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Jerzy A. Przeszowski

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OF THE LIGHT-FRONT

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Jerzy A. Przesowski
Institute of Fundamental Technological Research
Polish Academy of Sciences
ul. Świątokrzyska 11/21, 00-049 Warsaw, Poland
email: jprzeszo@ippt.gov.pl

Gauge Conditions in the Canonical Hamiltonian Formulation of the Light-Front Quantum Electrodynamics

Abstract

We report here the status of different gauge conditions in the canonical formulation of quantum electrodynamics on light-front surfaces. We start with the massive vector fields as pedagogical models where all basic concepts and possible problems manifestly appear. Several gauge choices are considered for both the infinite and the finite volume formulation of massless gauge field electrodynamics. We obtain the perturbative Feynman rules in the first approach and the quantum Hamiltonian for all sectors in the second approach. Different space-time dimensions are discussed in all models where they crucially change the physical meaning. Generally, fermions are considered as the charged matter fields but also one simple 1+1 dimensional model is discussed for scalar fields. Finally the perspectives for further research projects are discussed.

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Part I

Introduction

Half a century ago Dirac [1] has proposed 3 different forms of relativistic dynamics depending on the types of surfaces where independent modes were initiated. The first possibility when a space-like surface is chosen (named by Dirac *instant form*) has been used most frequently so far and is usually called **equal-time** quantization (ET). The second choice is to take a surface of a single light wave (called by Dirac *front form*). This kind of quantization will be discussed in this work and is commonly referred to as **light-front** formalism (LF). The third possibility is to take a branch of hyperbolic surface $x^\mu x_\mu = \kappa^2$, $x_0 > 0$ (named by Dirac *point form*). Though a single attempt in this last approach can hardly be found, we quote also this choice because very frequently the LF formalism is erroneously called *light-cone quantization* - a name which correctly should be connected with a special case (when $\kappa^2 \rightarrow 0$) of the point form of dynamics [1]. Then it took almost 20 years before Dirac's idea of front form of dynamics was applied by physicists.

At an ET surface, any two *different* points are spatially separated, therefore fields at these points are naturally independent quantities. For a LF surface things are different because for any point, there is a null-direction and all points lying along this direction are light-like separated, thus fields at these points may have nonvanishing commutators. For all other directions on a LF surface, points are spatially separated and fields at these points are independent. This property of the LF formalism has been used to infer the behaviour of light-cone commutators [2], [3] from the respective LF commutators. When Weinberg considered the scalar field theory in the **infinite momentum frame** (IMF) [4], he has found great simplifications of the "old-fashioned" time-ordered perturbation theory. Then the close relation between both the IMF formalism [5], [6], [7], [8], [9], [10], [11], [12] and the LF formalism [13], [14], [15], [16], [17], [18] was established, and these two names were quite frequently interchangeably used. Since then, a lot of successful attempts to phenomenology has been done [19], most of them being based on the LF perturbative calculations [20].

Another possibility started over 10 years ago when Brodsky and Pauli [21] have introduced so-called **discrete light-cone quantisation** (DLCQ), where periodic boundary conditions have been imposed on fields in a finite volume of the LF surface. This approach aimed at the nonperturbative approach opened a broad scope of both the theoretical and numerical studies. Another important attempts in the nonperturbative directions are the renormalization of Hamiltonians [22] and the relativistic bound-state problem [23]. Though, at the time, they are beyond the scope of our paper, the next steps that one can take starting from our analysis may be made precisely in these directions.

Main topic of this paper is the canonical description of **quantum electrodynamics** (QED) within the LF approach. This model is best known in the standard ET formulation but its LF analysis has also been studied in many papers since the first formulations in [7], [9], [14]. However, one important question of gauge fixing conditions, which can be imposed of gauge field potentials, has been not solved so far. Nearly all papers use, seemingly the only possible, the gauge fixing condition¹ $A^+ = A_- = 0$ named the **light-cone gauge** (LC-gauge) or the IMF-gauge. Evidently such a choice simplifies enormously many problems, specially those which appear in the sectors of charged matter fields. Also this gauge allows for the propagation of only physical photons in the sector of gauge fields. However, there is a minor inconsistency in the LC-gauge, because the Feynman perturbative rules

¹Notation is given in Appendix A.1.

found in the ET quantization [24] have the causal Mandelstam-Leibbrandt (ML) [25] prescription

$$\frac{1}{(n \cdot k + i\epsilon \text{sgn}(n^0 \cdot k))^n}$$

for the spurious poles of gauge field propagator, while the corresponding rules found in the LF quantization give the Cauchy Principal Value (CPV) prescription for these poles. While for Abelian models both prescriptions are equally correct then non-Abelian models need exclusively the ML-prescription [26]. This important caution should be kept in mind during the present discussion where Abelian interactions don't falsify the CPV prescription but only prepare a future developments with non-Abelian interactions.

In the canonical LF approach one chooses one null-direction, usually x^+ , as a parameter of dynamical evolution and treats all ∂_+ operators as 'time' derivatives. Therefore the ET and the LF descriptions of the same relativistic models formally may look quite different, specially the structure of their canonical conjugate momenta. Here we will not dogmatically follow Dirac's general method of canonical description for constraint system [27]. But to the contrary, we will rather focus our attention on important physical points. First, we will consider as *truly constrained* variables only those fields which satisfy non-dynamical equations (i.e. with no ∂_+ terms). Such equations can be *formally* solved for non-dynamical fields and therefore, non-dynamical and dependent variables are understood as *synonyms* here. By definition, all other fields are considered as dynamical and independent variables. Second, canonical momenta for dynamical fields are introduced and canonical Hamiltonians are defined via the Legendre transformation. If necessary, the terms which contain ∂_+ will be expressed by means of corresponding canonical momenta and only in this way the conjugate momenta can appear in the canonical LF Hamiltonian. For all variables which appear in the Hamiltonian we have dynamical equations of motion which are of the first order with respect to ∂_+ , and we demand that these Hamilton equations follow from the canonical Hamiltonian by means of the classical Poisson-Dirac brackets. Next, having a consistent classical canonical structure one can *define* a consistent quantum theory where Poisson-Dirac brackets are replaced by quantum commutation relations, and the Heisenberg equations for quantum field operators have the same form (up to some ordering of noncommuting operators) as the corresponding classical counterparts. Third, for the interacting system of gauge and matter fields one can carry the above canonical method in steps, starting with the gauge field sector where all interactions with matter are described by linear couplings to arbitrary external currents. Having eliminated all gauge constraints one can write an *equivalent* Lagrangian which describes the gauge sector by means of fewer unconstrained fields. Finally the canonical quantization of complete system can be carried out quite easily having the constraints for gauge and matter fields transparently separated and solved.²

Our paper is organized as follows. In part II a pedagogical model of massive vector electrodynamics with fermion matter fields is analysed. Though this is not a true gauge field model, it is discussed in three cases of no gauge fixing, LF Weyl and Lorentz gauges. In part III the QED with fermions is presented for several different gauges, and perturbation Feynman rules for practical S-matrix calculations are given in each case. Also QED for charged scalar fields is presented in 1+1 dimensions for the LF Weyl gauge. In part IV we analyse the DLCQ approach to QED with periodic boundary conditions for gauge fields and antiperiodic conditions for fermion fields imposed on the finite volume LF. Also here several gauge conditions are discussed and the canonical analysis is carried out subsequently in different sectors for various gauge field modes and fermion fields. In Conclusions all results are generally discussed special attention being paid to the further developments. The Appendices contain all notations and definitions of different functions used throughout the main text. Most of these results have not been published so far, otherwise due references to original papers are given in the paper.

²This method of quantization is similar to that proposed by Jackiw [28].

Part II

Massive Electrodynamics

1 Theory without gauge fixing condition

Our first model, which we analyse along the lines indicated in the Introduction, will be the electrodynamics of massive vector field B_μ with the mass term m^2 . Its kinetic term has the Maxwell form built with $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ and it couples with fermion fields via the electrodynamic current term $g\bar{\psi}\gamma^\mu\psi B_\mu$. Though this model has been consistently quantized on LF by Yan over 25 years ago [11] via the Schwinger action principle [29], we decided to derive his results in our original method of quantization for constrained systems. This model is even quite a pedagogical example because it contains all obstacles and restrictions which later will be encountered in more physical cases.

1.1 Maxwell theory with a mass term

We start our analysis with the following Lagrangian density:

$$\begin{aligned}\mathcal{L}_{mass} &= -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{m^2}{2}B_\mu B^\mu + \bar{\psi}(i\partial_\mu\gamma^\mu - eB_\mu\gamma^\mu + M)\psi \\ &= (\partial_+ B_i - \partial_i B_+) (\partial_- B_i - \partial_i B_-) + \frac{1}{2}(\partial_+ B_- - \partial_+ B_-)^2 - \frac{1}{4}(\partial_i B_j - \partial_j B_i)^2 \\ &\quad + \frac{m^2}{2}(2B_+ B_- - B_i^2) + \bar{\psi}(i\partial_\mu\gamma^\mu - eB_\mu\gamma^\mu + M)\psi ,\end{aligned}\tag{1.1}$$

where the explicit LF coordinates are introduced for the vector fields. It is known that only a half of the fermion degrees of freedom are dynamical fields, therefore in order to separate the vector field constraints from the fermion field ones, we have decided to start with a subsystem where the fermion contribution is solely described by external currents j^μ coupled linearly with B_μ . Because in such a simplified model the dynamics of fermions is omitted, one cannot argue in favour of the external current conservation law $\partial_\mu j^\mu = 0$. Even more we stress that all components of the current should be taken as independent *classical* quantities which allow for taking functional derivatives with respect to them³

$$\frac{\delta j^\mu(x)}{\delta j^\nu(y)} = \delta^\mu_\nu \delta(x - y).\tag{1.2}$$

For such a simplified model one can easier find a set of dynamical independent field variables which identically satisfy the constraints and have correct dynamical properties. Later we can write an *equivalent Lagrangian* with these independent modes and external currents j^μ , and then by reintroducing fermion kinetic terms we end up with the *complete equivalent Lagrangian* and finally, quantize a complete system without any effort. However this scenario is valid only if the condition of mutual independence of fermion and vector fields is satisfied. Otherwise some obstacles will appear and (fortunately or not) this situation will appear in our first model.

1.1.1 Vector fields with external currents

Our first task will be to analyse a subsystem of vector fields, so we start with the Lagrangian density

$$\mathcal{L}_B^{mass} = (\partial_+ B_i - \partial_i B_+) (\partial_- B_i - \partial_i B_-) + \frac{1}{2}(\partial_+ B_- - \partial_- B_+)^2 - \frac{1}{4}(\partial_i B_j - \partial_j B_i)^2$$

³This property of external currents is crucial in checking the consistency of the simplified model with the one we have started with.

$$+ \frac{m^2}{2}(2B_+B_- - B_i^2) + B_\mu j^\mu, \quad (1.3)$$

and taking the variations of vector fields B_μ we generate the Euler-Lagrange equations

$$\partial_+(\partial_+B_- - \partial_-B_+ - \partial_iB_i) = (m^2 - \Delta_\perp)B_+ + j^-, \quad (1.4.a)$$

$$-\partial_-(\partial_+B_- - \partial_-B_+ + \partial_iB_i) = (m^2 - \Delta_\perp)B_- + j^+, \quad (1.4.b)$$

$$(2\partial_+\partial_- - \Delta_\perp + m^2)B_i = \partial_i(\partial_+B_- + \partial_-B_+ - \partial_jB_j) + j^i. \quad (1.4.c)$$

As the consistency condition for these equations we have

$$\partial_+B_- + \partial_-B_+ - \partial_jB_j = -\frac{1}{m^2}(\partial_+j^+ + \partial_-j^- + \partial_i j^i) \quad (1.5)$$

so, together with (1.4.a), we end up with two independent equations with ∂_+B_- - but the excess of dynamical equations indicates the presence of constraints. This observation will become transparent after introducing the canonical momentum Π^- . Generally at LF, one introduces canonical momenta Π^μ as the functional derivatives of Lagrangian by ∂_+B_μ , respectively. In our present case this gives

$$\Pi^- = \partial_+B_- - \partial_-B_+, \quad (1.6.a)$$

$$\Pi^i = \partial_-B_i - \partial_iB_-, \quad (1.6.b)$$

$$\Pi^+ = 0, \quad (1.6.c)$$

which according to Dirac's nomenclature indicates the presence of *primary constraints* (1.6.b) and (1.6.c) because they do not determine ∂_+B_i and ∂_+B_+ . Constraint $\Pi^+ \approx 0$ is characteristic also for the gauge fields and here the component B_+ is a dependent variable which can be determined from Eq.(1.5) as⁴

$$B_+ = -\frac{1}{2\partial_-} * \left[\Pi^- - \partial_iB_i + \frac{1}{m^2}(\partial_+j^+ + \partial_-j^- + \partial_i j^i) \right]. \quad (1.7)$$

Other constraints $\Pi^i - \partial_-B_i + \partial_iB_- \approx 0$ regularly appear on LF for covariant relativistic fields and they can be ignored only after the canonical Hamiltonian is calculated

$$\begin{aligned} \mathcal{H}_B^{\text{can}} &= \Pi^- \partial_+B_- + \Pi^i \partial_+B_i - \mathcal{L}_B \\ &= \frac{1}{2}(\Pi^-)^2 + \frac{1}{4}(\partial_iB_j - \partial_jB_i)^2 + \frac{1}{2}m^2B_i^2 - B_-j^- - B_jj^i \\ &\quad + B_+ \left[-\partial_-(\Pi^- + \partial_iB_i) + (\Delta_\perp - m^2)B_- - j^+ \right]. \end{aligned} \quad (1.8)$$

We notice that the expression in the square bracket identically vanishes due to Eq.(1.4.b) which determines the field component B_-

$$B_- = \frac{1}{\Delta_\perp - m^2} * \left[\partial_-(\Pi^- + \partial_iB_i) + j^+ \right]. \quad (1.9)$$

For other dynamical field components we have the equations of motion

$$(2\partial_+\partial_- - \Delta_\perp + m^2)\Pi^- = -\partial_+j^+ + \partial_-j^-, \quad (1.10.a)$$

$$(2\partial_+\partial_- - \Delta_\perp + m^2)B_i = -\frac{\partial_i}{m^2}(\partial_+j^+ + \partial_-j^- + \partial_jj^j) + j^i, \quad (1.10.b)$$

⁴Integral operators and their convolutions which appear hereafter are introduced in Appendix B.3.

which contain the troublesome expression $\partial_+ j^+$. Here, contrary to the complete theory with fermion dynamics, we cannot impose the external current conservation for curing this problem but rather introduce new field variables

$$\Pi = \Pi^- + \frac{1}{2\partial_-} * j^+, \quad (1.11.a)$$

$$\tilde{B}_i = B_i + \frac{\partial_i}{m^2} \frac{1}{2\partial_-} * j^+, \quad (1.11.b)$$

which satisfy the dynamical equations

$$(2\partial_+ \partial_- - \Delta_\perp + m^2) \Pi = \partial_- j^- + (\Delta_\perp - m^2) \frac{1}{2\partial_-} * j^+, \quad (1.12.a)$$

$$(2\partial_+ \partial_- - \Delta_\perp + m^2) \tilde{B}_i = -\frac{\partial_i}{m^2} \left(\partial_- j^- + \partial_k j^k + (\Delta_\perp - m^2) \frac{1}{2\partial_-} * j^+ \right) + j^i. \quad (1.12.b)$$

Now the independent dynamical equations have the form of Hamilton equations and the Hamiltonian density can be rewritten in terms of these new fields

$$\begin{aligned} \mathcal{H}_{can} = & \frac{1}{2} \left(\Pi - \frac{1}{2\partial_-} * j^+ \right)^2 + \frac{1}{4} (\partial_i \tilde{B}_j - \partial_j \tilde{B}_i)^2 + \frac{1}{2} m^2 \left(\tilde{B}_i - \frac{\partial_i}{m^2} \frac{1}{2\partial_-} * j^+ \right)^2 \\ & - j^- \left[\frac{1}{\Delta_\perp - m^2} * \partial_- (\Pi + \partial_i \tilde{B}_i) - \frac{j^+}{2m^2} \right] - j^i \left(\tilde{B}_i - \frac{\partial_i}{m^2} \frac{1}{2\partial_-} * j^+ \right). \end{aligned} \quad (1.13)$$

Next, one can demand that Eqs.(1.12.a and 1.12.b) follow canonically from this Hamiltonian and this produces Dirac brackets⁵ at LF

$$2\partial_-^{\pm} \left\{ \tilde{\Pi}(x^+, \vec{x}), \tilde{\Pi}(x^+, \vec{y}) \right\}_{DB} = -(\Delta_\perp - m^2) \delta^3(\vec{x} - \vec{y}), \quad (1.14.a)$$

$$2\partial_-^{\pm} \left\{ \tilde{\Pi}(x^+, \vec{x}), \tilde{B}_i(x^+, \vec{y}) \right\}_{DB} = 0, \quad (1.14.b)$$

$$2\partial_-^{\pm} \left\{ \tilde{B}_i(x^+, \vec{x}), \tilde{B}_j(x^+, \vec{y}) \right\}_{DB} = - \left(\delta_{ij} - \frac{\partial_i \partial_j}{m^2} \right) \delta^3(\vec{x} - \vec{y}). \quad (1.14.c)$$

In this way we have found a canonical structure with Hamiltonian and brackets which is almost ready for further canonical quantization, however, before we will make this next step, we should check its consistence with the primary Lagrangian (1.3). When we functionally differentiate the primary Lagrangian with respect to the external currents j^μ , we obtain primary fields B_μ . Similarly, differentiation of the consistent canonical Hamiltonian should produce $-B_\mu$ but in this case Eq.(1.13) do not correctly reproduce the expression for B_+ (1.7). Physically this means that influence of the fermion fields on the massive vector fields cannot be reduced here to the presence of charged matter currents, but contrary, these two kinds of fields have non-vanishing mixed brackets at LF. Keeping this limitation in mind one can yet introduce *the effective Lagrangian* for vector field sector. First we parameterize the primary vector fields as

$$B_+ = \partial_+ \frac{\phi}{m} + \frac{1}{\partial_-} * \left[m\phi + \partial_i C_i + \frac{1}{\partial_-} * j^+ \right], \quad (1.15.a)$$

$$B_- = \partial_- \frac{\phi}{m}, \quad (1.15.b)$$

$$B_i = \partial_i \frac{\phi}{m} + C_i, \quad (1.15.c)$$

⁵Though in our analysis we didn't follow Dirac's procedure, we have decided to call these brackets Dirac ones in order to stress their consistency with all constraints present in the primary description of the model.

where new fields ϕ and C_i satisfy equations of motion

$$(2\partial_+\partial_- - \Delta_\perp + m^2)\phi = -\frac{1}{m}(\partial_+j_- + \partial_-j_+ - \partial_i j_i) - m\frac{1}{\partial_-} * j^+, \quad (1.16.a)$$

$$(2\partial_+\partial_- - \Delta_\perp + m^2)C_i = j^i + \partial_i \frac{1}{\partial_-} * j^+. \quad (1.16.b)$$

When this parameterization is introduced into the primary Lagrangian (1.3), we end up with the expression

$$\begin{aligned} \mathcal{L}_B^{eff} &= \partial_+ C_i \partial_- C_i - \frac{1}{2}(\partial_i C_j)^2 - \frac{m^2}{2} C_i^2 \\ &+ \partial_+ \phi \partial_- \phi - \frac{1}{2}(\partial_i \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{1}{2} \left(\frac{1}{\partial_-} * j^+ \right)^2 + \frac{\partial_- \phi}{m} j^- + \frac{\partial_+ \phi}{m} j^+ + \frac{\partial_i \phi}{m} j^i \\ &+ C_i j^i - (m\phi + \partial_i C_i) \frac{1}{\partial_-} * j^+, \end{aligned} \quad (1.17)$$

where evidently fields C_i and ϕ are independent modes. We stress that now there are no constraints in this effective Lagrangian and one can reintroduce the fermion kinetic terms and substitute external currents by the fermion currents. Next the canonical quantization leads to the same results as in [11], especially the mixed commutators for the ϕ field and the fermion fields are nonzero.

1.2 Gross-Treiman model

More than 25 years ago, Gross and Treiman have proposed a modified model for describing the interaction of massive vector fields with fermions [30]. They have been specially interested in the commutators with a smoother behaviour near the light-cone, and their model can be described by the following modified Lagrangian:

$$\begin{aligned} \mathcal{L}_{GT} &= (\partial_+ V_i - \partial_i V_+) (\partial_- V_i - \partial_i V_-) + \frac{1}{2} (\partial_+ V_- - \partial_+ V_-)^2 - \frac{1}{4} (\partial_i V_j - \partial_j V_i)^2 \\ &+ \frac{m^2}{2} \left(V_\mu - \frac{\partial_\mu \phi}{m} \right) \left(V^\mu - \frac{\partial^\mu \phi}{m} \right) + \bar{\psi} (i \partial_\mu \gamma^\mu - e V_\mu \gamma^\mu + M) \psi - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2} \phi^2, \end{aligned} \quad (1.18)$$

where another scalar field ϕ has been added with apparently incorrect signs in quadratic terms. Now we can take the sector of vector and scalar fields by omitting the fermion dynamics

$$\begin{aligned} \mathcal{L}_{GT}^{V\phi} &= (\partial_+ V_i - \partial_i V_+) (\partial_- V_i - \partial_i V_-) + \frac{1}{2} (\partial_+ V_- - \partial_+ V_-)^2 - \frac{1}{4} (\partial_i V_j - \partial_j V_i)^2 \\ &+ \frac{m^2}{2} \left(V_\mu - \frac{\partial_\mu \phi}{m} \right) \left(V^\mu - \frac{\partial^\mu \phi}{m} \right) + V_\mu j^\mu - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2} \phi^2, \end{aligned} \quad (1.19)$$

and then we change variables

$$V_\mu = B_\mu + \frac{\partial_\mu \phi}{m}. \quad (1.20)$$

We notice that the Lagrangian (1.19) splits into two parts $\mathcal{L}_{GT}^{V\phi} = \mathcal{L}_B + \mathcal{L}_\phi$, where \mathcal{L}_B is the previously analysed Lagrangian (1.3) and \mathcal{L}_ϕ describes solely negative metric scalar field

$$\mathcal{L}_\phi = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2} \phi^2 + \frac{\partial_\mu \phi}{m} j^\mu. \quad (1.21)$$

Therefore we need to analyze only this last contribution of ϕ fields here and then adopt previous results for B_μ fields. The Euler-Lagrange equations

$$(2\partial_+\partial_- - \Delta_\perp + m^2)\phi = \frac{1}{m}(\partial_+j^+ + \partial_-j^- + \partial_i j^i), \quad (1.22)$$

suggest the change of field variables

$$\phi = \tilde{\phi} + \frac{1}{2m} \frac{1}{\partial_-} * j^+, \quad (1.23)$$

which removes $\partial_+ j^+$ term from the equation of motion

$$(2\partial_+ \partial_- - \Delta_\perp + m^2) \tilde{\phi} = \frac{1}{m} \left((\Delta_\perp - m^2) \frac{1}{2\partial_-} * j^+ + \partial_- j^- + \partial_i j^i \right). \quad (1.24)$$

If we introduce the canonical momentum for the ϕ field

$$\pi_\phi = -\partial_- \phi + \frac{j^+}{m}, \quad (1.25)$$

then the canonical Hamiltonian density is

$$\begin{aligned} \mathcal{H}_\phi^{\text{can}} &= \pi_\phi \partial_+ \phi - \mathcal{L}_\phi = -\frac{1}{2} (\partial_i \phi)^2 - \frac{m^2}{2} \phi^2 + \frac{\phi}{m} (\partial_- j^- + \partial_i j^i) \\ &= -\frac{1}{2} \left(\partial_i \tilde{\phi} + \frac{\partial_i}{m} \frac{1}{2\partial_-} * j^+ \right)^2 - \frac{m^2}{2} \left(\tilde{\phi} + \frac{1}{m} \frac{1}{2\partial_-} * j^+ \right)^2 \\ &\quad + \frac{1}{m} \left(\tilde{\phi} + \frac{1}{m} \frac{1}{2\partial_-} * j^+ \right) (\partial_- j^- + \partial_i j^i), \end{aligned} \quad (1.26)$$

and we find the Dirac bracket at LF

$$2\partial_-^x \left\{ \tilde{\phi}(x^+, \vec{x}), \tilde{\phi}(x^+, \vec{y}) \right\}_{DB} = \delta^3(\vec{x} - \vec{y}). \quad (1.27)$$

At last we may add the above results to those found previously and we write down the density of effective Hamiltonian as

$$\begin{aligned} \mathcal{H}_{GT}^{\text{eff}} &= \mathcal{H}_B^{\text{eff}} + \mathcal{H}_\phi^{\text{can}} = -\frac{1}{2} (\partial_i \tilde{\phi})^2 - \frac{m^2}{2} \tilde{\phi}^2 + \frac{1}{2} \Pi^2 + \frac{1}{4} (\partial_i \tilde{B}_i - \partial_j \tilde{B}_j)^2 + \frac{m^2}{2} \tilde{B}_i^2 \\ &\quad - j^- \partial_- \left[\frac{1}{\Delta_\perp - m^2} * (\Pi + \partial_i \tilde{B}_i) + \frac{\tilde{\phi}}{m} \right] \\ &\quad - j^i \left(\tilde{B}_i + \partial_i \frac{1}{m} \tilde{\phi} \right) - j^+ \frac{1}{2\partial_-} * \left[\frac{\Delta_\perp - m^2}{m} \tilde{\phi} - \Pi + \partial_i \tilde{B}_i \right], \end{aligned} \quad (1.28)$$

and nonzero Dirac brackets at LF for all independent field variables

$$2\partial_-^x \left\{ \Pi(x^+, \vec{x}), \Pi(x^+, \vec{y}) \right\}_{DB} = (\Delta_\perp - m^2) \delta^3(\vec{x} - \vec{y}), \quad (1.29.a)$$

$$2\partial_-^x \left\{ \tilde{B}_i(x^+, \vec{x}), \tilde{B}_j(x^+, \vec{y}) \right\}_{DB} = - \left(\delta_{ij} - \frac{\partial_i \partial_j}{m^2} \right) \delta^3(\vec{x} - \vec{y}), \quad (1.29.b)$$

$$2\partial_-^x \left\{ \tilde{\phi}(\vec{x}), \tilde{\phi}(\vec{y}) \right\}_{DB} = \delta(\vec{x} - \vec{y}). \quad (1.29.c)$$

Next, we can easily check the consistency of this effective Hamiltonian with the starting point in (1.19) - in both cases currents couple linearly to V_μ fields⁶

$$V_i = B_i + \partial_i \frac{\phi}{m} = \tilde{B}_i + \partial_i \frac{\tilde{\phi}}{m}, \quad (1.30.a)$$

$$V_- = B_- + \partial_- \frac{\phi}{m} = \partial_- \left[\frac{1}{\Delta_\perp - m^2} * (\Pi + \partial_i \tilde{B}_i) + \frac{\tilde{\phi}}{m} \right], \quad (1.30.b)$$

$$V_+ = B_+ + \partial_+ \frac{\phi}{m} = \frac{1}{2\partial_-} * \left[\frac{\Delta_\perp - m^2}{m} \tilde{\phi} - \Pi + \partial_i \tilde{B}_i \right]. \quad (1.30.c)$$

⁶In the case of V_+ one needs to use the equation of motion (1.24) in order to have the term $\partial_+ \tilde{\phi}$.

From these results we expect that the above structure of brackets will survive also in the interacting theory when complete dynamics of fermions will be reintroduced. In order to achieve this aim most easily, we build here the equivalent Lagrangian

$$\begin{aligned} \bar{\mathcal{L}}_{GT}^{V\phi} &= \partial_+ \Pi \frac{1}{\Delta_\perp - m^2} * \partial_- \Pi + \partial_+ \bar{B}_i \left(\delta_{ij} - \partial_i \partial_j \frac{1}{\Delta_\perp - m^2} * \right) \partial_- \bar{B}_i \\ &\quad - \partial_+ \bar{\phi} \partial_- \bar{\phi} - \mathcal{H}_{GT}^{eff} \end{aligned} \quad (1.31)$$

which is nonlocal at LF but directly leads to Dirac brackets (1.29.a, 1.29.b, 1.29.c) and equations of motion (1.12.a, 1.12.b, 1.26) are generated as the Euler-Lagrange equations.

1.2.1 Interaction with fermion fields

Having found the effective equivalent prescription of vector and scalar fields, we can add fermion kinetic terms and study the complete Gross-Treiman model

$$\begin{aligned} \mathcal{L}_{GT}^{eff} &= \partial_+ \Pi \frac{1}{\Delta_\perp - m^2} * \partial_- \Pi + \partial_+ \bar{B}_i \left(\delta_{ij} - \partial_i \partial_j \frac{1}{\Delta_\perp - m^2} * \right) \partial_- \bar{B}_i - \partial_+ \bar{\phi} \partial_- \bar{\phi} \\ &\quad + \frac{1}{2} (\partial_i \bar{\phi})^2 + \frac{m^2}{2} \bar{\phi}^2 - \frac{1}{2} \Pi^2 - \frac{1}{4} (\partial_i \bar{B}_j - \partial_j \bar{B}_i)^2 - \frac{m^2}{2} \bar{B}_i^2 + i\sqrt{2} \psi_+^\dagger \partial_+ \psi_+ \\ &\quad + i\sqrt{2} \psi_-^\dagger \partial_- \psi_- - M \left(\psi_-^\dagger \gamma^0 \psi_+ + \psi_+^\dagger \gamma^0 \psi_- \right) - e\sqrt{2} \psi_-^\dagger \psi_- \partial_- \left[\frac{1}{\Delta_\perp - m^2} * (\Pi + \partial_i \bar{B}_i) + \frac{\bar{\phi}}{m} \right] \\ &\quad - e \left(\psi_-^\dagger \alpha^i \psi_+ + \psi_+^\dagger \alpha^i \psi_- \right) \left(\bar{B}_i + \partial_i \frac{1}{m} \bar{\phi} \right) - e\sqrt{2} \psi_+^\dagger \psi_+ \frac{1}{2\partial_-} * \left[\frac{\Delta_\perp - m^2}{m} \bar{\phi} - \Pi + \partial_i \bar{B}_i \right] \\ &\quad + \psi_-^\dagger \alpha^i \partial_i \psi_+ + \psi_+^\dagger \alpha^i \partial_i \psi_- , \end{aligned} \quad (1.32)$$

where we have introduced two components of fermion fields $\psi_\pm = \Lambda_\pm \psi^7$. The components ψ_- and ψ_-^\dagger of fermion fields satisfy the non-dynamical Euler-Lagrange equations

$$\sqrt{2} [i\partial_- - e(\partial_- \Phi)] \psi_- = \xi \equiv \left[M\gamma^0 - i\partial_i \alpha^i + e\alpha^i \left(B_i + \frac{\partial_i \bar{\phi}}{m} \right) \right] \psi_+ , \quad (1.33.a)$$

$$\sqrt{2} [-i\partial_- - e(\partial_- \Phi)] \psi_-^\dagger = \xi^\dagger \equiv \psi_+ \left[M\gamma^0 + i\bar{\partial}_i \alpha^i + e\alpha^i \left(B_i + \frac{\partial_i \bar{\phi}}{m} \right) \right] , \quad (1.33.b)$$

$$\Phi = \left[\frac{1}{\Delta_\perp - m^2} * (\Pi + \partial_i \bar{B}_i) + \frac{\bar{\phi}}{m} \right] , \quad (1.33.c)$$

where additional notations ξ , ξ^\dagger and Π are introduced for convenience and clarity. Next we can write the formal solutions for these non-dynamical fermion components,

$$\psi_- = \frac{1}{\sqrt{2}} e^{-ie\Phi} \frac{1}{i\partial_-} * \left(e^{ie\Phi} \xi \right) , \quad (1.34.a)$$

$$\psi_-^\dagger = \frac{1}{\sqrt{2}} \left(\xi^\dagger e^{-ie\Phi} \right) * \frac{1}{i\partial_-} e^{ie\Phi} . \quad (1.34.b)$$

In this way there are only dynamical fields left and their quantization is straightforward. The Hamiltonian density has the following form:

$$\begin{aligned} \mathcal{H}_{GT}^{eff} &= -\frac{1}{2} (\partial_i \bar{\phi})^2 - \frac{m^2}{2} \bar{\phi}^2 + \frac{1}{2} \Pi^2 + \frac{1}{4} (\partial_i \bar{B}_j - \partial_j \bar{B}_i)^2 + \frac{m^2}{2} \bar{B}_i^2 \\ &\quad + \frac{1}{\sqrt{2}} \xi^\dagger e^{-ie\Phi} \frac{1}{i\partial_-} * \left(e^{ie\Phi} \xi \right) + e\sqrt{2} \psi_+^\dagger \psi_+ \frac{1}{2\partial_-} * \left[\frac{\Delta_\perp - m^2}{m} \bar{\phi} - \Pi + \partial_i \bar{B}_i \right] , \end{aligned} \quad (1.35)$$

⁷See Appendix A.2 for the complete LF notation of the Dirac matrices and the projection operators Λ_\pm .

while the non-vanishing (anti)commutators at LF have the expected form

$$2\partial_-^\varepsilon \left[\Pi(x^+, \bar{x}), \Pi(x^+, \bar{y}) \right] = i(\Delta_\perp - m^2)\delta^3(\bar{x} - \bar{y}), \quad (1.36.a)$$

$$2\partial_-^\varepsilon \left[\tilde{B}_i(x^+, \bar{x}), \tilde{B}_j(x^+, \bar{y}) \right] = -i \left(\delta_{ij} - \frac{\partial_i \partial_j}{m^2} \right) \delta^3(\bar{x} - \bar{y}), \quad (1.36.b)$$

$$2\partial_-^\varepsilon \left[\tilde{\phi}(x^+, \bar{x}), \tilde{\phi}(x^+, \bar{y}) \right] = i\delta(\bar{x} - \bar{y}), \quad (1.36.c)$$

$$\left\{ \psi_+^\dagger(x^+, \bar{x}), \psi_+(x^+, \bar{y}) \right\} = \frac{1}{\sqrt{2}} \Lambda_+ \delta(\bar{x} - \bar{y}). \quad (1.36.d)$$

The quantum theory defined above is formulated in a natural way in the Heisenberg representation, where the whole dynamics of the system is connected with the quantum field operators. For perturbative calculations the interaction picture is a more convenient choice, but here we will not proceed in this direction, leaving this important issue to a more physically relevant models which will be discussed later. Rather we end up our discussion of the present model with pointing out its two interesting properties. The first one is connected with the scalar field $\tilde{\phi}$ which evidently interacts with fermion fields

$$\begin{aligned} (2\partial_+ \partial_- - \Delta_\perp + m^2)\tilde{\phi} &= -\frac{e}{m}(\Delta_\perp - m^2)\frac{1}{2\partial_-} * \left(\sqrt{2}\psi_+^\dagger \psi_+ \right) \\ &- \frac{ie\sqrt{2}}{2m} \left[\xi^\dagger e^{-ie\Phi} \frac{1}{i\partial_-} * (e^{ie\Phi} \xi) - (\xi^\dagger e^{-ie\Phi}) * \frac{1}{i\partial_-} e^{ie\Phi} \xi \right] \\ &- \frac{e\sqrt{2}}{2m} \partial_i \left[\psi_+^\dagger e^{-ie\Phi} \alpha^i \frac{1}{i\partial_-} * (e^{ie\Phi} \xi) + (\xi^\dagger e^{-ie\Phi}) * \frac{1}{i\partial_-} e^{ie\Phi} \alpha^i \psi_+ \right]. \end{aligned} \quad (1.37)$$

However, one can define another field

$$\phi = \tilde{\phi} - \frac{e}{m}\sqrt{2}\frac{1}{2\partial_-} * \left(\psi_+^\dagger \psi_+ \right) \quad (1.38)$$

which, due to equations of the fermion fields

$$\begin{aligned} i\partial_+ \psi_+ &= e\psi_+ \frac{1}{2\partial_-} * \left[\frac{\Delta_\perp - m^2}{m} \tilde{\phi} - \Pi + \partial_i \tilde{B}_i \right] \\ &+ \frac{1}{2} \left[M\gamma^0 - i\partial_i \alpha^i + e\alpha^i \left(B_i + \frac{\partial_i \tilde{\phi}}{m} \right) \right] e^{-ie\Phi} \frac{1}{i\partial_-} * (e^{ie\Phi} \xi), \end{aligned} \quad (1.39.a)$$

$$\begin{aligned} -i\partial_+ \psi_+^\dagger &= e\psi_+^\dagger \frac{1}{2\partial_-} * \left[\frac{\Delta_\perp - m^2}{m} \tilde{\phi} - \Pi + \partial_i \tilde{B}_i \right] \\ &+ \frac{1}{2} \left(\xi^\dagger e^{-ie\Phi} \right) * \frac{1}{i\partial_-} e^{ie\Phi} \left[M\gamma^0 + i\partial_i \alpha^i + e\alpha^i \left(B_i + \frac{\partial_i \tilde{\phi}}{m} \right) \right], \end{aligned} \quad (1.39.b)$$

will satisfy the free field equation of motion

$$(2\partial_+ \partial_- - \Delta_\perp + m^2)\phi = 0. \quad (1.40)$$

However, the price for such a free evolution has to be paid at the level of LF commutators where now the non-vanishing relations with ϕ are

$$2\partial_-^\varepsilon \left[\phi(x^+, \bar{x}), \phi(x^+, \bar{y}) \right] = i\delta^3(\bar{x} - \bar{y}), \quad (1.41.a)$$

$$2\partial_-^\varepsilon \left[\phi(x^+, \bar{x}), \psi_+(x^+, \bar{y}) \right] = \frac{e}{m} \psi_+(x^+, \bar{x}) \delta^3(\bar{x} - \bar{y}), \quad (1.41.b)$$

$$2\partial_-^\varepsilon \left[\phi(x^+, \bar{x}), \psi_+^\dagger(x^+, \bar{y}) \right] = -\frac{e}{m} \psi_+^\dagger(x^+, \bar{x}) \delta^3(\bar{x} - \bar{y}). \quad (1.41.c)$$

The second property is connected with the components of the vector V_μ :

$$V_i = \tilde{B}_i + \partial_i \frac{\tilde{\phi}}{m}, \quad (1.42.a)$$

$$V_- = \partial_- \left[\frac{1}{\Delta_\perp - m^2} * (\Pi + \partial_i \tilde{B}_i) + \frac{\tilde{\phi}}{m} \right], \quad (1.42.b)$$

$$V_+ = \frac{1}{2\partial_-} * \left[\frac{\Delta_\perp - m^2}{m} \tilde{\phi} - \Pi + \partial_i \tilde{B}_i \right], \quad (1.42.c)$$

which satisfy equations of motion

$$(2\partial_+ \partial_- - \Delta_\perp + m^2)V_+ = e\sqrt{2}\psi_+^\dagger \psi_+, \quad (1.43.a)$$

$$(2\partial_+ \partial_- - \Delta_\perp + m^2)V_- = \frac{e}{\sqrt{2}} \left(\xi^\dagger e^{-ie\Phi} \right) * \frac{1}{i\partial_-} \frac{1}{i\partial_-} * \left(e^{ie\Phi} \xi \right), \quad (1.43.b)$$

$$(2\partial_+ \partial_- - \Delta_\perp + m^2)V_i = \frac{e}{\sqrt{2}} \left[\psi_+^\dagger \alpha^i e^{-ie\Phi} \frac{1}{i\partial_-} * \left(e^{ie\Phi} \xi \right) + \left(e^{-ie\Phi} \xi^\dagger \right) * \frac{1}{i\partial_-} e^{ie\Phi} \alpha^i \psi_+ \right], \quad (1.43.c)$$

and have the only non-vanishing commutation relations

$$2\partial_-^2 \left[V_\mu(x^+, \vec{x}), V_\nu(x^+, \vec{y}) \right] = -ig_{\mu\nu} \delta^3(\vec{x} - \vec{y}). \quad (1.44)$$

Therefore V_μ can be interpreted as the massive vector field in the so-called Feynman gauge. We see that modification of the massive theory, which solves the problems connected with singular behaviour of fields on LF, can be interpreted in terms of gauge condition, though for non-vanishing mass $m^2 \neq 0$, this model is *not a true gauge* theory. Therefore we suggest that also other modifications by means of truly gauge fixing terms may be worth to be studied in the canonical LF formulation of massive electrodynamics.

2 Lorentz covariant gauge

In the previous section we have learned that addition of the extra scalar degrees of freedom can change LF commutators to the form proper for the Feynman gauge, which is a special case of the Lorentz covariant gauges. General covariant gauge is implemented into Lagrangian by means of the scalar Lagrange multiplier field Λ and it takes the form $\partial_\mu B^\mu = -\alpha\Lambda$. For $\alpha = 0$ it becomes the Lorentz gauge and for $\alpha = 1$ it becomes the Feynman gauge.

As previously, we start with the classical fields theory defined by the Lagrangian density

$$\begin{aligned} \mathcal{L}_{mass}^{cov} &= (\partial_+ B_i - \partial_i B_+) (\partial_- B_i - \partial_i B_-) + \frac{1}{2} (\partial_+ B_- - \partial_- B_+)^2 - \frac{1}{4} (\partial_i B_j - \partial_j B_i)^2 \\ &+ m^2 \left(B_- B_+ - \frac{1}{2} B_i^2 \right) + \bar{\psi} (i\gamma^\mu \partial_\mu - e\gamma^\mu B_\mu - M) \psi \\ &+ \Lambda (\partial_+ B_- + \partial_- B_+ - \partial_i B_i) + \frac{\alpha}{2} \Lambda^2. \end{aligned} \quad (2.1)$$

We expect that the canonical structure of vector fields B_μ is independent of the fermion fields, therefore we will study the sector of vector fields with external arbitrary currents j^μ first.

2.1 Vector field sector

Omitting the fermion kinetic terms in (2.1) and inserting j^μ in the place of $e\bar{\psi}\gamma\psi$ we obtain

$$\begin{aligned} \mathcal{L}_{jmass}^{\text{cov}} &= (\partial_+ B_i - \partial_i B_+) (\partial_- B_i - \partial_i B_-) + \frac{1}{2} (\partial_+ B_- - \partial_- B_+)^2 - \frac{1}{4} (\partial_i B_j - \partial_j B_i)^2 \\ &+ m^2 \left(B_- B_+ - \frac{1}{2} B_i^2 \right) + B_\mu j^\mu + \Lambda (\partial_+ B_- + \partial_- B_+ - \partial_i B_i) + \frac{\alpha}{2} \Lambda^2, \end{aligned} \quad (2.2)$$

and next we generate the Euler-Lagrange equations for all field variables

$$(2\partial_+ \partial_- - \Delta_\perp + m^2) B_- = (1 - \alpha) \partial_- \Lambda - j^+, \quad (2.3.a)$$

$$(2\partial_+ \partial_- - \Delta_\perp + m^2) B_+ = (1 - \alpha) \partial_+ \Lambda - j^-, \quad (2.3.b)$$

$$(2\partial_+ \partial_- - \Delta_\perp + m^2) B_i = (1 - \alpha) \partial_i \Lambda + j^i, \quad (2.3.c)$$

$$\partial_+ B_- = -\partial_- B_+ + \partial_i B_i - \alpha \Lambda. \quad (2.3.d)$$

The consistency condition for these equations has the form of a dynamical equation for Λ

$$(2\partial_+ \partial_- - \Delta_\perp + \alpha m^2) \Lambda = \partial_+ j^+ + \partial_- j^- + \partial_i j^i, \quad (2.4)$$

and we see that in order to have no imaginary mass tachyon, we must keep the parameter $\alpha \geq 0$. Just like in the previous section, we have two different equations with $\partial_+ B_-$ (2.3.a) and (2.3.d) so there is a constraint

$$2\partial_- (\partial_i B_j - \alpha \Lambda - \partial_- B_+) = (\Delta_\perp - m^2) B_- + (1 - \alpha) \partial_- \Lambda - j^+. \quad (2.5)$$

If we define the canonical momentum conjugated to the field B_-

$$\Pi^- = \partial_+ B_- - \partial_- B_+ + \Lambda \quad (2.6)$$

then, using the gauge condition (2.3.d) we obtain another constraint

$$2\partial_- B_+ = \partial_i B_i - \Pi^- + (1 - \alpha) \Lambda. \quad (2.7)$$

In the dynamical equation for Λ (2.4) there is the $\partial_+ j^+$ term and in order to remove it, we redefine the Lagrange multiplier field

$$\lambda = \Lambda - \frac{1}{2\partial_-} * j^+. \quad (2.8)$$

In this manner we have selected the set of fields (Π^-, B_i, λ) which satisfy dynamical equations of motion

$$(2\partial_+ \partial_- - \Delta_\perp + \alpha m^2) \lambda = (\Delta_\perp - \alpha m^2) \frac{1}{2\partial_-} * j^+ + \partial_- j^- + \partial_i j^i, \quad (2.9.a)$$

$$(2\partial_+ \partial_- - \Delta_\perp + m^2) \Pi^- = m^2 (1 - \alpha) \left(\lambda + \frac{1}{2\partial_-} * j^+ \right) + 2\partial_- j^- + \partial_i j^i, \quad (2.9.b)$$

$$(2\partial_+ \partial_- - \Delta_\perp + m^2) B_i = (1 - \alpha) \partial_i \lambda + j^i + \frac{1 - \alpha}{2} \frac{1}{\partial_-} * \partial_i j^+, \quad (2.9.c)$$

so they are independent canonical variables. Also we have dependent fields B_- and B_+ :

$$B_- = \partial_- \frac{1}{\Delta_\perp - m^2} * (\Pi^- + \partial_i B_i - 2\lambda), \quad (2.10.a)$$

$$B_+ = \frac{1}{2\partial_-} * \left[\partial_i B_i - \Pi^- + (1 - \alpha) \lambda + \frac{1 - \alpha}{2} \frac{1}{\partial_-} * j^+ \right]. \quad (2.10.b)$$

Though the other canonical momenta apparently represent primary constraints (according to Dirac's nomenclature)

$$\Pi^i(x) - \partial_- A_i(x) + \partial_i A_-(x) \approx 0, \quad (2.11.a)$$

$$\Pi^+(x) \approx 0, \quad (2.11.b)$$

$$\Pi_\lambda(x) \approx 0, \quad (2.11.c)$$

we notice that they are absent from the canonical Hamiltonian density⁸

$$\begin{aligned} \mathcal{H}_{can} = & (\partial_+ B_-) \Pi^- + (\partial_+ B_i) \Pi^i - \mathcal{L} = \frac{1}{2} \left(\Pi^- - \lambda - \frac{1}{2\partial_-} * j^+ \right)^2 + \frac{1}{4} (\partial_i B_j - \partial_j B_i)^2 + \frac{m^2}{2} B_i^2 \\ & + \left(\lambda + \frac{1}{2\partial_-} * j^+ \right) \partial_i B_i - j^- \frac{1}{\Delta_\perp - m^2} * \partial_- (\Pi^- + \partial_i B_i - 2\lambda) - B_i j^i \\ & - \frac{\alpha}{2} \left(\lambda + \frac{1}{2\partial_-} * j^+ \right)^2. \end{aligned} \quad (2.12)$$

Next we construct Eqs. (2.9.a), (2.9.b), (2.9.c) as Hamilton equations by imposing the following Dirac brackets⁹ on independent variables at LF:

$$2\partial_-^\mp \{B_i(\vec{x}), \Pi^-(\vec{y})\}_{DB} = \partial_i^\mp \delta^3(\vec{x} - \vec{y}), \quad (2.13.a)$$

$$2\partial_-^\mp \{\Pi^-(\vec{x}), \Pi^-(\vec{y})\}_{DB} = \Delta_\perp \delta^3(\vec{x} - \vec{y}), \quad (2.13.b)$$

$$2\partial_-^\mp \{B_i(\vec{x}), B_j(\vec{y})\}_{DB} = -\delta_{ij} \delta^3(\vec{x} - \vec{y}), \quad (2.13.c)$$

$$2\partial_-^\mp \{\lambda(\vec{x}), B_j(\vec{y})\}_{DB} = -\partial_i^\mp \delta^3(\vec{x} - \vec{y}), \quad (2.13.d)$$

$$2\partial_-^\mp \{\lambda(\vec{x}), \lambda(\vec{y})\}_{DB} = m^2 \delta^3(\vec{x} - \vec{y}), \quad (2.13.e)$$

$$2\partial_-^\mp \{\lambda(\vec{x}), \Pi^-(\vec{y})\}_{DB} = m^2 \delta^3(\vec{x} - \vec{y}), \quad (2.13.f)$$

while all other brackets vanish. The structure of these brackets evidently indicates that our independent canonical variables are not independent modes yet. Therefore as independent modes we propose to take the following linear but nonlocal combinations of fields:

$$Q = \frac{\lambda}{m}, \quad (2.14.a)$$

$$\varphi = \frac{m}{\Delta_\perp - m^2} * (\Pi^- - 2\lambda + \partial_i B_i) - \frac{1}{m} \lambda, \quad (2.14.b)$$

$$C_i = B_i - \partial_i \frac{1}{\Delta_\perp - m^2} * (\Pi^- - 2\lambda + \partial_j B_j). \quad (2.14.c)$$

They satisfy the equations of motion

$$(2\partial_+ \partial_- - \Delta_\perp + m^2) \varphi = -\frac{\Delta_\perp + m^2}{m} \frac{1}{2\partial_-} * j^+ - \frac{1}{m} (\partial_- j^- + \partial_i j^i), \quad (2.15.a)$$

$$(2\partial_+ \partial_- - \Delta_\perp + \alpha m^2) Q = \frac{\Delta_\perp - \alpha m^2}{m} \frac{1}{2\partial_-} * j^+ + \frac{1}{m} (\partial_- j^- + \partial_i j^i), \quad (2.15.b)$$

$$(2\partial_+ \partial_- - \Delta_\perp + m^2) C_i = j^i + \partial_i \frac{1}{\partial_-} * j^+, \quad (2.15.c)$$

⁸We have used relations (2.10.a) and (2.10.b) to remove dependent variables from the Hamiltonian.

⁹Though we have not used Dirac's procedure of quantization, these brackets are evidently consistent with all constraints, therefore we call them *Dirac brackets*.

and their Dirac brackets have a diagonal form

$$2\partial_-^x \{C_i(\vec{x}), C_j(\vec{y})\}_{DB} = -\delta_{ij}\delta^3(\vec{x} - \vec{y}), \quad (2.16.a)$$

$$2\partial_-^x \{\varphi(\vec{x}), \varphi(\vec{y})\}_{DB} = -\delta^3(\vec{x} - \vec{y}), \quad (2.16.b)$$

$$2\partial_-^x \{Q(\vec{x}), Q(\vec{y})\}_{DB} = \delta^3(\vec{x} - \vec{y}). \quad (2.16.c)$$

The Hamiltonian density, when expressed in terms of the independent modes, looks like

$$\begin{aligned} \mathcal{H}_{jcov}^{eff} &= \frac{1-\alpha}{2} \left(mQ + \frac{1}{2\partial_-} * j^+ \right)^2 + \frac{1}{2} \left(\partial_i C_i + \frac{1}{\partial_-} * j^+ \right)^2 + \frac{1}{4} (\partial_i C_j - \partial_j C_i)^2 + \frac{m^2}{2} C_i^2 \\ &- \frac{1}{2} (\partial_i Q)^2 - \frac{m^2}{2} Q^2 + \frac{1}{2} (\partial_i \varphi)^2 + \frac{m^2}{2} \varphi^2 + \frac{\varphi + Q}{m} (\partial_- j^- + \partial_i j^i) - j^i C_i \\ &+ \left[\frac{\Delta_\perp - m^2}{m} Q + \frac{\Delta_\perp + m^2}{m} \varphi - \frac{1}{\partial_-} * j^+ \right] \frac{1}{2\partial_-} * j^+ \end{aligned} \quad (2.17)$$

and, in spite of the presence of terms quadratic in currents j^+ , it describes the same physical system as the primary Lagrangian with constraints. This allows us to believe that the canonical structure for independent modes of vector fields, that we have just formulated, will survive in the complete theory with fermion fields. In order to incorporate our hitherto obtained results into the full interacting theory, we construct the effective Lagrangian density by means of another Legendre transformation

$$\begin{aligned} \mathcal{L}_{eff}^{jmass} &= \partial_- C_i \partial_+ C_i - \partial_+ Q \partial_- Q + \partial_- \varphi \partial_+ \varphi - \mathcal{H}_{jcov}^{eff} = \\ &= \partial_- C_i \partial_+ C_i - \partial_+ Q \partial_- Q + \partial_- \varphi \partial_+ \varphi - \frac{1-\alpha}{2} \left(mQ + \frac{1}{2\partial_-} * j^+ \right)^2 - \frac{1}{2} \left(\partial_i C_i + \frac{1}{\partial_-} * j^+ \right)^2 \\ &- \frac{1}{4} (\partial_i C_j - \partial_j C_i)^2 - \frac{m^2}{2} C_i^2 + \frac{1}{2} (\partial_i Q)^2 + \frac{m^2}{2} Q^2 - \frac{1}{2} (\partial_i \varphi)^2 - \frac{m^2}{2} \varphi^2 + j^i C_i \\ &- \left[\frac{\Delta_\perp - m^2}{m} Q + \frac{\Delta_\perp + m^2}{m} \varphi - \frac{1}{\partial_-} * j^+ \right] \frac{1}{2\partial_-} * j^+ - \frac{\varphi + Q}{m} (\partial_- j^- + \partial_i j^i). \end{aligned} \quad (2.18)$$

One can easily check that the brackets (2.16.a-2.16.c) and the equations of motion (2.15.a-2.15.c) will follow directly from this effective Lagrangian within the canonical procedure without constraints¹⁰

2.2 Interactions with fermions

Our next step, from the effective Lagrangian for the gauge sector to the full interacting theory with fermions, can be done easily by inserting fermion currents instead of j^μ and addition of the fermion kinetic terms

$$\begin{aligned} \mathcal{L}_{eff}^{QED} &= \partial_- C_i \partial_+ C_i - \partial_+ Q \partial_- Q + \partial_- \varphi \partial_+ \varphi + i\sqrt{2}\psi_+^\dagger \partial_+ \psi_+ + \sqrt{2}\psi_-^\dagger \left[i\partial_- - e\partial_- \left(\frac{\varphi + Q}{m} \right) \right] \psi_- \\ &- \frac{1-\alpha}{2} \left(mQ - \frac{1}{2\partial_-} * J^+ \right)^2 - \frac{1}{2} \left(\partial_i C_i - \frac{1}{\partial_-} * J^+ \right)^2 \\ &- \frac{1}{4} (\partial_i C_j - \partial_j C_i)^2 - \frac{m^2}{2} C_i^2 + \frac{1}{2} (\partial_i Q)^2 + \frac{m^2}{2} Q^2 - \frac{1}{2} (\partial_i \varphi)^2 - \frac{m^2}{2} \varphi^2 \\ &+ \left[\frac{\Delta_\perp - m^2}{m} Q + \frac{\Delta_\perp + m^2}{m} \varphi + \frac{1}{\partial_-} * J^+ \right] \frac{1}{2\partial_-} * J^+ - \xi^\dagger \psi_- - \psi_-^\dagger \xi, \end{aligned} \quad (2.19)$$

¹⁰Strictly speaking there would be trivial primary constraints which usually appear for covariant relativistic fields at LF.

where

$$\xi = (-i\partial_i\alpha^i + M\beta)\psi_+ + e\left[C_i + \partial_i\left(\frac{\varphi+Q}{m}\right)\right]\alpha^i\psi_+, \quad (2.20.a)$$

$$\xi^\dagger = (i\partial_i\psi_+^\dagger\alpha^i + M\psi_+^\dagger\beta) + e\psi_+^\dagger\alpha^i\left[C_i + \partial_i\left(\frac{\varphi+Q}{m}\right)\right], \quad (2.20.b)$$

$$J^+ = e\sqrt{2}\psi_+^\dagger\psi_+. \quad (2.20.c)$$

As usually in the LF formulation, ψ_- and ψ_-^\dagger fermions are non-dynamical and can be expressed in terms of dynamical fermions ψ_+ and ψ_+^\dagger

$$\psi_- = \frac{1}{\sqrt{2}} \frac{1}{i\partial_- - e\partial_- \left(\frac{\varphi+Q}{m}\right)} * \xi, \quad (2.21.a)$$

$$\psi_-^\dagger = \frac{1}{\sqrt{2}} \xi^\dagger * \frac{1}{i\partial_- - e\partial_- \left(\frac{\varphi+Q}{m}\right)}, \quad (2.21.b)$$

and one obtains the nonlocal Hamiltonian density expressed solely in terms of dynamical independent modes

$$\begin{aligned} \mathcal{H}_{total} &= \frac{1-\alpha}{2} \left(mQ - \frac{1}{2\partial_-} * J^+\right)^2 + \frac{1}{2} \left(\partial_i C_i - \frac{1}{\partial_-} * J^+\right)^2 + \frac{1}{4} (\partial_i C_j - \partial_j C_i)^2 + \frac{m^2}{2} C_i^2 \\ &- \frac{1}{2} (\partial_i Q)^2 - \frac{m^2}{2} Q^2 + \frac{1}{2} (\partial_i \varphi)^2 + \frac{m^2}{2} \varphi^2 + \frac{1}{\sqrt{2}} \xi^\dagger \frac{1}{i\partial_- - e\partial_- \left(\frac{\varphi+Q}{m}\right)} * \xi \\ &- \left[\frac{\Delta_\perp - m^2}{m} Q + \frac{\Delta_\perp + m^2}{m} \varphi + \frac{1}{\partial_-} * J^+ \right] \frac{1}{2\partial_-} * J^+. \end{aligned} \quad (2.22)$$

As we have expected, the non-vanishing equal x^+ (anti)commutators have the forms compatible with the former vector field brackets

$$2\partial_-^\pm [C_i(\bar{x}), C_j(\bar{y})] = -i\delta_{ij}\delta^3(\bar{x} - \bar{y}), \quad (2.23.a)$$

$$2\partial_-^\pm [\varphi(\bar{x}), \varphi(\bar{y})] = -i\delta^3(\bar{x} - \bar{y}), \quad (2.23.b)$$

$$2\partial_-^\pm [Q(\bar{x}), Q(\bar{y})] = i\delta^3(\bar{x} - \bar{y}), \quad (2.23.c)$$

$$\left\{ \psi_+^\dagger(\bar{x}), \psi_+(\bar{y}) \right\} = \frac{1}{\sqrt{2}} \Lambda + \delta^3(\bar{x} - \bar{y}), \quad (2.23.d)$$

and, just like in the ET approach to the covariant gauge condition, there are negative metric excitations here.¹¹ From these relations one can derive the dynamical equations for the interacting system¹²

$$\begin{aligned} (2\partial_+\partial_- - \Delta_\perp + m^2)\varphi &= +\frac{\sqrt{2}e}{2m}\partial_- \left[\xi^\dagger * \frac{1}{i\partial_- - e\left(\frac{\varphi+Q}{m}\right)} \frac{1}{i\partial_- - e\left(\frac{\varphi+Q}{m}\right)} * \xi \right] \\ &+ \frac{\sqrt{2}e}{2m}\partial_i \left[\psi_+^\dagger\alpha^i \frac{1}{i\partial_- - e\left(\frac{\varphi+Q}{m}\right)} * \xi + \xi^\dagger * \frac{1}{i\partial_- - e\left(\frac{\varphi+Q}{m}\right)} \alpha^i\psi_+ \right] \end{aligned}$$

¹¹Opposite signs in (2.23.b) and (2.23.c) allow the combination of fields $\varphi+Q$ to commute with itself on LF and this simplifies considerably the consistent definition of the integral operator $[i\partial_- - e\partial_- \left(\frac{\varphi+Q}{m}\right)]^{-1}$.

¹²Here we do not discuss explicitly the ordering problems and the proper definition of singular products for noncommuting operators. These very important aspects of the complete definition of Quantum Field Theory remain out of the scope of this work where we are ultimately interested in the Feynman rules for perturbative calculations.

$$+ \frac{\Delta_{\perp} + m^2}{m} \frac{1}{2\partial_{-}} * J^{+}, \quad (2.24.a)$$

$$(2\hat{\omega}_{-}\partial_{-} - \Delta_{\perp} + \alpha m^2)Q = -\frac{\sqrt{2}e}{2m}\partial_{-} \left[\xi^{\dagger} * \frac{1}{i\partial_{-} - e\left(\frac{\varphi+Q}{m}\right)} \frac{1}{i\partial_{-} - e\left(\frac{\varphi+Q}{m}\right)} * \xi \right] \\ - \frac{\sqrt{2}e}{2m}\partial_i \left[\psi_{+}^{\dagger} \alpha^i \frac{1}{i\partial_{-} - e\left(\frac{\varphi+Q}{m}\right)} * \xi + \xi^{\dagger} * \frac{1}{i\partial_{-} - e\left(\frac{\varphi+Q}{m}\right)} \alpha^i \psi_{+} \right] \\ - \frac{\Delta_{\perp} - \alpha m^2}{m} \frac{1}{2\partial_{-}} * J^{+}, \quad (2.24.b)$$

$$(2)_{+}\partial_{-} - \Delta_{\perp} + m^2)C_i = -\frac{\sqrt{2}e}{2m} \left[\psi_{+}^{\dagger} \alpha^i \frac{1}{i\partial_{-} - e\left(\frac{\varphi+Q}{m}\right)} * \xi + \xi^{\dagger} * \frac{1}{i\partial_{-} - e\left(\frac{\varphi+Q}{m}\right)} \alpha^i \psi_{+} \right] \\ - \partial_i \frac{1}{\partial_{-}} * J^{+}, \quad (2.24.c)$$

$$i\sqrt{2}\partial_{+}\psi = \frac{1}{\sqrt{2}} \left[-i\partial_i \alpha^i + M\beta + e\alpha^i C_i + \alpha^i \partial_i \left(\frac{\varphi+Q}{m} \right) \right] \frac{1}{i\partial_{-} - e\partial_{-}\left(\frac{\varphi+Q}{m}\right)} * \xi \\ + \frac{e\sqrt{2}}{2} \frac{1}{\partial_{-}} * \left[\frac{\Delta_{\perp} - \alpha m^2}{m} Q + \frac{\Delta_{\perp} + m^2}{m} \varphi + \partial_i C_i - (1-\alpha) \frac{1}{\partial_{-}} * J^{+} \right] \psi_{+}, \quad (2.24.d)$$

$$-i\sqrt{2}\partial_{+}\psi^{\dagger} = \frac{1}{\sqrt{2}} \xi^{\dagger} * \frac{1}{i\partial_{-} - e\partial_{-}\left(\frac{\varphi+Q}{m}\right)} \left[i\bar{\partial}_i \alpha^i + M\beta + e\alpha^i C_i + \alpha^i \partial_i \left(\frac{\varphi+Q}{m} \right) \right] \\ + \frac{e\sqrt{2}}{2} \psi_{+}^{\dagger} \frac{1}{\partial_{-}} * \left[\frac{\Delta_{\perp} - \alpha m^2}{m} Q + \frac{\Delta_{\perp} + m^2}{m} \varphi + \partial_i C_i - (1-\alpha) \frac{1}{\partial_{-}} * J^{+} \right]. \quad (2.24.e)$$

These equations show that one can introduce another quantum field Λ

$$\Lambda = mQ + e \frac{1}{2\partial_{-}} * J^{+}, \quad (2.25)$$

which satisfies the free field equation

$$(2\partial_{+}\partial_{-} - \Delta_{\perp} + \alpha m^2)\Lambda = 0, \quad (2.26)$$

but at the price of having extra non-vanishing commutators on LF

$$[\psi_{+}(\vec{x}), \Lambda(\vec{y})] = e\delta^3(\vec{x} - \vec{y})\psi_{+}(\vec{x}), \quad (2.27.a)$$

$$[\psi_{+}^{\dagger}(\vec{x}), \Lambda(\vec{y})] = -e\delta^3(\vec{x} - \vec{y})\psi_{+}^{\dagger}(\vec{x}). \quad (2.27.b)$$

This alternative, of having either the non-free field Q which commutes with fermions or the free field Λ which satisfies (2.27.a) and (2.27.b), is a specific feature of the LF formulation - contrary to the ET case where Λ is the only opportunity.

The mere presence of q-number commutators would destroy a simple picture of perturbative calculations based on Feynman diagrams and Wick contractions, therefore in the next subsection we will build the perturbation theory taking Q as an independent field and going to the interaction picture, where only c-number terms will appear as contractions for all fields.

2.3 Perturbation theory

The perturbative calculations of the S-matrix elements are most easily performed in the interaction picture where all quantum field operators have free dynamics while interactions appear in the evolution of quantum states. Below we will work in this representation; however for clarity we will omit subscripts I for the field operators. Here the interaction representation is defined by taking the free Hamiltonian H_0 as the limit of the total Hamiltonian (2.22)

$$\begin{aligned} \mathcal{H}_0 = \lim_{\epsilon \rightarrow 0} \mathcal{H}_{total} &= \frac{1}{2}(\partial_i C_i)^2 + \frac{1}{4}(\partial_i C_j - \partial_j C_i)^2 + \frac{m^2}{2}C_i^2 - \frac{1}{2}(\partial_i Q)^2 - \alpha \frac{m^2}{2}Q^2 \\ &+ \frac{1}{2}(\partial_i \varphi)^2 + \frac{m^2}{2}\varphi^2 + \frac{1}{\sqrt{2}}\xi_0^\dagger \frac{1}{i\partial_-} * \xi_0, \end{aligned} \quad (2.28)$$

where

$$\xi_0 = (-i\partial_i \alpha^i + M\beta) \psi_+, \quad (2.29.a)$$

$$\xi_0^\dagger = (i\partial_i \psi_+^\dagger \alpha^i + M\psi_+^\dagger \beta). \quad (2.29.b)$$

Next the interaction Hamiltonian, which is defined as the difference of these two Hamiltonians, can be written, for the later convenience, as a sum of two contributions

$$\mathcal{H}_{int} = \mathcal{H}_{total} - \mathcal{H}_{int} = \mathcal{H}_{int}^1 + \mathcal{H}_{int}^2. \quad (2.30)$$

The first part describes the interaction of fermion and LF spatial components of the vector field

$$\begin{aligned} \mathcal{H}_{int}^1 &= \frac{1}{\sqrt{2}}\xi_0^\dagger \left(\frac{1}{i\partial_- - \epsilon\partial_- \phi} - \frac{1}{i\partial_-} \right) * \xi_0 \\ &+ \frac{e}{\sqrt{2}}\psi_+^\dagger \alpha^i (C_i + \partial_i \phi) \frac{1}{i\partial_- - \epsilon\partial_- \phi} * \xi_0 + \frac{e}{\sqrt{2}}\xi_0^\dagger \frac{1}{i\partial_- - \epsilon\partial_- \phi} * \alpha^i (C_i + \partial_i \phi) \psi_+ \\ &+ \frac{e^2}{\sqrt{2}}\psi_+^\dagger \alpha^j (C_j + \partial_j \phi) \frac{1}{i\partial_- - \epsilon\partial_- \phi} * \alpha^i (C_i + \partial_i \phi) \psi_+, \end{aligned} \quad (2.31.a)$$

while the second one is connected with the current J^+

$$\mathcal{H}_{int}^2 = -(2\partial_i C_i + m\varphi - \alpha m Q + \Delta_\perp \phi) \frac{1}{2\partial_-} * J^+ - \frac{1-\alpha}{8} J^+ \frac{1}{\partial_-^2} * J^+, \quad (2.31.b)$$

where we have introduced another notation

$$\phi = \frac{\varphi + Q}{m}. \quad (2.31.c)$$

2.3.1 Field operators in the interaction representation

From the free Hamiltonian (2.28) and the equal- x^+ (anti)commutation relations one derives the free field equations

$$(2\partial_+ \partial_- - \Delta_\perp + m^2)C_i = 0, \quad (2.32.a)$$

$$(2\partial_+ \partial_- - \Delta_\perp + m^2)\varphi = 0, \quad (2.32.b)$$

$$(2\partial_+ \partial_- - \Delta_\perp + \alpha m^2)Q = 0, \quad (2.32.c)$$

$$(2\partial_+ \partial_- - \Delta_\perp + M^2)\psi_+ = 0, \quad (2.32.d)$$

$$(2\partial_+ \partial_- - \Delta_\perp + M^2)\psi_+^\dagger = 0. \quad (2.32.e)$$

Such free fields have their Fourier representations for all x^+ and for vector field modes¹³ we write

$$C_i(x) = \int_{-\infty}^{\infty} \frac{d^2 k_{\perp}}{(2\pi)^2} \int_0^{\infty} \frac{dk_-}{4\pi k_-} \left[e^{-ik \cdot x} c_i(\vec{k}) + e^{+ik \cdot x} c_i^{\dagger}(\vec{k}) \right]_{k_+ = \frac{k_{\perp}^2 + m^2}{2k_-}}, \quad (2.33.a)$$

$$Q(x) = \int_{-\infty}^{\infty} \frac{d^2 k_{\perp}}{(2\pi)^2} \int_0^{\infty} \frac{dk_-}{4\pi k_-} \left[e^{-ik \cdot x} q(\vec{k}) + e^{+ik \cdot x} q^{\dagger}(\vec{k}) \right]_{k_+ = \frac{k_{\perp}^2 + \alpha m^2}{2k_-}}, \quad (2.33.b)$$

$$\varphi(x) = \int_{-\infty}^{\infty} \frac{d^2 k_{\perp}}{(2\pi)^2} \int_0^{\infty} \frac{dk_-}{4\pi k_-} \left[e^{-ik \cdot x} p(\vec{k}) + e^{+ik \cdot x} p^{\dagger}(\vec{k}) \right]_{k_+ = \frac{k_{\perp}^2 + m^2}{2k_-}}, \quad (2.33.c)$$

where the creation and annihilation operators have non-vanishing commutators

$$[q(\vec{k}), q^{\dagger}(\vec{k}')] = -[p(\vec{k}), p^{\dagger}(\vec{k}')] = (2\pi)^3 2k_- \delta^3(\vec{k} - \vec{k}'), \quad (2.34.a)$$

$$[c_i(\vec{k}), c_j^{\dagger}(\vec{k}')] = (2\pi)^3 2k_- \delta_{ij} \delta^3(\vec{k} - \vec{k}'). \quad (2.34.b)$$

Now it is an easy exercise to calculate the chronological (in x^+) products of these free field operators

$$\langle 0 | T^+ C_i(x) C_j(y) | 0 \rangle = \delta_{ij} \Delta_F(x - y, m^2), \quad (2.35.a)$$

$$\langle 0 | T^+ \varphi(x) \varphi(y) | 0 \rangle = \Delta_F(x - y, m^2), \quad (2.35.b)$$

$$\langle 0 | T^+ Q(x) Q(y) | 0 \rangle = -\Delta_F(x - y, \alpha m^2), \quad (2.35.c)$$

where the covariant Feynman massive propagator function $\Delta_F(x, \mu^2)$ is defined in Appendix B.1. In the interaction Hamiltonian, the linear combinations of independent vector modes

$$\bar{B}_+ = \frac{1}{2\partial_-} * [2\partial_- C_i + m\varphi - \alpha m Q + \Delta_{\perp} \phi], \quad (2.36.a)$$

$$\bar{B}_- = \partial_- \phi, \quad (2.36.b)$$

$$\bar{B}_i = C_i(x) + \partial_i \phi, \quad (2.36.c)$$

are coupled with the fermion currents, therefore in the Dyson-Wick perturbation procedure we effectively encounter contractions given by the chronological products of \bar{B}_{μ} . Now after some algebra we find

$$\begin{aligned} \langle 0 | T \bar{B}_{\mu}(x) \bar{B}_{\nu}(y) | 0 \rangle &= \left[-g_{\mu\nu} \Delta_F(x - y, m^2) + \partial_{\mu}^{\tau} \partial_{\nu}^{\tau} \frac{\Delta_F(x - y, m^2) - \Delta_F(x - y, \alpha m^2)}{m^2} \right] \\ &+ i g_{\mu-} g_{\nu-} \frac{1 - \alpha}{4} \frac{1}{(\partial_-)^2} * \delta(x - y), \end{aligned} \quad (2.37)$$

where the last non-covariant contribution arises when one uses the equation

$$(2\partial_+ \partial_- - \Delta_{\perp} + \mu^2) \Delta_F(x, \mu^2) = -i\delta^4(x) \quad (2.38)$$

in transforming the result into the form with second derivatives ∂_+ of the propagator functions.

In later discussion it will be very convenient to reintroduce the dependent fermion fields

$$\psi_- = \frac{1}{\sqrt{2}} \frac{1}{i\partial_-} * (-i\partial_i \alpha^i + M\beta) \psi_+, \quad (2.39.a)$$

$$\psi_-^{\dagger} = \frac{1}{\sqrt{2}} \left(i\partial_i \psi_+^{\dagger} \alpha^i + M \psi_+^{\dagger} \beta \right) * \frac{1}{i\partial_-}, \quad (2.39.b)$$

¹³The case of free fermion fields was given by Yan [10] therefore we will omit them here and only quote all the needed results.

and consider the complete spinor fields $\psi = \psi_+ + \psi_-$ and $\psi^\dagger = \psi_+^\dagger + \psi_-^\dagger$. Taking the well known chronological product for independent fermions [10]

$$\left\langle 0 \left| T \psi_+(x) \psi_+^\dagger(y) \right| 0 \right\rangle = i\sqrt{2}\Lambda_+ \partial_x^\tau \Delta_F(x-y, M^2), \quad (2.40)$$

we derive the chronological products for dependent fermion fields

$$\left\langle 0 \left| T^+ \psi_-(x) \psi_+^\dagger(y) \right| 0 \right\rangle + \left\langle 0 \left| T \psi_+(x) \psi_-^\dagger(y) \right| 0 \right\rangle = (i\partial_x^\tau \gamma^i + M) \gamma^0 \Delta_F(x-y, M^2), \quad (2.40.a)$$

$$\left\langle 0 \left| T^+ \psi_-(x) \psi_-^\dagger(y) \right| 0 \right\rangle = i\partial_x^\tau \gamma^+ \gamma^0 \Delta_F(x-y, M^2) - \gamma^+ \gamma^0 \frac{1}{2\partial_-} * \delta(x-y), \quad (2.40.b)$$

and finally for the complete fermion fields

$$\left\langle 0 \left| T^+ \psi(x) \psi^\dagger(y) \right| 0 \right\rangle = (i\partial_\mu^\tau \gamma^\mu + M) \gamma^0 \Delta_F(x-y, M^2) - \gamma^+ \gamma^0 \frac{1}{2\partial_-} * \delta(x-y), \quad (2.41.a)$$

or

$$\left\langle 0 \left| T \psi(x) \bar{\psi}(y) \right| 0 \right\rangle = (i\partial_\mu^\tau \gamma^\mu + M) \Delta_F(x-y, M^2) - \gamma^+ \frac{1}{2\partial_-} * \delta(x-y). \quad (2.41.b)$$

Now we may re-express the interaction Hamiltonian in terms of \bar{B}_μ , ψ and ψ^\dagger . First we take \mathcal{H}_{int}^1 and, using the identity

$$\frac{1}{i\partial_- - e\bar{B}_-} - \frac{1}{i\partial_-} = \left(\frac{1}{i\partial_- - e\bar{B}_-} e\bar{B}_- \right) * \frac{1}{i\partial_-} = \frac{1}{i\partial_-} * \left(e\bar{B}_- \frac{1}{i\partial_- - e\bar{B}_-} \right) \quad (2.42)$$

we write it as

$$\begin{aligned} H_{int}^1 &= \frac{1}{\sqrt{2}} \xi_0^\dagger * \frac{1}{i\partial_-} e\bar{B}_- * \left(1 + \frac{1}{i\partial_- - e\bar{B}_-} e\bar{B}_- \right) * \frac{1}{i\partial_-} * \xi_0 \\ &+ \frac{e}{\sqrt{2}} \psi_+^\dagger \alpha^i \bar{B}_i * \left(1 + \frac{1}{i\partial_- - e\bar{B}_-} e\bar{B}_- \right) * \frac{1}{i\partial_-} * \xi_0 \\ &+ \frac{e}{\sqrt{2}} \xi_0^\dagger * \frac{1}{i\partial_-} * \left(1 + \frac{1}{i\partial_- - e\bar{B}_-} e\bar{B}_- \right) * \alpha^i \bar{B}_i \psi_+ \\ &+ \frac{e}{\sqrt{2}} \psi_+^\dagger \alpha^i \bar{B}_i * \frac{1}{i\partial_- - e\bar{B}_-} * \alpha^j \bar{B}_j \psi_+. \end{aligned} \quad (2.43)$$

Then ξ_0 and ξ_0^\dagger can be expressed in terms of ψ_- and ψ_-^\dagger fields and finally, we arrive at the factorized form

$$H_{int}^1 = \bar{\psi} e \left(\gamma^- \bar{B}_- + \gamma^i \bar{B}_i \right) * \left[1 + \frac{e}{2} \gamma^+ \frac{1}{i\partial_- - e\bar{B}_-} \left(\gamma^- \bar{B}_- + \gamma^j \bar{B}_j \right) \right] * \psi. \quad (2.44)$$

The second part of the interaction Hamiltonian is much simpler

$$\mathcal{H}_{int}^2 = \bar{\psi} e \gamma^+ \bar{B}_+ \psi - \frac{1-\alpha}{8} J^+ \frac{1}{\partial_-^2} * J^+, \quad (2.45)$$

where for simplicity, in the last term we have left the notation J^+ for fermion current.

These formulas show that here the noncovariant terms appear both in the propagators and the interaction Hamiltonians, contrary to the equal-time results where all the corresponding expressions are explicitly covariant. However, following the analysis by Yan [10], one can hope that also here these non-covariant terms will cancel in pairs (2.31.b) with (2.37) and (2.41.b) with (2.44).

2.3.2 LF perturbative calculations of the S-matrix elements

Here we will check the above conjuncture by studying the formal structure of perturbative calculations. The functional techniques [31], [10] are quite useful for this purpose because they allow us to analyse contractions of gauge and fermion fields separately. First we check the contractions of vector field \bar{B}_+ and treat all other components of vector fields and the fermion fields as classical objects for time being. The Wick theorem for transformation of chronological products into the normal products says

$$T^+ \exp \left\{ -i \int j^+ \bar{B}_+ \right\} \exp \left\{ i \frac{1-\alpha}{8} \int j^+ \frac{1}{\partial_-^2} * j^+ \right\} = \exp \left\{ \frac{i}{2} \int j^+ \mathcal{D}_{++} j^+ \right\} : \exp \left\{ -i \int j^+ \bar{B}_+ \right\} : \quad (2.46)$$

where

$$\mathcal{D}_{++}(x) = i \langle 0 | T \bar{B}_+(x) \bar{B}_+(y) | 0 \rangle + \frac{1-\alpha}{4} \frac{1}{\partial_-^2} * \delta(x) = i \partial_+^x \partial_+^y \frac{\Delta_F(x-y, m^2) - \Delta_F(x-y, \alpha m^2)}{m^2}, \quad (2.47)$$

and as usually, colons denote the normal product. Therefore we can simultaneously omit the instantaneous current interaction in \mathcal{H}_{int}^2 and take the covariant propagator for contractions of all vector field components

$$\begin{aligned} \mathcal{D}_{F\mu\nu}(x) &= -g_{\mu\nu} \Delta_F(x-y, m^2) + \partial_\mu^x \partial_\nu^y \frac{\Delta_F(x-y, m^2) - \Delta_F(x-y, \alpha m^2)}{m^2} = \\ &= i \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ik(x)}}{2k_+ k_- - k_\perp^2 - m^2 + i\epsilon} \left[-g_{\mu\nu} + (1-\alpha) \frac{k_\mu k_\nu}{2k_+ k_- - k_\perp^2 - \alpha m^2 + i\epsilon} \right]. \end{aligned} \quad (2.48)$$

Thus effectively the complete interaction Hamiltonian is bilinear in fermion fields

$$H_{int} = \bar{\psi} * e\gamma^\mu \bar{B}_\mu \left[1 + \frac{e}{2} \gamma^+ \frac{1}{i\partial_- - e\bar{B}_-} e\gamma^\nu \bar{B}_\nu \right] * \psi = \bar{\psi} * \mathcal{V}[\bar{B}_\mu] * \psi. \quad (2.49)$$

Next when all the vector fields are kept as c-numbers b_μ , one can easily study the contractions of fermion fields [31], [10]

$$T^+ \exp -i\bar{\psi} * \mathcal{V}[b_\mu] * \psi = \exp [\text{Tr} \ln (1 - \bar{S}_F * \mathcal{V})] : \exp [-i\bar{\psi} * \mathcal{V} * (1 - \bar{S}_F * \mathcal{V})^{-1} * \psi] : \quad (2.50)$$

where now

$$i\bar{S}_F(x, y) = \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = iS_F(x-y) - \gamma^+ \frac{1}{2\partial_-} * \delta(x-y) \quad (2.51.a)$$

$$iS_F(x) = \left(i\gamma^\mu \partial_\mu^x + M \right) \Delta_F(x, M^2). \quad (2.51.b)$$

One can check that the following factorization property holds

$$\begin{aligned} 1 - \bar{S}_F * \mathcal{V} &= 1 - \left(S_F - \frac{\gamma^+}{2i\partial_-} \right) * e\gamma^\mu b_\mu \left[1 + \frac{\gamma^+}{2} \frac{1}{i\partial_- - eb_-} * e\gamma^\nu b_\nu \right] \\ &= (1 - S_F e\gamma^\mu b_\mu) * \left[1 + \frac{\gamma^+}{2} \frac{1}{i\partial_- - eb_-} * e\gamma^\nu b_\nu \right] \\ &= (1 - S_F * e\gamma^\mu b_\mu) * \mathcal{V}_2, \end{aligned} \quad (2.52)$$

and then it is easy to notice that the non-covariant factor H_2 disappears from the normal product part of Eq.(2.50). It still formally remains in the closed loop contribution

$$\exp \text{Tr} \ln \left(1 + \frac{\gamma^+}{2} \frac{1}{i\partial_- - eb_-} * e\gamma^\mu b_\mu \right) \approx \exp \text{Tr} \ln \left(1 + \frac{1}{i\partial_- - eb_-} * eb_- \right).$$

However, the last expression can be shown to be independent of b_- .¹⁴ In this manner we have shown that the formal perturbative series based on the noncovariant interaction Hamiltonian (2.50) and the canonical noncovariant fermion propagator (2.51.a) will give the same result as the calculation based on the covariant interaction Hamiltonian $\bar{\psi} * e\gamma^\mu b_\mu * \psi$ and the covariant fermion propagator S_F

$$T^+ \exp -i\bar{\psi} * \mathcal{V}[b_\mu] * \psi = \exp [\text{Tr} \ln (1 - S_F * e\gamma^\mu b_\mu)]: \exp \left[-i\bar{\psi} * (1 - S_F * e\gamma^\nu b_\nu)^{-1} * \psi \right] : \quad (2.53)$$

provided all divergent expressions are properly regularized.

3 LF Weyl gauge

Another gauge condition which we have chosen for the massive vector fields is the LF Weyl gauge $B_+ = B_- = 0$. This gauge condition is particularly suitable for the canonical procedure because it explicitly removes this component of vector field whose canonically conjugated momentum is zero. Contrary to the previous covariant gauge, the LF Weyl gauge can be strongly implemented in the Lagrangian density thus reducing the number of field variables.

3.1 Vector fields sector

First we take the model of massive vector fields coupled to the external currents and impose explicitly the LF-Weyl gauge condition. We see that the covariant mass term takes the form $-\frac{1}{2}m^2(B_i)^2$

$$\mathcal{L}_{jWeyl}^{mass} = \partial_+ B_i (\partial_- B_i - \partial_i B_-) + \frac{1}{2} (\partial_+ B_-)^2 - \frac{1}{4} (\partial_i B_j - \partial_j B_i)^2 - \frac{m^2}{2} B_i^2 + B_- j^- + B_i j^i \quad (3.1)$$

and all Euler-Lagrange equations are dynamical

$$(2\partial_+ \partial_- - \Delta_\perp + m^2) B_i = \partial_i (\partial_+ B_- - \partial_i B_i) + j^i, \quad (3.2.a)$$

$$\partial_+ (\partial_+ B_- - \partial_i B_i) = j^-. \quad (3.2.b)$$

Thus if we rewrite these equations in the first-order form

$$(2\partial_+ \partial_- - \Delta_\perp) B_i = \partial_i \Pi + j^i, \quad (3.3.a)$$

$$\partial_+ \Pi = j^-, \quad (3.3.b)$$

$$\partial_+ B_- = \Pi + \partial_i B_i, \quad (3.3.c)$$

and find the canonical Hamiltonian¹⁵

$$\mathcal{H}_{can}^{mass} = \Pi^- \partial_+ B_- + \Pi^i \partial_+ B_i - \mathcal{L}_{jWeyl}^{mass} = \frac{1}{2} (\Pi)^2 + \frac{1}{2} (\partial_i B_j)^2 + \Pi \partial_i B_i + \frac{m^2}{2} B_i^2 - B_i j^i - B_- j^-, \quad (3.4)$$

¹⁴One can proved perturbatively showing that the closed loop diagrams disappear, but in the present model, from $b_- = \partial_- \phi$, one also gets

$$[i\partial_- - e(\partial_- \phi)]^{-1}(z, y) = \exp ie\phi(z)(i\partial_-)^{-1}(z, y) \exp -ie\phi(y)$$

and its functional determinant is evidently independent of ϕ .

¹⁵The canonical momenta are simple here

$$\Pi^- = \partial_+ B_- - \partial_i B_i,$$

$$\Pi^i = \partial_- B_i - \partial_i B_-.$$

then the Poisson brackets will follow immediately

$$\left\{2\partial_- B_i(x^+, \vec{x}), B_j(x^+, \vec{y})\right\}_{PB} = -\delta_{ij}\delta^3(\vec{x} - \vec{y}), \quad (3.5.a)$$

$$\left\{B_-(x^+, \vec{x}), \Pi(x^+, \vec{y})\right\}_{PB} = \delta^3(\vec{x} - \vec{y}). \quad (3.5.b)$$

Just as in the previous case, our independent canonical variables (B_i, B_-) are not independent modes. Rather we should take their linear combinations

$$C_i = B_i + \partial_i \frac{1}{\Delta_\perp - m^2} * \Pi, \quad (3.6.a)$$

$$C_- = B_- - \partial_- \frac{1}{\Delta_\perp - m^2} * \left[2\partial_j B_j + \Delta_\perp \frac{1}{\Delta_\perp - m^2} * \Pi\right], \quad (3.6.b)$$

which satisfy separated equations of motion¹⁶

$$\left(2\partial_+ \partial_- - \Delta_\perp + m^2\right) C_i = j^i - 2\partial_i \frac{1}{m^2 - \Delta_\perp} * \partial_- j^-, \quad (3.7.a)$$

$$\partial_+ \Pi = j^-, \quad (3.7.b)$$

$$\begin{aligned} \partial_+ C_- &= -m^2 \frac{1}{\Delta_\perp - m^2} * \Pi - \frac{1}{\Delta_\perp - m^2} * \partial_i j^i \\ &\quad - \Delta_\perp \frac{1}{\Delta_\perp - m^2} * \left(\frac{1}{\Delta_\perp - m^2} * \partial_- j^-\right), \end{aligned} \quad (3.7.c)$$

and still have diagonal Poisson brackets

$$\left\{2\partial_- C_i(x^+, \vec{x}), C_j(x^+, \vec{y})\right\}_{PB} = -\delta_{ij}\delta^3(\vec{x} - \vec{y}), \quad (3.8.a)$$

$$\left\{C_-(x^+, \vec{x}), \Pi(x^+, \vec{y})\right\}_{PB} = \delta^3(\vec{x} - \vec{y}). \quad (3.8.b)$$

At last we can express the Hamiltonian density (3.4) in terms of independent modes

$$\begin{aligned} \mathcal{H}_{total}^{mass} &= \frac{1}{2}(\partial_i C_j)^2 + \frac{m^2}{2} C_i^2 - \frac{m^2}{2} \Pi \frac{1}{\Delta_\perp - m^2} * \Pi - C_- j^- \\ &\quad - \Pi \frac{1}{\Delta_\perp - m^2} * \left[\partial_i j^i + \Delta_\perp \frac{1}{\Delta_\perp - m^2} * \partial_- j^-\right] - C_i \left[j^i - 2\partial_i \frac{1}{\Delta_\perp - m^2} * \partial_- j^-\right] \end{aligned} \quad (3.9)$$

and notice that no instantaneous interaction of currents occurs in this model. We see that the fermion field contribution is quite similar to that discussed in the previous case of covariant gauge and no modification of the perturbative vector field propagators should appear here. Therefore all we need to know here are the free vector field propagators.

3.2 Free quantum fields

We restrict our discussion to the free field case because in the interaction representation field operators have free dynamics. From the canonical analysis we take the form of canonical commutators at LF

$$\left[2\partial_- C_i(\vec{x}), C_j(\vec{y})\right] = -i\delta_{ij}\delta^3(\vec{x} - \vec{y}), \quad (3.10.a)$$

$$\left[C_-(\vec{x}), \Pi(\vec{y})\right] = i\delta^3(\vec{x} - \vec{y}), \quad (3.10.b)$$

and the free Hamiltonian density

$$\mathcal{H}_0^{mass} = \frac{1}{2}(\partial_i C_j)^2 + \frac{m^2}{2} C_i^2 - \frac{m^2}{2} \Pi \frac{1}{\Delta_\perp - m^2} * \Pi. \quad (3.11)$$

¹⁶The field C_- is a *noncovariant multipole field* which is characteristic for noncovariant gauge conditions.

They generate free field equations

$$(2\partial_+\partial_- - \Delta_\perp + m^2)C_i = 0, \quad (3.12.a)$$

$$\partial_+\Pi = 0, \quad (3.12.b)$$

$$\partial_+C_- = -m^2 \frac{1}{\Delta_\perp - m^2} * \Pi. \quad (3.12.c)$$

All fields, the covariant C_i and the noncovariant (Π, C_-) have their Fourier representations for all x^+

$$C_i(x) = \int_{-\infty}^{\infty} \frac{d^2k_\perp}{(2\pi)^2} \int_0^\infty \frac{dk_-}{4\pi k_-} \left[e^{-ik \cdot x} c_i(\vec{k}) + e^{+ik \cdot x} c_i^\dagger(\vec{k}) \right]_{k_+ = \frac{k_\perp^2 + m^2}{2k_-}}, \quad (3.13.a)$$

$$\Pi(\vec{x}) = \int_{-\infty}^{\infty} \frac{d^2k_\perp}{(2\pi)^2} \int_0^\infty \frac{dk_-}{2\pi} \left[e^{-i\vec{k} \cdot \vec{x}} p(\vec{k}) + e^{+i\vec{k} \cdot \vec{x}} p^\dagger(\vec{k}) \right], \quad (3.13.b)$$

$$C_-(x^+, \vec{x}) = -m^2 x^+ \frac{1}{\Delta_\perp - m^2} * \Pi(\vec{x}) \\ + \int_{-\infty}^{\infty} \frac{d^2k_\perp}{(2\pi)^2} \int_0^\infty \frac{dk_-}{2\pi} \left[e^{-i\vec{k} \cdot \vec{x}} c_-(\vec{k}) + e^{+i\vec{k} \cdot \vec{x}} c_-^\dagger(\vec{k}) \right], \quad (3.13.c)$$

and the commutator relations for the creation and annihilation operators are

$$[c_-(\vec{k}), p^\dagger(\vec{k}')] = [c_-^\dagger(\vec{k}), p(\vec{k}')] = i(2\pi)^3 \delta^3(\vec{k} - \vec{k}') \quad (3.14.a)$$

$$[c_i(\vec{k}), c_j^\dagger(\vec{k}')] = (2\pi)^3 2k_- \delta_{ij} \delta^3(\vec{k} - \vec{k}'). \quad (3.14.b)$$

Now a quite straightforward calculation gives the chronological products for independent modes

$$\langle 0|T C_i(x)C_j(y)|0 \rangle = \delta_{ij} \Delta_F(x-y), \quad (3.15.a)$$

$$\langle 0|T C_-(x)\Pi(y)|0 \rangle = E_F^-(x_L - y_L) \delta^2(x_\perp - y_\perp), \quad (3.15.b)$$

$$\langle 0|T C_-(x)C_-(y)|0 \rangle = -m^2 E_F^-(x_L - y_L) \frac{1}{\Delta_\perp - m^2} * \delta^2(x_\perp - y_\perp), \quad (3.15.c)$$

where all other propagators vanish¹⁷. We notice that the causal propagation of noncovariant fields (Π, C_-) takes place only in the LF longitudinal directions $x_L = (x_+, x_-)$ and in the transverse directions $x_\perp = (x_2, x_3)$ these propagators are either local (3.15.b) or are given by the modified inverse Laplace operator (3.15.c). This means that their causal properties i.e. the ML-prescription which appears in them, do not depend on the space-time dimensionality, and thus have no direct infrared singularities¹⁸.

Finally, in order to find the propagators for primary vector fields, we use the relations

$$B_i = C_i - \partial_i \frac{1}{\Delta_\perp - m^2} * \Pi, \quad (3.16.a)$$

$$B_- = C_- + \partial_- \frac{1}{\Delta_\perp - m^2} * \left[2\partial_j C_j - \Delta_\perp \frac{1}{\Delta_\perp - m^2} * \Pi \right], \quad (3.16.b)$$

and then after some algebra we obtain the expressions

$$\langle 0|T B_i(x)B_j(x)|0 \rangle = \delta_{ij} \Delta_F(x-y, m^2), \quad (3.17.a)$$

¹⁷ Definitions and properties of the noncovariant propagator functions $E_F^{1,2}(x)$ are given in Appendix B.1.

¹⁸ Evidently the infrared singularity would appear in the integral operator $(\Delta_\perp - m^2)^{-1}$ when the limit $m^2 \rightarrow 0$ is taken in two transverse dimensions.

$$\begin{aligned}
\langle 0|T B_-(x)B_-(y)|0 \rangle &= \partial_i^x \frac{1}{\Delta_\perp - m^2} * \left[2\partial_-^x \Delta_F(x-y) + E_F^1(x_L - y_L)\delta^2(x_\perp - y_\perp) \right] \\
&= \partial_i^x \int_0^{x^+ - y^-} d\xi \Delta_F(\xi, \vec{x} - \vec{y}), \tag{3.17.b}
\end{aligned}$$

$$\begin{aligned}
\langle 0|T B_-(x)B_-(y)|0 \rangle &= 2\partial_-^x \Delta_\perp \frac{1}{(\Delta_\perp - m^2)^2} * \left[2\partial_-^x \Delta_F(x-y) + E_F^1(x_L - y_L)\delta^2(x_\perp - y_\perp) \right] \\
&\quad - m^2 E_F^2(x_L - y_L) \frac{1}{\Delta_\perp - m^2} * \delta^2(x_\perp - y_\perp) \\
&= 2\partial_-^x \int_0^{x^+ - y^-} d\xi \Delta_F(\xi, \vec{x} - \vec{y}) + m^2 \int_0^{x^+ - y^-} d\xi \int_0^\xi d\eta \Delta_F(\eta, \vec{x} - \vec{y}), \tag{3.17.c}
\end{aligned}$$

with the successively increasing number of independent modes contributions¹⁹. These all components have a concise form in their Fourier representation

$$\begin{aligned}
\langle 0|T B_\mu(x)B_\nu(y)|0 \rangle &= i \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ik \cdot (x-y)}}{k^2 - m^2 + i\epsilon} \left(-g_{\mu\nu} + \frac{k_\nu N_\mu + k_\mu N_\nu}{k_+ + i\epsilon' \text{sgn}(k_-)} \right. \\
&\quad \left. + m^2 \frac{N_\mu N_\nu}{[k_+ + i\epsilon' \text{sgn}(k_-)]^2} \right), \tag{3.18}
\end{aligned}$$

where we have introduced the LF Weyl gauge vector $N_\mu = (N_- = 1, N_+ = 0, N_\perp = 0)$ which chooses the LF-Weyl gauge condition $N^\mu B_\mu = B_+ = 0$. In this way we have obtained the causal ML-prescription for the spurious poles in the vector field propagator via the canonical quantization procedure on a single LF surface of quantization. We see that the noncovariant modes Π and C_- are inevitable for this encouraging result. Thus we can speculate that the canonical procedure at a single LF cannot lead to the ML-prescription for the LC-gauge, where all noncovariant nonphysical modes are excluded.

¹⁹In the second and the third case we have used the property (B.9) of the propagator functions $\Delta_F(x)$ i

Part III

Light Front QED

4 Class of LF Weyl gauges

Now we begin the discussion of the Abelian gauge field models where the gauge transformation is a local symmetry of the theory. Therefore now it is not a problem of having smoother behaviour of canonical LF commutators but the fundamental property of a gauge field prescription which makes us choose some gauge fixing condition. The first choice that we discuss here is the class of LF Weyl gauge conditions which is a generalization of the exact LF Weyl gauge $A_+ = 0$. It is usually introduced into the Lagrangian density by the so-called *gauge fixing term*

$$\frac{1}{2\alpha} (N^\mu A_\mu)^2, \quad (4.1)$$

where in our case the LF Weyl gauge vector is $N^\mu = (N^+ = 1, N^- = 0, N^\perp = 0)$. This term is very convenient in the path-integral approach to the quantization of gauge field theories [32] because it constitutes a non-singular quadratic part of gauge field Lagrangian thus allowing for immediate inversion. However the path-integral procedure gives no prescription for spurious poles in the gauge field propagator which should have the form

$$\begin{aligned} D_{\mu\nu}(x) &= i \langle 0 | T A_\mu(x) A_\nu(0) | 0 \rangle \\ &= \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x} \left\{ \frac{1}{2k_+ k_- - k_\perp^2 + i\epsilon} \left[g_{\mu\nu} - \frac{N_\mu k_\nu + N_\nu k_\mu}{[k_+ + i\epsilon' \text{sgn}(k_-)]} \right] - \alpha \frac{k_\mu k_\nu}{[k_+ + i\epsilon' \text{sgn}(k_-)]^2} \right\} \end{aligned} \quad (4.2)$$

with the causal ML-prescription $[k_- + i\epsilon' \text{sgn}(k_+)]^{-1,2}$. We see that while the above propagator is finite for $\alpha \rightarrow 0$ where it describes the exact LF Weyl gauge, the expression (4.1) is ill-defined for $\alpha = 0$. Therefore we choose another form of the gauge fixing condition via the Lagrange multiplier field Λ

$$\Lambda A_+ - \frac{\alpha}{2} \Lambda^2, \quad (4.3)$$

which is equivalent to the previous choice as long as $\alpha \neq 0$. Evidently it is also regular for $\alpha = 0$ when it describes the exact LF-Weyl gauge $A_+ = 0$. In the present case Λ is not a dynamical field, contrary to the case of covariant gauges. Below we will consecutively discuss the sector of gauge fields in two distinct cases of 1+1 and D+1 dimensions.

4.1 Gauge field sector in 1+1 dimensions

We want to discuss first the case of 1+1 dimensions where the infrared singularities in the transverse momenta for $k_\perp^2 = 0$ are excluded from the considerations. Thus we can focus our attention on singularities in the longitudinal momentum k_- . Just as we did for previous models, also here we start from the sector of gauge fields coupled linearly to external currents j^μ

$$\mathcal{L}_{\text{QED}}^{1+1} = \frac{1}{2} (\partial_+ A_- - \partial_- A_+)^2 + A_- j^- + A_+ j^+ + \Lambda A_+ - \frac{\alpha}{2} \Lambda^2, \quad (4.4)$$

and the Euler-Lagrange equations are

$$\partial_+ (\partial_+ A_- - \partial_- A_+) = j^-, \quad (4.5.a)$$

$$\partial_- (\partial_+ A_- - \partial_- A_+) = -j^+ - \Lambda, \quad (4.5.b)$$

$$A_+ = \alpha \Lambda. \quad (4.5.c)$$

Without going into details, we give the canonical Hamiltonian structure which follows from the above equations after removing non-dynamical field variables (A_+, Λ) . The canonical Hamiltonian density

$$\mathcal{H}_{\alpha Weyl}^{can,1+1} = = \frac{1}{2} \Pi^- (1 - \alpha \partial_-^2) \Pi^- - A_- j^- + \alpha j^+ \partial_- \Pi^- + \frac{\alpha}{2} (j^+)^2, \quad (4.6)$$

and non-vanishing commutator at LF

$$\left[\Pi(x^+, x^-), A_-(x^+, y^-) \right] = -i \delta(x^- - y^-). \quad (4.7)$$

generate effective dynamical equations

$$\partial_+ \Pi^- = j^- \quad (4.8.a)$$

$$\partial_+ A_- = (1 - \alpha \partial_-^2) \Pi^- - \alpha \partial_- j^+. \quad (4.8.b)$$

We stress that this quantum theory describes a larger system than the physical excitations. This is due to the lack of the Gauss law as an equation of motion

$$G = \partial_- (\partial_+ A_- - \partial_- A_+) + j^+ = \partial_- \Pi^- + j^+ = 0. \quad (4.9)$$

Evidently one cannot *strongly* impose the condition $G = 0$ because this would be in a conflict with the commutator (4.7); thus one has to implement it *weakly* as a condition on states [42]

$$\langle phys^+ | G(x) | phys \rangle = 0. \quad (4.10)$$

The interaction part of the Hamiltonian (4.6) indicates that external currents are coupled with the linear combinations of fields

$$\bar{A}_- = A_-, \quad (4.11.a)$$

$$\bar{A}_+ = -\alpha \partial_- \Pi^-, \quad (4.11.b)$$

and also here there is an instantaneous interaction term $\frac{\alpha}{2} j^+ j^+$ which, during the Wick contraction procedure, would modify the perturbative propagator from the canonical chronological product to the following expression:

$$D_{\mu\nu}(x-y) = i \langle 0 | T^+ \bar{A}_\mu(x) \bar{A}_\nu(y) | 0 \rangle - \alpha g_{-\mu} g_{-\nu} \delta^2(x-y). \quad (4.12)$$

First we calculate the chronological product for independent modes in the free field case when they can be represented by

$$\Pi^-(x) = \pi(x^-), \quad (4.13.a)$$

$$A_-(x) = x^+ (1 - \alpha \partial_-^2) \pi(x^-) + a_-(x^-), \quad (4.13.b)$$

with the Fourier representation

$$a_-(x^-) = \int_0^\infty \frac{dk_-}{2\pi} \left[e^{-ik_- x^-} a(k_-) + e^{+ik_- x^-} a^\dagger(k_-) \right], \quad (4.14.a)$$

$$\pi(x^-) = \int_0^\infty \frac{dk_-}{2\pi} \left[e^{-ik_- x^-} p(k_-) + e^{+ik_- x^-} p^\dagger(k_-) \right], \quad (4.14.b)$$

and the commutation relations for the creation and annihilation operators²⁰

$$\left[a_-(k_-), p^\dagger(k'_-) \right] = \left[a_-^\dagger(k_-), p(k'_-) \right] = i 2\pi \delta(k_- - k'_-). \quad (4.15)$$

²⁰The weak Gauss law condition (4.10) shows that for free fields, all physical states are created by the $p(k)^\dagger$ operators and hence they have zero norm. This means that no physical photons are present in 1+1 dimensions.

Now the free propagators are

$$\langle 0|T^+ \Pi^-(x)\Pi^-(y)|0\rangle = 0, \quad (4.16.a)$$

$$\langle 0|T^+ A_-(x)\Pi^-(y)|0\rangle = E_F^1(x-y), \quad (4.16.b)$$

$$\langle 0|T^+ A_-(x)A_-(y)|0\rangle = (1-\alpha\partial_-^2)E_F^2(x-y), \quad (4.16.c)$$

where noncovariant functions $E_F^1(x)$ and $E_F^2(x)$ are defined in Appendix B.1. The perturbative gauge field propagator has the causal form

$$D_{\mu\nu}(x) = \int \frac{d^2k}{(2\pi)^2} e^{-ikx} \left\{ \frac{1}{2k_+k_- + i\epsilon} \left[g_{\mu\nu} - \frac{N_\mu k_\nu + N_\nu k_\mu}{[k_+ + i\epsilon' \text{sgn}(k_-)]} \right] - \alpha \frac{k_\mu k_\nu}{[k_+ + i\epsilon' \text{sgn}(k_-)]^2} \right\}, \quad (4.17)$$

and simultaneously the current interaction term is omitted in the interaction Hamiltonian. Then one could incorporate the fermion interaction, but we will not discuss it here because this would be a mere repetition of the relevant considerations from the subsection (2.3.2) with a rather evident neglect of transverse coordinates and components.

We conclude that in 1+1 dimensions, the class of the LF Weyl gauges effectively leads to the causal form of the perturbative Feynman rules though at the canonical level it contains non-covariant terms which ultimately cancel in pairs.

4.2 Gauge field sector in D+1 dimensions

As a physically relevant model we take the higher-dimensional case where excitations of physical photons are possible. However if we would take 3+1 dimensions then we would encounter infrared singularities connected with the inverse Laplace operator in 2 transverse directions. Therefore we have decided to use here the dimensional regularization of this singularity by working with $d = D - 1 > 2$ transverse coordinates $x_\perp = (x_2, x_3, \dots, x_D)$. Thus we would like to start with the Lagrangian density

$$\begin{aligned} \mathcal{L}_{\text{Weyl}} = & (\partial_+ A_i - \partial_i A_+) (\partial_- A_i - \partial_i A_-) + \frac{1}{2} (\partial_+ A_- - \partial_- A_+)^2 - \frac{1}{4} (\partial_i A_j - \partial_j A_i)^2 \\ & + A_+ \Lambda - \frac{\alpha}{2} \Lambda^2 + A_+ j^+ + A_- j^- + A_i j^i, \end{aligned} \quad (4.18)$$

which generates the Euler-Lagrange equations

$$\partial_+ (\partial_+ A_- - \partial_- A_+ - \partial_j A_j) = -\Delta_\perp^d A_+ + j^-, \quad (4.19.a)$$

$$-\partial_- (\partial_+ A_- - \partial_- A_+ + \partial_j A_j) = -\Delta_\perp^d A_- + j^+ + \Lambda, \quad (4.19.b)$$

$$(2\partial_+ \partial_- - \Delta_\perp^d) A_i = \partial_i (\partial_+ A_- + \partial_- A_+ - \partial_j A_j) + j^i, \quad (4.19.c)$$

$$A_+ = \alpha \Lambda, \quad (4.19.d)$$

where $\Delta_\perp^d = (\partial_j)^2$ denotes the Laplace operator in $d > 2$ dimensions. As the independent modes we take the modified canonical momentum

$$\Pi = \partial_+ A_- - \partial_i A_i - \partial_- A_+ \quad (4.20.a)$$

and the gauge fields

$$C_i = A_i - \partial_i \frac{1}{\Delta_\perp^d} * (\Pi + 2\partial_j A_j), \quad (4.20.b)$$

$$C_- = A_- - \partial_- \frac{1}{\Delta_\perp^d} * (\Pi + 2\partial_j A_j) \quad (4.20.c)$$

thus the dynamical equations of motion are

$$(2\partial_+ \partial_- - \Delta_\perp) C_i = j^i - 2\partial_i \frac{1}{\Delta_\perp^d} * (\partial_- j^- + \partial_j j^j), \quad (4.21.a)$$

$$\partial_+ C_- = -\frac{1}{\Delta_\perp^d} * (\partial_i j^i + \partial_- j^-), \quad (4.21.b)$$

$$\partial_+ \Pi = -\alpha(\Delta_\perp^d)^2 C_- + j^- + \alpha \Delta_\perp^d j^+. \quad (4.21.c)$$

The canonical Hamiltonian density can be expressed in terms of the independent modes

$$\begin{aligned} \mathcal{H}_{\alpha Weyl}^{eff} &= \frac{1}{2}(\partial_i C_j)^2 + \frac{\alpha}{2} [\Delta_\perp^d C_- - j^+]^2 - \Pi \frac{1}{\Delta_\perp^d} * [\partial_i j^i + \partial_- j^-] \\ &- C_i \left[j^i - 2\partial_i \frac{1}{\Delta_\perp^d} * (\partial_- j^- + \partial_j j^j) \right] - C_- j^-, \end{aligned} \quad (4.22)$$

and the non-vanishing commutators at LF are

$$[\Pi(x^+, \vec{x}), C_-(x^+, \vec{y})] = -i\delta^{d+1}(\vec{x} - \vec{y}), \quad (4.23.a)$$

$$[2\partial_- C_i(x^+, \vec{x}), C_j(x^+, \vec{y})] = -i\delta_{ij}\delta^{d+1}(\vec{x} - \vec{y}). \quad (4.23.b)$$

Just like in the low-dimensional case, we have the instantaneous current interaction which, during the Wick contractions, modifies the perturbative gauge field propagator from the chronological product form

$$D_{\mu\nu}(x-y) = i\langle 0|T^+ \bar{A}_\mu(x)\bar{A}_\nu(y)|0\rangle - \alpha g_{-\mu} g_{-\nu} \delta^{d+2}(x-y), \quad (4.24)$$

where we have introduced the notation

$$\bar{A}_i = C_i - \partial_i \frac{1}{\Delta_\perp^d} * (\Pi + 2\partial_j C_j), \quad (4.25.a)$$

$$\bar{A}_- = C_- - \partial_- \frac{1}{\Delta_\perp^d} * (\Pi + 2\partial_j C_j), \quad (4.25.b)$$

$$\bar{A}_+ = -\alpha \Delta_\perp^d C_-. \quad (4.25.c)$$

The free propagators for independent modes are calculated directly from the Fourier representations

$$C_-(x) = c_-(\vec{x}) = \int_{-\infty}^{\infty} \frac{d^d k_\perp}{(2\pi)^d} \int_0^\infty \frac{dk_-}{2\pi} \left[e^{-i\vec{k}\cdot\vec{x}} a(\vec{k}) + e^{+i\vec{k}\cdot\vec{x}} a^\dagger(\vec{k}) \right], \quad (4.26.a)$$

$$C_i(x) = \int_{-\infty}^{\infty} \frac{d^d k_\perp}{(2\pi)^d} \int_0^\infty \frac{dk_-}{2\pi} \frac{1}{2k_-} \left[e^{-ik\cdot x} c_i(\vec{k}) + e^{+ik\cdot x} c_i^\dagger(\vec{k}) \right]_{k_+ = \frac{k^2}{2k_-}}, \quad (4.26.b)$$

$$\Pi(x) = \pi(\vec{x}) - \alpha x^+ (\Delta_\perp^d)^2 c_-(\vec{x}), \quad (4.26.c)$$

$$\pi(\vec{x}) = \int_{-\infty}^{\infty} \frac{d^d k_\perp}{(2\pi)^d} \int_0^\infty \frac{dk_-}{2\pi} \left[e^{-i\vec{k}\cdot\vec{x}} p(\vec{k}) + e^{+i\vec{k}\cdot\vec{x}} p^\dagger(\vec{k}) \right], \quad (4.26.d)$$

where the non-vanishing commutators for the creation and annihilation operators are²¹

$$[a(\vec{k}), p^\dagger(\vec{k}')] = [a^\dagger(\vec{k}), p(\vec{k}')] = i(2\pi)^{d+1} \delta^{d+1}(\vec{k} - \vec{k}'), \quad (4.27.a)$$

$$[c_i(\vec{k}), c_j^\dagger(\vec{k}')] = (2\pi)^{d+1} 2k_- \delta_{ij} \delta^{d+1}(\vec{k} - \vec{k}'), \quad (4.27.b)$$

²¹We recognize that the present operators a^\dagger and p^\dagger are trivial generalizations of the respective operators in 1+1 dimensions. Free field physical states, selected by the weak Gauss law, can be created by c_i^\dagger and p^\dagger excitations, where the former would be the positive norm photon states while the latter would be the accompanying zero norm states.

where $\delta^{d+1}(\vec{k}) = \delta^d(k_\perp)\delta(k_-)$. Determination of free propagators for independent modes is rather simple

$$\langle 0|T C_-(x)\Pi(y)|0\rangle = E_F^1(x_L - y_L)\delta^d(x_\perp - y_\perp), \quad (4.28.a)$$

$$\langle 0|T \Pi(x)\Pi(y)|0\rangle = \alpha(\Delta_\perp^d)^2 E_F^2(x_L - y_L)\delta^d(x_\perp - y_\perp), \quad (4.28.b)$$

$$\langle 0|T C_i(x)C_j(y)|0\rangle = \delta_{ij}D_F^{d+2}(x - y), \quad (4.28.c)$$

where the covariant Feynman propagator function $D_F^{d+2}(x)$ is defined in Appendix B.1, while the calculation of the perturbative gauge field propagator is quite tedious; here is the final result presented in concise Fourier representation

$$D_{\mu\nu}(x - y) = \int \frac{d^{d+2}k}{(2\pi)^{d+2}} e^{-ik \cdot (x-y)} \left\{ \frac{1}{k^2 + i\epsilon} \left[g_{\mu\nu} - \frac{(k_\mu N_\nu + k_\nu N_\mu)}{k_+ + i\epsilon' \operatorname{sgn}(k_-)} \right] - \alpha \frac{k_\mu k_\nu}{[k_+ + i\epsilon' \operatorname{sgn}(k_-)]^2} \right\}, \quad (4.29)$$

which evidently has a regular limit $d \rightarrow 2$, where it gives the expected result (4.2).

5 General axial gauge

In this section we would like to discuss the general axial gauge condition imposed on the gauge field potential $n^\mu A_\mu = 0$, where the axial gauge vector has the form $n^+ = 1, n^- = -\alpha, n^\perp = 0$. This general choice will allow us to analyse and compare within the LF canonical formalism different gauge conditions:

- the LF-Weyl - for $\alpha = 0$;
- the temporal Minkowski - for $\alpha = -1$;
- the spatial Minkowski - for $\alpha = 1$.

Also the verification whether the limit $\alpha \rightarrow \pm\infty$, can be considered as a possible limiting procedure leading to the LC gauge $A_- = 0$. The present form of the gauge condition $n^\mu A_\mu = A_+ - \alpha A_- = 0$ can be implemented either explicitly or via the Lagrange multiplier field; below we take the first possibility. Also here we discuss two cases in 1+1 dimensions and in 3+1 dimensions because they have quite different physical interpretations. Again we deal explicitly only with the gauge field sector because the discussion of the fermions and interactions with them would follow along the lines described in Section 2.

5.1 Gauge fields in 1+1 dimensions

Now the Lagrangian density is greatly simplified

$$\mathcal{L}_{g_{\text{axi}}}^{1+1} = \frac{1}{2} (\partial_+ A_- - \alpha \partial_- A_-)^2 - A_- (j^- + \alpha j^+), \quad (5.1)$$

and generates only one Euler-Lagrange equation

$$(\partial_+ - \alpha \partial_-)^2 A_- = j^- + \alpha j^+, \quad (5.2.a)$$

which is equivalent to the system of the first order equations

$$\partial_+ \Pi = \alpha \partial_- \Pi + j^- + \alpha j^+, \quad (5.3.a)$$

$$\partial_+ A_- = \Pi + \alpha \partial_- A_-. \quad (5.3.b)$$

When these equations are assumed as the Hamilton equations with the Hamiltonian density

$$\mathcal{H}_{can}^{1+1} = \Pi^- \partial_+ A_- - \mathcal{L}_{temp} = \frac{1}{2}(\Pi^-)^2 - \alpha A_- \partial_- \Pi^- + -\alpha(\partial_+ A_-)^2 - A_-(j^- + \alpha j^+), \quad (5.4)$$

then the non-vanishing Poisson bracket is canonical and the quantum commutator is

$$[\Pi(x^-), A_-(y^-)] = -i\delta(x^- - y^-). \quad (5.5)$$

Inspecting the Hamiltonian we find no instantaneous interactions of currents, thus the perturbative and free propagators will coincide here. The quantum free fields can be given as

$$\Pi(x) = \pi(\alpha x^+ + x^-), \quad (5.6.a)$$

$$A_-(x) = a_-(\alpha x^+ + x^-) + x^+ \pi(\alpha x^+ + x^-), \quad (5.6.b)$$

where the fields $\pi(x)$ and $a_-(x)$ were defined by (4.14.a) and (4.14.b) in Subsection 4.1. Therefore the whole discussion given there applies also here and in order to avoid unnecessary repetitions, we just take the results

$$\langle 0|T A_-(x)\Pi(y)|0\rangle = E_F^1[x^+ - y^-, \alpha(x^+ - y^+) + x^- - y^-], \quad (5.7.a)$$

$$\langle 0|T A_-(x)A_-(y)|0\rangle = E_F^2[x^+ - y^-, \alpha(x^+ - y^+) + x^- - y^-]. \quad (5.7.b)$$

Next we easily find the form of the perturbative propagators

$$D_{--}(x) = \langle 0|T A_-(x)A_-(0)|0\rangle = i \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} e^{-ikx} \frac{1}{[k_+ - \alpha k_- + i\epsilon \operatorname{sgn}(k_-)]^2}, \quad (5.8.a)$$

$$D_{+-}(x) = \alpha \langle 0|T A_-(x)A_-(0)|0\rangle = i \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} e^{-ikx} \frac{\alpha}{[k_+ - \alpha k_- + i\epsilon \operatorname{sgn}(k_-)]^2}, \quad (5.8.b)$$

$$D_{++}(x) = \alpha^2 \langle 0|T A_-(x)A_-(0)|0\rangle = i \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} e^{-ikx} \frac{\alpha^2}{[k_+ - \alpha k_- + i\epsilon \operatorname{sgn}(k_-)]^2}, \quad (5.8.c)$$

which can be written concisely as the Fourier integral

$$D_{\mu\nu}(x) = i \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} \frac{e^{-ikx}}{2k_+ k_- + i\epsilon} \left[-g_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{k_+ - \alpha k_- + i\epsilon \operatorname{sgn}(k_-)} - n^2 \frac{k_\mu k_\nu}{[k_+ - \alpha k_- + i\epsilon \operatorname{sgn}(k_-)]^2} \right]. \quad (5.9)$$

We see that this propagator exists for all finite values of α with the ML-prescription for spurious poles. Thus even for the spatial axial gauge we can produce a gauge field propagator with the causal spurious poles when quantizing canonically at LF, contrary to the ET formalism where such case is not possible [33]. When one takes the limit $\alpha \rightarrow \pm\infty$, the causal nature of spurious poles in the above propagator is lost and one again has the LC-gauge case with the CPV poles.

5.2 Gauge fields in 3+1 dimensions

Now we would like to analyse the general axial gauge condition in the more physically relevant case of 3+1 dimensions. Here again the gauge condition is implemented explicitly in the Lagrangian density

$$\begin{aligned} \mathcal{L}_{\text{alpha}} = & (\partial_+ A_i - \alpha \partial_i A_-)(\partial_- A_i - \partial_i A_-) + \frac{1}{2}(\partial_+ A_- - \alpha \partial_- A_-)^2 - \frac{1}{4}(\partial_i A_j - \partial_j A_i)^2 \\ & + A_-(j^- + \alpha j^+) + A_i j^i, \end{aligned} \quad (5.9)$$

where the external currents j^μ describe interactions with the charged matter. The canonical analysis starts with the Euler-Lagrange equations

$$\left[(\partial_+ - \alpha \partial_-)^2 + 2\alpha \Delta_\perp \right] A_- = (\partial_+ + \alpha \partial_-) \partial_i A_i + j^- + \alpha j^+, \quad (5.10.a)$$

$$(2\partial_+ \partial_- - \Delta_\perp) A_i = \partial_i [(\partial_+ + \alpha \partial_-) A_- - \partial_j A_j] + j^i, \quad (5.10.b)$$

which are equivalent to the Hamilton equations

$$\partial_+ \Pi = \alpha \partial_- (\Pi + 2\partial_i A_i) - 2\alpha \Delta_\perp A_- + j^- + \alpha j^+, \quad (5.11.a)$$

$$(2\partial_+ \partial_- - \Delta_\perp) A_i = \partial_i (\Pi + 2\alpha \partial_- A_-) + j^i, \quad (5.11.b)$$

$$\partial_+ A_- = \Pi + \partial_i A_i + \alpha \partial_- A_-. \quad (5.11.c)$$

The canonical Hamiltonian density is

$$\mathcal{H}_{\text{can}} = \frac{1}{2}(\Pi)^2 + \frac{1}{2}(\partial_i A_j)^2 + \Pi \partial_i A_i - \alpha A_- \partial_- (\Pi + 2\partial_i A_i) - \alpha (\partial_i A_-)^2 - A_- (j^- + \alpha j^+) - A_i j^i \quad (5.12)$$

with the Dirac brackets

$$\{\Pi(\vec{x}), A_-(\vec{y})\}_{DB} = -\delta^3(\vec{x} - \vec{y}), \quad (5.13.a)$$

$$\{2\partial_- A_i(\vec{x}), A_j(\vec{y})\}_{DB} = -\delta_{ij} \delta^3(\vec{x} - \vec{y}), \quad (5.13.b)$$

while other brackets vanish. When trying to separate these independent variables into the independent modes we encounter the problem of inverting the differential operator at LF $\Delta_\alpha^- = 2\alpha \partial_-^2 - \Delta_\perp$ which, only for $\alpha < 0$, is an elliptic operator with the regular Green function (defined in Appendix B.3). Thus we have to choose only negative values of α which is equivalent to the selection of temporal axial gauges.²² Now we define independent modes

$$\Lambda = \Pi - 2\alpha \partial_-^2 \frac{1}{\Delta_\alpha^-} * [\Pi - 2\alpha \partial_- A_- + 2\partial_j A_j], \quad (5.14.a)$$

$$C_- = A_- + \partial_- \frac{1}{\Delta_\alpha^-} * [\Pi - 2\alpha \partial_- A_- + 2\partial_j A_j], \quad (5.14.b)$$

$$C_i = A_i + \partial_i \frac{1}{\Delta_\alpha^-} * [\Pi - 2\alpha \partial_- A_- + 2\partial_j A_j], \quad (5.14.c)$$

which satisfy dynamical equations of motion

$$(\partial_+ - \alpha \partial_-) C_- = \frac{1}{\Delta_\alpha^-} * [\partial_k j^k + \partial_- (j^- + \alpha j^+)], \quad (5.15.a)$$

$$(\partial_+ - \alpha \partial_-) \Lambda = 2\alpha \Delta_\alpha^- C_- + j^- + \alpha j^+ - 2\alpha \partial_- \frac{1}{\Delta_\alpha^-} * [\partial_k j^k + \partial_- (j^- + \alpha j^+)], \quad (5.15.b)$$

$$(2\partial_+ \partial_- - \Delta_\perp) C_i = j^i + 2\partial_i \frac{1}{\Delta_\alpha^-} * [\partial_k j^k + \partial_- (j^- + \alpha j^+)], \quad (5.15.c)$$

²²The cases of spatial gauges ($\alpha > 0$) will not be discussed here and are left for future investigations, while the null gauge ($\alpha = 0$) is singular at 3+1 dimensions and needs some infrared regularizations.

and have non-vanishing commutators at LF

$$2\partial_x^- [C_i(\vec{x}), C_j(\vec{y})] = -i\delta_{ij}\delta^3(\vec{x} - \vec{y}), \quad (5.16.a)$$

$$[\Lambda(\vec{x}), C_-(\vec{y})] = -i\delta^3(\vec{x} - \vec{y}). \quad (5.16.b)$$

The Hamiltonian density (5.12), when expressed in term of these modes

$$\begin{aligned} \mathcal{H}_{can} = & \frac{1}{2}(\partial_t C_j)^2 + \alpha\Lambda\partial_- C_- - \alpha C_- \Delta_\alpha^- C_- - j^\dagger \left[C_i + \partial_i \frac{1}{\Delta_\alpha^-} * [\Lambda - 2\alpha\partial_- C_- + 2\partial_j C_j] \right] \\ & - (j^- + \alpha j^+) \left[C_- + \partial_- \frac{1}{\Delta_\alpha^-} * [\Lambda - 2\alpha\partial_- C_- + 2\partial_j C_j], \right] \end{aligned} \quad (5.17)$$

contains no direct interactions of currents, therefore the perturbative propagators are the chronological product of free fields. For free fields we write

$$C_-(x) = c_-(\alpha x^+ + x^-, x_\perp), \quad (5.18.a)$$

$$\Lambda(x) = 2\alpha x^+ \Delta_\alpha^- c_-(\alpha x^+ + x^-, x_\perp) + \lambda(\alpha x^+ + x^-, x_\perp), \quad (5.18.b)$$

$$C_i(x) = c_i(x), \quad (5.18.c)$$

where

$$c_-(\vec{x}) = \int_{-\infty}^{\infty} \frac{d^2 k_\perp}{(2\pi)^2} \int_0^{\infty} \frac{dk_-}{2\pi} \left[e^{-i\vec{k}\cdot\vec{x}} a(\vec{k}) + e^{+i\vec{k}\cdot\vec{x}} a^\dagger(\vec{k}) \right], \quad (5.19)$$

$$\lambda(\vec{x}) = \int_{-\infty}^{\infty} \frac{d^2 k_\perp}{(2\pi)^2} \int_0^{\infty} \frac{dk_-}{2\pi} \left[e^{-i\vec{k}\cdot\vec{x}} p(\vec{k}) + e^{+i\vec{k}\cdot\vec{x}} p^\dagger(\vec{k}) \right], \quad (5.20)$$

$$c_i(x) = \int_{-\infty}^{\infty} \frac{d^2 k_\perp}{(2\pi)^2} \int_0^{\infty} \frac{dk_-}{2\pi 2k_-} \left[e^{-ik\cdot x} c_i(\vec{k}) + e^{+ik\cdot x} c_i^\dagger(\vec{k}) \right]_{k_+ = \frac{k^2}{2k_-}}, \quad (5.21)$$

with the commutators for creation and annihilation operators

$$[a(\vec{k}), p^\dagger(\vec{k}')] = [a^\dagger(\vec{k}), p(\vec{k}')] = i(2\pi)^3 \delta^3(\vec{k} - \vec{k}'), \quad (5.22)$$

$$[c_i(\vec{k}), c_j^\dagger(\vec{k}')] = (2\pi)^3 2k_- \delta_{ij} \delta^3(\vec{k} - \vec{k}'), \quad (5.23)$$

while other commutators vanish. Now we easily find the relevant chronological products

$$\langle 0|T C_-(x)\Lambda(y)|0\rangle = E_{\alpha F}^1(x_L - y_L)\delta^2(x_\perp - y_\perp), \quad (5.24.a)$$

$$\langle 0|T \Lambda(x)\Lambda(y)|0\rangle = -2\alpha\Delta_\alpha^- E_{\alpha F}^2(x_L - y_L)\delta^2(x_\perp - y_\perp), \quad (5.24.b)$$

$$\langle 0|T C_i(x)C_j(y)|0\rangle = \delta_{ij} D_F^1(x - y), \quad (5.24.c)$$

where $E_{\alpha F}^{1,2}(x_L) = E_F^{1,2}(x^+, x^- + \alpha x^+)$. The perturbative propagators contain the following linear combinations of independent modes:

$$\Pi = \Lambda - 2\alpha\partial_-^2 \frac{1}{\Delta_\alpha^-} * [\Lambda - 2\alpha\partial_- C_- + 2\partial_j C_j], \quad (5.25.a)$$

$$A_- = C_- + \partial_- \frac{1}{\Delta_\alpha^-} * [\Lambda - 2\alpha\partial_- C_- + 2\partial_j C_j], \quad (5.25.b)$$

$$A_i = C_i + \partial_i \frac{1}{\Delta_\alpha^-} * [\Lambda - 2\alpha\partial_- C_- + 2\partial_j C_j], \quad (5.25.c)$$

and we find after some algebra

$$\langle 0|T A_-(x)A_-(y)|0\rangle = i2\partial_-^x \partial_-^y \left(D_F * E_{\alpha F}^2 \right) (x-y), \quad (5.26.a)$$

$$\langle 0|T A_i(x)A_-(y)|0\rangle = i2\partial_i^x (\alpha\partial_-^y + \partial_+^y) \left(D_F * E_{\alpha F}^2 \right) (x-y), \quad (5.26.b)$$

$$\langle 0|T A_i(x)A_j(y)|0\rangle = \delta_{ij} D_F(x-y) + i2\alpha\partial_i^x \partial_j^y \left(D_F * E_{\alpha F}^2 \right) (x-y). \quad (5.26.c)$$

Next, using the relation $A_+ = \alpha A_-$, we find the Fourier representation for all components of the gauge field propagator

$$\langle 0|T A_\mu(x)A_\nu(y)|0\rangle = i \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x-y)} \left\{ \frac{1}{2k_+ k_- - k_\perp^2 + i\epsilon} \left[-g_{\mu\nu} + \frac{(k_\mu n_\nu + k_\nu n_\mu)}{k_+ - \alpha k_- + i\epsilon' \operatorname{sgn}(k_-)} \right] - n^2 \frac{k_\mu k_\nu}{[k_+ - \alpha k_- + i\epsilon' \operatorname{sgn}(k_-)]^2} \right\}. \quad (5.27)$$

Thus we have found the perturbative gauge field propagator with the ML-prescription for spurious poles for all temporal gauges. Now the LF Weyl gauge can be understood as the limit $\alpha \rightarrow 0$ taken in (5.27). One can compare all LF Weyl propagators obtained via different routines: the massive electrodynamics (3.18) when $m^2 \rightarrow 0$, the class of the LF Weyl gauges (4.29) when $\alpha \rightarrow 0$, and the above general axial gauge (5.27) when $\alpha \rightarrow 0$ leading always to the same result

$$\langle 0|T A_\mu(x)A_\nu(y)|0\rangle_{LFWeyl} = i \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ik \cdot (x-y)}}{2k_+ k_- - k_\perp^2 + i\epsilon} \left[-g_{\mu\nu} + \frac{(k_\mu N_\nu + k_\nu N_\mu)}{k_+ + i\epsilon' \operatorname{sgn}(k_-)} \right]. \quad (5.28)$$

The other passage limit to null gauge, when $\alpha \rightarrow \pm\infty$ changes again the ML-prescription into the CPV for the LC-gauge propagator.

6 Flow covariant gauge

The analysis presented in Section 2 can be considered as an infrared regularized model which in the limit $m^2 \rightarrow 0$ leads to the QED for the class of covariant Lorentz gauges $\partial_\mu A^\mu = \alpha\Lambda$. Therefore here, instead of repeating the previous analysis for explicitly massless vector gauge fields, we decided to study another class of gauge conditions: the flow covariant gauge $\partial_+ A_- + \alpha\partial_- A_+ - \alpha\partial_\perp A_\perp = 0$ which for $\alpha = 1$ produces the results for the Lorentz gauge while for $\alpha = 0$ it describes the LC-gauge $A_- = 0$.²³

6.1 Model in 1+1 dimensions

First we would like to discuss the electrodynamics with charged fermion fields in 1+1 dimensions which is classically described by the Lagrangian density

$$\mathcal{L}_{flow}^{1+1QED} = \frac{1}{2} (\partial_+ A_- - \partial_- A_+)^2 + \bar{\psi} (i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - M) \psi + \Lambda (\partial_+ A_- + \alpha\partial_- A_+), \quad (6.1)$$

where the Lagrange multiplier field Λ implements the flow covariant gauge. Here we explicitly see that the flow gauge is attainable, by some suitable gauge transformation, for all values of the gauge parameter $\alpha \neq 1$. Thus we expect that we may encounter expressions singular at $\alpha = -1$ during the canonical procedure and in final results.

²³Strictly speaking it leads to the modified LC-gauge $\partial_+ A_- = 0$ which should have the same perturbative Feynman rules as the LC-gauge.

6.1.1 Gauge field sector

According to our previous routine, we start with the sector of gauge field coupled with the arbitrary external sources j^μ

$$\mathcal{L}_{mLor}^{1+1} = \frac{1}{2} (\partial_+ A_- - \partial_- A_+)^2 + A_- j^- + A_+ j^+ + \Lambda (\partial_+ A_- + \alpha \partial_- A_+) , \quad (6.2)$$

and here we have the Euler-Lagrange equations for the gauge potential and the Lagrange multiplier fields

$$\partial_+ (\partial_+ A_- - \partial_- A_+ + \Lambda) = j^- , \quad (6.3.a)$$

$$\partial_- (\partial_- A_+ - \partial_+ A_- + \alpha \Lambda) = j^+ , \quad (6.3.b)$$

$$\partial_+ A_- = -\alpha \partial_- A_+ . \quad (6.3.c)$$

However two fields can be parameterized

$$\Lambda = \frac{1}{1+\alpha} \left(\Pi^- + \frac{1}{\partial_-} * j^+ \right) , \quad (6.4.a)$$

$$A_+ = \frac{\alpha}{(1+\alpha)^2} \frac{1}{\partial_-} \left(\alpha \Pi^- - \frac{1}{\partial_-} * j^+ \right) , \quad (6.4.b)$$

by means of the canonical conjugate momentum field Π^- and effectively there are only two dynamical equations of motion:

$$\partial_+ \Pi^- = j^- , \quad (6.5.a)$$

$$\partial_+ A_- = \frac{\alpha}{(1+\alpha)^2} \left(\alpha \Pi^- - \frac{1}{\partial_-} * j^+ \right) , \quad (6.5.b)$$

with the canonical Poisson-Dirac bracket at LF

$$\{A_-(x^+, x^-), \Pi^-(x^+, y^-)\}_{DB} = \delta(x^- - y^-) , \quad (6.6)$$

while the canonical Hamiltonian density is

$$\mathcal{H}_{flow}^{1+1can} = \Pi^- \partial_- A_+ - \mathcal{L}_{flow}^{1+1} = \frac{1}{2(1+\alpha)^2} \left(\alpha \Pi^- - \frac{1}{\partial_-} * j^+ \right)^2 - A_- j^- . \quad (6.7)$$

Now we can give the effective Lagrangian density for the gauge field sector

$$\mathcal{L}_{flow}^{1+1eff} = \Pi^- \partial_+ A_- - \frac{1}{2(1+\alpha)^2} \left(\alpha \Pi^- - \frac{1}{\partial_-} * j^+ \right)^2 + A_- j^- , \quad (6.8)$$

and then easily incorporate fermions back into the dynamical system.

6.1.2 Interaction with fermion fields

When the gauge fields are described by effective independent variables, the system containing complete dynamics of fermions and their interaction with gauge fields is given by the Lagrangian density

$$\begin{aligned} \tilde{\mathcal{L}}_{flow}^{1+1QED} &= \Pi^- \partial_+ A_- + \sqrt{2} i \psi_+^\dagger \partial_+ \psi_+ - \frac{1}{2(1+\alpha)^2} \left(\alpha \Pi^- + e\sqrt{2} \frac{1}{\partial_-} * (\psi_+^\dagger \psi_+) \right)^2 \\ &+ \sqrt{2} i \psi_-^\dagger \partial_- \psi_- - e\sqrt{2} \psi_-^\dagger \psi_- A_- - M \left(\psi_-^\dagger \gamma^0 \psi_+ + \psi_+^\dagger \gamma^0 \psi_- \right) . \end{aligned} \quad (6.9)$$

As it usually happens in the LF formalism, the fermion fields ψ_- i ψ_-^\dagger are non-dynamical and as dependent variables can be expressed by other fields

$$\psi_- = \frac{1}{\sqrt{2}} \frac{1}{i\partial_- - eA_-} * M\gamma^0\psi_+, \quad (6.10.a)$$

$$\psi_-^\dagger = M \frac{1}{\sqrt{2}} \xi^\dagger * \frac{1}{i\partial_- - eA_-} \gamma^0. \quad (6.10.b)$$

In this way, we have arrived at the Hamiltonian density which depends solely on the independent dynamical fields (which are already independent modes)

$$\mathcal{H}_{flow}^{1+1QED} = \frac{1}{2(1+\alpha)^2} \left(\alpha\Pi^- + e\sqrt{2} \frac{1}{\partial_-} * (\psi_+^\dagger\psi_+) \right)^2 + \frac{M^2}{\sqrt{2}} \psi_+^\dagger \frac{1}{i\partial_- - eA_-} * \psi_+. \quad (6.11)$$

Now the canonical quantization is immediate, one takes the properly ordered expression (6.11) as the quantum Hamiltonian density and non-vanishing (anti)commutation relations

$$\left[\Pi^-(x^+, x^-), A_-(x^+, y^-) \right] = i\delta(x^- - y^-), \quad (6.12.a)$$

$$\left\{ \psi_+^\dagger(x^+, x^-), \psi_+(x^+, y^-) \right\} = \frac{\Lambda_+}{\sqrt{2}} \delta(x^- - y^-). \quad (6.12.b)$$

Here the equations for quantum operators will have the same functional form as their classical counterparts²⁴

$$\partial_+ \Pi^- = e \frac{M^2}{\sqrt{2}} \psi_+^\dagger * \frac{1}{i\partial_- - eA_-} - \frac{1}{i\partial_- - eA_-} * \psi_+, \quad (6.13.a)$$

$$\partial_+ A_- = \frac{\alpha}{(1+\alpha)^2} \left[\alpha\Pi^- + e\sqrt{2} \frac{1}{\partial_-} * (\psi_+^\dagger\psi_+) \right], \quad (6.13.b)$$

$$i\sqrt{2}\partial_+ \psi_+ = \frac{M^2}{\sqrt{2}} \frac{1}{i\partial_- - eA_-} * \psi_+^\dagger - \frac{e\sqrt{2}}{(1+\alpha)^2} \psi_+^\dagger \frac{1}{\partial_-} * \left[\alpha\Pi^- + e\sqrt{2} \frac{1}{\partial_-} * (\psi_+^\dagger\psi_+) \right], \quad (6.13.c)$$

$$-i\sqrt{2}\partial_+ \psi_+^\dagger = \frac{M^2}{\sqrt{2}} \psi_+^\dagger * \frac{1}{i\partial_- - eA_-} - \frac{e\sqrt{2}}{(1+\alpha)^2} \psi_+^\dagger \frac{1}{\partial_-} * \left[\alpha\Pi^- + e\sqrt{2} \frac{1}{\partial_-} * (\psi_+^\dagger\psi_+) \right]. \quad (6.13.d)$$

6.1.3 Perturbation theory

In the interaction representation, the field operators have free dynamics generated by the free Hamiltonian which in the present model is

$$\mathcal{H}_0^{1+1flow} = \frac{\alpha^2}{2(1+\alpha)^2} (\Pi^-)^2 + \frac{M^2}{\sqrt{2}} \psi_+^\dagger \frac{1}{i\partial_-} * \psi_+. \quad (6.14)$$

The remaining part of the Hamiltonian is taken as the interaction Hamiltonian

$$\begin{aligned} \mathcal{H}_{int}^{1+1QED} &= \mathcal{H}_{flow}^{1+1QED} - \mathcal{H}_0^{1+1flow} = e \frac{\sqrt{2}\alpha^2}{(1+\alpha)^2} \Pi^- \frac{1}{\partial_-} * (\psi_+^\dagger\psi_+) \\ &+ e^2 \frac{1}{(1+\alpha)^2} \left(\frac{1}{\partial_-} * (\psi_+^\dagger\psi_+) \right)^2 - \frac{M^2}{\sqrt{2}} \psi_+^\dagger \left[\frac{1}{i\partial_-} - \frac{1}{i\partial_- - eA_-} \right] * \psi_+, \end{aligned} \quad (6.15)$$

²⁴The expression $\psi_+^\dagger(x)\psi_+(x)$ is singular for quantum fields operators and needs some regularization. If one takes the splitting-point method which is compatible with the local gauge symmetry $\lim_{\eta \rightarrow 0} \psi_+^\dagger(x^+, x^- - \eta)e^{-ie \int_{x^- - \eta}^{x^+} d\epsilon A_-(x^+, \epsilon)} \psi_+(x^- + \eta)$, then the equations for Π^- , ψ_+ i ψ_+^\dagger are modified. However, this would lead us beyond the scope of this paper and no such regularization will be discussed hereafter.

and we see that there is a direct instantaneous interaction of currents which will modify the perturbative propagators from the primary form of Wick's contractions of effective gauge field potentials

$$\mathcal{D}_{\mu\nu}(x-y) = i \langle 0 | T \bar{A}_\mu(x) \bar{A}_\nu(y) | 0 \rangle + \frac{g_\mu - g_\nu}{(1+\alpha)^2} \frac{1}{\partial_-^2} * \delta(x-y) \quad (6.16)$$

where

$$\bar{A}_- = A_- , \quad (6.17.a)$$

$$\bar{A}_+ = -\frac{\alpha}{(1+\alpha)^2} \frac{1}{\partial_-} * \Pi^- . \quad (6.17.b)$$

We need the free propagator for independent modes and they can be obtained quite easily

$$\langle 0 | T^+ \Pi^-(x) A_-(y) | 0 \rangle = E_F^1(x-y) , \quad (6.18.a)$$

$$\langle 0 | T^+ A_-(x) A_-(y) | 0 \rangle = \frac{\alpha^2}{(1+\alpha)^2} E_F^2(x-y) , \quad (6.18.b)$$

$$\left\langle 0 \left| T^+ \psi_+(x) \psi_+^\dagger(y) \right| 0 \right\rangle = i\sqrt{2} \Lambda_+ \partial_-^x \Delta_F^2(x-y, M^2) , \quad (6.18.c)$$

where $\Delta_F^2(x, M^2)$ is given in Appendix B.1. Then one finds the perturbative propagator for the gauge fields

$$D_{--}(x) = \frac{\alpha^2}{(1+\alpha)^2} E_F^2(x) , \quad (6.19.a)$$

$$D_{+-}(x) = -\frac{\alpha}{(1+\alpha)^2} \frac{1}{\partial_-} * E_F^1(x) , \quad (6.19.b)$$

$$D_{++}(x) = \frac{1}{(1+\alpha)^2} \frac{1}{\partial_-^2} * \delta(x-y) , \quad (6.19.c)$$

which has non-causal structure for (+) components. This is connected with the infrared singularities of covariant massless fields in the 1+1 dimensional LF formulation [34]. Specially the LC-gauge limit $\alpha \rightarrow 0$ produces the gauge field propagator with the CPV prescription for spurious pole and no causal pole at all.

6.2 Higher-dimensional model

The failure of the 1+1 dimensional approach towards perturbative propagators with causal poles indicates the danger of the infrared singularities also in the 3+1 dimensions. Therefore here we introduce the dimensional regularization by taking $d = D - 1 > 2$ transverse coordinates $\mathbf{x}_\perp = \{x_2, \dots, x_{d+1}\}$. The gauge field sector described by the Lagrangian density

$$\begin{aligned} \mathcal{L}_{flow}^{D+1 \text{ gauge}} &= (\partial_+ A_i - \partial_i A_+) (\partial_- A_i - \partial_i A_-) + \frac{1}{2} (\partial_+ A_- - \partial_- A_+)^2 - \frac{1}{4} (\partial_i A_j - \partial_j A_i)^2 \\ &+ A_- j^- + A_+ j^+ + A_i j^i + \Lambda (\partial_+ A_- + \alpha \partial_- A_+ - \alpha \partial_i A_i) , \end{aligned} \quad (6.20)$$

still contains constraints; therefore, omitting all details which will be presented elsewhere, we can use the effective description

$$\begin{aligned} \tilde{\mathcal{L}}_{flow}^{D+1 \text{ gauge}} &= \partial_- C_i \partial_+ C_i - \frac{1}{2} (\partial_i C_j)^2 - (1+\alpha) \partial_+ \lambda \partial_- \phi + \alpha \partial_i \lambda \partial_i \phi \\ &- \frac{1}{2} \left(\alpha \lambda - \frac{1}{1+\alpha} \frac{1}{\partial_-} * j^+ \right)^2 - \left(\partial_i C_i + \Delta_\perp \phi \frac{\alpha}{1+\alpha} \right) \frac{1}{\partial_-} * j^+ \\ &- \phi (\partial_- j^- + \partial_i j^i) + j^i C_i \end{aligned} \quad (6.21)$$

where the independent modes (C_i, ϕ, λ) are defined by primary fields

$$\phi = \frac{1}{\Delta_{\perp}} * \left[(1 + \alpha)(\partial_i A_i - \partial_- A_+) - \alpha \Lambda - \frac{1}{\partial_-} * j^+ \right], \quad (6.22.a)$$

$$\lambda = \Lambda - \frac{1}{1 + \alpha} \frac{1}{\partial_-} * j^+, \quad (6.22.b)$$

$$C_i = A_i - \partial_i \phi. \quad (6.22.c)$$

Next we add directly the fermion contributions and as the starting point for the quantization of the complete model, we take the effective Lagrangian density

$$\begin{aligned} \mathcal{L}_{flow}^{D+1} = & \partial_- C_i \partial_+ C_i - (1 + \alpha) \partial_+ \lambda \partial_- \phi + i\sqrt{2} \psi_+^\dagger \partial_+ \psi_+ + \sqrt{2} \psi_-^\dagger (i\partial_- - e\partial_- \phi) \psi_- \\ & - \frac{1}{2} (\partial_i C_j)^2 + \alpha \partial_i \lambda \partial_i \phi + \frac{1}{2} \left(\alpha \lambda + \frac{1}{1 + \alpha} \frac{1}{\partial_-} * J^+ \right)^2 \\ & + \left(\partial_i C_i + \Delta_{\perp} \phi \frac{\alpha}{1 + \alpha} \right) \frac{1}{\partial_-} * J^+ - \xi^\dagger \psi_- - \psi_-^\dagger \xi, \end{aligned} \quad (6.23)$$

where

$$\xi = (-i\partial_i \alpha^i + M\beta) \psi_+ + e(C_i + \partial_i \phi) \alpha^i \psi_+, \quad (6.24.a)$$

$$\xi^\dagger = (i\partial_i \psi_+^\dagger \alpha^i + M\psi_+^\dagger \beta) + e\psi_+^\dagger \alpha^i (C_i + \partial_i \phi), \quad (6.24.b)$$

$$J^+ = e\sqrt{2} \psi_+^\dagger \psi_+. \quad (6.24.c)$$

After removing the dependent fermion fields ψ_- and ψ_-^\dagger

$$\psi_- = \frac{1}{\sqrt{2}} \frac{1}{i\partial_- - e\partial_- \phi} * \xi, \quad (6.25.a)$$

$$\psi_-^\dagger = \frac{1}{\sqrt{2}} \xi^\dagger * \frac{1}{i\partial_- - e\partial_- \phi}, \quad (6.25.b)$$

we have the Hamiltonian density

$$\begin{aligned} \mathcal{H}_{flow}^{D+1} = & \frac{1}{2} (\partial_i C_j)^2 - \alpha \partial_i \lambda \partial_i \phi + \frac{1}{2} \left(\alpha \lambda + \frac{1}{1 + \alpha} \frac{1}{\partial_-} * J^+ \right)^2 \\ & - \left(\partial_i C_i + \Delta_{\perp} \phi \frac{\alpha}{1 + \alpha} \right) \frac{1}{\partial_-} * J^+ + \frac{1}{\sqrt{2}} \xi^\dagger \frac{1}{i\partial_- - e\partial_- \phi} * \xi, \end{aligned} \quad (6.26)$$

which depends only on the independent modes, and the non-vanishing (anti)commutators at LF are

$$2\partial_-^x [C_i(\vec{x}), C_j(\vec{y})] = -i\delta_{ij} \delta^3(\vec{x} - \vec{y}), \quad (6.27.a)$$

$$(1 + \alpha) \partial_-^x [\phi(\vec{x}), \lambda(\vec{y})] = i\delta^3(\vec{x} - \vec{y}), \quad (6.27.b)$$

$$\left\{ \psi_+^\dagger(\vec{x}), \psi_+(\vec{y}) \right\} = \frac{1}{\sqrt{2}} \Lambda_+ \delta^3(\vec{x} - \vec{y}). \quad (6.27.c)$$

Therefore we have the field equations

$$\begin{aligned} (2\partial_+ \partial_- - \Delta_{\perp}) C_i = & -e \frac{1}{\sqrt{2}} \xi^\dagger * \frac{1}{i\partial_- - e\partial_- \phi} \alpha^i \psi_+ - e \frac{1}{\sqrt{2}} \psi_+^\dagger \alpha^i \frac{1}{i\partial_- - e\partial_- \phi} * \xi \\ & - \frac{1}{\partial_-} * \partial_i J^+, \end{aligned} \quad (6.28.a)$$

$$[(1 + \alpha)\partial_+\partial_- - \alpha\Delta_\perp]\phi = \alpha^2\lambda + \frac{\alpha}{1 + \alpha}\frac{1}{\partial_-} * J^+, \quad (6.28.b)$$

$$\begin{aligned} [(1 + \alpha)\partial_+\partial_- - \alpha\Delta_\perp]\lambda &= -\frac{e}{\sqrt{2}}\partial_i \left[\xi^\dagger * \frac{1}{i\partial_- - e\partial_- \phi} \alpha^i \psi_+ + \psi_+^\dagger \alpha^i \frac{1}{i\partial_- - e\partial_- \phi} * \xi \right] \\ &\quad - \frac{e}{\sqrt{2}}\partial_- \left[\xi^\dagger * \frac{1}{i\partial_- - e\partial_- \phi} \frac{1}{i\partial_- - e\partial_- \phi} * \xi \right] \\ &\quad - \Delta_\perp \frac{\alpha}{1 + \alpha} \frac{1}{\partial_-} * J^+, \end{aligned} \quad (6.28.c)$$

$$\begin{aligned} i\sqrt{2}\partial_+\psi &= \frac{1}{\sqrt{2}} \left[-i\partial_i \alpha^i + M\beta + e(C_i + \partial_i \phi) \alpha^i \right] \frac{1}{i\partial_- - e\partial_- \phi} * \xi \\ &\quad - e\sqrt{2} \frac{1}{1 + \alpha} \psi_+ \frac{1}{\partial_-} * \left(\alpha\lambda + \frac{1}{1 + \alpha} \frac{1}{\partial_-} * J^+ \right) \\ &\quad + e\sqrt{2} \psi_+ \frac{1}{\partial_-} * \left(\partial_i C_i + \Delta_\perp \phi \frac{\alpha}{1 + \alpha} \right), \end{aligned} \quad (6.28.d)$$

$$\begin{aligned} -i\sqrt{2}\partial_+\psi^\dagger &= \frac{1}{\sqrt{2}} \xi^\dagger * \left[i\bar{\partial}_i \alpha^i + M\beta + e(C_i + \partial_i \phi) \alpha^i \right] \frac{1}{i\partial_- - e\partial_- \phi} \\ &\quad - e\sqrt{2} \frac{1}{1 + \alpha} \psi_+^\dagger \frac{1}{\partial_-} * \left(\alpha\lambda + \frac{1}{1 + \alpha} \frac{1}{\partial_-} * J^+ \right) \\ &\quad + e\sqrt{2} \psi_+^\dagger \frac{1}{\partial_-} * \left(\partial_i C_i + \Delta_\perp \phi \frac{\alpha}{1 + \alpha} \right), \end{aligned} \quad (6.28.e)$$

which describe the quantum theory in the Heisenberg picture. We also notice that for the complete interacting theory one can define a free field

$$\Lambda = \lambda - \frac{1}{1 + \alpha} \frac{1}{\partial_-} * J^+, \quad (6.29)$$

which satisfies noncovariant dynamical equation

$$[(1 + \alpha)\partial_+\partial_- - \alpha\Delta_\perp]\Lambda = 0, \quad (6.30)$$

but has two extra non-vanishing q-number commutators

$$(1 + \alpha)[\partial_- \Lambda(\vec{x}), \psi_+(\vec{y})] = e\delta^3(\vec{x} - \vec{y})\psi_+(\vec{x}), \quad (6.31.a)$$

$$(1 + \alpha)\left[\partial_- \Lambda(\vec{x}), \psi_+^\dagger(\vec{y})\right] = -e\delta^3(\vec{x} - \vec{y})\psi_+^\dagger(\vec{x}). \quad (6.31.b)$$

These properties show that the Λ field can be taken for the specification of physical states via the condition

$$\langle phys | \Lambda(x) | phys \rangle = 0 \quad (6.32)$$

just like in the ET formalism [35], [36]. However in the perturbation calculations, the property of free propagation is far less important than the presence of q-number commutators which can be nontrivial obstacles for Wick's contractions. Therefore when one works with Λ , then one should take other fermion fields

$$\chi(x) = e^{ie\phi(x)}\psi_+(x) \quad (6.33.a)$$

$$\chi^\dagger(x) = \psi_+^\dagger(x)e^{-ie\phi(x)} \quad (6.33.b)$$

which already commute with Λ . However, for these new field operators the Hamiltonian density operator changes drastically (the field ϕ decouples from the fermion currents)

$$\mathcal{H}_{total}^{LC} = \frac{1}{2}(\partial_i C_j)^2 - \alpha\partial_i \Lambda \partial_i \phi + \frac{1}{2} \left(\alpha\lambda + \frac{1}{\partial_-} * J_x^+ \right)^2 - \partial_i C_i \frac{1}{\partial_-} * J_x^+ + \frac{1}{\sqrt{2}} \xi_{LC}^\dagger \frac{1}{i\partial_-} * \xi_{LC}, \quad (6.34)$$

where now we have

$$J_x^+ = \sqrt{2}\chi^\dagger\chi, \quad (6.35.a)$$

$$\xi_{LC} = (-i\partial_i\alpha^i + M\beta + eC_i\alpha^i)\chi, \quad (6.35.b)$$

$$\xi_{LC}^\dagger = (i\partial_i\chi^\dagger\alpha^i + M\chi^\dagger\beta) + e\chi^\dagger\alpha^i C_i. \quad (6.35.c)$$

One can verify that this new system describes the QED for the LC-gauge condition and the perturbative gauge field propagator would have the CPV-prescription for spurious poles. On the one hand this shows how different gauges can be linked together at the level of quantum field operators, while on the other hand this explains how easily the causal poles may be replaced by the CPV ones.

6.2.1 Perturbation theory

The perturbation theory is formulated in the interaction representation with the free Hamiltonian density

$$\mathcal{H}_0^{D+1} = \lim_{\epsilon \rightarrow 0} \mathcal{H}_{flow}^{D+1} = \frac{\alpha}{2}\lambda^2 + \frac{1}{2}(\partial_j C_i)^2 - \alpha\partial_i\phi\partial\lambda + \frac{1}{\sqrt{2}}\xi_0^\dagger \frac{1}{i\partial_-} * \xi_0, \quad (6.36)$$

where

$$\xi_0 = (-i\partial_i\alpha^i + M\beta)\psi_+, \quad (6.37.a)$$

$$\xi_0^\dagger = (i\partial_i\psi_+^\dagger\alpha^i + M\psi_+^\dagger\beta), \quad (6.37.b)$$

and the interaction Hamiltonian density can be divided into two parts

$$\mathcal{H}_{int}^{D+1} = \mathcal{H}_{flow}^{D+1} - \mathcal{H}_0^{D+1} = \mathcal{H}_{int}^1 + \mathcal{H}_{int}^2, \quad (6.38)$$

where

$$\begin{aligned} \mathcal{H}_{int}^1 &= \frac{1}{\sqrt{2}}\xi_0^\dagger \left(\frac{1}{i\partial_- - e\partial_- \phi} - \frac{1}{i\partial_-} \right) * \xi_0 \\ &+ \frac{e}{\sqrt{2}}\psi_+^\dagger\alpha^i(C_i + \partial_i\phi)\frac{1}{i\partial_- - e\partial_- \phi} * \xi_0 + \frac{e}{\sqrt{2}}\xi_0^\dagger\frac{1}{i\partial_- - e\partial_- \phi}\alpha^i * (C_i + \partial_i\phi)\psi_+ \\ &+ \frac{e^2}{\sqrt{2}}\psi_+^\dagger\alpha^j(C_j + \partial_j\phi)\frac{1}{i\partial_- - e\partial_- \phi}\alpha^i * (C_i + \partial_i\phi)\psi_+, \end{aligned} \quad (6.39.a)$$

$$\mathcal{H}_{int}^2 = -\left(\partial_i C_i + \frac{\alpha}{1+\alpha}\Delta_\perp\phi - \frac{\alpha}{1+\alpha}\lambda\right)\frac{1}{\partial_-} * J^+ + \frac{1}{2}\left(\frac{2}{1+\alpha}\right)^2\left(\frac{1}{2\partial_-} * J^+\right)^2. \quad (6.39.b)$$

We notice a close analogy between the present case and the model from Section 2, therefore here we will point out only the differences between these two cases. The first one comes from the free equations of motion

$$[(1+\alpha)\partial_+\partial_- - \alpha\Delta_\perp]\lambda = 0, \quad (6.40.a)$$

$$[(1+\alpha)\partial_+\partial_- - \alpha\Delta_\perp]\phi = \alpha^2\lambda, \quad (6.40.b)$$

where here we have the multipole non-covariant dynamical field ϕ which has no true Fourier representation. However, from the commutation relations (6.27.a), (6.27.b) and (6.27.c) one can derive the

form of chronological products

$$\langle 0 | TC_i(x) C_j(y) | 0 \rangle = \delta_{ij} D_F(x-y), \quad (6.41.a)$$

$$\langle 0 | T \phi(x) \lambda(y) | 0 \rangle = -G_{\alpha F}^1(x-y), \quad (6.41.b)$$

$$\langle 0 | T \phi(x) \phi(y) | 0 \rangle = \alpha^2 G_{\alpha F}^2(x-y), \quad (6.41.c)$$

$$\left\langle 0 \left| T \psi_+(x) \psi_+^\dagger(y) \right| 0 \right\rangle = i\sqrt{2} \Lambda_+ \partial_-^\epsilon \Delta_F(x-y, M^2), \quad (6.41.d)$$

where the covariant $D_F(x)$ and the non-covariant $G_{\alpha F}^{1,2}(x)$ massless Feynman propagator functions are defined in Appendix B.1. The combinations of independent modes which are coupled with fermion currents are

$$\bar{A}_+ = \frac{1}{\partial_-} * \left[\partial_i C_i - \frac{\alpha}{1+\alpha} \lambda + \frac{\alpha}{1+\alpha} \Delta_\perp \phi \right], \quad (6.42.a)$$

$$\bar{A}_- = \partial_- \phi, \quad (6.42.b)$$

$$\bar{A}_i = C_i + \partial_i \phi, \quad (6.42.c)$$

with the chronological products given by

$$\langle 0 | T \bar{A}_i(x) \bar{A}_j(y) | 0 \rangle = \delta_{ij} D_F(x-y) + \alpha^2 \partial_i^\epsilon \partial_j^\epsilon G_{\alpha F}^2(x-y), \quad (6.43.a)$$

$$\langle 0 | T \bar{A}_-(x) \bar{A}_j(y) | 0 \rangle = \alpha^2 \partial_-^\epsilon \partial_j^\epsilon G_{\alpha F}^2(x-y), \quad (6.43.b)$$

$$\langle 0 | T \bar{A}_-(x) \bar{A}_-(y) | 0 \rangle = \alpha^2 \partial_-^\epsilon \partial_-^\epsilon G_{\alpha F}^2(x-y), \quad (6.43.c)$$

$$\langle 0 | T \bar{A}_+(x) \bar{A}_-(y) | 0 \rangle = -\frac{\alpha}{1+\alpha} \left[G_{\alpha F}^1(x-y) + \alpha^2 \Delta_\perp G_{\alpha F}^2(x-y) \right], \quad (6.43.d)$$

$$\langle 0 | T \bar{A}_+(x) \bar{A}_i(y) | 0 \rangle = i(1-\alpha) \partial_+^\epsilon \partial_i^\epsilon (G_{\alpha F}^1 * D_F)(x-y) + \alpha^2 \partial_+^\epsilon \partial_i^\epsilon G_{\alpha F}^2(x-y), \quad (6.43.e)$$

$$\begin{aligned} \langle 0 | T \bar{A}_+(x) \bar{A}_+(y) | 0 \rangle &= i2(1-\alpha) \partial_+^\epsilon \partial_+^\epsilon (G_{\alpha F}^1 * D_F)(x-y) + \alpha^2 \partial_+^\epsilon \partial_+^\epsilon G_{\alpha F}^2(x-y) \\ &+ \frac{i}{(1+\alpha)^2} \partial_-^2 * \delta(x-y), \end{aligned} \quad (6.43.f)$$

where no explicit inverse Laplace operator appear; thus the limit $d \rightarrow 2$ can be taken already at the level of independent modes and the free propagators in 3+1 dimensions have the Fourier representation

$$\begin{aligned} \langle 0 | T \bar{A}_\mu(x) \bar{A}_\nu(y) | 0 \rangle &= i \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x-y)} \left\{ \frac{1}{2k_+ k_- - k_\perp^2 + i\epsilon} \left[-g_{\mu\nu} + \frac{k_+(1-\alpha)[n_\nu^{LC} k_\mu + n_\mu^{LC} k_\nu]}{(1+\alpha)k_+ k_- - \alpha k_\perp^2 + i\epsilon'} \right] \right. \\ &\left. + \frac{\alpha^2 k_\nu k_\mu}{[(1+\alpha)k_+ k_- - \alpha k_\perp^2 + i\epsilon']^2} - \frac{n_\nu^{LC} n_\mu^{LC}}{(1+\alpha)^2} \text{CPV} \frac{1}{k_\perp^2} \right\}, \end{aligned} \quad (6.44)$$

where $n_\pm^{LC} = 1, n_\mp^{LC} = 0$. However, the Wick contractions of gauge fields and direct interactions of currents will give rise to the modification of perturbative propagators

$$D_{\mu\nu}^{low}(x-y) = i \langle 0 | T \bar{A}_\mu(x) \bar{A}_\nu(y) | 0 \rangle + \frac{g_{\mu\nu} - g_{\nu\nu}}{(1+\alpha)^2 \partial_-^2} * \delta(x-y), \quad (6.45)$$

which no longer have any spurious pole with CPV prescription

$$\begin{aligned} D_{\mu\nu}^{low}(x) &= - \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x} \left\{ \frac{1}{2k_+ k_- - k_\perp^2 + i\epsilon} \left[-g_{\mu\nu} + \frac{k_+(1-\alpha)[n_\nu^{LC} k_\mu + n_\mu^{LC} k_\nu]}{(1+\alpha)k_+ k_- - \alpha k_\perp^2 + i\epsilon'} \right] \right. \\ &\left. + \frac{\alpha^2 k_\nu k_\mu}{[(1+\alpha)k_+ k_- - \alpha k_\perp^2 + i\epsilon']^2} \right\}. \end{aligned} \quad (6.46)$$

The cancellation of non-covariant terms during the Wick contractions of fermion fields is identical with that in Section 2, therefore, quoting those previous results, we say that the perturbative calculations based on the interaction Hamiltonians (6.39.a), (6.39.b) and canonical free propagators are equivalent to those with covariant interaction vertices and covariant perturbative propagators. In the perturbative gauge field propagator (6.46) with all causal poles we can take the appropriate limits to the Lorentz gauge ($\alpha \rightarrow 1$)

$$D_{\mu\nu}^{Lor}(x) = -\int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{1}{2k_+k_- - k_\perp^2 + i\epsilon} \left[-g_{\mu\nu} + \frac{k_\nu k_\mu}{[2k_+k_- - \alpha k_\perp^2 + i\epsilon]} \right] \quad (6.47)$$

and to the LC-gauge ($\alpha \rightarrow 0$)²⁵

$$D_{\mu\nu}^{flow}(x) = -\int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{1}{2k_+k_- - k_\perp^2 + i\epsilon} \left[-g_{\mu\nu} + \frac{k_+ [n_\nu^{LC} k_\mu + n_\mu^{LC} k_\nu]}{k_+ k_- + i\epsilon'} \right]. \quad (6.48)$$

When we compare (6.47) with the double limit ($\alpha \rightarrow 0$ and $m^2 \rightarrow 0$) of (2.48) we find that they are the same and also coincide with the ET propagator. The expression (6.48) represents the ML-prescription for the LC-gauge within the LF procedure. Contrary to other attempts [37], [38] which used two front surfaces for free gauge field sector, the present results are valid for the perturbative QED with fermions.

7 Electrodynamics of charged scalar fields

In previous sections we have discussed electrodynamics of charged fermions which describes the phenomena where electrons interact with photons. However it is also interesting to consider charged matter consisting of other fields. In the usual ET approach, scalar fields are treated as the simplest possible choice for matter fields. In the LF formalism, electrodynamics of scalar fields is far from being trivial. Below we will discuss only one choice of gauge fixing condition, the LF Weyl gauge in 1+1 dimensions, which has particularly strange properties. Electromagnetic currents built from scalar fields contain derivatives of matter fields $j_\mu^{scalar} = e\phi^\dagger \overleftrightarrow{\partial}_\mu \phi$, which are dramatically different from the fermion currents. At LF this means that the complete dynamics of scalar fields directly manifests itself in coupling with the gauge fields. While such phenomenon has been recognized a long ago [3], it is only here that a solution to this problem has been found.

7.1 LF Weyl gauge in 1+1 dimensions

When the LF Weyl gauge $A_+ = 0$ is strongly imposed on gauge fields, the electrodynamics of scalar fields is described by the Lagrangian density

$$\mathcal{L}_{Weyl}^{1+1\,scalar} = \frac{1}{2}(\partial_+ A_-)^2 + (\partial_- - ieA_-)\phi^\dagger \partial_+ \phi + \partial_+ \phi^\dagger (\partial_- + ieA_-)\phi - m^2 \phi^\dagger \phi \quad (7.1)$$

and its equations of motion (Euler-Lagrange equations) are

$$\partial_+^2 A_- = -ie(\phi^\dagger \partial_+ \phi - \partial_+ \phi^\dagger \phi), \quad (7.2.a)$$

$$2(\partial_- + ieA_-)\partial_+ \phi = -ie\phi \partial_+ A_- - m^2 \phi, \quad (7.2.b)$$

$$2(\partial_- - ieA_-)\partial_+ \phi^\dagger = ie\phi^\dagger \partial_+ A_- - m^2 \phi^\dagger. \quad (7.2.c)$$

²⁵The ML-prescription is written here in the equivalent (in the sense of distributions) form $\frac{1}{[k_\perp]_{ML}} = \frac{k_+}{k_+ k_- + i\epsilon'}$.

This means that all fields are dynamical variables, but as usually, the Gauss law

$$G = \partial_- \partial_+ A_- - ie \left(\phi^\dagger \partial_- \phi - \phi \partial_- \phi^\dagger \right) + 2e^2 \phi^\dagger \phi A_- \quad (7.3)$$

is missing and has to be added as an extra postulate of the physical quantum theory. The canonical momenta conjugated to all dynamical fields

$$\Pi^- = \partial_+ A_- , \quad (7.4.a)$$

$$\Pi_\phi = (\partial_- - ieA_-) \phi^\dagger , \quad (7.4.b)$$

$$\Pi_{\phi^\dagger} = (\partial_- + ieA_-) \phi , \quad (7.4.c)$$

show the presence of the second class primary constraints (according to Dirac's nomenclature) which are characteristic for LF dynamics of relativistic particles. However, contrary to our previous treatment of similar constraints, due to the presence of other fields we cannot discard (7.4.b and 7.4.c) as superficial constraints. This makes a sharp distinction between these expressions and canonical momenta for A_i in QED or even for free scalar fields. In our case, Dirac's method of quantization for constrained systems would give nonzero brackets (commutators) for scalar fields and Π^- or between two momenta Π^- . All these brackets and also the one for scalar fields would depend functionally on A_- . Thus everyone should understand those who decided to choose the LC-gauge $A_- = 0$ where all above problems totally disappear.

Here we follow our method of dealing with constrained system where we pay special attention to equations of motion. Eqs. (7.2.a-7.2.c) can be rewritten as the first order equations in ∂_+

$$\partial_+ \Pi^- = -ie \left(\phi^\dagger \partial_+ \phi - \partial_+ \phi^\dagger \phi \right) , \quad (7.5.a)$$

$$2(i\partial_- - eA_-) \partial_+ \phi = e\phi \Pi^- - im^2 \phi , \quad (7.5.b)$$

$$2(i\partial_- + eA_-) \partial_+ \phi^\dagger = -e\phi^\dagger \Pi^- - im^2 \phi^\dagger , \quad (7.5.c)$$

$$\partial_+ A_- = \Pi^- . \quad (7.5.d)$$

Then using the Green function $(i\partial_- + eA_-)^{-1}[x^-, y^-]^{26}$ for the covariant partial derivative

$$(i\partial_- - eA_-)^x (i\partial_- - eA_-)^{-1}[x^-, y^-] = -(i\partial_- + eA_-)^y (i\partial_- - eA_-)^{-1}[x^-, y^-] = \delta(x^- - y^-) , \quad (7.6)$$

we can transform these equations into the form of Hamilton equations of motion

$$\partial_+ \phi(x) = \frac{1}{2} \int dy^- (i\partial_- - eA_-)^{-1}[x^-, y^-] (e\phi \Pi^- - im^2 \phi)(x^+, y^-) , \quad (7.7.a)$$

$$\partial_+ \phi^\dagger(x) = \frac{1}{2} \int dy^- (e\phi^\dagger \Pi^- + im^2 \phi^\dagger)(x^+, y^-) (i\partial_- - eA_-)^{-1}[y^-, x^-] , \quad (7.7.b)$$

$$\begin{aligned} \partial_+ \Pi^-(x) &= \frac{e}{2} \phi^\dagger(x) \int dy^- (i\partial_- - eA_-)^{-1}[x^-, y^-] (-ie\phi \Pi^- - m^2 \phi)(x^+, y^-) \\ &\quad + \frac{e}{2} \phi(x) \int dy^- (ie\phi^\dagger \Pi^- - m^2 \phi^\dagger)(x^+, y^-) (i\partial_- - eA_-)^{-1}[y^-, x^-] , \end{aligned} \quad (7.7.c)$$

$$\partial_+ A_-(x) = \Pi^-(x) , \quad (7.7.d)$$

or in the self-explanatory matrix notation (which will be used hereafter)

$$\partial_+ \phi = \frac{1}{2} (i\partial_- - eA_-)^{-1} * (e\phi \Pi^- - im^2 \phi) , \quad (7.8.a)$$

²⁶The perturbative definition of this Green function and the notation for convolutions of integral operators, which is specially suitable for the further analysis, are given in Appendix B.2.

$$\partial_+ \phi^\dagger = \frac{1}{2} (e\phi^\dagger \Pi^- + im^2 \phi^\dagger) * (i\partial_- - eA_-)^{-1}, \quad (7.8.b)$$

$$\partial_+ \Pi^- = \frac{e}{2} \phi^\dagger (i\partial_- - eA_-)^{-1} * (-ie\phi \Pi^- - m^2 \phi) + \frac{e}{2} (ie\phi^\dagger \Pi^- - m^2 \phi^\dagger) * (i\partial_- - eA_-)^{-1} \phi, \quad (7.8.c)$$

$$\partial_+ A_- = \Pi^-. \quad (7.8.d)$$

Now we encounter a real problem because, on the one hand, these equations evidently describe interacting fields, while on the other hand, the canonical Hamiltonian density has a free field form

$$\mathcal{H}_{can}^{QED} = \Pi^- \partial_+ A_- + \Pi_{\phi^\dagger} \partial_+ \phi^\dagger + \Pi_\phi \partial_+ \phi - \mathcal{L}_{Weyl} = \frac{1}{2} (\Pi^-)^2 + m^2 \phi^\dagger \phi \quad (7.9)$$

and does not depend on the coupling constant e . Even more, the generator of translations in the direction x^- is not a kinematical operator but depends on interactions

$$\begin{aligned} P_- &= \int dy^- \left[\Pi^- \partial_- A_- + \Pi_{\phi^\dagger} \partial_- \phi^\dagger + \Pi_\phi \partial_- \phi \right] \\ &= \int dy^- \left[\Pi^- \partial_- A_- + (\partial_- - ieA_-) \phi \partial_- \phi^\dagger + (\partial_- - ieA_-) \phi^\dagger \partial_- \phi \right]. \end{aligned} \quad (7.10)$$

These observations are apparently in a conflict with the original Dirac analysis of the front form of dynamics [1]. The only way out of these problems is to allow the Dirac brackets to contain interactions and it really happens here because we have non-vanishing expressions

$$\left\{ \Pi^-(x^+, x^-), A_-(x^+, y^-) \right\}_{DB} = -\delta(x^- - y^-), \quad (7.11.a)$$

$$\left\{ \Pi^-(x^+, x^-), \phi(x^+, y^-) \right\}_{DB} = -\frac{e}{2} \phi(x^+, x^-) (i\partial_- - eA_-)^{-1} [x^-, y^-], \quad (7.11.b)$$

$$\left\{ \Pi^-(x^+, x^-), \phi^\dagger(x^+, y^-) \right\}_{DB} = -\frac{e}{2} \phi^\dagger(x^+, x^-) (i\partial_- - eA_-)^{-1} [x^-, y^-], \quad (7.11.c)$$

$$\left\{ \phi(x^+, x^-), \phi^\dagger(x^+, y^-) \right\}_{DB} = -\frac{i}{2} (i\partial_- - eA_-)^{-1} [x^-, y^-], \quad (7.11.d)$$

$$\begin{aligned} \left\{ \Pi^-(x^+, x^-), \Pi^-(x^+, y^-) \right\}_{DB} &= -i \frac{e^2}{2} \phi^\dagger(x^+, x^-) (i\partial_- - eA_-)^{-1} [x^-, y^-] \phi(x^+, y^-) \\ &+ i \frac{e^2}{2} \phi^\dagger(x^+, y^-) (i\partial_- - eA_-)^{-1} [y^-, x^-] \phi(x^+, x^-). \end{aligned} \quad (7.11.e)$$

Now it is not difficult to show that these brackets lead to the correct equations of motion for all field variables (7.7.c-7.7.b) and also give the expected translations in the direction x^-

$$\left\{ \Pi^-(x), P_-(x^+) \right\}_{DB} = \partial_- \Pi^-(x), \quad (7.12.a)$$

$$\left\{ A_-(x), P_-(x^+) \right\}_{DB} = \partial_- A_-(x), \quad (7.12.b)$$

$$\left\{ \phi(x), P_-(x^+) \right\}_{DB} = \partial_- \phi(x), \quad (7.12.c)$$

$$\left\{ \phi^\dagger(x), P_-(x^+) \right\}_{DB} = \partial_- \phi^\dagger(x). \quad (7.12.d)$$

If one would like to take these brackets as a basis for respective quantum commutators, while defining the canonical quantum theory, then one would end up with a hopeless problem of the perturbative calculations. When the interaction is not located in the Hamiltonian but in commutators, then the definition of the interaction representation is not unique, if at all possible.

7.2 Dressed scalar fields

In our previous analysis we have found that LF dynamics allows for using various forms of fields which differ in both the evolution equations and the commutator relations (e.g. the Lagrange multiplier fields in Section 6). Thus we expect that there are also various forms of scalar fields, one of them may have free commutators. Usually at LF, the dressed scalar fields have the field-dependent phase factor like

$$\tilde{\phi} = \exp \left\{ -i e (\partial_-)^{-1} * A_- \right\} \phi, \quad (7.13.a)$$

$$\tilde{\phi}^\dagger = \phi^\dagger \exp \left\{ i e A_- * (\partial_-)^{-1} \right\}. \quad (7.13.b)$$

However this really means a gauge transformation to the LC-gauge and evidently is not a true solution for the LF Weyl gauge problem. Therefore we take another possibility and define a new scalar field φ

$$\phi = \mathcal{W}_{-1}[\hat{a}] * \varphi, \quad (7.14)$$

where the integral operator $\mathcal{W}_{-1}[\hat{a}]$ is defined in Appendix B.2. For simplicity, the other scalar field ϕ^\dagger is not changed.²⁷ Now one can easily check that for the pair of scalar fields φ and ϕ^\dagger , their Dirac bracket has already a free form

$$\left\{ \varphi(x^+, x^-), \phi^\dagger(x^+, y^-) \right\}_{DB} = -\frac{i}{2} (i\partial_-)^{-1} (x^- - y^-). \quad (7.15)$$

Encouraged by this result we find the scalar field contribution to the Lagrangian

$$\begin{aligned} L^{*scalar} &= \partial_+ \phi^\dagger * (\partial_- + ieA_-) \phi + (\partial_- - ieA_-) \phi^\dagger * \partial_+ \phi - m^2 \phi^\dagger * \phi = \\ &= \partial_- \phi^\dagger * \partial_+ \varphi + \partial_+ \phi^\dagger * \partial_- \varphi - m^2 \phi^\dagger * \mathcal{W}_{-1}[\hat{a}] * \varphi \\ &\quad + ie \phi^\dagger * \partial_+ A_- \mathcal{W}_{-1}[\hat{a}] * \varphi \end{aligned} \quad (7.16)$$

and then calculate the canonical momenta conjugated to scalar fields

$$\Pi_\varphi = \partial_- \phi^\dagger, \quad (7.17.a)$$

$$\Pi_{\phi^\dagger} = \partial_- \varphi, \quad (7.17.b)$$

which are already free. This means that we have again trivial primary constraints for these momenta which can be discarded in the Hamiltonian approach. Also the third momentum (for A_- gauge field) has been changed

$$\Pi = \partial_+ A_- - \partial_- A_+ + ie \phi^\dagger \mathcal{W}_{-1}[\hat{a}] * \varphi, \quad (7.18)$$

but there are no constraints connected with this mode, so this change introduces no problems into further analysis. Now the complete Hamiltonian for the LF Weyl gauge is

$$\tilde{H}_{can}^{QED} = \frac{1}{2} \int dx^- \left(\Pi - ie \phi^\dagger \mathcal{W}_{-1}[\hat{a}] * \varphi \right)^2 + m^2 \phi^\dagger * \mathcal{W}_{-1}[\hat{a}] * \varphi \quad (7.19)$$

and evidently it contains interactions, while the translation generator

$$P_- = \frac{1}{2} \int dx^- \Pi * \partial_- A_- + 2\partial_- \phi^\dagger * \partial_- \varphi \quad (7.20)$$

²⁷Also another choices of dressed scalar fields are possible, specially the symmetrical choice for both fields. These issues will be discussed elsewhere.

is kinematical. Finally, we give nonzero commutators at LF

$$2\partial_-^x \left[\phi^\dagger(x^+, x^-), \varphi(x^+, y^-) \right] = i\delta(x^- - y^-), \quad (7.21.a)$$

$$\left[\Pi(x^+, x^-), A_-(x^+, y^-) \right] = -i\delta(x^- - y^-), \quad (7.21.b)$$

which also have canonical free forms. Therefore we see that the redefinition (7.14) of the scalar field leads to the canonical description of scalar QED at the LF.

7.3 Perturbation theory

Now we can define the interaction representation with the free evolution of field operators which is given by the free Hamiltonian H_0

$$H_0 = \frac{1}{2} \int dx^- \Pi^2 + m^2 \phi^\dagger \star \phi. \quad (7.22)$$

The evolution of states will be given by the interaction Hamiltonian H_I

$$\begin{aligned} H_I = \tilde{H}_{can}^{QED} - H_0 &= \frac{1}{2} \left(ie\phi_0^\dagger \mathcal{W}_{-1}[\hat{a}] \star \phi_{-1} \right)^2 - ie\phi_0^\dagger \Pi \star \mathcal{W}_{-1}[\hat{a}] \star \phi_{-1} \\ &+ m^2 \phi_0^\dagger \star (\mathcal{W}_{-1}[\hat{a}] - 1) \star \phi_{-1}. \end{aligned} \quad (7.23)$$

At low orders (in e) of the LF perturbation, different contributions generated by the above Hamiltonian formally sum up to the covariant results, which can be obtained within the ET approach.²⁸ However, if we want to prove the equivalence of both perturbations for all orders then we need to follow the methods used earlier in Section 2. In order to separate the effects of scalar and gauge field contractions, we divide the interaction Hamiltonian into two parts

$$H_I = H_I^1 + H_I^2, \quad (7.24.a)$$

$$H_I^1 = \phi_0^\dagger \star \left\{ -ie\Pi + m^2\hat{a} \right\} \star \mathcal{W}_{-1}[\hat{a}] \star \phi_{-1}, \quad (7.24.b)$$

$$H_I^2 = \frac{1}{2} \int dx^- \left(ie\phi_0^\dagger \mathcal{W}_{-1}[\hat{a}] \star \phi_{-1} \right)^2 = \frac{1}{2} \int dx^- J_{\Pi}^2. \quad (7.24.c)$$

The functional form of the Wick theorem [31]

$$T \exp -i \int \sigma J_{\Pi} = : \exp -i \int J_{\Pi} \left(\sigma + i \frac{\delta}{\delta \sigma} \right) : = \exp -\frac{i}{2} \int J_{\Pi}^2 : \exp -i \int \sigma J_{\Pi} :, \quad (7.25)$$

indicates that the Hamiltonian H_I^2 can be equivalently substituted by the linear term $\int \sigma J_{\Pi}$, where the new field σ has non-zero Wick's contractions

$$\langle 0 | T^+ \sigma(x) \sigma(y) | 0 \rangle = i\delta^2(x - y). \quad (7.26)$$

Then from the canonical propagators

$$\langle 0 | T^+ \Pi(x) \Pi(y) | 0 \rangle = 0, \quad (7.27.a)$$

$$\langle 0 | T^+ A_-(x) \Pi(y) | 0 \rangle = E_F^{-1}(x - y), \quad (7.27.b)$$

²⁸The ET interaction Hamiltonian also contains *one* noncovariant term which cancels the noncovariant part of the second time derivative of gauge field propagator [39].

we find the Wick contractions for the modified field $\tilde{\Pi}^- = \Pi^- + \sigma$

$$\langle 0 | T^+ \tilde{\Pi}^-(x) \tilde{\Pi}^-(y) | 0 \rangle = i\delta^2(x-y), \quad (7.28.a)$$

$$\langle 0 | T^+ A_-(x) \tilde{\Pi}^-(y) | 0 \rangle = E_F^-(x-y). \quad (7.28.b)$$

Thus effectively we can take the equivalent interaction Hamiltonian which is bilinear in scalar fields

$$H_I^{eff} = \phi_0^\dagger * \{-ie\tilde{\Pi} + m^2\tilde{a}\} * \mathcal{W}_{-1}[\tilde{a}] * \varphi = \phi_0^\dagger * \tilde{H}_{cov} * \mathcal{W}_{-1}[\tilde{a}] * \varphi. \quad (7.29)$$

Having the Wick contraction for the pair of scalar fields

$$\langle 0 | T^+ \phi^\dagger(x) \varphi(y) | 0 \rangle = \Delta_F(x-y, m^2) \quad (7.30)$$

we can write the functional form of the Wick contractions for scalar fields

$$\begin{aligned} T^+ \exp -i \left\{ \phi^\dagger * \tilde{H}_{cov} * \mathcal{W}_{-1}[\tilde{a}] * \varphi \right\} &= \\ &= \exp -i \left\{ \left(\phi^\dagger + \frac{\delta}{\delta\varphi} * \Delta_F \right) * \tilde{H}_{cov} * \mathcal{W}_{-1}[\tilde{a}] * \left(\varphi + \Delta_F * \frac{\delta}{\delta\phi^\dagger} \right) \right\} = \\ &= : \exp -i \left\{ \phi^\dagger * \tilde{H}_{cov} * \mathcal{W}_{-1}[\tilde{a}] * \left(1 + i\Delta_F * \tilde{H}_{cov} * \mathcal{W}_{-1}[\tilde{a}] \right)^{-1} * \varphi \right\} : \times \\ &\quad \times \exp -\text{Tr} \ln \left(1 + i\Delta_F * \tilde{H}_{cov} * \mathcal{W}_{-1}[\tilde{a}] \right). \end{aligned} \quad (7.31)$$

In \tilde{H}_{cov} there is the integral operator $m^2(i\partial_-)^{-1}$ which has a convolution either with the field ϕ^\dagger

$$\phi_0^\dagger * m^2(i\partial_-)^{-1} = -2i\partial_+ \phi_0^\dagger \quad (7.32)$$

or with the propagator function Δ_F

$$\Delta_F * m^2(i\partial_-)^{-1} = -2i\partial_+ \Delta_F \bar{\partial}_+ + i(i\partial_-)^{-1}. \quad (7.33)$$

Now it is quite easy to check the following factorization:

$$1 - i\Delta_F * \tilde{H}_{cov} * \mathcal{W}_{-1}[\tilde{a}] = (1 - i\Delta_F * H_{cov}) * \mathcal{W}_{-1}[\tilde{a}], \quad (7.34)$$

where

$$H_{cov} = -ie\bar{\Pi} - 2ie\bar{\partial}_+ A_-, \quad (7.35)$$

and next we can write the expression for the effective scalar field contractions

$$\begin{aligned} T^+ \exp -i \left\{ \phi^\dagger * \tilde{H}_{cov} * \mathcal{W}_{-1}[\tilde{a}] * \varphi \right\} &= : \exp -i \left\{ \phi^\dagger * H_{cov} * \left(1 + \Delta_F * H_{cov} \right)^{-1} * \varphi \right\} : \times \\ &\quad \times \exp -\text{Tr} \ln \left(1 + i\Delta_F * H_{cov} \right) \times \exp -\text{Tr} \ln \mathcal{W}_{-1}[\tilde{a}]. \end{aligned} \quad (7.36)$$

In this way, we have found the effective Feynman rules for scalar and gauge fields in the usual form of perturbative propagators (7.30) and (5.28), respectively. The vertices are given by

$$H_{cov} = -ie\partial_+ A_- - 2ie\bar{\partial}_+ A_- \quad (7.37)$$

so at any vertex we have the factor $(2p_+ - k_+)$, where k_+ is the momentum of gauge field line and p_+ is the momentum of scalar field line at its ϕ^\dagger end.²⁹ It is very interesting that here for the scalar field QED all noncovariant contributions boil down to the closed loop factor

$$\exp -\text{Tr} \ln \mathcal{W}_{-1}[\tilde{a}] \approx \exp \text{Tr} \ln [i\partial_- - eA_-] \quad (7.38)$$

and according to the same arguments that we have used for the fermion field case, we can omit these contributions completely.

²⁹Due to the momentum conservation at each vertex, this is equivalent to the symmetrical rules where momenta of both scalar lines are taken, for example see [45].

Part IV

Finite volume QED

8 LF Weyl gauge

8.1 The DLCQ method

Notation for models in a finite volume of LF are generally given in [40]. Below, for convenience, we present some of the basic steps of the DLCQ method underlying these points which are different from [40].

We choose the space of LF as a "hypertorus": $-L < x^- < L$ $-L_\perp < x_\perp < L_\perp$ and impose periodic boundary conditions for boson fields. For fermions we choose antiperiodic boundary conditions and this choice means that, contrary to bosons, fermions will solely have modes with all nonzero components of momentum.

Zero modes of boson fields can be discussed using the classification introduced in [41]. We denote the full gauge field as $V_\mu(\vec{x})$ and this can be expanded in Fourier modes. We distinguish, respectively, the *simple zero mode*

$$\int_{-L}^{+L} \frac{dx^-}{2L} V_\mu(x^-, x_\perp) \quad (8.1)$$

and the *normal mode* gauge field:

$$A_\mu(\vec{x}) \equiv V_\mu(\vec{x}) - \int_{-L}^{+L} \frac{dx^-}{2L} V_\mu(x^-, x_\perp). \quad (8.2)$$

The latter degrees of freedom are known to represent the usual propagating photons in the light-cone representation, and as such we reserve the symbol A_μ to denote them. From the simple zero mode one can build the totally space-independent *global zero modes*

$$q_\mu = \int_{-L}^L \int_{-L_\perp}^{L_\perp} \frac{dx^- d^2 x_\perp}{8LL_\perp^2} V_\mu(\vec{x}), \quad (8.3)$$

where in future we shall write d^3x for $dx^- d^2x_\perp$ and suppress the limits of integration. Evidently, the q_μ are 0 + 1 dimensional fields, namely quantum mechanical variables - thus the notation q . Finally, the simple and global zero modes can be used to build modes with *no* x^- -dependence but *no constant* part in x_\perp , i.e. the *proper zero modes*:

$$a_\mu(x_\perp) = \int_{-L}^{+L} \frac{dx^-}{2L} V_\mu(x^-, x_\perp) - q_\mu. \quad (8.4)$$

When the decomposition of modes is complete we shall refer to one of the normal A_μ , proper zero mode a_μ and global zero mode q_μ sectors.

For all the above sectors one needs the corresponding delta functions. We adopt the notation that the *periodic* three-dimensional delta function is represented as $\delta^{(3)}(\vec{x} - \vec{y})$, which includes the zero modes. For the *antiperiodic* delta function, a subscript 'a' is appended: $\delta_a^{(3)}(\vec{x} - \vec{y})$. The explicit difference between these two objects can be easily seen by expanding in discrete Fourier modes. Next we distinguish the delta functions appropriate for each mode sector for periodic functions. Thus in the normal mode sector we must subtract the x^- independent part of $\delta^{(3)}$, and so define

$$\delta_n^{(3)}(\vec{x} - \vec{y}) \equiv \delta^{(3)}(\vec{x} - \vec{y}) - \frac{1}{2L} \delta^{(2)}(x_\perp - y_\perp). \quad (8.5)$$

In the proper zero mode sector we must subtract the overall two-dimensional volume factor in defining the relevant delta distribution:

$$\delta_p^{(2)}(x_\perp - y_\perp) \equiv \delta^{(2)}(x_\perp - y_\perp) - \frac{1}{4L_\perp^2}. \quad (8.6)$$

When reaching a canonical formulation, the QED Lagrangian, expressed in terms of the complete fields V_μ , ψ and ψ^\dagger , takes the standard form

$$L = \int d^3x \left[-\frac{1}{4}(\partial_\nu V_\nu - \partial_\nu V_\nu)(\partial^\mu V^\nu - \partial^\nu V^\mu) + \bar{\psi}(i\gamma^\mu \partial_\mu - e\gamma^\mu V_\mu - M)\psi \right]. \quad (8.7)$$

Once the boson field is decomposed into its different sectors, $V_\mu(\vec{x}) = A_\mu(\vec{x}) + a_\mu(x_\perp) + q_\mu$, the Lagrangian Eq.(8.7) breaks into three parts

$$L = L_{nm} + L_{pzm} + L_{gzm}, \quad (8.8)$$

where

$$L_{nm} = \int d^3x \left[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \bar{\psi}(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - M)\psi \right], \quad (8.9)$$

$$L_{pzm} = \int d^3x \left[-\frac{1}{4}(\partial_\mu a_\nu - \partial_\nu a_\mu)(\partial^\mu a^\nu - \partial^\nu a^\mu) - e\bar{\psi}\gamma^\mu \psi a_\mu \right], \quad (8.10)$$

$$L_{gzm} = (8LL_\perp^2)^{\frac{1}{2}}(\partial_+ q_-)^2 - eq_\mu \int d^3x \bar{\psi}\gamma^\mu \psi. \quad (8.11)$$

These formulas in turn can be simplified by decomposing the total electromagnetic current, $\mathcal{J}^\mu = -e\bar{\psi}\gamma^\mu\psi$, into its normal mode and proper and global zero mode parts: $\mathcal{J}^\mu = J_{nm}^\mu + J_{pzm}^\mu + Q^\mu$.³⁰ Fermion current, while being bilinear in fermion fields, satisfies periodic boundary conditions and its global zero modes are given by

$$Q^\pm = -e\sqrt{2} \int \frac{d^3x}{8L L_\perp^2} \psi_\pm^\dagger(\vec{x})\psi_\pm(\vec{x}), \quad (8.12.a)$$

$$Q^i = -e \int \frac{d^3x}{8L L_\perp^2} \left[\psi_+^\dagger(\vec{x})\alpha^i\psi_-(\vec{x}) + \psi_-^\dagger(\vec{x})\alpha^i\psi_+(\vec{x}) \right]. \quad (8.12.b)$$

The proper zero modes are

$$J_{pzm}^\pm(x_\perp) = -e\sqrt{2} \int_{-L}^L \frac{dx^-}{2L} \psi_\pm^\dagger(\vec{x})\psi_\pm(\vec{x}) - Q^\pm, \quad (8.13.a)$$

$$J_{pzm}^i(x_\perp) = -e \int_{-L}^L \frac{dx^-}{2L} \left[\psi_+^\dagger(\vec{x})\alpha^i\psi_-(\vec{x}) + \psi_-^\dagger(\vec{x})\alpha^i\psi_+(\vec{x}) \right] - Q^i. \quad (8.13.b)$$

Finally we can give the normal modes for fermion currents as follows:

$$J_{nm}^\pm(\vec{x}) = -e\sqrt{2}\psi_\pm^\dagger(\vec{x})\psi_\pm(\vec{x}) - J_{pzm}^\pm(x_\perp) - Q^\pm, \quad (8.14.a)$$

$$J_{nm}^i(\vec{x}) = -e \left[\psi_+^\dagger(\vec{x})\alpha^i\psi_-(\vec{x}) + \psi_-^\dagger(\vec{x})\alpha^i\psi_+(\vec{x}) \right] - J_{pzm}^i(x_\perp) - Q^i. \quad (8.14.b)$$

³⁰This notation is different from [40] in order to avoid collision with external currents j^μ and fermion currents J^μ .

8.2 Canonical formalism for LF Weyl gauge

The Weyl gauge condition can be imposed strongly, namely $A_+ = a_+ = q_+ = 0$ at the classical level. As usual, this means that Gauss' law (given explicitly later) appears as a constraint to be imposed on the physical states. Our procedure for carrying the canonical procedure is as follows: we will analyse different subsystems, where only one field sector is treated in terms of independent degrees of freedom while the remaining fields are regarded as non-dynamical external fields and/or currents. Then we will exchange non-dynamical modes for effective interactions of dynamical ones and will give a simpler (though nonlocal) Lagrangian where only dynamical fields are present. This will result in a sequence of equivalent effective Lagrangians which contain fewer modes but have the same Euler-Lagrange equations as those which are generated by the primary Lagrangian, provided the constraint equations are implemented for non-dynamical fields. This procedure is based on the observation that different Lagrangians can lead to the same system of Euler-Lagrange equations though they may have very different constraint structure. One can feel free to choose the most suitable one for carrying out the canonical quantization procedure.

8.2.1 Proper zero mode sector

The Lagrangian Eq.(8.10) can be written explicitly in LF coordinates

$$L_{Weyl}^{pzm} = \int d^3x \left[-\partial_i a_- \partial_+ a_i + \frac{1}{2}(\partial_+ a_-)^2 - \frac{1}{4}(\partial_i a_j - \partial_j a_i)^2 + J_{pzm}^+ a_\perp + J_{pzm}^- a_- \right], \quad (8.15)$$

where the Weyl gauge condition $a_+ = 0$ has been explicitly imposed. Because we are interested here in boson fields we will treat fermion currents as arbitrary external currents, however without introducing any distinction in notation we hope that this will cause no misunderstandings. Thus the Lagrangian (8.15) leads to the classical equations of motion

$$\partial_+^2 a_- = \partial_i \partial_+ a_i + J_{pzm}^-, \quad (8.16.a)$$

$$-\partial_i \partial_+ a_- = \Delta_\perp a_i - \partial_i \partial_k a_k + J_{pzm}^i, \quad (8.16.b)$$

which impose the parameterization of the field a_i

$$a_i = -\frac{1}{\Delta_\perp} * \left(J_{pzm}^i + \frac{1}{2L} \partial_i \pi \right). \quad (8.17)$$

This new field π has the Dirac bracket

$$\{\pi(x_\perp), a_-(y_\perp)\}_{DB} = -\delta_p^{(2)}(x_\perp - y_\perp), \quad (8.18)$$

which is the only nonzero bracket in this sector. The canonical Hamiltonian contains the effective nonlocal terms

$$H_{Weyl}^{pzm} = \frac{2L}{2} \int d^2x_\perp J_{pzm}^i \frac{1}{\Delta_\perp} * J_{pzm}^i - \int d^2x_\perp \pi \frac{1}{\Delta_\perp} * \partial_i J_{pzm}^i - 2L \int d^2x_\perp a_- J_{pzm}^-. \quad (8.19)$$

One can check that the effective equations of motion which follow from the above Hamiltonian and bracket agree with the Euler-Lagrange Eqs.(8.16.a, 8.16.b). Therefore our Hamiltonian and brackets describe the same classical system as the primary Lagrangian Eq.(8.15) and one can give an equivalent Lagrangian density

$$L_{eff}^{pzm} = \frac{1}{2L} \pi \partial_+ a_- - \mathcal{H}_D^{pzm} = \frac{1}{2L} \pi \partial_+ a_- + \frac{1}{2L} \pi \frac{1}{\Delta_\perp} * \partial_i J_{pzm}^i + a_- J_{pzm}^- - \frac{1}{2} J_{pzm}^i \frac{1}{\Delta_\perp} * J_{pzm}^i, \quad (8.20)$$

which directly leads to the correct bracket and equations of motion.

8.2.2 Normal mode sector

In the second step, we analyse the sector of normal modes of gauge field potentials A_μ , treating normal modes of electromagnetic currents J_{nm}^μ as arbitrary external sources. From the Lagrangian (8.9) we take these terms which contain A_μ

$$\mathcal{L}_{W_{eyl}}^{nm} = \partial_+ A_i (\partial_- A_i - \partial_i A_-) + \frac{1}{2} (\partial_+ A_-)^2 - \frac{1}{4} (\partial_i A_j - \partial_j A_i)^2 + A_- J^- + A_i J^i, \quad (8.21)$$

and find the Euler-Lagrange equations of motion

$$\partial_+ (\partial_+ A_- - \partial_i A_i) = J_{nm}^-, \quad (8.22.a)$$

$$(2\partial_+ \partial_- - \Delta_\perp) A_i = \partial_i (\partial_+ A_- - \partial_j A_j) + J_{nm}^i. \quad (8.22.b)$$

We see that these equations have the same structure as the respective equations in Section 4 in the limit $\alpha \rightarrow 0$, therefore we can adopt our previous results here. Thus we take a new field $\Pi = \partial_+ A_- - \partial_i A_i$, and get the diagonal structure of Dirac brackets

$$\{\bar{A}_-(\vec{x}), \Pi(\vec{y})\}_{DB} = \delta_n^{(3)}(\vec{x} - \vec{y}), \quad (8.23.a)$$

$$2\partial_-^x \{A_i(\vec{x}), A_j(\vec{y})\}_{DB} = -\delta_{ij} \delta_n^{(3)}(\vec{x} - \vec{y}), \quad (8.23.b)$$

while all other brackets vanish. The canonical Hamiltonian

$$\mathcal{H}_{W_{eyl}}^{nm} = \frac{1}{2} (\Pi)^2 + \Pi \partial_i A_i + \frac{1}{2} (\partial_i A_j)^2 - A_- J_{nm}^- - A_i J_{nm}^i \quad (8.24)$$

generates equations of motion which are equivalent to the previous ones and can be used for defining an effective Lagrangian

$$\begin{aligned} \mathcal{L}_{W_{eyl}}^{nm} &= \partial_+ A_i \partial_- A_i + \Pi \partial_+ A_- - \mathcal{H}_{W_{eyl}}^{nm} \\ &= \partial_+ A_i \partial_- A_i - \frac{1}{2} (\partial_i A_j)^2 - \frac{1}{2} (\Pi)^2 + \Pi (\partial_+ A_- - \partial_i A_i) + A_- J_{nm}^- + A_i J_{nm}^i. \end{aligned} \quad (8.25)$$

Having analysed the canonical structure of the gauge field sector, one can substitute the Lagrangian Eq.(8.10) by Eq.(8.20) and the boson part of Eq.(8.9) by Eq.(8.25) and instead of the total Lagrangian Eq.(8.8), one can work with the effective Lagrangian

$$\begin{aligned} \tilde{\mathcal{L}}_{W_{eyl}}^{eff} &= \partial_+ A_i \partial_- A_i - \frac{1}{2} (\partial_i A_j)^2 - \frac{1}{2} (\Pi)^2 + \Pi (\partial_+ A_- - \partial_i A_i) + \frac{1}{2L} \pi \partial_+ a_- \\ &+ \frac{1}{2} (\partial_+ q_-)^2 + i\sqrt{2} \psi_+^\dagger \partial_+ \psi_+ + i\sqrt{2} \psi_-^\dagger \partial_- \psi_- + i\psi_-^\dagger \alpha^i \partial_i \psi_+ + i\psi_+^\dagger \alpha^i \partial_i \psi_- \\ &- M \psi_+^\dagger \gamma^0 \psi_- - M \psi_-^\dagger \gamma^0 \psi_+ - e\sqrt{2} \psi_-^\dagger \psi_- V_- \\ &- eV'_i (\psi_+^\dagger \alpha^i \psi_- + \psi_-^\dagger \alpha^i \psi_+) - \frac{1}{2} J_{pzm}^i \left(\frac{1}{\Delta_\perp} * J_{pzm}^i \right). \end{aligned} \quad (8.26)$$

In this expression,

$$V'_i = A_i - \frac{1}{2L} \partial_i \frac{1}{\Delta_\perp} * \pi + q_i, \quad (8.27)$$

namely it is the original V_i but with its proper zero mode a_i expressed as in Eq.(8.17) and the new "current-current" interaction subtracted.³¹ The decomposition of V_- remains unchanged.

³¹This term has been explicitly reintroduced as the last term in Eq.(8.26).

8.2.3 Fermion sector

In the next step, we take the fermion part of the effective Lagrangian Eq.(8.26)

$$\begin{aligned} \mathcal{L}_{Weyl}^{fer} &= i\sqrt{2}\psi_+^\dagger\partial_+\psi_+ + i\sqrt{2}\psi_-^\dagger\partial_-\psi_- + i\psi_-^\dagger\alpha^i\partial_i\psi_+ + i\psi_+^\dagger\alpha^i\partial_i\psi_- \\ &- M\psi_+^\dagger\gamma^0\psi_- - M\psi_-^\dagger\gamma^0\psi_+ - e\sqrt{2}\psi_-^\dagger\psi_- V_- \\ &- eV'_+ (\psi_+^\dagger\alpha^i\psi_- + \psi_-^\dagger\alpha^i\psi_+) - \frac{1}{2}j^i (\mathcal{G}_{(\perp)}[0] * j^i) . \end{aligned} \quad (8.28)$$

The non-dynamical modes are the fermion field ψ_- and the global zero mode q_i . These are determined by the following differential equation and global integral condition:

$$(i\sqrt{2}\partial_- - e\sqrt{2}V_-) \psi_- = -i\alpha^i\partial_i\psi_+ + M\gamma^0\psi_+ + e\alpha^i \left(V_i - \frac{1}{\Delta_\perp} * J_{pzm}^i \right) \psi_+ , \quad (8.29)$$

$$0 = Q^i = \frac{1}{8LL_\perp^2} \int d^3x (\psi_+^\dagger\alpha^i\psi_- + \psi_-^\dagger\alpha^i\psi_+) (\vec{x}) . \quad (8.30)$$

The first equation leads to

$$\psi_- (\vec{x}) = \frac{1}{\sqrt{2}} \frac{1}{i\partial_- - eV_-} * \xi (\vec{x}) - \frac{e}{\sqrt{2}} \frac{1}{\Delta_\perp} * (J_{pzm}^i(x_\perp) - q_i) \alpha^i \frac{1}{i\partial_- - eV_-} * \psi_+ (\vec{x}) , \quad (8.31)$$

where

$$\xi (\vec{x}) = \left[M\gamma^0 - i\alpha^i\partial_i + e\alpha^i A_i (\vec{x}) - e\alpha^i \frac{1}{2L} \partial_i \left(\frac{1}{\Delta_\perp} * \pi \right) (\vec{x}) \right] \psi_+ (\vec{x}) . \quad (8.32)$$

Note that these are not yet the solutions for the dependent fermion field ψ_- because these fields appear also on the right-hand side in the zero mode currents J_{pzm}^i . However, one can introduce them into the definition of $J_{pzm}^i + Q^i$ (see Eqs.(8.12.b,8.13.b))

$$J_{pzm}^i(x_\perp) + Q^i = -\frac{e}{2L} \Gamma^i(x_\perp) + \frac{e^2}{2L} \mathcal{M}^{ik}(x_\perp) \left[\left(\frac{1}{\Delta_\perp} * J_{pzm}^k \right) (x_\perp) - q_k \right] , \quad (8.33)$$

where

$$\begin{aligned} \Gamma^i(x_\perp) &= \frac{1}{\sqrt{2}} \int dx^- \psi_+^\dagger (\vec{x}) \alpha^i \left(\frac{1}{i\partial_- - eV_-} * \xi \right) (\vec{x}) \\ &+ \frac{1}{\sqrt{2}} \int dx^- \xi^\dagger (\vec{x}) \alpha^i \left(\frac{1}{i\partial_- - eV_-} * \psi_+ \right) (\vec{x}) , \end{aligned} \quad (8.34.a)$$

$$\mathcal{M}^{ij}(x_\perp) = \delta^{ij} \sqrt{2} \int dx^- \psi_+^\dagger (\vec{x}) \left(\frac{1}{i\partial_- - eV_-} * \psi_+ \right) (\vec{x}) = \delta^{ij} \mathcal{M}^2(x_\perp) . \quad (8.34.b)$$

Now, from the constraint $Q^i = 0$ one gets the differential equation

$$\left[\Delta_\perp - \frac{e^2}{2L} \mathcal{M}^2(x_\perp) \right] \left[\left(\frac{1}{\Delta_\perp} * J_{pzm}^i \right) (x_\perp) - q_i \right] = -\frac{e}{2L} \Gamma^i(x_\perp) , \quad (8.35)$$

which has a formal solution

$$\left(\frac{1}{\Delta_\perp} * J_{pzm}^i \right) (x_\perp) - q_i = -\frac{e}{2L} \int d^2y_\perp \mathcal{G}_{(\perp)}[x_\perp, y_\perp; \mathcal{M}^2] \Gamma^i(y_\perp) \quad (8.36)$$

in terms of the functional $\mathcal{G}_{(\perp)}[x_\perp, y_\perp; \mathcal{M}^2]$ introduced in Eq.(B.26). Finally, we may express the non-dynamical fermion field as

$$\psi_- (\vec{x}) = \frac{1}{\sqrt{2}} \left(\frac{1}{i\partial_- - eV_-} * \xi \right) (\vec{x}) + \frac{e^2}{2\sqrt{2}L} \left(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \Gamma^i \right) (x_\perp) \left(\frac{1}{i\partial_- - eV_-} * \psi_+ \right) (\vec{x}) , \quad (8.37)$$

whereby we obtain

$$H_D^{fer} = \frac{1}{\sqrt{2}} \int d^3x \xi^\dagger(\vec{x}) \left(\frac{1}{i\partial_- - eV_-} * \xi \right) (\vec{x}) + \frac{1}{2} \frac{e^2}{2L} \int d^2x_\perp \Gamma^i(x_\perp) \left(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \Gamma^i \right) (x_\perp) \quad (8.38)$$

as the Dirac Hamiltonian for unconstrained fields. Just as before we can give the effective classical Lagrangian for the fermions

$$\begin{aligned} \mathcal{L}_{eff}^{fer} &= (\partial_+ \psi_+) \Pi_{\psi_+} - \mathcal{H}_D^{fer} = i\sqrt{2} \psi_+^\dagger \partial_+ \psi_+ \\ &- \frac{1}{\sqrt{2}} \xi^\dagger \left(\frac{1}{i\partial_- - eV_-} * \xi \right) - \frac{1}{2} \frac{e^2}{4L^2} \Gamma^i \left(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \Gamma^i \right). \end{aligned} \quad (8.39)$$

8.3 Quantum theory

Having eliminated all non-dynamical fields we may now proceed by substituting the fermion part of Eq.(8.26) by that given in Eq.(8.39). We thereby obtain the total effective Lagrangian

$$\begin{aligned} \mathcal{L}_{eff} &= \partial_+ A_i \partial_- A_i - \frac{1}{2} (\partial_i A_j)^2 - \frac{1}{2} (\Pi)^2 + \Pi (\partial_+ A_- - \partial_i A_i) - \frac{1}{2L} \pi \partial_+ a_- + \frac{1}{2} (\partial_+ q_-)^2 \\ &+ i\sqrt{2} \psi_+^\dagger \partial_i \psi_+ - \frac{1}{\sqrt{2}} \xi^\dagger \left(\frac{1}{i\partial_- - eV_-} * \xi \right) - \frac{1}{2} \frac{e^2}{4L^2} \Gamma^i \left(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \Gamma^i \right) \end{aligned} \quad (8.40)$$

as the starting point for the canonical quantisation procedure. It generates the Euler-Lagrange equations of motion which agree with the dynamical equations of the primary Lagrangian Eq.(8.7) with all non-dynamical equations (formally) implemented. The canonical quantisation is simple here and one gets the equal- x^+ quantum commutation relations

$$[\Pi(\vec{x}), A_-(\vec{y})] = -i\delta_n^{(3)}(\vec{x} - \vec{y}), \quad (8.41.a)$$

$$\{\psi_+^\dagger(\vec{x}), \psi_+(\vec{y})\} = \frac{1}{\sqrt{2}} \Lambda_a \delta_a^{(3)}(\vec{x} - \vec{y}), \quad (8.41.b)$$

$$[\pi(x_\perp), a_-(y_\perp)] = -i\delta_p^{(2)}(x_\perp - y_\perp), \quad (8.41.c)$$

$$2\partial_-^\pm [A_i(\vec{x}), A_j(\vec{y})] = -i\delta_{ij} \delta_n^{(3)}(\vec{x} - \vec{y}), \quad (8.41.d)$$

$$[p^-, q_-] = -i, \quad (8.41.e)$$

and the quantum Hamiltonian, like \mathcal{L}_{eff} , which comes from \mathcal{H}_D^{pzm} , \mathcal{H}_D^{nm} and H_D^{fer} ,

$$\begin{aligned} H_{eff} &= \int d^3x \left[\frac{1}{2} (\Pi(\vec{x}))^2 + \Pi \partial_i A_i(\vec{x}) + \frac{1}{2} (\partial_i A_j(\vec{x}))^2 \right] + \frac{1}{16LL_1^2} (p^-)^2 \\ &+ \frac{1}{\sqrt{2}} \int d^3x \xi^\dagger(\vec{x}) \left(\frac{1}{i\partial_- - eV_-} * \xi \right) (\vec{x}) + \frac{1}{2} \frac{e^2}{2L} \int d^2x_\perp \Gamma^i(x_\perp) \left(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \Gamma^i \right) (x_\perp), \end{aligned} \quad (8.42)$$

tacitly defining the ordering. One can show that, due to the effective equations of motion, the Gauss law operator

$$G(\vec{x}) = \partial_- \Pi(\vec{x}) + 2\partial_- \partial_i A_i(\vec{x}) - \Delta_\perp A_-(\vec{x}) - \Delta_\perp a_-(x_\perp) - e\sqrt{2} \psi_+^\dagger(\vec{x}) \psi_+(\vec{x}) \quad (8.43)$$

is x^+ -independent. It leads to a classical first class constraint, $G \simeq 0$; namely it must annihilate physical states in the quantum theory. Furthermore, it is intimately connected to the residual gauge

symmetry

$$A_i(\vec{x})^h = \Omega_h A_i(\vec{x}) \Omega_h^\dagger = A_i(\vec{x}) - \partial_i h(\vec{x}), \quad (8.44.a)$$

$$A_-(\vec{x})^h = \Omega_h A_-(\vec{x}) \Omega_h^\dagger = A_-(\vec{x}) - \partial_- h(\vec{x}), \quad (8.44.b)$$

$$\Pi(\vec{x})^h = \Omega_h \Pi(\vec{x}) \Omega_h^\dagger = \Pi(\vec{x}) + \Delta_\perp h(\vec{x}), \quad (8.44.c)$$

$$\psi_+(\vec{x})^h = \Omega_h \psi_+(\vec{x}) \Omega_h^\dagger = e^{ieh(\vec{x})} \psi_+(\vec{x}), \quad (8.44.d)$$

$$a_-(x_\perp)^h = \Omega_h a_-(x_\perp) \Omega_h^\dagger = a_-(x_\perp), \quad (8.44.e)$$

$$\pi(x_\perp)^h = \Omega_h \pi(x_\perp) \Omega_h^\dagger = \pi(x_\perp) + \Delta_\perp \int_{-L}^L dy^- h(y^-, x_\perp), \quad (8.44.f)$$

$$p^{-h} = \Omega_h p^- \Omega_h^\dagger = p^-, \quad (8.44.g)$$

where

$$\Omega_h = e^{i \int d^3x h(x) G(x)}. \quad (8.45)$$

8.3.1 Translation Generators

With the effective Lagrangian density Eq.(8.40) one can calculate the canonical momentum-energy tensor

$$\begin{aligned} T^{\nu\mu} &= \frac{\delta \mathcal{L}_{eff}}{\delta (\partial_\nu A_i)} \partial^\mu A_i + \partial^\mu \psi_+ \frac{\delta \mathcal{L}_{eff}}{\delta (\partial_\nu \psi_+)} + \frac{1}{2L} \frac{\delta \mathcal{L}_{eff}}{\delta (\partial_\nu \pi)} \partial^\mu \pi \\ &+ \frac{1}{2L} \frac{\delta \mathcal{L}_{eff}}{\delta (\partial_\nu a_-)} \partial^\mu a_- + \frac{1}{8LL^2} \frac{\delta \mathcal{L}_{eff}}{\delta (\partial_\nu q_-)} \partial^\mu q_- - g^{\nu\mu} \mathcal{L}_{eff}. \end{aligned} \quad (8.46)$$

Then from the generators of translations $P^\mu = \int d^3x T^{\mu 0}(\vec{x})$, the spatial translations are

$$P^i = \int d^3x \left[-\partial_- A_k \partial_i A_k - \Pi \partial_i A_- - i\sqrt{2} \psi_+^\dagger \partial_i \psi_+ - \frac{1}{2L} \pi \partial_i a_- \right], \quad (8.47)$$

$$P^+ = \int d^3x \left[\partial_- A_k \partial_- A_k + \Pi \partial_- A_- + i\sqrt{2} \psi_+^\dagger \partial_- \psi_+ \right]. \quad (8.48)$$

Now from the (anti)commutation relations Eqs.(8.41.a-8.41.c) one can recover the correct Heisenberg relations for all dynamical quantum fields $\varphi_J = (A_k, A_-, \Pi, \pi, a_-)$

$$\partial_i \varphi_J = -i [P^i, \varphi_J], \quad (8.49.a)$$

$$\partial_- \varphi_J = i [P^+, \varphi_J], \quad (8.49.b)$$

and this confirms the translation invariance of QED in the Weyl gauge. We note that the generators P^+ and P^i are not invariant under the residual gauge transformation with gauge function $h(\vec{x})$:

$$P_h^+ = P^+ + \int d^3x G(\vec{x}) \partial_- h(\vec{x}), \quad (8.50.a)$$

$$P_h^i = P^i - \int d^3x G(\vec{x}) \partial_i h(\vec{x}), \quad (8.50.b)$$

and this is connected with the lack of gauge invariance of the canonical energy-momentum tensor. We return to this below.

8.3.2 Implementation of Gauss' Law

As we have mentioned earlier, the physical quantum gauge system has to satisfy Gauss' law [42]. However this cannot be implemented *strongly* as the condition for quantum field operators which would be incompatible with the commutation relations but rather as the condition for physical states

$$G(\vec{x})|\text{phys}\rangle = 0. \quad (8.51)$$

In [40] the method of quantum mechanical gauge fixing [43] has been used and details of this formalism are given therein. Here we will give only some most important results. Two gauge transformations are defined with the help of the Gauss law operator

$$U_1[\vartheta] = \exp\left(-i \int d^3x g(\vec{x})\vartheta[\vec{x}; A_-]\right), \quad (8.52.a)$$

$$U_2[\eta] = \exp\left(i \int d^2x_\perp \rho_2(x_\perp)\eta[x_\perp; \pi]\right), \quad (8.52.b)$$

where

$$\vartheta[\vec{x}; A_-] = \left(\frac{1}{\partial_-} * A_-\right)(\vec{x}), \quad (8.53.a)$$

$$g(\vec{x}) \equiv G(\vec{x}) - \partial_- \Pi(\vec{x}) = 2\partial_- \partial_i A_i - \Delta_\perp A_- - \Delta_\perp a_- - e\sqrt{2}\psi_+^\dagger \psi_+, \quad (8.53.b)$$

$$\eta[x_\perp; \pi] = -\frac{1}{2L} \left(\frac{1}{\Delta_\perp} * \pi\right)(x_\perp), \quad (8.53.c)$$

$$\rho_2 = \frac{e\sqrt{2}}{2L} \int dx^- \psi_+^\dagger \psi_+. \quad (8.53.d)$$

They allow us to express the Gauss law condition (8.51) in two pairs

$$\Pi|\text{phys}'\rangle = 0, \quad (8.54.a)$$

$$\int \frac{dx^-}{2L} g(\vec{x})|\text{phys}'\rangle = 0, \quad (8.54.b)$$

where

$$|\text{phys}'\rangle \equiv U_1|\text{phys}\rangle, \quad (8.55)$$

and

$$a_-(x_\perp)|\text{phys}''\rangle = 0, \quad (8.56.a)$$

$$Q|\text{phys}''\rangle = \int d^2x_\perp \rho_2(x_\perp)|\text{phys}''\rangle = 0, \quad (8.56.b)$$

where

$$|\text{phys}''\rangle \equiv U_2|\text{phys}'\rangle. \quad (8.57)$$

Here we have the neutrality condition (8.56.b) as the only one constraint which survives from the infinite number of primary constraints which have no physical meaning.

Also the *physical Hamiltonian* can be defined as it acts on physical states

$$U_2 U_1 H U_1^\dagger U_2^\dagger |\text{phys}''\rangle = H_{\text{in}}^{W\text{eyl}} |\text{phys}''\rangle \quad (8.58)$$

and due to (8.54.a and 8.56.a) the field operators Π and a_- can be omitted as cyclic variables

$$\begin{aligned}
H_{fin}^{Weyl} &= \int d^3x \left[\frac{1}{2} \left(\partial_j A_j - \sqrt{2}e \frac{1}{\Delta_-} * (\psi_+^\dagger \psi_+) \right)^2 + \frac{1}{4} (\partial_i A_j - \partial_j A_i)^2 \right] \\
&+ \frac{1}{16LL^2} (p^-)^2 + \frac{1}{\sqrt{2}} \int d^3x \chi^\dagger(\vec{x}) \left(\frac{1}{i\partial_- - e \left[q'_- - \frac{1}{\Delta_-} * \rho_2 \right]} * \chi \right) (\vec{x}) \\
&+ \frac{1}{2} \frac{e^2}{2L} \int d^2x_\perp \Gamma''^n(x_\perp) \left(\mathcal{G}_{(\perp)}[\mathcal{M}''^2] * \Gamma''^n \right) (x_\perp). \tag{8.59}
\end{aligned}$$

The operators Γ'' and \mathcal{M}'' follow directly from the respective operators Γ and \mathcal{M} in which ξ is replaced by

$$\chi = [m\gamma_0 - i\alpha^i \partial_i + e\alpha^i A_i] \psi_+ \tag{8.60}$$

and V_- by

$$q'_- - \frac{1}{\Delta_-} * \rho_2, \tag{8.61}$$

where

$$q'_- = q_- - \frac{e}{2L} \frac{1}{\Delta_-} * \delta^2(0). \tag{8.62}$$

This redefinition of q'_- can be viewed as an infinite renormalization of the field, which does not change the commutators.

Similarly for the generator of other translations we have

$$U_2 U_1 P^i U_1^\dagger U_2^\dagger = - \int d^3x \left(\partial_- A_j \partial_i A_j + i\sqrt{2} \psi_+^\dagger \partial_i \psi_+ \right) \equiv P_{fin}^i, \tag{8.63.a}$$

$$U_2 U_1 P^+ U_1^\dagger U_2^\dagger = \int d^3x \left(\partial_- A_j \partial_- A_j + i\sqrt{2} \psi_+^\dagger \partial_- \psi_+ \right) \equiv P_{fin}^+, \tag{8.63.b}$$

which means their invariance in the subspace of physical states.

Also we stress that in the physical operators, the dependence on the proper zero mode a_- has totally disappeared which contradicts the naive interpretation of this mode as a gauge-invariant physical field.

9 Covariant gauge

In the finite volume LF the Lorentz covariant gauge condition $\partial_\mu V^\mu = 0$ would be equivalent to three independent conditions for each sector

$$\partial_\mu V^\mu = 0 \implies \begin{cases} \partial_+ A_- + \partial_- A_+ - \partial_i A_i = 0 \\ \partial_+ a_- - \partial_i a_i = 0 \\ \partial_+ q_- = 0. \end{cases} \tag{9.1}$$

However the last condition for global zero modes is not *attainable* by means of any gauge transformation because mode q_- is *gauge-invariant*.³² Therefore here as the Lorentz covariant gauge we take the following conditions:

$$\partial_+ A_- + \partial_- A_+ - \partial_i A_i = 0, \tag{9.2.a}$$

$$\partial_+ a_- - \partial_i a_i = 0, \tag{9.2.b}$$

$$q_+ = 0, \tag{9.2.c}$$

³²This resembles the situation in the LC-gauge in the sector of zero modes, where conditions $a_- = q_- = 0$ are not attainable as gauge conditions [41], [44].

with the LF Weyl gauge for global zero modes. In Lagrangians for different gauge field modes the first two conditions will be implemented by means of the Lagrange multipliers Λ_{nm} and Λ_{pzm} , respectively, while the third one will be imposed explicitly.

9.1 Normal mode sector

In the sector of normal modes we start with the Lagrangian density

$$\begin{aligned} \mathcal{L}_{cov}^{nm} = & (\partial_+ A_i - \partial_i A_+) (\partial_- A_i - \partial_i A_-) + \frac{1}{2} (\partial_+ A_- - \partial_- A_+)^2 - \frac{1}{4} (\partial_i A_j - \partial_j A_i)^2 \\ & + A_- J_{nm}^- + A_+ J_{nm}^+ + A_i J_{nm}^i + \Lambda_{nm} (\partial_+ A_- + \partial_- A_+ - \partial_\perp A_\perp). \end{aligned} \quad (9.3)$$

Next we introduce the canonical momentum conjugated with A_-

$$\Pi^- = \partial_+ A_- - \partial_- A_+ + \Lambda_{nm}, \quad (9.4)$$

and then modify the Lagrange multiplier field

$$\lambda_{nm} = \Lambda_{nm} - \frac{1}{2\partial_-} * J_{nm}^+, \quad (9.5)$$

in order to separate dynamical equations

$$[2\partial_+ \partial_- - \Delta_\perp] \lambda_{nm} = \Delta_\perp \frac{1}{2\partial_-} * J_{nm}^+ + \partial_- J_{nm}^- + \partial_i J_{nm}^i, \quad (9.6.a)$$

$$(2\partial_+ \partial_- - \Delta_\perp) \Pi^- = 2\partial_- J_{nm}^- + \partial_i J_{nm}^i, \quad (9.6.b)$$

$$(2\partial_+ \partial_- - \Delta_\perp) A_i = \partial_i \lambda_{nm} + J_{nm}^i + \frac{1}{2\partial_-} * J_{nm}^+, \quad (9.6.c)$$

from the constraints

$$A_- = \frac{1}{\Delta_\perp} * \partial_- [\Pi^- + \partial_i A_i - 2\lambda_{nm}], \quad (9.7.a)$$

$$A_+ = \frac{1}{2\partial_-} * \left[\partial_i A_i + \lambda_{nm} - \Pi^- + \frac{1}{2\partial_-} * J_{nm}^+ \right]. \quad (9.7.b)$$

From the canonical Hamiltonian density

$$\begin{aligned} \mathcal{H}_{cov}^{nm} = & (\partial_+ A_-) \Pi^- + (\partial_+ A_i) \Pi^i - \mathcal{L} = \frac{1}{2} \left(\Pi^- - \lambda_{nm} - \frac{1}{2\partial_-} * J_{nm}^+ \right)^2 + \frac{1}{4} (\partial_i A_j - \partial_j A_i)^2 \\ & + \left(\lambda_{nm} + \frac{1}{2\partial_-} * J_{nm}^+ \right) \partial_i A_i - \partial_- [\Pi^- + \partial_i A_i - 2\lambda_{nm}] \frac{1}{\Delta_\perp} * J_{nm}^- - A_i J_{nm}^i \end{aligned} \quad (9.8)$$

and the dynamical equations of motion we find the Dirac brackets

$$2\partial_-^x \{A_i(\vec{x}), \Pi^-(\vec{y})\}_{DB} = \partial_i^x \delta^3(\vec{x} - \vec{y}), \quad (9.9.a)$$

$$2\partial_-^x \{\Pi^-(\vec{x}), \Pi^-(\vec{y})\}_{DB} = \Delta_\perp \delta^3(\vec{x} - \vec{y}), \quad (9.9.b)$$

$$2\partial_-^x \{A_i(\vec{x}), A_j(\vec{y})\}_{DB} = -\delta_{ij} \delta^3(\vec{x} - \vec{y}), \quad (9.9.c)$$

$$2\partial_-^x \{\lambda_{nm}(\vec{x}), A_i(\vec{y})\}_{DB} = -\partial_i^x \delta^3(\vec{x} - \vec{y}), \quad (9.9.d)$$

while all other brackets vanish. However, another choice of field variables

$$\phi = \frac{1}{\Delta_\perp} * (\Pi^- - 2\lambda_{nm} + \partial_i A_i), \quad (9.10.a)$$

$$C_i = A_i - \partial_i \frac{1}{\Delta_\perp} * (\Pi + \partial_j A_j) = A_i - \partial_i \phi, \quad (9.10.b)$$

leads to a simpler structure of Dirac brackets

$$2\partial_-^x \{C_i(\vec{x}), C_j(\vec{y})\}_{DB} = -\delta_{ij}\delta^3(\vec{x} - \vec{y}), \quad (9.11.a)$$

$$2\partial_-^x \{\phi(\vec{x}), \lambda_{nm}(\vec{y})\}_{DB} = \delta^3(\vec{x} - \vec{y}), \quad (9.11.b)$$

and dynamical equations are separated

$$[2\partial_+\partial_- - \Delta_\perp]\phi = \lambda_{nm} - \frac{1}{2\partial_-} * J_{nm}^+, \quad (9.12.a)$$

$$[2\partial_+\partial_- - \Delta_\perp]\lambda_{nm} = \Delta_\perp \frac{1}{2\partial_-} * J_{nm}^+ + \partial_- J_{nm}^- + \partial_i J_{nm}^i, \quad (9.12.b)$$

$$(2\partial_+\partial_- - \Delta_\perp)C_i = J_{nm}^i + \partial_i \frac{1}{2\partial_-} * J_{nm}^+. \quad (9.12.c)$$

Next from the Hamiltonian density

$$\begin{aligned} \mathcal{H}_{cov}^{nm} &= \frac{1}{2} \left(\lambda_{nm} - \partial_i C_i - \frac{1}{2\partial_-} * J_{nm}^+ \right)^2 + \frac{1}{4} (\partial_i C_j - \partial_j C_i)^2 + \\ &+ (\Delta_\perp \phi + \partial_i C_i) \left(\lambda_{nm} + \frac{1}{2\partial_-} * J_{nm}^+ \right) + \phi (\partial_- J_{nm}^- + \partial_i J_{nm}^i) - J_{nm}^i C_i \end{aligned} \quad (9.13)$$

and the diagonal Dirac brackets (9.11.a, 9.11.b) we can construct the effective Lagrangian density for the sector of normal modes

$$\begin{aligned} \mathcal{L}_{cov}^{nm, eff} &= \partial_- C_i \partial_+ C_i - \frac{1}{2} (\partial_i C_j)^2 - 2\partial_+ \lambda_{nm} \partial_- \phi + \partial_i \lambda_{nm} \partial_i \phi - \frac{1}{2} \left(\lambda_{nm} - \frac{1}{2\partial_-} * J_{nm}^+ \right)^2 \\ &- (2\partial_i C_i + \Delta_\perp \phi) \frac{1}{2\partial_-} * J_{nm}^+ - \phi (\partial_- J_{nm}^- + \partial_i J_{nm}^i) + J_{nm}^i C_i. \end{aligned} \quad (9.14)$$

9.2 Proper zero mode sector

This sector is specially interesting because both its Lagrangian density

$$\mathcal{L}_{cov}^{pzm} = \partial_i a_- (\partial_i a_+ - \partial_+ a_i) + \frac{1}{2} (\partial_+ a_-)^2 - \frac{1}{4} (\partial_i a_j - \partial_j a_i)^2 + a_\mu J_{pzm}^\mu + \Lambda_{pzm} (\partial_+ a_- - \partial_i a_i) \quad (9.15)$$

and the Euler-Lagrange equations

$$\partial_+^2 a_- = \partial_i (\partial_+ a_i - \partial_i a_+) + J_{pzm}^- - \partial_+ \Lambda_{pzm}, \quad (9.16.a)$$

$$-\partial_+ \partial_i a_- = \Delta_\perp a_i - \partial_i \partial_j a^j + J_{pzm}^i + \partial_i \Lambda_{pzm}, \quad (9.16.b)$$

$$0 = -\Delta_\perp a_- + J_{pzm}^+, \quad (9.16.c)$$

$$\partial_+ a_- = \partial_i a_i, \quad (9.16.d)$$

apparently indicate dynamical modes. However, a closer inspection shows that all fields are non-dynamical and explicitly depend on arbitrary external currents J_{pzm}^μ

$$a_\mu = \frac{1}{\Delta_\perp} * (J_{pzm}^\mu + \partial_\mu \Lambda_{pzm}), \quad (9.17.a)$$

$$\Lambda_{pzm} = -\frac{1}{\Delta_\perp} * (\partial_+ J_{pzm}^+ + \partial_i J_{pzm}^i). \quad (9.17.b)$$

Therefore there is no canonical structure here and this resembles the massive QED in Section 1. All we can do now is to find the effective Lagrangian which properly couples the dependent gauge potentials (9.17.a) with external currents. The best choice is

$$\mathcal{L}_{cov}^{pzm, eff} = J_{pzm}^- \frac{1}{\Delta_\perp} J_{pzm}^+ + \frac{1}{2} \Lambda_{pzm}^2 - \partial_\mu \Lambda_{pzm} \frac{1}{\Delta_\perp} J_{pzm}^\mu - \frac{1}{2} J_{pzm}^i \frac{1}{\Delta_\perp} J_{pzm}^i, \quad (9.18)$$

where the non-dynamical field Λ_{pzm} is left in order to avoid the term $\partial_+ J_{pzm}^+$.

9.3 Fermion field sector

Gathering our effective Lagrangians (9.14) and (9.18) we have the total effective Lagrangian density for QED in the Lorentz covariant gauge

$$\begin{aligned} \mathcal{L}_{cov}^{eff} = & \partial_- C_i \partial_+ C_i - \frac{1}{2} (\partial_i C_j)^2 - 2\partial_+ \lambda_{nm} \partial_- \phi + \partial_i \lambda_{nm} \partial_i \phi - \frac{1}{2} \lambda_{nm}^2 - \frac{1}{2} \left(\frac{1}{2\partial_-} * J_{nm}^+ \right)^2 \\ & + \frac{1}{2} \Lambda_{pzm}^2 + i\sqrt{2} \psi_+^\dagger \partial_+ \psi_+ + i\sqrt{2} \psi_-^\dagger \partial_- \psi_- + i\psi_-^\dagger \alpha^i \partial_i \psi_+ + i\psi_+^\dagger \alpha^i \partial_i \psi_- \\ & - M \psi_+^\dagger \gamma^0 \psi_- - M \psi_-^\dagger \gamma^0 \psi_+ - e\sqrt{2} \psi_-^\dagger \psi_- V_- - e\sqrt{2} \psi_+^\dagger \psi_+ V_+ \\ & - eV_i \left(\psi_+^\dagger \alpha^i \psi_- + \psi_-^\dagger \alpha^i \psi_+ \right) - \frac{1}{2} J_{pzm}^i \frac{1}{\Delta_\perp} * J_{pzm}^i + J_{pzm}^- \frac{1}{\Delta_\perp} * J_{pzm}^+ , \end{aligned} \quad (9.19)$$

where the gauge potential V_μ is given by

$$V_i = C_i + \partial_i \phi + \partial_i \frac{1}{\Delta_\perp} * \Lambda_{pzm} + q_i , \quad (9.20.a)$$

$$V_- = \partial_- \phi + q_- , \quad (9.20.b)$$

$$V_+ = \frac{1}{2\partial_-} * (2\partial_i C_i + \Delta_\perp \phi - \lambda_{nm}) + \partial_+ \frac{1}{\Delta_\perp} * \Lambda_{pzm} . \quad (9.20.c)$$

Though the equations of motion for fermion fields lead effectively to the condition $\Lambda_{pzm} = 0$ we would like to remove this field before analysing fermion sector.³³ This can be achieved by the redefinition of fermion fields

$$\psi'_\pm = \exp \left(-ie \frac{1}{\Delta_\perp} * \Lambda_{pzm} \right) \psi_\pm , \quad (9.21.a)$$

$$\psi'^\dagger_\pm = \psi^\dagger_\pm \exp \left(ie \frac{1}{\Delta_\perp} * \Lambda_{pzm} \right) , \quad (9.21.b)$$

which in (9.19) changes gauge potentials into V_μ^{cov34}

$$V_i^{cov} = C_i + \partial_i \phi + q_i , \quad (9.22.a)$$

$$V_-^{cov} = \partial_- \phi + q_- , \quad (9.22.b)$$

$$V_+^{cov} = \frac{1}{2\partial_-} * (2\partial_i C_i + \Delta_\perp \phi - \lambda_{nm}) . \quad (9.22.c)$$

Thus the non-dynamical field Λ_{pzm} appears only in the term $\frac{1}{2} \Lambda_{pzm}^2$ and its equation of motion is $\Lambda_{pzm} = 0$. Therefore we can completely disregard this field in the further analysis .

The fermion Lagrangian density

$$\mathcal{L}_{cov}^{fer} = i\sqrt{2} \psi'_+{}^\dagger \partial_+ \psi'_+ + i\sqrt{2} \psi'_-{}^\dagger \partial_- \psi'_- + i\psi'_-{}^\dagger \alpha^i \partial_i \psi'_+ + i\psi'_+{}^\dagger \alpha^i \partial_i \psi'_- - \frac{1}{2} \left(\frac{1}{2\partial_-} * J_{nm}^+ \right)^2$$

³³Its presence would generate the extra primary condition which contains fermion fields

$$\Pi^{\Lambda_{pzm}} = \frac{\delta \mathcal{L}_{cov}^{fer}}{\delta \partial_+ \Lambda_{pzm}} = -e\sqrt{2} \frac{1}{\Delta_\perp} * \int dx^- (\psi'_+{}^\dagger \psi_+)$$

³⁴Strictly speaking these gauge potentials *do not* satisfy the Lorentz covariant gauge condition but rather

$$\partial^\mu V_\mu^{cov} = \frac{1}{4\partial_-} * J_{nm}^+ .$$

However the direct interactions of currents compensates the above non-vanishing current term.

$$\begin{aligned}
& - M \psi'_+ \dagger \gamma^0 \psi'_- - M \psi'_- \dagger \gamma^0 \psi'_+ - e\sqrt{2} \psi'_- \dagger \psi'_- V^{\text{cov}}_- - e\sqrt{2} \psi'_+ \dagger \psi'_+ V^{\text{cov}}_+ \\
& - eV^{\text{cov}}_i \left(\psi'_+ \dagger \alpha^i \psi'_- + \psi'_- \dagger \alpha^i \psi'_+ \right) - \frac{1}{2} J'^i_{pzm} \frac{1}{\Delta_\perp} * J'^i_{pzm} + J'^i_{pzm} \frac{1}{\Delta_\perp} * J'^i_{pzm} , \quad (9.23)
\end{aligned}$$

where all currents are built from the new primed fermion fields, generates constraint equations for dependent fermion fields ψ'_- and global zero modes q_i

$$\begin{aligned}
\left(i\sqrt{2}\partial_- - e\sqrt{2}V_-^{\text{cov}} - \frac{e\sqrt{2}}{2L} \frac{1}{\Delta_\perp} * J'^i_{pzm} \right) \psi'_- &= -i\alpha^i \partial_i \psi'_+ + m\gamma^0 \psi'_+ \\
&+ e\alpha^i \left(V_i^{\text{cov}} - \frac{1}{\Delta_\perp} * J'^i_{pzm} \right) \psi'_+ , \quad (9.24.a)
\end{aligned}$$

$$0 = \int d^3x \left(\psi'^\dagger_+ \alpha^i \psi'_- + \psi'^\dagger_- \alpha^i \psi'_+ \right)(x) , \quad (9.24.b)$$

with a similar structure as their counterparts for the LF Weyl gauge. Therefore we can incorporate those respective results from Section 8 and write

$$\psi'_-(\vec{x}) = \frac{1}{\sqrt{2}} \left(\frac{1}{i\partial_- - eV'_-} * \xi' \right) (\vec{x}) + \frac{e^2}{2\sqrt{2}L} \left(\mathcal{G}_{(\perp)}[\mathcal{M}'^2] * \Gamma'^i \right) (x_\perp) \alpha^i \left(\frac{1}{i\partial_- - eV'_-} * \psi'_+ \right) (\vec{x}) , \quad (9.25.a)$$

$$\frac{1}{\Delta_\perp} * J'^i_{pzm} - q_i = -\frac{e}{2L} \mathcal{G}_{(\perp)}[\mathcal{M}'^2] * \Gamma'^i . \quad (9.25.b)$$

Then we take the expression for currents

$$J'^i_{pzm} = -\frac{e}{2L} \Gamma'^i - \frac{e^3}{(2L)^2} \mathcal{M}'^2 \mathcal{G}_{(\perp)}[\mathcal{M}'^2] * \Gamma'^i , \quad (9.26)$$

where

$$\begin{aligned}
\Gamma'^i(x_\perp) &= \frac{1}{\sqrt{2}} \int dx^- \psi'^\dagger_+(\vec{x}) \alpha^i \frac{1}{i\partial_- - eV'_-} * \xi'(\vec{x}) \\
&+ \frac{1}{\sqrt{2}} \int dx^- \xi'^\dagger(\vec{x}) * \frac{1}{i\partial_- - eV'_-} \alpha^i \psi'_+(\vec{x}) , \quad (9.27.a)
\end{aligned}$$

$$\mathcal{M}'^2(x_\perp) = \sqrt{2} \int dx^- \psi'^\dagger_+(\vec{x}) \frac{1}{i\partial_- - eV''_-} * \psi'_+(\vec{x}) , \quad (9.27.b)$$

and

$$V'_- = \partial_- \phi + q_- - e\sqrt{2} \frac{1}{\Delta_\perp} * J'^i_{pzm} , \quad (9.28.a)$$

$$\xi' = \left[M\gamma^0 - i\alpha^i \partial_i + e\alpha^i (C_i + \partial_i \phi) \right] \psi'_+ . \quad (9.28.b)$$

Next we find the effective Hamiltonian for fermion sector

$$\begin{aligned}
H_{\text{cov}}^{J'et} &= \frac{1}{\sqrt{2}} \int d^3x \xi'^\dagger(\vec{x}) \frac{1}{i\partial_- - eV'_-} * \xi'(\vec{x}) + \frac{1}{2} \int d^3x \left(\frac{1}{\Delta_\perp} * J'^i_{nm} \right)^2 \\
&+ \frac{1}{2} \frac{e^2}{2L} \int d^2x_\perp \Gamma'^i(x_\perp) \mathcal{G}_{(\perp)}[\mathcal{M}'^2] * \Gamma'^i(x_\perp) + e\sqrt{2} \psi'_+ \dagger \psi'_+ V^{\text{cov}}_+ , \quad (9.29)
\end{aligned}$$

and finally we write the expression for effective total Lagrangian density

$$\begin{aligned}
L_{cov}^{eff,tot} &= \partial_- C_i \partial_+ C_i - \frac{1}{2} (\partial_i C_j)^2 - 2\partial_+ \lambda_{nm} \partial_- \phi + \partial_i \lambda_{nm} \partial_i \phi - \frac{1}{2} \lambda_{nm}^2 - \frac{1}{2} \left(\frac{1}{2\partial_-} * J_{nm}^+ \right)^2 \\
&+ \frac{1}{2} (\partial_+ q_-)^2 + i\sqrt{2} \psi_+^\dagger \partial_+ \psi_+' - e\sqrt{2} \psi_+^\dagger \psi_+' V_+^{cov} - \frac{1}{\sqrt{2}} \xi'^\dagger \frac{1}{i\partial_- - eV_-'} * \xi' \\
&- \frac{1}{2} \frac{e^2}{4L^2} \Gamma^i(x_\perp) \mathcal{G}_{(\perp)}[\mathcal{M}'^2] * \Gamma^i(x_\perp) .
\end{aligned} \tag{9.30}$$

9.4 Quantum theory and physical states

When all constraints are solved at the classical level, then the canonical quantization is straightforward - one writes the (anti)commutation relations

$$2\partial_-^\pm [C_i(\vec{x}), C_j(\vec{y})] = -i\delta_{ij} \delta_n^{(3)}(\vec{x} - \vec{y}) , \tag{9.31.a}$$

$$2\partial_-^\pm [\phi(\vec{x}), \lambda_{nm}(\vec{y})] = i\delta_n^{(3)}(\vec{x} - \vec{y}) , \tag{9.31.b}$$

$$\{\psi_+'^\dagger(\vec{x}), \psi_+'(\vec{y})\} = \frac{1}{\sqrt{2}} \Lambda_+ \delta_n^{(3)}(\vec{x} - \vec{y}) , \tag{9.31.c}$$

$$[p^-, q_-] = -i , \tag{9.31.d}$$

while other relations vanish, and takes the quantum Hamiltonian in its canonical form

$$\begin{aligned}
H_{quan}^{Lor} &= \int d^3x \left[\frac{1}{2} (\partial_i C_j)^2 - \partial_i \lambda_{nm} \partial_i \phi + \frac{1}{2} \lambda_{nm}^2 + \frac{1}{2} \left(\frac{1}{2\partial_-} * J_{nm}^+ \right)^2 + e\sqrt{2} \psi_+'^\dagger \psi_+' V_+^{cov} \right] \\
&+ \frac{1}{16LL_\perp^2} (p^-)^2 + \frac{1}{\sqrt{2}} \int d^3x \xi'^\dagger(\vec{x}) \left(\frac{1}{i\partial_- - eV_-'} * \xi' \right) (\vec{x}) \\
&+ \frac{1}{2} \frac{e^2}{2L} \int d^2x_\perp \Gamma^i(x_\perp) \left(\mathcal{G}_{(\perp)}[\mathcal{M}'^2] * \Gamma^i \right) (x_\perp) ,
\end{aligned} \tag{9.32}$$

The presence of extra *nonphysical* modes ϕ and λ_{nm} indicates that the Hilbert space contains also non-physical states. In order to select the physical states we will follow the analysis presented in Section 6 and introduce first the quantum field Λ_{nm}

$$\Lambda_{nm} = \lambda_{nm} + \frac{1}{2\partial_-} * J_{nm}^+ , \tag{9.33}$$

which is a free field even for the fully interacting system

$$(2\partial_+ \partial_- - \Delta_\perp) \Lambda_{nm} = 0 \tag{9.34}$$

and has nonzero commutators

$$2\partial_-^\pm [\phi(\vec{x}), \Lambda_{nm}(\vec{y})] = i\delta_n^{(3)}(\vec{x} - \vec{y}) , \tag{9.35.a}$$

$$2\partial_-^\pm [\Lambda_{nm}(\vec{x}), \psi_+'(\vec{y})] = e\psi_+'(\vec{x}) \delta_n^{(3)}(\vec{x} - \vec{y}) , \tag{9.35.b}$$

$$2\partial_-^\pm \left[\Lambda_{nm}(\vec{x}), \psi_+'^\dagger(\vec{y}) \right] = -e\psi_+'^\dagger(\vec{x}) \delta_n^{(3)}(\vec{x} - \vec{y}) . \tag{9.35.c}$$

The q-number commutators (9.35.b) and (9.35.c) would be very cumbersome in further analysis, therefore we introduce the dressed *physical* fermion fields

$$\psi_{phys}(x) = \exp ie\phi(x) \psi_+(x) , \tag{9.36.a}$$

$$\psi_{phys}^\dagger(x) = \psi_+'^\dagger(x) \exp -ie\phi(x) , \tag{9.36.b}$$

which already commute with Λ_{nm} at LF. Next the quantum Hamiltonian can be expressed in terms of these new fields

$$\begin{aligned}
H_{\text{quas}}^{\text{Lor}} &= \int d^3x \left[\frac{1}{2} (\partial_i C_j)^2 - \partial_i \Lambda_{nm} \partial_i \phi + \frac{1}{2} \left(\Lambda_{nm} - \frac{1}{\partial_-} * J_{phys}^+ \right)^2 + e\sqrt{2} \psi_{phys}^\dagger \psi_{phys} V_{phys}^{phys} \right] \\
&+ \frac{1}{16LL_1^2} (p^-)^2 + \frac{1}{\sqrt{2}} \int d^3x \xi_{phys}^\dagger(\vec{x}) \left(\frac{1}{i\partial_- - eV_{phys}^{phys}} * \xi_{phys} \right) (\vec{x}) \\
&+ \frac{1}{2} \frac{e^2}{2L} \int d^2x_\perp \Gamma_{phys}^i(x_\perp) \left(\mathcal{G}_{(\perp)} [\mathcal{M}_{phys}^2] * \Gamma_{phys}^i \right) (x_\perp), \tag{9.37}
\end{aligned}$$

where

$$\begin{aligned}
\Gamma_{phys}^i(x_\perp) &= \frac{1}{\sqrt{2}} \int dx^- \psi_{phys}^\dagger(\vec{x}) \alpha^i \frac{1}{i\partial_- - eV_{phys}^{phys}} * \xi_{phys}(\vec{x}) \\
&+ \frac{1}{\sqrt{2}} \int dx^- \xi_{phys}^\dagger(\vec{x}) * \frac{1}{i\partial_- - eV_{phys}^{phys}} \alpha^i \psi_{phys}(\vec{x}), \tag{9.38.a}
\end{aligned}$$

$$\mathcal{M}_{phys}^2(x_\perp) = \sqrt{2} \int dx^- \psi_{phys}^\dagger(\vec{x}) \frac{1}{\partial_- - eV_{phys}^{phys}} * \psi_{phys}(\vec{x}), \tag{9.38.b}$$

and

$$J_{phys}^+ = \sqrt{2} \psi_{phys}^\dagger \psi_{phys}, \tag{9.39.a}$$

$$V_{phys}^+ = \frac{1}{\partial_-} * \partial_i C_i, \tag{9.39.b}$$

$$V_{phys}^- = q_- - e\sqrt{2} \frac{1}{\Delta_\perp} * J_{phys}^+, \tag{9.39.c}$$

$$\xi_{phys} = [m\gamma^0 - i\alpha^i \partial_i + e\alpha^i C_i] \psi_{phys} \tag{9.39.d}$$

Now we are in the position to give the condition for physical states

$$\langle phys' | \Lambda_{nm}(x) | phys \rangle = 0. \tag{9.40}$$

This means that the excitations of ϕ cannot appear in physical states, while an arbitrary excitations of Λ_{nm} (all having zero norm) can accompany physical photons. The Hamiltonian which effectively acts on physical states has the form

$$\begin{aligned}
H_{phys}^{\text{Lor}} &= \int d^3x \left[\frac{1}{4} (\partial_i C_j - \partial_j C_i)^2 + \frac{1}{2} \left(\partial_i C_i - \frac{1}{\partial_-} * J_{phys}^+ \right)^2 \right] \\
&+ \frac{1}{16LL_1^2} (p^-)^2 + \frac{1}{\sqrt{2}} \int d^3x \xi_{phys}^\dagger(\vec{x}) \left(\frac{1}{i\partial_- - eV_{phys}^{phys}} * \xi_{phys} \right) (\vec{x}) \\
&+ \frac{1}{2} \frac{e^2}{2L} \int d^2x_\perp \Gamma_{phys}^i(x_\perp) \left(\mathcal{G}_{(\perp)} [\mathcal{M}_{phys}^2] * \Gamma_{phys}^i \right) (x_\perp), \tag{9.41}
\end{aligned}$$

and is identical with the Hamiltonian (8.59) obtained for the LF Weyl gauge after the implementation of Gauss' Law. Thus we can take the above gauge-independent expression as *the physical Hamiltonian* for QED at the finite volume LF.

10 Transverse Coulomb gauge

When the Coulomb condition is imposed for transverse components of gauge potentials $\partial_i V_i = 0$ it effectively introduces constraints for two sectors

$$\partial_i V_i(\vec{x}) = 0 \implies \begin{cases} \partial_i A_i(\vec{x}) = 0 \\ \partial_i a_i(x_\perp) = 0. \end{cases} \quad (10.1)$$

Therefore we have to add some gauge condition for the global zero modes and we choose the LF Weyl gauge $q_+ = 0$. Because all these conditions are non-dynamical, we will impose them explicitly via the suitable decomposition of vector gauge fields into transverse and longitudinal parts.

10.1 Proper zero mode sector

As usually, we start with the Lagrangian density

$$\mathcal{L}_{Coul}^{pzm} = \partial_i a_- \partial_i a_+ + \frac{1}{2} (\partial_+ a_-)^2 - \frac{1}{2} (\partial_i a_j^T)^2 + J_{pzm}^+ a_+ + J_{pzm}^i a_i^T + J_{pzm}^- a_-, \quad (10.2)$$

where the transverse components of fields are defined as

$$a_i^T \stackrel{df}{=} \left(\delta_{ik} - \partial_i \partial_k \frac{1}{\Delta_\perp} \right) a_k. \quad (10.3)$$

The Euler-Lagrange equations

$$\partial_+^2 a_- = -\Delta a_+ + J_{pzm}^-, \quad (10.4.a)$$

$$-\partial_i \partial_+ a_- = \Delta_\perp a_i^T + J_{pzm}^i, \quad (10.4.b)$$

$$0 = -\Delta_\perp a_- + J_{pzm}^+, \quad (10.4.c)$$

can be explicitly solved in terms of external currents

$$a_- = \frac{1}{\Delta_\perp} * J_{pzm}^+, \quad (10.5.a)$$

$$a_i = -\frac{1}{\Delta_\perp} * J_{pzm}^i + \partial_i \frac{1}{\Delta_\perp} * \left(\frac{1}{\Delta_\perp} * \partial_j J_{pzm}^j \right), \quad (10.5.b)$$

$$a_+ = \frac{1}{\Delta_\perp} * J_{pzm}^- - \partial_+ \frac{1}{\Delta_\perp} * \left(\frac{1}{\Delta_\perp} * J_{pzm}^+ \right). \quad (10.5.c)$$

So here again there are no independent proper zero modes and there is no canonical structure in this sector. Thus we only write the effective Lagrangian

$$\begin{aligned} \mathcal{L}_{Coul}^{pzm\ eff} &= J_{pzm}^+ \frac{1}{\Delta_\perp} * J_{pzm}^- - \frac{1}{2} J_{pzm}^i \frac{1}{\Delta_\perp} * J_{pzm}^i - \frac{1}{2} \left(\frac{1}{\Delta_\perp} * \partial_i J_{pzm}^i \right)^2 \\ &- \frac{1}{2} (\Lambda_{pzm})^2 - \partial_+ \Lambda_{pzm} \frac{1}{\Delta_\perp} * J_{pzm}^+, \end{aligned} \quad (10.6)$$

where the subsidiary non-dynamical field Λ_{pzm} has been introduced to avoid the explicit presence of $\partial_+ J_{pzm}^+$.

10.2 Normal mode sector

Normal modes are described by the Lagrangian density

$$\begin{aligned} \mathcal{L}_{Coul}^{nm} = & \partial_i A_+ \partial_i A_- + \partial_- A_i^{Tr} \partial_+ A_i^{Tr} + \frac{1}{2} (\partial_+ A_- - \partial_- A_+)^2 \\ & - \frac{1}{2} (\partial_i A_j^{Tr})^2 + A_- j^- + A_+ j^+ + A_i^{Tr} J_{nm}^{Tr i}, \end{aligned} \quad (10.7)$$

which generates the Euler-Lagrange equations

$$(2\partial_+ \partial_- - \Delta_\perp) A_- = \partial_- (\partial_+ A_- + \partial_- A_+) - J_{nm}^+, \quad (10.8.a)$$

$$(2\partial_+ \partial_- - \Delta_\perp) A_+ = \partial_+ (\partial_+ A_- + \partial_- A_+) - J_{nm}^-, \quad (10.8.b)$$

$$(2\partial_+ \partial_- - \Delta_\perp) A_i^{Tr} = J_{nm}^{Tr i}. \quad (10.8.c)$$

These equations describe dynamical evolution but contain also the constraints. To see this most easily we parameterize two gauge field potentials A_\pm by the single field Φ

$$A_- = \partial_- \Phi, \quad (10.9.a)$$

$$A_+ = -\frac{1}{\partial_-} * \left(\Delta_\perp \Phi - \frac{1}{\partial_-} * J_{nm}^+ \right) + \partial_+ \Phi, \quad (10.9.b)$$

which satisfies the dynamical equation of motion

$$(2\partial_+ \partial_- - \Delta_\perp) \Phi = -\frac{1}{\partial_-} * J_{nm}^+ + \frac{1}{\Delta_\perp} * (\partial_- J_{nm}^- + \partial_+ J_{nm}^+). \quad (10.10)$$

In the canonical analysis the last equations would need a modification for removing the term $\partial_+ J_{nm}^+$. However this would introduce such a term in the definition of A_+ , therefore, just like in the case of massive QED in Section 1, there is no unconstrained canonical structure consistent with the primary constrained system. Thus all we can find is the effective Lagrangian density which contains dynamical fields Φ and A_i^{Tr}

$$\begin{aligned} \mathcal{L}_{Coul}^{nm\ eff} = & \partial_+ \partial_i \Phi \partial_- \partial_i \Phi - \frac{1}{2} \left(\Delta_\perp \Phi - \frac{1}{\partial_-} * J_{nm}^+ \right)^2 + \partial_- \Phi J_{nm}^- + \partial_+ \Phi J_{nm}^+ \\ & + \partial_- A_i^{Tr} \partial_+ A_i^{Tr} - \frac{1}{2} (\partial_i A_j^{Tr})^2 + A_i^{Tr} J_{nm}^{Tr i}. \end{aligned} \quad (10.11)$$

Combining two effective Lagrangians (10.6) and (10.11) we obtain the total effective Lagrangian for the complete system with fermion fields

$$\begin{aligned} \mathcal{L}_{Coul}^{eff} = & \partial_+ A_i^{Tr} \partial_- A_i^{Tr} - \frac{1}{2} (\partial_i A_j^{Tr})^2 + \partial_+ \partial_i \Phi \partial_- \partial_i \Phi - \frac{1}{2} \left(\Delta_\perp \Phi - \frac{1}{\partial_-} * J_{nm}^+ \right)^2 \\ & + \frac{1}{2} (\partial_+ q_-)^2 + i\sqrt{2} \psi_+^\dagger \partial_+ \psi_+ + i\sqrt{2} \psi_-^\dagger \partial_- \psi_- + i\psi_-^\dagger \alpha^i \partial_i \psi_+ + i\psi_+^\dagger \alpha^i \partial_i \psi_- \\ & - M \psi_+^\dagger \gamma^0 \psi_- - M \psi_-^\dagger \gamma^0 \psi_+ - e\sqrt{2} \psi_+^\dagger \psi_+ V_+^{Coul} - e\sqrt{2} \psi_-^\dagger \psi_- V_-^{Coul} \\ & - eV_i^{Coul} (\psi_+^\dagger \alpha^i \psi_- + \psi_-^\dagger \alpha^i \psi_+) - \frac{1}{2} J_{pzm}^i \left(\frac{1}{\Delta_\perp} * J_{pzm}^i \right) - \frac{1}{2} \left(\frac{1}{\Delta_\perp} * \partial_i J_{pzm}^i \right)^2 \\ & - J_{pzm}^- \frac{1}{\Delta_\perp} * J_{pzm}^+ - \frac{1}{2} (\Lambda_{pzm})^2, \end{aligned} \quad (10.12)$$

where

$$V_i^{Coul} = A_i^{Tr} + q_i, \quad (10.13.a)$$

$$V_-^{Coul} = \partial_- \Phi + q_- - \frac{1}{\Delta_\perp} * J_{nm}^+, \quad (10.13.b)$$

$$V_+^{Coul} = \partial_+ \Phi - \partial_+ \frac{1}{\Delta_\perp} * \Lambda_{pzm}. \quad (10.13.c)$$

10.3 Fermion field sector

Before writing down the Lagrangian for the fermion field we have decided to introduce the *dressed* fermion field

$$\tilde{\psi}_\pm(\vec{x}) = \psi_\pm(\vec{x}) \exp \left\{ -ie \frac{1}{\Delta_\perp} * \Lambda_{pzm}(x_\perp) \right\} \quad (10.14)$$

and this leads to

$$\begin{aligned} \mathcal{L}_{Coul}^{fer} &= i\sqrt{2}\tilde{\psi}_+^\dagger \partial_+ \tilde{\psi}_+ + i\sqrt{2}\tilde{\psi}_+^\dagger \partial_- \tilde{\psi}_- + i\tilde{\psi}_+^\dagger \alpha^i \partial_i \tilde{\psi}_+ + i\tilde{\psi}_+^\dagger \alpha^i \partial_i \tilde{\psi}_- \\ &- m\tilde{\psi}_+^\dagger \gamma^0 \tilde{\psi}_- - m\tilde{\psi}_-^\dagger \gamma^0 \tilde{\psi}_+ - e\sqrt{2}\tilde{\psi}_+^\dagger \tilde{\psi}_- V_-^{Coul} - e\sqrt{2}\tilde{\psi}_+^\dagger \tilde{\psi}_+ \tilde{V}'_+ \\ &- eV_i^{Coul} (\tilde{\psi}_+^\dagger \alpha^i \tilde{\psi}_- + \tilde{\psi}_-^\dagger \alpha^i \tilde{\psi}_+) - \frac{1}{2} \tilde{J}_{pzm}^i \left(\frac{1}{\partial_\perp} * \tilde{J}_{pzm}^i \right) \\ &- \frac{1}{2} \left(\Lambda_{pzm} - \frac{1}{\Delta_\perp} * \partial_i \tilde{J}_{pzm}^i \right)^2 - \frac{1}{2} \left(\Delta_\perp \Phi - \frac{1}{\partial_-} * \tilde{J}^+ \right)^2, \end{aligned} \quad (10.15)$$

where

$$\tilde{V}'_+ = \partial_+ \Phi. \quad (10.16)$$

Now we have the same form of constraints for dependent fermions as in the case of LF Weyl gauge in Section 8

$$(i\sqrt{2}\partial_- - e\sqrt{2}V_-^{Coul}) \tilde{\psi}_- = -i\alpha^i \partial_i \tilde{\psi}_+ + M\gamma^0 \tilde{\psi}_+ + e\alpha^i \left(V_i^{Coul} - \frac{1}{\Delta_\perp} * \tilde{J}_{pzm}^i \right) \tilde{\psi}_+, \quad (10.17.a)$$

$$0 = \int d^3x (\tilde{\psi}_+^\dagger \alpha^i \tilde{\psi}_- + \tilde{\psi}_-^\dagger \alpha^i \tilde{\psi}_+)(x), \quad (10.17.b)$$

thus we can adopt those results directly and only the equation for Λ_{pzm} is new

$$\Delta_\perp \Lambda_{pzm} = -\partial_i \tilde{J}_{pzm}^i. \quad (10.18)$$

We easily obtain

$$\frac{1}{\Delta_\perp} \tilde{J}_{pzm}^i - q_i = -\frac{e}{2L} G_\perp [\tilde{\mathcal{M}}^2] * \tilde{\Gamma}^i \quad (10.19)$$

and further the effective Lagrangian density for fermions takes the following form:

$$\mathcal{L}_{eff}^{fer} = i\sqrt{2}\tilde{\psi}_+^\dagger \partial_+ \tilde{\psi}_+ - e\sqrt{2}\tilde{\psi}_+^\dagger \tilde{\psi}_+ \tilde{V}'_+ - \frac{1}{\sqrt{2}} \tilde{\xi}^\dagger \left(\frac{1}{i\partial_- - eV_-^{Coul}} * \tilde{\xi} \right) - \frac{1}{24L^2} \tilde{\Gamma}^i \left(\mathcal{G}_{(\perp)}[\tilde{\mathcal{M}}_{Coul}^2] * \tilde{\Gamma}^i \right), \quad (10.20)$$

where

$$\begin{aligned} \tilde{\xi}(x) &= [M\gamma^0 - i\alpha^i \partial_i + e\alpha^i V_i^{Coul}(x)] \tilde{\psi}_+(x), \\ \tilde{\Gamma}^i(x_\perp) &= \frac{1}{\sqrt{2}} \int dx^- \tilde{\psi}_+^\dagger(x) \alpha^i \left(\frac{1}{i\partial_- - eV_-^{Coul}} * \tilde{\xi} \right) (\vec{x}) \end{aligned} \quad (10.21)$$

$$+ \frac{1}{\sqrt{2}} \int dx^- \bar{\xi}^\dagger(x) \alpha^i \left(\frac{1}{i\partial_- - eV_-^{Coul}} * \tilde{\psi}_+ \right) (\bar{x}), \quad (10.22)$$

$$\bar{\mathcal{M}}_{Coul}^2(x_\perp) = \sqrt{2} \int dx^- \bar{\psi}_+^\dagger(x) \left(\frac{1}{i\partial_- - eV_-^{Coul}} * \tilde{\psi}_+ \right) (\bar{x}). \quad (10.23)$$

The equation for the subsidiary dependent field Λ_{pzm}

$$\Delta_\perp \Lambda_{pzm} = -\frac{e}{2L} \partial_i \mathcal{G}_{(\perp)} [\bar{\mathcal{M}}_{Coul}^2] * \bar{\Gamma}^i \quad (10.24)$$

can be uniquely solved but we really do not need this solution. Since Λ_{pzm} has disappeared from the effective Lagrangian (10.20), we can omit it in further steps.

Our analysis of constraints for the complete QED in the Coulomb gauge leads to the final effective Lagrangian density

$$\begin{aligned} \mathcal{L}_{eff}^{Coul} &= \partial_+ A_i^{Tr} \partial_- A_i^{Tr} - \frac{1}{2} (\partial_i A_j^{Tr})^2 + \partial_+ \partial_i \Phi \partial_- \partial_i \Phi - \frac{1}{2} \left(\Delta_\perp \Phi - e\sqrt{2} \frac{1}{\partial_-} * (\tilde{\psi}_+^\dagger \tilde{\psi}_+) \right)^2 \\ &+ \frac{1}{2} (\partial_+ q_-)^2 + i\sqrt{2} \psi_+^\dagger \partial_+ \psi_+ - e\sqrt{2} \psi_+^\dagger \psi_+ \partial_+ \Phi \\ &+ \frac{1}{\sqrt{2}} (\xi^\dagger + e\psi_+^\dagger \alpha^i \partial_i \phi) \frac{1}{i\partial_- - eV_-^{Coul}} * (\xi + e\psi_+ \partial_i \phi) \\ &- \frac{1}{2} \frac{e^2}{4L^2} (\Gamma^i + e\mathcal{M}_{Coul}^2 \partial_i \phi) \mathcal{G}_{(\perp)} [\mathcal{M}^2] * (\Gamma^i + e\mathcal{M}_{Coul}^2 \partial_i \phi). \end{aligned} \quad (10.25)$$

10.4 Canonical quantization

From the classical effective Lagrangian (10.25) we can infer the structure of quantum (anti)commutation relations for all independent fields. Besides the canonical relations

$$2\partial_-^\pm [A_i^{Tr}(\bar{x}), A_j A_i^{Tr}(\bar{y})] = -i\delta_{ij} A_i^{Tr} \delta_n^{(3)}(\bar{x} - \bar{y}), \quad (10.26.a)$$

$$2\partial_-^\pm [\Phi(\bar{x}), \Phi(\bar{y})] = i\Delta_\perp \delta_n^{(3)}(\bar{x} - \bar{y}), \quad (10.26.b)$$

$$\{\psi_+^\dagger(\bar{x}), \psi_+'(\bar{y})\} = \frac{1}{\sqrt{2}} \Lambda^{(+)} \delta_n^{(3)}(\bar{x} - \bar{y}), \quad (10.26.c)$$

$$[p^-, q_-] = -i, \quad (10.26.d)$$

we find also two noncanonical commutators between Φ and the fermion fields

$$2\partial_-^\pm [\Phi(x_0^+, \bar{x}), \tilde{\psi}_+(x_0^+, \bar{y})] = e\delta_p^3(\bar{x} - \bar{y}), \quad (10.27.a)$$

$$2\partial_-^\pm [\Phi(x_0^+, \bar{x}), \tilde{\psi}_+'(x_0^+, \bar{y})] = -e\delta_p^3(\bar{x} - \bar{y}), \quad (10.27.b)$$

which come from the term $\psi_+^\dagger \psi_+ \partial_+ \Phi$. The quantum Hamiltonian is assumed in the form:

$$\begin{aligned} H_{quan}^{Coul} &= \int d^3x \left[\frac{1}{2} (\partial_i A_j^{Tr})^2 + \frac{1}{2} \left(\Delta_\perp \Phi - e\sqrt{2} \frac{1}{\partial_-} * (\tilde{\psi}_+^\dagger \tilde{\psi}_+) \right)^2 \right] \\ &+ \frac{1}{16LL^2} (p^-)^2 + \frac{1}{\sqrt{2}} \int d^3x \xi_{Coul}^\dagger(\bar{x}) \left(\frac{1}{i\partial_- - eV_-^{Coul}} * \xi_{Coul} \right) (\bar{x}) \\ &+ \frac{1}{2} \frac{e^2}{2L} \int d^2x_\perp \Gamma_{Coul}^i(x_\perp) (\mathcal{G}_{(\perp)} [\mathcal{M}_{Coul}^2] * \Gamma_{Coul}^i)(x_\perp) \end{aligned} \quad (10.28)$$

tacitly defining proper ordering of non-commuting propagators. Thus we see that the transverse Coulomb gauge condition has the same number of independent excitations as the physical subsystems defined either in the LF Weyl or in the Lorentz covariant gauges. Here one encounters a strange phenomenon that one boson field does not commute with fermion fields and this is connected with the rather extravagant choice of gauge which imposes the constraint on transverse gauge fields which are known to describe two physical photon excitations at LF. This non-commutativity is the lowest price we can pay for such careless choice of gauge fixing condition and it can be solved if we introduce another dressed *physical* fermion fields

$$\psi_{phys}(\vec{x}) = \exp ie\Phi(\vec{x})\tilde{\psi}_+(\vec{x}) \quad (10.29.a)$$

$$\psi_{phys}^\dagger(\vec{x}) = \tilde{\psi}_+^\dagger(\vec{x})\exp -ie\Phi(\vec{x}), \quad (10.29.b)$$

which already commute with Φ fields. Now the quantum Hamiltonian (10.28) takes the form

$$\begin{aligned} H_{quan}^{phys} &= \int d^3x \left[\frac{1}{2} (\partial_i A_j^{Tr})^2 + \frac{1}{2} \left(\Delta_\perp \Phi - e \frac{1}{\partial_-} * \sqrt{2} \psi_{phys}^\dagger \psi_{phys} \right)^2 \right] \\ &+ \frac{1}{16LL_\perp^2} (p^-)^2 + \frac{1}{\sqrt{2}} \int d^3x \xi_{phys}^\dagger(\vec{x}) \left(\frac{1}{i\partial_- - eV_-^{phys}} * \xi_{phys} \right) (\vec{x}) \\ &+ \frac{1}{2L} \int d^2x_\perp \Gamma_{phys}^i(x_\perp) \left(\mathcal{G}_\perp [M_{phys}^2] * \Gamma_{phys}^i \right) (x_\perp), \end{aligned} \quad (10.30)$$

where

$$V_i^{phys}(x) = A_i^{Tr}(x) - \partial_i \Phi(x) + q_i(x^+), \quad (10.31)$$

$$V_-^{phys}(x) = q_-(x^+) - \frac{e\sqrt{2}}{2L} \frac{1}{\Delta_\perp} * \int dx^- \psi_{phys}^\dagger(x) \psi_{phys}(x), \quad (10.32)$$

$$\xi_{phys}(x) = \left[M\gamma^0 - i\alpha^i \partial_i + e\alpha^i V_i^{phys}(x) \right] \psi_{phys}(x), \quad (10.33)$$

$$\begin{aligned} \Gamma_{phys}^i(x_\perp) &= \frac{1}{\sqrt{2}} \int dx^- \psi_{phys}^\dagger(x) \alpha^i \left(\frac{1}{i\partial_- - eV_-^{phys}} * \xi_{phys} \right) (x) \\ &+ \frac{1}{\sqrt{2}} \int dx^- \xi_{phys}^\dagger(x) \alpha^i \left(\frac{1}{i\partial_- - eV_-^{phys}} * \psi_{phys} \right) (x), \end{aligned} \quad (10.34)$$

$$\mathcal{M}_{phys}^2(x_\perp) = \sqrt{2} \int dx^- \psi_{phys}^\dagger(x) \left(\frac{1}{i\partial_- - eV_-^{phys}} * \psi_{phys} \right) (x), \quad (10.35)$$

and agrees with the former expressions for the physical Hamiltonians in the LF Weyl gauge (8.59) and the Lorentz covariant gauge (9.41).

10.5 Physical gauge conditions

The physical field variables that we have just found after redefinitions of the fields can be obtained directly from the following gauge conditions:

$$A_- = 0, \quad (10.36.a)$$

$$\partial_+ a_- = \partial_\perp a_\perp, \quad (10.36.b)$$

$$q_+ = 0. \quad (10.36.c)$$

Only the sector of normal modes has not been analysed by us so far [44], therefore below we sketch some crucial points. From the Lagrangian density

$$\mathcal{L}^{nm} = (\partial_+ A_i - \partial_i A_+) \partial_- A_i + \frac{1}{2} (\partial_- A_+)^2 - \frac{1}{4} (\partial_i A_j - \partial_j A_i)^2 + A_+ J_{nm}^+ + A_i J_{nm}^i, \quad (10.37)$$

where the gauge fixing condition $A_- = 0$ has been explicitly implemented, we derive the Euler-Lagrange equations

$$\partial_- (\partial_- A_+ - \partial_i A_i) = J_{nm}^+ , \quad (10.38.a)$$

$$(2\partial_+ \partial_- - \Delta_\perp) A_i = \partial_i (\partial_- A_+ - \partial_j A_j) - J_{nm}^i . \quad (10.38.b)$$

Evidently, A_+ is not a dynamical variable and its equation of motion (10.38.a) can be solved as

$$A_+ = \frac{1}{2\partial_-} * \left(\partial_i A_i + \frac{1}{2\partial_-} * J_{nm}^+ \right) . \quad (10.39)$$

Only one Dirac bracket is nonzero

$$2\partial_-^\varepsilon \{A_i(\vec{x}), A_j(\vec{y})\}_{DB} = -\delta_{ij} \delta^3(\vec{x} - \vec{y}) \quad (10.40.a)$$

and the Dirac Hamiltonian is

$$\mathcal{H}_D = \frac{1}{2} (\partial_i A_i + \frac{1}{\partial_-} * J_{nm}^+)^2 + \frac{1}{4} (\partial_i A_j - \partial_j A_i)^2 - A_i J_{nm}^i . \quad (10.41)$$

These results can be incorporated into the whole procedure of equivalent Lagrangians if the above results are transformed into the effective Lagrangian density for the normal mode part

$$\mathcal{L}_{eff}^{nm} = \partial_+ A_i \partial_- A_i - \frac{1}{2} (\partial_i A_i + \frac{1}{\partial_-} * J_{nm}^+)^2 - \frac{1}{4} (\partial_i A_j - \partial_j A_i)^2 + A_i J_{nm}^i . \quad (10.42)$$

Then following the analysis from Subsection 9.2 and (with trivial modifications) from Subsection 9.3, one finds the commutator relations for physical fields (10.26.a - 10.26.d) and the physical Hamiltonian (10.30).

Conclusions and perspectives

In this paper we have presented a novel method of canonical quantisation for constrained systems when applying it to the analysis of QED at LF. Separating different sectors of fields, we have been able to find effective description of them avoiding an explicit implementation of Dirac's procedure for the whole system of entangled constraints. Different gauge conditions proved to be legitimate choices for the quantum gauge field system interacting with fermions.

Feynman rules for perturbative calculations have been found and they have common properties. Generally, the canonical propagators for gauge fields have additional noncovariant terms which behave non-causally i.e. in the Fourier representation they are given solely by the CPV poles. These terms are cancelled by the direct interaction of currents which canonically appear in the interaction Hamiltonians. The fermion parts of interaction Hamiltonians are factorized and the non-covariant factors cancel with the additional non-covariant term in the fermion field propagator at LF. In this manner there are two equivalent settings of Feynman perturbative rules, the first (canonical) rules can contain non-causal and non-covariant terms, the second (effective) rules are causal and covariant.

The LC-gauge has been investigated as the limit of the general axial gauge and the flow covariant gauge. The ML-prescription for the LC-gauge propagator has been derived only in the second choice and these results hold for the interactions with fermions.

In the DLCQ method three choices of gauge conditions have been canonically analysed. In the cases of the LF Weyl gauge, the Lorentz covariant gauge and the transverse Coulomb gauge, the same physical Hamiltonian has been found though via different methods of Gauss' law implementation, the Lautrup-Nakanishi condition and the redefinition of fermion fields, respectively. Finally, the set of *physical gauge conditions*, which straightforwardly leads to the above physical Hamiltonian, has been proposed.

All the above results have been derived for fermion currents. The problems which can appear for other charged matter fields have been discussed in the simplest model of 1+1 dimensional LF-Weyl QED with scalar fields. A reasonable analysis of this model has been formulated starting with the nonlocal redefinition of one scalar field at LF. For these new fields, the canonical analysis and structure of perturbative calculations have been studied. Both methods and results were similar to those for fermion fields.

We have found that the presence of field derivatives in the matter currents (as in the scalar case) can be neutralized at LF by a suitable nonlocal redefinition of these fields. We expect that this is a general property of the LF formulation and can also be applied for more physically relevant models. Therefore we plan to analyse, first, the QED for charged vector fields and then the non-Abelian interactions. Especially the latter case will ultimately settle the status of the LC-gauge at LF, first of all the question of proper prescription for spurious poles within the canonical LF formulation.

A LF Notation

A.1 Coordinates

In 3+1 dimensions we define longitudinal coordinates $x^\pm = \frac{x^0 \pm x^1}{\sqrt{2}}$ and take x^+ as the dynamical evolution parameter. We denote transverse components $x_\perp = (x^2, x^3)$ by Latin indices (i, j, \dots) . Similarly we define components of any 4-vector and a scalar product of two 4-vectors decomposes as $A \cdot B = A^+ B^- + A^- B^+ - A^i B^i$. The metric has non-vanishing components $g_{+-} = 1, g_{ij} = -\delta_{ij}$. Partial derivatives are defined as $\partial_\pm = \partial/\partial x^\pm$, $\partial_i = \partial/\partial x^i$. Tensor components are defined analogously e.g. $T^{\pm\mu} = \frac{1}{\sqrt{2}}(T^{0\mu} \pm T^{1\mu})$ and summation over repeated indices is understood.

Also we introduce a vector notation for components $\vec{x} = (x^-, x_\perp)$ which lie on a light-front surface $x^+ = \text{const.}$ and for momenta associated with them $\vec{k} = (k_-, k_\perp)$. Scalar product of such 3-vectors decomposes as $\vec{k} \cdot \vec{x} = k_- x^- - k_i x_i$.

In the D+1 dimensions the notation generalizes trivially with the number of transverse directions changing from 2 to d.

A.2 Dirac matrices

The Dirac matrices γ^μ satisfy anticommutation relation

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \quad (\text{A.1})$$

where their components are defined analogously to coordinates e.g. $\gamma^\pm = \frac{\gamma^0 \pm \gamma^1}{\sqrt{2}}$. Thus γ^\pm are nilpotent matrices $(\gamma^\pm)^2 = 0$.

For the projection operators

$$\Lambda_\pm = \frac{1}{\sqrt{2}} \gamma^0 \gamma^\pm = \frac{1}{2} \gamma^\mp \gamma^\pm, \quad (\text{A.2})$$

we have useful relations

$$\Lambda_\pm \Lambda_\pm = \Lambda_\pm, \quad (\text{A.3.a})$$

$$\Lambda_+ \Lambda_- = \Lambda_- \Lambda_+ = 0, \quad (\text{A.3.b})$$

$$\Lambda_+ + \Lambda_- = 1, \quad (\text{A.3.c})$$

$$\gamma^\pm \Lambda_\mp = \Lambda_\pm \gamma^\mp = 0, \quad (\text{A.3.d})$$

$$\gamma^\pm \Lambda_\pm = \Lambda_\pm \gamma^\pm, \quad (\text{A.3.e})$$

$$\gamma^0 \Lambda_\pm = \Lambda_\mp \gamma^0, \quad (\text{A.3.f})$$

$$\gamma^i \Lambda_\pm = \Lambda_\pm \gamma^i. \quad (\text{A.3.g})$$

In many places we also use the standard notation [45]

$$\gamma^0 = \beta, \quad (\text{A.4.a})$$

$$\gamma^0 \gamma^i = \alpha^i. \quad (\text{A.4.b})$$

B Green Functions

B.1 Feynman propagator functions

In 1+1 dimensions we define noncovariant Feynman Green functions $E_F^1(x)$ and $E_F^2(x)$

$$E_F^1(x) \stackrel{\text{df}}{=} i \int_0^\infty \frac{dk_-}{2\pi} \left[\Theta(x^+) e^{-ik_- x^-} - \Theta(-x^+) e^{ik_- x^-} \right]$$

$$= \frac{1}{2\pi} \left(\frac{\Theta(x^+)}{x^- - i\epsilon} + \frac{\Theta(-x^+)}{x^- + i\epsilon} \right), \quad (\text{B.5.a})$$

$$E_F^2(x) \stackrel{df}{=} -\frac{x^+}{2\pi} \left(\frac{\Theta(x^+)}{x^- - i\epsilon} + \frac{\Theta(-x^+)}{x^- + i\epsilon} \right), \quad (\text{B.5.b})$$

which can be also represented by the 2-dimensional Fourier integrals

$$E_F^1(x) = -\int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} e^{-ik \cdot x} \frac{1}{k_+ + i\epsilon \operatorname{sgn}(k_-)}, \quad (\text{B.6.a})$$

$$E_F^2(x) = -i \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} e^{-ik \cdot x} \frac{1}{[k_+ + i\epsilon \operatorname{sgn}(k_-)]^2}. \quad (\text{B.6.b})$$

In D+1 dimensions we define covariant Feynman Green function as:

$$\begin{aligned} D_F^{d+2}(x) &\stackrel{df}{=} \int_{-\infty}^{\infty} \frac{d^d k_{\perp}}{(2\pi)^d} \int_0^{\infty} \frac{dk_-}{2\pi 2k_-} \left[\Theta(x^+) e^{-ik \cdot x} + \Theta(-x^+) e^{+ik \cdot x} \right]_{k_+ = \frac{k_{\perp}^2}{2k_-}} \\ &= i \int \frac{d^{d+2}k}{(2\pi)^{d+2}} \frac{e^{-ik(x-y)}}{k^2 + i\epsilon}. \end{aligned} \quad (\text{B.7})$$

The Feynman propagator function for the massive fields in 3+1 dimensions are given by

$$\Delta_F(x, M^2) = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot x}}{2k_+ k_- - k_{\perp}^2 - M^2 + i\epsilon}, \quad (\text{B.8.a})$$

thus the functions for massless fields can be defined as the limits

$$D_F(x) = \lim_{m^2 \rightarrow 0} \Delta_F(x, M^2) = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot x}}{2k_+ k_- - k_{\perp}^2 + i\epsilon}, \quad (\text{B.8.b})$$

$$E_F(x) = -\lim_{m^2 \rightarrow 0} \frac{\partial}{\partial m^2} \Delta_F(x, m^2) = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot x}}{(2k_+ k_- - k_{\perp}^2 + i\epsilon)^2}. \quad (\text{B.8.c})$$

In the main text we use the following property

$$\begin{aligned} [2\partial^{\bar{x}} D_F^{d+2}(x) + E_F^1(x_L) \delta^d(x_{\perp})] &= (-i) \int_{-\infty}^{\infty} \frac{d^d k_{\perp}}{(2\pi)^d} \int_0^{\infty} \frac{dk_-}{2\pi} \left[\Theta(x^+) \left(e^{-i\frac{k_{\perp}^2}{2k_-} x^+} - 1 \right) e^{-i\bar{k} \cdot \bar{x}} \right. \\ &\quad \left. + \Theta(-x^+) \left(e^{i\frac{k_{\perp}^2}{2k_-} x^+} - 1 \right) e^{-i\bar{k} \cdot \bar{x}} \right] = -\Delta_{\perp} \int_0^{x^+} d\xi D_F^{d+2}(\xi, \bar{x}) \\ &= -\Delta_{\perp} \int \frac{d^{d+2}k}{(2\pi)^{d+2}} \frac{e^{-ik(x-y)}}{k^2 + i\epsilon} \frac{1}{k_+ + i\epsilon \operatorname{sgn}(k_-)}, \end{aligned} \quad (\text{B.9})$$

and a similar property holds also for the massive case

$$[2\partial^{\bar{x}} \Delta_F(x, m^2) + E_F^1(x_L) \delta^2(x_{\perp})] = -(\Delta_{\perp} - m^2) \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} \frac{1}{k_+ + i\epsilon \operatorname{sgn}(k_-)}. \quad (\text{B.10})$$

The noncovariant Feynman propagator functions in 3+1 dimensions are given by

$$G_{\alpha F}^1(x) = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot x}}{(1 + \alpha)k_+ k_- - \alpha k_{\perp}^2 + i\epsilon}, \quad (\text{B.11.a})$$

$$G_{\alpha F}^2(x) = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot x}}{[(1 + \alpha)k_+ k_- - \alpha k_{\perp}^2 + i\epsilon]^2}. \quad (\text{B.11.b})$$

B.2 Integral operators in x^- direction

We use the basic inversion of ∂_- defined by the Fourier integral

$$(\partial_-)^{-1}(x^- - y^-) = i \int_{-\infty}^{\infty} dk \exp -ik_-(x^- - y^-) \text{CPV} \frac{1}{k_-} = \frac{1}{2} \text{sgn}(x^- - y^-). \quad (\text{B.12})$$

However, for the fermion and scalar fields it is more convenient to use another Green function $(i\partial_-)^{-1}(x^- - y^-) = -i(\partial_-)^{-1}(x^- - y^-) = -i\frac{1}{2}\text{sgn}(x^- - y^-)$, which under the complex conjugation behaves like a Hermitian matrix

$$\left[(i\partial_-)^{-1}(x^- - y^-) \right]^* = (i\partial_-)^{-1}(y^- - x^-). \quad (\text{B.13})$$

Therefore, the Green function for the covariant derivative $(i\partial_- - eA_-)^{-1}$ can be given in the terms of integral operators

$$(i\partial_- - eA_-)^{-1}[x^-, y^-] = \int dz^- (i\partial_-)^{-1}(x^- - z^-) \mathcal{W}_{-1}[z^-, y^-; \hat{a}] \quad (\text{B.14.a})$$

$$= \int dz^- \mathcal{W}_{-1}[x^-, z^-; \hat{a}^\dagger] (i\partial_-)^{-1}(z^- - y^-), \quad (\text{B.14.b})$$

where

$$\mathcal{W}_{-1}[x^-, y^-; \hat{a}] = \delta(x^- - y^-) + \sum_{k=1}^{\infty} (\hat{a})^k [x^-, y^-], \quad (\text{B.15.a})$$

$$\hat{a}[x^-, y^-] = eA_-(x^-) (i\partial_-)^{-1}(x^- - y^-), \quad (\text{B.15.b})$$

$$(\hat{a})^n [x^-, y^-] = \int dz^- (\hat{a})^{n-1} [x^-, z^-] \hat{a}[z^-, y^-] = \int dz^- \hat{a}[x^-, z^-] (\hat{a})^{n-1} [z^-, y^-], \quad (\text{B.15.c})$$

$$\hat{a}^\dagger [x^-, y^-] = e(i\partial_-)^{-1}(x^- - y^-) A_-(y^-), \quad (\text{B.15.d})$$

$$(\hat{a}^\dagger)^n [x^-, y^-] = \int dz^- (\hat{a}^\dagger)^{n-1} [x^-, z^-] \hat{a}^\dagger [z^-, y^-] = \int dz^- \hat{a}^\dagger [x^-, z^-] (\hat{a}^\dagger)^{n-1} [z^-, y^-]. \quad (\text{B.15.e})$$

One can easily check that under the complex conjugation these expressions behave as follows:

$$((\hat{a})^n [x^-, y^-])^* = (\hat{a}^\dagger)^n [y^-, x^-] \quad (\text{B.16.a})$$

$$((\hat{a}^\dagger)^n [x^-, y^-])^* = (\hat{a})^n [y^-, x^-] \quad (\text{B.16.b})$$

$$(\mathcal{W}_{-1}[x^-, y^-; \hat{a}])^* = \mathcal{W}_{-1}[y^-, x^-; \hat{a}^\dagger] \quad (\text{B.16.c})$$

$$(\mathcal{W}_{-1}[x^-, y^-; \hat{a}^\dagger])^* = \mathcal{W}_{-1}[y^-, x^-; \hat{a}] \quad (\text{B.16.d})$$

$$\left((i\partial_- - eA_-)^{-1}[x^-, y^-] \right)^* = (i\partial_- - eA_-)^{-1}[y^-, x^-]. \quad (\text{B.16.e})$$

If all two-argument expressions are treated as generalized (infinite dimensional) matrices and their convolutions are denoted by asterisks then one can introduce the self-explanatory matrix notation

$$(i\partial_- - eA_-)^{-1} = (i\partial_-)^{-1} * \mathcal{W}_{-1}[\hat{a}] = \mathcal{W}_{-1}[\hat{a}^\dagger] * (i\partial_-)^{-1}, \quad (\text{B.17})$$

where

$$\mathcal{W}_{-1}[\hat{a}] = 1 + \sum_{k=1}^{\infty} (\hat{a})^k, \quad (\text{B.18.a})$$

$$\hat{a} = eA_-(i\partial_-)^{-1}, \quad (\text{B.18.b})$$

$$(\hat{a})^n = (\hat{a})^{n-1} * \hat{a} = \hat{a} * (\hat{a})^{n-1}, \quad (\text{B.18.c})$$

$$\hat{a}^\dagger = e(i\partial_-)^{-1} A_-, \quad (\text{B.18.d})$$

$$(\hat{a}^\dagger)^n = (\hat{a}^\dagger)^{n-1} * \hat{a}^\dagger = \hat{a}^\dagger * (\hat{a}^\dagger)^{n-1}. \quad (\text{B.18.e})$$

Also one can check the following property

$$i\partial_- \mathcal{W}_{-1}[\hat{a}^\dagger]_* = i\partial_- + eA_- \mathcal{W}_{-1}[\hat{a}^\dagger]_* , \quad (\text{B.19})$$

so the integral operator $\mathcal{W}_{-1}[\hat{a}^\dagger]$ effectively transforms the covariant derivative into the partial derivative

$$(i\partial_- - eA_-)\mathcal{W}_{-1}[\hat{a}^\dagger]_* = i\partial_- . \quad (\text{B.20})$$

Another simple calculation shows that

$$(i\partial_- - eA_-)(i\partial_-)^{-1} * \mathcal{W}_{-1}[\hat{a}] = \mathcal{W}_{-1}[\hat{a}] - \hat{a} * \mathcal{W}_{-1}[\hat{a}] = 1 \quad (\text{B.21})$$

and this constitutes the proof that the integral operator defined by (B.17) is the Green function for the covariant derivative operator $(i\partial_- - eA_-)$. All above results can be easily transformed to the quantum theory, where the real-valued field $A_-(x^-)$ is substituted by the Hermitian operator and the complex conjugation is replaced by the Hermitian conjugation.

All charged fields can be easily incorporated into the matrix notation, where and the scalar field ϕ and fermion field ψ are treated as the one-column matrices while their Hermitian conjugated counterparts ϕ^\dagger and ψ^\dagger as the one-row matrices, respectively.

B.3 Inverse Laplace operators

The Green function for the Laplace operator is well defined for $d > 2$ dimensions

$$[\Delta_\perp^d]^{-1}(x_\perp) = - \int \frac{d^d k_\perp}{(2\pi)^d} \frac{e^{ik_\perp x_\perp}}{k_\perp^2} = - \frac{1}{2^{2-d/2}} \frac{1}{(2\pi)^{d/2}} \frac{1}{(x_\perp)^{d/2-1}} \Gamma(d/2 - 1) \quad (\text{B.22})$$

and for $d = 2$ it is singular. From [46] we know that there are also other regularizations, strictly in $d = 2$ dimensions. In the paper, we use the massive regularization - when the pole at $k_\perp^2 = 0$ is shifted by the mass parameter m^2

$$[\Delta_\perp]^{-1}(x_\perp) \rightarrow [\Delta_\perp - m^2]^{-1}(x_\perp) = - \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{e^{ik_\perp x_\perp}}{k_\perp^2 + m^2} = - \frac{1}{2\pi} K_0(m\sqrt{x_\perp^2}), \quad (\text{B.23})$$

which naturally appears for the massive electrodynamics.

Another regularized Green function appears in Section 5

$$\frac{1}{\Delta_\alpha}(\vec{x}) = \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{dk^-}{2\pi} \frac{e^{-i\vec{k} \cdot \vec{x}}}{k_\perp^2 - 2\alpha k^-} \quad (\text{B.24})$$

and for $\alpha < 0$ there is no ambiguity in the integrand.

B.4 Finite volume Green functions

One can safely invert differential operators such as ∂_- and ∂_\perp^2 in terms of well-defined Green's functions, taking care of the respective mode sector in which the operator acts. With the covariant derivative $D_\mu = \partial_\mu + ieV_\mu$, we define the operator-valued Green's function to (iD_-) :

$$(iD_-^x) \frac{1}{i\partial_- - eV_-}[\vec{x}, \vec{y}] \equiv \delta^{(3)}(\vec{x} - \vec{y}) - [Sub.] , \quad (\text{B.25})$$

where [Sub.] denotes possible subtractions corresponding to zero eigenvalues of the operator in question. A nonperturbative construction for this particular Green's function is presented in Appendix A of [40]. The Green's function $\mathcal{G}_{(\perp)}[x_{\perp}, y_{\perp}; \mathcal{O}]$ is defined by the relation

$$[\Delta_{\perp}^{\mp} - \frac{e^2}{2L}\mathcal{O}(x_{\perp})]\mathcal{G}_{(\perp)}[x_{\perp}, y_{\perp}; \mathcal{O}] \equiv \delta^{(2)}(x_{\perp} - y_{\perp}), \quad (\text{B.26})$$

where $\Delta_{\perp} \equiv \partial_{\perp}^2$, and now \mathcal{O} is some field operator of mass dimension one. Eq.(B.26) can be elucidated order by order in perturbation theory for which one uses the basic inversion of Δ_{\perp} . A nonperturbative definition is however nontrivial and therefore the above operation is, at best, merely formal and its concrete implementation remains an open problem.

The convolutions of the above Green's functions are also denoted by asterisks.

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