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On the Effective Reflection Properties of the Randomly Segmented Elastic Bar.

Abstract.

In the paper transition of the wave pulses through bars with random properties is investigated. It is assumed that the bar is built of several homogeneous segments. It is permitted that the lengths of the segments as well as their material parameters are random variables. The overall properties of such bars are studied. The discussion of obtained results is illustrated with numerical calculations.

1. Introduction.

Engineering structures performing their functions undergo external or internal excitations which generate vibrations or waves in their elements. Depending on the desired behavior of the structures such phenomena are utilized for its proper work or must be eliminated to avoid its damage. The problem of special interest is propagation of wave pulses through elements of structures. Such a phenomenon takes place in a number of working machines or mechanical tools like rammers and drilling rods, where the wave pulse is applied for machine utilities. In some other situations (dumpers, absorbers, fixing elements) the wave pulse plays a destructive role. In both cases it is reasonable to design the elements of structure in the optimal way, depending on its functioning.

As it is known, the wave pulse is an (elastic) disturbance of the medium, travelling in space, of limited duration at time and transporting energy. The transmission effect of the wave pulse depends not only of the properties of the element of structure but also of the duration and shape of the pulse itself. In the design procedure these two factors must be taken into account.

In this paper we analyse the particular problem of the longitudinally polarized wave pulse propagation in bars - the elements of structure mathematically modeled by the spatially one-dimensional partial differential equations. In the model we neglect internal and external dumping (friction) as well as thermal effects. In spite of the fact that such models extremely simplify the reality, they (mostly in deterministic case) have been widely analysed in literature, both analytically and numerically, giving sufficient theoretical support for experiment design (cf. e.g. [1], [2]). The purpose of our paper is to study such models of the elements where their properties are regarded as random.

Starting from the randomized model of the bar we try to obtain its overall properties as the element transporting wave pulses and, what it follows, the energy. We use both the analytical and numerical tools in our investigation.

To describe the model of the bar let us assume that it occupies the part of real number axis x starting from $x=0$ to $x=d$.

The environment is regarded as a couple of semi-infinite bars expanding from minus infinity to $x=0$ and from $x=d$ to infinity. Moreover, we assume that the investigated bar has some internal structure; it consists of several (e.g. N) segments of the lengths h_j, $\sum_{j=1}^{N} h_j = d$, being themselves homogeneous bars. In our model it is permitted that both the length of the segment as well as its material parameters can be random variables. In such a case also the length of the bar d is a random variable.

The problem of the propagation of wave pulses is posed in such a structure. It is assumed that the longitudinally polarized pulse $f(x,t)$ is coming from the left environment and reaching the front end of the bar $x=0$ at the instant of time $t=0$. Then the pulse partially reflects from the interface and partially transmits to the first segment of the bar. Going on, the transmitted pulse reflects and transmits at all interfaces of the segments; moreover, we have also reverberation of all the reflected waves on the panels already passed by the wave front. This multiply reflections and transmission process makes that the global picture is very complicated. Trying to simplify this situation we replace the originally continuous model with some discrete-continuous one, what is the assumption which do not disturb the global description of the pulse but neglects some local physical effects in the near surrounding of the interfaces of the segments.

The mathematical analysis of the wave pulse transmission is significantly simplified when one goes in the description of the problem from the space-time domain to the space-spectrum one, dealing with the Fourier transform of the wave field with respect to temporal variable t. Then the exciting wave pulse is $f(x,\omega)$ and the governing equations are the ordinary differential ones. Such an approach was for example effectively utilized in papers of Lundberg and coauthors (e.g. [1], [2]); it is also efficient in multi-dimensional problem, e.g. wave propagation in layered media (e.g. Kennett [3]).

The following section is devoted to the formulation of the problem of the wave pulse propagation through the segmented bar with the use of the spectral method. Then we consider the periodic case where the bar is built of a series of the identical couples of segments and the stochastic model. In both cases we obtain the overall reflection properties of the bar, in stochastic case using the law of large numbers for the product of random matrices. In the last section we illustrate the analytical results with the numerical example.

2. Wave pulses in the segmented elastic bar.

Wave propagating along the length of an elastic bar of a constant cross-section is described by the system of two differential equations:

$$\frac{\partial \sigma}{\partial x} = A\, \rho\, \frac{\partial v}{\partial t}$$

$$\frac{\partial v}{\partial x} = \frac{1}{A\eta} \frac{\partial \sigma}{\partial t} \tag{2.1}$$

where

σ denotes the stress,

v is the particle velocity in the medium

and

A is the area of the perpendicular cross-section of the bar,

ρ is the density of the material,

η is the Young modulus,

and x, t are respectively, the spatial variable along the length of the bar and time.

Introducing the matrix notation we can rewrite equation (2.1) in the following form:

$$\frac{\partial}{\partial x}\begin{bmatrix}\sigma\\ v\end{bmatrix} = \begin{bmatrix}0 & A\rho\\ \frac{1}{A\eta} & 0\end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix}\sigma\\ v\end{bmatrix}. \tag{2.2}$$

After substitution of the wave velocity in the equation (2.2):

$$c = \sqrt{\frac{\eta}{\rho}} \tag{2.3}$$

we transform it to the following form:

$$\frac{\partial}{\partial x}\begin{bmatrix}\sigma\\ v\end{bmatrix} = \begin{bmatrix}0 & A\sqrt{\rho\eta}\\ \frac{1}{A\sqrt{\rho\eta}} & 0\end{bmatrix} c\,\frac{\partial}{\partial t}\begin{bmatrix}\sigma\\ v\end{bmatrix}. \tag{2.4}$$

Let us introduce, instead of the the spatial variable x, new independent variable being the wave travel time from 0 to x, defined as:

$$\xi = \int_0^x \frac{1}{c}\,dx' = \frac{x}{c} \tag{2.5}$$

If we moreover define the impedance Z as:

$$Z = A\sqrt{\rho\eta} \tag{2.6}$$

then the wave equation (2.4) takes the form:

$$\frac{\partial}{\partial \xi}\begin{bmatrix}\sigma\\ \nu\end{bmatrix} = \begin{bmatrix}0 & Z\\ \frac{1}{Z} & 0\end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix}\sigma\\ \nu\end{bmatrix}. \tag{2.7}$$

Introducing in eq. (2.7) the vectorial notation we have

$$\frac{\partial}{\partial \xi}\,\mathbf{s} = \mathbf{Q}\,\frac{\partial}{\partial t}\,\mathbf{s} \tag{2.8}$$

where

$$\mathbf{Q} = \begin{bmatrix}0 & Z\\ \frac{1}{Z} & 0\end{bmatrix}, \qquad \mathbf{s} = \begin{bmatrix}\sigma\\ \nu\end{bmatrix}. \tag{2.9}$$

Finally we apply in eq. (2.8) the Fourier transform with respect to time t:

$$\hat{\mathbf{s}}(\xi,\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\,\mathbf{s}(\xi,t)\,dt, \qquad \mathbf{s}(\xi,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,\hat{\mathbf{s}}(\xi,\omega)\,d\omega, \tag{2.10}$$

and reach the following ordinary differential equation for the transformed wave fields:

$$\frac{d}{d\xi}\,\hat{\mathbf{s}} = i\omega\,\mathbf{Q}\,\hat{\mathbf{s}} \tag{2.11}$$

with some initial condition $\hat{\mathbf{s}}(0,\omega)$.
Its solution in an uniform bar can be represented in the following form:

$$\hat{\mathbf{s}}(\xi,\omega) = \mathbf{P}(\xi,\omega)\,\hat{\mathbf{s}}(0,\omega) \tag{2.12}$$

where $\mathbf{P}(\xi,\omega)$ is the solution of the following matrix differential equation:

$$\frac{d}{d\xi}\,\mathbf{P} = i\omega\,\mathbf{Q}\,\mathbf{P}, \qquad \mathbf{P}(0,\omega) = \mathbf{Id}. \tag{2.13}$$

Such a solution has the following form:

$$\mathbf{P} = e^{i\omega Q\xi}, \tag{2.14}$$

or explicitly:

$$\mathbf{P} = \frac{1}{2}\begin{bmatrix} e^{i\omega\xi} + e^{-i\omega\xi} & Z\left(-e^{i\omega\xi} + e^{-i\omega\xi}\right) \\ \frac{1}{Z}\left(-e^{i\omega\xi} + e^{-i\omega\xi}\right) & e^{i\omega\xi} + e^{-i\omega\xi} \end{bmatrix}. \tag{2.15}$$

Expressing the exponent in above formula in terms of sine and cosine functions we obtain:

$$\mathbf{P} = \frac{1}{2}\begin{bmatrix} \cos\omega\xi & -iZ\sin\omega\xi \\ -\frac{i}{Z}\sin\omega\xi & \cos\omega\xi \end{bmatrix}. \tag{2.16}$$

This idealized situation complicates when the travelling wave pulse reaches a surface of discontinuity where some material parameters of the bar change. Then the wave is partially reflected and partially transmitted and at two sides of the interface of two homogeneous panels of the bar the amplitude of the wave and its phase jump. However, the values of these parameters at both sides of the interface point ξ are connected by the continuity condition of the stress and velocity field:

$$\hat{\mathbf{s}}(\xi^-,\omega) = \hat{\mathbf{s}}(\xi^+,\omega) \tag{2.17}$$

Restricting our interest to the stress only we can eliminate the velocity field $\hat{v}$ for the equation (2.11). The resulting equation for $\hat{\sigma}$ is:

$$\frac{d^2}{d\xi^2}\hat{\sigma} + \omega^2\hat{\sigma} = 0 \tag{2.18}$$

and it has the solution

$$\hat{\sigma}(\xi,\omega) = \hat{\sigma}_I(\omega)\, e^{-i\omega\xi} + \hat{\sigma}_R(\omega)\, e^{i\omega\xi} \tag{2.19}$$

where

$\hat{\sigma}_I(\omega)$ is the amplitude of the right-going (incident) wave,

$\hat{\sigma}_R(\omega)$ is the amplitude of the left-going (reflected) wave.

Then, since

$$\frac{d}{d\xi}\hat{v}(\xi,\omega) = i\omega\,\frac{1}{Z}\,\hat{\sigma}(\xi,\omega) \tag{2.20}$$

we find the velocity field $\hat{v}(\xi,\omega)$:

$$\hat{v}(\xi,\omega) = \frac{1}{Z}\left(- \hat{\sigma}_I(\omega)\, e^{-i\omega\xi} + \hat{\sigma}_R(\omega)\, e^{i\omega\xi} \right). \tag{2.21}$$

The continuity condition (2.18) at point ξ for two media indexed with numbers 1 and 2 respectively, is:

$$\begin{gathered} \hat{\sigma}^1_I(\omega)\, e^{-i\omega\xi} + \hat{\sigma}^1_R(\omega)\, e^{i\omega\xi} = \hat{\sigma}^2_I(\omega)\, e^{-i\omega\xi} + \hat{\sigma}^2_R(\omega)\, e^{i\omega\xi} \\ \frac{1}{Z_1}\left(- \hat{\sigma}^1_I(\omega)\, e^{-i\omega\xi} + \hat{\sigma}^1_R(\omega)\, e^{i\omega\xi} \right) = \frac{1}{Z_2}\left(- \hat{\sigma}^2_I(\omega)\, e^{-i\omega\xi} + \hat{\sigma}^2_R(\omega)\, e^{i\omega\xi} \right) \end{gathered} \tag{2.22}$$

Assume now that the bar is built of N homogeneous panels; in j-th panel the impedance is Z_j, the wave traveling time through it is h_j. The beginning of the bar is located at the point 0; the following points of interface of the panels are in the travel time domain $\xi_j = \sum_{k=1}^{j} h_k$. Assume also that the wave pulse $\hat{\sigma}^0_I\, e^{-i\omega\xi}$ comes from the surrounding media (with the impedance indexed by 0) to the front end of the bar. Then it generates the reflected pulse in media 0 and transmitted wave in panel 1. Going on, due to the sequence of interfaces of the panels (discontinuity points) we have the right and the left-going waves of the following form:

$$\Phi_I(\omega,\xi) = \hat{\sigma}^j_I\, e^{-i\omega\xi}, \qquad \Phi_R(\omega,\xi) = \hat{\sigma}^j_R\, e^{i\omega\xi}, \tag{2.23}$$

at the panel with the travel time h_j, that is for $\xi \in (\xi_{j-1}, \xi_j)$. In the surrounding media behind the bar there is a right-going (transmitted wave) of the form:

$$\Phi_R(\omega,\xi) = \hat{\vartheta}_R^{N+1} e^{i\omega\xi}, \tag{2.24}$$

This means that the continuity conditions at the interfaces of the strata written down in the matrix form, for the particular point ξ_{j-1} are:

$$\begin{bmatrix} 1 & 1 \\ -\dfrac{1}{Z_{j-1}} & \dfrac{1}{Z_{j-1}} \end{bmatrix} \begin{bmatrix} \hat{\vartheta}_I^{j-1} e^{-i\omega\xi_{j-1}} \\ \hat{\vartheta}_R^{j-1} e^{i\omega\xi_{j-1}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\dfrac{1}{Z_j} & \dfrac{1}{Z_j} \end{bmatrix} \begin{bmatrix} \hat{\vartheta}_I^{j} e^{-i\omega\xi_{j-1}} \\ \hat{\vartheta}_R^{j} e^{i\omega\xi_{j-1}} \end{bmatrix} \tag{2.25}$$

Let us introduce new variables:

$$\hat{f}_I^j = \hat{\vartheta}_I^j e^{-i\omega\xi_{j-1}}, \qquad \hat{f}_R^j = \hat{\vartheta}_R^j e^{i\omega\xi_{j-1}}. \tag{2.26}$$

Since $\xi_{j-1} = \xi_j - h_j$ (or $\xi_j = \xi_{j-1} + h_j$) for $j=1,2,....,N$; $\xi_0 = 0$, $h_0 = 0$,
we obtain :

$$\xi_{j-2} = \xi_{j-1} - h_{j-1}, \qquad \xi_{j-1} = \xi_{j-2} + h_{j-1} \tag{2.27}$$

and

$$\hat{\vartheta}_I^{j-1} e^{-i\omega\xi_{j-1}} = \hat{\vartheta}_I^{j-1} e^{-i\omega\xi_{j-2}} e^{-i\omega h_{j-1}} = \hat{f}_I^{j-1} e^{-i\omega h_{j-1}} \tag{2.28}$$

$$\hat{\vartheta}_R^{j-1} e^{i\omega\xi_{j-1}} = \hat{\vartheta}_R^{j-1} e^{i\omega\xi_{j-2}} e^{i\omega h_{j-1}} = \hat{f}_R^{j-1} e^{i\omega h_{j-1}} \tag{2.29}$$

and the continuity equation takes the form:

$$\begin{bmatrix} 1 & 1 \\ -\dfrac{1}{Z_{j-1}} & \dfrac{1}{Z_{j-1}} \end{bmatrix} \begin{bmatrix} \hat{f}_I^{j-1} e^{-i\omega h_{j-1}} \\ \hat{f}_R^{j-1} e^{i\omega h_{j-1}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\dfrac{1}{Z_j} & \dfrac{1}{Z_j} \end{bmatrix} \begin{bmatrix} \hat{f}_I^{j} \\ \hat{f}_R^{j} \end{bmatrix} \tag{2.30}$$

or

$$\begin{bmatrix} 1 & 1 \\ -\dfrac{1}{Z_{j-1}} & \dfrac{1}{Z_{j-1}} \end{bmatrix} \begin{bmatrix} e^{-i\omega h_{j-1}} & 0 \\ 0 & e^{i\omega h_{j-1}} \end{bmatrix} \begin{bmatrix} \hat{f}_I^{j-1} \\ \hat{f}_R^{j-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\dfrac{1}{Z_j} & \dfrac{1}{Z_j} \end{bmatrix} \begin{bmatrix} \hat{f}_I^{j} \\ \hat{f}_R^{j} \end{bmatrix}. \tag{2.31}$$

Solving this equation with respect to (j-1)-th variable we obtain:

$$\begin{bmatrix} \hat{f}_I^{j-1} \\ \hat{f}_R^{j-1} \end{bmatrix} = \begin{bmatrix} e^{i\omega h_{j-1}} & 0 \\ 0 & e^{-i\omega h_{j-1}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -\dfrac{1}{Z_{j-1}} & \dfrac{1}{Z_{j-1}} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ -\dfrac{1}{Z_j} & \dfrac{1}{Z_j} \end{bmatrix} \begin{bmatrix} \hat{f}_I^{j} \\ \hat{f}_R^{j} \end{bmatrix}. \tag{2.32}$$

Let us denote:

$$\mathbf{A}_j = \begin{bmatrix} 1 & 1 \\ -\dfrac{1}{Z_j} & \dfrac{1}{Z_j} \end{bmatrix}, \qquad \mathbf{E}_j = \begin{bmatrix} e^{i\omega h_{j-1}} & 0 \\ 0 & e^{-i\omega h_{j-1}} \end{bmatrix}, \qquad \hat{\mathbf{F}}^j = \begin{bmatrix} \hat{f}_I^{j} \\ \hat{f}_R^{j} \end{bmatrix}. \tag{2.33}$$

Then the transition equation takes the following simple form:

$$\hat{\mathbf{F}}^{j-1} = \mathbf{E}_{j-1}\, \mathbf{A}_{j-1}^{-1}\, \mathbf{A}_j\, \hat{\mathbf{F}}^j. \tag{2.34}$$

Considering the bar consisting of N elements we have ($\hat{\mathbf{F}}^0$ corresponds to the left environment, $\hat{\mathbf{F}}^{N+1}$ to the right environment):

$$\hat{\mathbf{F}}^0 = \mathbf{E}_0\, \mathbf{A}_0^{-1}\, \mathbf{A}_1\, \mathbf{E}_1\, \mathbf{A}_1^{-1} \cdots \mathbf{A}_N\, \mathbf{E}_N\, \mathbf{A}_N^{-1}\, \mathbf{A}_{N+1}\, \hat{\mathbf{F}}^{N+1}. \tag{2.35}$$

Since $\mathbf{E}_0 \equiv \mathbf{Id}$, the equation can be written in the following form:

$$\hat{\mathbf{F}}^0 = \mathbf{A}_0^{-1} \prod_{j=1}^{N} \mathbf{A}_j\, \mathbf{E}_j\, \mathbf{A}_j^{-1} \cdot \mathbf{A}_{N+1}\, \hat{\mathbf{F}}^{N+1}. \tag{2.36}$$

We can write the transition matrix $\mathbf{A}_j\, \mathbf{E}_j\, \mathbf{A}_j^{-1}$ explicitly. Since:

$$\mathbf{A}_j^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{Z_j}{2} \\ \frac{1}{2} & \frac{Z_j}{2} \end{bmatrix} \tag{2.37}$$

we obtain:

$$\mathbf{A}_j\ \mathbf{E}_j\ \mathbf{A}_j^{-1} = \begin{bmatrix} 1 & 1 \\ -\frac{1}{Z_j} & \frac{1}{Z_j} \end{bmatrix} \begin{bmatrix} e^{i\omega h_{j-1}} & 0 \\ 0 & e^{-i\omega h_{j-1}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{Z_j}{2} \\ \frac{1}{2} & \frac{Z_j}{2} \end{bmatrix} = \tag{2.38}$$

$$= \begin{bmatrix} \cos \omega h_j & -iZ_j \sin \omega h_j \\ -i\frac{1}{Z_j} \sin \omega h_j & \cos \omega h_j \end{bmatrix}.$$

The transition matrix $\mathbf{A}_j\ \mathbf{E}_j\ \mathbf{A}_j^{-1}$ has also the real representation; it can be obtained by multiplying the matrix by matrix **R** and its inverse defined as:

$$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \qquad \mathbf{R}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}. \tag{2.39}$$

Then the transition matrix is:

$$\mathbf{M}_j = \mathbf{R}^{-1}\ \mathbf{A}_j\ \mathbf{E}_j\ \mathbf{A}_j^{-1}\ \mathbf{R} = \begin{bmatrix} \cos \omega h_j & Z_j \sin \omega h_j \\ -\frac{1}{Z_j} \sin \omega h_j & \cos \omega h_j \end{bmatrix} \tag{2.40}$$

and the equation for all the bar becomes:

$$\hat{\mathbf{F}}^0 = \mathbf{A}_0^{-1}\ \mathbf{R} \prod_{j=1}^{N} \mathbf{M}_j\ \mathbf{R}^{-1} \cdot \mathbf{A}_{N+1}\ \hat{\mathbf{F}}^{N+1}. \tag{2.41}$$

Coming back to the original amplitudes $\hat{\sigma}_I^0$, $\hat{\sigma}_R^0$, $\hat{\sigma}_I^{N+1}$ (since in the right half-space there is no reflected wave, we have $\hat{\vartheta}_R^{N+1} = 0$), using the definition we have:

$$\hat{f}_I^0 = \hat{\vartheta}_I^0, \qquad \hat{f}_R^0 = \hat{\vartheta}_R^0, \qquad \hat{f}_I^{N+1} = \hat{\vartheta}_I^{N+1} e^{-i\omega\xi_N} = \hat{\vartheta}_I^{N+1} exp\left\{ -i\omega \sum_{j=1}^{N} h_j \right\}, \tag{2.42}$$

and the equation becomes:

$$\begin{bmatrix} \hat{\vartheta}_I^0 \\ \hat{\vartheta}_R^0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -iZ_0 \\ 1 & iZ_0 \end{bmatrix} \prod_{j=1}^{N} \mathbf{M}_j \begin{bmatrix} 1 & 1 \\ \dfrac{i}{Z_{N+1}} & \dfrac{-i}{Z_{N+1}} \end{bmatrix} \cdot \begin{bmatrix} \hat{\vartheta}_I^{N+1} e^{-i\omega\Sigma h_j} \\ 0 \end{bmatrix} \tag{2.43}$$

or

$$\begin{bmatrix} \hat{\vartheta}_I^0 \\ \hat{\vartheta}_R^0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -iZ_0 \\ 1 & iZ_0 \end{bmatrix} \prod_{j=1}^{N} \mathbf{M}_j \begin{bmatrix} 1 \\ \dfrac{i}{Z_{N+1}} \end{bmatrix} \hat{\vartheta}_I^{N+1} exp\left\{ -i\omega \sum_{j=1}^{N} h_j \right\}. \tag{2.44}$$

3. The model of the periodic bar.

Consider now a periodic case, that is such that the bar consists of two kinds of elements , the couples of which are located periodically. Let $N = 2K$ and the transition matrices through the layers are:

$$\mathbf{M}_i = \mathbf{M}_1 \text{ for } i = 1,3, \dots , 2K - 1,$$
$$\mathbf{M}_i = \mathbf{M}_2 \text{ for } i = 2,4, \dots , 2K,$$

where

$$\mathbf{M}_1 = \mathbf{R}^{-1} \mathbf{A}_1 \mathbf{E}_1 \mathbf{A}_1^{-1} \mathbf{R} = \begin{bmatrix} cos\ \omega h_1 & Z_1\ sin\ \omega h_1 \\ -\dfrac{1}{Z_1} sin\ \omega h_1 & cos\ \omega h_1 \end{bmatrix}. \tag{3.1}$$

$$\mathbf{M}_2 = \mathbf{R}^{-1}\,\mathbf{A}_2\,\mathbf{E}_2\,\mathbf{A}_2^{-1}\,\mathbf{R} = \begin{bmatrix} \cos\omega h_2 & Z_2 \sin\omega h_2 \\ -\dfrac{1}{Z_2}\sin\omega h_2 & \cos\omega h_2 \end{bmatrix}. \tag{3.2}$$

The travel time periods h_1 and h_2 are related to the lengths of the panels l_1 and l_2 according to formula (2.5) as:

$$h_1 = \frac{l_1}{c_1} = \sqrt{\frac{\rho_1}{\eta_1}}\, l_1 = \frac{A\rho_1}{Z_1}\, l_1, \qquad h_2 = \frac{l_2}{c_2} = \sqrt{\frac{\rho_2}{\eta_2}}\, l_2 = \frac{A\rho_2}{Z_2}\, l_2 \tag{3.3}$$

Then we can introduce the transfer matrix through the couple of layers:

$$\mathbf{M} = \mathbf{M}_1\,\mathbf{M}_2 = \tag{3.4}$$

$$\begin{bmatrix} \cos\omega h_1 \cos\omega h_2 - \dfrac{Z_1}{Z_2}\sin\omega h_1 \sin\omega h_2 & Z_1 \sin\omega h_1 \cos\omega h_2 + Z_2 \cos\omega h_1 \sin\omega h_2 \\ -\dfrac{\sin\omega h_1 \cos\omega h_2}{Z_1} - \dfrac{\cos\omega h_1 \sin\omega h_2}{Z_2} & \cos\omega h_1 \cos\omega h_2 - \dfrac{Z_2}{Z_1}\sin\omega h_1 \sin\omega h_2 \end{bmatrix}$$

and transform the equation for the amplitudes to the following form:

$$\begin{bmatrix} \hat{\vartheta}_I^0 \\ \hat{\vartheta}_R^0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -iZ_0 \\ 1 & iZ_0 \end{bmatrix} \mathbf{M}^K \begin{bmatrix} 1 \\ \dfrac{i}{Z_{fin}} \end{bmatrix} \hat{\vartheta}_I^{fin}\, e^{-i\omega K(h_1+h_2)}. \tag{3.5}$$

Consider now a limit case where the number of panels in the bar tends to infinity but their lengths tend to zero in such a way that the length of the bar as well as the ratio of the lengths of its components remain constant:

$$K \to \infty\,, \qquad l_1 = \frac{L_1}{K}\,, \qquad l_2 = \frac{L_2}{K}. \tag{3.6}$$

and, what it follows:

$$h_1 = \frac{H_1}{K}, \qquad\qquad h_2 = \frac{H_2}{K}. \tag{3.7}$$

where

$$H_1 = \frac{A\rho_1}{Z_1} L_1, \qquad\qquad H_2 = \frac{A\rho_2}{Z_2} L_2 \tag{3.8}$$

are the travel time periods through all the segments within the bar made, respectively, of material 1 and 2.

It can be shown that in the limit the reflection properties of the bar are equivalent to the properties of some uniform homogenized bar with the effective parameters (impedance and travel time from the beginning of the bar to its end). Their numerical values can be calculated analogously to the effective properties of the stratified slab obtained in [4,5]. To calculate them let us substitute the assumed values of $h_1 = \dfrac{H_1}{K}$ and $h_2 = \dfrac{H_2}{K}$ to the formula (3.3) for the transition matrix **M**. Expanding the matrix **M** into the power series we have:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{K} \begin{bmatrix} 0 & Z_1\omega\, H_1 + Z_2\omega\, H_2 \\ -\dfrac{\omega\, H_1}{Z_1} - \dfrac{\omega\, H_2}{Z_2} & 0 \end{bmatrix} + o\left(\frac{1}{K}\right) \tag{3.9}$$

or

$$\mathbf{M} = \mathbf{A} + \frac{1}{K}\mathbf{B} + o\left(\frac{1}{K}\right). \tag{3.10}$$

where

$$\mathbf{A} = \mathbf{Id} \tag{3.11}$$

and

$$\mathbf{B} = \begin{bmatrix} 0 & Z_1\omega\, H_1 + Z_2\omega\, H_2 \\ -\dfrac{\omega\, H_1}{Z_1} - \dfrac{\omega\, H_2}{Z_2} & 0 \end{bmatrix} \tag{3.12}$$

Calculating the limit

$$\lim_{K \longrightarrow \infty} \mathbf{M}^{K} = \lim_{K \longrightarrow \infty} \left(\mathbf{Id} + \frac{1}{K}\mathbf{B} + o\left(\frac{1}{K}\right) \right)^{K} = e^{\mathbf{B}}, \tag{3.13}$$

we obtain the effective transition matrix for the homogenized bar:

$$e^{\mathbf{B}} = \frac{1}{2} \begin{bmatrix} e^{-i\omega a} + e^{i\omega a} & ib\left(e^{-i\omega a} - e^{i\omega a}\right) \\ \frac{1}{ib}\left(e^{-i\omega a} - e^{i\omega a}\right) & e^{-i\omega a} + e^{i\omega a} \end{bmatrix}, \tag{3.14}$$

where the effective travel time through the bar a and the effective impedance b are equal to:

$$a = \sqrt{\frac{\left(H_1 Z_1 + H_2 Z_2\right)\left(H_1 Z_2 + H_2 Z_1\right)}{Z_1\, Z_2}}, \tag{3.15}$$

$$b = \sqrt{Z_1\, Z_2\, \frac{H_1 Z_1 + H_2 Z_2}{H_1 Z_2 + H_2 Z_1}}. \tag{3.16}$$

The transition matrix can be also written down in the real numbers form:

$$e^{\mathbf{B}} = \begin{bmatrix} \cos \omega a & b \sin \omega a \\ -\frac{1}{b} \sin \omega a & \cos \omega a \end{bmatrix}. \tag{3.17}$$

The effective parameters a and b can be also expressed in terms of the lengths L_1 and L_2 instead of the travel time moments. Substituting expressions (3.8) into (3.15) and (3.16) we obtain:

$$a = A\sqrt{\left(L_1\rho_1 + L_2\rho_2\right)\left(\frac{L_1\rho_1}{Z_1^2} + \frac{L_2\rho_2}{Z_2^2}\right)} \tag{3.18}$$

$$b = \sqrt{\frac{L_1\rho_1 + L_2\rho_2}{\frac{L_1\rho_1}{Z_1^2} + \frac{L_2\rho_2}{Z_2^2}}} \tag{3.19}$$

Moreover, substituting in (3.18) and (3.19) the definition of impedance (2.6) we obtain a and b expressed in terms of material parameters and lengths:

$$a = \sqrt{\left(L_1\rho_1 + L_2\rho_2\right)\left(\frac{L_1}{\eta_1} + \frac{L_2}{\eta_2}\right)} \tag{3.20}$$

$$b = A\sqrt{\frac{L_1\rho_1 + L_2\rho_2}{\frac{L_1}{\eta_1} + \frac{L_2}{\eta_2}}} \tag{3.21}$$

The above formulae prove that the effective parameters of the material in the considered dynamical (time-dependent) model are the same as in static or harmonic one (see [5]).

Substituting the effective transfer matrix into the equation for the amplitudes of the waves (3.5) we obtain following final equation:

$$\begin{bmatrix} \hat{\sigma}_I^0 \\ \hat{\sigma}_R^0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -iZ_0 \\ 1 & iZ_0 \end{bmatrix}\begin{bmatrix} \cos \omega a & b \sin \omega a \\ -\frac{1}{b}\sin \omega a & \cos \omega a \end{bmatrix}\begin{bmatrix} 1 \\ \frac{i}{Z_{fin}} \end{bmatrix}\hat{\sigma}_I^{fin}\, e^{-i\omega(H_1+H_2)}. \tag{3.22}$$

Multiplying the matrices in (3.22) we obtain the expressions for the amplitudes:

$$\hat{\sigma}_I^0(\omega) = \frac{1}{2}\left[\left(1 + \frac{Z_0}{Z_{fin}}\right)\cos \omega a + i\left(\frac{b}{Z_{fin}} + \frac{Z_0}{b}\right)\sin \omega a\right]\hat{\sigma}_I^{fin}(\omega)\, e^{-i\omega(H_1+H_2)} \tag{3.23}$$

$$\hat{\vartheta}_R^0(\omega) = \frac{1}{2}\left[\left(1 - \frac{Z_0}{Z_{fin}}\right) \cos \omega a + i\left(\frac{b}{Z_{fin}} - \frac{Z_0}{b}\right) \sin \omega a\right] \hat{\vartheta}_I^{fin}(\omega)\, e^{-i\omega(H_1+H_2)} \tag{3.24}$$

Posing the problem we have assumed that $\hat{\vartheta}_I^0(\omega)$ is the known quantity in our model. Therefore we can express the unknown amplitudes of the reflected and transmitted wave with the use of $\hat{\vartheta}_I^0(\omega)$ in the following form:

$$\hat{\vartheta}_R^0(\omega) = \frac{\left[\left(1 - \frac{Z_0}{Z_{fin}}\right) \cos \omega a + i\left(\frac{b}{Z_{fin}} - \frac{Z_0}{b}\right) \sin \omega a\right] \hat{\vartheta}_I^0(\omega)}{\left[\left(1 + \frac{Z_0}{Z_{fin}}\right) \cos \omega a + i\left(\frac{b}{Z_{fin}} + \frac{Z_0}{b}\right) \sin \omega a\right]} \tag{3.25}$$

$$\hat{\vartheta}_I^{fin}(\omega) = \frac{2\, \hat{\vartheta}_I^0(\omega)\, e^{i\omega(H_1+H_2)}}{\left[\left(1 + \frac{Z_0}{Z_{fin}}\right) \cos \omega a + i\left(\frac{b}{Z_{fin}} + \frac{Z_0}{b}\right) \sin \omega a\right]} \tag{3.26}$$

Calculating the inverse Fourier transform of the expressions (3.25) and (3.26) we obtain the behavior (the shape) of the reflected and the transmitted pulses in time.

4. The model of the bar with random properties.

In the model of the bar considered in the previous section it is assumed that both the material parameters and the geometrical dimensions of the bar are deterministic. In reality, since the element of the structure is built in factory conditions (some tolerances in dimensions, the pieces of the material selected from some bigger sample, etc.), such quantities should be regarded as random variables.

Then, the waves generated by the incident deterministic wave pulse prove to have stochastic properties and must be regarded as stochastic processes.

In practice we are mostly interested in some averaged properties of such transition phenomena - average wave amplitudes, average transmitted (reflected) energy and - in some limit case - the overall properties of the bar. In this section we apply the law of large numbers for the product of random matrices (cf.[6]) to obtain the effective transmission properties of the bar built of large number of segments with random properties.

The equations (2.44) satisfied by the reflected and transmitted waves are valid also in the case when the material parameters and the lengths of the segments are random variables. Therefore, for every finite number of elements in the bar the obtained equations (with an appropriate stochastic interpretation) can describe the wave field. The situation complicates a bit in the limit case when the number of segments tends to infinity. However, the law of large numbers makes that the problem can be successfully solved.

Assume that the bar is built of $2K$ segments with the lengths $l_1(\gamma)$, $l_2(\gamma)$, ... ,$l_{2K}(\gamma)$, where $l_i(\gamma)$, $i=1,2,\ ...\ ,2K$ are random variables. In the above $\gamma\in\Gamma$ is an elementary event and $(\Gamma,\mathcal{F},\mathcal{P})$ is the complete probabilistic space (cf.[7]). Assume additionally that the material parameters of the segments and the areas of their cross-sections $\left(\rho_{2j-1}(\gamma),\ \eta_{2j-1}(\gamma),\ A_{2j-1}(\gamma),\ \rho_{2j}(\gamma),\ \eta_{2j}(\gamma),\ A_{2j}(\gamma)\right)$ are, as the vector random variables, independent and identically distributed for $j=1,2,...,2K$. Moreover, we assume that the lengths of the segments have the following particular property:

$$\left(\ l_{2j-1}(\gamma),\ l_{2j}(\gamma)\ \right) = \left(\frac{L_{2j-1}(\gamma)}{2K},\ \frac{L_{2j}(\gamma)}{2K}\right), \tag{4.1}$$

for $j=1,2,...,K$ are independent, identically distributed two-dimensional random variables with

$$E\left\{\ L_{2j-1}(\gamma)\ \right\} = L_1, \qquad\qquad E\left\{\ L_{2j}(\gamma)\ \right\} = L_2, \tag{4.2}$$

for $j=1,2,...,K$. In this particular case the equation (2.44) for the Fourier transform of the amplitudes takes the following form:

$$\begin{bmatrix} \vartheta_I^0(\omega,\gamma) \\ \vartheta_R^0(\omega,\gamma) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -iZ_0 \\ 1 & iZ_0 \end{bmatrix}_j \prod_{j=1}^{K} \mathbf{M}_j(\omega,\gamma) \begin{bmatrix} 1 \\ \dfrac{i}{Z_{2K+1}} \end{bmatrix} \vartheta_I^{2K+1}(\omega,\gamma)\ exp\left\{ -i\omega \sum_{jj=1}^{2K} h_j(\gamma) \right\}, \tag{4.3}$$

where $h_j(\gamma)$ are the randomized counterparts of the travel time defined in (2.5), $\mathbf{M}_j(\omega,\gamma)$ are the randomized transfer matrices through the couple of layers defined in (3.4):

$$\mathbf{M}_j(\omega,\gamma) = \tag{4.4}$$

$$\left[\begin{matrix} \cos\omega h_{2j-1}(\gamma)\ \cos\omega h_{2j}(\gamma) - \dfrac{Z_{2j-1}(\gamma)}{Z_{2j}(\gamma)} \sin\omega h_{2j-1}(\gamma)\ \sin\omega h_{2j}(\gamma), \\ -\dfrac{\sin\omega h_{2j-1}(\gamma)\ \cos\omega h_{2j}(\gamma)}{Z_{2j-1}(\gamma)} - \dfrac{\cos\omega h_{2j-1}(\gamma)\ \sin\omega h_{2j}(\gamma)}{Z_{2j}(\gamma)}, \end{matrix} \right.$$

$$\left. \begin{matrix} Z_{2j-1}(\gamma)\ \sin\omega h_{2j-1}(\gamma)\ \cos\omega h_{2j}(\gamma) + Z_{2j}(\gamma)\ \cos\omega h_{2j-1}(\gamma)\ \sin\omega h_{2j}(\gamma) \\ \cos\omega h_{2j-1}(\gamma)\ \cos\omega h_{2j}(\gamma) - \dfrac{Z_{2j}(\gamma)}{Z_{2j-1}(\gamma)} \sin\omega h_1(\gamma)\ \sin\omega h_{2j-1}(\gamma)\ \sin\omega h_{2j}(\gamma) \end{matrix} \right]$$

for j=1,2,...,K, with

$$Z_i(\gamma) = A_i(\gamma) \sqrt{\rho_i(\gamma)\ \eta_i(\gamma)}\ ,\ i=1,2,...,2K. \tag{4.5}$$

To study the asymptotic behavior of the randomized equation for the amplitudes of the waves we apply the law of large numbers for the products of random matrices obtained in [5]. This theorem can be written in the following form.

Consider the sequence of the products of real random matrices

$$\mathbf{P}_K(\gamma) = \prod_{j=1}^{K} \mathbf{M}_{j,K}(\gamma). \tag{4.6}$$

It is assumed that for K tending to infinity the matrices $\mathbf{M}_{j,K}$ can be represented as

$$\mathbf{M}_{j,K}(\gamma) = \mathbf{Id} + \frac{1}{K}\,\mathbf{B}_{j,K}(\gamma) + \mathbf{R}_j(K,\gamma), \tag{4.7}$$

where $\mathbf{B}_{j,K}(\gamma)$ for $j=1,2,\dots,K$ are independent, identically distributed random matrices, integrable with respect to probability measure $\mathscr{P}$ and $|\mathbf{R}_j(K,\gamma)| = o(K^{-1})$ for large K. Under these conditions the law of large numbers takes place and

$$\lim_{K \longrightarrow \infty} \mathbf{P}_K(\gamma) = exp\left[E\left\{ \mathbf{B}_{j,K}(\gamma) \right\} \right], \tag{4.8}$$

in the sense of convergence in distribution of all the vectors obtained from multiplication of the random matrix by an arbitrary deterministic vector.

To analyse the limit case of the propagation of wave pulses through the bar built of the segments we decompose the transition matrix defined in (4.4) under the assumption (4.1) ($h_j(\gamma)$ are connected with $l_j(\gamma)$ by the formula analogous to (3.8)) with respect to the powers of $1/K$:

$$\mathbf{M}_j(\omega,\gamma) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \tag{4.9}$$

$$+ \frac{1}{K}\begin{bmatrix} 0 & Z_{2j-1}(\gamma)\omega\, H_{2j-1}(\gamma) + Z_{2j}(\gamma)\omega\, H_{2j}(\gamma) \\ -\dfrac{\omega\, H_{2j-1}(\gamma)}{Z_{2j-1}(\gamma)} - \dfrac{\omega\, H_{2j}(\gamma)}{Z_{2j}(\gamma)} & 0 \end{bmatrix}$$

$$+ o\left(\frac{1}{K}\right)$$

The matrices $\mathbf{B}_j$ required in formula (4.8) are defined as:

$$\mathbf{B}_j = \begin{bmatrix} 0 & Z_{2j-1}(\gamma)\omega\, H_{2j-1}(\gamma) + Z_{2j}(\gamma)\omega\, H_{2j}(\gamma) \\ -\dfrac{\omega\, H_{2j-1}(\gamma)}{Z_{2j-1}(\gamma)} - \dfrac{\omega\, H_{2j}(\gamma)}{Z_{2j}(\gamma)} & 0 \end{bmatrix} \tag{4.10}$$

and their common average value is

$$E\left\{ \mathbf{B}_j \right\} = \omega \begin{bmatrix} 0 & E\left\{ Z_1(\gamma)H_1(\gamma) \right\} + E\left\{ Z_2(\gamma)H_2(\gamma) \right\} \\ -E\left\{ \dfrac{H_1(\gamma)}{Z_1(\gamma)} \right\} - E\left\{ \dfrac{H_2(\gamma)}{Z_2(\gamma)} \right\} & 0 \end{bmatrix} \tag{4.11}$$

The matrix $e^{E\{\mathbf{B}_j\}}$ is of the form analogous to (3.14):

$$e^{E\{\mathbf{B}_j\}} = \frac{1}{2} \begin{bmatrix} e^{-i\omega a} + e^{i\omega a} & ib\left(e^{-i\omega a} - e^{i\omega a}\right) \\ \dfrac{1}{ib}\left(e^{-i\omega a} - e^{i\omega a}\right) & e^{-i\omega a} + e^{i\omega a} \end{bmatrix}, \tag{4.12}$$

where at present the effective travel time through the bar a and the effective impedance b are equal to:

$$a = \sqrt{\left(E\left\{Z_1(\gamma)H_1(\gamma)\right\} + E\left\{Z_2(\gamma)H_2(\gamma)\right\}\right)\left(E\left\{\frac{H_1(\gamma)}{Z_1(\gamma)}\right\} + E\left\{\frac{H_2(\gamma)}{Z_2(\gamma)}\right\}\right)}\ , \tag{4.13}$$

$$b = \sqrt{\frac{E\left\{ Z_1(\gamma)H_1(\gamma) \right\} + E\left\{ Z_2(\gamma)H_2(\gamma) \right\}}{E\left\{ \dfrac{H_1(\gamma)}{Z_1(\gamma)} \right\} + E\left\{ \dfrac{H_2(\gamma)}{Z_2(\gamma)} \right\}}} \tag{4.14}$$

Analogously to (3.17) the transition matrix can be written in the real form:

$$e^{E\{\mathbf{B}_j\}} = \begin{bmatrix} \cos \omega a & b \sin \omega a \\ -\dfrac{1}{b} \sin \omega a & \cos \omega a \end{bmatrix}. \tag{4.15}$$

We can also find the randomized counterparts of formulae (3.18-3.21) where the effective parameters a and b are expressed in terms of the lengths L_1 and L_2 and the material parameters of the segments. We have;

$$a = \sqrt{\left[E\left\{\rho_1(\gamma)A_1(\gamma)L_1(\gamma)\right\} + E\left\{\rho_2(\gamma)A_2(\gamma)L_2(\gamma)\right\}\right]} \times$$

$$\times \sqrt{\left[E\left\{\frac{\rho_1(\gamma)A_1(\gamma)L_1(\gamma)}{Z_1^2(\gamma)}\right\} + E\left\{\frac{\rho_2(\gamma)A_2(\gamma)L_2(\gamma)}{Z_2^2(\gamma)}\right\}\right]} \tag{4.16}$$

$$b = \sqrt{\frac{E\left\{\rho_1(\gamma)A_1(\gamma)L_1(\gamma)\right\} + E\left\{\rho_2(\gamma)A_2(\gamma)L_2(\gamma)\right\}}{E\left\{\frac{\rho_1(\gamma)A_1(\gamma)L_1(\gamma)}{Z_1^2(\gamma)}\right\} + E\left\{\frac{\rho_2(\gamma)A_2(\gamma)L_2(\gamma)}{Z_2^2(\gamma)}\right\}}} \tag{4.17}$$

or

$$a = \sqrt{\left[E\left\{\rho_1(\gamma)A_1(\gamma)L_1(\gamma)\right\} + E\left\{\rho_2(\gamma)A_2(\gamma)L_2(\gamma)\right\}\right]} \times$$

$$\times \sqrt{E\left\{\frac{L_1(\gamma)}{\eta_1(\gamma)A_1(\gamma)}\right\} + E\left\{\frac{L_2(\gamma)}{\eta_2(\gamma)A_2(\gamma)}\right\}} \tag{4.18}$$

$$b = \sqrt{\frac{E\left\{\rho_1(\gamma)A_1(\gamma)L_1(\gamma)\right\} + E\left\{\rho_2(\gamma)A_2(\gamma)L_2(\gamma)\right\}}{E\left\{\frac{L_1(\gamma)}{\eta_1(\gamma)A_1(\gamma)}\right\} + E\left\{\frac{L_2(\gamma)}{\eta_2(\gamma)A_2(\gamma)}\right\}}} \tag{4.19}$$

The above formulae can be easily generalized on the case where the bar is built of more than two kinds of material - the period of the segments (in our stochastic sense) is e.g. k. Then the transfer matrix for the homogenized bar is also of the form (4.12) or (4.15) but the effective travel time a and the effective impedance b are defined as

$$a = \sqrt{\sum_{i=1}^{k} E\left\{\rho_i(\gamma) A_i(\gamma) L_i(\gamma)\right\}} \sqrt{\sum_{i=1}^{k} E\left\{\frac{L_i(\gamma)}{\eta_i(\gamma) A_i(\gamma)}\right\}} \tag{4.20}$$

$$b = \sqrt{\frac{\sum_{i=1}^{k} E\left\{\rho_i(\gamma) A_i(\gamma) L_i(\gamma)\right\}}{\sum_{i=1}^{k} E\left\{\frac{L_i(\gamma)}{\eta_i(\gamma) A_i(\gamma)}\right\}}} \tag{4.21}$$

The expressions for the amplitudes are analogous to (3.25)-(3.26) with the obtained above parameters a and b.

5. Illustrative example and discussion.

In the considerations of this section let us concentrate on the reflected pulses characterized for the homogenized (uniform) bar by formula (3.25) and for the segmented bar by an analogous formula obtained directly from equation (3.5):

$$\hat{\sigma}_R^0(\omega) = \frac{\left[\left(M_{11}^K - M_{22}^K \frac{Z_0}{Z_{fin}}\right) + i\left(M_{12}^K \frac{1}{Z_{fin}} + M_{21}^K Z_0\right)\right] \hat{\sigma}_I^0(\omega)}{\left[\left(M_{11}^K + M_{22}^K \frac{Z_0}{Z_{fin}}\right) + i\left(M_{12}^K \frac{1}{Z_{fin}} - M_{21}^K Z_0\right)\right]} \tag{5.1}$$

where $\mathbf{M}_{ij}^k$ is the ij-th element of the matrix $\mathbf{M}^k$. Calculating the inverse Fourier transform of the above expression we obtain the shape of the reflected pulse in the temporal domain and taking its absolute value - the changes in time of the amplitude of the reflected wave.

As it is seen from the formula (3.25) or (5.1), the reflected pulse is the function of two components: the initial pulse, characterized by $\hat{\theta}_I^0(\omega)$ and the material properties of the dynamical system, characterized by the remaining part of the formulae. The purpose of this section is the numerical studying of the convergence of the segmented bar to the homogenized one when the number of segments tends to infinity. This fact determines the material (geometrical) part of the equation for the reflected pulse. Up to our decision is the determination of the shape of the initial pulse $\hat{\theta}_I^0(\omega)$, which in this particular problem plays the role of a testing tool. Therefore we can select such a form of the initial pulse which gives us the possibility to inspect with the best accuracy what happens inside the reflecting bar. The rectangular pulse seems to have such a property - it starts rapidly and has finite duration. This makes that we know when the pulse reflected at a given interface starts and how long it continues.

Writing this in terms of the formulae, we assume that the initial rectangular pulse has the following form:

$$f(t) = \begin{cases} \beta, & t \in (0,\alpha) \\ 0, & \text{otherwise}' \end{cases} \tag{5.2}$$

where β is the value of the amplitude of the pulse (the displacement of the material) and α is the duration of the pulse (starting at time t=0). Then the Fourier transform of the pulse required in the formula (5.1) is the following function of the spectral parameter $\omega \in (-\infty,\infty)$:

$$F(\omega) = \alpha\beta \left(\frac{\sin\frac{\omega\alpha}{2}}{\frac{\omega\alpha}{2}} \right) exp\left\{-i\frac{\omega\alpha}{2}\right\}, \tag{5.3}$$

that is in equation (5.1) we substitute

$$\hat{\theta}_I^0(\omega) = F(\omega) \tag{5.4}$$

The fact that the rectangular initial pulse gives very distinct shape of the reflected pulse is very important since even in the case of homogeneous reflecting bar its shape can be quite complicated. For example, for some fixed parameters of the system we have the amplitude of the reflected pulse as it is shown at Figures 1a-d. We assumed in calculations that the impedance of the surrounding medium is $Z_0=Z_{fin}=1.0$, impedance of the reflecting bar is $b=2.828427$, the amplitude of the incident pulse is $\beta=1.0$ and its time of duration α is also equal to 1. The parameter which moderates the picture is the travel time of the wave through the reflecting bar. It is seen that the ratio of the duration of the pulse and the travel time through the bar has the strong effect on the shape of the reflected pulse. We see how, in the case of the bar short comparing to the duration of the exciting pulse, the multiple reflection on the interfaces of the materials summarize giving the final amplitude of the pulse. The much more complicated situation is when the reflecting bar consists of several segments. In this example we wish to present how the homogenization procedure, theoretically analysed in the previous sections, works in the case of the concrete bar.

Assume that our periodic bar analysed in Section 3 is built of two kinds of material with impedances, respectively, $Z_1=4.0$ and $Z_2=2.0$. We assume that the surrounding medium has the impedance equal to 1 (that is in our formula $Z_0=Z_{fin}=1.0$). Moreover, we assume that thickness of the bar is finite and both materials participate in it in such a way, that the travel time through both of them is the same, equal to 1. (in (3.1-3.8) $H_1=H_2=1.0$). For such parameters in our periodic model analysed in Section 3, we can calculate the numerical values of the effective travel time and the effective impedance of the homogenized medium according to the formulae (3.15)-(3.16). The result is

$$a = 2.12132, \tag{5.5}$$

$$b = 2.828427. \tag{5.6}$$

It is seen that the effective travel time is higher than the sum of the travel times through its components.

To know something about the reflection process in the case when the number of segments in the bar is finite we must perform the numerical calculations. In particular, to find the amplitude of the reflected pulse, we must calculate the inverse Fourier transform of the expression defined in (5.1), which in the numerical case is the discrete Fourier transform (see [8]). The algorithm of the numerical calculation of the amplitude of the reflected pulse is the following.

As it was assumed in Section 2, we consider the Fourier transform defined for the function $f(t)$ as:

$$F(\omega) = \int f(t)\, e^{-i\omega t}\, dt \tag{5.7}$$

Then the continuous inverse transform is defined as

$$f(t) = \frac{1}{2\pi} \int F(\omega)\, e^{i\omega t}\, d\omega \tag{5.8}$$

or in the discrete form:

$$f(t) = \frac{1}{2\pi} \sum (TD_n)\, e^{i\omega t}\, \Delta\omega \tag{5.9}$$

where

$$D_n = F(\omega_n), \quad \omega_n = n\Delta\omega, \quad \Delta\omega = \frac{2\pi}{T}, \tag{5.10}$$

and T is a constant regarded as a period of the temporal function $f(t)$.

Assume that we have given the continuous Fourier transform $F(\omega)$. We are sampling it on the interval $(-R,R)$ transforming the continuous function to the equivalent discrete form. The number of the sample points on the interval $(0,R)$ is assumed as N; the total number of the points is $NN=2N$. Under the above assumption, the sampling interval $\Delta\omega$ is equal:

$$\Delta\omega = \frac{R}{N}, \tag{5.11}$$

and, what it follows, the sample points are defined as:

$$\omega_n = n\Delta\omega = n\,\frac{R}{N}, \quad \text{for} \quad n=0,\pm 1,\pm 2, \quad ,\pm N. \tag{5.12}$$

By definition,

$$D_n = \overline{D_{-n}},\tag{5.13}$$

where the overbar denotes the complex conjugate of the number.

For the numerical calculations, the sample points of $F(\omega)$ are located in the complex vector **F** of the length $NN=2N$ in a specific way. We substitute:

$$\mathbf{F}_1 = D_0,\ \mathbf{F}_2 = D_1, \ldots,\ \mathbf{F}_{N+1} = D_{\pm N},\ \mathbf{F}_{N+2} = D_{-N+1},\ \ldots,\ \mathbf{F}_{2N} = D_{-1}\tag{5.14}$$

where it was assumed the periodicity of the Fourier transform; the values of the transform at the ends of the interval of sampling are considered as equal - they are located at the element $\mathbf{F}_{N+1}$ of the discrete transform vector.

To restore the function $f(t)$ we must calculate the discrete inverse Fourier transform defined as:

$$f(t_m) = f_m = \frac{1}{T} \sum_{n=0}^{2N-1} D_n\, e^{i\omega_n t_m} = \frac{1}{T} \sum_{n=0}^{2N-1} D_n\, e^{i2\pi nm/2N},\tag{5.15}$$

where the following definitions of t_m and ω_n have been used:

$$t_m = m\,\Delta T, \qquad \omega_n = \frac{2\pi n}{2N\Delta T} = \frac{2\pi n}{T}\,.\tag{5.16}$$

To connect the required quantity T describing time (the period of the function being transformed) with the dimension of the spectral domain of the transform we use the expressions for ω_n. Comparing ω_n from (5.12) and (5.16) we obtain:

$$\frac{nR}{N} = \frac{2\pi n}{2N\,\Delta T}\,,\tag{5.17}$$

Then the time step in temporal discretization is:

$$\Delta T = \frac{\pi}{R}\,,\tag{5.18}$$

and the period T is defined as:

$$T = 2\ N\ \Delta T = \frac{2\pi N}{R} \tag{5.19}$$

For the numerical calculations of the discrete Fourier transform defined above we applied the procedure of the Fast Fourier Transform taken from the book [9], where we assumed that the number of points $N = 2^{16} = 65536$.

As we mentioned, in our numerical example we consider the convergence of the segmented bar to the homogenized one when the number of segments K tends to infinity. In contradiction to the harmonic waves where the reflection coefficient plays the role of a measure of the convergence (see [10]), in the case of pulses defining the adequate measure is very difficult. Therefore to show the convergence we present the shape of the reflected pulses for a given number of segments in the bar (under the constrain that the total share of the materials within the bar remains constant). At Figures 2-10 there are shown the amplitudes of the reflected pulse for, respectively, 1, 2, 4, 10, 20, 30, 40, 50, 100 periodic couples of segments (figures marked with the letter *a* are for Z_1=4.0, Z_2=2.0, figures with *b* - for Z_1=2.0, Z_2=4.0). Figure 11 shows the reflected pulse when the bar is built of 500 periodic couples of segments (for such large number of segments the picture does not depend on the order of the materials). Finally, at Figure 12 the reflected pulse for the homogenized bar is presented (calculated according to the formula (3.25)).

The first conclusion we can draw from this sequence of plots is that the homogenization (the procedure regarded in theoretical considerations as an asymptotic phenomenon) really takes place. It is seen, that even for very small number of layers (e.g. 10) some concentration of pulses, similar to the case of the homogenized bar, takes place. In this case we can predict the location of the pulses at time but their amplitude still remains unknown. Increasing the density of stratification in the bar we reach to the result closed to the asymptotic limit. One can follow the speed of the convergence studying the presented set of pictures.

Another conclusion observed from our calculations is that the reflected pulse for small number of segments in the bar is strongly dependent on the fact, which material is the first in the periodic couple of segments. This dependence is stronger for the later reflected pulses. It practically disappears for the high number of segments in the bar.

The presented calculations are performed for the case of periodic non-random bar. Randomness of the properties of the bar makes the picture much more complicated. The convergence of the reflected pulse to the picture given in the homogenized case (which is, as it was shown in Section 4, very similar to the periodic one) is much slower with respect to the growing number of segments in the bar. The preparation of the adequate illustrative pictures needs a lot of time-consuming computations and therefore in this paper we restrict ourselves to the periodic problem.

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a=1.0, b=2.828427

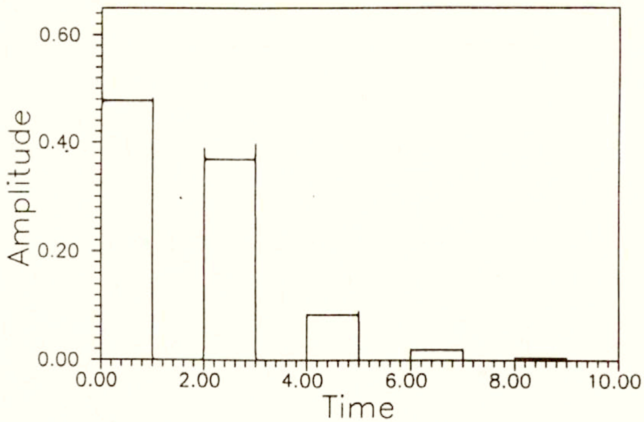

Fig.1a

a=0.5, b=2.828427

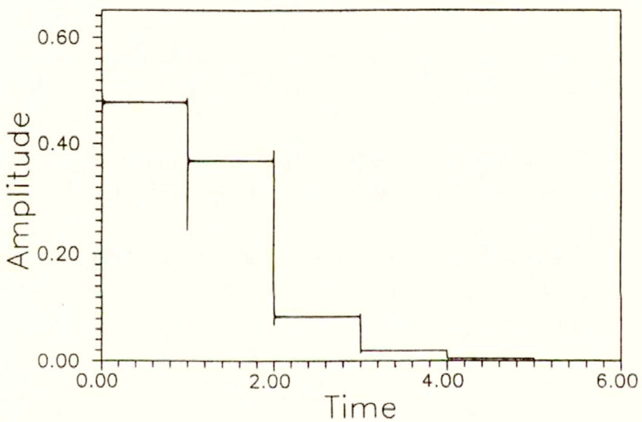

Fig.1b

a=0.2, b=2.828427

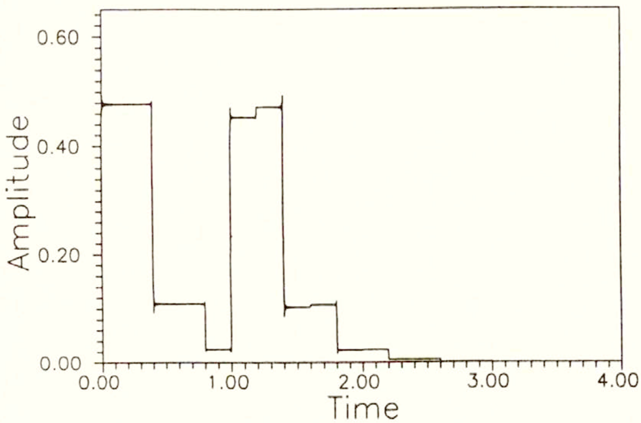

Fig.1c

a=0.05, b=2.828427

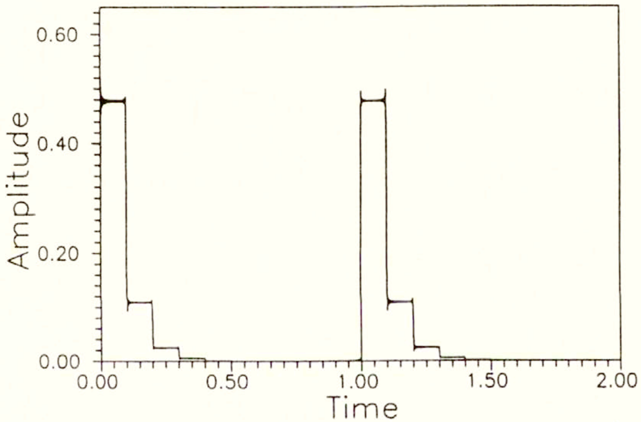

Fig.1d

1 couple of layers, $z_1=4.0$, $z_2=2.0$

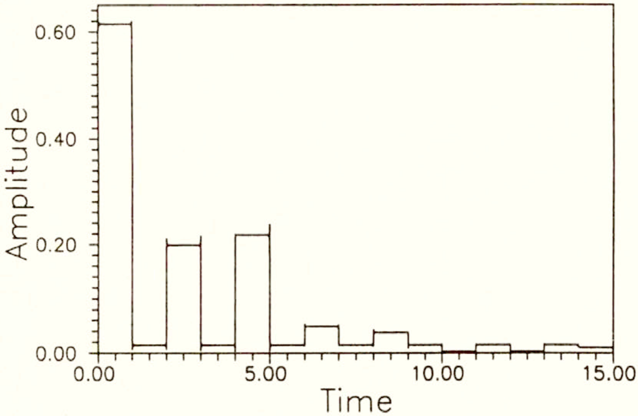

Fig.2a

1 couple of layers, $z_1=2.0$, $z_2=4.0$

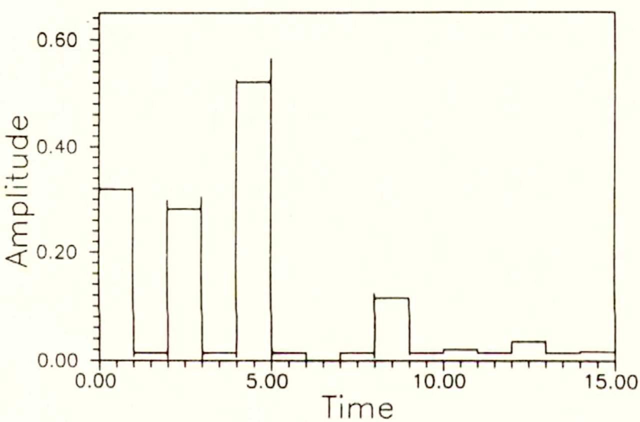

Fig.2b

2 couples of layers, z1=4.0, z2=2.0

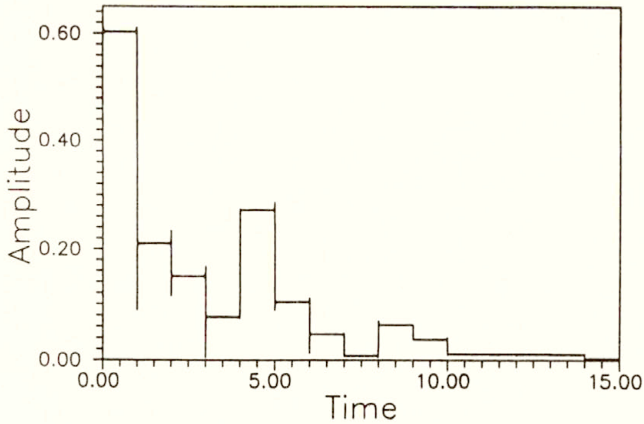
0.60
0.40
0.20
0.00
Amplitude
0.00
5.00
10.00
15.00
Time

Fig.3a

2 couples of layers, z1=2.0, z2=4.0

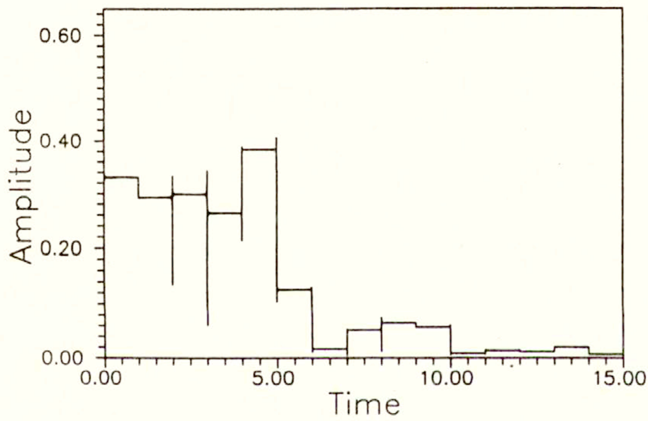
0.60
0.40
0.20
0.00
Amplitude
0.00
5.00
10.00
15.00
Time

Fig.3b

4 couples of layers, $z_1=4.0$, $z_2=2.0$

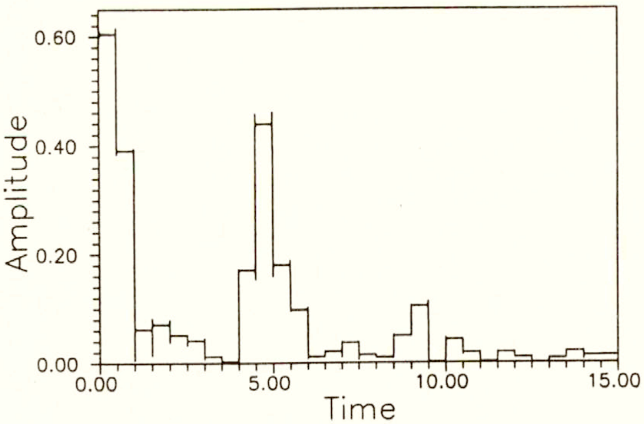

Fig.4a

4 couples of layers, $z_1=2.0$, $z_2=4.0$

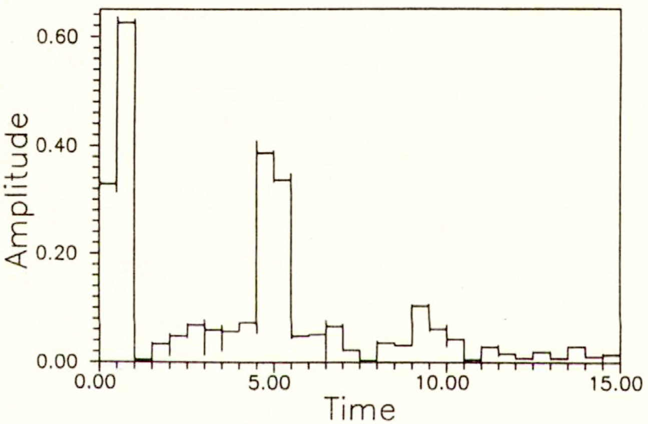

Fig.4b

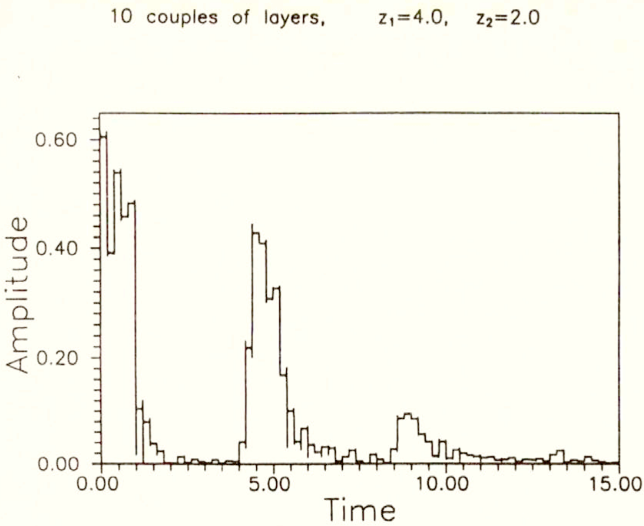
10 couples of layers, z1=4.0, z2=2.0
0.60
0.40
0.20
0.00
Amplitude
0.00
5.00
10.00
15.00
Time

Fig.5a

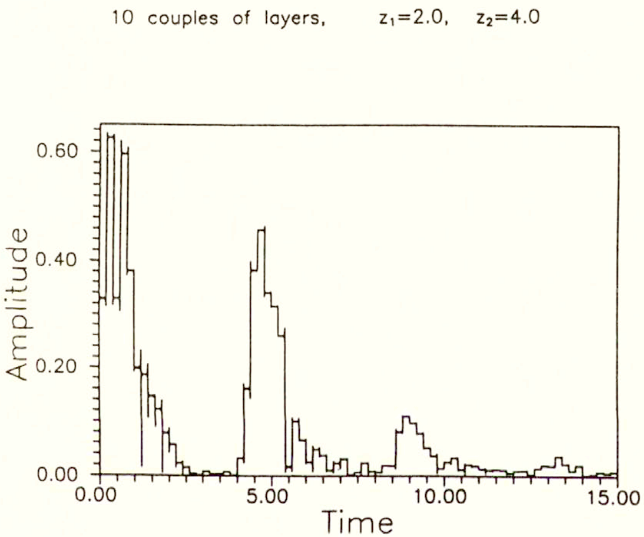
10 couples of layers, z1=2.0, z2=4.0
0.60
0.40
0.20
0.00
Amplitude
0.00
5.00
10.00
15.00
Time

Fig.5b

20 couples of layers, $z_1=4.0$, $z_2=2.0$

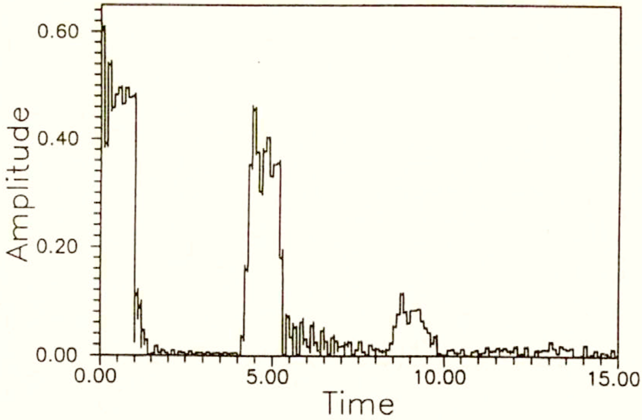

Fig.6a

20 couples of layers, $z_1=2.0$, $z_2=4.0$

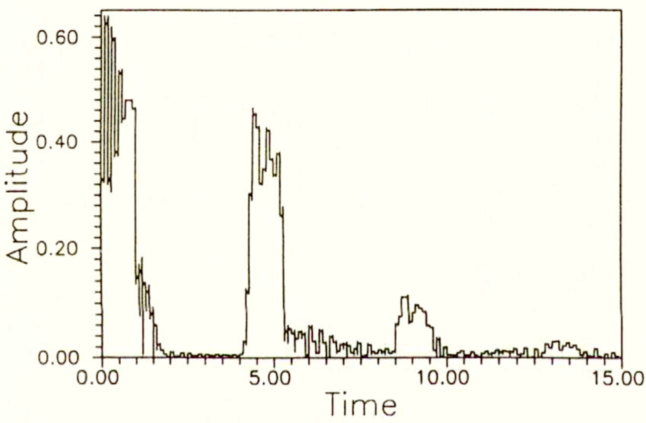

Fig.6b

30 couples of layers, $z_1=4.0$, $z_2=2.0$

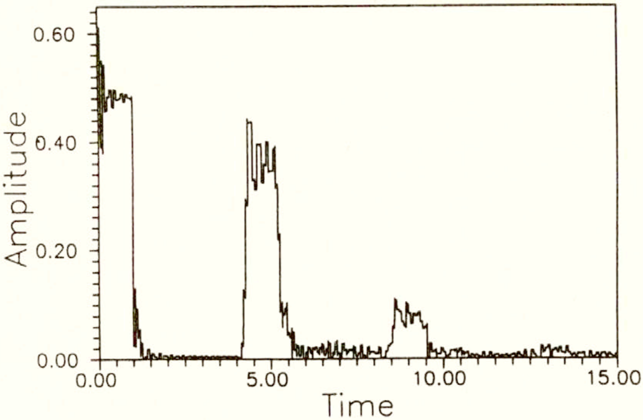

Fig.7a

30 couples of layers, $z_1=2.0$, $z_2=4.0$

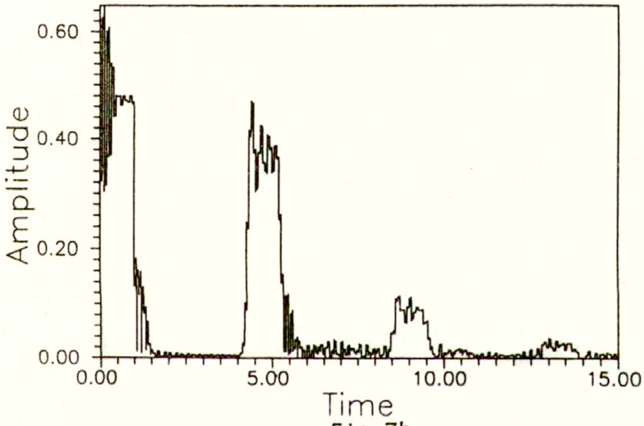

Fig.7b

40 couples of layers, $z_1=4.0$, $z_2=2.0$

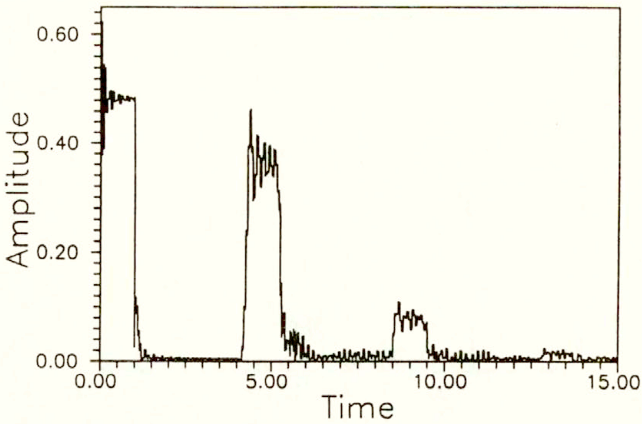

Fig.8a

40 couples of layers, $z_1=2.0$, $z_2=4.0$

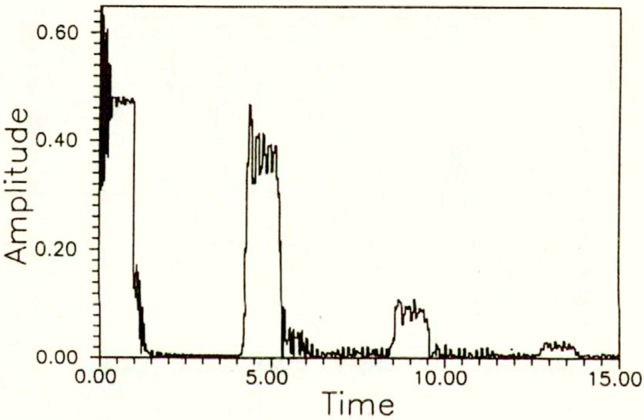

Fig.8b

50 couples of layers, $z_1=4.0$, $z_2=2.0$

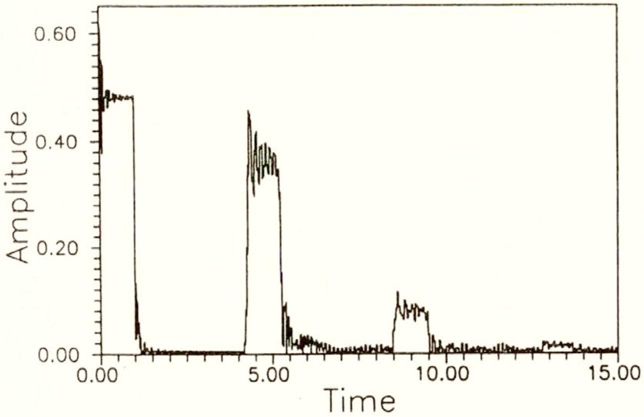

Fig.9a

50 couples of layers, $z_1=2.0$, $z_2=4.0$

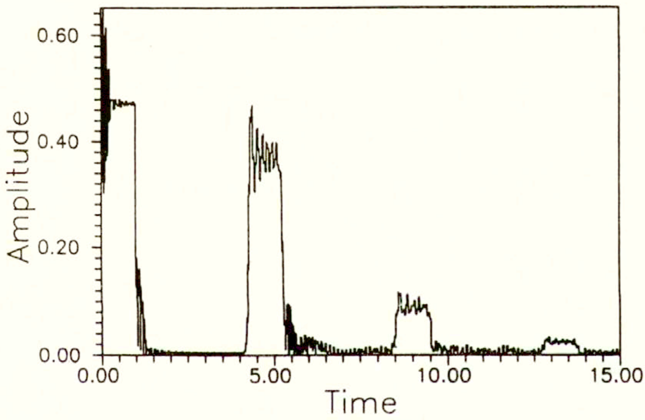

Fig.9b

100 couples of layers, $z_1=4.0$, $z_2=2.0$

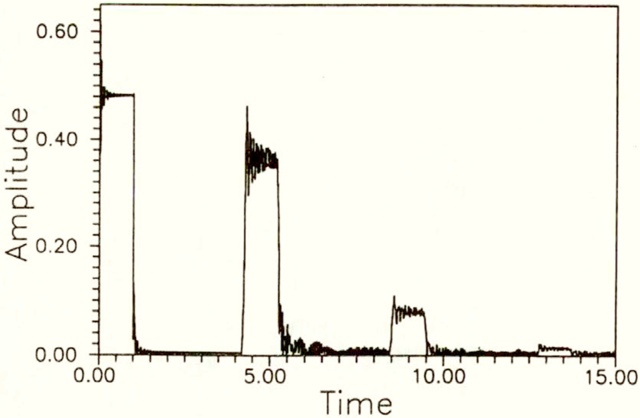

Fig.10a

100 couples of layers, $z_1=2.0$, $z_2=4.0$

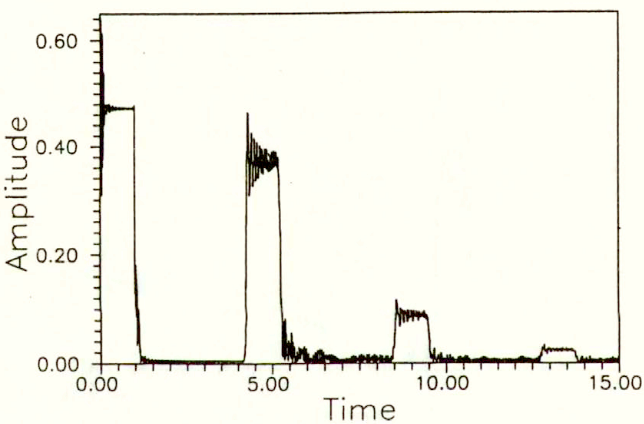

Fig.10b

500 couples of layers, $z_1=4.0$, $z_2=2.0$

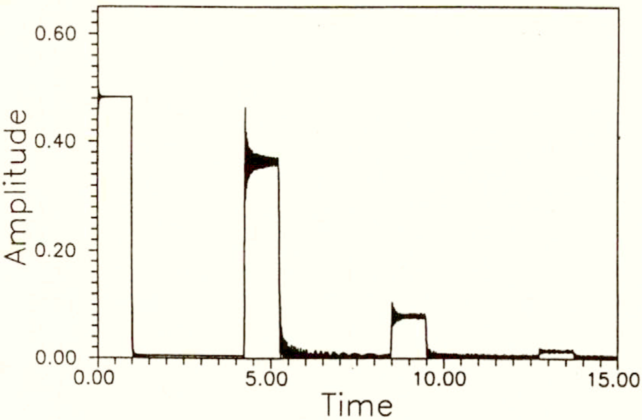

Fig.11

Homogenized bar, $z_1=4.0$, $z_2=2.0$ ($a=2.12132$, $b=2.828427$)

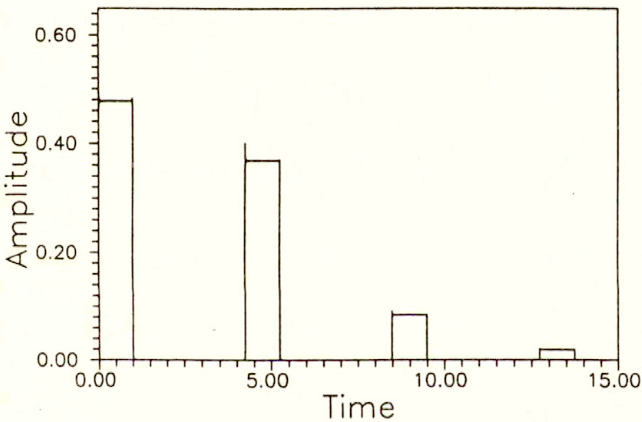

Fig.12