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Design of adaptive structures

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New approach to structural redesign and sensitivity analysis, based on so-called Virtual Distortion Method, is presented. The sensitivity analysis carried out simultaneously with the march approach to an optimally redesigned structure is applied to adaptive structures (equipped with so-called structural fuses, e.g. yielding when overloaded in a controlled way). Maximisation of energy dissipation is chosen as the objective function for structures optimally adapting to overloading. A general formulation of the problem is discussed. Some simple numerical examples are included to illustrate theoretical considerations and to verify results of structural remodelling reached on two different ways. The first approach is based on the automatic tracing active constraints concept, while the second one makes use of the VDM based sensitivity analysis in the gradient optimisation method.

1. Introduction

Adaptive structures (structures equipped with controllable semi-active dissipaters (after [5]), so called *structural fuses*) with highest ability of adaptation to extremal overloading are discussed. The quasistatic formulation of this problem allows developing effective numerical tools necessary for farther considerations concerning dynamic problem of optimal design for the best structural crash-worthiness (see [2]). The structures with the highest impact absorption properties can be designed in this way. The proposed optimal design method combines sensitivity analysis with remodelling process, allowing approach (with material distribution as well as stress limits controlled) to an optimally redesigned structure. So called Virtual Distortion Method (see [1]), leading to analytical formulae for gradient calculations, has bee used in numerically efficient algorithm.

2. VDM based structural remodelling and sensitivity analysis

Let us concentrate on the sensitivity analysis for the truss structure under progressive collapse process due to extremal load applied. The superposition of virtual, plastic-like distortions β_i^o, simulating non-linear member behaviour, with distortions ε_i^o, modelling modifications of design variables (e.g. A_i), turns out to be productive in this case.

The strains and stresses, calculated with respect to initial cross-sections, can be expressed as follows (see [1], [4]):

$$\sigma_i' = E_i\left(\varepsilon_i - \varepsilon_i^o - \beta_i^o\right) = E_i\left(\varepsilon_i^L + \sum_j\left(D_{ij}-\delta_{ij}\right)\varepsilon_j^o + \sum_k\left(D_{ik}-\delta_{ik}\right)\beta_k^o\right),$$
$$\varepsilon_i = \varepsilon_i^L + \sum_j D_{ij}\varepsilon_j^o + \sum_k D_{ik}\beta_k^o \tag{1}$$

where D_{ij} denote deformations caused in the members i by the unit virtual distortions ε_j^o generated in members j. The corresponding derivatives take the following form:

$$\frac{d\sigma_i'}{d\varepsilon_j^o} = E_i(D_{ij}-\delta_{ij}), \qquad \frac{d\sigma_i'}{d\beta_k^o} = E_i(D_{ik}-\delta_{ik}),$$
$$\frac{d\varepsilon_i}{d\varepsilon_j^o} = D_{ij}, \qquad \frac{d\varepsilon_i}{d\beta_k^o} = D_{ik}. \tag{2}$$

The subscripts j and k in the above formulae run through all modified and plastified members, respectively. Taking advantage of two expressions for the internal forces applied to so called *distorted* (with modification of material distribution modelled through virtual distortions) and*modified* (with redesigned cross-sections from A_i' to A_i , without imposing virtual distortions) structure:

$$P_i = E_i A_i'(\varepsilon_i - \varepsilon_i^o - \beta_i^o),$$
$$P_i = E_i A_i\left(\varepsilon_i - \beta_i^o\right). \tag{3}$$

(where components of ε_i^o, β_i^o are non-zero only in distorted or plastified members, respectively), the following formula can be derived:

$$A_i\left(\varepsilon_i^L + \sum_j D_{ij}\varepsilon_j^o + \sum_k\left(D_{ik}-\delta_{ik}\right)\beta_k^o\right)$$
$$= A_i'\left(\varepsilon_i^L + \sum_j\left(D_{ij}-\delta_{ij}\right)\varepsilon_j^o + \sum_k\left(D_{ik}-\delta_{ik}\right)\beta_k^o\right), \tag{4}$$

what can be expressed alternatively:

$$\sum_j\left[A_i'\left(D_{ij}-\delta_{ij}\right) - A_iD_{ij}\right]\varepsilon_j^o + \sum_k\left[\left(A_i'-A_i\right)\left(D_{ik}-\delta_{ik}\right)\right]\beta_k^o = \left(A_i - A_i^o\right)\varepsilon_i^L. \tag{5}$$

Calculation of the derivative with respect to A_m leads to:

$$\delta_{im}\left[\varepsilon_i^L + \sum_j D_{ij}\varepsilon_j^o + \sum_k\left(D_{ik}-\delta_{ik}\right)\beta_k^o\right] + A_i\left(\sum_j D_{ij}\frac{\partial\varepsilon_j^o}{\partial A_m}\right.$$
$$\left.+\sum_k\left(D_{ik}-\delta_{ik}\right)\frac{\partial\beta_k^o}{\partial A_m}\right) = A_i'\left(\sum_j\left(D_{ij}-\delta_{ij}\right)\frac{\partial\varepsilon_j^o}{\partial A_m} + \sum_k\left(D_{ik}-\delta_{ik}\right)\frac{\partial\beta_k^o}{\partial A_m}\right). \tag{6}$$

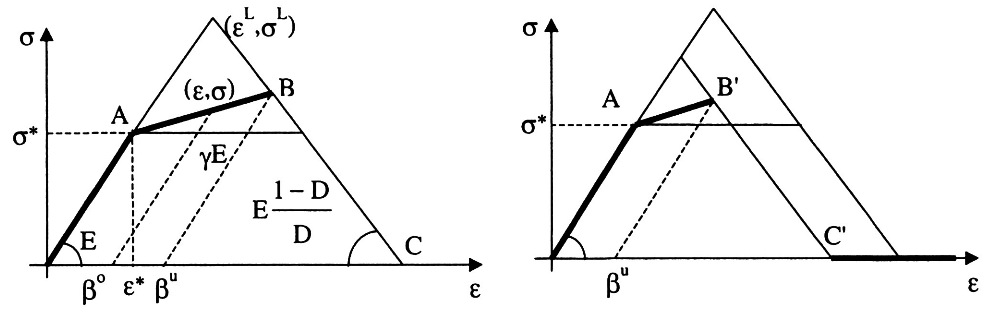


FIGURE 1. Yield criterion for the modified structure.

After rearranging the above formula, we have:

$$\sum_j \left[A_i' \left(D_{ij} - \delta_{ij}\right) - A_i D_{ij}\right] \frac{\partial \varepsilon_j^o}{\partial A_m} + \sum_k \left[\left(A_i' - A_i\right)\left(D_{ik} - \delta_{ik}\right)\right] \frac{\partial \beta_k^o}{\partial A_m} = \left(\varepsilon_i - \beta_i^o\right) \delta_{im}. \tag{7}$$

The associated conditions for derivatives $\partial \beta_i^o / \partial A_i$ and $\partial \varepsilon_j^o / \partial A_m$ can be determined from the yield criterion (cf. Fig. 1a), written for the *modified structure* (with modified cross-sections A_i):

$$\sigma_i - \sigma^* = \gamma_i E_i \left(\varepsilon_i - \varepsilon^*\right). \tag{8}$$

For the *modified structure*, where ε^o affects the stress formula in an implicit way through modified deformations (cf. Eqs. (1) for *distorted structure*), we get the following strains and stresses, with respect to remodelled cross-sections A_i:

$$\begin{aligned} \sigma_i &= E_i \left(\varepsilon_i - \beta_i^o\right) = E_i \left(\varepsilon_i^L + \sum_j D_{ij} \varepsilon_j^o + \sum_k \left(D_{ik} - \delta_{ik}\right) \beta_k^o\right), \\ \varepsilon_i &= \varepsilon_i^L + \sum_j D_{ij} \varepsilon_j^o + \sum_k D_{ik} \beta_k^o. \end{aligned} \tag{9}$$

Substituting (9) to (8) we obtain:

$$\begin{gathered} \sum_k B_{lk} \beta_k^o + \sum_j \left(1 - \gamma_l\right) D_{lj} \varepsilon_j^o = -\left(1 - \gamma_l\right)\left(\varepsilon_l^L - \varepsilon^*\right) \\ \text{where:} \quad B_{lk} = \left(1 - \gamma_l\right) D_{lk} - \delta_{lk}. \end{gathered} \tag{10}$$

Indices l and k run through plastified members and j runs through distorted members. The matrix B (so-called simulation matrix in collapse analysis) is non-positive definite. The mechanical interpretation of VDM simulation requires that all diagonal elements of B are non-positive. Therefore the following constraint imposed on the softening parameters:

$$\gamma_k \geqslant -\frac{1 - D_{kk}}{D_{kk}}$$

for all members k has to be satisfied, to get correct solution through the VDM approach. If a member does not satisfy the above constraint, its contribution to the stress distribution drops to zero and we have to follow line BC (Fig. 1a) to reach the corresponding local stress vanishing (point C).

If a limit β^u imposed on plastic-like distortions $|\beta^o| \leqslant \beta^u$ is violated during structural adaptation process (cf. Fig. 1b), then the following equation of the line B′C′ has to be used to simulate stress vanishing in the active member:

$$\begin{gathered}\sigma_i = -E_i \frac{1 - D_{ii}}{D_{ii}} (\varepsilon_i - \tilde{\varepsilon}_i) \\ \text{where:} \quad \tilde{\varepsilon}_i = \frac{\sigma^*}{E_i (1 - D_{ii})} + \frac{\gamma_i + (1 - \gamma_i)(1 - D_{ii})}{(1 - \gamma_i)(1 - D_{ii})} \beta^u. \end{gathered} \tag{11}$$

Now, calculating derivatives with respect to A_m we can get from (10) the following set of l' equations:

$$\sum_k \left[(1 - \gamma_l) D_{lk} - \delta_{lk}\right] \frac{\partial \beta_k^o}{\partial A_m} + \sum_j (1 - \gamma_l) D_{lj} \frac{\partial \varepsilon_j^o}{\partial A_m} = 0 \tag{12}$$

where l' denotes the number of plastified members and $l, k = 1, 2, \ldots, l'$.

Finally, to calculate the sensitivities (for example, with respect to modifications of material distribution) for elasto-plastic structure:

$$\begin{aligned}\frac{\partial \sigma_i'}{\partial A_m} &= \sum_j \frac{\partial \sigma_i'}{\partial \varepsilon_j^0} \frac{\partial \varepsilon_j^o}{\partial A_m} + \sum_k \frac{\partial \sigma_i'}{\partial \beta_k^0} \frac{\partial \beta_k^o}{\partial A_m} \\ &= \sum_j E_i D_{ij} \frac{\partial \varepsilon_j^o}{\partial A_m} + \sum_k E_i (D_{ik} - \delta_{ik}) \frac{\partial \beta_k^o}{\partial A_m}, \\ \frac{\partial \varepsilon_i}{\partial A_m} &= \sum_j \frac{\partial \varepsilon_i}{\partial \varepsilon_j^o} \frac{\partial \varepsilon_j^o}{\partial A_m} + \sum_k \frac{\partial \varepsilon_i}{\partial \beta_k^o} \frac{\partial \beta_k^o}{\partial A_m} = \sum_j D_{ij} \frac{\partial \varepsilon_j^o}{\partial A_m} + \sum_k D_{ik} \frac{\partial \beta_k^o}{\partial A_m},\end{aligned} \tag{13}$$

the partial derivatives determined by l' equations (12) and m equations (7) (for each chosen design variable $\mu_m = A_m/A_m'$) have to be determined from the following set (15) of equations:

$$\begin{array}{l} m\{ \\ \\ l\{ \end{array} \begin{bmatrix} (1 - \mu_i) D_{ij} - \delta_{ij} & \cdots & (1 - \mu_i) D_{ij} \\ \vdots & & \vdots \\ (1 - \gamma_i) D_{ij} & \cdots & (1 - \gamma_i) D_{ij} - \delta_{ij} \end{bmatrix} \begin{bmatrix} \varepsilon_j^o \\ \vdots \\ \beta_j^o \end{bmatrix} = \begin{bmatrix} -(1 - \mu_i) \varepsilon_i^L \\ \vdots \\ -(1 - \gamma_i)(\varepsilon_i^L - \varepsilon_i^*) \end{bmatrix}, \tag{14}$$

$$\begin{array}{l} m\{ \\ \\ l\{ \end{array}\begin{bmatrix} (1-\mu_i)D_{ij}-\delta_{ij} & \cdots & (1-\mu_i)D_{ij} \\ \vdots & & \vdots \\ (1-\gamma_i)D_{ij} & \cdots & (1-\gamma_i)D_{ij}-\delta_{ij} \end{bmatrix} \begin{bmatrix} \dfrac{d\varepsilon^o_j}{d\mu_k} \\ \vdots \\ \dfrac{d\beta^o_j}{d\mu_k} \end{bmatrix} = \begin{bmatrix} \dfrac{(\varepsilon_i-\beta^o_i)}{A'_i}\delta_{ik} \\ \vdots \\ 0 \end{bmatrix}. \tag{15}$$

On the other hand, the set (14), with the same main matrix, describes the virtual distortion fields simulating modified structure.

The above formulae can be, for example, applied to the gradient based optimal remodelling processes of adaptive structures. The plastic-like behaviour (simulated through β^o) corresponds to the performance of actuators, while the material redistribution modified during the redesign process is modelled through virtual distortions ε^o. The gradients computed from Eqs. 15 allow calculation of gradients (13) and finally, the gradient of an objective function. Then, corrections for the material distribution leading to reduction of the objective function can be performed, the corresponding modifications of virtual distortions can be determined from (14) and again new gradients can be computed from (15). Following this algorithm we can approach step by step the minimum of the objective function.

If stress limits $\sigma^*_k \leqslant \sigma^u$ will be considered as design variables, rather than material redistribution, the gradient formulae (15) will take the following form:

$$\begin{array}{l} m\{ \\ \\ l\{ \end{array}\begin{bmatrix} (1-\mu_i)D_{ij}-\delta_{ij} & \cdots & (1-\mu_i)D_{ij} \\ \vdots & & \vdots \\ (1-\gamma_i)D_{ij} & \cdots & (1-\gamma_i)D_{ij}-\delta_{ij} \end{bmatrix} \begin{bmatrix} \dfrac{d\varepsilon^o_j}{d\sigma^*_k} \\ \vdots \\ \dfrac{d\beta^o_j}{d\sigma^*_k} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \dfrac{(1-\gamma_i)}{E_i}\delta_{ik} \end{bmatrix}. \tag{15a}$$

Note, that the elements with the plastic like distortions exhausted ($\beta^o_i = \beta^u$), the equation derived from (11), by substituting σ_i and ε_i from (1), should replace equation $(14)_2$. Also, its corresponding derivatives should substitute the equations $(15)_2$ and $(15\text{a})_2$.

3. Design of adaptive structures for maximal energy dissipation general formulation

The optimal design problem leading to maximal structural ability of adaptation to overloading can be now formulated (e.g. for the ideal elasto-plastic case, with hardening) as requirement of maximisation of the global energy dissipation:

$$\max U$$

where

$$U = \begin{cases} \displaystyle\sum_i \frac{1}{2}\left(\sigma^*_i + \sigma_i\right)\beta^o_i \mu_i A'_i l_i & \text{if} \quad \beta^o_i \leqslant \beta^u, \\ \displaystyle\sum_i \left(\sigma^*_i + \frac{\gamma_i E_i}{1-\gamma_i}\beta^u\right)\beta^u \mu_i A'_i l_i & \text{if} \quad \beta^o_i > \beta^u \end{cases} \tag{16}$$

subjects to

$$
\begin{gathered}
\mu_i = \frac{\varepsilon_i - \varepsilon_i^o}{\varepsilon_i} \geqslant 0, \\
\sum_i \mu_i A_i' l_i = V_o, \\
|\sigma_i - \sigma_i^*| \leqslant |\gamma_i E_i (\varepsilon_i - \varepsilon_i^*)|, \\
|\sigma_i| \leqslant \left| \frac{\sigma_i^*}{E_i (1 - D_{ii})} + \frac{\gamma_i + (1-\gamma_i)(1 - D_{ii})}{(1-\gamma_i)(1 - D_{ii})} \beta^u \right|, \\
\sigma_i \beta_i^o \geqslant 0,
\end{gathered}
\tag{17}
$$

where: σ_i and ε_i are expressed by the formula (1), V_o denotes the constant material volume and σ_i^*, μ_i are control parameters (together with the associated virtual distortions ε_i^o, β_i^o). The parameters μ_i are responsible for material redistribution, while σ_i^* control the best adaptation of the yield stress limits of dissipative devices. β^u denotes the maximal stroke of structural fuses. The conditions $(17)_4$ constrains $\sigma - \varepsilon$ response to points below the line B$'$C$'$ (if the dissipative stroke β^u is exhausted), shown in Fig. 1. The gradients of the objective function and the side constraints with respect to the control parameters μ_i and σ_i^* can be calculated (through virtual distortions) making use of the formulae analogous to (15). In the consequence, an efficient numerical algorithm for the gradient based optimal redesign process can be proposed. Heaving control parameters modified (due to gradient calculations) in the iterative process, the virtual distortions ε_i^o, β_i^o can be updated solving Eqs. (14). The above variational problem formulation leads to the solution corresponding to analysis of the structure composed of members with piece-wise-linear constitutive characteristics shown (with bold line) in Fig. 1.

Let us now discuss (in the following section) particular cases of the structural redesign problem.

4. Remodelling effect

4.1. Gradient based remodelling of elastic structures

Assuming constant plastic-like properties of dissipaters ($\sigma_i^* = \text{const}$) the above *adaptive structure design* problem (16, 17) can be solved in two steps:

(i) elastic step, and

(ii) elasto-plastic step.

The first one (let us call it Rem1) can be formulated as follows:

$$\max U$$

where

$$U = \sum_i \frac{1}{2} E_i \varepsilon_i^2 \mu_i A_i' l_i \tag{18}$$

subjects to

$$\begin{aligned}\mu_i &= \frac{\varepsilon_i - \varepsilon_i^o}{\varepsilon_i} \geqslant 0,\\ \sum_i \mu_i A_i' l_i &= V_o,\end{aligned} \tag{19}$$

where ε_i is determined by $(1)_2$ (with $\beta \equiv 0$) and leads to an isostatic substructure with uniformly stressed elements (up to some σ^*).

The second, elasto-plastic step allows the isostatic structure determined in the first step to yield plastically up to the limit $\beta_i^o = \beta^u$ what means total maximum of energy dissipation $\mathrm{U} = \mathrm{V}_o \sigma^* \beta^u$.

Note, that other isostatic substructures can be uniformly stressed to lower stresses than the above σ^*. For example, the well known problem (see [4]) of the stiffest substructure minimising the objective function (18) gives the lowest possible uniformly distributed stresses.

The isostatic substructure with maximal compliance defined by (18), (19) can be determined through the gradient based optimisation procedure. The equations (14), (15) take the following form in the remodelling case:

$$\begin{aligned}\left[(1-\mu_i)\,D_{ij} - \delta_{ij}\right]\varepsilon_j^o &= -\,(1-\mu_i)\,\varepsilon_i^L,\\ \left[(1-\mu_i)\,D_{ij} - \delta_{ij}\right]\frac{d\varepsilon_j^o}{d\mu_k} &= \frac{\varepsilon_i}{A_i'}\delta_{ik},\end{aligned} \tag{20}$$

and the gradient of the auxiliary objective function (18) with added side constraint $(19)_2$ multiplied by the Lagrange coefficient λ can be calculated:

$$\frac{dU'}{d\mu_i} = \frac{\partial U'}{\partial \mu_i} + \sum_k \sum_j \frac{\partial U'}{\partial \varepsilon_k}\frac{\partial \varepsilon_k}{\partial \varepsilon_j^o}\frac{\partial \varepsilon_j^o}{\partial \mu_i} + \lambda A_i' l_i. \tag{21}$$

Substituting expression $(1)_2$ for $\varepsilon_i(\varepsilon_j^o)$ to (18) and performing partial derivatives, the above equation leads to the following form:

$$\frac{dU'}{d\mu_i} = \frac{1}{2}E_i\varepsilon_i^2 A_i' l_i + \sum_k \sum_j E_k \varepsilon_k \mu_k A_k' l_k D_{kj}\frac{\partial \varepsilon_j^o}{\partial \mu_i} + \lambda A_i' l_i \tag{22}$$

where the partial derivative $\frac{\partial \varepsilon_j^o}{\partial \mu_i}$ has to be determined in advance solving linear equations $(20)_2$.

Adding obvious derivative with respect to the parameter λ:

$$\frac{dU'}{d\lambda} = \sum_j \mu_j A_j' l_j - V_o \tag{23}$$

the gradient based optimization procedure solving the optimal remodelling problem (18), (19) can be performed according to the algorithm shown in Table 1.

TABLE 1. Gradient based remodelling procedure.

Calculate $\dfrac{d\varepsilon_j^o}{d\mu_k}$ from $(20)_2$.

Calculate $\dfrac{dU'}{d\mu_i}$ from (22) and $\dfrac{dU'}{d\lambda}$ from (23).

Perform modification of the design parameters μ_i, λ:

$$\mu_i = \mu_i + \delta_\mu \frac{dU'}{d\mu_i},$$

$$\lambda = \lambda + \delta_\lambda \frac{dU'}{d\lambda}.$$

Determine virtual distortions ε_j^o simulating remodelled structure from $(20)_1$.

Update strains from $(1)_2$.

4.2. Two remodelling procedures: Rem1 and Rem2

Modifying the optimal design problem (18), (19) requiring minimisation rather than maximisation of the objective function U, the well known problem (let us call it Rem2) of the stiffest structure can be formulated and solved through the gradient based procedure analogous to that shown in Table 1.

Alternatively, the same results can be achieved on the base of remodelling process prescribed by the following two steps (cf. [4]):

- Iterative tracing of uniformly distributed local strains $\tilde{\varepsilon}$ applying modifications of local cross-sections $\mu_i^{\rm new}/\mu_i$ (where $\mu^{\rm new}$ is a new cross-section to be applied in the next iterative step) according to the modification rule shown in Fig. 2. An isostatic substructure can be expected as the result of this procedure in the case of one load state.

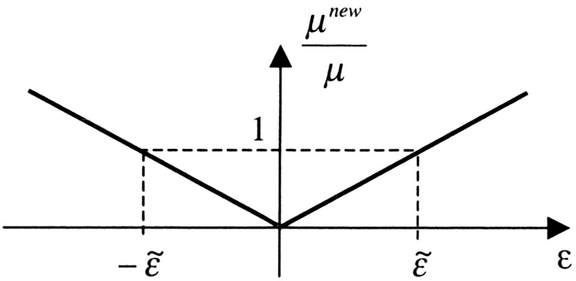


FIGURE 2. Cross-section modification rule for the Tracing Uniform Strains procedure of remodelling.

- Scaling cross-sections μ_i to reach the initial material volume V_o. Modified, however still uniformly distributed strains will be obtained (for the isostatic substructure) in this way.

The Tracing Uniform Strains procedure is numerically more efficient than the gradient based remodelling approach.

Both of these procedures (the gradient based as well as the Tracing Uniform Strains one) can be generalised for a multi-load case.

4.3. Example: elasto-plastic case

Let us illustrate the discussed above cases on the simple example shown in Fig. 3a.

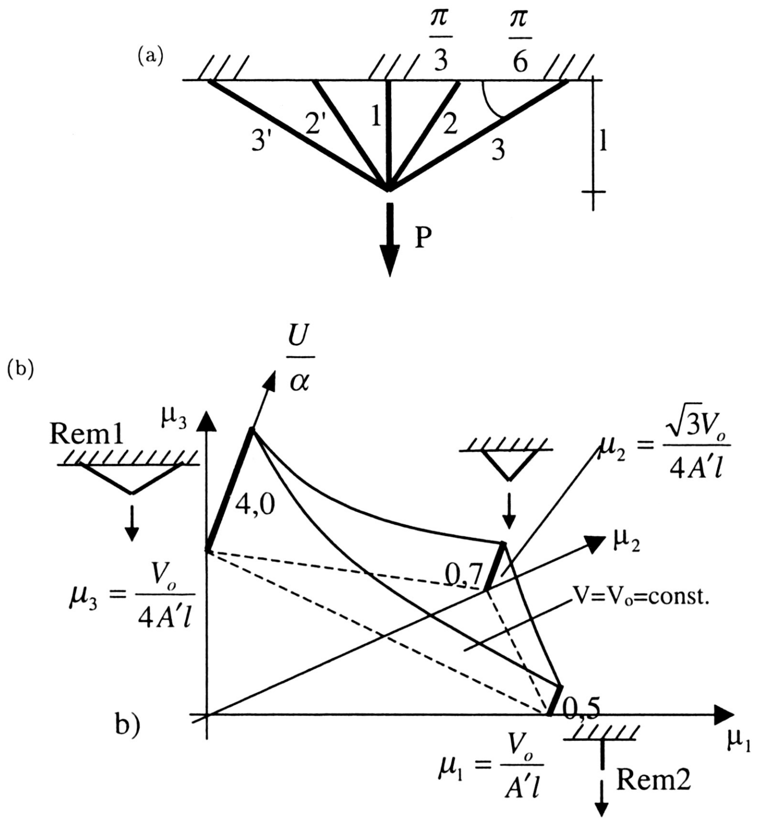


FIGURE 3. Simple example of remodelled structure.

The objective function $U(\mu_1, \mu_2, \mu_3)$ and three isostatic substructures as particular cases are shown in Fig. 3b. We can see that the objective function for the isostatic substructure Rem1 is eight times higher than for the substructure Rem2. Assuming now the ideal elasto-plastic behaviour of members, when some stress-limit σ^*, $\sigma^* \leqslant \sigma^u$ is reached (where σ^u is the yield stress for the material used) the energy absorption capacities: $U = V_o \sigma^* \beta^u$ for the two particular designs (Rem1, Rem2) and various load levels (P) are shown in Fig. 4.

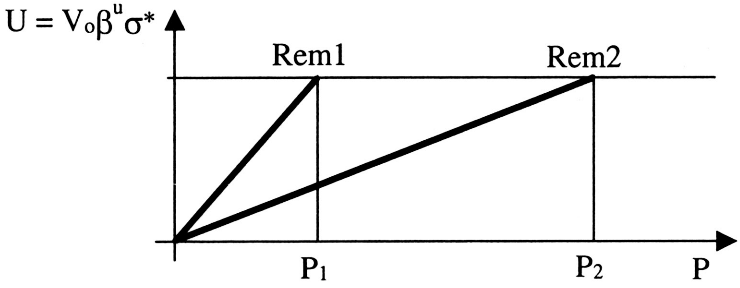


FIGURE 4. Energy Absorbtion Capacity for Designes Rem1 and Rem2.

Because of contradiction between two properties:

(i) the energy absorbtion capacity and

(ii) the load capacity,

real applications needs compromise solutions. To determine them there is still room (apart from the general approach described in Sec. 3 for simpler, easier to apply quasi-optimal solutions. One of them can be based on design composed of two particulars, isostatic substructures Rem1 and Rem2.

Designing the structure for several loads, proportional participation of the solution Rem2 in the final design should be chosen as minimum but satisfying that the structure sustains maximal expected load. Then, the real-time use of the adaptive structure requires adjusting of the stress limits σ^* for redundant elements from Rem2 (as the response for identified load P) to maximise the global energy dissipation.

5. Structural adaptation effect

5.1. Gradient based approach

In the case of fixed geometry ($\mu_i = \text{const} = 1$) the problem (16), (17) is reduced to searching for optimal distribution of the yield stress levels σ_i^*:

$$\max U$$

where

$$U = \begin{cases} \displaystyle\sum_i \frac{1}{2}(\sigma_i^* + \sigma_i)\beta_i^o A_i l_i, & \text{if} \quad \beta_i^o \leqslant \beta^u, \\ \displaystyle\sum_i \left(\sigma_i^* + \frac{\gamma_i E_i}{1-\gamma_i}\beta^u\right)\beta^u A_i l_i, & \text{if} \quad \beta_i^o > \beta^u. \end{cases} \tag{24}$$

subjects to

$$\sigma_i - \sigma_i^* \leqslant \gamma_i E_i(\varepsilon_i - \varepsilon_i^*),$$
$$|\sigma_i| \leqslant \left| \frac{\sigma^*}{E_i(1-D_{ii})} + \frac{\gamma_i + (1-\gamma_i)(1-D_{ii})}{(1-\gamma_i)(1-D_{ii})}\beta^u \right|, \tag{25}$$

where: σ_i and ε_i are expressed by the formula (1) and σ_i^* are control parameters (together with the associated virtual distortions β_i^o, while $\varepsilon_i^o \equiv 0$). The parameters σ_i^* control the best adaptation of the yield stress limits of dissipative devices. β^u denotes the maximal stroke of structural fuses. The conditions $(6)_1$ and $(6)_2$ constrain $\sigma - \varepsilon$ responses to points below the lines AB in Fig. 1a. and B′C′ in Fig. 1b (if the dissipative stroke β^u is exhausted), respectively. The hardening coefficients γ can be also treated as design variables, however, let us assume here that it is a small, positive parameter.

Postulated maximisation of energy dissipation formulated as the problem (5), (6) can be realised trough the gradient based optimisation algorithm shown in Table 1. The linear objective function (5) and linear constraints (6) limiting convex area lead to the solution determined by active constraints (6) describing natural behaviour of elasto-plastic-brittle structure with controllable stress limits and with the following constitutive relations:

$$\begin{aligned} &\sigma_i - \sigma_i^* = \gamma_i E_i(\varepsilon_i - \varepsilon_i^*), && \text{if} \quad |\sigma_i| > \sigma_i^*, \\ &\sigma_i = 0, && \text{if} \quad |\sigma_i| \geqslant \left| \frac{\sigma_i^*}{E_i(1-D_{ii})} + \frac{\gamma_i + (1-\gamma_i)(1-D_{ii})}{(1-\gamma_i)(1-D_{ii})}\beta^u \right|. \end{aligned} \tag{26}$$

Determining virtual distortions simulating plastic-like behaviour of adaptive elements ($i \in$B) and fracture-like behaviour of elements with exhausted adaptation ability ($\beta_i^o > \beta^u$, $i \in$B) the following set of equations has to be solved (substituting (1) to (7), cf. (4)):

$$\begin{bmatrix} (1-\gamma_i)D_{ik} - \delta_{ik} & \vdots & (1-\gamma_i)D_{ik} \\ \cdots & & \cdots \\ D_{ik} & \vdots & D_{ik} - \delta_{ik} \end{bmatrix} \begin{bmatrix} \beta_i^o \\ \cdots \\ \beta'^o_i \end{bmatrix} = \begin{bmatrix} (1-\gamma_i)(\varepsilon_i^* - \varepsilon_i^L) \\ \cdots \\ 0 \end{bmatrix} \begin{matrix} \} i \in B \backslash B' \\ \\ \} i \in B' \end{matrix} . \tag{27}$$

Searching for the best corrections of the control parameters σ_i^* (cf. Table 2) the following gradient of the objective function (25) is required:

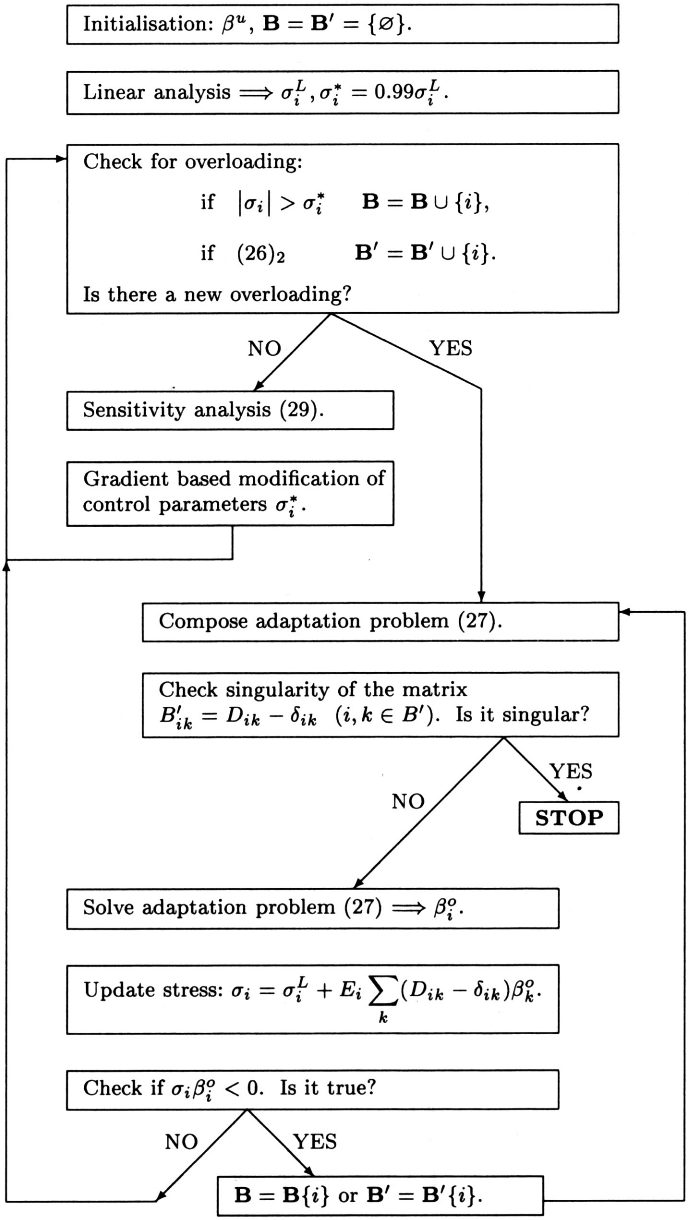
$$\frac{dU}{d\sigma_i^*} = \frac{\partial U}{\partial \sigma_i^*} + \sum_k \left[\frac{\partial U}{\partial \beta_k^o} + \sum_l \frac{\partial U}{\partial \sigma_l} \frac{\partial \sigma_l}{\partial \beta_k^o} \right] \frac{\partial \beta_k^o}{\partial \sigma_i^*} \tag{28}$$

where:

$$\frac{\partial U}{\partial \sigma_i^*} = \begin{cases} \frac{1}{2} A_i l_i \beta_i^o, & \text{if} \quad \beta_i^o \leqslant \beta^u, \\ A_i l_i \beta_i^u, & \text{if} \quad \beta_i^o > \beta^u, \end{cases}$$

$$\frac{\partial U}{\partial \beta_k^o} = \begin{cases} \sum\limits_l A_l l_l E_l \beta_l^o (D_{lk} - \delta_{lk}) + \frac{1}{2}(\sigma_k^* + \sigma_k) A_k l_k, & \text{if} \quad \beta_k^o \leqslant \beta^u, \\ 0, & \text{if} \quad \beta_k^o > \beta^u, \end{cases}$$

TABLE 2. Flow-chart of the algorithm of optimal adaptation to overloading.



$$\frac{\partial U}{\partial \sigma_l} = \begin{cases} \frac{1}{2} A_l l_l \beta_l^o, & \text{if} \quad \beta_l^o \leqslant \beta^u, \\ 0, & \text{if} \quad \beta_l^o > \beta^u, \end{cases}$$

$$\frac{\partial \sigma_l}{\partial \beta_k^o} = E_l \left(D_{lk} - \delta_{lk} \right)$$

and $\frac{\partial \beta_k^o}{\partial \sigma_i^*}$ can be computed from the following set of equations derived differentiating (27):

$$\begin{bmatrix} (1-\gamma_i) D_{ik} - \delta_{ik} & \vdots & (1-\gamma_i) D_{ik} \\ \cdots & & \cdots \\ D_{ik} & \vdots & D_{ik} - \delta_{ik} \end{bmatrix} \begin{bmatrix} \frac{\partial \beta_k^o}{\partial \sigma_m^*} \\ \cdots \\ \frac{\partial \beta'^o_k}{\partial \sigma_m^*} \end{bmatrix} = \begin{bmatrix} \frac{1-\gamma_i}{E_i} \delta_{im} \\ \cdots \\ 0 \end{bmatrix}. \tag{29}$$

Note, that going to the full dynamic analysis, the influence matrix D_{ij} has to be replaced with so called dynamic influence matrix [2] $D_{ij}(t-\tau)$ determining the strain evolution caused in the truss member i and in the time instance $\tau \geqslant 0$, due to the unit *virtual distortion* impulse $\Delta\varepsilon_j^o(\tau) = 1$ generated in the member j in the time instant τ. $\Delta\varepsilon_j^o(t) = \dot{\varepsilon}_j^o(t)\Delta t$ and i runs through all members of truss structure. Note that the matrix $\mathbf{D}$ stores information about the entire structure properties (including boundary conditions) and describes dynamic (not static) structural response for locally generated impulse of virtual distortion. In this case, the main matrix of the set of equations (14) does not become singular, even for number of adaptive members higher than the structural redundancy k.

In particular case, when limitations β^u do not interfere our solution, the constraints $(26)_2$ can be disregarded. Assuming also no hardening ($\gamma = 0$), the objective function (5) takes the following form:

$$U = \sum_i \sigma_i^* \beta_i^o A_i l_i \tag{30}$$

Then, the gradient expression (28) takes the following simpler form:

$$\frac{dU}{d\sigma_i^*} = \beta_i^o A_i l_i + \sum_k \sigma_k^* A_k l_k \left[E_k \left(D_{ki} - \delta_{ki} \right) \right]^{-1} \tag{31}$$

where $[\,]^{-1}$ denotes the inversion of the matrix $E_k(D_{ki} - \delta_{ki})$. Let us apply the above formula in gradient based approach to determine the optimal distribution of the control parameters σ^*. We can see that the contribution of adaptive elements into the energy dissipation depends on:

(i) the current stage of plastic-like distortions development,
(ii) the current yield stress level distribution, and
(iii) the compliance of the set of adaptive members determined by the inverse matrix $[\,]^{-1}$.

5.2. Heuristic, compromise solutions

Let us now illustrate the discussed above case on the simple example discussed in Secs. 4 and 3, however, allowing additionally possibility of switching "*on*" and "*off*" structural members during the loading process.

Assume remodelling (composed of particular solutions Rem1 and Rem2) selected for a range of various load P intensities (Fig. 5a).

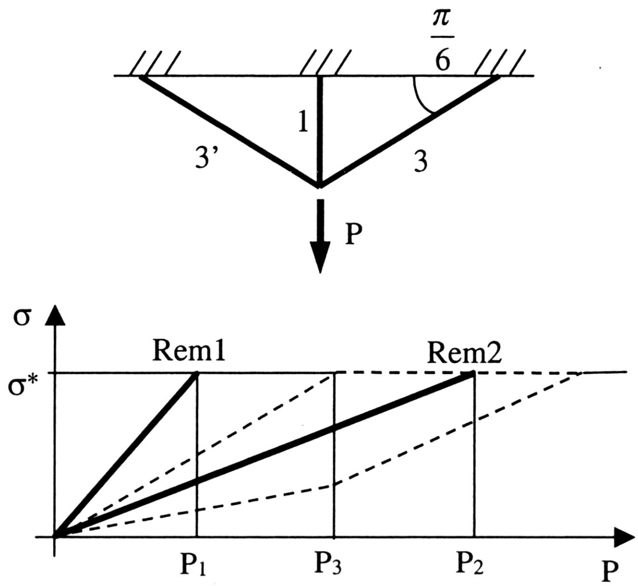


FIGURE 5. Energy Absorbtion Capacity for the composed (Rem1 + Rem2) structure (dotted lines).

An easy applicable heuristic optimal solution for the problem of maximal energy dissipation (with switching option) can be proposed. Postulating maximal possible contribution of the substructure Rem1 (the most energy absorbing one) into the dissipation process the three particular cases of load intensity shown in Fig. 6 should be taken into account.

In the first discussed case (Fig. 6a) application of stress limits: $\sigma_1^* = 0$ and σ_3^* shown in the figure guarantees energy dissipation $U = \sigma_3^* \beta^u V_3$.

In the second case (Fig. 6b) application of switching "*on*" the member 1 during the loading process and σ_1^*, σ_3^* marked in the figure gives maximal energy dissipation.

Finally, in the third case (Fig. 6c) switching "*on*" the member 3 and switching "*off*" the member 1 at the same instant gives the desired result.

The above example demonstrates (apart from the fact that the final results depend on proportion between substructures Rem1 and Rem2 combined into the compromise solution) that allowing switching option can significantly improve the dissipation effect. From the other side it demonstrate also that heuristic methods can be quite useful in solving these optimal design problems.

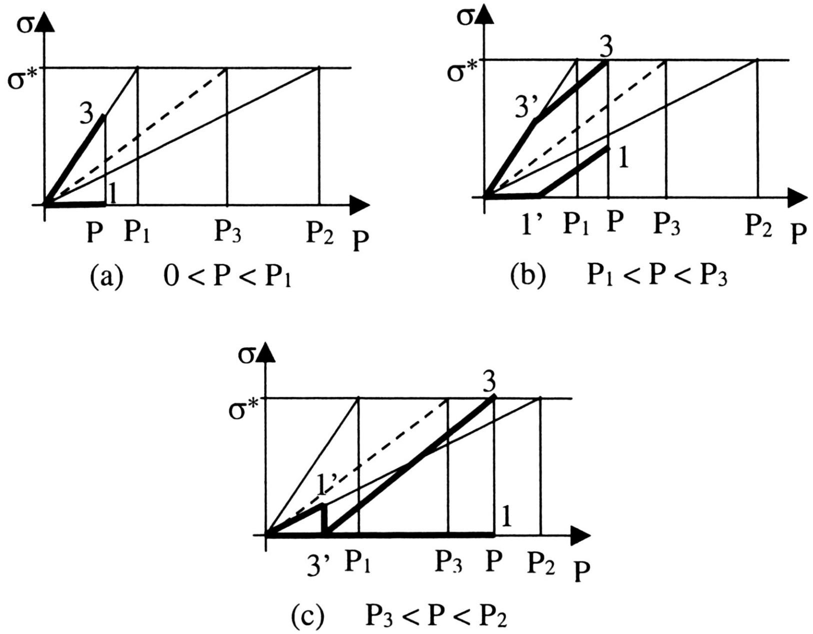


FIGURE 6. Particular cases of control strategies with switching option.

5.3. Numerical example

The hyper-static truss structure shown in Fig. 7 has been used to demonstrate the structural adaptation procedure. Assuming fixed structural geometry ($\mu_i = 1 = \text{const}$, $A = 0.0201$, $E = 10^8$, $\beta^u = 0.01$) the optimal stress limits σ_i^* for each member have been determined (for "detected" extreme load P) using the algorithm described in Sec. 3.3. To make the problem differentiable for all structural elements (plastified as well as still elastic), elasto-plastic behaviour with small hardening (small $\gamma = 0.01$) has been applied.

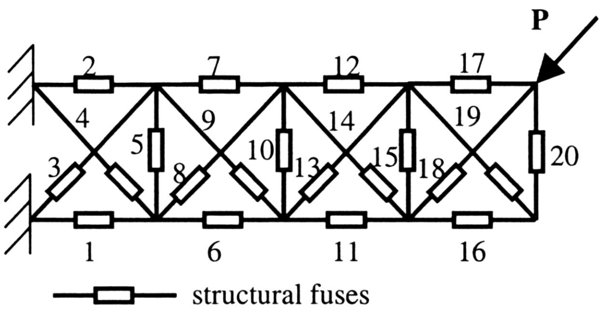


FIGURE 7. Adaptive truss structure example.

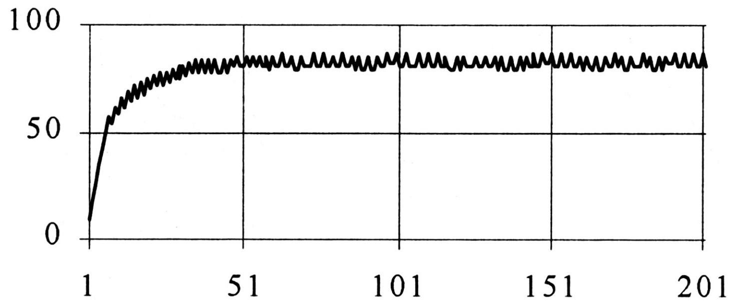


FIGURE 8. The objective function iteration [kJ].

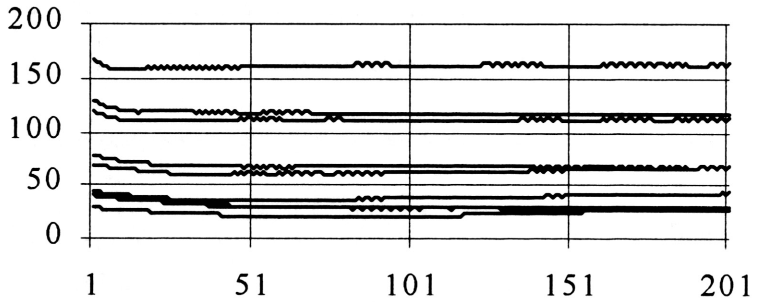


FIGURE 9. Iteration of yield stress limits [MPa] in remaining members.

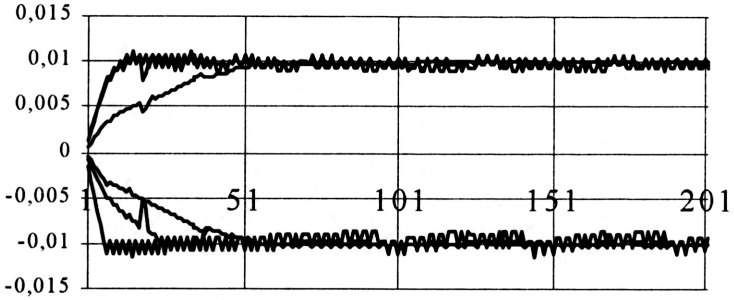


FIGURE 10. Iteration of plastic-like distortions in remaining members.

The results of the optimisation process have been shown in Figs. 3-6. The stress limits (Fig. 9) and plastic-like distortions' (Fig. 10) iteration show that the final result can be realised as the iso-static substructure demonstrated in Fig. 6a. However, this result describes a local maximum of the objective function (see Fig. 8 for its iteration, $f = 85\,\mathrm{kJ}$). To find the improved solution ($f = 104\,\mathrm{kJ}$, cf. Fig. 11) some special treatment (on-line corrections of the optimisation procedure) has to be applied. The solution demonstrated in Fig. 11 is close to the result (obtained through another, numerically expensive approach) reported in [3].

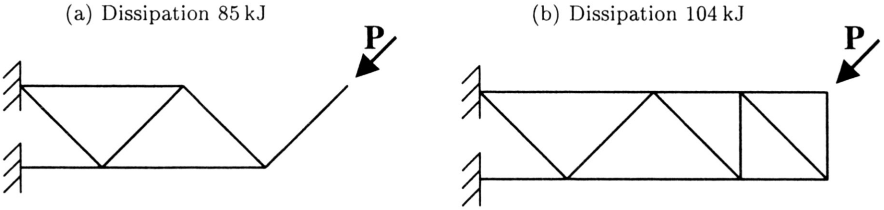


FIGURE 11. The best iso-static substructure: (a) local maximum, (b) improved solution.

6. Conclusions

It can be demonstrated (e.q. through numerical tests) that the particular, *optimal remodelling* solution discussed in section 4 will be reached also for the general *adaptive structure redesign* problem (16), (17) (for both: σ_i^* and μ_i control parameters) when only one loading state is considered. However, normally, we have to take into account several possibles extremal loading scenarios, and that is why more complex results are normally expected. The gradient- based approach can be applied also to the above, general redesign problem. However, this formulation can be substituted, in the first approximation, by the following, simpler, decomposed, two-steps problem:

(i) *Structural remodelling – Multiload Case* and

(ii) *Structural Multiload Adaptation.*

The *Tracing Active Constraints* algorithm (mentioned above, Sec. 4) can be generalised for several load states (describing all possible extremal loading states). Assuming $\sigma_i^* = \sigma^u = \mathrm{const}$ the material redistribution μ_i can be determined. Each structural element is fully loaded ($|\sigma_i| = \sigma^u$) at least in one load state, but the structure is no more isostatic. The structure with fixed geometry (determined in the first step (i)) can be now optimally adapted (σ_i^* controlled) to particular (detected in real time) load state using the following approach of structural multi-load adaptation.

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