COUPLED DYNAMICS THERMOVISCOELASTIC PROBLEM

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In the present study a closed solution of coupled dynamics thermoviscoelastic problem for finite body is obtained. The solution is of the form of spectral expansion to the biorthogonal eigenfunction system of non-self-adjoined differential pencil, generated by the initial–boundary value problem under consideration. The representation of spectral expansion is obtained by special non-symmetrical integral transformation [1,2].

Consider the coupled equations of viscoelastic motion and heat conduction in cylindrical coordinate system (r, φ, z):

(1) $$\begin{pmatrix} \mathcal{L}_1 & -\gamma\mathcal{L}_2 \\ 0 & \nabla^2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} \mathcal{L}_1' & 0 \\ -\eta\mathcal{L}_3 & -1/\kappa \end{pmatrix} \frac{\partial}{\partial t}\mathbf{y} + \begin{pmatrix} -\rho\mathcal{E} & 0 \\ 0 & 0 \end{pmatrix} \frac{\partial^2}{\partial t}\mathbf{y} = \mathbf{f},$$

wherein $\mathbf{f} = (-X_r, -X_\varphi, -X_z, -\omega)$ is prescribed vector-function, defined by volumetric force and heat sources intensity, $\mathcal{E}$ is identity operator, $\mathcal{L}_1, \dots \mathcal{L}_3$ are the following differential operators:

$$\mathcal{L}_1 = \begin{pmatrix} \mu(\nabla^2 - \frac{1}{r^2}) + (K + \frac{\mu}{3})\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) & \frac{K+\mu/3}{r}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right) - \frac{2\mu}{r^2}\frac{\partial}{\partial\varphi} & (K + \frac{\mu}{3})\frac{\partial^2}{\partial r\partial z} \\ \frac{2\mu}{r^2}\frac{\partial}{\partial\varphi} + \frac{K+\mu/3}{r}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) & \mu(\nabla^2 - \frac{1}{r^2}) + \frac{K+\mu/3}{r^2}\frac{\partial^2}{\partial\varphi^2} & \frac{K+\mu/3}{r}\frac{\partial^2}{\partial\varphi\partial z} \\ (K + \frac{\mu}{3})\frac{\partial}{\partial z}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) & \frac{K+\mu/3}{r}\frac{\partial^2}{\partial z\partial\varphi} & \mu\nabla^2 + (K + \frac{\mu}{3})\frac{\partial^2}{\partial z^2} \end{pmatrix},$$

$$\mathcal{L}_1' = \mu' \begin{pmatrix} \nabla^2 - \frac{1}{r^2} + \frac{1}{3}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) & \frac{1}{3r}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right) - \frac{2}{r^2}\frac{\partial}{\partial\varphi} & \frac{1}{3}\frac{\partial^2}{\partial r\partial z} \\ \frac{2}{r^2}\frac{\partial}{\partial\varphi} + \frac{1}{3r}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) & \nabla^2 - \frac{1}{r^2} + \frac{1}{3r^2}\frac{\partial^2}{\partial\varphi^2} & \frac{1}{3r}\frac{\partial^2}{\partial\varphi\partial z} \\ \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) & \frac{1}{3r}\frac{\partial^2}{\partial z\partial\varphi} & \nabla^2 + \frac{1}{3}\frac{\partial^2}{\partial z^2} \end{pmatrix},$$

$$\mathcal{L}_2 = \left(\frac{\partial}{\partial r} \quad \frac{1}{r}\frac{\partial}{\partial\varphi} \quad \frac{\partial}{\partial z}\right)^{\mathrm{T}}, \ \mathcal{L}_3 = \left(\frac{\partial}{\partial r} + \frac{1}{r} \quad \frac{1}{r}\frac{\partial}{\partial\varphi} \quad \frac{\partial}{\partial z}\right), \ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2},$$

K, μ are the elastic modulus; γ, η are the thermomechanical constants; κ is the thermal conductivity coefficient, ρ is the density, μ' is the viscosity modulus.

The boundary conditions $\mathcal{D}$ are arbitrary on lateral area and have some restrictions on end faces (to admit the separation of variables, see [3]):

(2) $$\mathcal{D} = \{\mathbf{y} | \mathbf{y} \in L_2^4, \mathcal{B}\mathbf{y} = 0, \mathbf{y} = O(1)\},$$

$$\mathcal{B}\mathbf{y} = \begin{pmatrix} \mathcal{B}_1\mathbf{y}\big|_{r=R} \\ \mathcal{B}_2\mathbf{y}\big|_{z=0} \\ \mathcal{B}_2\mathbf{y}\big|_{z=H} \\ [\mathbf{y}]_0^{2\pi} \end{pmatrix}, \quad \mathcal{B}_1 = \begin{pmatrix} \vartheta\frac{\partial}{\partial r} + \frac{\lambda}{r} & \frac{\lambda}{r}\frac{\partial}{\partial\varphi} & \lambda\frac{\partial}{\partial z} & -\gamma \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} & 0 \\ \frac{1}{r}\frac{\partial}{\partial\varphi} & \frac{\partial}{\partial r} - \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial r} \end{pmatrix}, \quad \mathcal{B}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{1}{r}\frac{\partial}{\partial\varphi} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial r} \end{pmatrix},$$

wherein $[\mathbf{y}]_0^{2\pi} = \mathbf{y}\big|_{\varphi=0} - \mathbf{y}\big|_{\varphi=2\pi}$, $\vartheta = 4\left(\mu + \mu'\partial/\partial t\right)/3 + K$, $\lambda = K - 2\left(\mu + \mu'\partial/\partial t\right)/3$). Initial values are defined by initial distributions of temperature, displacements and velocities.

The obtained solutions of problem (1), (2) are of the form of spectral expansions based on complete biorthogonal sets of eigenfunctions (and perforce associated functions), corresponding to the conjugate pairs of matrix operator pencils $\mathcal{L}_\nu$, $\mathcal{L}_{\overline{\nu}}^*$:

$$\mathcal{L}_\nu = \mathcal{A}_0 + \mathcal{A}_1\nu + \mathcal{A}_2\nu^2, \quad \mathcal{L}_\nu^* = \mathcal{A}_0^* + \mathcal{A}_1^*\overline{\nu} + \mathcal{A}_2\overline{\nu}^2,$$

$$\mathcal{A}_0^* = \begin{pmatrix} \mathcal{L}_1 & 0 \\ \gamma\mathcal{L}_3 & \nabla^2 \end{pmatrix}, \quad \mathcal{A}_1^* = \begin{pmatrix} \mathcal{L}_1' & \eta\mathcal{L}_2 \\ 0 & -1/\kappa \end{pmatrix}.$$

Here $\mathcal{A}_i^*$ are conjugate to $\mathcal{A}_i$ differential operators, defined in the domain $\mathcal{D}^*$, that defined by boundary operator $\mathcal{B}^*$:

$$\mathcal{B}^*\mathbf{y} = \begin{pmatrix} \mathcal{B}_1^*\mathbf{y}\big|_{r=R} \\ \mathcal{B}_2\mathbf{y}\big|_{z=0} \\ \mathcal{B}_2\mathbf{y}\big|_{z=H} \\ [\mathbf{y}]_0^{2\pi} \end{pmatrix}, \quad \mathcal{B}_1^* = \begin{pmatrix} \vartheta\frac{\partial}{\partial r} + \frac{\lambda}{r} & \frac{\lambda}{r}\frac{\partial}{\partial\varphi} & \lambda\frac{\partial}{\partial z} & \nu\eta \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} & 0 \\ \frac{1}{r}\frac{\partial}{\partial\varphi} & \frac{\partial}{\partial r} - \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial r} \end{pmatrix}.$$

The coefficients of expansions referred to as transforms one can obtain by applying direct integral transformation to (1), resulting in the reduction initial boundary value problem to the sequence of initial problems for ODEs in image space [1,2]. It enable us to represent the solution of (1), (2) as follows:

$$\mathbf{y} = \sum_{i=1}^{\infty} \Bigg[\big(\langle \mathcal{A}_1^*\mathbf{k}_i^* + \overline{\nu}_i\mathcal{A}_2^*\mathbf{k}_i^*, \mathbf{y}_0 \rangle + \langle \mathcal{A}_2^*\mathbf{k}_i^*, \dot{\mathbf{y}}_0 \rangle \big) \mathbf{exp}(\overline{\nu}_i t) + \\ + \int_0^t \langle \mathbf{f}(\tau), \mathbf{k}_i^* \rangle \, \mathbf{exp}\big(\overline{\nu}_i(t-\tau)\big)\, d\tau \Bigg] \mathbf{k}_i \mathcal{Q}_i^{-1}. \tag{3}$$

One can found $\mathbf{k}_i$, $\mathbf{k}_i^*$ by solving the coupled set of boundary eigenvalue problems:

$$\mathcal{L}_\nu\mathbf{k} = 0 \quad (\mathbf{k} \in \mathcal{D}), \quad \mathcal{L}_{\overline{\nu}}^*\mathbf{k}^* = 0 \quad (\mathbf{k}^* \in \mathcal{D}^*).$$

In equation (3) $\mathcal{Q}_\nu$ is the normalizing matrix and ν_i $(i = 1, \ldots, \infty)$ are the elements of pencil discrete spectrum. The constructible representation of normalizing matrix $\mathcal{Q}_\nu$ and the exact method for evaluation of corresponding quadratures are described in [4, 5]. Note, that biorthogonal relations [3] here are in the form

$$\left\langle \mathcal{A}_1\mathbf{k}_i, \mathbf{k}_j^* \right\rangle + (\nu_i + \nu_j) \left\langle \mathcal{A}_2\mathbf{k}_i, \mathbf{k}_j^* \right\rangle = 0, \quad \left\langle \mathcal{A}_0\mathbf{k}_i, \mathbf{k}_j^* \right\rangle - \nu_i\nu_j \left\langle \mathcal{A}_2\mathbf{k}_i, \mathbf{k}_j^* \right\rangle = 0.$$

It is important to note, that, unlike well-known transformation technique (Laplace transform, etc.), that uses numerical approach for inversion, proposed method admit to obtain solution in closed analytical form and to develop effective algorithmic realization of computer simulation. It usability for the analysis of the non-stationary, high frequency loadings on several particular examples is elucidated [1–4].

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