MULTIMATERIALS WITH SHELL-LIKE REINFORCEMENT

***A.L. Bessoud*[1,2]*, F. Krasucki*[1] *and M. Serpilli*[2]**

[1]*Institut de Mathématiques et de Modélisation de Montpellier – UMR5149, Université Montpellier II France*

[2]*Laboratoire de Mécanique et Génie Civil – UMR 5508, Université Montpellier II, France*

After the pioneering works of Pham Huy-Sanchez [1], and Caillerie [2], the thin inclusion of a third material between two other ones when the rigidity properties of the inclusion are highly contrasted with respect to those of the surrounding material has been deeply investigated. More recently, Chapelle-Ferent [3] in order to justify some methods used in FEM approximation have studied the asymptotic behavior of a shell-like inclusion of $\frac{1}{\varepsilon^p}$-rigidity ($p = 1$ or $p = 3$) in a 3D domain. In a slightly different geometrical and mechanical context, Bessoud et al. [4] have studied the behavior of a ε-thin 3D layer of $\frac{1}{\varepsilon}$-rigidity. We study a new situation where the shell-like thin layer is obtained by the translation in the normal direction of a general 2D surface. Using a system of curvilinear coordinates we deduce the formal limit problem for the two cases $p = 1$ and $p = 3$. We obtain the same limit problems as in [3], also if the kinematical assumptions for the physical problem are not the same. Indeed in [3] the authors a priori assume a shell-like energy in the thin layer. One must stress that the well-posedness of the limit problems is essentially linked to the shell inhibition phenomena [5]. For the well-posedness of the flexural and membrane shell models see e.g.[6], [5]. When ω is planar and in the isotropic case, the surface energy term can be interpreted as the membranal energy of a Kirchhoff-Love plate ($p = 1$) and as the flexural energy of a Kirchhoff-Love plate ($p = 3$).

1. Shell-like inclusion : asymptotic behavior

In the three-dimensional Euclidean space $\mathcal{E}^3$ referred to the Cartesian coordinate frame $(O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, let Ω^+ and Ω^- be two disjoint open domains with smooth boundaries $\partial\Omega^+$ and $\partial\Omega^-$. Let $\omega = \{\partial\Omega^+ \cap \partial\Omega^-\}^\circ$, which is assumed to be a domain in $\mathbb{R}^2$ having a positive two-dimensional measure and let $y = (y_\alpha)$ denote a generic point of $\overline{\omega}$. Let $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ be an injective mapping such that the vectors $\boldsymbol{a}_\alpha(y) := \partial_\alpha \boldsymbol{\theta}(y)$ form the covariant basis of the tangent plane to the surface $S := \boldsymbol{\theta}(\overline{\omega})$ at the point $\boldsymbol{\theta}(y)$; the two vectors $\boldsymbol{a}^\alpha(y)$ of the tangent plane, defined by the relations $\boldsymbol{a}^\alpha(y) \cdot \boldsymbol{a}_\beta(y) = \delta^\alpha_\beta$, form its contravariant basis. Also let $\boldsymbol{a}_3(y) = \boldsymbol{a}^3(y)$ be the unit normal vector to S. Let $\Omega^{m,\varepsilon} := \omega \times] - \varepsilon, \varepsilon[$, with $\Gamma^{\pm,\varepsilon} := \omega \times \{\pm\varepsilon\}$. Let x^ε denote the generic point in the set $\overline{\Omega}^{m,\varepsilon}$, with $x^\varepsilon_\alpha = y_\alpha$. We consider a shell-like domain with middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ and thickness $2\varepsilon > 0$, whose reference configuration is the image $\boldsymbol{\Theta}^{m,\varepsilon}(\overline{\Omega}^{m,\varepsilon}) \subset \mathbb{R}^3$ of the set $\overline{\Omega}^{m,\varepsilon}$ through the mapping $\boldsymbol{\Theta}^{m,\varepsilon} : \overline{\Omega}^{m,\varepsilon} \to \mathbb{R}^3$ given by $\boldsymbol{\Theta}^{m,\varepsilon}(x^\varepsilon) := \boldsymbol{\theta}(y) + x^\varepsilon_3 \boldsymbol{a}_3(y),$ for all $x^\varepsilon = (y, x^\varepsilon_3) = (y_1, y_2, x^\varepsilon_3) \in \overline{\Omega}^{m,\varepsilon}$. Moreover, we suppose that there exists an immersion $\boldsymbol{\Theta}^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ defined as follows :

$$\boldsymbol{\Theta}^\varepsilon := \begin{cases} \boldsymbol{\Theta}^{\pm,\varepsilon} \text{ on } \overline{\Omega}^{\pm,\varepsilon} \\ \boldsymbol{\Theta}^{m,\varepsilon} \text{ on } \overline{\Omega}^{m,\varepsilon} \end{cases}, \ \boldsymbol{\Theta}^{\pm,\varepsilon}(\Gamma^{\pm,\varepsilon}) = \boldsymbol{\Theta}^{m,\varepsilon}(\Gamma^{\pm,\varepsilon}),$$

with $\boldsymbol{\Theta}^{\pm,\varepsilon} : \overline{\Omega}^{\pm,\varepsilon} \to \mathbb{R}^3$ immersions over $\overline{\Omega}^{\pm,\varepsilon}$ defining the curvilinear coordinates on $\overline{\Omega}^{\pm,\varepsilon}$. We insert the intermediate shell-like layer moving the image $\boldsymbol{\Theta}^{+,\varepsilon}(\overline{\Omega}^{+,\varepsilon}) \subset \mathbb{R}^3$ of the set $\overline{\Omega}^{+,\varepsilon}$, (resp. $\boldsymbol{\Theta}^{-,\varepsilon}(\Omega^{-,\varepsilon})$) in the $\boldsymbol{a_3}(\mathbf{y})$ (resp. $-\boldsymbol{a_3}(\mathbf{y})$) direction of an amount equal to $\varepsilon > 0$, the small dimensionless real parameter. The structure is clamped on $\Gamma_0 \subset (\partial\Omega^\varepsilon \setminus \Gamma^{m,\varepsilon})$ and $\Gamma^{m,\varepsilon} := \partial\omega \times] - \varepsilon, \varepsilon[$ is traction free. We suppose that the materials occupying Ω^ε are linearly elastic and isotropic. Let

$$V^\varepsilon = \{(\boldsymbol{V}, \boldsymbol{v}) \in H^1(\Omega^\varepsilon; \mathbb{R}^3) \times H^1(\Omega^{m,\varepsilon}; \mathbb{R}^3);\ \boldsymbol{V}_{|\Omega^{m,\varepsilon}} = \boldsymbol{v};\ \boldsymbol{V}_{|\Gamma_0} = 0\}.$$

The physical variational problem in these curvilinear coordinates on the variable domain Ω^ε is

$$(1)\qquad \begin{cases} \text{Find } (\boldsymbol{U}^\varepsilon, \boldsymbol{u}^\varepsilon) \in V^\varepsilon \text{ such that for all } (\mathbf{V}^\varepsilon, \boldsymbol{v}^\varepsilon) \in V^\varepsilon \\ A^{-,\varepsilon}(\boldsymbol{U}^\varepsilon, \boldsymbol{V}^\varepsilon) + A^{+,\varepsilon}(\boldsymbol{U}^\varepsilon, \boldsymbol{V}^\varepsilon) + A^{m,\varepsilon}(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon) = L(\boldsymbol{V}^\varepsilon), \end{cases}$$

where $A^{\pm,\varepsilon}(\boldsymbol{U}^\varepsilon, \boldsymbol{V}^\varepsilon)$ and $A^{m,\varepsilon}(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon)$ are the bilinear form associated with the elastic behavior of the domain. In order to study the asymptotic behavior of the physical problem (1), we apply the usual change of variable, which transforms Ω^ε into a fixed domain Ω.

Now, the leading terms $(\boldsymbol{U}^0, \boldsymbol{u}^0)$ of the asymptotic expansion satisfy the following limit problems :

1. $p = 1$:

$$\begin{cases} \text{Find } (\boldsymbol{U}^0, \boldsymbol{u}^0) \in V_M \text{ such that for all } (\boldsymbol{V}, \boldsymbol{v}) \in V_M \\ A^-(\boldsymbol{U}^0, \boldsymbol{V}) + A^+(\boldsymbol{U}^0, \boldsymbol{V}) + A_M^m(\boldsymbol{u}^0, \boldsymbol{v}) = L(\boldsymbol{V}), \end{cases}$$

where

$$V_M = \{(\boldsymbol{V}, \boldsymbol{v}) \in H^1(\Omega; \mathbb{R}^3) \times H^1(\omega; \mathbb{R}^3);\ \boldsymbol{V}_{|\omega} = \boldsymbol{v},\ \boldsymbol{V}_{|\Gamma_0} = 0\},$$

$A_M^m(\boldsymbol{u}, \boldsymbol{v}) = \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{u})\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}\ dy$ is the bilinear form associated with the membrane behavior of the shell, $a^{\alpha\beta\sigma\tau}$ are the contravariant components of the elasticity tensor of the shell and $\gamma_{\alpha\beta}(\boldsymbol{u})$ are the covariant components of the change of metric tensor.

2. $p = 3$:

$$\begin{cases} \text{Find } (\boldsymbol{U}^0, \boldsymbol{u}^0) \in V_F \text{ such that for all } (\boldsymbol{V}, \boldsymbol{v}) \in V_F \\ A^-(\boldsymbol{U}^0, \boldsymbol{V}) + A^+(\boldsymbol{U}^0, \boldsymbol{V}) + A_F^m(\boldsymbol{u}^0, \boldsymbol{v}) = L(\boldsymbol{V}), \end{cases}$$

where

$$V_F = \{(\boldsymbol{V}, \boldsymbol{v}) \in H^1(\Omega; \mathbb{R}^3) \times H^2(\omega; \mathbb{R}^3);\ \boldsymbol{V}_{|\omega} = \boldsymbol{v},\ \boldsymbol{V}_{|\Gamma_0} = 0,\ \gamma_{\alpha\beta}(\boldsymbol{v}) = 0 \text{ in } \omega\},$$

$A_F^m(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{u})\rho_{\alpha\beta}(\boldsymbol{v})\sqrt{a}\ dy$ is the bilinear form associated with the flexural behavior of the shell and $\rho_{\alpha\beta}(\boldsymbol{u})$ are the covariant components of the change of curvature tensor.

[1] H. Pham Huy, E. Sanchez-Palencia, Phénomène de transmission à travers des couches minces de conductivité élevée, J. Math. Anal. Appl. 47 (1974) 284–309.

[2] D. Caillerie, The effect of a thin inclusion of high rigidity in an elastic body, Math. Methods Appl. Sci. 2 (1980) 251–270.

[3] D. Chapelle, A. Ferent, Modeling of the inclusion of a reinforcing sheet within a 3D medium, Math. Models Methods Appl. Sci. 13 (2003) 573–595.

[4] A.L. Bessoud, F. Krasucki, G. Michaille, Multi-materials with strong interface : variational modelings, (submitted).

[5] J. Sanchez-Hubert, E. Sanchez-Palencia, Coques élastiques minces : propriétés asymptotiques, Masson, Paris, 1997.

[6] P.G. Ciarlet, Mathematical Elasticity, vol. III : Theory of shells, Studies in mathematics and its applications, North-Holland, Amsterdam, 2000.

[7] G. Geymonat, F. Krasucki, S. Lenci, Mathematical analysis of a bonded joint with a soft thin adhesive, Math. Mech. Solids 4 (1999) 201–225.