RECOVERING THE BIPOTENTIAL OF AN IMPLICIT STANDARD MATERIAL BY FITZPATRICK'S METHOD

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1. Implicit Standard Materials

The mechanical behavior of many materials can be modeled by a constitutive law deriving from a convex lower semi-continuous (lsc) **potential** Φ. A stress-like variable y is related to a strain-like variable x equivalently by one of the three following conditions:
(i) $y \in \partial\Phi(x)$ i.e. y belongs to the subdifferential of Φ at x ($\forall\xi, \Phi(\xi) \geq \Phi(x) + \langle \xi - x, y\rangle$)
(ii) $x \in \partial\Phi^\star(y)$ i.e. x belongs to the subdifferential of $\Phi^\star$ at y ($\forall\eta, \Phi^\star(\eta) \geq \Phi^\star(y) + \langle x, \eta - y\rangle$)
(iii) $\Phi(x) + \Phi^\star(y) = \langle x, y\rangle$.

The brackets enclosing x and y denote the duality product between x and y. Condition (iii) can be regarded as an extremal case of the **Fenchel-Young inequality** $\Phi(x) + \Phi^\star(y) \geq \langle x, y\rangle$ derived directly from the definition of the **Legendre-Fenchel-Moreau functional transformation** [10]

$$\Phi^\star(y) = \sup_x(\langle x, y\rangle - \Phi(x)).$$

Such materials are called **"Generalized Standard Materials"** [8]. However, there exist materials, clays for example [5], whose behavior cannot be modeled by a convex lsc potential. In this case, the constitutive law is called **non-associated**. Giving up the sum decomposition in (iii), Gery de Saxcé [5] succeeded in modeling the behavior of a new class of materials, the **"Implicit Standard Materials"**. These materials are characterized by a bipotential $b(x, y)$, as stated in the following section.

2. Bipotentials

A function $b(x, y)$ satisfying the conditions:
(i) $b(x, y)$ is convex and lsc in x (ii) $b(x, y)$ is convex and lsc in y (iii) $b(x, y) \geq \langle x, y\rangle$
is called **bipotential** [4]. When the constitutive law of a material can be expressed indifferently by any of the following three conditions:
(iv) $y \in \partial_x b(x, y)$ i.e. y belongs to the subdifferential of the function $\xi \mapsto b(\xi, y)$ at $\xi = x$
(v) $x \in \partial_y b(x, y)$ i.e. x belongs to the subdifferential of the function $\eta \mapsto b(x, \eta)$ at $\eta = y$
(vi) $b(x, y) = \langle x, y\rangle$
this law is said to admit the bipotential b, and the material is referred as "Standard Implicit".

The "Generalized Standard Materials" are special "Implicit Standard Materials" with **separable bipotentials** of the type $b(x, y) = \Phi(x) + \Phi^\star(y)$, for which condition (iii) is nothing else than the Fenchel-Young inequality.

3. Parallelism of two vectors

As a start point to exhibit the bipotential modeling the Coulomb dry friction ([5], [6]), let us consider the constitutive law enacting that two vectors x and y of an Hilbert space H have the same orientation. This constitutive law is not maximal monotone and therefore cannot be described by a convex lsc potential. Nevertheless, one can express this law by making equal the product of the norms with the duality product: $||x||\ ||y|| = \langle x, y\rangle$. We can remark that the function $b(x, y) = ||x||\ ||y||$

satisfies the conditions (i), (ii) and (iii) of Section 2, the last one being true thanks to the Cauchy-Schwarz-Buniakovsky inequality. The equivalence of the three conditions (iv), (v) and (vi) is due to the following property of the norm in a Hilbert space: the subdifferential of the norm at x is equal to the closed unit ball if $x = 0$ and is reduced to $\left\{ \frac{x}{||x||} \right\}$ if $x \neq 0$.

4. Representing a constitutive law by a function

For representing a **maximal monotone multifunction** $x \longmapsto y \in Tx \subset H$, S. Fitzpatrick ([3],[7]) introduced the global convex lsc function

$$F(x, y) = \langle x, y \rangle - \inf_{y' \in Tx'} \langle x' - x, y' - y \rangle.$$

Since T is maximal monotone, the above infimum $\inf_{y' \in Tx'} \langle x' - x, y' - y \rangle$ is non-positive and its equality to 0 holds if and only if $y \in Tx$. Therefore $F(x, y)$ is bounded from below by the duality product $\langle x, y \rangle$, and we recover the conditions of Section 2 for F to be a bipotential representing T.

Thus, in case of maximal monotonicity of the constitutive law, a bipotential can be constructed as a **Fitzpatrick function** ([1],[2],[9]). But, does Fitzpatrick's method work for non monotone constitutive laws?

In this lecture we will present two examples. The first one concerns the linear monotone explicit law $y = Ax$ with $S = \frac{A+A^T}{2}$ as a positive-definite linear mapping. The second one is devoted to the non monotone implicit law discussed in Section 3.

5. References

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