VISCOUS INCOMPRESSIBLE FLOW IN POROUS MEDIA

R. Wojnar
Institute of Fundamental Technological Research PAS, Warszawa

1. General

Our aim is to extend the Darcy law to a range of higher speeds of flow, and to derive a number of properties of such a flow. First, we note that the laminar flows occur for the large values of Reynolds number. Next, it is shown that viscosity scaling for small capillaries in a porous medium is not related to the Reynolds number, and the Darcy law, applicable not only to the stokesian seepage, is obtained using the Navier–Stokes equations for the steady case. Finally, a non-homogeneous porous medium, consisting of two different porous components is selected to show that for such a composite so called the Dykhne hypotheses are satisfied and a square root formula for the effective permeability is obtained.

2. The laminar flow

Consider steady flow in a pipe of arbitrary cross-section, the same along the whole length of the pipe. Let **v** denote the velocity, p – pressure, η - viscosity. Moreover, let t denote the time and **x** – the position. We take the axis of the pipe as the x_3 axis. The fluid velocity is along the x_3 axis, and is a function of x_1 and x_2 only. We have $\partial v_i/\partial t = 0$, v_1=v_2=0 and $v_3 \equiv v$. Hence, the left-hand side of the Navier-Stokes equation vanishes. If η is constant then $\partial p/\partial x_1 = \partial p/\partial x_2 = 0$ and

$$\frac{\partial p}{\partial x_3} = \eta\left(\frac{\partial^2 v}{\partial x_1{}^2} + \frac{\partial^2 v}{\partial x_2{}^2}\right) \qquad (*)$$

In general, fluid flow in a pipe crosses the threshold from laminar to turbulent flow when Reynolds number R reaches about 2000, $R=\rho u d/\eta$; ρ – the fluid density, u – the mean velocity over the pipe cross-section, and d – its mean diameter. For the water (ρ=1 g/cm^3, η =0.01 g/cm s) flowing in a pipe with the diameter d =1mm we reach such value of R with the mean velocity u=2m/s. Laminar flow has actually been observed even to Reynolds number $R \approx 50\,000$, what gives u=50m/s. The velocity of blood in aorta (in pulsatile regime) is of the order u=4 m/s.

3. Scaling in laminar flow

Let the cross-section of a pipe be an equilateral triangle of side a. We put $x = x_1, y = x_2$ and $z = x_3$. The solution of the equation $(*)$ is

$$v = -\frac{2}{\sqrt{3}}\frac{H}{a}\frac{1}{\eta}\frac{dp}{dz} \quad \text{where} \quad H = y\left[\left(\frac{\sqrt{3}}{2}a - y\right)^2 - 3x^2\right]. \quad \text{Hence} \quad Q = \frac{\sqrt{3}}{320}a^4\frac{1}{\eta}\frac{dp}{dz}$$

and Q denotes the discharge, it is the volume of fluid passing each second through the pipe.

Next, we divide each side of cross-section into two equal parts, introduce into the parallel rigid walls with infinitesimal thickness, and obtain four smaller pipes similar to the original one. After n such divisions Q_t=$Q/2^{2n}$ and Q_t vanishes as number of divisions n goes to infinity. To conserve the total discharge we should reduce the viscosity of fluid by factor ε^2, where $\varepsilon = 1/2^n$. In reality, instead of η it is the pressure gradient which scaled .

4. Homogenisation of stationary laminar flow in porous composite

Consider stationary laminar flow in a porous medium of dimension L with periodic structure (elementary cell with dimension l) and introduce the fraction $\varepsilon = l / L$. According to an asymptotic development metod we put for the pressure p^{ε} and velocity $\boldsymbol{v}^{\varepsilon}$ the expansions

$$p^{\varepsilon} = p^{(0)}(x,y) + \varepsilon p^{(1)}(x,y) + \varepsilon^2 p^{(2)}(x,y) + \ldots \qquad v^{\varepsilon} = v^{(0)}(x,y) + \varepsilon v^{(1)}(x,y) + \varepsilon^2 v^{(2)}(x,y) + \ldots,$$

where $y = x / \varepsilon$, substitute to the laminar flow equation (*), and compare terms at the same power of ε. Term with ε^{-1} provides $\partial p^{(0)} / \partial y_3 = 0$ what means $p^{(0)} = p^{(0)}(x)$. To satisfy equation with power ε^0, we put $p^{(0)} = \xi(y)\left(-\partial p^{(0)} / \partial x\right)$ and $v^{(0)} = \chi(y)\left(-\partial p^{(0)} / \partial x\right)$ where χ satisfies

$$\left(\frac{\partial^2 \chi}{\partial {y_1}^2} + \frac{\partial^2 \chi}{\partial {y_2}^2}\right) = -1 + \frac{\partial \xi}{\partial y_3}$$

After averaging the velocity over the elementary cell we get the Darcy law

$$\left\langle v^{(0)} \right\rangle = \left\langle \chi(y) \right\rangle \left(-\frac{\partial p^{(0)}}{\partial x_3} \right)$$

derived not from the Stokes but from the Navier–Stokes equation for the steady laminar flow.

5. Stationary flow in two-dimensional two-component porous composite

The solid part of the system contains two overlapping domains of distinctly different permeabilities, K_1 and K_2. In geology, the low permeability medium corresponds to block matrix with primary porosity, surrounded by fractures, and the high permeability continuum corresponds to rock fractures (secondary porosity). In biology we observe, for example, pores of different size in plant tissues or in animal bones, cortical and trabecular. If such systems are planar and the following Dykhne assumptions are satisfied: (*i*) considered fields are 2-dimensional, (*ii*) the flow is stationary and has the potential, (*iii*) statistical symmetry and isotropy of the composite is assured, then the square root formula for the effective property holds.

Define vector **f** as 2-dimensional gradient of pressure field, $f_{\alpha} = -\partial p / \partial x_{\alpha}$ where $\alpha = 1,2$. The Darcy law has the form $v_{\alpha} = K\, f_{\alpha}$ where K is a permeability. On the another hand, curl of **f** as of the potential vector, vanishes, it is $f_{2,1} - f_{1,2} = 0$ and the assumption that the flow is incompressible gives $v_{\alpha,\alpha} = 0$ or $f_{1,1} + f_{2,2} = 0$. Thus the conditions of Dykhne are satisfied and $K^{eff} = \sqrt{K_1 K_2}$. This is the formula for effective permeability if the domains with permeabilities K_1 and K_2 are statistically equivalent. It gives the effective values also in the case when the Hagen-Poiseuille flow and Darcy flow are mixed together.

6. Aknowledgements

This work was supported by the grant KBN No 4 T07A 003 27.

7. References

[1] L. Landau i E. Lifszic (1958). *Mechanika ośrodków ciągłych*, tłumaczył z ros. Roman Żelazny, PWN, Warszawa.
[2] É. Sanchez-Palencia (1980). *Non-homogeneous media and vibration theory*, Springer, Berlin.
[3] A. M. Dykhne (1970). Conductivity of a two–dimensional two–phase system, *Zhurnal Eksperimentalnoi i Teoreticheskoi Fiziki* **59**(7), 110-115.