SOME PROBLEMS CONCERNING THE DEFORMATION OF ANISOTROPIC COSSERAT ELASTIC SHELLS

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1. Introduction

We investigate the deformation of loaded cylindrical anisotropic elastic shells, in the framework of the Cosserat theory. Within the linear theory, we approach the relaxed Saint–Venant's problem and the problem of Truesdell.

The theory of Cosserat shells is an interesting approach to the mechanics of elastic shell–like bodies, in which the thin three–dimensional body is modelled as a two–dimensional continuum (i.e. a surface) endowed with a deformable director assigned to every point. For a detailed analysis of the theory of Cosserat surfaces and its relation with other shell theories, we refer to the classical monograph of Naghdi [1] and the modern book of Rubin [2].

Due to its importance in engineering, the Saint–Venant's problem has been studied in many articles in the context of classical theories of shells or in the theory of Cosserat surfaces [3]. For isotropic and homogeneous Cosserat shells, the solution of Saint–Venant's relaxed problem was given in [4].

In the present work, we consider anisotropic and inhomogeneous cylindrical Cosserat shells. The cylindrical surfaces can be open or closed, and the cross–section is not necessarily circular. We assume that the constitutive coefficients of the Cosserat shell are independent of the axial coordinate.

2. The relaxed Saint–Venant's problem

For any Cosserat shell, we denote by $\boldsymbol{r}$ and $\boldsymbol{d}$ the position vector and the director assigned to every point of the deformed surface. Let $\boldsymbol{R}$ and $\boldsymbol{D}$ designate the position vector and the director fields associated to the reference configuration $\mathcal{S}$ of the Cosserat surface. Then, the (infinitesimal) displacement $\boldsymbol{u}$ and director displacement $\boldsymbol{\delta}$ are defined by

$$(1) \qquad \boldsymbol{u} = \boldsymbol{r} - \boldsymbol{R}, \qquad \boldsymbol{\delta} = \boldsymbol{d} - \boldsymbol{D}.$$

We consider that the reference configuration $\mathcal{S}$ is a general cylindrical surface (open or closed), and we denote by z and s the axial coordinate and the circumferential coordinate on $\mathcal{S}$, respectively.

The well–known Saint–Venant's problem consists in determining the equilibrium of such shells under the action of prescribed contact forces and contact director couples distributed over its end edges. In the relaxed formulation of this problem, we consider that the terminal loads are given in the form of the resultant forces and resultant moments acting on the end edges.

We determine a solution of the relaxed Saint–Venant's problem for anisotropic Cosserat shells using the method established by Ieşan [5] in the context of three–dimensional elasticity. Our solution is presented in the form of the displacement field $v = (\boldsymbol{u}, \boldsymbol{\delta})$ and it is expressed in terms of the solutions to some auxiliary boundary–value problems for ordinary differential equations (called the cross–section plane problems).

In order to obtain the solution, we separate the relaxed Saint–Venant's problem into two problems: (P_1) the extension–bending–torsion problem, and (P_2) the flexure problem.

First, we search for a solution $v = (\boldsymbol{u}, \boldsymbol{\delta})$ of the problem (P_1) such that $\partial v/\partial z$ is a rigid body displacement field of the Cosserat shell. As in the three-dimensional theory, this solution is determined in terms of four constants, say a_1 , a_2 , a_3 and a_4 , which can be interpreted as the global measures of axial curvature, axial strain and twist. We denote the solution of the extension–bending–torsion problem by $v = v\{a_1, a_2, a_3, a_4\}$, indicating thus its dependence on the constants a_k .

For the flexure problem (P_2), we obtain a solution of the form

$$v = \int_0^z v\{b_1, b_2, b_3, b_4\}\, \mathrm{d}z + v\{c_1, c_2, c_3, c_4\} + w(s), \tag{2}$$

where $\{b_1, b_2, b_3, b_4\}$ and $\{c_1, c_2, c_3, c_4\}$ are constants, while $w(s)$ is a displacement field depending only on s, which are determined in the paper.

3. Truesdell's problem

We notice that the solution obtained for the relaxed Saint–Venant's problem in the theory of Cosserat shells possesses some properties which are analogous to the characteristic properties of the classical Saint–Venant's solution for cylinders. For instance, we prove that our solutions can be characterized as minimizers of the strain energy on certain classes of solutions (in correlation with the corresponding three–dimensional results for cylinders, see e.g. [5, 6]).

Further, we extend this analogy and derive a solution for the problem of Truesdell for anisotropic cylindrical shells. In [7], Truesdell proposed the following problem for the torsion of elastic cylinders: to define the functional $\tau(\cdot)$ on the set of all solutions $\boldsymbol{u}$ of the torsion problem, corresponding to a scalar torque $\mathcal{M}$, such that

$$\tau(\boldsymbol{u}) = -\frac{\mathcal{M}}{\mu D} \ , \tag{3}$$

where μD is the torsional rigidity of the cylinder. Podio–Guidugli [8] rephrased the problem for extension and bending, while Ieşan [5] considered the flexure of elastic cylinders.

We present a solution of Truesdell's problem formulated for the extension–bending–torsion problem and for the flexure problem of anisotropic cylindrical shells.

Examples are given for orthotropic cylindrical shells and for the special case of Cosserat plates.

4. References

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