical relation, the Bellman's equation. Knowing the solution to the Bellman's equation on can find the optimal strategy in the form of a *closed loop system*.

The second method, due to Pontryagin and his coworkers (see [49, 193]), leads to necessary conditions on the optimal *open–loop strategy* formulated in the form of the so-called maximum principle.

Let us pass to the Bellman's equation. Assume that the state space E of a control system is an open subset of $\mathbb{R}^n$ and let the set U of control parameters be included in $\mathbb{R}^m$. Furthermore, the functions $\mathbf{f}$, g and G are continuous on $E \times U$ and E respectively; moreover g is nonnegative.

Theorem 18: *Assume that a real function W defined and continuous on $[0,T] \times E$ is of class C^1 on $(0,T) \times E$ and satisfies the* Bellman equation

$$\frac{\partial W(\mathbf{y}^0, t)}{\partial t} = \inf_{\mathbf{u} \in U} \left\{ g(\mathbf{y}^0, \mathbf{u}) + \langle W_{\mathbf{y}^0, t}(\mathbf{y}^0, t), \mathbf{f}(\mathbf{y}^0, \mathbf{u}) \rangle \right\}, \tag{246}$$

where $(\mathbf{y}^0, t) \in E \times (0,T)$, with the boundary condition

$$W(\mathbf{y}^0, 0) = G(\mathbf{y}^0), \quad \mathbf{y}^0 \in E.$$

(i) If $\mathbf{u}(\cdot)$ is a control and $\mathbf{y}(\cdot)$ the corresponding absolutely continuous, E–valued solution of (242) then

$$J_T\big(\mathbf{y}^0, \mathbf{u}(\cdot)\big) \geq W(\mathbf{y}^0, T).$$

(ii) Assume that for certain function $\hat{\mathbf{v}}\colon [0,T] \times E \to U$

$$g\big(\mathbf{y}^0, \hat{\mathbf{v}}(\mathbf{y}^0, t)\big) + \big\langle W_{\mathbf{y}^0}(\mathbf{y}^0, t), \mathbf{f}\big(\mathbf{y}^0, \hat{\mathbf{v}}(\mathbf{y}^0, t)\big)\big\rangle \leq g(\mathbf{y}^0, \mathbf{u}) + \big\langle W_{\mathbf{y}^0}(\mathbf{y}^0, t), \mathbf{f}(\mathbf{y}^0, \mathbf{u})\big\rangle,$$
$$\mathbf{y}^0 \in E, \quad t \in (0,T), \quad \mathbf{u} \in U,$$

and that $\hat{\mathbf{y}}$ is an absolutely continuous E-valued solution of the equation

$$\frac{d\hat{\mathbf{y}}(t)}{dt} = \mathbf{f}\big(\hat{\mathbf{y}}(t), \hat{\mathbf{v}}(T-t, \hat{\mathbf{y}}(t))\big), \quad t \in [0,T].$$

Then, for the control $\hat{\mathbf{u}}(t) = \hat{\mathbf{v}}(T-t, \hat{\mathbf{y}}(t))$, $t \in [0,T]$,

$$J_T(\mathbf{y}^0, \hat{\mathbf{u}}(\cdot)) = W(\mathbf{y}^0, T).$$

For the proof, the reader is referred to Fattorini [49] and Zabczyk [193]. Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing (the scalar product) in $\mathbb{R}^n$ and $W_{\mathbf{y}} = \partial W / \partial \mathbf{y}$.

For the control problem on the infinite time interval (infinite horizon) we have the following easy consequence of the last theorem.

Corollary 1. Let g be a nonnegative, continuous function and assume that there exists a nonnegative function W, defined on E and of class C^1, which satisfies the equation.

$$\inf_{\mathbf{u} \in U} \{ g(\mathbf{y}^0, \mathbf{u}) + \langle W_{\mathbf{y}^0}(\mathbf{y}^0), \mathbf{f}(\mathbf{y}^0, \mathbf{u}) \rangle \} = 0, \quad \mathbf{y}^0 \in E.$$

If for a strategy (input) $\mathbf{u}(\cdot)$ and the corresponding output $\mathbf{y}$, $\lim_{t \to +\infty} W(\mathbf{y}(t)) = 0$ then

$$J(\mathbf{y}^0, \mathbf{u}(\cdot)) \geq W(\mathbf{y}^0).$$

If $\hat{\mathbf{v}}$: $F \to U$ is a mapping such that

$$g(\mathbf{y}^0, \hat{\mathbf{v}}(\mathbf{y}^0)) + \langle W_{\mathbf{y}^0}(\mathbf{y}^0), \mathbf{f}(\mathbf{y}^0, \hat{\mathbf{v}}(\mathbf{y}^0)) \rangle = 0 \quad \text{for} \quad \mathbf{y}^0 \in E.$$

and $\hat{\mathbf{y}}$ is an absolutely continuous, E-valued solution of the equation

$$\frac{d\hat{\mathbf{y}}(t)}{dt} = \mathbf{f}(\hat{\mathbf{y}}(t), \hat{\mathbf{v}}(\hat{\mathbf{y}}(t))), \quad t \geq 0,$$

for which $\lim_{t \to +\infty} W(\hat{\mathbf{y}}(t)) = 0$, then

$$J(\hat{\mathbf{y}}^0, \hat{\mathbf{u}}(\cdot)) = W(\mathbf{y}^0).$$

Proof. It is sufficient to apply Theorem 18 with

$$G(\mathbf{y}^0) = W(\mathbf{y}^0), \quad \mathbf{y}^0 \in E.$$

6.2. The linear regulator problem and the Ricatti equation

Consider now a special case of problem (242), (243) when the system is linear:

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{u}, \quad \mathbf{y}(0) = \mathbf{y}^0 \in \mathbb{R}^n. \tag{247}$$

Here $\mathbf{A} \in M(n,m)$, $\mathbf{B} \in M(n,m)$ and the state space $E = \mathbb{R}^n$ whilst the set of control parameters $U = \mathbb{R}^m$, cf. [193].

The cost functional is assumed in the following form

$$J_T = \int_0^T [\langle \mathbf{Q}\mathbf{y}(s), \mathbf{y}(s) \rangle + \langle \mathbf{R}\mathbf{u}(s), \mathbf{u}(s) \rangle] ds + \langle P_0 \mathbf{y}(T), \mathbf{y}(T) \rangle, \tag{247a}$$

where $\mathbf{Q} \in M(n,n)$, $R \in M(m,m)$, $P_0 \in M(n,n)$ are symmetric, nonnegative matrices and the matrix $\mathbf{R}$ is a positive definite. The problem of minimizing (247a) for a linear system (247) is called the *linear regulator problem* or the *linear-quadratic problem.*

The form of optimal solution to (247) and (247a) is strongly connected with the following *matrix Ricatti equation*, cf. also [123, 127],

$$\dot{\mathbf{P}} = \mathbf{Q} + \mathbf{PA} + \mathbf{A}^*\mathbf{P} - \mathbf{PBR}^{-1}\mathbf{B}^*\mathbf{P}, \quad \mathbf{P}(0) = \mathbf{P}_0, \tag{248}$$

in which $\mathbf{P}(t)$, $t \in [0,T]$ is the unknown function with values in $M(n,n)$; $\mathbf{A}^*$ denotes the transpose matrix of $\mathbf{A}$. For $n = 1$, Eq. (248) reduces to the Count Ricatti equation

$$\dot{p}(t) = -r(t) - 2a(t)p(t) + b^2(t)p^2(t).$$

We have [193].

Theorem 19: *Equation (248) has a unique global solution* $\mathbf{P}(t), t \geq 0$. *For arbitrary* $t \geq 0$ *the matrix* $\mathbf{P}(t)$ *is symmetric and nonnegative definite. The minimal value of the functional (247a) is equal to* $\langle \mathbf{P}(T)\mathbf{y}^0, \mathbf{y}^0 \rangle$ *and the optimal control is of the form*

$$\hat{\mathbf{u}}(t) = -\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P}(T-t)\hat{\mathbf{y}}(t), \quad t \in [0,T],$$

where

$$\dot{\hat{\mathbf{y}}}(t) = (\mathbf{A} - \mathbf{BR}^{-1}\mathbf{B}^*\mathbf{P}(T-t))\hat{\mathbf{y}}(t), \quad t \in [0,T], \hat{\mathbf{y}}(0) = \mathbf{y}^0.$$

For the proof the reader is referred to [193]. However, we observe that the function $W(\mathbf{y}^0, t) = \langle \mathbf{P}(t)\mathbf{y}^0, \mathbf{y}^0 \rangle$, $t \in [0,T]$, $\mathbf{y}^0 \in \mathbb{R}^n$, is a solution to the Bellman equation (246) associated with the linear regulator problem (247), (247a).

6.3. The linear regulator and stabilization

The infinite time horizon case and the solution of the linear regulator problem suggest an important way to stabilize linear systems. It is connected with the *algebraic Riccati equation*:

$$\mathbf{Q} + \mathbf{PA} + \mathbf{A}^*\mathbf{P} - \mathbf{PBR}^{-1}\mathbf{B}^*\mathbf{P} = \mathbf{0}, \quad \mathbf{P} \geq 0, \tag{249}$$

in which the unknown is a nonnegative definite matrix $\mathbf{P}$. If $\tilde{\mathbf{P}}$ is a solution to (249) and $\tilde{\mathbf{P}} \leq \mathbf{P}$ for all other solutions $\mathbf{P}$, then $\tilde{\mathbf{P}}$ is called a *minimal solution* of (249). For arbitrary control $\mathbf{u}(\cdot)$ defined on $[0, +\infty]$ we introduce the functional

$$J(\mathbf{y}^0, \mathbf{u}) = \int_0^{+\infty} \{\langle \mathbf{Qy}(t), \mathbf{y}(t) \rangle + \langle \mathbf{Ru}(t), \mathbf{u}(t) \rangle\} dt, \tag{250}$$

where $\mathbf{y}$ is a solution to the control system (247).

Now we are in a position to formulate

Theorem 20: *If there exists a nonnegative solution* $\mathbf{P}$ *of Eq. (249) then there also exists a unique minimal solution* $\tilde{\mathbf{P}}$ *of 249, and the control* $\tilde{\mathbf{u}}$ *given in the feedback form*

$$\tilde{\mathbf{u}}(t) = -\mathbf{R}^{-1}\mathbf{B}^*\tilde{\mathbf{P}}\mathbf{y}(t), \quad t \geq 0,$$

minimizes functional (250). Moreover, the minimal value of the cost functional is equal to

$$\langle \tilde{\mathbf{P}}\mathbf{y}^0, \mathbf{y}^0 \rangle.$$

For the proof the reader is referred to [193].

Prior to passing to stabilizability we need some definitions. Consider a linear system

$$\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(0) = \mathbf{y}^0 \in \mathbb{R}^n, \tag{251}$$

and an *observation relation*

$$\mathbf{w}(t) = \mathbf{C}\mathbf{y}(t), \quad t \geq 0 \tag{252}$$

where $\mathbf{C} \in M(k, n)$.

Classically, the solution of (251) has the following form

$$\mathbf{y}(t) = \mathbf{S}(t)\mathbf{y}^0 + \int_0^t \mathbf{S}(t-s)\mathbf{B}\mathbf{u}(s)ds,$$

where

$$\mathbf{S}(t) = \exp t\mathbf{A} = e^{t\mathbf{A}} = \sum_{n=1}^{\infty} \frac{\mathbf{A}^n}{n!} t^n, \quad t \geq 0.$$

A generalization to infinite–dimensional systems is presented in Sec. 6.5 and involves semigroups of operators.

Observability

Assume that $\mathbf{B} = \mathbf{0}$. Then the system (251) reduces to

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}, \quad z(0) = \mathbf{y}^0. \tag{253}$$

The observation relation remains unchanged:

$$\mathbf{w} = \mathbf{C}\mathbf{z}. \tag{253a}$$

The solution of (253) is denoted by $\mathbf{z}(\mathbf{y}^0, t)$, $t \geq 0$. We have

$$\mathbf{z}(\mathbf{y}^0, t) = \mathbf{S}(t)\mathbf{y}^0, \quad y^0 \in \mathbb{R}^n.$$

The system (253), (253a) or the pair $(\mathbf{A}, \mathbf{C})$ is said to be *observable* if for arbitrary $\mathbf{y}^0 \in \mathbb{R}^n$, $\mathbf{y}^0 \neq 0$, there exists $t > 0$ such that

$$\mathbf{w}(t) = \mathbf{C}\mathbf{z}(\mathbf{y}^0, t) \neq 0.$$

If for a given $T > 0$ and for arbitrary $\mathbf{y}^0 \neq 0$ there exists $t \in [0, T]$ with the above property, then the system (253), (253a) or the pair $(\mathbf{A}, \mathbf{C})$ are said to be observable at time T.

For more details the reader is referred to Fattorini [49], Klamka [82], Zabczyk [193].

Stable linear systems

The system (253) is called *stable* if for arbitrary $\mathbf{y}^0 \in \mathbb{R}^n$

$$\mathbf{z}(\mathbf{y}^0, t) \to \mathbf{0} \text{ as } t \to +\infty.$$

One often says that the matrix $\mathbf{A}$ is stable.

For linear finite–dimensional system the following theorem holds true.

Theorem 21: *Let* $\mathbf{A} \in M(n,n)$. *The following conditions are equivalent:*

(i) $\mathbf{z}(\mathbf{y}^0, t) \to \mathbf{0}$ *as* $t \to +\infty$, *for arbitrary* $\mathbf{y}^0 \in \mathbb{R}^n$.

(ii) $\mathbf{z}(\mathbf{y}^0, t) \to \mathbf{0}$ *exponentially as* $t \to +\infty$, *for arbitrary* $\mathbf{y}^0 \in \mathbb{R}^n$.

(iii) $\omega(\mathbf{A}) = \sup\{Re\lambda | \lambda \in \sigma(\mathbf{A})\} < 0$.

(iv) $\int\limits_0^{+\infty} |\mathbf{z}(\mathbf{y}^0, t)|^2 dt < +\infty$, *for arbitrary* $\mathbf{y}^0 \in \mathbb{R}^n$.

The proof is given in [193]. We observe that the last theorem is no longer valid for infinite–dimensional system. We recall that $\sigma(\mathbf{A})$ denotes the spectrum of $\mathbf{A}$.

Stabilizability and controllability

The system

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{u}, \quad \mathbf{y}(0) = \mathbf{y}^0 \in \mathbb{R}^n \tag{254}$$

is said to be *stabilizable* or the pair $(\mathbf{A}, \mathbf{B})$ is stabilizable if there exist a matrix $\mathbf{K} \in M(m,n)$ such that the matrix $\mathbf{A} + \mathbf{BK}$ is stable. Consequently if the pair $(\mathbf{A}, \mathbf{B})$ is stabilizable and a control $\mathbf{u}(\cdot)$ is given in the feedback form

$$\mathbf{u}(t) = \mathbf{K}\mathbf{y}(t), \quad t \geq 0,$$

then all solutions of the equation

$$\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{BK}\mathbf{y}(t) = (\mathbf{A} + \mathbf{BK})\mathbf{y}(t), \quad \mathbf{y}(0) = \mathbf{y}^0, t \geq 0,$$

tend to zero as $t \to +\infty$.

The system (254) *completely stabilizable* (exponentially stablilizable) if and only if for arbitrary $\omega > 0$ there exists a matrix $\mathbf{K}$ and a constant $M > 0$ such that for an arbitrary solution $\mathbf{y}(\mathbf{y}^0, t)$, $t \geq 0$, of (254)

$$\left|\mathbf{y}(\mathbf{y}^0, t)\right| \leq M e^{-\omega t} \left|\mathbf{y}^0\right|, \quad t \geq 0.$$

Exponential stabilizability of (254) is equivalent to controllability. The controllability means that the system (254) can be driven from an arbitrary initial state $\mathbf{y}^0$ to a desired final state $\mathbf{y}_T$.

Detectability

A pair $(\mathbf{A}, \mathbf{C})$ is said to be detectable if there exists a matrix $\mathbf{L} \in M(n,k)$ such that the matrix $\mathbf{A} + \mathbf{LC}$ is stable. We note that observability implies detectability. Indeed if the pair $(\mathbf{A}, \mathbf{C})$ is observable then $(\mathbf{A}^*, \mathbf{C}^*)$ is controllable and there exists a matrix $\mathbf{K}$ such that $\mathbf{A}^* + C^*\mathbf{K}$ is a stable matrix. Therefore the matrix $\mathbf{A} + \mathbf{K}^*\mathbf{C}$ is stable and is enough to set $\mathbf{L} = \mathbf{K}^*$.

The notions of observability, stabilizability and detectability can easily be extended to infinite–dimensional spaces. Then A, B, C, K and L are linear operators acting in suitable function spaces.

Remark 19. The book by Trentelman *et al.* [196] provides an extensive treatment of the theory of feedback control design for linear, finite–dimensional, time–invariant state space systems with inputs and outputs. Particularly, the H_∞ control problem and its application to problems of robust stabilization have been extensively discussed.

6.4. Mathematical background for Lyapunov–based control

In the study of stabilization of finite– and infinite–dimensional systems the methods based on Lyapunov function are often used. The book by Quieroz *et al.* [167] is entirely concerned with Lyapunov–based techniques as mechanisms for developing different nonlinear control structures for mechanical systems like nonlinear discrete systems with friction, cables and strings, beams. In essence, the methods used are based on the following four lemmas.

Lemma 8: *Let $V(t) \in \mathbb{R}$ be a nonnegative function of time on $[0, +\infty)$ that satisfies the differential inequality*

$$\dot{V}(t) \leq -\gamma V(t)$$

where γ is a positive constant. Then

$$V(t) \leq V(0) \exp(-\gamma t), \quad \forall\, t \in [0, +\infty).$$

Here V(t) is the Lyapunov function, usually an energy–like function.

Lemma 9: *Let $V(t)$ be a nonnegative function of time on $[0, +\infty)$ that satisfies the differential inequality*

$$\dot{V} \leq -\gamma V + \epsilon,$$

where γ and ϵ are positive constants. Then

$$V(t) \leq V(0) \exp(-\gamma t) + \frac{\epsilon}{\gamma}\big(1 - \exp(-\gamma t)\big), \quad \forall\, t \in [0, \infty).$$

Lemma 10: *Let $V(t)$ be a nonnegative function of time on $[0, +\infty)$. If*

(i) $\dot{V} \leq -f(t)$, where $f(t) \geq 0$,

(ii) $f(t)$ is uniformly continuous or $\dot{f} \in L^\infty(0, \infty)$

then

$$\lim_{t \to +\infty} f(t) = 0$$

Lemma 11: *Let $r(t)$, $e(t) \in \mathbb{R}$ be functions of time on $[0, +\infty)$. Given the differential equation*

$$r(t) = \dot{e}(t) + \alpha e(t),$$

if $r(t)$ is exponentially stable in the sense that

$$|r(t)| \leq \beta_0 \exp(-\beta_1 t),$$

where β_0, β_1 are positive constants, then $e(t)$ and $\dot{e}(t)$ are exponentially stable in the sense that

$$|e(t)| \leq \exp(-\alpha t)|e(0)| + \frac{\beta_0}{\alpha - \beta_1}\big[\exp(-\beta_1 t) - \exp(-\alpha t)\big],$$

$$|\dot{e}(t)| \leq \alpha \exp(-\alpha t)|e(0)| + \beta_0 \exp(-\beta_1 t) + \frac{\alpha\beta_0}{\alpha - \beta_1}\big[\exp(-\beta_1 t) - \exp(-\alpha t)\big].$$

Remark 20. Recently Liu and Zuazua [152] used multiplier techniques and Lyapunov methods to show that the energy of the thermoelastic system decays to zero at an exponential or polynomial rate. The boundary feedback is nonlinear.

6.5. Elements of the theory of semigroups of operators

The semigroup approach is a convenient tool in the study of infinite–dimensional systems, cf. Fattorini [49], Lasiecka [118], Lasiecka and Triggiani [127], Zabczyk [193]. In the present section we shall only provide some suitable results pertaining to semigroups of operators. For a more complete theory the reader is referred to Yosida [188]. An application to the controllability of plates will be provided in Sec. 7.

From the Sec. 6.3 we know that the solutions to finite–dimensional linear system

$$\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(0) = \mathbf{y}^0 \in \mathbb{R}^n, \quad t \geq 0$$

are given by the formula

$$\mathbf{y}(t) = \mathbf{S}(t)\mathbf{y}^0 + \int_0^t \mathbf{S}(t-s)\mathbf{B}\mathbf{u}(s)ds, \quad t \geq 0,$$

where

$$\mathbf{S}(t) = e^{t\mathbf{A}}, \quad t \geq 0,$$

is the fundamental solution of the equation

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}, \quad \mathbf{z}(0) = \mathbf{y}^0 \in \mathbb{R}^n.$$

We observe that a matrix function $\mathbf{S}(t)$, $t \geq 0$ is the fundamental solution of the last equation if and only if it is a continuous solution of the matrix equation

$$\mathbf{S}(t+s) = \mathbf{S}(t)\mathbf{S}(s), \quad t, s \geq 0, \quad \mathbf{S}(0) = \mathbf{I}.$$

This leads us to the following generalization.

Let E be a Banach space. A semigroup of operators is an arbitrary family of bounded linear operators $S(t) : E \to E, t \geq 0$, satisfying:

(i) $S(t+s) = S(t)S(s), \quad t, s \geq 0, \quad S(0) = I.$

(ii) $\lim_{t\to 0} S(t)x = x, \forall x \in E.$

Such a semigroup is called C_0*–semigroup of operators* or strongly continuous semigroup.

In the finite–dimensional case the matrix $\mathbf{A}$ in the equation $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}$ is identical with the derivative of $\mathbf{S}(t)$ at 0:

$$\frac{d\mathbf{S}(0)}{dt} = \lim_{h\to 0} \frac{\mathbf{S}(h) - \mathbf{I}}{h} = \mathbf{A} \tag{255}$$

In the general case, the counterpart of $\mathbf{A}$ is called the infinitesimal operator or the generator of $\mathbf{S}(t), t \geq 0$. In general $D(A) \subset E$. The generator A is given by the formula

$$Ax = \lim_{h\to 0} \frac{S(h)x - x}{h}, \quad x \in D(A).$$

Example 3. If A is a bounded linear operator on E then the family

$$S(t) = e^{t\mathbf{A}} = \sum_{m=0}^{\infty} \frac{t^m}{m!} A^m, \quad t \geq 0,$$

is a semigroup with generator A. Other examples are given in Yosida [188] and Zabczyk [193].

Two basic theorems

Theorem 22 (the Hille-Yosida theorem): *Let A be a closed linear operator defined on a dense set contained in Banach space E. If there exists $\omega \in \mathbb{R}$ and $M \geq 1$ such that, for arbitrary $\lambda > \omega$, the operator $\lambda I - A$ has an inverse $R(\lambda) = (\lambda I - A)^{-1}$ satisfying*

$$||R^m(\lambda)|| \leq \frac{M}{(\lambda - \omega)^m}, \quad \textit{for} \quad m = 1, 2, \ldots,$$

then A is the infinitesimal generator of a semigroup $S(t), t \geq 0$ on E such that

$$||S(t)|| \leq M e^{\omega t}, \quad t > 0.$$

For the proof the reader is referred to Yosida [188] and Zabczyk [193]. We recall that the family of operators $(\lambda I - A)^{-1}$, $\lambda > \omega$, is called the resolvent of A.

For control theory, and particularly stabilizability, of great importance is a result due to Philips, see [188, 193].

Theorem 23: *If an operator generates a semigroup on a Banach space E and $K : E \to E$ is bounded linear operator then the operator $A + K$ with the domain identical to $D(A)$ is also a generator.*

Remark 21. Of practical importance, particularly for the study of control problems, are also holomorphic or analytic semigroups, cf. Fattorini [49], Lasiecka and Triggani [127], Yosida [188]. An analytic semigroup is a holomorphic continuation of a strongly continuous semigroup to a sector of a complex plane containing the positive t-axis.

6.6. The infinite–dimensional estimator and compensator

Consider an abstract, linear infinite–dimenasional system

$$\begin{aligned} \dot{y}(t) &= Ay(t) + Bu(t) \quad \text{on} \quad (\operatorname{dom} A^*)', \\ y_{ob}(t) &= Cy(t), \\ y(0) &= y^0 \in H. \end{aligned} \tag{256}$$

Let H be a Hilbert space, $\operatorname{dom} A^*$ denotes the domain of the adjoint operator A^* of A. We assume that A generates a C_0–semigroup, not necessarily analytic, cf. Banks *et al.* [15], Lasiecka and Triggani [127]. Such a system would arise, for instance when modeling a weakly damped structural system or a structural system with a coupled hyperbolic component, e.g. a structural acoustic system. It is also assumed that B is unbounded and C is bounded with $B : U \to (\operatorname{dom} A^*)'$ and $C \in L(H, Y)$. The spaces U, Y denote the control and observation spaces respectively. The notation $L(H, Y)$ denotes the space of linear and bounded (continuous) operators from U to Y. In a typical application involving the control of structural vibrations using piezoceramic actuators, u denotes the voltage to an actuator and B is unbounded due to the discontinuous geometry of the patches which leads to external applied moments in the structure.

To guarantee that the system (256) is well posed on H, it is assumed that B satisifies the regularity constraint

$$\int_0^T \left|\left|B^* e^{t\mathbf{A}} z\right|\right|_U^2 dt \leq M_T ||z||_H^2, \quad z \in \operatorname{dom} A^*,$$

where M_T is a positive constant, and

$$\langle Bu, f\rangle_H = \langle u, B^* f\rangle_U, \quad u \in U, f \in \operatorname{dom} B^* \subset \operatorname{dom} A^*.$$

If there exist operators $K \in L(H,U)$ and $F \in L(Y,H)$ such that $A+BK$ and $A-FC$ generate exponentially stable semigroups, then the *infinite dimensional estimator* or *observer*

$$\begin{aligned} \dot{y}_c(t) &= Ay_c(t) + Bu(t) + K\left[y_{ob}(t) - Cy_c(t)\right], \\ y_c(0) &= y_c^0, \end{aligned} \tag{257}$$

with the dynamic feedback law

$$u(t) = Fy_c(t), \tag{258}$$

exponentially stabilizes the original system (256). The combination of the estimator (257) and feedback law (258) is sometimes referred to as the *compensator*.

We recall that (A, B) is said to be *stabilizable* if there exists an operator F such that $A + BF$ generates an exponentially stable semigroup on V', i.e.

$$\left|\left|e^{t(A+BF)}\right|\right|_{L(V')} \leq Me^{-\omega t} \quad \text{for} \quad M \geq 1,\ \omega > 0.$$

Here $L(V') = L(V', V')$.

Similarly the pair (A, C) is said to be *detectable* if there exists an operator $K \in L(Y, V')$ such that $A - KC$ generates an exponentially stable semigroup on V'.

The following theorem characterizes the compensator, cf. [15, 79].

Theorem 24: *Let $\mathfrak{H} = H \times H$. The operator $A_c : D(A_c) \in H \to \mathfrak{H}$ given by*

$$A_c = \begin{bmatrix} A & BF \\ KC & A + BF - KC \end{bmatrix}$$

with $D(A_c) = \{(x,y) \in \mathfrak{H} | Ax + BFy \in H, KCx + (A + BF - KC)y \in H\}$ generates an analytic and exponentially stable semigroup on $\mathfrak{H}$. Moreover, the reconstruction error satisifes the bound

$$||y(t) - y_c(t)||_H \leq Me^{-\omega t}||y^0 - y_c^0||_H,$$

where M and ω are positive constants.

7. Plates

Among the structures like beams, membranes, plates, shells and junctions we shall only consider some control problems for plates and shells. Junctions will also be mentioned. Another class of control problems is linked with solid–fluid interactions and structural accoustics, cf. [118, 127].

Plates are two-dimensional structures, very important from the engineering point of view, cf. [30, 31, 137, 143]. It is thus not surprising that they were the subject of many papers. Earlier results on controllability and stabilization of simplified and realistic plate models are summarized in the books by Komornik [94], Lagnese [103], Lagnese and Lions [109], Lions [144, 145], cf. also Jaffard [77], Komornik [83, 87, 91], Krabs *et al.* [100], Lasiecka and Triggiani [125]. All these papers and books are confined to homogeneous and *isotropic* plates. The first result on exact controllability of homogeneous *anisotropic* Kirchhoff plates is due to Telega and Bielski [183], where the Dirichlet boundary controllability was considered.

In this section we are going to review recent results and those not covered by the books and papers just cited.

The reader should be aware that our terminology is rather typical for structural mechanics; for instance we use the notion dynamic Kirchhoff plate model with or without rotational inertia term.

7.1. Exact Controllability

Komornik (see [180]) solved the problem of exact boundary controllability for a simplified model of circular plate (Petrovsky equation posed on an open ball in $\mathbb{R}^n, n \geq 1$). Of practical interest is obviously the case $n = 2$. This problem was investigated by using the HUM combined with certain result from nonharmonic analysis and an asymptotic formula:

$$c_{m,k,1} = m + bm^{1/3} + O(1) \quad \text{as} \quad m \to \infty,$$

where $c_{m,k,1}$ is the smallest positive zero of

$$kJ_m(x) + xJ'(x).$$

Here m, k are two arbitrary real numbers and $J_m(x)$ is the Bessel function, being a nontrivial solution of the ordinary differential equation:

$$x^2y'' + xy' + (x^2 - m^2)y = 0 \quad \text{in} \quad (0, \infty).$$

We recall that now the solution of the homogeneous Petrovsky equation (with imposed initial conditions) involves Bessel's functions.

Burg [23] studied the exact controllability of a simplified plate model in a domain containing strictly convex obstacles. A control acts only on the trace of the Laplacian on the exterior boundary of the domain. It was shown that for *any time* $T > 0$, any initial data $(w^0(0), \dot{w}^1(0)) \in H_0^{1+\varepsilon} \times H^{-1+\varepsilon}$ $(\varepsilon > 0)$ the plate can be controlled in time T. The problem was reduced to a problem of exponential decay of energy. The author essentially exploited microlocal analysis and geometrical approach due to Lebeau, cf. Bardos *et al.* [16] and [144, Appendice 2].

The semilinear simplified plate equation with Neumann boundary control was investigated by Liu [148]; this author studied the Dirichlet boundary control. Let $\Omega \subset \mathbb{R}^n$ ($n = 2$ for real plates) be a bounded domain with suitably smooth boundary $\Gamma = \partial\Omega$ (say C^3). By χ_{Σ_0} we denote the characteristic function of a subset Σ_0 of $\Sigma = \Gamma \times (0, T)$ where $T > 0$; $g(w)$ is a given function. The semilinear plate equation studied by Liu [148] is specified by

$$\ddot{w} + \Delta^2 w + \alpha w + g(w) = 0, \quad \text{in} \quad Q = \Omega \times (0,T),$$
$$w = 0, \quad \frac{\partial w}{\partial n} = f\chi_{\Sigma_0}, \quad \text{on} \quad \Sigma, \tag{259}$$
$$w(0) = w^0, \quad \dot{w}(0) = w^1, \quad \text{in} \quad \Omega.$$

We observe that the term $\alpha w + g(w)$ may be due to interaction with a nonlinear, deformable foundation.

The following super-linear assumption on g is made:

Assumption (H). Assume that $g \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$ and $g(0) = 0$, and assume that there exist constants $k > 0$ and $p > 1$ such that

$$|g'(z)| \leq k|z|^{p-1}, \quad z \in \mathbb{R}, \tag{260}$$

with

$$1 < p < 2 \quad \text{for} \quad 1 \leq n \leq 4 \quad \text{or} \quad 1 < p \leq 1 + \frac{4}{n} \quad \text{for} \quad n \geq 5. \tag{261}$$

We observe that one could try to include the term αw in the nonlinear function $g(w)$. Then, however, condition (260) applied to $\alpha w + g(w)$ would require $\alpha = 0$.

Let $\mathcal{C}$ be the set of all initial states (w^0, w^1) in a suitable Hilbert space (to be specified in the theorem below), each of which can be steered to rest by a controller f. The set $\mathcal{C}$ is called the *set of null controllability.*

Definition 2: *The system is said to be locally controllable if the set $\mathcal{C}$ of null controllability contains an open neighborhood of* 0 *in a suitable Hilbert space.*

Remark 22. From Definition 2 follows the definition of local controllability for a control process in $\mathbb{R}^n$.

Now we are in a position to formulate the main result due to Liu [148].

Theorem 25: *Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary Γ of class C^3. Let $T > 0$. Assume (H) holds. Further, if $n \geq 5$ and $p = 1 + \frac{4}{n}$, then suppose that $g'(z)$ is continuous on $\mathbb{R}$. Then, the system (259) is locally controllable in $L^2(\Omega) \times H^{-2}(\Omega)$. That is, there exists a neighborhood $\mathfrak{O}$ of $(0,0)$ in $L^2(\Omega \times H^{-2}(\Omega)$ such that for any $(w^0, w^1) \in \mathfrak{O}$ there exists a control $f \in L^2(\Sigma(x^0))$ which brings the system to rest.*

We recall that, cf. Sec. 3

$$\Sigma(\mathbf{x}^0) = \Gamma(\mathbf{x}^0) \times (0,T),$$

where $\mathbf{x}^0 \in \mathbb{R}^n$ and

$$\Gamma(\mathbf{x}^0) = \{\mathbf{x} \in \Gamma \mid \boldsymbol{m}(\mathbf{x}) \cdot \boldsymbol{n}(\mathbf{x}) > 0\}, \quad \mathbf{m}(\mathbf{x}) = (x_k - x_k^0).$$

Remark 23. In the proof of the last theorem one has to distinguish two cases. Schauder's fixed point theorem can be applied in the case where $1 < p < 2$ for $1 \leq n \leq 4$ or $1 < p < 1 + \frac{4}{n}$ for $n \geq 5$. Unfortunately, Schauder's fixed point theorem cannot be applied in the case where $p = 1 + \frac{4}{n}$ because the corresponding nonlinear operator may not be compact. Instead, in the second case the inverse function theorem was used.

7.2. Stabilization

Horn and Lasiecka [74] studied the boundary stabilization of the isotropic Kirchhoff plate with and without the rotational inertia term.

The first system is given by

$$\begin{aligned} \ddot{w} - \gamma\Delta\ddot{w} + \Delta^2 w = 0 \quad &\text{in} \quad \Omega \times \mathbb{R}^+, \\ w = 0 \quad &\text{on} \quad \Gamma \times \mathbb{R}^+, \\ \Delta w = -\frac{\partial \dot{w}}{\partial n} \quad &\text{on} \quad \Gamma \times \mathbb{R}^+, \\ w(0) = w^0,\ \dot{w}(0) = w^1 \quad &\text{in} \quad \Omega. \end{aligned} \tag{262}$$

The control function u is thus given by: $u = -\frac{\partial \dot{w}}{\partial n}$. The parameter γ is positive and proportional to the square of the thickness of the plate. For thin plate it is thus assumed to be small.

The second problem is obtained from (262) by deleting the rotational inertia term:

$$\begin{aligned} \ddot{w} + \Delta^2 w = 0 \quad &\text{in} \quad \Omega \times \mathbb{R}^+, \\ w = 0 \quad &\text{on} \quad \Gamma \times \mathbb{R}^+, \\ \Delta w = -\frac{\partial \dot{w}}{\partial n} \quad &\text{on} \quad \Gamma \times \mathbb{R}^+, \\ w(0) = w^0, \quad \dot{w}(0) = w^1 \quad &\text{in} \quad \Omega. \end{aligned} \tag{263}$$

In fact, system (263) is the limit of (262) when $\gamma \to 0$. We recall that model (262) is of hyperbolic type (with finite speed of propagation), while model (263) is of Petrovsky type (with infinite speed of propagation).

The set Ω is an open and bounded domain in $\mathbb{R}^2$ with sufficiently smooth, say C^∞, boundary Γ. In fact, it suffices to assume that the boundary Γ is C^4. With C^∞ regularity of Γ the estimates at the level of pseudodifferential calculus are less involved.

The energy corresponding to system (262) is defined by

$$\begin{aligned} E_\gamma(t) &= \frac{1}{2}\left[\|\dot{w}(t)\|_{L^2(\Omega)} + \gamma\|\nabla\dot{w}(t)\|^2_{L^2(\Omega)} + \|\Delta w(t)\|^2_{L^2(\Omega)}\right] \\ &= \frac{1}{2}\left[\|\dot{w}(t)\|_{H^1_{0,\gamma}(\Omega)} + \|\Delta w(t)\|^2_{L^2(\Omega)}\right] \end{aligned} \tag{264}$$

where $H^1_{0,\gamma}(\Omega)$ denotes the Hilbert space $H^1_0(\Omega)$ with norm

$$\|f\|^2_{H^1_{0,\gamma}(\Omega)} = \|f\|^2_{L^2(\Omega)} + \gamma\|\nabla f\|^2_{L^2(\Omega)}. \tag{265}$$

The uniform stabilization result for system (262) is formulated as follows.

Theorem 26: *The feedback system (262) is exponentially (uniformly) stable on the space $H^2(\Omega) \times H^1_{0,\gamma}(\Omega)$; i.e. there exists constants $c > 0$, $\omega > 0$, such that*

$$\|w(t)\|^2_{H^2(\Omega)} + \|\dot{w}(t)\|^2_{H^1_{0,\gamma}(\Omega)} \le ce^{-\omega t}\left[\|w^0\|^2_{H^2(\Omega)} + \|w^1\|^2_{H^1_{0,\gamma}(\Omega)}\right]. \tag{266}$$

Moreover, the constants c and ω are independent of γ.

Remark 24. Similar uniform stabilization theorem holds for system (263); this system is stable on the space $H^2(\Omega) \times L^2(\Omega)$.

The proof of the last theorem uses the semigroup approach and multiplier methods. This part is rather standard. However, nonstandard is the derivation of two estimates for the traces of the solution of (262) on the boundary. Derivation of these estimates requires both microlocal analysis and special regularity properties of a pseudodifferential (abstract) Schrödinger equation, cf. also Burg [23], Lasiecka and Triggiani [125] for the case $n \geq 2$.

Let us set

$$\Sigma_T = \Gamma \times (0, T).$$

Lemma 12: *Let w be the solution to (262) and let $0 < \alpha < T/2$. Then w satisfies the following inequality*

$$\left\| \frac{\partial}{\partial n} \frac{\partial w}{\partial \tau} \right\|^2_{L^2(\alpha, T-\alpha; \Gamma)} \leq c \left\{ \left\| \frac{\partial \dot{w}}{\partial n} \right\|^2_{L^2(\Sigma_T)} + \|w\|^2_{L^2(0,T; H^{3/2+\varepsilon}(\Omega))} \right\}, \tag{267}$$

where $0 < \varepsilon < \frac{1}{2}$ and c is independent of γ.

Here $\frac{\partial w}{\partial \tau}$ denotes the tangential trace of w.

Lemma 13: *Let w be the solution to (262) and α and ε be as above. Then w satisfies the following inequality*

$$\left\| \frac{\partial (\Delta w)}{\partial n} \right\|^2_{H^{-1}(\alpha, T-\alpha; L^2(\Gamma))}$$

$$\leq c_T \left\{ (1+\gamma) \left\| \frac{\partial \dot{w}}{\partial n} \right\|^2_{L^2(\Sigma_T)} + (1+\gamma) \|w\|^2_{L^2(0,T; H^{3/2+\varepsilon}(\Omega))} \right\}, \tag{268}$$

where c_T is independent of γ and $H^{-1}(\alpha, T-\alpha; L^2(\Gamma))$ is the dual (pivotal) to the space $H^1(\alpha, T-\alpha; L^2(\Gamma))$

Remark 25. Several other results on decay of solutions of Petrovsky systems were provided by Komornik [94, 180] and Rao [168]. The approach used in similar to the one sketched in Sec. 3.3. Rao [168] proved strong stability for a plate model which is clamped on one part of the boundary and rimmed along the other with a flange that has mass and moment inertia of the boundary. The system is strongly but not uniformly stable (only rational decay rate was proved). In contrast, a simplified model in which the bending moment of inertia of the boundary is absent is exponentially stable.

Ji and Lasiecka [80] extended the stabilization results concerning the systems (262), (263) to *semilinear* equations:

(i)

$$\begin{gathered} \ddot{w} - \gamma \Delta \ddot{w} + \Delta^2 w + f(w) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\ w = 0, \quad \Delta w = -g_1\left(\frac{\partial \dot{w}}{\partial n}\right) \quad \text{on} \quad \Gamma \times \mathbb{R}^+, \\ w(0) = w^0, \quad \dot{w}(0) = w^1 \quad \text{in} \quad \Omega; \end{gathered} \tag{269}$$

(ii)

$$\ddot{w} + \Delta^2 w + f(w) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+,$$
$$w = 0, \quad \Delta w = -g_1\left(\frac{\partial \dot{w}}{\partial n}\right) \quad \text{on} \quad \Gamma \times \mathbb{R}^+, \tag{270}$$
$$w(0) = w^0, \quad \dot{w}(0) = w^1 \quad \text{in} \quad \Omega.$$

Under appropriate assumptions on f and g_1, strong,rational and exponential stability were proved, cf. also Aassila [1]. The last author considered the following damped Petrovsky system

$$\ddot{w} + \Delta^2 w - g(\Delta \dot{w}) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+,$$
$$w = \Delta w = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}^+, \tag{271}$$
$$w(0) = w^0, \quad \dot{w}(0) = w^1 \quad \text{in} \quad \Omega,$$

and proved the strong asymptotic stability. The function g in Eq. $(271)_1$ satisfies conditions similar to those satisfied by g appearing in wave equation, see Sec. 3.3.

From Sec. 3.3 we know that stabilization problems in the presence of unilateral conditions lead to the study of the differential inclusion [180, 194], or more generally, to [110, 112, 118, 180],

$$\ddot{u} + Au + B\partial\psi B^* \dot{u} \ni 0. \tag{272}$$

Obviously, inclusion (272) has to completed with initial conditions. The available results solve the question of strong stability; the problem of exponential stability seems to remain open. Moreover, Coulomb friction is not covered by the proposed formalism. We also note that Conrad and Pierre (see [180, 194]) studied strong stability of rectangular Kirchhoff plate (without the rotational inertia), where the external forces act at the points $(p_i, q_i) \in \Omega$, $i = 1, \ldots, l$. In the case of one pointwise actuator we have

$$\psi(v) = \frac{1}{2} v^2(p, q),$$

where $\psi : V \to \mathbb{R}^+$, and

$$V = \{v \in H^2(\Omega) \mid v = 0 \quad \text{on} \quad \Gamma\}.$$

Then

$$\partial\psi(v) = v(p, q)\delta_{pq}.$$

Obviously δ_{pq} denotes Dirac mass at the point (p, q).

Ji and Lasiecka [79] studied the following abstract model

$$\dot{y} = Ay + Bu \quad \text{in} \quad [D(A^*)]',$$
$$y(0) = y_0 \in H, \tag{273}$$
$$y_{ob} = Cy,$$

where $[D(A^*)]'$ is the dual of $D(A^*)$ with respect to the H-topology, A is a generator of an analytic semigroup defined on a Hilbert space H, B is the control operator, and

C is the observation. Both control and observation are modelled by fully unbounded operators. Under certain hypotheses on A, B and C there exists an infinite-dimensional compensator y_c, the solution of, cf. also [15]

$$\dot{y}_c = (A + BF - KC)z + KCy, \qquad y_c(0) = y_c^0, \tag{274}$$

such that the feedback control

$$u = Fy_c, \tag{275}$$

exponentially stabilizes (273). The linear operators F, K appear in the stabilizability-detectability assumption.

The study was motivated by recent applications of "smart material" technology in the context of control and stabilization. Smart actuators and sensors such as piezoceramic patches, piezoelectric devices are modelled by unbounded operators, cf. Banks *et al.* [15].

The main result of Ji and Lasiecka [79] provides a construction of the partially observed stabilizing feedback. Elaboration of approximating schemes requires many assumptions on approximation of operators A, B, C, F and K. The examples concerning simplified models of Kirchhoff plates deal with: (i) a structurally damped plate with boundary control and point observation, and (ii) a structurally damped plate with point control and point observations. The second model arises in the context of smart sensors and actuators when the control action is exerted via voltage applied to the piezoceramic patches which are bonded to the plate.

The next particular case of (273) was considered by Hendrickson [69] who considered a Kirchhoff plate with the rotational inertia term; a boundary control acts in the form of the bending moment. The boundary observation is specified by, cf. also [70],

$$y_{ob}(x,t) = C(w, \dot{w}) = \frac{\partial w(x,t)}{\partial n} \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+.$$

This author developed an algorithm that constructs a finite-dimensional compensator that produces near optimal performance when applied to the original dynamics. The algorithm includes a regularization of the original problem and an approximation of the regularized problem. Particularly, the case $\Omega = (0,1)$ was considered, using the space of Hermite cubics defined on a uniform mesh of Ω. The results of calculations were presented in the form of figures and tables.

7.3. Junction and transmission problems

Let us first introduce some notations, cf. Nicaise [160–162],

$$\Gamma = \{x \in (-1,1)^3 \mid x_2 = 0,\ 0 < x_1 < 1\},$$

$$\Omega_1 = \{x \in \mathbb{R}^3 \mid x_2 = 0,\ 0 < x_1 < 2 \text{ and } -1 < x_3 < 1\},$$

$$\Omega = (-1,1)\backslash\bar{\Gamma}.$$

Γ and Ω_1 may be identified with open sets $(0,1) \times (-1,1)$ and $(0,2) \times (-1,1)$ of $\mathbb{R}^2$, respectively. We observe that Ω is the unit cube with a slit along the half-plane $x_2 = 0$, $x_1 \geq 0$. The slit Γ of Ω has two faces: Γ_+ and Γ_-, consequently we denote by $\gamma_+ u$

(respectively $\gamma_- u$) the trace of a function u on Γ from above (respectively from below) in Ω. The boundary $\partial\Omega$ of Ω is decomposed as follows:

$$\Gamma_1 = \{x \in \partial\Omega : |x_3| = 1\},$$

$$\Gamma_2 = \partial\Omega \backslash \{\bar{\Gamma} \backslash \Gamma_1\}.$$

It is also convenient to split up Γ_1 and Γ_2 into their plane faces:

$$\Gamma_1 = \bigcup_{k=1,2} \bar{\Gamma}_{1k}, \qquad \Gamma_2 = \bigcup_{k=1}^{5} \bar{\Gamma}_{2k}.$$

We denote by γ_{ik} the trace operator on the face Γ_{ik} in Ω.

Let us set $\boldsymbol{U} = (\boldsymbol{u}, w)$, where $\boldsymbol{u}$ is the displacement vector of the cube with the slit Γ whilst w denotes the deflection of Ω_1. Both the cube Ω and the plate Ω_1 are made of the same homogeneous and isotropic linear elastic material. With such a coupled system we can associate a selfadjoint operator A [162]:

$$\boldsymbol{A}\boldsymbol{U} = \{ -\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}),\ \varrho\Delta^2 w + \{\gamma_- \sigma_{22}(\boldsymbol{u}) - \gamma_+ \sigma_{22}(\boldsymbol{u})\}\chi_\Gamma \},$$

where $\varrho = \frac{8}{3}\frac{\mu(\lambda+\mu)}{\lambda+2\mu}$, and

$$\sigma_{ij}(\boldsymbol{u}) = 2\mu e_{ij}(\boldsymbol{u}) + \lambda e_{kk}(\boldsymbol{u})\delta_{ij}.$$

As usual, χ_Γ denotes the characteristic function of Γ.

Nicaise [161, 162] assumed that $w \in H_0^2(\Omega_1)$. This author proved obtained an exact controllability result for such a coupled linear elastic system. The control functions act on $\Sigma_{1k} = \Gamma_{1k} \times (0,T)$, $\Sigma_{2k} = \Gamma_{2k} \times (0,T)$ and – in a special manner – on Ω_1. The Hilbert Uniqueness Method was adopted to solve this controllability problem. The main difficulty lies in the fact that the 3D-part of the weak solution of the stationary problem has never the regularity $H^{3/2+\varepsilon}$, for some $\varepsilon > 0$. Fortunately, since it has only edge singularities along the bottom of the crack, a proper choice of the multiplier enables us to use the HUM.

Reissner–Mindlin plate model is more general than Kirchhoff model. Now the (average) rotation angles φ_α of cross sections are independent of the transverse deflection w; in the Kirchhoff theory we have $\varphi_\alpha = -w_{,\alpha}$. The Reissner–Mindlin model is not restricted to thin plates, it also describes *moderately thick* plates. The problem of exact controllability via boundary controls of isotropic Reissner–Mindlin plate with *piecewise constant elastic moduli* was considered by Lagnese [106]. The domain Ω is such as in Fig. 1. Physically justified transmission conditions were assumed along S_0. Only Dirichlet boundary control was examined, though other boundary conditions are also possible.

7.4. Geometrically nonlinear plates

Until now we have considered geometrically linear plate models. Such models admit only small transversal displacements, i.e. small with respect to the thickness. To cope with transversal displacements of the order of thickness, geometrically nonlinear models have to be employed. Among the most popular nonlinear models is the von Kármán plate model and its variants, cf. Ciarlet and Rabier [31], Lewiński and Telega [143] and the references therein. The reader should be aware that the von Kármán plate theory describes plates with *moderately large* deflections, and not, as is often claimed in relevant mathematical literature, large deflections. The characteristic feature of von Kármán plate is a coupling between stretching and bending.

Let $\boldsymbol{u} = (u_\alpha)$, $\alpha = 1, 2$, be the in-plane displacement vector and let w denote, as previously, the deflection of the plate. The strain measures for von Kármán plates are defined by

$$\epsilon_{\alpha\beta}(\boldsymbol{u}, w) = e_{\alpha\beta} + \frac{1}{2} w_{,\alpha} w_{,\beta}, \tag{276}$$

$$\kappa_{\alpha\beta}(w) = -w_{,\alpha\beta}, \tag{277}$$

where $w_{,\alpha} = \frac{\partial w}{\partial x_\alpha}$, etc.

The constitutive relationships are specified by

$$N_{\alpha\beta}(\boldsymbol{u}, w) = A_{\alpha\beta\lambda\mu} \epsilon_{\lambda\mu}(\boldsymbol{u}, w), \tag{278}$$

$$M_{\alpha\beta}(w) = D_{\alpha\beta\lambda\mu} \kappa_{\lambda\mu}(\boldsymbol{u}, w). \tag{279}$$

Here $\boldsymbol{N} = (N_{\alpha\beta})$ is the membrane force tensor while $\boldsymbol{M} = (M_{\alpha\beta})$ is the moment tensor. For isotropic von Kármán plates the fourth-order tensors are isotropic. Only isotropic von Kármán plates have been investigated in the literature on controllability and stabilization.

For earlier results on stabilization of von Kármán plates the reader is referred to Lagnese [103, Chap. 5]. At that time, the question of uniqueness and regularity of solutions of the dynamic system have remained open. These questions have been solved several years later, see below.

Tucsnak (see [180]) investigated the asymptotic behaviour of the solution of the following system

$$\begin{gathered}
\ddot{w} + \Delta^2 w - b\Big(\int_\Omega |\nabla w|^2\, dx\Big)\Delta w + a(x)\dot{w} = 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\
w = \frac{\partial w}{\partial n} = 0, \quad \text{on} \quad \Sigma = \Gamma \times \mathbb{R}^+, \\
w(0) = w^0, \quad \dot{w}(0) = w^1, \quad \text{in} \quad \Omega,
\end{gathered} \tag{280}$$

where $\Omega \subset \mathbb{R}^n$ is an open, regular and bounded set, b is a positive constant and $a \in L^\infty(\Omega)$, $a(x) \geq 0$ a.e. in Ω.

If $n = 2$ the system (280) represents the *Berger approximation* of the von Kármán equations, cf. Leissa [137].

Using the standard semigroup techniques one easily proves the existence and uniqueness of strong and weak solutions of the system (280) on a finite time interval $[0, T]$.

The energy functional is defined by

$$E(t) = \frac{1}{2}\left[\|\dot{w}(t)\|^2 + \|\Delta w(t)\|^2 + \frac{b}{4}\|\nabla w(t)\|^2\right]. \tag{281}$$

Here $\|\cdot\|$ stands for the L^2-norm. Simple calculation yields

$$\dot{E}(t) = -\int_\Omega a(x)[\dot{w}(x,t)]^2\,dx \leq 0.$$

To prove the exponential decay:

$$\exists\ c,\omega \geq 0, \quad E(t) \leq ce^{-\omega t}E(0), \quad \forall\ t \geq 0, \tag{282}$$

two cases were studied:

(i) There exist an open set $\mathfrak{O} \subset \Omega$ and $a_0 > 0$ such that

$$\mathfrak{O} \subset \{x \in \Omega\ s.t.\, a(x) \geq a_0\}.$$

If $\mathfrak{O} = \Omega$, the damping acts in the whole domain.

(ii) $\mathfrak{O}$ is a neighbourhood of the boundary Γ.

Under appropriate, rather strong assumptions, (282) can also be proved.

Introducing the Airy stress function φ such that

$$N_{11} = \frac{\partial^2\varphi}{\partial x_2^2}, \qquad N_{22} = \frac{\partial^2\varphi}{\partial x_1^2}, \qquad N_{12} = N_{21} = -\frac{\partial^2\varphi}{\partial x_1 \partial x_2},$$

one can consider von Kármán equations in the variables $w(x,t)$ and $\varphi(w(x,t))$.

Bradley and Lasiecka [21] considered the following von Kármán system with the rotational inertia term

$$\begin{aligned}
\ddot{w} - \gamma\Delta\ddot{w} + \Delta^2 w + b(x)\dot{w} &= [w,\varphi(w)] \quad \text{in} \quad Q,\\
w = \frac{\partial w}{\partial n} &= 0 \quad \text{on} \quad \Sigma_0,\\
\Delta w + (1-\mu)B_1 w &= -\frac{\partial \dot{w}}{\partial n} \quad \text{on} \quad \Sigma_1,\\
\frac{\partial \Delta w}{\partial n} + (1-\mu)\tilde{B}_2 w - \gamma\frac{\partial \ddot{w}}{\partial n} &= \dot{w} - \frac{\partial^2 \dot{w}}{\partial \tau^2} \quad \text{on} \quad \Sigma_1,\\
w(0) = w^0, \quad \dot{w}(0) &= w^1 \quad \text{in} \quad \Omega.
\end{aligned} \tag{283}$$

Here $Q = \Omega \times (0,T)$ and $\Sigma_\alpha = \Gamma_\alpha \times (0,T)$, $\alpha = 0,1$; Ω is a bounded open domain in $\mathbb{R}^2$ with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 a relatively open, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$. The function $b \in L^\infty(\Omega)$ and satisfies $b(x) > 0$ a. e. in Ω, $0 < \mu < \frac{1}{2}$ is the Poisson ratio and the boundary operators B_1 and $\tilde{B}_2$ are given by

$$\begin{aligned}
B_1 w &= 2n_1 n_2 w_{x_1x_2} - n^2 w_{x_2x_2} - n^2 w_{x_1x_1},\\
\tilde{B}_2 w &= \frac{\partial}{\partial \tau}\left[(n_1^2 - n_2^2)w_{x_1x_2} + n_1 n_2(w_{x_2x_2} - w_{x_1x_1})\right],
\end{aligned} \tag{284}$$

where $\tau = (\tau_1,\tau_2)$ denotes the tangential direction; $w_{x_1x_1} = \frac{\partial^2 w}{\partial x_1^2}$, etc.

Moreover, $\varphi(w)$ satisfies the system of equations

$$\begin{aligned} \Delta^2\varphi &= -[w,w], \\ \varphi = \frac{\partial\varphi}{\partial n} &= 0 \quad \text{on} \quad \Gamma\times(0,\infty), \end{aligned} \tag{285}$$

where

$$[\varphi,\psi] = \frac{\partial^2\varphi}{\partial x_1^2}\frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial x_2^2}\frac{\partial^2\psi}{\partial x_1^2} - 2\frac{\partial^2\varphi}{\partial x_1\partial x_2}\frac{\partial^2\psi}{\partial x_1\partial x_2}.$$

The energy functional is defined by

$$E(t) = \frac{1}{2}\int_\Omega \left[|\dot w|^2 + \gamma|\nabla\dot w|^2 + |\Delta\varphi(w)|^2\right]dx + \frac{1}{2}a(w,w), \tag{286}$$

where

$$a(w,v) = \int_\Omega \left[\Delta w\Delta v + (1-\mu)(2w_{x_1x_2}v_{x_1x_2} - w_{x_1x_1}v_{x_2x_2} - w_{x_2x_2}v_{x_1x_1})\right]dx.$$

We observe that in order to treat the control function $\dot w$ acting along γ_1 as forces one has either to treat the system as primarily nondimensionalized or multiply it by a proper coefficient. Similar comment is pertinent to the control function $\frac{\partial\dot w}{\partial n}$ and $\frac{\partial^2\dot w}{\partial\tau^2}$, treated as being moments.

The main result obtained by Bradley and Lasiecka [21] is formulated as follows.

Theorem 27: *Assume that $\Gamma_0\neq\emptyset$, and that there exists $\mathbf{x}^0\in\mathbb{R}^2$ such that*

$$(\mathbf{x}-\mathbf{x}^0)\cdot\mathbf{n}(\mathbf{x})\le 0 \quad \text{for} \quad \mathbf{x}\in\Gamma_0.$$

Then for any initial data, $(w^0,w^1)\in\mathfrak{H} = H^2_{\Gamma_0}(\Omega)\times H^1_{\Gamma_0}(\Omega)$, there exists a constant c and a constant $\omega = \omega(\|w^0\|_{H^2_{\Gamma_0}(\Omega)}, \|w^1\|_{H^1_{\Gamma_0}(\Omega)})$, such that the solution pair of (283) satisfies

$$\|(w(t),\dot w(t))\|_{\mathfrak{H}} \le ce^{-\omega t}\|(w^0,w^1)\|_{\mathfrak{H}}. \tag{287}$$

The constant c depends on the size of the initial data measured in $\mathfrak{H}$-norm.

The space $H^2_{\Gamma_0}(\Omega)$ is defined as follows

$$H^2_{\Gamma_0}(\Omega) = \left\{w\in H^2(\Omega)\;\middle|\; w = \frac{\partial w}{\partial n} = 0 \text{ on } \Gamma_0\right\}. \tag{288}$$

We observe that no geometric conditions are imposed on the controlled portion of the boundary, i.e., on Γ_1. In order to dispense with geometric conditions, sharp regularity results for the traces of the corresponding linear problem were used, cf. Lasiecka and Trigianni [125], and Lemmas 12, 13.

Remark 26.

(i) Both strong and weak solutions to (283) exist and are unique.

(ii) Horn and Lasiecka [75] extended the last theorem to *nonlinear* boundary feedback controls and $\Gamma_0 = \emptyset$. More precisely, it was shown that the energy function decays to zero at a uniform rate which is independent of the value of the parameter γ in $(283)_1$. In this case, no geometric conditions are imposed on Γ. Under rather standard assumptions on the control functions a solution exists and is unique.

(iii) A von Kármán system *without* the rotational inertia ($\gamma = 0$) and nonlinear feedback boundary controls was investigated by Horn and Lasiecka [73], Favini *et al.* [47].

The results obtained in the last two papers also solved a long-standing open problem of uniqueness of weak solutions to von Kármán systems without the regularizing term $\gamma\Delta\ddot{w}$. Horn and Lasiecka [73] based their proof on a nonlinear Galerkin method. On the other hand, Favini *et al.* [47] achieved deeper results proving "sharp" regularity of the Airy's stress function. The existence and uniqueness of *intermediate* solutions was also proved; intermediate means here solutions which belong to fractional spaces and are intermediate between strong and weak solutions.

Remark 27. In a series of papers Lasiecka [113–115] considered the asymptotic behaviour of solutions to von Kármán systems in the absence of "light" interior damping, say in $(283)_1$ (i.e., $b \equiv 0$), or/and one of boundary controls. The existence of attractors for three von Kármán systems was established.

The second (and more natural approach) to the study of von Kármán systems consists in direct formulation in the variables $u_\alpha(x,t)$ and $w(x,t)$.

Puel and Tucsnak [165] considered the following von Kármán system with linear boundary controls

$$
\begin{aligned}
&\ddot{\boldsymbol{u}} - \operatorname{div} \boldsymbol{N}(\boldsymbol{u}, w) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+,\\
&\ddot{w} - \gamma\Delta\ddot{w} + D\Delta^2 w - \operatorname{div}\big(\boldsymbol{N}(\boldsymbol{u}, w)\nabla w\big) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+,\\
&\boldsymbol{u} = \boldsymbol{0}, \quad w = \frac{\partial w}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}^+,\\
&\boldsymbol{N}(\boldsymbol{u}, w)\boldsymbol{n} = \boldsymbol{g} \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+,\\
&D\big[\Delta w + (1-\mu)B_1 w\big] = -M_s \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+,\\
&D\left[\frac{\partial \Delta w}{\partial \boldsymbol{n}} + (1-\mu)\frac{\partial B_2 w}{\partial s}\right] - \gamma\frac{\partial \ddot{w}}{\partial \boldsymbol{n}} - \big(\boldsymbol{N}(\boldsymbol{u}, w)\boldsymbol{n}\big)\cdot\nabla w = -\frac{\partial M_n}{\partial s} - h \\
&\qquad\qquad \text{on} \quad \Gamma_1 \times \mathbb{R}^+,\\
&\boldsymbol{u}(0) = \boldsymbol{u}^0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}^1, \quad w(0) = w^0, \quad \dot{w}(0) = w^1 \quad \text{in} \quad \Omega.
\end{aligned}
\tag{289}
$$

Here $\Omega \subset \mathbb{R}^2$ is a bounded sufficiently smooth domain, $\Gamma_0 \subset \Gamma$, $\Gamma_1 = \Gamma \backslash \Gamma_0$; D denotes the flexural rigidity of the plate. The plate is isotropic and $N_{\alpha\beta}(\boldsymbol{u}, w)$, B_1 are given by Eqs. (278), $(284)_1$, respectively. The boundary operator B_2 is given by

$$
B_2 w = (n_1^2 - n_2^2) w_{x_1 x_2} + n_1 n_2 (w_{x_2 x_2} - w_{x_1 x_1}). \tag{290}
$$

The quantities g, h, M_s and M_n are the boundary controls of the system. They correspond to the stretching in the plane of the plate, the effect of transverse shear force, the bending moment around the tangential vector to Γ and the twisting moment about the normal to Γ, respectively. The control functions were assumed in the following form

$$g = -a(\boldsymbol{m}\cdot\boldsymbol{n})\dot{\boldsymbol{u}} - b\left(\frac{\partial u_2}{\partial s} - \frac{\partial u_1}{\partial s}\right), \quad \text{where} \quad a, b > 0,$$

$$M_s = a(\boldsymbol{m}\cdot\boldsymbol{n})\frac{\partial \dot{w}}{\partial \boldsymbol{n}}, \tag{291}$$

$$\frac{\partial M_n}{\partial s} + h = -a(\boldsymbol{m}\cdot\boldsymbol{n})\dot{w} + a\frac{\partial}{\partial s}\left[(\boldsymbol{m}\cdot\boldsymbol{n})\frac{\partial \dot{w}}{\partial \boldsymbol{n}}\right].$$

The boundary Γ of Ω satisfies the standard assumption:

$$\boldsymbol{m}(x)\cdot\boldsymbol{n}(x) \leq 0, \quad \text{if} \quad x \in \Gamma_0, \qquad \boldsymbol{m}(x)\cdot\boldsymbol{n}(x) \geq 0, \quad \text{if} \quad x \in \Gamma_1.$$

The energy of the system is now defined by, cf. (286),

$$E(t) = \frac{1}{2}\Big[\|\dot{\boldsymbol{u}}(t)\|^2 + \|\dot{w}(t)\|^2 + \gamma\|\nabla\dot{w}(t)\|^2 + a\big(w(t), w(t)\big) + \big\langle N_{\alpha\beta}\big(\boldsymbol{u}(t), w(t)\big)\epsilon_{\alpha\beta}\big(\boldsymbol{u}(t), w(t)\big)\big\rangle\Big], \tag{292}$$

where $\epsilon_{\alpha\beta}(\boldsymbol{u}, w)$ is defined by (276), and

$$a(w,v) = D\int_\Omega \left[\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 v}{\partial x_2^2} + \mu\left(\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}\right) + 2(1-\mu)\frac{\partial^2 w}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2}\right] dx. \tag{293}$$

Puel and Tucsnak [165] proved the *exponential decay* of the energy $E(t)$ for any *strong solution* of (289) with the feedback law (291).

In an accompanying paper [166], these authors proved the existence and uniqueness of strong solutions to (289) with (291). Lasiecka [116] extended the last result proving also the existence and uniqueness of weak (variational) solutions as well as the well-posedness in intermediate spaces, provided that on Γ_1 the damping is nonlinear. Supplementing the l.h.s. of Eq. $(289)_1$ with $b_1\dot{\boldsymbol{u}}$, and $(289)_2$ with $b_2\dot{w}$ and making appropriate assumptions on the nonlinear feedback functions, Lasiecka [117] proved uniform energy decay rates for such a model without the geometric assumption on Γ_1. This result holds for strong solutions; b_1 and b_2 being nonnegative operators on L^2. Similar results is valid for weak solutions in the case of *linear damping functions*.

The lectures by Lasiecka [118] offer an abstract approach applicable to the study of stabilization of von Kármán plates.

8. Shells

Shells are important engineering structures where geometry plays an important role, cf. Leissa [138], Lewiński and Telega [143] and the references therein. Thus it is not surprising that the number of significant and rigorous results related to controllability and stabilization of shells is rather modest. Indeed, we shall see that no controllability result seems to be available for a general model of isotropic Koiter shell. We recall that the Koiter shell model is very popular in structural mechanics. This model is geometrically linear.

8.1. Membrane shells

Consider axially symmetric vibrations of a spherical membrane with the radius R of the middle surface and the opening angle θ_0. The meridional and radial components of the displacement vector $\boldsymbol{u} = (u(\theta,t), w(\theta,t))$ satisfy the system

$$\begin{gathered} a\ddot{u} - \left(\frac{(u\sin\theta)'}{\sin\theta}\right)' - (1-\nu)u + (1+\nu)w' = 0 \quad \text{in} \quad (0,\theta_0)\times(0,T), \\ a\ddot{w} - \frac{(1+\nu)}{\sin\theta}(u\sin\theta)' + 2(1+\nu)w = 0 \quad \text{in} \quad (0,\theta_0)\times(0,T), \\ u(0,t) = 0, \qquad u(\theta_0,t) = g(t), \\ \boldsymbol{u}(\theta,0) = \boldsymbol{u}^0, \qquad \dot{\boldsymbol{u}}(\theta,0) = \boldsymbol{u}^1, \end{gathered} \tag{294}$$

where $a = \varrho R^2(1-\nu^2)/E$, $\nu \in (-1, 1/2)$, ϱ is the density of the material and the prime denotes the derivative with respect to θ.

We set

$$L^2 = L^2(0,\theta_0;\sin\theta\,d\theta) = \left\{ f \;\middle|\; \int_0^{\theta_0} |f|^2 \sin\theta\,d\theta < +\infty \right\},$$

$$\mathcal{U} = \left\{ u \;\middle|\; \frac{\partial u}{\partial\theta}, u\cot\theta \in L^2(0,\theta_0;\sin\theta\,d\theta),\ u(0) = u(\theta_0) = 0 \right\}.$$

It can be shown that this membrane problem is *not*, in general *exactly controllable* for any $\{\boldsymbol{u}^0, \boldsymbol{u}^1\} \in (L^2\times L^2)\times(\mathcal{U}\times L^2)$, cf. Geymonat and Valente [63], Loreti and Valente [153], Valente [186]. Indeed, for the hemispherical membrane there exists a subsequence of eigenfunctions $\boldsymbol{u}_n(\theta) = (u_n(\theta), w_n(\theta))$ such that

$$\lim_{n\to\infty}\left[u_n\left(\frac{\pi}{2}\right) - (1+\nu)w_n\left(\frac{\pi}{2}\right)\right] = 0.$$

Then the sequence $\{\boldsymbol{u}_n, 0\}$ with $\|\boldsymbol{u}_n\|_{\mathcal{U}\times L^2} = 1$ (initial data for the homogeneous problem associated with (294)) does not satisfy the necessary (and sufficient) condition of exact controllability.

In contrast, the partial controllability is possible. More precisely, one can find a control function g such that the final conditions:

$$u(\theta,T) = u_T^0, \qquad \dot{u}(\theta,T) = u_T^1,$$

are satisfied. In other words, we can only control one component of the displacement vector. To solve the problem of partial controllability, Loreti and Valente [153] used the RHUM , cf. Lions [145]. This result was achieved by a generalization of the Ingham theorem, cf. Avdonin and Ivanov [10, 11]. We observe that the problem considered is of space dimension one.

To study more general membrane shell model, Valente [186] used the *relaxed spectral exact controllability*. The physical meaning of this concept was not clarified.

8.2. Geometrically linear shells

Geometrically linear shell models describe shells for which the strain measures are linear functions of displacements, cf. Leissa [138], Lewiński and Telega [143].

Consider first a thin, shallow spherical shell. Its reference configuration in spherical coordinates (r,θ,ϕ) is the region: $r \in [R-h, R+h]$, $\theta \in [0,\theta_0]$, $\phi \in [0,2\pi)$, where h is the half-thickness of the shell, R is the middle surface radius and $\theta_0 < \pi$ is the opening angle. A spherical shallow shell has two characteristic parameters, thinness and shallowness defined respectively by $\alpha = h/R\sin\theta_0$ and $\beta = (R - R\cos\theta_0)/R\sin\theta_0$. For a thin and shallow spherical shell we take $r = R\sin\theta_0$ and suppose $\alpha = h/r \ll 1$ and $\beta \ll 1$. The latter condition implies θ_0 sufficiently small. Within the framework of Koiter's linear shell model we arrive at the shallow shell approximation by introducing the coordinate $\vartheta = R\theta$ and by replacing $\cot\theta$ by $\frac{1}{\theta}$. Then, the axially symmetric vibrations for the meridional and radial middle surface displacements (u,w) can be written in the following form

$$\begin{gathered}\ddot{u} + \frac{e}{R}\ddot{v} - L(u) - \frac{e}{R}L(v) + \frac{1+\mu}{R}w' = 0 \quad \text{in} \quad (0,\vartheta_0)\times(0,T),\\ \ddot{w} - \frac{e}{R}(\ddot{v}r)' + \frac{e}{\vartheta}[L(v)\vartheta]' - \frac{1+\mu}{\vartheta R}(u\vartheta)' + \frac{2(1+\mu)}{R^2}w = 0 \quad \text{in} \quad (0,\vartheta_0)\times(0,T),\end{gathered} \tag{295}$$

where $v \equiv u/R + w'$, $L(u) = u'' + u'/\vartheta - u'/\vartheta^2$, $\vartheta_0 = R\theta_0$, $e = h^2/3$, and μ denotes Poisson's ratio; the prime symbol "$'$" denotes the differentiation with respect to ϑ. The boundary conditions at $\vartheta = 0$ are specified by

$$u = w' = L(v) = 0, \quad \vartheta = 0, \quad t > 0. \tag{296}$$

The following *dissipative* boundary conditions are assumed at $\vartheta = \vartheta_0$

$$\begin{gathered}u' - \frac{1+\mu}{R}w + \mu\frac{u}{\vartheta_0} = -\dot{u} - u, \quad \vartheta = \vartheta_0, \quad t > 0,\\ ev' = -\dot{v}, \quad \vartheta = \vartheta_0, \quad t > 0,\\ eL(v) - e\ddot{v} = \dot{w}, \quad \vartheta = \vartheta_0, \quad t > 0.\end{gathered} \tag{297}$$

The initial conditions are given by

$$u(0,\vartheta) = u^0, \quad \dot{u}(0,\vartheta) = u^1, \quad w(0,\vartheta) = w^0, \quad \dot{w}(0,\vartheta) = w^1, \quad \vartheta \in (0,\vartheta_0). \tag{298}$$

Lasiecka and Triggiani [126] studied the well-posedness of the dissipative mixed problem (295)-(298). A unique solution was shown to exist provided that:

$$u^0 \in \mathcal{U}^1_\vartheta, \quad w^0 \in \mathcal{W}^2_\vartheta, \quad (u^1, w^1) \in \mathcal{V}^1_\vartheta,$$

where

(i) $\mathcal{U}^1_\vartheta(0,\vartheta_0) = \left\{ u \;\middle|\; \frac{u}{\sqrt{\vartheta}},\ u'\sqrt{\vartheta} \in L^2(0,\vartheta_0),\ u(0)=0 \right\}$, with norm

$$\|u\|_{\mathcal{U}^1_\vartheta} = \left\{ \int_0^{\vartheta_0} \left[\frac{u^2}{\vartheta} + (u')^2\vartheta \right] d\vartheta \right\}^{1/2} ;$$

(ii) $W^2_\vartheta(0,\vartheta_0) = \left\{ w \;\middle|\; w\sqrt{\vartheta} \in L^2(0,\vartheta_0),\ w' \in \mathcal{U}^1_\vartheta(0,\vartheta_0) \right\}$, with norm

$$\|w\|_{W^2_\vartheta} = \left\{ \int_0^{\vartheta_0} w^2\vartheta\, d\vartheta + \|w'\|^2_{\mathcal{U}^1_\vartheta} \right\}^{1/2} = \left\{ \int_0^{\vartheta_0} \left[w^2\vartheta + \frac{(w')^2}{\vartheta} + (w'')^2\vartheta \right] d\vartheta \right\}^{1/2} ;$$

(iii) $\mathcal{V}^1_\vartheta(0,\vartheta_0) = \left\{ (u,w) \in L^2_\vartheta(0,\vartheta_0)^2 \;\middle|\; v = \frac{u}{R} + w' \in L^2_\vartheta(0,\vartheta_0),\right.$ or, equivalently, $\left. w' \in L^2_\vartheta(0,\vartheta_0) \right\}$, with norm

$$\|(u,w)\|_{\mathcal{V}^1_\vartheta} = \left\{ \|u\|^2_{L^2_\vartheta} + \|w\|^2_{l^2_\vartheta} + e\|v\|^2_{L^2_\vartheta} \right\}^{1/2}$$

The space $L^2_\vartheta(0,\vartheta_0)$ is defined by

$$L^2_\vartheta(0,\vartheta_0) = \left\{ u \,|\, u\sqrt{\vartheta} \in L^2(0,\vartheta_0) \right\},$$

with norm

$$\|u\|_{L^2_\vartheta} = \left\{ \int_0^{\vartheta_0} u^2\vartheta\, d\vartheta \right\}^{1/2} .$$

The well-posedness (and regularity), was assessed by using the semigroup approach.

The energy of the shallow shell considered is defined by

$$E(t) = E_k(t) + E_p(t), \tag{299}$$

where

$$E_k(t) = \frac{1}{2}\int_0^{\vartheta_0} (\dot{u}^2 + \dot{w}^2 + e\dot{v}^2)\vartheta\, d\vartheta, \tag{300}$$

$$\begin{aligned} E_p(t) = \frac{1}{2}e\int_0^{\vartheta_0} \left[(v')^2\vartheta + \frac{v^2}{\vartheta} \right] d\vartheta + \frac{1-\mu}{2}\int_0^{\vartheta_0} \left[(u' - \frac{w}{R})^2\vartheta + (\frac{u}{\vartheta} - \frac{w}{R})^2\vartheta \right] d\vartheta \\ + \frac{\mu}{2}\int_0^{\vartheta_0} \left[(u' - \frac{w}{R})\sqrt{\vartheta} + (\frac{u}{\vartheta} - \frac{w}{R})\sqrt{\vartheta} \right]^2 d\vartheta + \frac{1}{2}u^2(\vartheta_0)\vartheta_0. \end{aligned} \tag{301}$$

Lasiecka *et al.* [129] proved the exponential stability, provided that $\{u^0, w^0, u^1, w^1\} \in \mathcal{E}$, where

$$\mathcal{E} = \{(u,w),(u_1,w_1) \mid u \in \mathcal{U}^1_\vartheta,\ w \in \mathcal{W}^2_\vartheta,\ (u_1,w_1) \in \mathcal{V}^1_\vartheta\} = (\mathcal{U}^1_\vartheta \times \mathcal{W}^2_\vartheta) \times \mathcal{V}^1_\vartheta.$$

Consequently, we obtain, by using a result of Russell [174] (see also Komornik [94]), a corresponding exact controllability result.

The problem of exact controllability of general model of Koiter shells made of isotropic materials was examined by Miara and Valente [157]. In this case the strain measures are given by

$$\gamma_{\alpha\beta}(\boldsymbol{u},w) = u_{(\alpha|\beta)} - b_{\alpha\beta}w, \tag{302}$$

$$\varrho_{\alpha\beta}(\boldsymbol{u},w) = -(w_{|\alpha\beta} - b^\lambda_\alpha b_{\lambda\beta}w + b^\lambda_{\alpha|\beta}u_\lambda + b^\lambda_\alpha u_{\lambda|\beta} + b^\lambda_{\lambda|\alpha}). \tag{303}$$

Here

$$u_{\alpha|\beta} = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}),$$

and $u_{\alpha|\beta}$ denotes the covariant derivative of u_α; $b_{\alpha\beta}$ are the covariant components of the second fundamental form of the middle surface of the shell. This surface is of class C^4. The displacement vector is (u_α, w), where $\boldsymbol{u}$ is tangent to the middle surface and w is the displacement along the normal to this surface.

The control problem of a linear elastic Koiter shell studied by Miara and Valente [157] can be represented as follows

$$\begin{gathered}\ddot{\boldsymbol{U}} + A\boldsymbol{U} = \boldsymbol{0} \quad \text{in} \quad Q = \Omega \times (0,T),\\ u_\alpha = v_\alpha, \quad w = 0, \quad \frac{\partial w}{\partial \boldsymbol{n}} = v \quad \text{on} \quad \Sigma = \Gamma \times (0,T),\\ \boldsymbol{U}(0) = (u^0_\alpha, w^0), \quad \dot{\boldsymbol{U}}(0) = (u^1_\alpha, w^1),\end{gathered} \tag{304}$$

where $\boldsymbol{U} = (u_\alpha, w)$; now Ω is a bounded open connected subset of $\mathbb{R}^2$ with a boundary of class C^4. The middle surface S of the shell is the image of Ω; more precisely, $S = \boldsymbol{\Phi}(\Omega)$, where $\boldsymbol{\Phi} \in C^4(\Omega)^3$, is an injective mapping. The function $\boldsymbol{v} = (v_\alpha, v) \in L^2(\Sigma)^3$ is the control. The operator A is of the form $A = A_M + \frac{h^2}{2}A_F$, where A_M is the membrane part whilst A_F is the flexural part; h is related to the half-thickness of the shell. To prove the exact controllability, the method HUM was used.

Remark 28. The last exact controllability result with rather high regularity of the boundary hold for shells close to shallow ones. The last restriction is necessary for the indirect inequality to hold. General setting remains open.

8.3. Geometrically nonlinear shells

For geometrically nonlinear shells the strain measures are nonlinear functions of displacements. There are many models of such shells, mathematically poorly investigated even in the case of statics. Among the simplest is the model of a shallow shell, which for $b_{\alpha\beta} \equiv 0$ reduces to the von Kármán model of plates, cf. formulae (276), (277).

Probably, the first result on the stabilization of a nonlinear shell is due to Lasiecka and Valente [128]. The nonlinear model of a thin and shallow elastic spherical shell studied by these authors is described by the following nonlinear system of equations in u and w (representing, as for the linear model, meridional and radial displacements) and defined in $Q = (0, \vartheta_0) \times \mathbb{R}^+$

$$e_1\ddot{u} + \frac{e_1 e}{R}\ddot{V} + b_1^2(\vartheta)\dot{u} - L(u) + \frac{1+\mu}{R}w' - \frac{e}{R}L(V) \\ - \left[V' + \frac{1}{2}V\left(\frac{1-\mu}{\vartheta}\right)\right]V + \frac{V\xi}{R} = 0, \quad (305)$$

$$e_1\ddot{w} - e_1 e\frac{\partial^2}{\partial t^2}\left(V' + \frac{V}{\vartheta}\right) + b_2^2(\varrho)\dot{w} + \frac{e}{\vartheta}\left[L(V)\vartheta\right]' - \frac{1+\mu}{\vartheta}\left(\frac{u}{R}\vartheta\right)' \\ + \frac{2(1+\mu)w}{2} - \frac{1}{2}\frac{V^2}{R}(1+\mu) - \frac{(V\xi\vartheta)'}{\vartheta} = 0. \quad (306)$$

Here $V = \frac{u}{R} + w'$, $\xi = u' - \frac{w}{R} + \frac{1}{2}V^2 + \mu\left(\frac{u}{\vartheta} - \frac{u}{R}\right)$, e_1 is a positive (material) constant. The functions $b_\alpha \in L^\infty(0, \vartheta_0)$, $\alpha = 1, 2$, represent a light damping in the system and they are assumed to be positive on $(0, \vartheta_0)$ (i.e., $b_\alpha > 0$ a. e. in $\vartheta \in [0, \vartheta_0]$).

The boundary conditions applied to the edge of the shell are:

(i) at $\vartheta = 0$

$$u = 0, \quad w = 0, \quad \dot{w} = 0, \quad (307)$$

(ii) at $\vartheta = \vartheta_0$

$$\xi = g_1, \quad eV' = g_2, \quad eL(V) - V\xi - e\ddot{V} = g_3. \quad (308)$$

A possible choice of nonlinear feedback control is

$$\begin{aligned} g_1 &= -f_1(\dot{u}) - u, \\ g_2 &= -f_2(\dot{V}), \\ g_3 &= -f_3(\dot{w}), \end{aligned} \quad (309)$$

where the functions f_i, $i = 1, 2, 3$, are assumed to be continuous, monotone increasing and such that $f_i(0) = 0$, $c_1 s^2 \le f_i(s)s \le c_2 s^2$ for $|s| \ge s_0 > 0$, where s_0 is a given positive constant.

The considered nonlinear problem is a difficult one, due to the intrinsic lack of uniqueness for the stationary (zero load) model (unlike the plate case). The uniform stabilization result of Lasiecka and Valente [128] state that if the half-thickness of the plate satisfies

$$h \ge c_0\vartheta_0\Delta_0, \quad \text{where} \quad \Delta_0 = \frac{\vartheta_0}{R} \ll 1, \quad (310)$$

where c_0 is a positive constant independent of other parameters in the equation, then

$$E(t) \le S(t), \quad \text{for} \quad t \ge T_0. \quad (311)$$

Here $S(t) \to 0$ as $t \to \infty$ and it is determined as a solution of the following nonlinear ODE

$$\begin{aligned} \dot{S}(t) + q(S(t)) &= 0, \\ S(0) &= E(0), \end{aligned} \quad (312)$$

with a monotone increasing function q; obviously $E(t)$ is the energy functional.

We observe that the restrictions imposed by (310) are confirmed by the static nonlinear theory which predicts, for small values of the thickness parameter, the existence of multiple equilibrium states (everted states).

A special case of the model (305), (306) was considered by Bradley and Lasiecka [22] where the control functions are given by

$$g_1 = \dot{u} - u, \qquad g_2 = -\frac{\partial w'}{\partial t}, \qquad g_3 = \dot{w}. \tag{313}$$

Under the "shallowness condition" the exponential decay of the energy was shown.

A general nonlinear shallow shall model was considered by Lasiecka and Marchand [120]. The strain measure are then defined by

$$\begin{aligned} \gamma_{\alpha\beta}(\boldsymbol{u}, w) &= u_{\alpha|\beta} - b_{\alpha\beta} w + \frac{1}{2} w_{,\alpha} w_{,\beta}, \\ \varrho_{\alpha\beta}(\boldsymbol{u}, w) &= \eta_{\alpha\beta}(w) = -w_{|\alpha\beta}. \end{aligned} \tag{314}$$

The feedback functions are nonlinear and no growth conditions were imposed at the origin. The energy of the system obeys the estimate of the type (311). The existence and uniqueness of solutions to the considered dynamic shell was shown by an application of the nonlinear Galerkin method.

9. Control of stochastic systems

In contrast to many papers on control of finite - dimensional stochastic systems we know of only a few papers in the infinite - dimensional case. The aim of the present section is to present the results for the latter case. For papers on the finite-dimensional systems the reader is referred to [27, 42, 48, 176, 192].

9.1. Selected mathematical preliminaries

We presume that the reader is familiar with the theory of probability. Our main aim here is to introduce the notion of stochastic integrals and stochastic differential equations, cf. [163].

Definition 3: *A* random variable *is an $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}^n$, where $(\Omega, \mathcal{F}, P)$ is a (complete) probability space. (Thus $\mathcal{F}$ is a σ-algebra of subsets of Ω, P is a probability measure on Ω, assigning values in $[0,1]$ to each member $\mathcal{F}$ and if B is a Borel set in $\mathbb{R}^n$ then $X^{-1}(B) \in \mathcal{F}$). Every random variable induces a measure μ on $\mathbb{R}^n$, defined by*

$$\mu_X(B) = P(X^{-1}(B)).$$

μ_X is called the distribution of X.

Let $X : \Omega \to \mathbb{R}$ be a random variable. The distribution function F of X is defined by

$$F(x) = P[X \leq x].$$

F has the following properties:

(i) $0 \leq F \leq 1$, $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$;

(ii) F is non-decreasing;

(iii) F is right-continuous, i.e. $F(x) = \lim_{h \to 0, h > 0} F(x+h)$.

If $\int_\Omega |X(\omega)| dP(\omega) < \infty$ then

$$E[X] := \int_\Omega X(\omega) dP(\omega) = \int_{\mathbb{R}^n} x d\mu_X(x)$$

is called the *expectation* of X(w.r.to P = with respect to P).

Let $g : \mathbb{R} \to \mathbb{R}$ be measurable such that $E[|g(X)|] < \infty$. Then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) dF(x),$$

where the integral on the right is interpreted in the Lebesgue–Stieltjes sense.

Let $p(x) > 0$ be a measurable function on $\mathbb{R}$. We say that X *has the density* p if

$$F(x) = \int_{-\infty}^{x} p(y) dy \quad \text{for all } x.$$

Definition 4: *A random variable $X : \Omega \to \mathbb{R}$ is* normal *if the distribution of X has a density of the form*

$$p_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right),$$

where $\sigma > 0$ and m are constants. In other words,

$$P[X \in G] = \int_G p_X(x) dx, \quad \textit{for all Borel sets } G \subset \mathbb{R}.$$

Then

$$E[X] = \int_\Omega X dP = \int_{\mathbb{R}} x p_X(x) dx = m,$$

$$\operatorname{var}[X] = E[(X-m)^2] = \int_{\mathbb{R}} (x-m)^2 p_X(x) dx = \sigma^2.$$

More generally, a random variable $X : \Omega \to \mathbb{R}^n$ is called (multi)normal $N(m, \mathbf{C})$ *if the distribution of X has a density of the form*

$$p_X(x_1, \ldots, x_n) = \frac{\sqrt{|\mathbf{A}|}}{(2\pi)^{n/2}} \cdot \exp\left[-\frac{1}{2} \sum_{j,k=1}^{n} (x_j - m_j) a_{jk} (x_k - m_k)\right],$$

where $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{R}^n$ *and* $\mathbf{C}^{-1} = \mathbf{A} = [a_{ij}]$ *is a positive definite matrix. Then*

$$E[X] = \mathbf{m},$$

and

$$\mathbf{A}^{-1} = \mathbf{C} = [c_{ij}]$$

is the covariance *matrix of* X, *i.e.,*

$$c_{ij} = E\big[(X_i - m_i)(X_j - m_j)\big].$$

Definition 5: *Two subsets* $A, B \in \mathcal{F}$ *are called* independent *if*

$$P(A \cap B) = P(A) \cdot P(B).$$

A collection $\mathcal{A} = \{\mathcal{H}_i; i \in I\}$ *of families* $\mathcal{H}_i$ *of measurable sets is* independent *if*

$$P(H_{i_1} \cap \ldots \cap H_{i_k}) = P(H_{i_1}) \ldots P(H_{i_k})$$

for all choices of $H_{i_1} \in \mathcal{H}_{i_1}, \dots, H_{i_k} \in \mathcal{H}_{i_k}$. *The* σ*-algebra* $\mathcal{H}_X$ *induced by a random variable* X *is*

$$\mathcal{H}_X = \{X^{-1}(B); B \in \mathcal{B}\},$$

where $\mathcal{B}$ *is the Borel* σ*-algebra on* $\mathbb{R}^n$. *A collection of random variables* $\{X_i; i \in I\}$ *is* independent *if the collection of induced* σ*-algebras* $\mathcal{H}_{X_i}$ *is independent.*

Definition 6: *A* stochastic process *is a parametrized collection of random variables*

$$\{X_t\}_{t \in T}$$

defined on a probability space $\{\Omega, \mathcal{F}, P\}$ *and assuming values in* $\mathbb{R}^n$.

The parameter space T is usually the halfline $[0, \infty)$, but it may also be an interval $[a, b]$, the nonnegative integers and even subsets of $\mathbb{R}^n$ for $n \geq 1$. Note that for each $t \in T$ fixed we have a random variable

$$\omega \to X_t(\omega), \quad \omega \in \Omega.$$

On the other hand, fixing $\omega \in \Omega$ we can consider the function

$$t \to X_t(\omega), \quad t \in T$$

which is called a *path* of X_t.

Sometimes it is convenient to write $X(t, \omega)$ instead of $X_t(\omega)$. Thus we may also regard the process as a function of two variables

$$(t, \omega) \to X(t, \omega)$$

from $T \times \Omega$ into $\mathbb{R}^n$. Such a point of view is important, since in stochastic analysis it is crucial to have $X(t, \omega)$ jointly measurable in (t, ω).

Remark 29. Random variables may also assume values in spaces more general than $\mathbb{R}^n$, say in Hilbert or Banach spaces, cf. [195].

The (finite-dimensional) *distributions* of the process $X = \{X_t\}_{t \in T}$ are the measures $\mu_{t_1, \dots, t_k}$ defined on $\mathbb{R}^{nk}$, $k = 1, 2, \dots$, by

$$\mu_{t_1, \dots, t_k}(F_1 \times F_2 \times \ldots \times F_k) = P[X_{t_1} \in F_1, \dots, X_{t_k} \in F_k], \quad t_i \in T.$$

Here $F_1, \dots, F_k$ denote Borel subsets of $\mathbb{R}^n$.

Brownian motion or Wiener stochastic process

In 1828 the Scottish botanist R. Brown observed that pollen grains suspended in a liquid performed an irregular motion. To describe the motion mathematically it is natural to use the concept of a stochastic process $B_t(\omega)$, often denoted by $w_t(\omega)$, interpreted as the position at time t of the pollen grain ω. Consider an n-dimensional counterpart.

Fix $\mathbf{x} \in \mathbb{R}^n$ and define

$$p(t, \mathbf{x}, \mathbf{y}) = (2\pi t)^{-n/2} \cdot \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{2t}\right) \quad \text{for} \quad y \in \mathbb{R}^n, \quad t > 0.$$

We observe that

$$\int_{\mathbb{R}^n} p(t, \mathbf{x}, \mathbf{y}) d\mathbf{y} = 1, \quad \text{for all } \ t \geq 0.$$

It can be shown that there exists a probability space $(\Omega, \mathcal{F}, P^{\mathbf{x}})$ and a stochastic process $\{B_t\}_{t\geq 0}$ on Ω such that the finite-dimensional distributions of B_t are given by

$$P^{\mathbf{x}}(B_{t_1} \in F_1, \ldots, B_{t_k} \in F_k) = \int_{F_1 \times \ldots \times F_k} p(t_1, \mathbf{x}, \mathbf{x}_1) \ldots p(t_k - t_{k-1}, \mathbf{x}_{k-1}, \mathbf{x}_k) d\mathbf{x}_1 \ldots d\mathbf{x}_k.$$

Such a process is called a Brownian motion starting at $\mathbf{x}$. We observe that $P^{\mathbf{x}}(B_0 = \mathbf{x}) = 1$.

Basic properties of Brownian motion

(i) B_t is a Gaussian process, i.e. for all $0 \leq t_1 \leq \ldots \leq t_k$ the random variable $= (B_{t_1}, \ldots, B_{t_k}) \in \mathbb{R}^{nk}$ has a (multi)normal distribution.

(ii) A stochastic process X_t is called stationary if $\{X_t\}$ has the same distribution as $\{X_{t+h}\}$ for any $h > 0$. Brownian motion $\{B_t\}$ has stationary increments, i.e. the process $\{B_{t+h} - B_t\}_{h\geq 0}$ has the same distribution for all t.

(iii) $\{B_t\}$ has independent increments, i.e. $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}}$ are independent for all $0 \leq t_1 < t_2 < \ldots < t_k$.

Indeed, it suffices to prove that

$$E^{\mathbf{x}}\left[(B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})\right] = 0 \quad \text{when} \quad t_i < t_j.$$

We recall that random variables are independent if they are uncorrelated.

The continuity of Brownian motion

Question: Is $t \to B_t(\omega)$ continuous for almost all ω? Stated like this the question does not make sense, because the set $H = \{\omega \mid t \to B_t(\omega) \text{ is continuous}\}$, is not measurable with respect to the Borel σ-algebra $\mathcal{B}$ on $(\mathbb{R}^n)^{[0,\infty]}$ (H involves an uncountable number of t's). Fortunately, if modified slightly the question can be given a positive answer.

Definition 7: *Suppose that $\{X_t\}$ and $\{Y_t\}$ are stochastic processes on $(\Omega, \mathcal{F}, P)$. Then we say that $\{X_t\}$* is a version *of $\{Y_t\}$ if*

$$P(\{\omega \mid X_t(\omega) = Y_t(\omega)\}) = 1 \ \textit{ for all } \ t.$$

Note that if X_t is a version of Y_t, then X_t and Y_t have the same finite-dimensional distributions. Two such processes are undistinguishable; however, their path properties may be different.

Theorem 28 (Kolmogorov's continuity theorem): *Suppose that the process $X = \{X_t\}_{t>0}$ satisfies the following condition: for all $T > 0$ there exist positive constants α, β, D such that*

$$E\left[|X_t - X_s|^\alpha\right] \le D|t-s|^{1+\beta}; \quad 0 \le s,\ t \le T.$$

Then there exists a continuous version of X.

For Brownian motion B_t it is not difficult to prove that

$$E^{\mathrm{x}}\left[|B_t - B_s|^4\right] = 3n^2|t-s|^2.$$

Thus now $\alpha = 4$, $\beta = 1$ and $D = 3n^2$. Consequently, from now on we assume that B_t is a continuous version.

Condtional expectations

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}^n$ be a random variable such that $E[|X|] < \infty$. If $\mathcal{H} \subset \mathcal{F}$ is a σ-algebra, then the *conditional expectation* of X given $\mathcal{H}$, denoted by $E[X|\mathcal{H}]$, is defined as follows.

Definition 8: *$E[X|\mathcal{H}]$ is the (almost surely unique) function from Ω to $\mathbb{R}^n$ satisfying:*

(i) $E[X|\mathcal{H}]$ is $\mathcal{H}$-measurable,

(ii) $\int\limits_H E[X|\mathcal{H}]dP = \int\limits_H XdP$, for all $H \in \mathcal{H}$.

We observe that the existence and uniqueness of $E[X|\mathcal{H}]$ comes from the Radon–Nikodym theorem: Let μ be the measure on $\mathcal{H}$ defined by

$$\mu(H) = \int\limits_H XdP, \quad H \in \mathcal{H}.$$

Then μ is absolutely continuous with respect to $P|\mathcal{H}$, so there exists a $P|\mathcal{H}$-unique, $\mathcal{H}$-measurable function F on Ω such that

$$\mu(H) = \int\limits_H FdP \quad \text{for all} \quad H \in \mathcal{H}.$$

Thus $E[X|\mathcal{H}] := F$ and this function is unique a.s. (almost surely) w.r.to (with respect to) the measure $P|\mathcal{H}$.

Let us now pass to basic properties of the conditional expectation.

Proposition 4. Suppose $Y : \Omega \to \mathbb{R}^n$ is another random variable with $E[|Y|] < \infty$ and let $a, b \in \mathbb{R}$. Then

(a) $E[aX + bY|\mathcal{H}] = aE[X|\mathcal{H}] + bE[Y|H]$,

(b) $E[E[X|\mathcal{H}]] = E[X]$,

(c) $E[X|\mathcal{H}] = X$, if X is $\mathcal{H}$-measurable,
(d) $E[X|\mathcal{H}] = E[X]$, if X is independent of $\mathcal{H}$,
(e) $E[Y \cdot X|\mathcal{H}] = Y \cdot E[X|\mathcal{H}]$, if Y is $\mathcal{H}$-measurable,

where the dot denotes the inner product in $\mathbb{R}^n$.

Proof. We shall only prove (d) and (e).
(d) If X is independent of $\mathcal{H}$ we have for $H \in \mathcal{H}$:

$$\int_H X dP = \int_\Omega \chi_H X dP = \int_\Omega X dP \int_\Omega \chi_H dP = E[X]P(H),$$

so the constant $E[X]$ satisfies (i) and (ii). Here χ_H denotes the characteristic function of H.
(e) We first establish the result in the case when $Y = \chi_H$, for some $H \in \mathcal{H}$. Then for all $G \in \mathcal{H}$

$$\int_G Y \cdot E[X|\mathcal{H}]dP = \int_{G\cap H} E[X|\mathcal{H}]dP = \int_{G\cap H} X dP = \int_G YXdP,$$

so $Y \cdot [X|\mathcal{H}]$ satisfies both (i) and (ii). Similarly we obtain that the result is true if Y is a simple function

$$Y = \sum_{j=1}^{m} c_j \chi_{H_j}, \quad \text{where} \quad H_j \in \mathcal{H}.$$

The result in the general case then follows by approximating Y by such simple functions.

Proposition 5. Let $\mathcal{G}$, $\mathcal{H}$ be σ-algebras such that $\mathcal{G} \subset \mathcal{H}$. Then

$$E\big[X|\mathcal{G}\big] = E\big[E[X|\mathcal{H}]|\mathcal{G}\big].$$

Proof. If $G \in \mathcal{G}$ then $G \in \mathcal{H}$ and therefore

$$\int_G E\big[X|\mathcal{H}\big]dP = \int_G XdP.$$

Hence $E[E[X|\mathcal{H}]|\mathcal{G}] = E[X|\mathcal{G}]$ by uniqueness. □

Martingales

Definition 9: *Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. A family $\{f_j\}_{j\in J}$ of real, measurable functions f_j on Ω is called* uniformly integrable *if*

$$\lim_{n\to\infty} \left(\sup_{j\in J} \left\{ \int_{\{|f_j|>n\}} |f_j| dP \right\} \right) = 0.$$

One of the most useful tests for uniform integrability is obtained by using the following concept.

Definition 10: *A function* $\Psi : [0,\infty) \to [0,\infty)$ *is called a u.i. (uniformly integrable)* test function *if* Ψ *is increasing, convex and*

$$\lim_{x\to\infty} \frac{\Psi(x)}{x} = +\infty.$$

For instance, the function $\Psi(x) = x^p$ is a u.i. test function if $p > 1$, but not if $p = 1$. The justification for the name in the last definition is the following result.

Proposition 6. The family $\{f_j\}_{j\in J}$ is uniformly integrable if and only if there is a u.i. test function Ψ such that

$$\sup_{j\in J}\left\{ \int \Psi(|f_j|)dP \right\} < +\infty.$$

□

One major reason for the usefulness of uniform integrability is the following result, which may be regarded as the ultimate generalization of the various convergence theorems in integration theory.

Theorem 29: *Suppose* $\{f_k\}_{k\in\mathbb{N}}$ *is a sequence of real measurable functions on* Ω *such that*

$$\lim_{k\to\infty} f_k(\omega) = f(\omega) \quad \text{for a.a. } \omega\in\Omega.$$

Then the following conditions are equivalent:

(1) $\{f_k\}$ *is uniformly integrable,*

(2) $f\in L^1(\Omega)$ *and* $f_k \to f$ *in* $L^1(\Omega)$, *i.e.,*

$$\int\limits_{\Omega} |f_k - f| dP \to 0 \quad \text{as} \quad k\to\infty.$$

An important application of uniform integrability is within the convergence theorems for *martingales.*

Definition 11: *Let* $(\Omega, \mathcal{N}, P)$ *be a probability space and let* $\{\mathcal{N}_t\}_{t\geq 0}$ *be an increasing family of* σ*-algebras,* $\mathcal{N}_t \subset \mathcal{N}$ *for all* t. *A stochastic process* $N_t : \Omega \to \mathbb{R}$ *is called a* supermartingale *(w.r.to* $\{\mathcal{N}_t\}$*) if* N_t *is* $\mathcal{N}_t$*-adapted,* $E[|N_t|] < \infty$ *for all* t *and*

$$N_t \geq E\big[N_s|\mathcal{N}_t\big] \quad \text{for all} \quad s > t. \tag{315}$$

Similarly, if (315) holds with the inequality reversed for all $s > t$, then N_t is called a *submartingale.* And if (315) holds with equality then N_t is called a *martingale.*

We will assume that each $\mathcal{N}_t$ contains all the null sets of $\mathcal{N}$, that $t \to N_t(\omega)$ is right continuous for a.a. ω and that $\{\mathcal{N}_t\}$ is right continuous, in the sense that $\mathcal{N}_t = \bigcap_{s>t} \mathcal{N}_s$ for all $t \geq 0$.

Definition 12: *Let* $\{\mathcal{N}_t\}_{t\geq 0}$ *be an increasing family of* σ*-algebras of subsets of* Ω. *A process* $g(t,\omega) : [0,\infty)\times\Omega \to \mathbb{R}^n$ *is called* $\mathcal{N}_t$*-adapted if for each* $t \geq 0$ *the function*

$$\omega \to g(t,\omega)$$

is $\mathcal{N}_t$*-measurable.*

Thus the process $h_1(t,\omega) = B_{t/2}(\omega)$ is $\mathcal{F}_t$-adapted, while $h_2(t,\omega) = B_{2t}(\omega)$ is not.

Definition 13: *Let $B_t(\omega)$ be n-dimensional Brownian motion. Then we define $\mathcal{F}_t = \mathcal{F}_t^{(n)}$ to be the σ-algebra generated by the random variables $B_s(\cdot)$, $s \leq t$. In other words, $\mathcal{F}_t$ is the smallest σ-algebra containing all sets of the form*

$$\{\omega | B_{t_1} \in F_1, \ldots, B_{t_k} \in F_k\},$$

where $k = 1, 2, \ldots$, $t_j \leq t$ and $F_j \subset \mathbb{R}^n$ are Borel sets (we assume that all sets of measure zero are included in $\mathcal{F}_t$).

One may think of $\mathcal{F}_t$ as "the history of B_s up to time t". A function $h(\omega)$ will be $\mathcal{F}_t$-measurable if and only if h can be written as the pointwise a.e. (almost everywhere) limit of sums of functions of the form

$$g_1(B_{t_1})g_2(B_{t_2}) \ldots g_k(B_{t_k}),$$

where $g_1, \ldots, g_k$ are Borel functions and $t_j < t$. Intuitively, that h is F_t-measurable means that the value of $h(\omega)$ can be decided from the values of $B_s(\omega)$ for $s \leq t$. For instance, $h_1(\omega) = B_{1/2}(\omega)$ is $\mathcal{F}_t$-measurable while $h_2(\omega) = B_{2t}(\omega)$ is not.

Ito integrals

Now we are in a position to introduce the Ito integral. More precisely, we want to define, for $0 \leq S < T$ and $f(t,\omega)$ given, the integral

$$\int_S^T f(t,\omega) dB_t(\omega).$$

Here $B_t(\omega)$ is 1-dimensional Brownian motion starting from the origin and f belongs to a wide class of functions, $f : [0,\infty] \times \Omega \to \mathbb{R}$. It is natural to approximate the function $f(t,\omega)$ by

$$\sum_j f(t_j^*, \omega) \chi_{(t_j, t_{j+1}]}(t)$$

where the points t_j^* belong to the intervals $[t_j, t_{j+1}]$ and then define $\int_S^T f(t,\omega) dB_t(\omega)$ as the limit (in a sense that will be explained) of $\sum_j f(t_j^*, \omega)[B_{t_{j+1}} - B_{t_j}](\omega)$ as $n \to \infty$. However, in contrast with the Riemann–Stieltjes integral it does make a difference here what points t_j^* we choose. The following two choices have turned out to be the most useful ones:

(1) $t_j^* = t_j$ (the left end point), which leads to the *Ito integrals*, denoted by

$$\int_S^T f(t,\omega) dB_t(\omega),$$

and

(2) $t_j^* = (t_j + t_{j+1})/2$ (the mid-point), which leads to the *Stratonovich integral*, denoted by

$$\int_S^T f(t,\omega) \circ dB_t(\omega).$$

We shall be interested only in the Ito integral. For a discussion of relations and distinctions between these two integrals the reader is referred to the book by Øksendal [163].

Definition 14: *Let $N = N(S,T)$ be the class of functions $f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$ such that*

(i) $(t,\omega) \to f(t,\omega)$ *is $\mathcal{B}\times\mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on* $[0,\infty)$.

(ii) $f(t,\omega)$ *is $\mathcal{F}_t$-adapted.*

(iii) $E\Big[\int_S^T f^2(t,\omega)dt\Big] < \infty$.

For functions $f \in N$ we will now show how to define the Ito integral

$$\mathcal{J}[f](\omega) = \int_S^T f(t,\omega)dB_t(\omega),$$

where B_t is 1-dimensional Brownian motion.

The idea is natural: First we define $\mathcal{J}[\phi]$ for a simple class of functions ϕ. Then we show that each $f \in N$ can be approximated (in an appropriate sense) by such ϕ's and we use to define $\int f dB$ as the limit $\int \phi dB$ as $\phi \to f$.

Let us pass to a concise presentation of details. A function $\phi \in N$ is called *elementary* if it has the form

$$\phi(t,\omega) = \sum_j a_j(\omega)\chi_{(t_j,t_{j+1}]}(t).$$

Note that since $\phi \in N$ each function a_j must be $\mathcal{F}_{t_j}$-measurable.

For elementary functions $\phi(t,\omega)$ we define the Ito integral

$$\int_S^T \phi(t,\omega)dB_t(\omega) = \sum_{j\geq 0} a_j(\omega)[B_{t_{j+1}} - B_{t_j}](\omega).$$

Lemma 14 (The Ito isometry): *If $phi(t,\omega)$ is bounded and elementary then*

$$E\Bigg[\bigg(\int_S^T \phi(t,\omega)dB_t(\omega)\bigg)^2\Bigg] = E\bigg[\int_s^T \phi^2(t,\omega)dt\bigg]. \tag{316}$$

The idea is now to use the isometry (316) to extend the definition from elementary functions to functions in N. This is done in several steps:

Step 1. Let $g \in N$ be bounded and $g(\cdot,\omega)$ continuous for each ω. Then there exist elementary functions $\phi_n \in N$, $n = 1,2,\ldots$, such that

$$E\bigg[\int_S^T (g-\phi)^2 dt\bigg] \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2. Let $h \in N$ be bounded. Then there exist bounded functions $g_n \in N$ such that $g_n(\cdot,\omega)$ is continuous for all ω and n, and

$$E\Big[\int_S^T (h-g_n)^2 dt\Big] \to 0.$$

Step 3. Let $f \in N$. Then there exists a sequence $\{h_n\} \subset N$ such that h_n is bounded for each n and

$$E\Big[\int_S^T (f-h_n)^2 dt\Big] \to 0 \quad \text{as} \quad n \to \infty.$$

Now we are in a position to complete the definition of the Ito integral

$$\int_S^T f(t,\omega) dB_t(\omega) \quad \text{for} \quad f \in N.$$

If $f \in N$ we choose, by Steps 1-3, elementary functions $\phi_n \in N$ such that

$$E\Big[\int_S^T (f-\phi_n)^2 dt\Big] \to 0 \quad \text{as} \quad n \to \infty.$$

Then define

$$\mathcal{J}[\varphi](\omega) := \int_S^T f(t,\omega) dB_t(\omega) := \lim_{n\to\infty} \int_S^T \phi_n(t,\omega) dB_t(\omega).$$

The limit exists as an element of $L^2(\Omega,P)$, since $\{\int_S^T \phi_n(t,\omega)dB_t(\omega)\}$ forms a Cauchy sequence in $L^2(\Omega,P)$, by the Ito isometry. Also by the same isometry the limit is independent of the sequence $\{\phi_n\}$ and

$$E\Big[\Big(\int_S^T f(t,\omega)dB_t\Big)^2\Big] = E\Big[\int_S^T f^2(t,\omega)dt\Big].$$

Example 4.

$$\int_0^T B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t.$$

For the proof the reader is referred to Øksendal [163].

Basic properties of the Ito integral

Theorem 30: *Let $f, g \in N(0,T)$ and let $0 \le S < U < T$. Then*

(i) $\int\limits_S^T f dB_t = \int\limits_S^U f dB_t + \int\limits_U^T f dB_t$ *for a.a.* $\omega \in \Omega$,

(ii) $\int\limits_S^T (cf+g) dB_t = c\int\limits_S^T f dB_t + \int\limits_S^T g dB_t$ *(c – constant) for a.a.* ω,

(iii) $E\Big[\int\limits_S^T (f dB_t\Big] = 0.$

Proof. This clearly holds for all elementary functions, so by taking limits we obtain this for all $f, g \in N(0,T)$. □

We recall that a stochastic process $\{M_t\}_{t\ge 0}$ on $(\Omega, \mathcal{F}, P)$ is called a *martingale* with respect to a nondecreasing sequence $\{\mathcal{M}_t\}_{t\ge 0}$ of σ-algebras if

(i) M_t is $\mathcal{M}_t$-measurable, for all t,

(ii) $E[|M_t|] < \infty$, for all t,

(iii) $E[M_s|\mathcal{M}_t] = M_t$, for all $s \ge t$.

Example 5. Brownian motion B_t is a martingale w.r.to the σ-algebras $\mathcal{F}_t$ generated by $\{B_s; s \le t\}$, because

$$E\big[|B_t|\big]^2 \le E\big[|B_t|^2\big] = |B_0|^2 + t,$$

and

$$E\big[B_s|\mathcal{F}\big] = E\big[B_s - B_t + B_t|\mathcal{F}_t\big] + E\big[B_s - B_t|\mathcal{F}_t\big] + E\big[B_t|\mathcal{F}_t\big] = 0 = B_t.$$

The integral can be chosen to depend continuously on t. Indeed, we have the following result.

Theorem 31: *Let $f \in N(0,T)$. Then there exists a t*-continuous version *of*

$$\int\limits_0^t f(s,\omega) dB_s(\omega), \quad 0 \le t \le T,$$

i.e. there exists a t-continuous stochastic process J_t on $(\Omega, \mathcal{F}, P)$ such that

$$P\Big[J_t = \int\limits_0^t f dB\Big] = 1 \quad \textit{for all} \quad t, \quad 0 \le t \le T.$$

The proof is given in [163].

Remark 30. The Ito integral exists for a class of functions larger than $N(S,T)$, cf. Øksendal [163].

Stochastic integrals

From the example

$$\int\limits_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$$

or

$$\frac{1}{2}B_t^2 = \frac{1}{2}t + \int_0^t B_s dB_s,$$

we see that the image of the integral $B_t = \int_0^t dB_s$ by the map $g(x) = \frac{1}{2}x^2$ is not again an Ito integral of the form

$$\int_0^t f(s,\omega) dB_s(\omega)$$

but a combination of a dB_s- and a ds-integral

$$\frac{1}{2}B_t^2 = \int_0^t \frac{1}{2}ds + \int_0^t B_s dB_s.$$

It turns out that if we introduce *stochastic integrals* a sums of a dB_s- and a ds- integral then this family of integrals is stable under smooth maps. Thus we introduce the following definition.

Definition 15: *Let B_t be 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$. A (1-dimensional)* stochastic integral *is a stochastic process X_t on $(\Omega, \mathcal{F}, P)$ of the form*

$$X_t = X_0 + \int_0^t u(s,\omega) ds + \int_0^t v(s,\omega) dB_s,$$

where v is such that

$$P\Big[\int_0^t v^2(s,\omega) ds < \infty \quad \text{for all} \quad t \geq 0\Big] = 1.$$

We also assume that u is $\mathcal{H}_t$-adapted and

$$P\Big[\int_0^t |u(s,\omega)| ds < \infty \quad \text{for all} \quad t \geq 0\Big] = 1.$$

Here $\mathcal{H}_t$, $t > 0$, is a non-decreasing family of σ-algebras such that

(a) B_t is a martingale with respect to $\mathcal{H}_t$,

(b) v_t is $\mathcal{H}_t$-adapted.

Stochastic differential equations

Consider the stochastic differential equation

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t) W_t, \tag{317}$$

where $b(t,x) \in \mathbb{R}$, $\sigma(t,x) \in \mathbb{R}$, while W_t is 1-dimensional "white noise". As we already know, the Ito interpretation of (317) is that X_t satisfies the stochastic integral equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \tag{318}$$

or in the differential form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t. \tag{319}$$

Therefore, to get from (317) to (319) we formally replace the white noise W_t by dB_t/dt in (317) and multiply by dt.

The question of existence and uniqueness of Eq. (319) has been studied in [163]. This result can be extended to the multidimensional stochastic differential equation of type (319).

9.2. Controllability of stochastic heat equation

Controllability of linear and nonlinear heat equations in the deterministic case was the subject of many papers, cf. [152, 194] and the references therein.

Here we shall consider a stochastic heat equation of the form [53, 169]:

$$\begin{aligned} \dot{\theta} - \Delta\theta = \chi_{\mathcal{O}} u + B(t)\dot{w}_t \quad &\text{in} \quad Q = D \times (0,T), \quad P - \text{a.s.}, \\ \theta = 0 \quad &\text{in} \quad \Sigma = \partial D \times (0,T), \quad P - \text{a.s.}, \\ \theta(0) = \theta^0 \quad &\text{in} \quad D, \quad P - \text{a.s.}, \end{aligned} \tag{320}$$

where u is the control function and $D \subset \mathbb{R}^n$ is a sufficiently regular, bounded open set with $\mathcal{O} \subset D$. We recall that $\chi_{\mathcal{O}}$, denotes characteristic function of the set $\mathcal{O}$ and $\dot{w} = \partial_t w_t$ is a Gaussian field, white noise in time. Obviously, θ is the temperature.

Caution: Since B is now an operator therefore the Brownian motion (Wiener process) is denoted by w_t.

Our aim is to prove that for general θ^0, θ_T and B, one can obtain find states $\theta(T)$ arbitrarily close to θ_T in quadratic mean by choosing u appropriately (an approximate controllability result). For an approximate controllability result in the deterministic case the reader is referred to [66, 194]. It will also be proven that if B is not random and in some sense small, then we can choose u such that $\theta(T) = 0$ (a null–controllability result).

9.2.1. An approximate controllability result. As previously, $(\Omega, \mathcal{F}, P)$ is a complete probability space. We set: $H = L^2(D)$, $V = H_0^1(D)$. Assume that a separable Hilbert space H_1 and a Wiener process w_t on $(\Omega, \mathcal{F}, P)$ with values in H_1 are given. This means that, cf. [195],

$$w_t = \sum_{k=1}^{\infty} \beta_t^k e_k, \quad \forall\, t \geq 0,$$

where the β_t^k are mutually independent real Wiener process satisfying

$$E[\beta_t^k]^2 = \lambda_k t, \qquad \sum_{k=1}^{\infty} \lambda_k < +\infty,$$

and $\{e_k\}_{k\in\mathbb{R}}$ is an orthonormal basis in H_1. We observe that w_t has not only a continuous version but has Hölder-continuous sample paths.

Let $\mathcal{B}$ be a Banach space. We denote by $I^2(0,T;\mathcal{B})$ the space formed by all stochastic processes $X \in L^2(\Omega \times (0,T), dP \bigotimes dt; \mathcal{B})$ which are $\mathcal{F}_t$-adapted a.e. in $(0,T)$, i. e. such that X_t is $\mathcal{F}_t$-measurable for almost all $t \in (0,T)$. Then $I^2(0,T;\mathcal{B})$ is a closed subspace of $L^2(\Omega \times (0,T), dP \bigotimes dt; \mathcal{B})$, cf. [195].

Assume that a stochastic process B is given, with $B \in L^2(0,T; L(H_1,H))$. Then the stochastic integral of B w.r.to w_t is defined by the formula

$$\int_0^t B(s)dw(s)dw_s = \sum_{k=1}^{\infty} \int_0^t B(s)d\beta_s^k, \quad \forall\, t \in [0,T]. \tag{321}$$

Now we write $B(s)$ instead of B_s. The convergence of the series in (321) is understood in the sense of $L^2(\Omega,\mathcal{F}_t;H)$. The stochastic integrals in the right hand side are defined as follows

$$\Big(\int_0^t B(s)e_k d\beta_s^k, h\Big) = \int_0^t \big(B(s)e_k, h\big)d\beta_s^k, \quad \forall\, h \in H,$$

where the latter are the usual Ito integrals w.r.to the real-valued process β_t^k; $(\cdot,\cdot)$ denotes the scalar product in H.

Let $\theta^0 \in H$ and set $A = \Delta$ (the usual Laplace operator). For each $u \in I^2(0,T;H)$ there exists exactly one solution $\theta(t;u)$ to the problem

$$\begin{gathered} \theta \in I^2(0,T;V) \cap L^2\big(\Omega; C^0([0,T];H)\big) \\ \theta(t) = \theta^0 + \int_0^t \big[A\theta(s) + \chi_{\mathcal{O}} u(s)\big]ds + \int_0^t B(s)dw_s \quad \forall\, t \in [0,T],\ P - \text{a.s. in } V'. \end{gathered} \tag{322}$$

Let $S(t)$ be the semigroup generated in H by A, with domain $D(A) = \{v \in V \mid Av \in H\}$. Then

$$\theta(t;v) = S(t)\theta^0 + \int_0^t S(t-s)\chi_{\mathcal{O}} u(s)\,ds + \int_0^t S(t-s)B(s)\,dw_s \quad \forall\, t \in [0,T]. \tag{323}$$

The first result is formulated as follows.

Theorem 32: *The linear manifold $H_T = \{\theta(T;u) \mid u \in I^2(0,T;H)\}$ is dense in the space $L^2(\Omega,\mathcal{F}_T;H)$.*

Proof. On account of (323) it is sufficient to verify that if $f \in L^2(\Omega,\mathcal{F}_T;H)$ and

$$E\Big(\int_0^T S(T-s)\chi_{\mathcal{O}} u(s)ds, f\Big) = 0 \quad \forall\, u \in I^2(0,T;H),$$

then $f = 0$. Let f be a function in $L^2(\Omega, \mathcal{F}_T; H)$ satisfying the last equation and assume that $\psi \in I^2(0,T;H)$ is a solution to

$$-\dot{\psi} - A\psi = 0 \quad \text{on} \quad Q,$$
$$\psi = 0 \quad \text{on} \quad \Sigma,$$
$$\psi(T) = f,$$

i.e. $\psi(t) = S(T-t)f$. Now it suffices to prove that $E[\psi(t)|\mathcal{F}_t] = 0$ for all $t \in (0,T)$. Indeed, this and the fact that

$$\mathcal{F}_t = \sigma\Big(\bigcup_{s<t} \mathcal{F}_s\Big) \quad \forall\, t > 0$$

imply

$$f = E\big[\psi(T)|\mathcal{F}_t\big] = 0.$$

It is known that

$$E\Big[\int_0^T (u(s), \chi_O \psi(s))ds\Big] = 0 \quad \forall\, u \in I^2(0,T;H).$$

Consequently, $\chi_O E[\psi|\mathcal{F}_t]$ is a stochastic process in $I^2(0,T;H)$ such that

$$E\Big[\int_0^T \big(u(s), \chi_O E\big[\psi(s)|\mathcal{F}_s\big]\big)ds\Big] = \int_0^T E\Big(E\big[\big(u(s), \chi_O\psi(s)\big)|\mathcal{F}_s\big]\Big)ds$$
$$= \int_0^T E\big[\big(u(s), \chi_O\psi(s)\big)ds\big] = 0 \quad \forall\, u \in I^2(0,T;H).$$

Thus we get

$$\chi_O E[\psi(t)|\mathcal{F}_t] = 0.$$

For each $t \in (0,T)$, $E[\psi(t)|\mathcal{F}_t] = S(T-t)E[f|\mathcal{F}_t]$ is real analytic in the variable $x \in D$. Hence, we must necessarily have $[\psi(t)|\mathcal{F}_t] = 0$ for all $t \in (0,T)$ and the theorem is proved.

Corollary 2. For all $\theta_T \in L^2(\Omega, \mathcal{F}_T; H)$, $\varepsilon > 0$ and $\delta > 0$, a control $u \in I^2(0,T;H)$ can be found such that

$$P\{\| \theta(T;u) - \theta_T \|_H < \varepsilon\} \geq 1 - \delta.$$

Remark 31. The existence of a control $u \in I^2(0,T;H)$ such that

$$P\{\| \theta(T;u) - \theta_T \|_H < \varepsilon\} = 1$$

remains an open question.

Remark 32. Theorem 32 can easily be extended to a more general case. Let $A \in L(V, V')$ be an operator of the form

$$A\theta = \partial_i(a_{ij}(x)\partial_j\theta) + \partial_i(b_i\theta)\theta) + c\theta,$$

where the coefficients satisfy

$$a_{ij} \in C^1(\overline{D}), \quad b_i, c \in L^\infty(D)$$

and the usual ellipticity condition

$$\exists\, \alpha > 0, \quad a_{ij}(x)\lambda_i\lambda_j \geq \alpha|\lambda|^2 \quad \forall\, \lambda \in \mathbb{R}^n, \quad \forall\, x \in D.$$

Then the corresponding linear manifold

$$H_T = \{\theta(T; u) \mid u \in I^2(0, T; H)\}$$

is dense in the space $L^2(\Omega, \mathcal{F}_T; H)$.

9.2.2. A null–controllability result. We are going to formulate a null–controllability result for (320). Let us fix a positive function $\gamma \in C^\infty(0, T)$ such that $\gamma(t) = t$ near $t = 0$ and $\gamma(t) = T - t$ near $t = T$. Furthermore, we assume that the hypotheses in last Corollary 2 hold, that B is not random and satisfies $B \in C^1([0, T]; L(H_1, H))$ and, also, that the support of $B(t)$ does not intersect $\mathcal{O}$ for all t.

Theorem 33: *There exist a positive function $\kappa = \kappa(x)$ such that if*

$$\int\limits_Q t\left[\gamma^{-1}(t) \parallel B \parallel^2_{L(H_1,H)} + \gamma^3(t) \parallel \partial_t B \parallel^2_{L(H_1,H)}\right] e^{2\kappa(x)/\gamma(x)} dx dt < \infty$$

then, for each $\theta^0 \in H$ there exists $u \in I^2(0, T; H)$ satisfying $\theta(T; u) = 0$.

For the proof the reader is referred to [53].

9.2.3. Stokes and quasi-Stokes systems. Let us introduce the space

$$\mathcal{V} = \{\mathbf{v} \in C_0^\infty(D)^n \mid \operatorname{div} \mathbf{v} = 0 \text{ in } D\}.$$

Now by V (resp. H) we denote the closure of $\mathcal{V}$ in $H_0^1(D^n)$ (resp. $L^2(D)^n$). The operator $A \in L(V, V')$ is given by

$$< A\mathbf{v}, \mathbf{z} >= -\int\limits_D \left[(\nabla \mathbf{v} \cdot \nabla \mathbf{z} + a_i(x) v_i \partial_i z_j + c_{ij}(x) v_i z_j\right] dx$$

for each $\mathbf{v}, \mathbf{z} \in V$. Here

$$a_i, c_{ij} \in L^\infty(D).$$

Viewed as an unbounded operator on H with the domain

$$D(A) = \{\mathbf{v} \in V \mid A\mathbf{v} \in H\},$$

a is the generator of a semigroup on H, again denoted by $S(t)$. Assume that

$$\mathbf{v}^0 \in H, \quad B \in I^2(0,T;L(H_1;H)).$$

For each $\mathbf{u} \in I^2(0,T;L^2(D)^n)$ there exists one and only one solution $\mathbf{v}(t;\mathbf{u})$ to the problem:

$$\mathbf{v} \in I^2(0,T;V) \cap L^2(\Omega, C^0([0,T];H)$$

$$\mathbf{v}(t) = \mathbf{v}^0 + \int_0^t \left[A\mathbf{v}(s) + \chi_{\mathcal{O}}\mathbf{u}(s)\right]ds + \int_0^t B(s)dw_s \quad \forall\, t \in [0,T].$$

In fact, if P_H denotes the orthogonal projector from $L^2(D)^n$ onto H (the Leray operator), then

$$\mathbf{v}(t;\mathbf{u}) = S(t)\mathbf{v}^0 + \int_0^t S(t-s)\left[P_H\left(\chi_{\mathcal{O}}\mathbf{u}(s)\right)\right]ds = \int_0^t S(t-s)B(s)dw_s.$$

Theorem 34: *With the notation introduced in this section, the linear manifold* $H_t = \{\mathbf{v}(T;\mathbf{u}) \mid \mathbf{u} \in I^2(0,T;L^2(D)^n)\}$ *is dense in the space* $L^2(\Omega, \mathcal{F}; H)$.

Remark 33. For the stochastic Stokes problem ($a_i = c_{ij} \equiv 0$) it is possible to prove that, for fixed $1 \le j \le n$, the set

$$\left\{\mathbf{v}(T;\mathbf{u}) \mid \mathbf{u} \in I^2\left(0,T;L^2(D)^n\right),\ u_j = 0\right\}$$

is also dense in $L^2(\Omega, \mathcal{F}_T; H)$.

Remark 34. For the stochastic $3D$ Stokes problem in a cylindrical domain $D = D_1 \times (0,L)$ one also has approximate controllability in a "generic" sense with respect to D_1 with controls in the set, cf. [53],

$$U_{ad} = \left\{\mathbf{u} \in I^2\left(0,T;L^2(D)^3\right) \mid u_1 = u_2 = 0\right\}.$$

9.3. Control of stochastic Burgers equation

9.3.1. Existence and uniqueness results. It is known that Burgers equation is not a good model for turbulence, cf. [29, 164] and the references therein. It does not display any chaos; even when a force is added to the right hand side all solutions converge to a unique stationary solution as time goes to infinity. The situation changes when the force is a random one. The stochastic Burgers equation has been used as a simple model for turbulence. The equation has also been proposed to study the dynamics of interfaces.

Da Prato *et al.* (see [29, 164, 180]) considered Burgers equation with a *random force* which is a *space-time white-noise* (or Brownian sheet, cf.[195])

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{2}\frac{\partial}{\partial x}\left(u^2(x,t)\right) + \frac{\partial^2 \tilde{W}}{\partial t \partial x}. \tag{324}$$

We recall that $\tilde{W}(x,t)$, $t \ge 0$, $x \in \mathbb{R}$ is a zero mean Gaussian process whose covariance function is given by

$$E\left[\tilde{W}(x,t)\tilde{W}(y,t)\right] = (t \wedge s)(x \wedge y); \quad t, s \ge 0,\ x, y \in \mathbb{R}.$$

Here $(a \wedge b) = \min\{a, b\}$.

Alternatively we can consider a *cylindrical Wiener process* W by setting

$$W(t) = \frac{\partial \tilde{W}}{\partial x} = \sum_{k=1}^{\infty} \beta_h e_h, \tag{325}$$

where $\{e_k\}$ is an orthonormal basis of $L^2(0,1)$ and $\{\beta_k\}$ is a sequence of mutually independent real Brownian motions in a fixed probability space $(\Omega, \mathcal{F}, \mathcal{P})$ adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. As we already know, Eq. (324) can be written as follows

$$du(x,t) = \left(\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{2}\frac{\partial}{\partial x} u^2(x,t) \right) dt + dW, \tag{326}$$

where $x \in [0,1]$ and $t \geq 0$. The last equation is supplemented with Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0, \tag{327}$$

and the initial condition

$$u(x,0) = u^0(x), \quad x \in [0,1]. \tag{328}$$

Problem (326)-(328) has a unique global solution [29, 164, 180]. Let us pass to the presentation of essential steps of the proof.

Local existence in time

We introduce the unbounded self-adjoint operator A on $L^2(0,1)$ by

$$Au = \frac{\partial^2 u}{\partial x^2}$$

for u on the domain

$$D(A) = \left\{ u \in H^2(0,1) \mid u(0) = u(1) = 0 \right\}.$$

We denote by e^{tA}, $t \geq 0$, the semigroup on $L^2(0,1)$ generated by A. It is known that e^{tA}, $t \geq 0$ has a natural extension that we still denote by e^{tA}, $t \geq 0$, as a contraction semigroup on $L^p(0,1)$ for any $p \geq 1$. Moreover, $\{e_k\}$ denotes the orthonormal system on $L^2(0,1)$ which diagonalizes A and $\{\lambda_k\}$ stands for the corresponding eigenvalues. We have

$$e_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad k = 1, 2, \ldots$$

and

$$\lambda_k = -\pi^2 k^2, \quad k = 1, 2, \ldots$$

Now we rewrite (326)-(328) in the form of the abstract differential stochastic equation

$$du = \left(Au + \frac{1}{2}\frac{\partial}{\partial x}\left(u^2\right) \right) dt + dW, \quad u(0) = u^0. \tag{329}$$

The solution to the linear problem

$$dz = Azdt + dW, \quad z(0) = u^0,$$

is unique and given by the so-called stochastic convolution [195]:

$$W_A(t) = \int_0^t e^{(t-s)A} dW(s).$$

It can be shown that W_A is a Gaussian process and it is mean-square continuous with values in $L^2(0,1)$. Moreover, W_A has a version which is a.s. for $\omega \in \Omega$, α–Hölder continuous with respect to (x,t) for any $\alpha \in [0,1/4)$.

We now set

$$v(t) = u(t) - W_A(t), \quad t \geq 0.$$

Then u satisfies (329) if and only if v is a solution to

$$\frac{dv}{dt} = Av + \frac{1}{2}\frac{\partial}{\partial x}(v + W_A)^2, \quad v(0) = u^0. \tag{330}$$

We may write (330) as follows

$$v(t) = e^{tA}u^0 + \frac{1}{2}\int_0^t e^{(t-s)A}\frac{\partial}{\partial x}(v + W_A)^2 ds. \tag{331}$$

Then if v satisfies (331) we say that it is a *mild solution* of (330).

Equation (331) is solved by a fixed point argument in the space $C\left([0,T^*]; L^p(0,1)\right)$ for $p > 1$ and for some $T^* > 0$. We set

$$V_p(q,T^*) = \left\{v \in C\left([0,T^*]; L^p(0,1)\right) \mid ||v(t)||_{L^p(0,1)} \leq q, \ \forall\, t \in [0,T^*]\right\},$$

and consider an initial datum $u^0 \in \mathcal{F}_0$–measurable and belonging to $L^p(0,1)$, $\omega \in \Omega$ a.s.

Proposition 7. For any $p \geq 2$ and $q > ||u^0||_{L^p(0,1)}$, there exists a stopping time $T^* > 0$ such that (331) has a unique solution in $V_p(q,T^*)$.

Global existence

Let

$$a_p = \sup_{t\in[0,T]} ||W_A t||_{L^p(0,1)}$$

Proposition 8. If $v \in C([0,T]; L^p(0,1))$ satisfies (331) then

$$||v(t)||_{L^p(0,1)} \leq C\left(a_\infty^2 + ||u^0||_{L^p(0,1)}\right)e^{(2pa_\infty+1)t}.$$

It is now easy to deduce from the last two propositions the global existence result.

Theorem 35: *Let u^0 be given which is $\mathcal{F}_0$–measurable and such that for some $p \geq 2$, $u^0 \in L^p(0,1)$ a.s. Then there exists a unique mild solution of Eq. (329), which belongs a.s. to $C([0,T]; L^p(0,1))$.*

9.3.2. Some model problems in flow control. This subsection is linked with Sec. 4.2.2. We will develop feedback procedures applicable to flow control. These procedures will be applied to the Burgers equation subject to *random forcing*. We follow the paper by Choi *et al.* [29].

Stationary channel flow

Consider a stationary channel flow, where $x = x_1$ is a streamwise direction and the walls are at $y = x_2 = \pm 1$. Let $\mathbf{v} = (v_1, v_2, v_3)$ denote the velocity vector of the fluid , and assume that the flow is controlled through blowing and suction at the wall, through the wall–normal velocity at the wall

$$g = v_{2|w}, \quad \text{w=wall}, \tag{332}$$

where

$$\int g\,dx\,dz = 0$$

is imposed so that the mass flux M is constant. As we already know (see Sec. 4) the stationary Navier–Stokes equations reduce to

$$\nu A\mathbf{v} + R(\mathbf{v}, g) = 0. \tag{333}$$

Here $\nu > 0$ is the kinematic viscosity, A is the so–called Stokes operator and R corresponds to the inertial and boundary terms and is a function of $\mathbf{v}$ and g.

A typical control problem for Eq. (333) is the following: find the best g such that some observation $\gamma = C\mathbf{v}$ achieves some desired value γ_d or is at least as close as possible to γ_d where C is a general linear or nonlinear operator, which may involve integrals of $\mathbf{u}$ and/or derivatives of $\mathbf{u}$.

The *cost function* could be, for instance

$$J(g) = \frac{1}{2} l ||g||^2 + \frac{1}{2} m ||C\mathbf{v} - \gamma_d||^2. \tag{334}$$

Here $m > 0$, $l \geq 0$ and $||\cdot|| = ||\cdot||_{L^2}$. The mathematical formulation of the problem means evaluating

$$\inf_g \{J(g) \mid \mathbf{v} \text{ solves } (333)\}$$

The control g can be unrestricted or restricted to some admissible set of controls $\mathcal{U}_{ad}$ due to the physical and technological limitations, cf. Sec. 4. We recall that the velocity $\mathbf{v}$ in (334) depends on the control g.

We recall that *feedback theory* involves constructing g as a function of $\mathbf{v}$ or some observation of $\mathbf{v}$. Although feedback schemes are mainly relevant to time–dependent problems, we can formulate such a scheme also for stationary problems.

Time–dependent channel flow

Now the *state equation* is the nonstationary Navier–Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + \nu A\mathbf{v} + R(\mathbf{v}, g) = 0. \tag{335}$$

The drag is essentially measured on average by $D = D(\mathbf{v})$:

$$D = \int \left[\left.\frac{\partial v_1}{\partial x_2}\right|_{x_2=-1} - \left.\frac{\partial v_1}{\partial x_2}\right|_{x_2=1} \right] dx_1 x_3. \tag{336}$$

If we choose to reduce a time average of the drag as expressed by (336), then a plausible cost functional would be

$$J(g) = \frac{l}{2}\frac{1}{T}\int_0^T \int_w |g|^2 dx_1 dx_3 dt + \frac{m}{2}\frac{1}{T}\int_0^T |D|^2 dt, \tag{336a}$$

where D is a function of g through $\mathbf{v}$ which itself is a function of g. The reader can now easily formulate the minimization of J subject to (335).

Suboptimal control and feedback procedures

As we already know an optimal control problem can be solved by a gradient algorithm.

1. Stationary problem

The gradient algorithm consists of computing the Gâteaux derivative $J'(g) = GJ(g)$ and using the following iterative process for the cost minimization:

$$g^{k+1} - g^k = -\rho J'(g^k), \tag{337}$$

where g^k is a member of a sequence of controls and ρ is the parameter of descent. It can be shown that

$$J'(g^k) = lg^k - m\left[G_g R(\mathbf{v}^k, g^k)\right]^* \zeta^k, \tag{337a}$$

where ζ is the adjoint state defined by the adjoint state equation:

$$\nu A^*\zeta + \left[G_{\mathbf{v}} R(\mathbf{v}, g)\right]^* \zeta = C^*(C\mathbf{v}(g) - \gamma_d). \tag{338}$$

Here $G_g R$ denotes the partial Gâteaux derivative of R with respect to g. Once g^k is known, compute the adjoint state ζ^k by solving (338) with $g = g^k$ and $\mathbf{v} = \mathbf{v}^k$. Obtain g^{k+1} from (337) using (337a). Then compute $\mathbf{v}^{k+1}$ by solving the state equation (333) with $\mathbf{v} = \mathbf{v}^{k+1}$, and continue until convergence.

Let us pass to *suboptimal feedback laws*. Such laws can be implemented by looking for the best feedback

$$g = g(\mathbf{v}), \tag{339}$$

in a particular class of functions corresponding to a suitable approximation of (339):

$$g = \alpha_0 + \alpha_1 g_1(\mathbf{v}), \tag{340}$$

where $g_1(\mathbf{v})$ is prescribed from physical intuition or experience, and α_0 and α_1 are determined through a control algorithm and thus have an implicit dependence on $\mathbf{v}$, i.e. $\alpha_0 = \alpha_0(\mathbf{v})$ and $\alpha_1 = \alpha_1(\mathbf{v})$. Set $\mathbf{e} = \{\alpha_0, \alpha_1\}$. The cost function is chosen to be a functional $\tilde{J}$ of α_0 and α_1

$$\tilde{J}(\mathbf{e}) = \frac{1}{2}l||\mathbf{e}||^2 + \frac{1}{2}m||C\mathbf{v} - \gamma_d||^2, \tag{341}$$

where $||\mathbf{e}||^2 = ||\alpha_0||^2 + ||\alpha_1||^2$. Now the gradient algorithm consists of constructing two sequences α_0^k, α_1^k recursively defined by

$$\alpha_0^{k+1} - \alpha_0^k = -\rho G_{\alpha_0}\tilde{J}(\alpha_0^k, \alpha_1^k), \qquad \alpha_1^{k+1} - \alpha_1^k = -\rho_1 G_{\alpha_1}\tilde{J}(\alpha_0^k, \alpha_1^k). \tag{342}$$

It can be shown that

$$\begin{aligned} G_{\alpha_0}\tilde{J}(\alpha_0^k, \alpha_1^k) &= l\alpha_0^k - m\left[G_g R(\mathbf{v}^k, g^k)\right]^* \zeta^k, \\ G_{\alpha_1}\tilde{J}(\alpha_0^k, \alpha_1^k) &= l\alpha_1^k - m g_1(\mathbf{v}^k)\left[G_g R(\mathbf{v}^k, g^k)\right]^* \zeta^k. \end{aligned} \tag{343}$$

Now the adjoint state ζ is defined through the following adjoint state equation

$$\left[\nu A + G_{\mathbf{v}}R + (G_g R)\alpha_1 G_{\mathbf{v}} g_1\right]^* \zeta = C^*(C\mathbf{v} - \gamma_d). \tag{344}$$

Once α_0^k and α_1^k are known, compute the adjoint state ζ^k by solving (338) with $g = g^k$ and $\mathbf{v} = \mathbf{v}^k$. Obtain α_0^{k+1} and α_1^{k+1} from (342) by using (343). Then compute $\mathbf{v}^{k+1}$ by solving the state equation (333) with g^{k+1} given by (340) and continue until convergence.

2. Time-dependent problem

The suboptimal procedure proposed in [29] consists of the following:

(i) discretize the state equation in time,

(ii) at each instant of time, the discretized equation is a stationary one to which the above procedure is applied, while the cost functional is an instantaneous version of (336a) (i.e. no time averaging, see (348) below).

This procedure means that, at each instant of time, we are directing the flow in a direction that produces the decay of instantaneous cost functional. Obviously, there is no reason to believe that the controls will be optimal, or even that the cost will actually decay in the long range. However, the numerical experiments conducted in the case of the stochastic Burgers equation and other model problems are promising.

Consider the evolution state equation (335); this could be the original Navier–Stokes equation for channel flow. For step (i) we consider here the Crank-Nicholson method:

$$\frac{\mathbf{v}^n - \mathbf{v}^{n-1}}{\Delta t} + \frac{1}{2}\nu\left(A\mathbf{v}^n + A\mathbf{v}^{n-1}\right) + \frac{1}{2}\left[R(\mathbf{v}^n, g^n) + R\left(\mathbf{v}^{n-1}, g^{n-1}\right)\right] = 0, \tag{345}$$

which we rewrite as

$$\mathcal{A}\mathbf{v} + \mathcal{R}^n(\mathbf{v}, g) = 0, \tag{346}$$

with $\mathbf{v} = \mathbf{v}^n$, $g = g^n$, and

$$\begin{aligned} \mathcal{A}\mathbf{v} &= \mathbf{v}^n + \frac{1}{2}\nu\Delta t A\mathbf{v}^n, \\ \mathcal{R}^n\left(\mathbf{v}^n, g^n\right) &= -\mathbf{v}^{n-1} + \frac{1}{2}\Delta t\left[\nu A\mathbf{v}^{n-1} + R(\mathbf{v}^n, g^n) + R\left(\mathbf{v}^{n-1}, g^{n-1}\right)\right]. \end{aligned} \tag{347}$$

At each step n, the functional is still given by (341)

$$J^n = J(g^n) = \frac{1}{2}l||g^n||^2 + \frac{1}{2}m||C\mathbf{v}^n - \gamma_d||^2, \tag{348}$$

with $\mathbf{v}^n$ a function of g^n through (346). For a sufficiently small Δt there exists a unique solution $\mathbf{v}^n$ to (346). Consequently, the difficulty of nonuniqueness of solution for (335) does not arise for (346).

The gradient algorithm (337) now reads:

$$g^{n,k+1} - g^{n,k} = -\rho J'(g^{n,k}), \tag{349}$$

where $g^{n,k}$ is a member of a sequence of controls at a given time step n, ρ is a parameter of descent and k is the iteration index at each time step. By Taylor's formula, for all n, k,

$$J(g^{n,k+1}) \leq J(g^{n,k}),$$

as $k \to \infty$, $g^{n,k}$ converges to g^n which achieves the minimum of J^n. However, it is not necessarily true that the minimum of J^n decreases as n increases. Now we also have

$$J'(g^{n,k}) = lg^{n,k} - \frac{1}{2}m\Delta t\left[G_g\mathcal{R}(\mathbf{v}^{n,k}, g^{n,k})\right]^* \zeta^{n,k}, \tag{350}$$

Thus once $g^{n,k}$ is known, we can compute the adjoint state $\zeta^{n,k}$ by solving the adjoint equation

$$\mathcal{A}^*\zeta^n + \left[G_{\mathbf{v}}\mathcal{R}(\mathbf{v}^n, g^n)\right]^*\zeta = C^*(C\mathbf{v}^n - \gamma_d), \tag{351}$$

with $g^n = g^{n,k}$ and $v^n = v^{n,k}$. Obtain $g^{n,k+1}$ from (349) using (350). Then compute $\mathbf{v}^{n,k+1}$ by solving the state equation (346) with $g^n = g^{n,k+1}$ and continue until convergence.

Suboptimal feedback laws for the time-dependent problem can be implemented in a manner similar to that described for the stationary problem.

9.3.3. Application to the stochastic Burgers equation. Choi *et al.* [29] applied the feedback control procedures just described to the Burgers equation subject to random forcing. This equation contains nonlinear convection and diffusion terms and its solution exhibits chaotic nature.

The Burgers equation with random forcing

Consider the randomly forced Burgers equation with no-slip boundary conditions

$$\begin{gathered}\frac{\partial \tilde{v}}{\partial t} + \frac{\partial}{\partial \tilde{x}}\frac{\tilde{v}^2}{2} = \nu\frac{\partial^2 \tilde{v}}{\partial x^2} + \tilde{w}(\tilde{x}, \tilde{t}), \quad 0 < \tilde{x} < L, \\ \tilde{v}(\tilde{x} = 0) = \tilde{v}(\tilde{x} = L) = 0\end{gathered} \tag{352}$$

where $\tilde{v}$ is the velocity, ν the kinematic viscosity, $\tilde{w}$ the random forcing and L the length of the computational domain. In the absence of forcing ($\tilde{w} = 0$) the solution of (352) decays to zero from any bounded initial state. We observe a small change in comparison with Eq. (324) from Sec. 9.3.1, where the nonlinear convection term has a different sign.

The forcing term $\tilde{w}$ is a white noise random process in $\tilde{x}$ with zero mean. The mean–square value of the dimensional forcing, σ^2, defines a velocity scale $U = (\sigma L)^{1/2}$. The Burgers equation in non–dimensional form using U and L as the typical velocity and length reads:

$$\begin{gathered}\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\frac{v^2}{2} = \frac{1}{\mathrm{Re}}\frac{\partial^2 v}{\partial x^2} + w(x,t), \quad 0 < \tilde{x} < 1,\\ v(x=0) = v(x=1) = 0\end{gathered} \tag{353}$$

where v, x, t are dimensionless quantities, Re is the Reynolds number UL/ν, and

$$\langle w \rangle_x = 0, \qquad \langle w^2 \rangle_x = 1. \tag{354}$$

Here $\langle \cdot \rangle_x$ denotes the average value over space. A Crank–Nicholson method in time and second–order centred differences in space were used by Choi *et al.* [29] to discretize (353). A Newton iterative method was used to solve the discretized nonlinear equation.

Feedback control procedures

Two types of feedback controls are investigated: distributed and boundary controls.

1. Distributed control

The non-dimensionalized Burgers equation with distributed control is

$$\begin{gathered}\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\frac{v^2}{2} = \frac{1}{\mathrm{Re}}\frac{\partial^2 v}{\partial x^2} + f(x,t) + w(x,t), \quad 0 < x < 1,\\ v(x,0) = v^0(x), \qquad v(0,t) = v(1,t) = 0.\end{gathered} \tag{355}$$

Here w is the random forcing and v^0 the initial data, an instantaneous solution of the Burgers equation with random forcing w and $f = 0$. The control input forcing f is of the form, cf. Sec. 9.3.2,

$$f = \alpha_0 + \alpha_1 f_1(v). \tag{356}$$

Note that α_0 and α_1 are not constant in time and space and they are continuously updated with the change of v. At each instant of time the cost functional considered is

$$J^n = J(\mathbf{e}^n) = \frac{1}{2} l_d ||\mathbf{e}^n||^2 + \frac{1}{2} m_d \int_0^1 \left(\frac{\partial v}{\partial x}\right)^2 dx, \tag{357}$$

where $||\mathbf{e}||^2 = ||\alpha_0||^2 + ||\alpha_1||^2$. Here we want to reduce the mean–square velocity gradient inside the domain at the expense of the control input. The detailed procedure of distributed control by body forces was described in the paper by Choi *et al.* [29].

2. Boundary control

The non-dimensionalized Burgers equation with boundary control is

$$\begin{gathered}\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\frac{v^2}{2} = \frac{1}{\mathrm{Re}}\frac{\partial^2 v}{\partial x^2} + w(x,t), \quad 0 < x < 1,\\ v(x,0) = v^0(x), \qquad v(0,t) = u_0(t), \qquad v(1,t) = u_1(t).\end{gathered} \tag{358}$$

Here v^0 is the initial data, which is an instantaneous solution of the Burgers equation with random forcing w and $u_0 = u_1 = 0$. The control input velocities at the boundary u_0 and u_1 are of the form

$$u_0 = \alpha_{0,0} + \alpha_{1,0} h_{1,0}(v), \qquad u_1 = \alpha_{0,1} + \alpha_{1,1} h_{1,1}(v). \tag{359}$$

At each instant of time, the instantaneous cost function considered is

$$J^n = J(\mathbf{e}^n) = \frac{1}{2} l_b ||\mathbf{e}^n||^2 + \frac{1}{2} m_b \left[\left(\frac{\partial v^n}{\partial x} \right) \Big|_{x=0} + \left(\frac{\partial v^n}{\partial x} \right) \Big|_{x=1} \right],$$

where $||\mathbf{e}||^2 = ||\boldsymbol{\alpha}_0||^2 + ||\boldsymbol{\alpha}_1||^2$, $\boldsymbol{\alpha}_0 = \{\alpha_{0,0}, \alpha_{0,1}\}$, $\boldsymbol{\alpha}_1 = \{\alpha_{1,0}, \alpha_{1,1}\}$. The detailed procedure of boundary control by boundary velocities was described in [29]. These

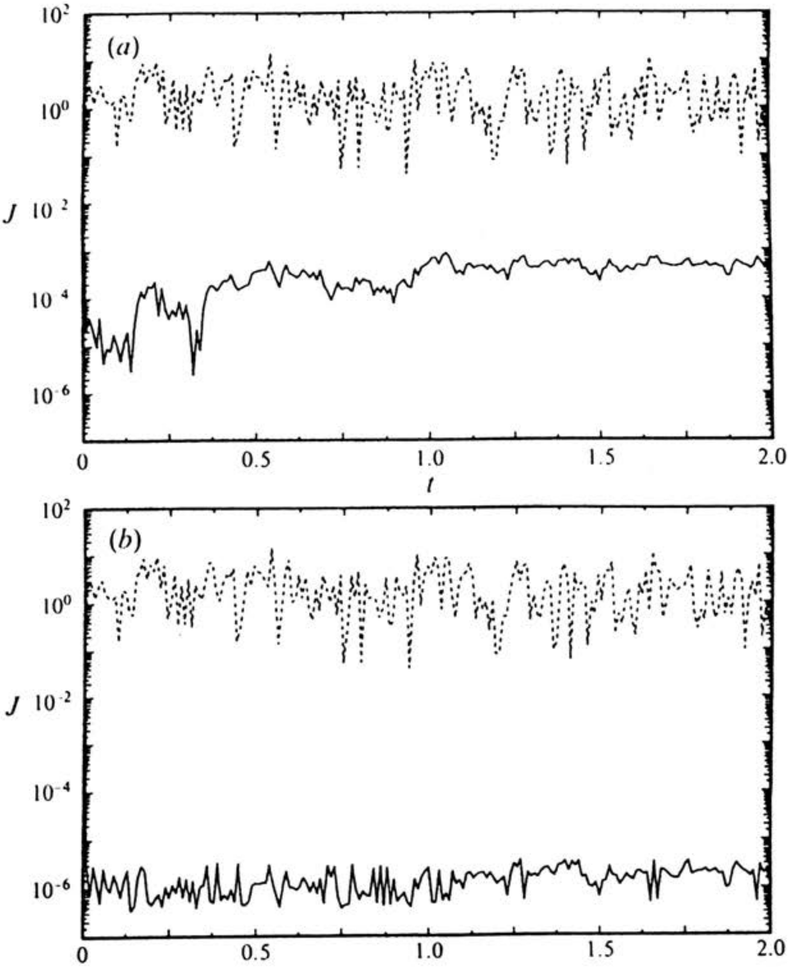

FIGURE 10. Time history of the cost: — with control, - - - without control; after Choi *et al.* [29]

authors provided also many numerical examples both for distributed and boundary control. One of such results for the boundary control is presented in Fig. 10.

Figure 10a was obtained for the following data: $l_b = 1$, $m_b = 1$ whilst Fig. 10b for $l_b = 0$ and $l_b = 0$, $m_b = 1$.

10. Final remarks

As we already know, theoretical foundations of controllability of parameter distributed systems are well established. The same cannot be said about stochastic systems. We know of only three papers concerned with control-theoretic aspects of linear stochastic systems [17, 18, 26].

Chan and Lau [26] described their results on (ϵ, δ) – stochastic controllability of linear systems provided that control inputs are restricted to some norm-bounded sets.

Bashirov and Karimov [17] and Bashirov and Mahmudov [18] assumed the set of all controls in the linear form. In the papers [17, 18] the notions of exact and approximate controllability was extended to stochastic systems. More detailed analysis was performed for linear partially observed stochastic systems. It was shown that the controllability of the primal stochastic system is linked with controllability of the corresponding deterministic system, cf. also Chan and Lau [26].

In this comprehensive paper a broad panorama of exact and approximate controllability as well as stabilization problems for a linear and nonlinear systems of applied mechanics has been presented. Some approximation methods and numerical algorithms have also been discussed. However, many important aspects have not been included (lack of space!). Particularly, we think here of controllability and stabilization of beams and coupled thermoelasticity equations, cf. [185, 194]. Our book [180] will cover all important physical and mechanical parameter distributed systems.

Having studied many papers on applied controllability we conclude that in many cases of distributed parameter systems we lack numerical algorithms, particularly robust ones.

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