SOLUTION OF THE CATTANEO-VERNOTTE BIO-HEAT TRANSFER EQUATION BY MEANS OF THE DUAL RECIPROCITY METHOD

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1. Governing equations

According to the newest opinions the heat conduction proceeding in the biological tissue domain should be described by the hyperbolic equation (Cattaneo and Vernotte equation [1]) in order to take into account its nonhomogeneous inner structure. So, the following bio-heat transfer equation is considered

$$c\left(\tau\frac{\partial^2 T(x,\ t)}{\partial t^2}+\frac{\partial T(x,\ t)}{\partial t}\right)=\lambda\nabla^2 T(x,\ t)+Q(x,\ t)+\tau\frac{\partial Q(x,\ t)}{\partial t}$$

where c, λ denote the volumetric specific heat and thermal conductivity of tissue, $Q(x, t)$ is the capacity of internal heat sources due to metabolism and blood perfusion, τ is the relaxation time (for biological tissue it is a value from the scope 20-35 s), T is the tissue temperature, x, t denote the spatial co-ordinates and time. The function $Q(x, t)$ is equal to

$$Q(x,\ t)=G_B c_B\left[T_B-T(x,\ t)\right]+Q_m$$

where G_B is the blood perfusion rate, c_B is the volumetric specific heat of blood, T_B is the artery temperature and Q_m is the metabolic heat source. It should be pointed out that for $\tau = 0$ the equation reduces to the well-known Pennes bio-heat equation.

The equation is supplemented by the boundary conditions

$$x\in\Gamma_1:\quad T(x,\ t)=T_b(x)$$
$$x\in\Gamma_2:\quad q(x,\ t+\tau)=-\lambda\,\mathbf{n}\cdot\nabla T(x,\ t)=q_b(x)$$

and initial ones

$$t=0:\quad T(x,\ t)=T_0,\quad \left.\frac{\partial T(x,\ t)}{\partial t}\right|_{t=0}=0$$

where Γ_1, Γ_2 are the surfaces limiting the domain, $q(x, t+\tau)$ is the boundary heat flux, $T_b(x)$, $q_b(x)$ are the known boundary temperature and the boundary heat flux and T_0 is the known initial temperature of the biological tissue.

2. Dual reciprocity boundary element method

For transition $t^{f-1}\rightarrow t^f$ the standard boundary element method leads to the integral equation [2]

$$B(\xi)T\left(\xi,t^f\right)+\int_\Gamma T^*(\xi,x)\,q\left(x,t^f\right)\mathrm{d}\Gamma=\int_\Gamma q^*(\xi,x)\,T\left(x,t^f\right)\mathrm{d}\Gamma-$$
$$\int_\Omega\left[\left(c+\tau G_B c_B\right)\frac{\partial T(x,t)}{\partial t}+c\,\tau\frac{\partial^2 T(x,t)}{\partial t^2}-G_B c_B\left[T_B-T(x,t)\right]-Q_m\right]_{t=t^f}T^*(\xi,x)\,\mathrm{d}\Omega$$

where ξ is the observation point, $B(\xi)\in(0,1)$, $T^*(\xi, x)$ is the fundamental solution, $q(x, t^f) = -\lambda\,\partial T(x, t^f)/\partial n$ is the heat flux, $q^*(\xi, x) = -\lambda\,\partial T^*(\xi, x)/\partial n$.

In the dual reciprocity method the following approximation is proposed [2]

$$\left[\left(c+\tau G_B c_B\right)\frac{\partial T(x,t)}{\partial t}+c\,\tau\frac{\partial^2 T(x,t)}{\partial t^2}-G_B c_B\left[T_B-T(x,t)\right]-Q_m\right]_{t=t^f}=\sum_{k=1}^{N+L}\lambda\, a_k\left(t^f\right)\nabla^2 U_k(x)$$

where $a_k(t^f)$ are unknown coefficients, $P_k(x)$ are approximating functions fulfilling the equations

$$P_k(x)=\lambda\nabla^2 U_k(x)$$

and $N + L$ corresponds to the total number of nodes, where N is the number of boundary nodes while L is the number of internal nodes. After the mathematical manipulations one obtains

$$B(\xi)T\left(\xi,t^f\right)+\int_\Gamma T^*(\xi,x)\,q\left(x,t^f\right)\mathrm{d}\Gamma=\int_\Gamma q^*(\xi,x)\,T\left(x,t^f\right)\mathrm{d}\Gamma+$$

$$\sum_{k=1}^{N+L}a_k\left(t^f\right)\left[B(\xi)U_k(\xi)+\int_\Gamma T^*(\xi,x)W_k(x)\,\mathrm{d}\Gamma-\int_\Gamma q^*(\xi,x)U_k(x)\,\mathrm{d}\Gamma\right]$$

where $W_k(x) = -\lambda\ \mathbf{n}\cdot\nabla U_k(x)$. This equation is solved in numerical way.

3. Example of computations

The biological tissue domain of dimensions 0.01 m × 0.01 m (L = 0.01 [m]) has been considered. The initial temperature of tissue equals T_0 = 37 °C. On the boundary $x_1 = 0$, $0 \le x_2 \le L$ the Dirichlet condition in the form $T_b(x_2) = 37 + (50 - T_0)\,x_2/L$ has been assumed, on the remaining part of the boundary the temperature T_b = 37 °C can be accepted. The input data have been taken from [1]. The boundary has been divided into N = 40 constant boundary elements, at the interior L = 100 internal nodes have been distinguished. Time step: Δt = 10 s.

In the Figures 1 and 2 the heating curves at three points (0.0035, 0.0035), (0.0055, 0.0055), (0.0075, 0.0075) from tissue domain for τ = 0 s (Pennes equation) and τ = 20 s (Cattaneo-Vernotte equation) are shown. The differences between the temperatures for these two models are visible.

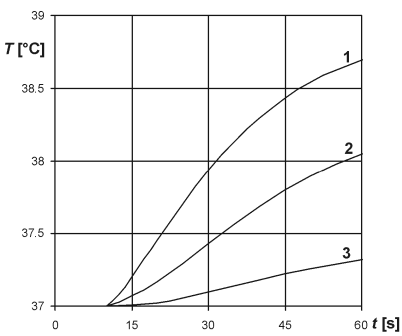

Fig. 1. Heating curves for τ = 0 s

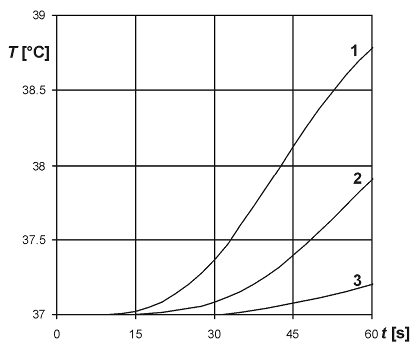

Fig. 2. Heating curves for τ = 20 s

4. References

[1] J. Liu and L.X. Xu (2000). Boundary information based diagnostics on the thermal states of biological bodies, *Journal of Heat and Mass Transfer*, 43, 2827–2839.

[2] P.W. Partridge, C.A. Brebbia, L.C. Wróbel (1992). *The dual reciprocity boundary element method*, CMP, London, New York.