FABRIC TENSOR AND STRENGTH SURFACE OF BONE-LIKE MATERIALS

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1. Abstract

The porous microstructure plays an important role in the damage resistance of bones, [1], [3]. The aim of the paper is to establish the strength criterion for the bone-like porous material, which takes into account the porous geometry explicitly, Fig.1. Firstly, we define a new fabric tensor based on the mathematical homogenization theory to separate geometrical effects from mechanical ones. Next, we construct the strength surface using the introduced fabric tensor.

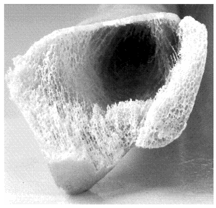

Fig.1. Anisotropic porous structure of a bone.

2. Fabric tensor based on the mathematical homogenization theory

The mathematical homogenization of periodic as well as stochastic media is based on the assumption that an inhomogeneous medium behaves as a homogeneous one, provided that macroscopic size *L*, is infinitely large as compared to the size *l* of its heterogeneities. The macroscopic behavior is described by effective (macroscopic) properties of a "homogenized" material, which are obtained in the case, when the small dimensionless parameter $\varepsilon = l/L$ tends to zero.

Let us assume that the local elastic properties described by the fourth order tensor, depend on the position $\mathbf{x}$ belonging to the space occupied by a material body. The material body is composed of two-phase inhomogeneous material in the following way: $\mathbf{C}(\mathbf{x}/\varepsilon) = \overset{(2)}{\mathbf{C}} + (\overset{(1)}{\mathbf{C}} - \overset{(2)}{\mathbf{C}})\chi(\mathbf{x}/\varepsilon)$. Here $\overset{(1)}{\mathbf{C}}$, $\overset{(2)}{\mathbf{C}}$ are elastic properties of components, and $\chi(\mathbf{y})$ denotes a characteristic function of the set occupied by the component denoted by the index (1). Effective properties of the composite are given by:

$$\mathbf{C}^{eff} = \overset{(2)}{\mathbf{C}} + (\overset{(1)}{\mathbf{C}} - \overset{(2)}{\mathbf{C}})\left\langle \chi \left(\mathbf{I}^{(4)} + \mathbf{\Gamma} * (\overset{(1)}{\mathbf{C}} - \overset{(2)}{\mathbf{C}})\,\chi \right)^{-1} \right\rangle,$$

where the components of 4th rank tensor $\Gamma_{ijkl}(y, y') = \partial^y_{(i}\partial^y_{j)} G_{(kl)}(y.y')$ compose the kernel of a integral operator. The operator can be defined by the Green function of the periodic boundary problem of elasticity for homogeneous material (2) if in a periodic structure of the composite was assumed. The brackets denote averaging over statistical ansamble or over the periodic cell. If $\overset{(1)}{\mathbf{C}} \to 0$, then the effective elastic properties of porous material with the skeleton described by $\overset{(2)}{\mathbf{C}}$ are given by the formula $\mathbf{C}^{eff} = \sqrt{\overset{(2)}{\mathbf{C}}} : \mathbf{T} : \sqrt{\overset{(2)}{\mathbf{C}}}$, where $\mathbf{T} = \mathbf{I}^{(4)} - \left\langle \chi \left(\mathbf{I}^{(4)} - \mathbf{A} * \chi \right)^{-1} \right\rangle$,

$\mathbf{A} = \sqrt[(2)]{\mathbf{C}} : \mathbf{\Gamma} : \sqrt[(2)]{\mathbf{C}}$. $\mathbf{I}^{(4)}$ - denotes unity in the space of the symmetric 4th rank tensors and two dots denote double contraction of the tensors. The tensor $\mathbf{T}$ is called the *fabric tensor* and it can be rewritten in the following form: $\mathbf{T} = \mathbf{I}^{(4)} - \int_0^1 \frac{d\boldsymbol{\mu}(x)}{1-x}$. The measure $d\boldsymbol{\mu}(x)$, namely their moments, describe geometry of the porous structure. In what follows, the index (2) is omitted. Let us define two tensors, $\mathbf{T}_{\mathrm{L}} = \sqrt{\mathbf{C}} : \mathbf{T} : \left(\sqrt{\mathbf{C}}\right)^{-1}$ and $\mathbf{T}_R = \left(\sqrt{\mathbf{C}}\right)^{-1} : \mathbf{T} : \sqrt{\mathbf{C}}$, which are called the left and right damage tensors, respectively. The names are justified by the relations $\boldsymbol{\sigma}^{\mathrm{eff}} = \mathbf{T}_{\mathrm{L}} : \boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}^{eff} = (\mathbf{T}_R)^{-1} : \boldsymbol{\varepsilon}$, where the first one is a stress relation between damaged (material with pores) and virgin material. It is assumed that strains are the same in damaged and virgin material and the second one is a strain relation between damaged and virgin material. Moreover, it is assumed that stresses are the same in both materials. The following relations also holds: $\mathbf{C}^{eff} = \mathbf{T}_L : \mathbf{C}$ and $\mathbf{T}_R = \mathbf{C}^{-1} : \mathbf{C}^{eff}$.

3. Strength surface

A homogenized failure criterion for an arbitrary two-phase elastic composite is formulated in [2]. The criterion incorporates an elegant first approximation to the microscopic stress fluctuation due to the interaction between the homogenized stress and the microstructure. To formulate homogenized criteria for trabecular bone, we assume that such criterion is known for the skeleton material i.e. compact bone, which is a component of composite. The criterion has the form of inequality for microstresses in the skeleton material: $\boldsymbol{\sigma}^{mikro} : \mathbf{\Pi} : \boldsymbol{\sigma}^{mikro} \leq 1$, where $\mathbf{\Pi}$ is given as 4^{th} rank positively definite tensor. Now, the strength criterion of the trabecular bone, expressed by the 4^{th} rank positively definite tensor, is given by the formulae

$$\mathbf{\Pi}^{eff} = (\mathbf{C}^{eff})^{-1} : \mathbf{C} : \mathbf{\Pi} : \mathbf{C} : [\nabla_{\mathbf{C}} \mathbf{C}^{eff}] : (\mathbf{C}^{eff})^{-1},$$

where $[\nabla_{\mathbf{C}} \mathbf{C}^{eff}]$ denotes so-called *phase gradient* of the effective tensor with respect to properties of compact bone material. It is the 8^{th} rank tensor. The tensors $\mathbf{C}^{eff}$, $(\mathbf{C}^{eff})^{-1}$ denote stiffness and compliance effective tensors of the trabecular bone, respectively. The phase gradient is obtained from the solutions of a so-called local problem. In the case of periodic structure it is called “problem on the periodic cell”. The criterion is applied to macroscopic stresses in the trabecular bone. The strength criterion is given by the inequality $\boldsymbol{\sigma}^{makro} : \mathbf{\Pi}^{eff} : \boldsymbol{\sigma}^{makro} \leq 1$, where the equality defines the strength surface. Using the dependence on fabric tensor of effective elastic tensor introduced above, the influence of geometry on the strength criterion is analyzed with various assumptions concerning microstructure of bones.

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