SAINT-VENANT'S PRINCIPLE IN MAGNETOELASTICITY

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Toupin's version of Saint-Venant's principle in linear elasticity is generalized to the case of linear magnetoelasticity. That is, it is shown that, for a straight prismatic bar made a linear magnetoelastic material end loaded by a self-equilibrated system at one end only, the internal energy stored in the portion of the bar which is beyond a distance s from the loaded end decreases exponentially with the distance s.

Mathematical versions of Saint-Venant's principle in linear elasticity due to Sternberg, Knowles, Zanaboni, Robinson and Toupin have been discussed by Gurtin [1] in his monograph. Later developments of the principle for Laplace's equation, isotropic, anisotropic, and composite plane elasticity, three-dimensional problems, nonlinear problems, and time-dependent problems are summarized in the review articles by Horgan and Knowles [2] and by Horgan [3]. In this paper we prove an analogue of Toupin's version of Saint-Venant's principle for linear magnetoelasticity. For a linear elastic homogeneous prismatic body of arbitrary length and cross-section loaded on one end only by an arbitrary system o self-equilibrated forces, Toupin [4] showed that the elastic energy U(s) stored in the part of the body which is beyond a distance s from the loaded end satisfies the inequality

(1) $$U(s) \leq U(0)\exp\left[-\frac{(s-l)}{s_c(l)}\right].$$

The characteristic decay length $s_c(l)$ depends upon the maximum and the minimum elastic moduli of the material and the smallest nonzero characteristic frequency of free vibration of a slice of the cylinder of length l. Inequalities similar to (1) have been obtained by Batra [5] for linear elastic piezoelectric prismatic bodies and by Borrelli & Patria [6] for a semi-infinite magnetoelastic cylinder on the asymptotic behaviour of the Dirichlet integral of the magnetic field and of the elastic energy.

Here we consider a linear theory of magnetoelasticity (for infinitesimal strain) in which only the ponderomotive force remains non-linear in presence of a magnetic field . We assume that the elastic body is homogeneous, isotropic and electrically conducting [7], [8], [9], [10].

Let the finite spatial region occupied by the magnetoelastic body be V, the boundary surface of V be S. the unit outward normal of S be n_i, and S be partitioned as

(2) $$S = S_u \cup S_T = S_E \cup S_B\,,\qquad 0 = S_u \cap S_T = S_E \cap S_B\,.$$

Physically, S_u, S_T are, respectively, parts of the boundary S on which mechanical displacements and tractions are prescribed. S_E is the part of S which is in contact with electrode, hence the tangential electric field vanishes on it, and S_B the parts of S on which the magnetic induction is prescribed. The governing equations and boundary conditions for static magnetoelasticity in rectangular Cartesian coordinates in SI units are:

$$\partial_i \breve{T}_{ij} = 0, \qquad \varepsilon_{ijk}\,\partial_j E_k = 0, \qquad \partial_k D_k = 0, \qquad in\;\; V,$$

$$\varepsilon_{ijk}\,\partial_j H_k = j_i\,, \quad \partial_k B_k = 0, \qquad in\;\; V,$$

$$j_k = \sigma E_k\,,\;\; D_k = \varepsilon E_k\,,\;\; B_k = \mu H_k\,, \qquad in\;\; V,$$

(3)
$$\breve{T}_{ij} = t_{ij} + T_{ij}\,,\quad t_{ij} = c_{ijkl}\,\varepsilon_{kl}\,,\quad T_{ij} = B_i H_j - \frac{1}{2}\delta_{ij} B_k H_k\,, \qquad in\;\; V,$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right), \qquad in\;\; V,$$

$$u_i = \tilde{u}_i \qquad on\;\; S_u\,, \qquad\qquad n_i \breve{T}_{ij} = \tilde{t}_j \quad on\;\; S_T\,,$$

$$\varepsilon_{ijk}\, n_j E_k = 0 \quad on\;\; S_E\,, \qquad\qquad n_i B_i = 0 \quad on\;\; S_B\,,$$

where u_i is the mechanical displacement, t_{ij} the mechanical stress tensor, T_{ij} the Maxwell stress tensor, ε_{ij} the strain tensor, E_k the electric field vector, D_k the electric displacement vector, H_k the magnetic field vector, j_k the current vector, B_k the magnetic induction vector, ε, μ, σ the electromagnetic material constants, c_{ijkl} the elastic moduli, ε_{ijk} the permutation tensor, δ_{ij} the unit tensor, ∂_k the spatial derivative, $\tilde{u}_i$ and $\tilde{t}_j$ are the prescribed boundary mechanical displacement and traction vectors.

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