RECOVERY OF DISPLACEMENT FIELDS FROM STRESS TENSOR FIELDS IN SHELL THEORY

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The stresses and displacements arising in an elastic shell in response to applied forces are modeled by a system of partial differential equations defined over a three-dimensional domain, representing the shell in its natural state (i.e., in absence of applied forces).

In the classical theory of shells, the displacement field is the primary unknown, while the stress tensor field inside the shell is a secondary unknown, given in terms of the displacement field by the constitutive law of the elastic material; see, e.g., Ciarlet [2]. By contrast, in the intrinsic theory of shells, the stress tensor field is the primary unknown, while the displacement field is a secondary unknown; see, e.g., Antman [1], Ciarlet et al. [3], Pietraszkiewicz et al. [6, 7], and Vallée [8]. One of the principal problems arising in the intrinsic theory of shells is to show that the displacement field can be recovered from the stress tensor field inside the shell. This presentation is dedicated to this problem.

Consider an elastic shell which in absence of applied forces occupies a domain contained in a thin neighborhood of a surface $S = \boldsymbol{\theta}(\omega)$, where $\omega \subset \mathbb{R}^2$ is a domain with a sufficiently smooth boundary and $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ is a sufficiently smooth immersion. Assume that the elastic material constituting the shell is homogeneous and isotropic, hence characterized by its two Lamé constants $\lambda > 0$ and $\mu > 0$. Finally assume that the shell is subjected to applied forces and that the shell is free, i.e., the displacement is not subjected to any boundary conditions.

As a mathematical model for this problem, we select the two-dimensional Koiter equations (see Koiter [5]). According to this model, the stresses inside the shell are related to the infinitesimal change of metric and change of curvature tensor fields of the surface S by a bijective linear function. As a consequence, recovering a displacement field $\boldsymbol{\eta} : \omega \to \mathbb{R}^3$ from the stress tensor field inside the shell amounts to recovering $\boldsymbol{\eta}$ from the infinitesimal change of metric and change of curvature tensor fields of the surface S, defined in what follows by their respective covariant components $\gamma_{\alpha\beta}$ and $\rho_{\alpha\beta}$. Here and in the sequel, Greek indices and exponents vary in the set $\{1, 2\}$ and the summation convention with respect to repeated indices and exponents is used.

Our main result is as follows (for details, see [4]). Assume that ω is simply connected. Let $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ be two symmetric matrix fields with components $\gamma_{\alpha\beta} \in L^2(\omega)$ and $\rho_{\alpha\beta} \in H^{-1}(\omega)$ that satisfy the following compatibility conditions, which we shall call the "Saint Venant equations on the surface S", viz.,

$$\begin{aligned}
&\gamma_{\sigma\alpha|\beta\tau} + \gamma_{\tau\beta|\alpha\sigma} - \gamma_{\tau\alpha|\beta\sigma} - \gamma_{\sigma\beta\alpha\tau} + R^{\nu}_{\cdot\alpha\sigma\tau}\gamma_{\beta\nu} - R^{\nu}_{\cdot\beta\sigma\tau}\gamma_{\alpha\nu} \\
&\qquad\qquad = b_{\tau\alpha}\rho_{\sigma\beta} + b_{\sigma\beta}\rho_{\tau\alpha} - b_{\sigma\alpha}\rho_{\tau\beta} - b_{\tau\beta}\rho_{\sigma\alpha}, \\
&\rho_{\sigma\alpha|\tau} - \rho_{\tau\alpha|\sigma} = b^{\nu}_{\sigma}(\gamma_{\alpha\nu|\tau} + \gamma_{\tau\nu|\alpha} - \gamma_{\tau\alpha|\nu}) - b^{\nu}_{\tau}(\gamma_{\alpha\nu|\sigma} + \gamma_{\sigma\nu|\alpha} - \gamma_{\sigma\alpha|\nu}).
\end{aligned}$$

Then there exists a vector field $\boldsymbol{\eta} : \omega \to \mathbb{R}^3$ of class H^1 such that the two fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ are respectively the linearized change of metric and linearized change of curvature tensors associated with the displacement field $\boldsymbol{\eta}$, in the sense that

$$\begin{aligned}
\gamma_{\alpha\beta} &= \frac{1}{2}(\partial_\alpha\boldsymbol{\eta}\cdot\partial_\beta\boldsymbol{\theta} + \partial_\alpha\boldsymbol{\theta}\cdot\partial_\beta\boldsymbol{\eta}) \quad \text{in } \omega, \\
\rho_{\alpha\beta} &= (\partial_{\alpha\beta}\boldsymbol{\eta} - \Gamma^{\nu}_{\alpha\beta}\partial_\nu\boldsymbol{\eta})\cdot\mathbf{a}_3 \qquad \text{in } \omega.
\end{aligned}$$

The functions $\gamma_{\alpha\beta|\sigma}$ and $\gamma_{\alpha\beta|\sigma\tau}$ denote respectively the first and the second covariant derivatives of the field $(\gamma_{\alpha\beta})$, $R^{\nu}_{\cdot\alpha\sigma\tau}$ denotes the components of the Riemann curvature tensor of the surface S, $b_{\alpha\beta}$ and b^{τ}_{σ} denote respectively the mixed components of the second fundamental form of the surface $S = \boldsymbol{\theta}(\omega)$, and $\mathbf{a}_3 := \dfrac{1}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|}\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}$.

The proof of this result furnishes an explicit algorithm for recovering the vector field $\boldsymbol{\eta}$ from the matrix fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$: one first solves the system

$$\begin{aligned}\lambda_{\alpha\beta|\sigma} + b_{\alpha\sigma}\lambda_{\beta} - b_{\beta\sigma}\lambda_{\alpha} &= \gamma_{\sigma\beta|\alpha} - \gamma_{\sigma\alpha|\beta},\\ \lambda_{\alpha|\sigma} + b^{\nu}_{\sigma}\lambda_{\alpha\nu} &= \rho_{\sigma\alpha} - b^{\nu}_{\sigma}\gamma_{\alpha\nu},\end{aligned}$$

where the unknowns are the antisymmetric matrix field $(\lambda_{\alpha\beta})$ and the vector field (λ_{α}) with components $\lambda_{\alpha\beta} \in L^2(\omega)$ and $\lambda_{\alpha} \in L^2(\omega)$; then one solves the system

$$\partial_{\alpha}\boldsymbol{\eta} = (\gamma_{\alpha\beta} + \lambda_{\alpha\beta})\mathbf{a}^{\beta} + \lambda_{\alpha}\mathbf{a}^{3} \text{ in } \omega,$$

where $\{\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3\}$ is the dual of the basis $\{\partial_1\boldsymbol{\theta}, \partial_2\boldsymbol{\theta}, \mathbf{a}_3\}$. The vector field $\boldsymbol{\eta} \in H^1(\omega; \mathbb{R}^3)$ found in this fashion has the desired properties.

Note that the first system has solutions because the matrix fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ satisfy the above Saint Venant equations on a surface and that the second system has solutions because the matrix fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ are symmetric.

These results may be viewed as the infinitesimal versions of the reconstruction of a surface from its fundamental forms, because the Saint Venant equations on a surface are nothing but the first order part with respect to ε of the Gauss and Codazzi-Mainardi equations associated with the immersion $(\boldsymbol{\theta} + \varepsilon\boldsymbol{\eta})$.

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