# MATRIX PADÉ BOUNDS ON EFFECTIVE TRANSPORT COEFFICIENTS OF ANISOTROPIC TWO-PHASE MEDIA

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The prediction of macroscopic coefficients $\mathbf{\Upsilon}$ of two-phase composites, if properties $\gamma_1$ and $\gamma_2$ and microstructures of their constituents are known, is one of the most important problems of mechanics of inhomogeneous media. Due to the difficulty of calculating of effective material constants $\mathbf{\Upsilon}$ exactly, there has been much of interest in obtaining bounds on $\mathbf{\Upsilon}$.

It is well known that effective transport coefficients $\mathbf{\Upsilon}$ of two-phase composites such as thermal and electrical conductivities, dielectric constants, magnetic permeabilities and diffusion coefficients have a matrix Stieltjes function representation $\mathbf{f}(z)$

$$(1)\qquad \mathbf{f}(z) = \frac{(\mathbf{\Upsilon} - \mathbf{I})}{z} = \int_0^1 \frac{d\gamma(u)}{1+zu}, \quad z \in \mathbb{C}\backslash(-\infty,-1), \quad d\gamma(u) \geq 0, \quad \mathbf{f}(-1) \leq \mathbf{I},$$

where $\mathbf{I}$ and $z = \frac{\gamma_1}{\gamma_2} - 1$ denote the unit matrix and the isotropic non-dimensional charakteristic of constituents. We assume that we know matrix coefficients $\mathbf{f}_j^{(k)}$ up to $p_j$ order in matrix Taylor expansions at $z_j$, $z = z_j \in \mathbb{C}\backslash(-\infty,-1), j = 1\ldots N$, i.e.

$$(2)\qquad \mathbf{f}_j^{(k)},\ j = 1,\ldots,N, k = 1,\ldots,p_j,$$

where

$$(3)\qquad \mathbf{f}(z_j) = \mathbf{f}_j^{(0)},\ \left.\frac{\partial \mathbf{f}(z_j)}{\partial z}\right|_{z=z_j} = \mathbf{f}_j^{(1)},\ \cdots,\ \left.\frac{\partial^{(p_j)} \mathbf{f}(z_j)}{\partial z^{(p_j)}}\right|_{z=z_j} = \mathbf{f}_j^{(p_j)}.$$

We seek the matrix function $\mathbf{F}_{n+2}(z;\alpha,\beta)$ estimating $\mathbf{f}(z)$ in the form a sum of simple matrix fractions given by:

$$(4)\qquad \mathbf{F}_{n+2}(z;\alpha,\beta) = \sum\nolimits_{k=1}^{K} \mathbf{A}_k^{\frac{1}{2}}(\alpha,\beta)(\mathbf{I} + z\mathbf{B}_k(\alpha,\beta))^{-1}\mathbf{A}_k^{\frac{1}{2}}(\alpha,\beta) + \alpha^{\frac{1}{2}}(\mathbf{I} + z\beta)^{-1}\alpha^{\frac{1}{2}},$$

where

$$(5)\qquad K = E((n+1)/2) \text{ and if } n \text{ even } \mathbf{B}_{n/2}(\alpha,\beta) > \mathbf{0} \text{ or if } n \text{ odd } \mathbf{B}_{n/2}(\alpha,\beta) = \mathbf{0}.$$

Here $\mathbf{A}_k(\alpha,\beta)$, $\alpha$, $\mathbf{B}_k(\alpha,\beta)$, $\beta$ are two-dimensional matrices satisfying matrices inequalities

$$(6)\qquad \mathbf{A}_k(\alpha,\beta) > \mathbf{0},\ \alpha > \mathbf{0},\ \mathbf{B}_k(\alpha,\beta) > \mathbf{0},\ \beta > \mathbf{0},$$

while $n$ denotes the number of independent input data given by (3). The coefficients $\mathbf{A}_k(\alpha,\beta)$ and $\mathbf{B}_k(\alpha,\beta)$ are determined by the assumption that $\mathbf{F}_{n+2}(z;\alpha,\beta)$ (matrix multipoint Padé approximant) and $\mathbf{f}(z)$ (matrix Stieltjes function) have matrix Taylor expansions coinciding up to $p_j$ order at $z_j$, $j = 1,\ldots,N$. The main results of this paper present the following matrix relations. By $\phi_{n+1}(z_0)$, $n = 1,2,...$, we denote the matrix bounds on $\mathbf{f}(z_0)$.

For $n = 0$

$$(7)\qquad \phi_1(z_0) = \left\{\alpha^{\frac{1}{2}}(\mathbf{I} + z_0\beta)^{-1}\alpha^{\frac{1}{2}};\ \alpha = (\mathbf{I} - \beta),\ \ \mathbf{0} \leq\ \beta \leq \mathbf{I}),\ (\mathbf{0} \leq \alpha \leq \mathbf{I},\ \beta = \mathbf{0})\right\}.$$

For $n = 1, 2, 3...$

(8) $$\phi_{n+1}(z_0) = \{\mathbf{F}_{n+2}(z_0, \alpha, \beta);\ \alpha = \alpha_n(\beta)\},$$

where

(9) $$\alpha_n(\beta) = \begin{cases} \alpha_{\mathbf{A}_n}(\beta), & \mathbf{0} \leq \beta \leq \beta_1^{(n)}, \\ \alpha_{\mathbf{F}_{n+2}}(\beta), & \beta_1^{(n)} \leq \beta \leq \beta_2^{(n)}, \end{cases} \quad \text{if } n \text{ is odd}$$

or

(10) $$\alpha_n(\beta) = \begin{cases} \alpha_{\mathbf{F}_{n+2}}(\beta), & \mathbf{0} \leq \beta \leq \beta_1^{(n)}, \\ \alpha_{\mathbf{A}_n}(\beta), & \beta_1^{(n)} \leq \beta \leq \beta_2^{(n)}, \end{cases} \quad \text{if } n \text{ is even.}$$

Here $\beta_1^{(n)},\ \beta_2^{(n)},\ ...,\ \beta_n^{(n)}$ are roots of the equation

(11) $$\alpha_{\mathbf{F}_{n+2}}(\beta) - \alpha_{A_n}(\beta) = \mathbf{0},\ n = 1, 2, 3, .\mathbf{0} = \beta_0^{(n)} < \beta_1^{(n)} < \beta_2^{(n)} < ... < \beta_n^{(n)} < \beta_{n+1}^{(n)} = \mathbf{I}.$$

The matrix functions appearing in (9) and (10)

$$\alpha = \alpha_{\mathbf{A}_n}(\beta) \text{ and } \alpha = \alpha_{\mathbf{F}_{n+2}}(\beta)$$

satisfy the matrix relations

(12) $$\mathbf{A}_n(\alpha, \beta) = \mathbf{0} \text{ and } \mathbf{F}_{n+2}(-1,\ \alpha, \beta) = \mathbf{I},$$

respectively. Coefficients $\mathbf{A}_n(\alpha, \beta)$ are determined by the system of equations

(13) $$\mathbf{f}(z) - \mathbf{F}_{n+2}(z, \alpha, \beta) = O((z - z_j)^{p_j}),\ j = 1, 2, ..., N,\ n = \sum\nolimits_{j=1}^{N} p_j.$$

The matrix Padé bounds $\phi_n(z_0)$,$n = 0, 1, 2, ...$determined by relations (7)-(13) are new. For the scalar case they coincide with the relevant ones reported in literature [1, 2]. Zero order bounds $\phi_1(z_0)$ on $\mathbf{f}(z_0)$ determined by (7) are calculated and depicted in Fig. 1.

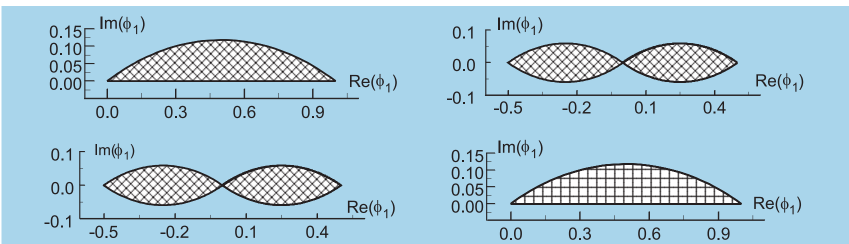


Fig. 1: Matric Padé bounds $\phi_1(z_0)$ on admissible values of a matrix Stieltjes function $\mathbf{f}(z_0)$, $z_0 = 1 - i$ representing the effective anisotropic transport coefficient $\boldsymbol{\Upsilon}$ of two-phase medium. The bounds $\phi_1(z_0)$ are calculated from one information only, i.e. $\mathbf{f}(-1) \leq \mathbf{I}$.

As an example of applications the effective conductivity of a rectangular array of cylinders is solved by means of matrix Padé bounds. Results are presented in a number of tables and graphs.

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