SIZE EFFECT OF AN ELLIPTIC INCLUSION IN ANTI-PLANE STRAIN COUPLE STRESS ELASTICITY

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1. Introduction

The application of higher order continuum theories, with size effect considerations, have recently been spread in the micro and nano-scale studies. One famous version of these theories, proposed by Mindlin[1], is the couple stress theory. This paper utilizes this theory to study the anti-plane problems of elliptic inclusions.

2. Solution of the governing equations

The governing field equation for the anti-plane problems of couple stress elasticity within a centrosymmetric isotropic material is given by

(1) $$\nabla^2 u_3 - \ell^2 \nabla^4 u_3 = 0,$$

where ℓ is the characteristic length and u_3 is the out of plane displacement, [2]. Concerned with the problems of elliptic cylindrical inclusions, the solution of Eq.(1) is sought in elliptic coordinates, (ξ, η) with $(x_1, x_2) = c_0(\cosh\xi\cos\eta, \sinh\xi\sin\eta)$, Fig.1. Where c_0 is a positive constant.

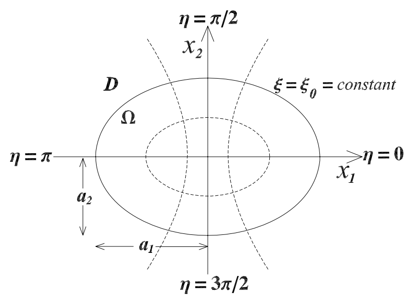

Figure 1. An elliptic domain within an infinite medium.

The general solution of this equation is taken as $\overline{u}_3 + \overline{\overline{u}}_3$, provided that

(2) $$\frac{\partial^2}{\partial\xi^2}\overline{u}_3 + \frac{\partial^2}{\partial\eta^2}\overline{u}_3 = 0, \quad \frac{\partial^2}{\partial\xi^2}\overline{\overline{u}}_3 + \frac{\partial^2}{\partial\eta^2}\overline{\overline{u}}_3 - \frac{c_0^2}{2\ell^2}(\cosh 2\xi - \cos 2\eta)\overline{\overline{u}}_3 = 0.$$

Consider an elliptic domain, Ω within an infinite medium, D, as shown in Fig.1. In the elliptic coordinate system the interface between Ω and D is described by $\xi = \xi_0$. The long and short semi-axes of Ω are denoted by a_1 and a_2, respectively. The general solution of Eq.(1), periodic in η, associated with the exterior and interior points of Ω are respectively given by $u_3^{(1)}$ and $u_3^{(2)}$ as:

(3) $$u_3^{(1)}(\xi,\eta) = \sum_{k=1}^{\infty}\left(\overline{a}_{k1}\cos k\eta + \overline{b}_{k1}\sin k\eta\right)(\cosh k\xi - \sinh k\xi) +$$
$$+ \overline{\overline{a}}_{01} ce_0(q,\eta) Ke_0(q,\xi) + \sum_{k=1}^{\infty}\overline{\overline{a}}_{k1} ce_k(q,\eta) Ke_k(q,\xi) + \overline{\overline{b}}_{k1} se_k(q,\eta) Ko_k(q,\xi),$$

and

$$(4)\quad u_3^{(2)}(\xi,\eta) = \sum_{k=1}^{\infty}\left(\overline{a}_{k2}\cos k\eta\cosh k\xi + \overline{b}_{k2}\sin k\eta\sinh k\xi\right) + \\ + \overline{\overline{a}}_{02}ce_0(q,\eta)Ie_0(q,\xi) + \sum_{k=1}^{\infty}\overline{\overline{a}}_{k2}ce_k(q,\eta)Ie_k(q,\xi) + \overline{\overline{b}}_{k2}se_k(q,\eta)Io_k(q,\xi),$$

where $q = -c_0^2/4\ell^2$. Here ce_k and se_k are the angular Mathieu functions and Ke_k, Ie_k, Ko_k and Io_k are the radial Bessel type Mathieu functions. Assume that the displacement field $u_3^* = 2e_{3i}^*x_i + e_{3ij}^*x_ix_j + ...$ is given inside Ω, where the summation is performed on $i,\ j = 1,\ 2$ and e_{3i}^*, e_{3ij}^*, ... stand for the eigenstrains [3]. The unknown coefficients in Eqs.(3-4) are determined through satisfaction of the following conditions on $\xi = \xi_0$,

$$(5)\quad \overline{T}_3^{(1)} = \overline{T}_3^{(2)},\quad \overline{M}_\eta^{(1)} = \overline{M}_\eta^{(2)},\quad u_3^{(1)} = u_3^{(2)} + u_3^*,\quad \sigma_{z\xi}^{(1)} = \sigma_{z\xi}^{(2)},$$

where $\overline{T}_3$ and $\overline{M}_\eta$ are the reduced traction components and σ_{ij} is the component of the stress tensor. The superscripts (1) and (2) over a field quantity implies that it is derived from the displacements, $u_3^{(1)}$ and $u_3^{(2)}$, respectively.

3. Numerical results and conclusion

Suppose Ω is an inclusion with uniform eigenstrain, $e_{13}^* = 1$. Ω and D are made of same material, and so they have the same shear modulus, μ and characteristic length, ℓ. To examine the size effect, various ratios for $\frac{\ell}{a_2}$ are considered. In a special case with $a_1 = a_2$, the results via the present formulation reduce to the results derived from the work of Lubarda [2]. For a case where $a_1 = 1.5a_2$, the shear stresses $\sigma_{\eta z}$ and $\sigma_{\xi z}$ just outside of the inclusion along the inclusion-matrix interface are shown in Fig.2. This figure verifies that the results of the present study approach the classical solutions as the inclusion dimensions grow. It is observed that $\sigma_{\xi z}$ attains its maximum at the end points of the long axis of the inclusion, while $\sigma_{\eta z}$ vanishes at these points.

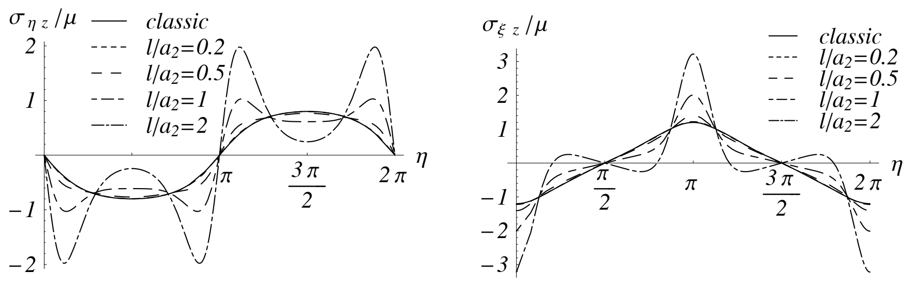

Figure 2. The stress distribution along the inclusion-matrix interface, approached from the matrix.

4. References

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