A WEAK FORMULATION FOR THE LARGE DEFORMATION CONTACT PROBLEM WITH COULOMB FRICTION

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1. Theoretical considerations

In this work, a weighted residual relationship involving both the displacements and a field of multipliers is proposed as the weak form of the large deformation contact problem with Coulomb friction in quasi-statics. It is shown that (i) the proposed weak form is equivalent to the strong form of the contact problem and (ii) the multipliers are equal to the contact tractions.

Consider two bodies, indexed by superscripts 1 and 2, undergoing motions $\boldsymbol{\phi}^{(1)}$ and $\boldsymbol{\phi}^{(2)}$ in the three-dimensional space during some time interval $[O, T]$. Let us assume that the bodies may come into contact with each other and formulate the contact problem using a Lagrangian description. The notations, rather standard, are summarized as follows. The reference configuration of the two bodies are represented by the regions $\Omega_o^{(1)}$ and $\Omega_o^{(2)}$. The prescribed body force per unit mass in body i is denoted $\mathbf{f}^{(i)}$. The boundary $S_o^{(i)}$ of body i is partitioned into three parts denoted $S_{oU}^{(i)}$, $S_{oT}^{(i)}$ and $S_{oc}^{(i)}$, where $S_{oU}^{(i)}$ and $S_{oT}^{(i)}$ are the parts where displacements and tractions are prescribed, respectively, and $S_{oc}^{(i)}$ is the part where contact potentially takes place. The stress state in body i is defined by the first Piola-Kirchhoff stress tensor $\mathbf{\Pi}^{(i)}$. The nominal traction vector at any point in $S_o^{(i)}$ with normal vector $\mathbf{N}^{(i)}$ is denoted $\mathbf{T}^{(i)} = \mathbf{\Pi}^{(i)}.\mathbf{N}^{(i)}$. The spatial counterparts of surface $S_o^{(i)}$ is denoted $S^{(i)}$.

Given a point $\mathbf{x} \in S_c^{(1)}$ one defines a contact point $\mathbf{y} \in S_c^{(2)}$ as the closest point to $\mathbf{x}$ via $\mathbf{y} = arg\min_{\mathbf{x}^{(2)} \in S_c^{(2)}} \|\mathbf{x} - \mathbf{x}^{(2)}\|$, and the proximity as $g = -\boldsymbol{\nu}(\mathbf{x} - \mathbf{y})$ where $\boldsymbol{\nu}$ is the outward normal at point $\mathbf{y}$. One also defines the point $\mathbf{X} \in S_{oc}^{(1)}$ related to point $\mathbf{x}$ in question by $\mathbf{x} = \boldsymbol{\phi}^{(1)}(\mathbf{X})$ and $\mathbf{Y}(\mathbf{X}) \in S_{oc}^{(2)}$ related to point $\mathbf{y}$ by $\mathbf{y} = \boldsymbol{\phi}^{(2)}(\mathbf{Y})$. The tangential kinematics is characterized by the slip velocity $\mathbf{V}_T = \dot{\eta}^\alpha \mathbf{a}_\alpha$ which is resolved in terms of the local spatial basis $(\mathbf{a}_1, \mathbf{a}_2)$ at point $\mathbf{y} \in S_c^{(2)}$. Likewise, the nominal traction vector at any point $\mathbf{X} \in S_{oc}^{(1)}$ is resolved as $\mathbf{T} = T_N \boldsymbol{\nu} - \mathbf{T}_T$. The coefficient of friction is μ.

The weak form proposed here is stated as a mixed relationship which involves both the displacement fields $\mathbf{U}^{(i)}$, $i \in \{1, 2\}$, defined in $\Omega_o^{(i)}$ and the multiplier fields λ_N and $\boldsymbol{\lambda}_T$ defined on $S_{oc}^{(1)}$. Accordingly, the weighting functions are the virtual displacements $\mathbf{U}^{(1)*}$, $\mathbf{U}^{(2)*}$, and the virtual multipliers λ_N^*, $\boldsymbol{\lambda}_T^*$. All the functions involved in the weak form are assumed to be regular enough for the integrations and differentiations to make sense. Two positive constants ϵ_N, ϵ_T being chosen, the weighted residual relationship is given in the following proposition.

PROPOSITION 1. $\forall t \in [O, T], \forall \mathbf{U}^{(1)*}, \forall \mathbf{U}^{(2)*}, \forall \lambda_N^*, \forall \boldsymbol{\lambda}_T^*$,

$$
\begin{aligned}
&\textstyle\sum_{i=1}^{2} \left\{ - \int_{\Omega_o^{(i)}} \mathbf{\Pi}^{(i)T} : \nabla_{\mathbf{X}^{(i)}} \mathbf{U}^{(i)*} d\Omega_o + \int_{\Omega_o^{(i)}} \mathbf{f}^{(i)} \mathbf{U}^{(i)*} d\Omega_o + \int_{S_{oU}^{(i)} \cup S_{oT}^{(i)}} \mathbf{T}^{(i)} \mathbf{U}^{(i)*} dS_o \right\} \\
&+ \int_{S_{oc}^{(1)}} \left[\langle \lambda_N + \epsilon_N g \rangle \boldsymbol{\nu} - \left(1 - \langle 1 - \tfrac{\mu \langle \lambda_N + \epsilon_N g \rangle}{\|\boldsymbol{\lambda}_T + \epsilon_T \mathbf{V}_T\|} \rangle \right) (\boldsymbol{\lambda}_T + \epsilon_T \mathbf{V}_T) \right] \\
&\left(\mathbf{U}^{(1)*}(\mathbf{X}) - \mathbf{U}^{(2)*}(\mathbf{Y}(\mathbf{X})) \right) dS_o + \int_{S_{oc}^{(1)}} \left\{ (\lambda_N - \langle \lambda_N + \epsilon_N g \rangle) \tfrac{\lambda_N^*}{\epsilon_N} + \right. \\
&\left. \left[\boldsymbol{\lambda}_T - \left(1 - \langle 1 - \tfrac{\mu \langle \lambda_N + \epsilon_N g \rangle}{\|\boldsymbol{\lambda}_T + \epsilon_T \mathbf{V}_T\|} \rangle \right) (\boldsymbol{\lambda}_T + \epsilon_T \mathbf{V}_T) \right] \tfrac{\boldsymbol{\lambda}_T^*}{\epsilon_T} \right\} dS_o = 0
\end{aligned}
\tag{1}
$$

where $\nabla_{\mathbf{X}^{(i)}} \mathbf{U}^{(i)*}$ is the gradient tensor of $\mathbf{U}^{(i)*}$ with respect to variables $\mathbf{X}^{(i)} \in \Omega_o^{(i)}$, $\langle . \rangle$ is the Macauley bracket: $\langle a \rangle = a$ if $a \geq 0$, $= 0$ if $a < 0$, and $\langle 1 - \frac{\mu\langle \lambda_N + \epsilon_N g\rangle}{\|\boldsymbol{\lambda}_T + \epsilon_T \mathbf{V}_T\|}\rangle$ must be replaced by 0 at any point on $S_{oc}^{(1)}$ where $\boldsymbol{\lambda}_T + \epsilon_T \mathbf{V}_T = \mathbf{0}$.

One can readily prove the following statement which means that the strong form implies the weak one: the solution fields of the strong problem - namely $(\mathbf{U}^{(1)}, \mathbf{U}^{(2)})$ in $\Omega_o^{(1)}, \Omega_o^{(2)}$ and $(T_N, \mathbf{T}_T, \mathbf{V}_T)$ on $S_{oc}^{(1)}$ - satisfy (1), provided that one makes in that relationship $\lambda_N = T_N$ and $\boldsymbol{\lambda}_T = \mathbf{T}_T$. Conversely, the following proposition shows that the weak form implies the strong one.

Proposition 2. By making some smoothness assumptions, which are not specified in this abstract, it can be shown that (1) implies at any time $t \in [O, T]$ the following local equations:

(a) The momentum balance equation for the two bodies 1 and 2.

(b) The following relation on the boundary portion $S_o^{(i)} \backslash S_{oc}^{(i)} = S_{oT}^{(i)} \cup S_{oU}^{(i)}$: $\mathbf{\Pi}^{(i)}\mathbf{N}^{(i)} = \mathbf{T}^{(i)}$.

(c) The equalities between the components of the nominal traction vectors and the multipliers on the contactor surface $S_{oc}^{(1)}$: $\forall \mathbf{X} \in S_{oc}^{(1)}, \mathbf{T}^{(1)} = \mathbf{\Pi}^{(1)}\mathbf{N}^{(1)} = \lambda_N \boldsymbol{\nu} - \boldsymbol{\lambda}_T. \Leftrightarrow T_N = \lambda_N$ and $\mathbf{T}_T = \boldsymbol{\lambda}_T$.

(d) The normal and tangential contact laws: $\forall \mathbf{X} \in S_{oc}^{(1)}, g(\mathbf{X}) \leq 0$ where

. if $g < 0$, then $T_N = 0, \mathbf{T}_T = \mathbf{0}$

. if $g = 0$, then $T_N \geq 0, \|\mathbf{T}_T\| \leq \mu T_N$ $\begin{cases} .\text{if } \|\mathbf{T}_T\| \leq \mu T_N, \text{then } \mathbf{V}_T = \mathbf{0} \text{ (stick)} \\ .\text{if } \|\mathbf{T}_T\| = \mu T_N, \text{then } \mathbf{V}_T \wedge \mathbf{T}_T = \mathbf{0}, \mathbf{V}_T.\mathbf{T}_T \geq 0 \text{ (slip)} \end{cases}$

(e) The following relationship which expresses the equilibrium of the traction vectors at the contact interface: $\forall \mathbf{X} \in S_{oc}^{(1)}, \forall \mathbf{Y}(\mathbf{X}) \in S_{oc}^{(2)}, \mathbf{T}^{(2)}(\mathbf{Y}) dS_{oc}^{(2)} = -\mathbf{T}^{(1)}(\mathbf{X}) dS_{oc}^{(1)}$, where $dS_{oc}^{(1)}$ is a differential reference area in $S_{oc}^{(1)}$ and $dS_{oc}^{(2)}$ its counterpart in $S_{oc}^{(2)}$.

2. Numerical examples

The weak form (1) has been discretized by means of the finite element method and the contact tangent stiffness obtained by appropriately linearizing the contact terms. Several numerical examples have been investigated on solid or membrane structures subjected to dead or following loads, with hyperelastic or finite elastoplastic material models. A typical example is shown here, concerning the contact between two hyperelastic beams, one of which being subjected at one tip to a follower force.

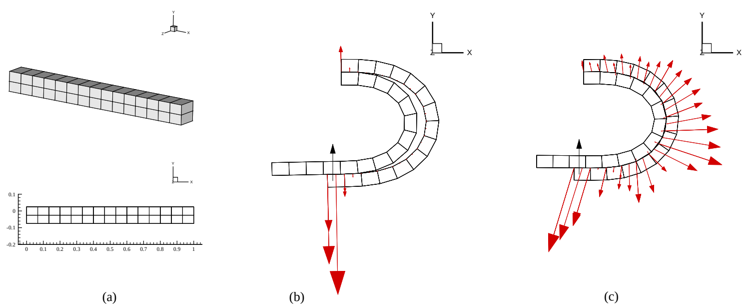

Figure 1. Contact between two beams. (a) Reference configuration, 3D view of the mesh and in-plane view. (b) Deformed shape and contact tractions at the last step in the frictionless case (c) Friction case with $\mu = 0.3$.

3. References

• P. Wriggers (2002) *Computational Contact Mechanics*, Wiley.

• T.A. Laursen (2003) *Computational Contact and Impact Mechanics*, Springer.

• G. Pietrzak and A. Curnier (1999). Large deformation frictional contact mechanics: continuum formulation and augmented Lagrangian treatment, *Comput. Methods Appl. Mech. Engrg.*, **177**, 351-381.