Contents of issue 3 vol. XLV

Polish Academy of Sciences

Institute of Fundamental Technological Research

Archives of Mechanics

Archiwum Mechaniki Stosowanej

volume 45 issue 3

Polish Scientific Publishers PWN

Warszawa 1993

ARCHIVES OF MECHANICS IS DEVOTED TO Theory of elasticity and plasticity • Theory of nonclassical continua • Physics of continuous media • Mechanics of discrete media • Nonlinear mechanics • Rheology • Fluid gas-mechanics • Rarefied gases • Thermodynamics

Printed in Poland, Editorial Office: Świętokrzyska 21,
00-049 Warszawa (Poland)

Wydanie I. Arkuszy drukarskich 7,25. Papier offset. kl. III 80 g. B1.
Druk ukończono w grudniu 1993 r.
Skład: *ArtGraph*
Druk i oprawa: Drukarnia Braci Grodzickich
Żabieniec k/Piaseczna, ul. Przelotowa 7

Arch. Mech., 45, 3, pp. 271–277, Warszawa 1993

Diffraction of interface waves by a bottom deformation

B. N. MANDAL and U. BASU (CALCUTTA)

THIS PAPER is concerned with diffraction of a train of time-harmonic progressive waves propagating at the interface between two laterally unbounded superposed fluids in the presence of a small cylindrical deformation of the bottom of the lower fluid, the upper fluid being extended infinitely upwards. A perturbational analysis is employed to obtain the first order correction to the transmission and reflection coefficients. In the absence of the upper fluid, known results for a single fluid are recovered.

1. Introduction

DUE TO THE PRESENCE of bottom deformation, a train of surface waves propagating in a laterally unbounded ocean experiences reflection by and transmission over the bottom. The reflection and transmission coefficients can be obtained approximately by a perturbational analysis assuming the deformation to be small (cf. MILES [1], MANDAL and BASU [2]). In the present paper this problem is generalised to two laterally unbounded superposed fluids, in which the upper fluid extends infinitely upwards and the lower fluid is of finite depth below the mean interface and has a small deformation in the form of a long cylinder extending in the lateral direction. A train of time-harmonic progressive interface waves is normally incident on the bottom deformation. It is then partially reflected and partially transmitted. By using the perturbational analysis employed by MANDAL and BASU [2] for the one-fluid case, the problem is reduced, up to the first order terms to a boundary value problem (BVP) whose solution is obtained by a Fourier transform. The first order corrections to the transmission and reflection coefficients are obtained. In the absence of the upper fluid, known results for a single fluid are recovered.

It may be mentioned here that, although two-fluid problems are not difficult to formulate mathematically within the framework of the linearised theory, the literature on two-fluid problems is rather limited. GORGOUI and KASSEM [3], RHODES-ROBINSON [4], KASSEM [5,6] studied generation of time-harmonic waves at the interface between two superposed fluids due to various types of basic singularities submerged in either of the two fluids. GORGOUI and KASSEM [7] also studied generation of short internal waves by a cylinder oscillating at the surface separating two infinite fluids. While the methods employed in [3, 5-7] are extensions of methods used in the corresponding one-fluid problems, RHODES-ROBINSON [4] used a splitting approach to handle a class of linearised interface wave problems for two superposed fluids occupying regions which are symmetric about their common interface. MANDAL and BANERJEA [8] recently used this splitting approach to solve the plane vertical wavemaker problem for two superposed infinite fluids. However, this approach is suitable only when the two fluids and the boundary conditions are symmetric about their common interface, and as such it cannot be applied if the lower fluid is of finite depth below the mean interface and the upper fluid extends infinitely upwards as is the case for the present problem. After formulating the problem mathematically as a BVP, Fourier transform technique is used to solve it approximately following the use of a perturbational method.

2. Statement and formulation of the problem

We consider two incompressible, inviscid and homogeneous fluids of densities ϱ_1 and ϱ_2 ($< \varrho_1$) with lower fluid of density ϱ_1 occupying the region $0 \leq y \leq h + \varepsilon c(x)$, $-\infty < x < \infty$ and upper fluid of density ϱ_2 in the region $y \leq 0, -\infty < x < \infty$, the (x, z)-plane being taken as the interface at rest and y-axis being taken vertically downwards. There is a small cylindrical deformation at the bottom of the lower fluid along the lateral boundary, and it is represented by

$$y = h + \varepsilon c(x),$$

where $c(x)$ is a continuous function of compact support so that $c(x) \to 0$ as $|x| \to \infty$, and ε is a small dimensionless quantity giving a measure of smallness of the bottom deformation. Far away from the deformation, the lower fluid is of uniform finite depth h below the mean interface.

Assuming linear theory and irrotational motion, a train of time-harmonic progressive interface waves can be represented by the undisturbed velocity potentials $\mathrm{Re}\{\varphi_0(x, y)e^{-i\sigma t}\}$ in the lower fluid and by $\mathrm{Re}\{\Psi_0(x, y)e^{-i\sigma t}\}$ in the upper fluid, where

$$\varphi_0(x, y) = \frac{\cosh k_0(h - y)}{\sinh k_0 h} e^{ik_0 x}, \tag{2.1}$$

$$\Psi_0(x, y) = -e^{k_0 y + ik_0 x}, \tag{2.2}$$

σ is the circular frequency, k_0 is the unique positive real zero of

$$\Delta(k) = K(\cosh kh + s \sinh kh) - (1 - s)k \sinh kh, \tag{2.3}$$

$K = \sigma^2/g$, g being the acceleration due to gravity and $s = \varrho_2/\varrho_1$. This wave train is normally incident upon the deformation. It then experiences reflection and transmission. The motions in the lower and upper fluids are described respectively by the velocity potentials $\mathrm{Re}\{\varphi(x, y)e^{-i\sigma t}\}$ and $\mathrm{Re}\{\Psi(x, y)e^{-i\sigma t}\}$. Then φ, Ψ satisfy the following coupled BVP:

$$\begin{aligned} &\nabla^2\varphi = 0, \quad 0 \leq y \leq h + \varepsilon c(x), \\ &\nabla^2\Psi = 0, \quad y \leq 0, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\varphi_y = \Psi_y \quad \text{on} \quad y = 0, \\ &K\varphi + \varphi_y = s(K\Psi + \Psi_y) \quad \text{on} \quad y = 0, \end{aligned} \tag{2.5}$$

$$\varphi_n = 0 \quad \text{on} \quad y = h + \varepsilon c(x), \tag{2.6}_1$$

where n denotes the inward normal to the bottom,

$$\nabla\Psi \to 0 \quad \text{as} \quad y \to -\infty. \tag{2.6}_2$$

Also, φ, Ψ satisfy the requirements at infinity that

$$\varphi \sim \begin{cases} T\varphi_0(x, y) & \text{as } x \to \infty, \\ \varphi_0(x, y) + R\varphi_0(-x, y) & \text{as } x \to -\infty \end{cases} \tag{2.7}_1$$

and

$$\Psi \sim \begin{cases} T\Psi_0(x, y) & \text{as } x \to \infty, \\ \Psi_0(x, y) + T\Psi_0(-x, y) & \text{as } x \to -\infty, \end{cases} \tag{2.7}_2$$

where T and R are, respectively, the transmission and reflection coefficients and have to be found.

Now the condition $(2.6)_1$ can be approximated up to the first order of ε (cf. [2]) as

$$-\varphi_y + \varepsilon \frac{d}{dx} c(x)\varphi_x + O(\varepsilon^2) = 0 \quad \text{on} \quad y = h\,. \tag{2.8}$$

We also note that in the absence of any deformation of the bottom, the interface wave train propagates without any hindrance and there is a total transmission. In view of this and of the approximate condition (2.8) we can assume a perturbation expansion for φ, Ψ, T, R in terms of ε as

$$\begin{aligned} \varphi(x,y) &= \varphi_0(x,y) + \varepsilon\varphi_1(x,y) + O(\varepsilon^2)\,,\\ \Psi(x,y) &= \Psi_0(x,y) + \varepsilon\Psi_1(x,y) + O(\varepsilon^2)\,,\\ T &= 1 + \varepsilon T_1 + O(\varepsilon^2)\,,\\ R &= \varepsilon R_1 + O(\varepsilon^2)\,, \end{aligned} \tag{2.9}$$

where φ_0, Ψ_0 are given in Eqs. (2.1) and (2.2), respectively. Utilising Eq. (2.9) in Eqs. (2.4), (2.5), (2.8), $(2.6)_2$, (2.7) we find that φ_1 and Ψ_1 satisfy

$$\begin{aligned} \nabla^2\varphi_1 &= 0, \quad 0 \le y \le h\,,\\ \nabla^2\Psi_1 &= 0, \quad y \le 0\,, \end{aligned} \tag{2.10}$$

$$\begin{aligned} \varphi_{1y} &= \Psi_{1y} \quad \text{on} \quad y = 0\,,\\ K\varphi_1 + \varphi_{1y} &= s(K\Psi_1 + \Psi_{1y}) \quad \text{on} \quad y = 0\,, \end{aligned} \tag{2.11}$$

$$\varphi_{1y} = \frac{ik_0}{\sinh k_0 h}\frac{d}{dx}\{c(x)e^{ik_0x}\} \equiv q(x), \quad \text{say, on} \quad y = h\,, \tag{2.12}$$

$$\nabla\Psi_1 \to 0 \quad \text{as} \quad y \to -\infty\,,$$

$$\varphi_1 \sim \begin{cases} T_1\varphi_0(x,y) & \text{as } x \to \infty\,,\\ R_1\varphi_0(-x,y) & \text{as } x \to -\infty \end{cases} \tag{$2.13)_1$}$$

and

$$\Psi_1 \sim \begin{cases} T_1\Psi_0(x,y) & \text{as } x \to \infty\,,\\ R_1\Psi_0(-x,y) & \text{as } x \to -\infty\,. \end{cases} \tag{$2.13)_2$}$$

It may be noted that T_1 and R_1 are the first-order corrections to the transmission and reflection coefficients and will be found in the sequel.

3. Solution for φ_1 and Ψ_1

The BVP for φ_1 and Ψ_1 described by Eqs. (2.10) to (2.13) is decoupled into two independent BVP's for φ_1 and Ψ_1 as follows:

BVP-I. To find φ_1 satisfying

$$\nabla^2\varphi_1 = 0, \quad 0 \le y \le h\,, \tag{3.1}$$

$$\varphi_{1y} = \eta(x) \quad \text{on} \quad y = 0\,, \tag{3.2}$$

where $\eta(x)$ is assumed to be known for the present,

$$\varphi_{1y} = q(x) \quad \text{on} \quad y = h\,, \tag{3.3}$$

and φ_1 satisfies the requirements at infinity that

$$\varphi_1 \sim \begin{cases} T_1\varphi_0(x,y) & \text{as} \quad x \to \infty\,, \\ R_1\varphi_0(-x,y) & \text{as} \quad x \to -\infty\,. \end{cases} \tag{3.4}$$

BVP-II. To find Ψ_1 satisfying

$$\nabla^2\Psi_1 = 0, \quad y \le 0\,, \tag{3.5}$$

$$\Psi_{1y} = \eta(x) \quad \text{on} \quad y = 0\,, \tag{3.6}$$

where $\eta(x)$ is the same as in the condition (3.2),

$$\nabla\Psi_1 \to 0 \quad \text{as} \quad y \to -\infty\,, \tag{3.7}$$

and Ψ_1 satisfies the infinity requirements that

$$\Psi_1 \sim \begin{cases} T_1\Psi_0(x,y) & \text{as } x \to \infty\,, \\ R_1\Psi_0(-x,y) & \text{as } x \to -\infty\,. \end{cases} \tag{3.8}$$

The condition $(2.11)_2$ takes the form

$$K(\varphi_1 - s\Psi_1) = -(1-s)\eta(x) \quad \text{on} \quad y = 0\,. \tag{3.9}$$

In order to solve the equation for φ_1 and Ψ_1 we use the Fourier transform defined by

$$\begin{aligned} \overline{\varphi}_1(\xi,y) &= \int_{-\infty}^{\infty} \varphi_1(x,y)e^{-i\xi x}\,dx\,, \\ \overline{\Psi}_1(\xi,y) &= \int_{-\infty}^{\infty} \Psi_1(x,y)e^{-i\xi x}\,dx\,. \end{aligned} \tag{3.10}$$

We also assume k_0 to have a small positive imaginary part so that φ_1, Ψ_1 decrease exponentially as $|x| \to \infty$. This ensures the existence of Fourier transforms of φ_1 and Ψ_1. Now $\overline{\varphi}_1(\xi,y)$ satisfies the BVP

$$\begin{aligned} &\overline{\varphi}_{1yy} - \xi^2\overline{\varphi}_1 = 0, \quad 0 \le y \le h\,, \\ &\overline{\varphi}_{1y} = \overline{\eta}(\xi) \quad \text{on} \quad y = 0\,, \\ &\overline{\varphi}_{1y} = \overline{q}(\xi) \quad \text{on} \quad y = h\,, \end{aligned} \tag{3.11}$$

where $\overline{\eta}$ and $\overline{q}$ are the Fourier transforms on $\eta(x)$ and $q(x)$, respectively. The $\overline{\varphi}_1$ has the solution given by

$$\overline{\varphi}_1(\xi,y) = \frac{\overline{q}(\xi)\cosh\xi y - \overline{\eta}(\xi)\cosh\xi(h-y)}{\xi\sinh\xi h}\,. \tag{3.12}$$

We note that the zeros of $\xi\sinh\xi h$ are not the poles of $\overline{\varphi}_1$.

Again, $\overline{\Psi}_1(\xi,y)$ satisfies

$$\begin{aligned} &\overline{\Psi}_{1yy} - \xi^2\overline{\Psi}_1 = 0, \quad y \le 0\,, \\ &\overline{\Psi}_{1y} = \overline{\eta}(\xi) \quad \text{on} \quad y = 0\,, \\ &\overline{\Psi}_{1y} \to 0 \quad \text{as} \quad y \to -\infty\,. \end{aligned} \tag{3.13}$$

Then $\overline{\Psi}_1$ has the solution given by

$$\overline{\Psi}_1(\xi,y) = \frac{\overline{\eta}(\xi)e^{|\xi|y}}{|\xi|}\,. \tag{3.14}$$

To find $\overline{\eta}$, we take the Fourier transform of Eq. (3.9) and obtain

$$K(\overline{\varphi}_1 - s\overline{\Psi}_1) = -(1-s)\overline{\eta} \quad \text{on} \quad y = 0\,. \tag{3.15}$$

Substituting Eqs. (3.12) and (3.14) in Eq. (3.15) we find

$$\overline{\eta}(\xi) = \frac{K\overline{q}(\xi)}{A(\xi)}\,, \tag{3.16}$$

where

$$A(\xi) = K(\cosh\xi h + s\,\mathrm{sgn}\,\xi \sinh\xi h) - (1-s)\xi\sinh\xi h\,. \tag{3.17}$$

Thus we obtain

$$\overline{\varphi}(\xi, y) = \frac{\overline{q}(\xi)}{\xi\sinh\xi h}\left\{\cosh\xi y - \frac{K}{A(\xi)}\cosh\xi(h-y)\right\} \tag{3.18}$$

and

$$\overline{\Psi}_1(\xi, y) = \frac{K\overline{q}(\xi)}{|\xi|A(\xi)}e^{|\xi|y}\,. \tag{3.19}$$

It may be noted that in Eq. (3.19), $\xi = 0$ is not a singularity because $\overline{q}(0) = 0$ since $c(x) \to 0$ as $|x| \to \infty$.

To find the Fourier inversion of Eqs. (3.18) and (3.19) we note that on the real axis, $A(\xi)$ has zeros at $\pm k_0$. As k_0 is assumed to have a small positive imaginary part, we write

$$\varphi_1(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\varphi}_1(\xi, y)e^{i\xi x}\,d\xi\,, \tag{3.20}$$

$$\Psi_1(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\Psi}_1(\xi, y)e^{i\xi x}\,d\xi\,, \tag{3.21}$$

where the path in each integral is slightly indented below the point $\xi = k_0$ and above $\xi = -k_0$. Then using Eqs. (3.18) and (3.19) we find

$$\begin{aligned}\varphi_1(x, y) = {} & \frac{1}{2\pi}\int_0^{\infty}\frac{\overline{q}(\xi)e^{i\xi x}}{\xi\sinh\xi h}\left\{\cosh\xi y - \frac{K}{\Delta(\xi)}\cosh\xi(h-y)\right\}d\xi \\ & + \frac{1}{2\pi}\int_0^{\infty}\frac{\overline{q}(-\xi)e^{-i\xi x}}{\xi\sinh\xi h}\left\{\cosh\xi y - \frac{K}{\Delta(\xi)}\cosh\xi(h-y)\right\}d\xi\end{aligned} \tag{3.22}$$

and

$$\Psi_1(x, y) = \frac{K}{2\pi}\left[\int_0^{\infty}\frac{\overline{q}(\xi)e^{\xi y + i\xi x}}{\xi\Delta(\xi)}d\xi + \int_0^{\infty}\frac{\overline{q}(-\xi)e^{\xi y - i\xi x}}{\xi\Delta(\xi)}\right], \tag{3.23}$$

where the path in each integral is indented below the pole at $\xi = k_0$.

To calculate T_1 we make $x \to \infty$ in either $\varphi_1(x, y)$ or $\Psi_1(x, y)$. As $x \to \infty$, the behaviour of φ_1 or Ψ_1 can be obtained from Eq. (3.22) or Eq. (3.23) by rotating the path of the first integrals into a contour in the first quadrant so that we must include the residue term at $\xi = k_0$. The path of the second integrals in Eq. (3.22) or Eq. (3.23) is rotated into a contour in the fourth quadrant, so that these integrals do not contribute as

$x \to \infty$. Then comparing with Eq. $(2.13)_1$ or Eq. $(2.13)_2$ we find

$$T_1 = -\frac{iK\bar{q}(k_0)}{k_0\Delta'(k_0)}.$$

But

$$\bar{q}(k_0) = -\frac{k_0^2}{\sinh^2 k_0 h}\int_{-\infty}^{\infty} c(x)\,dx,$$

so that we obtain

$$T_1 = -\frac{ik_0}{h + \dfrac{(1-s)\sinh^2 k_0 h}{K}}\int_{-\infty}^{\infty} c(x)\,dx. \tag{3.24}$$

In a similar way R_1 is obtained from the analysis of the behaviour of φ_1 or Ψ_1 in Eq. (3.22) or Eq. (3.23) as $x \to -\infty$ by rotating the path of the second integrals into a contour in the first quadrant, so that we must include the residue term at $\xi = k_0$. The path of the first integrals in Eq. (3.22) or Eq. (3.23) is rotated into a contour in the fourth quadrant so that these integrals do not contribute as $x \to -\infty$. Then comparing with Eq. $(2.13)_1$ or Eq. $(2.13)_2$ we find

$$R_1 = -\frac{iK\bar{q}(-k_0)}{k_0\Delta'(k_0)}.$$

But

$$\bar{q}(-k_0) = \frac{k_0^2}{\sinh^2 k_0 h}\int_{-\infty}^{\infty} c(x)e^{2ik_0x}\,dx,$$

so that we obtain

$$R_1 = \frac{ik_0}{h + \dfrac{(1-s)\sinh^2 k_0 h}{K}}\int_{-\infty}^{\infty} c(x)e^{2ik_0x}\,dx. \tag{3.25}$$

Once the functional form of the shape function $c(x)$ is known, T_1 and R_1 can be obtained explicitly.

4. Conclusion

The first order corrections to the reflection and transmission coefficients for the problem of diffraction of interface waves propagating at the interface of two superposed fluids in the presence of a small bottom deformation, was obtained by a perturbational technique. In the absence of the upper fluid ($s = 0$), the results for a single fluid are recovered. In particular, $|R_1|$ coincides with KREISEL'S [9] result. These results can be extended to include the effect of interfacial tension and also for oblique incidence of the interface wave train.

Acknowledgement

This work is partially supported by the CSIR research project No. 25 41/88/EMR II through the Calcutta Mathematical Society.

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Received June 2, 1992.

Arch. Mech., 45, 3, pp. 279–284, Warszawa 1993

Variational principles for the vibration of an elastic dielectric

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THIS PAPER presents two variational principles for the free vibration of a finite elastic dielectric with linear piezoelectromagnetism (dynamic piezoelectricity). They are in Rayleigh quotient form for the natural frequency. The variational principles are mixed in the sense that all field variables can be varied independently.

1. Introduction

THE THEORY of piezoelectromagnetism has fully dynamic electromagnetic fields. Reciprocity, uniqueness and minimum principles have been proved in [1]. The vibration of piezoelectromagnetic plates has been studied [2, 3] to consider the effect of radiation which is a phenomenon not included in the widely used quasi-static theory of piezoelectricity. A variational principle for piezoelectromagnetism is given in [4], which can be used to derive field equations for piezoelectromagnetism. A mixed variational principle for the field equations of piezoelectromagnetism is given in [5].

For linear quasi-static piezoelectricity, various variational principles for the natural frequency of free vibration has been summarized and developed systematically in [6]. It was shown that for each Legendre transform of the internal energy there exists a variational principle.

In this paper, two variational principles for the free vibration of a finite elastic dielectric with linear piezoelectromagnetism are presented. They are in Rayleigh quotient form for the natural frequency. The variational principles are mixed in the sense that all field variables can be varied independently. These variational principles generalize indirectly some of those in [6].

2. Governing equations

Let the finite spatial region occupied by the piezoelectromagnetic elastic dielectric be V, the boundary surface of V be S, the unit outward normal of S be n_i, and S be partitioned in the following ways

$$S_u \cup S_T = S_E \cup S_H = S,$$
$$S_u \cap S_T = S_E \cap S_H = 0.$$

The governing equations for the motion of a finite piezoelectromagnetic body in V with homogeneous boundary conditions on S are [4]

$$\begin{aligned} &T_{ji,j} = \rho \ddot{u}_i, \quad S_{ij} - \tfrac{1}{2}(u_{j,i} + u_{i,j}) = 0 \quad \text{in } V, \\ &D_{i,i} = 0, \quad -\varepsilon_{ijk} H_{k,j} = -\dot{D}_i \quad \text{in } V, \\ &B_{i,i} = 0, \quad -\varepsilon_{ijk} E_{k,j} = \dot{B}_i \quad \text{in } V, \end{aligned} \tag{2.1}$$

(2.1) [cont.]

$$
\begin{aligned}
&T_{ij} - (c_{ijkl}S_{kl} - e_{kij}E_k) = 0 \quad \text{in } V\,,\\
&-D_i - (-e_{ikl}S_{kl} - \varepsilon_{ik}E_k) = 0 \quad \text{in } V\,,\\
&H_i - \frac{1}{\mu_0}B_i = 0 \quad \text{in } V\,,\\
&u_i = 0 \quad \text{on } S_u, \qquad -T_{ji}n_j = 0 \quad \text{on } S_T\,,\\
&B_i n_i = 0, \qquad -\varepsilon_{ijk}n_j E_k = 0 \quad \text{on } S_E\,,\\
&D_i n_i = 0, \qquad -\varepsilon_{ijk}n_j H_k = 0 \quad \text{on } S_H\,,
\end{aligned}
$$

where ρ is mass density, T_{ij} stress, S_{ij} strain, u_i displacement, E_i electric field, D_i electric displacement, B_i magnetic induction, H_i magnetic field; μ_0 is the magnetic permeability of free space; c_{ijkl}, e_{ijkl} and ε_{ij} are all material constants; ε_{ijk} is the permutation tensor.

We note that the homogeneous electromagnetic boundary conditions represent the two common boundary conditions [7] of short circuit boundaries (electric wall, on which tangential **E** and normal **B** vanish) and open circuit boundary (magnetic wall, on which tangential **H** and normal **D** vanish).

For time harmonic motions, let

$$
\begin{aligned}
u_i(x,t) &= u_i(x)\cos\omega t, &&\\
T_{ij}(x,t) &= T_{ij}(x)\cos\omega t, \quad & S_{ij}(x,t) &= S_{ij}(x)\cos\omega t\,,\\
E_i(x,t) &= E_i(x)\cos\omega t, \quad & D_i(x,t) &= D_i(x)\cos\omega t\,,\\
H_i(x,t) &= H_i(x)\sin\omega t, \quad & B_i(x,t) &= B_i(x)\sin\omega t\,.
\end{aligned}
$$

Then Eqs. (2.1) becomes

(2.2)

$$
\begin{aligned}
&T_{ji,j} = -\omega^2\rho u_i, \qquad S_{ij} - \tfrac{1}{2}(u_{j,i} + u_{i,j}) = 0 \quad \text{in } V\,,\\
&D_{i,i} = 0, \qquad -\varepsilon_{ijk}H_{k,j} = \omega D_i \quad \text{in } V\,,\\
&B_{i,i} = 0, \qquad -\varepsilon_{ijk}E_{k,j} = \omega B_i \quad \text{in } V\,,\\
&T_{ij} - (c_{ijkl}S_{kl} - e_{kij}E_k) = 0 \quad \text{in } V\,,\\
&-D_i - (-e_{ikl}S_{kl} - \varepsilon_{ik}E_k) = 0 \quad \text{in } V\,,\\
&H_i - \frac{1}{\mu_0}B_i = 0 \quad \text{in } V\,,\\
&u_i = 0 \quad \text{on } S_u, \qquad -T_{ji}n_j = 0 \quad \text{on } S_T\,,\\
&B_i n_i = 0, \qquad -\varepsilon_{ijk}n_j E_k = 0 \quad \text{on } S_E\,,\\
&D_i n_i = 0, \qquad -\varepsilon_{ijk}n_j H_k = 0 \quad \text{on } S_H\,.
\end{aligned}
$$

Values of ω are sought from the conditions corresponding to which nontrivial solutions of u_i, v_i, S_{ij}, T_{ij}, E_i, D_i, H_i and B_i exist. Hence Eqs. (2.2) constitutes an eigenvalue problem. We note that in Eqs. (2.2) the two curl Maxwell's equations imply the two divergence Maxwell's equations. Similarly, the two tangential electromagnetic boundary conditions imply the two normal electromagnetic boundary conditions. For Eqs. (2.2), it is convenient to introduce the electric enthalpy density function [4]

$$
H(\mathbf{S},\mathbf{E},\mathbf{B}) = \tfrac{1}{2}c_{ijkl}S_{ij}S_{kl} - e_{ijk}E_iS_{jk} - \tfrac{1}{2}\varepsilon_{ij}E_iE_j + \tfrac{1}{2}\mu_0^{-1}B_iB_i \quad \text{in } V\,.
$$

Then Eqs. (2.2) can be written as

$$
\begin{gathered}
T_{ji,j} = -\omega^2 \rho u_i, \quad S_{ij} - \tfrac{1}{2}(u_{j,i} + u_{i,j}) = 0 \quad \text{in } V, \\
-\varepsilon_{ijk} H_{k,j} = \omega D_i, \quad -\varepsilon_{ijk} E_{k,j} = \omega B_i \quad \text{in } V, \\
T_{ij} - \frac{\partial H}{\partial S_{ij}} = 0, \quad -D_i - \frac{\partial H}{\partial E_i} = 0, \quad H_i - \frac{\partial H}{\partial B_i} = 0 \quad \text{in } V, \\
u_i = 0 \quad \text{on } S_u, \quad -T_{ji} n_j = 0 \quad \text{on } S_T, \\
-\varepsilon_{ijk} n_j E_k = 0 \quad \text{on } S_E, \quad -\varepsilon_{ijk} n_j H_k = 0 \quad \text{on } S_H.
\end{gathered} \tag{2.3}
$$

We now introduce the enthalpy function M from H through Legendre transform as follows,

$$
M(\mathbf{T}, \mathbf{E}, \mathbf{H}) = H(\mathbf{S}, \mathbf{E}, \mathbf{B}) - T_{ij} S_{ij} - H_i B_i,
$$

which generates the constitutive relations in the following form:

$$
S_{ij} = -\frac{\partial M}{\partial T_{ij}}, \quad D_i = -\frac{\partial M}{\partial E_i}, \quad B_i = -\frac{\partial M}{\partial H_i}
$$

and Eqs. (2.3) can be written as

$$
\begin{gathered}
T_{ji,j} = -\omega^2 \rho u_i, \quad S_{ij} - \tfrac{1}{2}(u_{j,i} + u_{i,j}) = 0 \quad \text{in } V, \\
-\varepsilon_{ijk} H_{k,j} = \omega D_i, \quad -\varepsilon_{ijk} E_{k,j} = \omega B_i \quad \text{in } V, \\
S_{ij} + \frac{\partial M}{\partial T_{ij}} = 0, \quad D_i + \frac{\partial M}{\partial E_i} = 0, \quad B_i + \frac{\partial M}{\partial H_i} = 0 \quad \text{in } V, \\
u_i = 0 \quad \text{on } S_u, \quad -T_{ji} n_j = 0 \quad \text{on } S_T, \\
-\varepsilon_{ijk} n_j E_k = 0 \quad \text{on } S_E, \quad -\varepsilon_{ijk} n_j H_k = 0 \quad \text{on } S_H,
\end{gathered} \tag{2.4}
$$

or, eliminating S_{ij}, D_i and B_i

$$
\begin{gathered}
T_{ji,j} = -\omega^2 \rho u_i, \quad -\tfrac{1}{2}(u_{j,i} + u_{i,j}) = \frac{\partial M}{\partial T_{ij}} \quad \text{in } V, \\
\varepsilon_{ijk} H_{k,j} = \omega \frac{\partial M}{\partial E_i}, \quad \varepsilon_{ijk} E_{k,j} = \omega \frac{\partial M}{\partial H_i} \quad \text{in } V, \\
u_i = 0 \quad \text{on } S_u, \quad -T_{ji} n_j = 0 \quad \text{on } S_T, \\
-\varepsilon_{ijk} n_j E_k = 0 \quad \text{on } S_E, \quad -\varepsilon_{ijk} n_j H_k = 0 \quad \text{on } S_H.
\end{gathered} \tag{2.5}
$$

Next we introduce (when $\omega \neq 0$) $v_i = \omega u_i$ so that Eqs. (2.5) can be written as a system as follows, which explicitly contains ω only, not ω^2:

$$
\begin{gathered}
T_{ji,j} = -\omega \rho v_i, \quad -\tfrac{1}{2}(v_{j,i} + v_{i,j}) = \omega \frac{\partial M}{\partial T_{ij}} \quad \text{in } V, \\
\varepsilon_{ijk} H_{k,j} = \omega \frac{\partial M}{\partial E_i}, \quad \varepsilon_{ijk} E_{k,j} = \omega \frac{\partial M}{\partial H_i} \quad \text{in } V, \\
v_i = 0 \quad \text{on } S_u, \quad -T_{ji} n_j = 0 \quad \text{on } S_T, \\
-\varepsilon_{ijk} n_j E_k = 0 \quad \text{on } S_E, \quad -\varepsilon_{ijk} n_j H_k = 0 \quad \text{on } S_H.
\end{gathered} \tag{2.6}
$$

3. A variational principle

In this section, we will give a variational formulation for the eigenvalue problem (2.6). Different to the variational formulations for the quasi-static case [6] which are for ω^2, the following variational principle is for ω. This is consistent with the corresponding variational principle for pure electromagnetic fields of a finite body [8].

Generally, for a fractional functional

$$\Pi = \frac{\Lambda}{\Gamma},$$

we have

$$\delta\Pi = \frac{1}{\Gamma^2}(\Gamma\delta\Lambda - \Lambda\delta\Gamma) = \frac{1}{\Gamma}(\delta\Lambda - \Pi\delta\Gamma).$$

Therefore $\delta\Pi = 0$ implies

$$\delta\Lambda - \Pi\delta\Gamma = 0.$$

We define

$$\begin{aligned}\Lambda_1(\mathbf{v},\mathbf{T},\mathbf{E},\mathbf{H}) = &\int_V (T_{ji,j}v_i + \tfrac{1}{2}\varepsilon_{ijk}H_{k,j}E_i + \tfrac{1}{2}\varepsilon_{ijk}E_{k,j}H_i)dV \\ &- \int_{S_T} T_{ji}n_j v_i dS - \int_{S_E} \tfrac{1}{2}\varepsilon_{ijk}n_j E_k H_i dS - \int_{S_H} \tfrac{1}{2}\varepsilon_{ijk}n_j H_k E_i dS \\ = &\int_V (T_{ji,j}v_i + \varepsilon_{ijk}H_{k,j}E_i)dV - \int_{S_T} T_{ji}n_j v_i dS - \int_{S_H} \varepsilon_{ijk}n_j H_k E_i dS \\ = &\int_V (T_{ji,j}v_i + \varepsilon_{ijk}E_{k,j}H_i)dV - \int_{S_T} T_{ji}n_j v_i dS - \int_{S_E} \varepsilon_{ijk}n_j E_k H_i dS,\end{aligned}$$

$$\Gamma_1(\mathbf{v},\mathbf{T},\mathbf{E},\mathbf{H}) = \int_V [-\tfrac{1}{2}\rho v_i v_i + M(\mathbf{T},\mathbf{E},\mathbf{H})]dV,$$

$$\Pi_1(\mathbf{v},\mathbf{T},\mathbf{E},\mathbf{H}) = \frac{\Lambda_1}{\Gamma_1}.$$

Then we have, after integration by parts,

$$\begin{aligned}\delta\Lambda_1 = &\int_V [T_{ji,i}\delta v_i - \tfrac{1}{2}(v_{j,i} + v_{i,j})\delta T_{ij} + \varepsilon_{ijk}H_{k,j}\delta E_i + \varepsilon_{ijk}E_{k,j}\delta H_i]dV \\ &+ \int_{S_u} v_i\delta T_{ji}n_j dS - \int_{S_T} T_{ji}n_j\delta v_i dS - \int_{S_E} \varepsilon_{ijk}n_j E_k\delta H_i dS - \int_{S_H} \varepsilon_{ijk}n_j H_k\delta E_i dS, \\ \delta\Gamma_1 = &\int_V \left(-\rho v_i\delta v_i + \frac{\partial M}{\partial T_{ij}}\delta T_{ij} + \frac{\partial M}{\partial E_i}\delta E_i + \frac{\partial M}{\partial H_i}\delta H_i\right)dV.\end{aligned}$$

Therefore, $\delta\Pi_1 = 0$ implies

$$\begin{aligned}&T_{ji,j} = -\Pi_1\rho v_i, \quad -\tfrac{1}{2}(v_{j,i} + v_{i,j}) = \Pi_1\frac{\partial M}{\partial T_{ij}} \quad \text{in } V, \\ &\varepsilon_{ijk}H_{k,j} = \Pi_1\frac{\partial M}{\partial E_i}, \quad \varepsilon_{ijk}E_{k,j} = \Pi_1\frac{\partial M}{\partial H_i} \quad \text{in } V,\end{aligned} \tag{3.1}$$

(3.1) [cont.)
$$v_i = 0 \quad \text{on } S_u, \qquad -T_{ji}n_j = 0 \quad \text{on } S_T,$$
$$-\varepsilon_{ijk}n_jE_k = 0 \quad \text{on } S_E, \qquad -\varepsilon_{ijk}n_jH_k = 0 \quad \text{on } S_H.$$

Comparing (3.1) with (2.6), we conclude that the stationarity condition of Π_1 gives the eigenvalue problem (2.6) with the stationary value of Π_1 as ω. This variational formulation is of mixed type in the sense that various mechanical and electromagnetic fields can vary independently and there are no constraints.

4. A more general variational principle

In [6] for quasi-static piezoelectricity, it was shown that for each Legendre transform of the electric enthalpy function H there exists a variational principle. The situation is similar for dynamic piezoelectricity. We will show another variational principle in the following and will not try to exhaust ali possible cases. First we introduce the internal energy function U

$$U(\mathbf{S}, \mathbf{D}, \mathbf{B}) = H(\mathbf{S}, \mathbf{E}, \mathbf{B}) + E_iD_i = M(\mathbf{T}, \mathbf{E}, \mathbf{H}) + T_{ij}S_{ij} + E_iD_i + H_iB_i,$$

which generates the constitutive relations in the following form:

$$T_{ij} = \frac{\partial U}{\partial S_{ij}}, \qquad E_i = \frac{\partial U}{\partial D_i}, \qquad H_i = \frac{\partial U}{\partial B_i}.$$

Then we consider the following functionals:

$$\begin{aligned}
\Lambda_2(\mathbf{v}, \mathbf{T}, \mathbf{S}, \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}) &= \int_V (T_{ji,j}v_i + \tfrac{1}{2}\varepsilon_{ijk}H_{k,j}E_i + \tfrac{1}{2}\varepsilon_{ijk}E_{k,j}H_i)dV \\
&\quad - \int_{S_T} T_{ji}n_jv_idS - \int_{S_E} \tfrac{1}{2}\varepsilon_{ijk}n_jE_kH_idS - \int_{S_H} \tfrac{1}{2}\varepsilon_{ijk}n_jH_kE_idS \\
&= \int_V (T_{ji,j}v_i + \varepsilon_{ijk}H_{k,j}E_i)dV - \int_{S_T} T_{ji}n_jv_idS - \int_{S_H} \varepsilon_{ijk}n_jH_kE_idS \\
&= \int_V (T_{ji,j}v_i + \varepsilon_{ijk}E_{k,j}H_i)dV - \int_{S_T} T_{ji}n_jv_idS - \int_{S_E} \varepsilon_{ijk}n_jE_kH_idS,
\end{aligned}$$

$$\Gamma_2(\mathbf{v}, \mathbf{T}, \mathbf{S}, \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}) = \int_V [-\tfrac{1}{2}\rho v_iv_i - T_{ij}S_{ij} - E_iD_i - H_iB_i + U(\mathbf{S}, \mathbf{D}, \mathbf{B})]dV,$$

$$\Pi_2(\mathbf{v}, \mathbf{T}, \mathbf{S}, \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}) = \frac{\Lambda_2}{\Gamma_2}.$$

We have, after integration by parts,

$$\begin{aligned}
\delta\Lambda_2 &= \int_V [T_{ji,i}\delta v_i - \tfrac{1}{2}(v_{j,i} + v_{i,j})\delta T_{ij} + \varepsilon_{ijk}H_{k,j}\delta E_i + \varepsilon_{ijk}E_{k,j}\delta H_i]dV \\
&\quad + \int_{S_u} v_i\delta T_{ji}n_jdS - \int_{S_T} T_{ji}n_j\delta v_idS - \int_{S_E} \varepsilon_{ijk}n_jE_k\delta H_idS - \int_{S_H} \varepsilon_{ijk}n_jH_k\delta E_idS,
\end{aligned}$$

$$\delta\Gamma_2 = \int_V \Big(-\rho v_i \delta v_i - T_{ij}\delta S_{ij} - S_{ij}\delta T_{ij} - E_i \delta D_i - D_i \delta E_i - H_i \delta B_i - B_i \delta H_i + \frac{\partial U}{\partial S_{ij}}\delta S_{ij} + \frac{\partial U}{\partial D_i}\delta D_i + \frac{\partial U}{\partial B_i}\delta B_i \Big) dV .$$

Therefore, $\delta \Pi_2 = 0$ implies

$$T_{ji,j} = -\Pi_2 \rho v_i, \quad -\tfrac{1}{2}(v_{j,i} + v_{i,j}) = -\Pi_2 S_{ij} \quad \text{in } V ,$$

$$\varepsilon_{ijk} H_{k,j} = -\Pi_2 D_i, \quad \varepsilon_{ijk} E_{k,j} = -\Pi_2 B_i \quad \text{in } V ,$$

$$0 = \Pi_2\Big(-T_{ij} + \frac{\partial U}{\partial S_{ij}} \Big), \quad 0 = \Pi_2\Big(-E_i + \frac{\partial U}{\partial D_i} \Big), \quad 0 = \Pi_2\Big(-H_i + \frac{\partial U}{\partial B_i} \Big) ,$$

$$v_i = 0 \quad \text{on } S_u, \quad -T_{ji} n_j = 0 \quad \text{on } S_T ,$$

$$-\varepsilon_{ijk} n_j E_k = 0 \quad \text{on } S_E, \quad -\varepsilon_{ijk} n_j H_k = 0 \quad \text{on } S_H ,$$

which is a system equivalent to (2.4) (when Π_2 or $\omega \neq 0$). Variational principles of other forms are possible. For quasi-static piezoelectricity four variational principles are shown in [6]. For dynamic piezoelectricity, we expect more varieties because of the presence of magnetic fields.

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Received April 15, 1993.

Arch. Mech., 45, 2, pp. 285–294, Warszawa 1993

Diffraction of SH-waves by a Griffith crack in nonhomogeneous elastic strip

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IN THIS PAPER the scattering of elastic SH-waves by a Griffith crack situated in an infinitely long inhomogeneous strip has been analyzed. Assuming that the shear modulus (μ) and density (ϱ) of the material vary in the vertical direction and applying Fourier transform, the mixed boundary value problem has been reduced to the solution of dual integral equations which finally has been reduced to the solution of a Fredholm integral equation of second kind. The numerical values of stress intensity factor and crack opening displacement have been illustrated graphically to show the effect of inhomogeneity of the material.

1. Introduction

THE NATURAL or artificial materials are usually inhomogeneous; so in recent years great attention has been given to the study of diffraction of elastic waves by cracks or obstacles in inhomogeneous media in view of their application in fracture mechanics. Many problems have been solved involving one or more cracks in an infinite homogeneous elastic medium. LOEBER and SIH [1] and MAL [2] have studied the problem of diffraction of elastic waves by a Griffith crack in an infinite medium. The problem of finite crack at the interface of two elastic half-spaces has been discussed by SRIVASTAVA *et al.* [3] and BOSTROM [4]. SINGH *et al.* [5, 6] considered the problem of scattering of a SH-wave by cracks or strips in a nonhomogeneous infinite elastic medium. Papers involving cracks located in an infinitely long elastic strip are very few. The problem of an infinite elastic strip containing an arbitrary number of unequal Griffith cracks, located parallel to its surfaces and opened by an arbitrary internal pressure, has been treated by ADAMS [7]. Finite crack perpendicular to the surface of the infinitely long elastic strip has been studied by CHEN [8] (for an impact load) and by SRIVASTAVA *et al.* [9] (for normally incident waves). Recently SHINDO *et al.* [10] considered the problem of impact response of a finite crack in an orthotropic strip. In our paper, the diffraction of normally incident SH-waves by a Griffith crack situated in an infinitely long inhomogeneous elastic strip has been discussed. The shear modulus (μ) and the density (ϱ) of the material have been assumed to vary in the vertical direction. Applying the Fourier transform, the mixed boundary value problem has been converted to the solution of dual integral equations. The dual integral equations have been finally reduced to a Fredholm integral equation of second kind by applying the Abel transform. Expressions for the stress intensity factor and crack opening displacement have been derived. The numerical values of stress intensity factor and crack opening displacement have been depicted by means of graphs to show the effect of material inhomogeneity.

2. Formulation of the problem

Consider the problem of diffraction of SH-waves by a Griffith crack in an inhomogeneous elastic strip of width $2h_1$. The crack is located in the region $-a \leq x_1 \leq a$, $-\infty < y_1 < \infty$, $z_1 = 0$ (Fig. 1). Normalizing all the lengths with respect to a and

putting $x_1/a = x$, $y_1/a = y$, $z_1/a = z$, $h_1/a = h$ it is found that the location of the crack is $-1 \leq x \leq 1$, $-\infty < y < \infty$, $z = 0$ referred to a Cartesian coordinate system (x, y, z). Let a plane harmonic SH-wave originating at $z = -\infty$ impinge on the crack normally to the x-axis. The variation of the shear modulus μ and the density ϱ is taken in the vertical (z) direction in such a manner that the shear velocity $(\mu_0/\varrho_0)^{1/2}$ is constant. The only non-vanishing y-component of the displacement which is independent of y is $v = v(x, z, t)$.

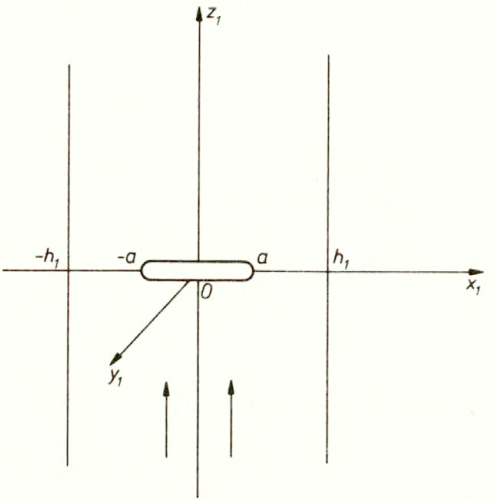

FIG. 1. Crack in the inhomogeneous strip.

The equation of motion is given by

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) = \varrho\frac{\partial^2 v}{\partial t^2}. \tag{2.1}$$

If we consider $v(x, z, t)$ in the form

$$v(x, z, t) = \frac{W(x, z, t)}{\sqrt{\mu(z)}}, \tag{2.2}$$

then

$$\mu\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial z^2}\right) + \frac{1}{2}\left[\frac{1}{2\mu}\left(\frac{\partial \mu}{\partial z}\right)^2 - \frac{\partial^2 \mu}{\partial z^2}\right]W = \varrho\frac{\partial^2 W}{\partial t^2}. \tag{2.3}$$

Putting $W(x, z, t) = F(x)G(z)e^{-i\omega t}$ and $\mu(z) = \mu_0 f(z)$, $\varrho(z) = \varrho_0 f(z)$ in Eq. (2.3) where μ_0, ϱ_0 are constants, such that $(\mu_0/\varrho_0)^{1/2} = c_2$ is the shear wave velocity, it is found that $F(x)$ and $G(z)$ satisfy the following equations

$$\frac{\partial^2 F}{\partial x^2} + n^2 F = 0, \tag{2.4}$$

$$\frac{\partial^2 G}{\partial z^2} + \left(\frac{a^2\omega^2}{c_2^2} - b^2 - n^2\right)G = 0, \tag{2.5}$$

provided $f(z)$ is of the form

$$-\frac{1}{4}\left(\frac{\partial f}{\partial z}/f\right)^2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial z^2}/f\right) = b^2, \tag{2.6}$$

where n and b are constants.

Let us assume $f(z)$ in the form

$$f(z) = \cosh^2(bz) \tag{2.7}$$

so that Eq. (2.6) is automatically satisfied.

Now the shear modulus $\mu(z)$ and density of the medium $\varrho(z)$ are

$$\mu = \mu_0 \cosh^2(bz), \quad \varrho = \varrho_0 \cosh^2(bz). \tag{2.8}$$

Using Eqs. (2.8), (2.2) and $W(x, z, t) = W(x, z)e^{-i\omega t}$, Eq. (2.1) takes the form

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial z^2} + k^2 W = 0, \quad k^2 = (k_2^2 - b^2), \quad k_2 = \frac{a\omega}{c_2}. \tag{2.9}$$

The displacement component $v^{(i)}(x, z, t)$ and stress $\tau^{(i)}(x, z, t)$ due to incident waves are given by

$$v^{(i)}(x, z, t) = \frac{A_0 e^{i(kz-\omega t)}}{\sqrt{\mu_0}\cosh(bz)} \tag{2.10}$$

and

$$\tau_{yz}^{(i)}(x, z, t) = A_0\sqrt{\mu_0}[ik\cosh(bz) - b\sinh(bz)]e^{i(kz-\omega t)}, \tag{2.11}$$

where A_0 is a constant.

Henceforth the time factor $e^{-i\omega t}$ will be suppressed in the sequel.

Solution of Eq. (2.9) is

$$W(x, z) = \int_0^\infty B_1(\xi)e^{-\beta z}\cos(\xi x)d\xi + \int_0^\infty C_1(\zeta)\cosh(\alpha x)\sin(\zeta z)d\zeta, \tag{2.12}$$

where

$$\begin{aligned} \alpha &= (\zeta^2 - k^2)^{1/2}, \quad \zeta > k, \quad \beta = (\xi^2 - k^2)^{1/2}, \quad \xi > k, \\ &= i(k^2 - \zeta^2)^{1/2}, \quad \zeta < k, \quad = -i(k^2 - \xi^2)^{1/2}, \quad \xi < k. \end{aligned}$$

Now displacement $v(x, z)$ and stresses $\tau_{yz}(x, z), \tau_{xy}(x, z)$ due to the scattered field are

$$v(x, z) = \frac{1}{\cosh(bz)}\Big[\int_0^\infty B(\xi)e^{-\beta z}\cos\xi x d\xi + \int_0^\infty C(\zeta)\cosh(\alpha x)\sin\zeta z d\zeta\Big], \tag{2.13}$$

$$\begin{aligned} \tau_{yz}(x, z) = -\mu_0 b\sinh(bz)\Big[\int_0^\infty B(\xi)e^{-\beta z}\cos\xi x\, d\xi \\ + \int_0^\infty C(\zeta)\cosh(\alpha x)\sin\zeta z d\zeta\Big] + \mu_0\cosh(bz) \\ \Big[-\int_0^\infty \beta B(\xi)e^{-\beta z}\cos\xi x\, d\xi + \int_0^\infty \zeta C(\zeta)\cosh(\alpha x)\cos\zeta z d\zeta\Big], \end{aligned} \tag{2.14}$$

$$\tau_{xy}(x,z) = \mu_0 \cosh(bz)\Big[- \int_0^\infty \xi B(\xi) e^{-\beta z} \sin \xi x \, d\xi + \int_0^\infty \alpha C(\zeta) \sinh(\alpha x) \sin \zeta z \, d\zeta \Big], \tag{2.15}$$

where

$$B(\xi) = \frac{1}{\sqrt{\mu_0}} B_1(\xi), \quad C(\zeta) = \frac{1}{\sqrt{\mu_0}} C_1(\zeta).$$

The boundary conditions are

$$\tau_{yz}(x,0) = -\tau_0, \qquad |x| \le 1, \tag{2.16}$$

$$v(x,0) = 0, \qquad 1 \le |x| \le h, \tag{2.17}$$

$$\tau_{xy}(\pm h, z) = 0, \qquad |z| < \infty, \tag{2.18}$$

where $\tau_0 = ikA_0\sqrt{\mu_0}$.

From the boundary condition (2.18) $C(\zeta)$ is found to be expressible in terms of $B(\xi)$ as follows:

$$C(\zeta) = \frac{2\zeta}{\pi\alpha \sinh(\alpha h)} \int_0^\infty \frac{\xi B(\xi) \sin(\xi h)}{\xi^2 + \alpha^2} \, d\xi. \tag{2.19}$$

Next, the use of Eq. (2.19) in the boundary condition (2.16) and (2.17) yields the following dual integral equations from which the unknown function $B(\xi)$ is to be determined:

$$\int_0^\infty \xi[1 + M(\xi)] B(\xi) \cos(\xi x) \, d\xi = p(x), \qquad |x| \le 1 \tag{2.20}$$

and

$$\int_0^\infty B(\xi) \cos(\xi x) \, d\xi = 0, \qquad 1 \le |x| \le h \tag{2.21}$$

where

$$M(\xi) = \left(\frac{\beta}{\xi} - 1\right), \tag{2.22}$$

$$p(x) = \frac{\tau_0}{\mu_0} + \frac{2}{\pi} \int_0^\infty \frac{\zeta^2 \cosh(\alpha x)}{\alpha \sinh(\alpha h)} \, d\zeta \int_0^\infty \frac{\xi B(\xi) \sin(\xi h)}{\xi^2 + \alpha^2} \, d\xi. \tag{2.23}$$

3. Method of solution

In order to solve the dual integral equations (2.20) and (2.21), $B(\xi)$ is taken in the form

$$B(\xi) = \frac{\tau_0}{\mu_0} \int_0^1 t\phi(t) J_0(\xi t) \, dt, \tag{3.1}$$

so that Eq. (2.21) is automatically satisfied.

Substitution of the value of $B(\xi)$ from Eq. (3.1) in Eq. (2.20), yields a Fredholm integral equation of second kind

$$\phi(t) + \int_0^1 u[L_1(u,t) + L_2(u,t)]\phi(u)\,du = 1, \tag{3.2}$$

where

$$L_1(u,t) = \int_0^\infty \xi M(\xi) J_0(\xi u) J_0(\xi t)\,d\xi, \tag{3.3}$$

$$L_2(u,t) = -\int_0^\infty \frac{\zeta^2 I_0(\alpha t) I_0(\alpha u) e^{-\alpha h}}{\alpha \sinh(\alpha h)}\,d\zeta. \tag{3.4}$$

Using contour integration technique [3], the infinite integral arising in the kernel $L_1(u,t)$ can be converted to a finite integral and is given by

$$\begin{aligned} L_1(u,t) &= -ik^2 \int_0^1 (1-\eta^2)^{1/2} J_0(k\eta t) H_0^{(1)}(k\eta u)\,d\eta, \quad u > t, \\ &= -ik^2 \int_0^1 (1-\eta^2)^{1/2} J_0(k\eta u) H_0^{(1)}(k\eta t)\,d\eta, \quad u < t. \end{aligned} \tag{3.5}$$

Now

$$\begin{aligned} L_2(u,t) &= \int_0^k \frac{\zeta^2 J_0(\alpha_1 t) J_0(\alpha_1 u) e^{i\alpha_1 h}}{\alpha_1 \sin(\alpha_1 h)}\,d\zeta - \int_k^\infty \frac{\zeta^2 I_0(\alpha t) I_0(\alpha u) e^{-\alpha h}}{\alpha \sinh(\alpha h)}\,d\zeta \\ &= \int_0^k \frac{\zeta^2}{\alpha_1} J_0(\alpha_1 t) J_0(\alpha_1 u) \operatorname{ctg}(\alpha_1 h)\,d\zeta + i \int_0^k \frac{\zeta^2}{\alpha_1} J_0(\alpha_1 t) J_0(\alpha_1 u)\,d\zeta \\ &\quad - \int_k^\infty \frac{\zeta^2 I_0(\alpha t) I_0(\alpha u) e^{-\alpha h}}{\alpha \sinh(\alpha h)}\,d\zeta, \end{aligned}$$

where

$$\alpha_1 = (k^2 - \zeta^2)^{1/2}.$$

Putting $\zeta^2 = k^2(1-y^2)$ in the first and second integrals and $\zeta^2 = k^2(1+y^2)$ in the third integral, it is found that

$$\begin{aligned} L_2(u,t) = k^2 \Big[&\int_0^1 (1-y^2)^{1/2} J_0(kyt) J_0(kyu) \operatorname{ctg}(kyh)\,dy \\ &+ i \int_0^1 (1-y^2)^{1/2} J_0(kyt) J_0(kyu)\,dy \\ &- \int_0^\infty (1+y^2)^{1/2} I_0(kyt) I_0(kyu) e^{-kyh} \operatorname{cosech}(kyh)\,dy \Big]. \end{aligned} \tag{3.6}$$

4. Stress intensity factor and crack opening displacement

From Eq. (2.14) the stress τ_{yz} on the plane $z = 0$ can be written as

$$\tau_{yz}(x,0) = \mu_0\Big[- \int_0^\infty \beta B(\xi)\cos\xi x\, d\xi + \int_0^\infty \zeta C(\zeta)\cosh(\alpha x)\, d\zeta\Big]. \tag{4.1}$$

Substituting the value of $C(\zeta)$ and $B(\xi)$ from Eqs. (2.19) and (3.1), the expression for the stress can finally be presented as

$$\tau_{yz}(x,0) = \frac{\tau_0 x}{(x^2-1)^{1/2}}\phi(1) + O(1), \quad |x| > 1.$$

Defining the stress intensity factor N by

$$N = \operatorname*{Lt}_{x\to 1^+}\left|\frac{(x-1)^{1/2}\tau_{yz}(x,0)}{\tau_0}\right|,$$

we obtain

$$N = \frac{1}{\sqrt{2}}|\phi(1)|. \tag{4.2}$$

Now the crack opening displacement $\Delta v(x,0) = v(x,0^+) - v(x,0^-)$ can be obtained from Eq. (2.13) as

$$\Delta v(x,0) = 2\int_0^\infty B(\xi)\cos(\xi x)\, d\xi, \quad |x| \le 1,$$

which, on substitution of the value of $B(\xi)$ from Eq. (3.1), takes the form

$$\Delta x(x,0) = \frac{2\tau_0}{\mu_0}\int_x^1 \frac{t\phi(t)}{(t^2-x^2)^{1/2}}\, dt, \quad |x| \le 1. \tag{4.3}$$

5. Numerical results and discussion

Using the method of FOX and GOODWIN [11], the Fredholm integral equation given by Eq. (3.2) has been solved numerically for different values of the material inhomogeneity parameters. In this method the integral in Eq. (3.2) has been represented at first by a quadrature formula involving the values of the desired function $\phi(t)$ at the pivotal points inside the specified range of integration, and then converted to a set of simultaneous linear algebraic equations; their solutions yield the first approximations to the requires pivotal values of $\phi(t)$. Applying the difference-correction technique, the first approximations have been improved. After solving the integral equation (3.2) numerically, the stress intensity factor N and the crack opening displacement $\mu_0\Delta v(x,0)/\tau_0$ have been calculated numerically and plotted separately against the dimensional frequency k_2 ($0.5 \le k_2 \le 1$) and dimensionless distance x ($0 \le x \le 1$), respectively, for different values of the material inhomogeneity parameter b and strip width $2h$.

In Fig. 2, the effect of the width of the strip on the stress intensity factor for a homogeneous material has been shown; the effect of inhomogeneity of the material on the stress intensity factor for different widths of the strip has been depicted in Figs. 3–5.

It is found that in both the homogeneous and nonhomogeneous cases, the effect of the strip width decreases with the increase of the frequency, and the graphs of the stress

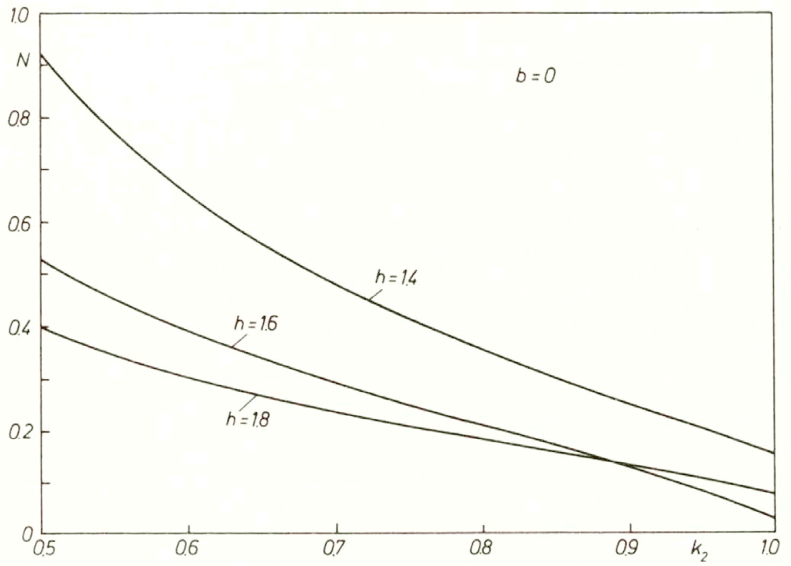

FIG. 2. Stress intensity factor N vs. dimensionless frequency k_2 for homogeneous medium ($b = 0$).

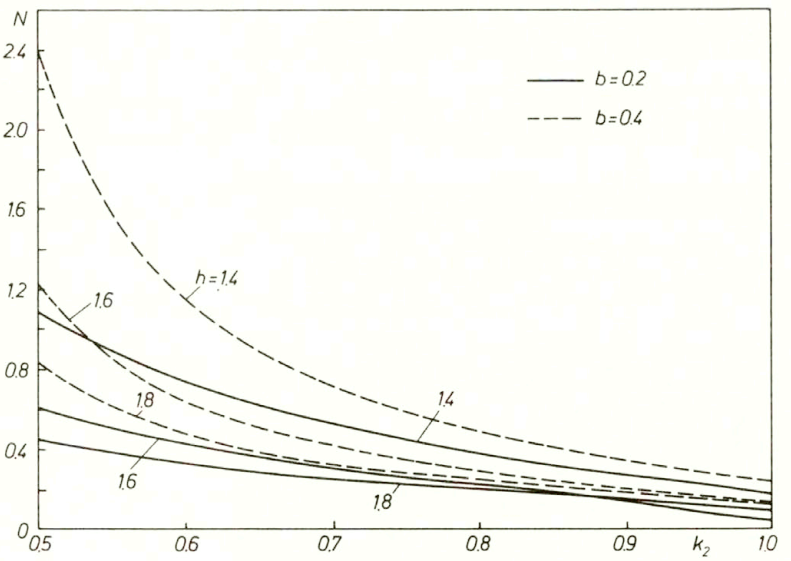

FIG. 3. Stress intensity factor N vs. dimensionless frequency k_2.

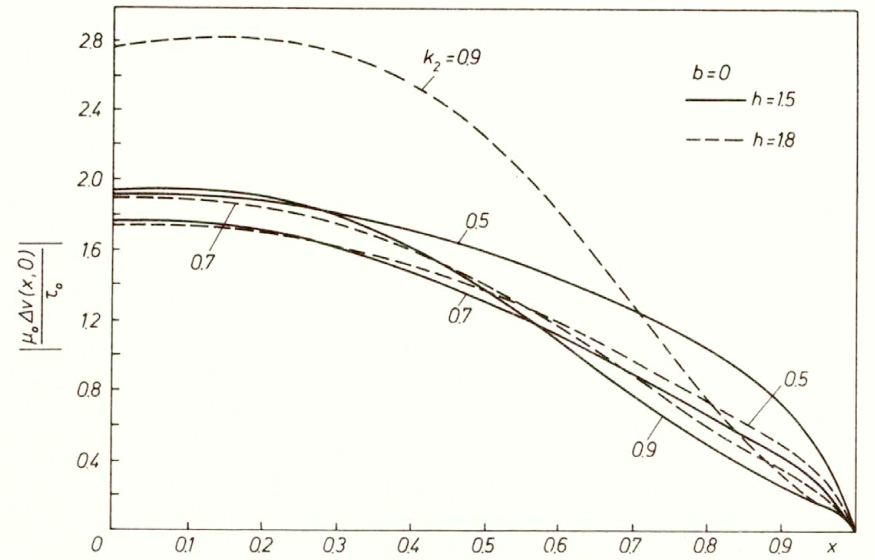

FIG. 4. Crack opening displacement vs. dimensionless distance $x (b = 0)$.

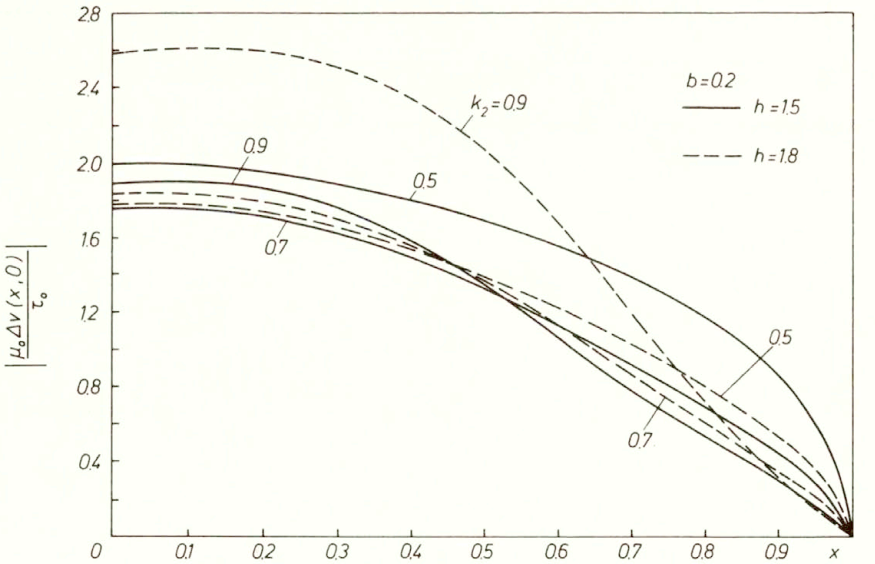

FIG. 5. Crack opening displacement vs. dimensionless distance $x (b = 0.2)$.

intensity factor N become flat with the increase of strip width $2h$. From Fig. 3 it is clear that the effect of inhomogeneity parameter b is prominent for low frequency k_2 and stress intensity factor is greater for higher values of the inhomogeneity parameter b.

In Figs. 4–8 the crack opening displacements against dimensionless distance x for different values of the material inhomogeneity parameter b and the strip width $2h$ have been illustrated by means of graphs. Case $b = 0$ corresponds to the homogeneous case (Fig. 4). From Figs. 4–6 it is seen that for a fixed value of inhomogeneity parameter b,

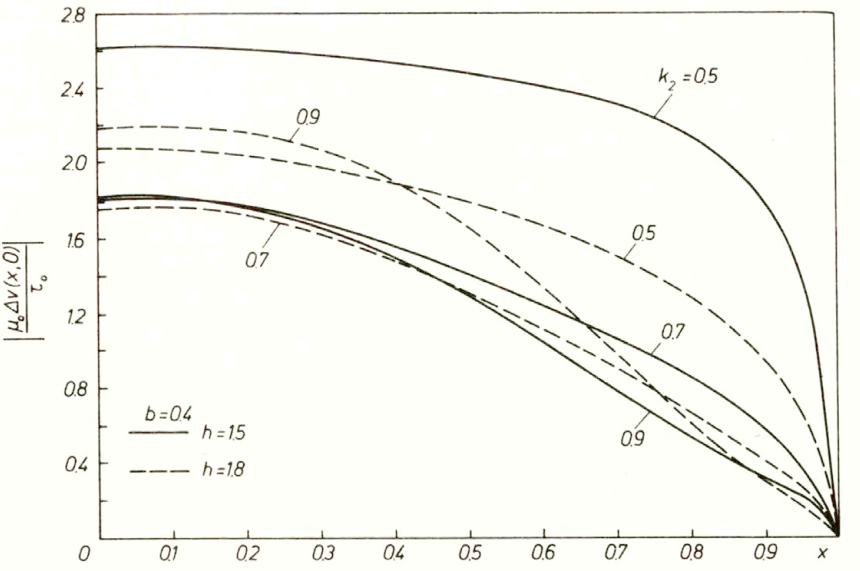

FIG. 6. Crack opening displacement vs. dimensionless distance $x (b = 0.4)$.

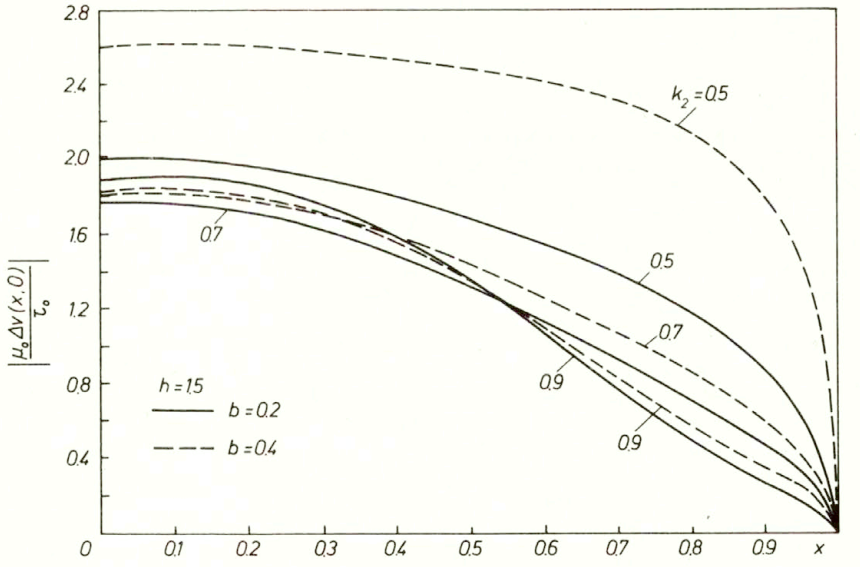

FIG. 7. Crack opening displacement vs. dimensionless distance $x (h = 1.5)$.

the crack opening displacement is greater for lower values of h when the frequencies are small, but the reverse effect is found for higher frequencies.

Next, in Figs. 7 and 8 we see that for a fixed value of h, the crack opening displacement is greater for higher values of the inhomogeneity parameter b when the frequencies are small, but for higher frequencies the effect is just reverse.

Finally it is found in all the cases that the crack opening displacement reaches its maximum at about $x = 0$, and then it gradually decreases and becomes zero at $x = 1$.

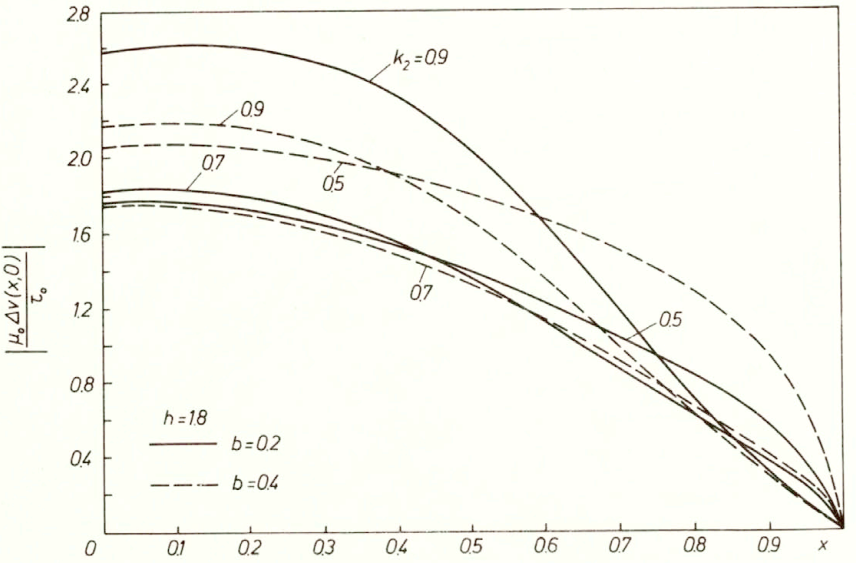

FIG. 8. Crack opening displacement vs. dimensionless distance $x(h = 1.8)$.

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Received April 21, 1993.

Arch. Mech., 45, 3, pp. 295–304, Warszawa 1993

Refined macro-dynamics of periodic structures (*)

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IN THIS CONTRIBUTION a certain non-asymptotic approach to the macro-dynamic modelling of micro-periodic composites is proposed. The obtained equations of macro-dynamics describe phenomena related to the micro-heterogeneity of materials, e.g. dispersion and scale effects. An example of applications to micro-vibration and wave propagation problems illustrates the usefulness of the proposed approach.

1. Introduction

AS IT IS KNOWN, by micromechanics of composite materials we mean a study of composite material behaviour wherein the *effects of the constituent materials are detected only as averaged apparent properties of the composite*, [1]. For periodic material structures which are made of a large number of repetitive micro-heterogeneous cells, the basis for averaging is a certain representative volume element (r.v.e.) of the structure. The equations of macro-mechanics can be obtained by the known asymptotic approaches to the macro-modelling of micro-heterogeneous materials, cf. [2–8], where inertial properties of a composite are uniquely determined by the averaged mass density. Hence, the asymptotic equations of macro-dynamics can be applied solely to problems in which the time-dependent excitations of the structure produce the wave-lengths much larger than the maximum length dimension of r.v.e. In order to eliminate this drawback, a certain non-asymptotic method of macro-modelling for dynamics of periodic structures will be proposed. The result of this approach will be referred to as a *refined macro-dynamics* of periodic material structures. The equations of refined macro-dynamics can also describe vibration and wave propagation problems with the wave-lengths of an order of the cell length dimensions.

The analysis will be restricted to the small displacement gradient theory and to the linear-elastic materials. The equations of refined macro-dynamics investigated in this paper will be obtained by simple calculations of averages and do not require any solution to boundary value problem on r.v.e. Hence, the proposed approach can be effectively applied to engineering problems.

Notations

Throughout the paper subscripts i, j, k, l run over the sequence 1, 2, 3 and are related to the Cartesian orthogonal coordinate system in the reference space. Indices a, b run over $1, \ldots, n$. The summation convention holds for i, j, k, l as well as for a, b unless otherwise stated. Points in the reference space $\mathbb{R}^3$ are denoted by $\mathbf{x}$, points in the region $Y \equiv (-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (-l_3/2, l_3/2)$ by $\mathbf{y}$. For any differentiable

(*) The research was supported by the Scientific Research Committee, Warsaw, under grant No. 3 3310 92 03.

function $f(\mathbf{x})$ we denote $f_{,i} \equiv \partial f/\partial x_i$. Symbol $\cong$ denotes an approximation admissible from a computational viewpoint. The time coordinate is denoted by τ, $\tau \in [\tau_0, \tau_f]$ and $\dot{u}(\cdot, \tau) \equiv \partial u/\partial \tau$. For an arbitrary integrable function $\phi(\cdot)$ defined almost everywhere in the region Ω in $\mathbb{R}^3$ we introduce the well-known averaging operator

$$\langle \phi \rangle(\mathbf{x}) \equiv \frac{1}{\mathrm{vol}(V)} \int_V \phi(\mathbf{x}+\mathbf{y}) dv(\mathbf{y}), \tag{1.1}$$

for every $\mathbf{x} \in \Omega_0$, where $\Omega_0 \equiv \{\mathbf{z} \in \mathbb{R}^3 : V(\mathbf{z}) \subset \Omega\}$ and $dv(\mathbf{y}) = dy_1 dy_2 dy_3$.

2. Preliminary concepts

Let the heterogeneous periodic material structure under consideration in its undeformed configuration occupy region Ω in the reference space, and $V \subset \Omega$ can be taken as a certain r.v.e. of the periodic structure under consideration. We assume that $l \equiv \max\{l_1, l_2, l_3\}$ is sufficiently small as compared to the minimum characteristic length dimension of Ω. Hence, we shall deal with a certain micro-periodic composite structure. Every element $V(\mathbf{x}) = \mathbf{x} + V$, such that $V(\mathbf{x}) \subset \Omega$, will be referred to as a cell of this structure. For the sake of simplicity we assume that $\overline{\Omega} = \cup \overline{V}(\mathbf{z})$, $\mathbf{z} \in \Lambda$, where Λ is the lattice of points in Ω which are centers of mutually disjoined cells $V(\mathbf{z})$.

In order to formulate basic hypotheses leading to macromodels of the micro-periodic body, we shall introduce two auxiliary concepts. The first of them is that of a V-macro function. A continuous function $F(\cdot)$ defined on Ω which, for every $\mathbf{x}, \mathbf{z} \in \Omega$ and $\mathbf{z} - \mathbf{x} \in V$, satisfies condition $F(\mathbf{x}) \cong F(\mathbf{z})$ is called a *continuous V-macro function*. Similarly, a continuous function $F(\cdot)$ having continuous derivatives up to k-th order, is called V-macro function if $F(\cdot)$, together with its derivatives up to k-th order, are continuous V-macro functions. Moreover, for every continuous V-macro function defined on Ω and for every integrable function $f(\cdot)$ defined on Ω we assume

$$\langle fF \rangle(\mathbf{x}) \cong \langle f \rangle(\mathbf{x}) F(\mathbf{x}) \tag{2.1}$$

for every $\mathbf{x} \in \Omega_0$. If $\langle f \rangle(\cdot)$ is a continuous V-macro function defined on Ω_0 and $F(\cdot)$ is V-macro function defined on Ω such that $F(\mathbf{x}) \cong \langle f \rangle(\mathbf{x}), \mathbf{x} \in \Omega_0$, then

$$\int_\Omega fG\, dv \cong \sum_{z \in \Lambda} \langle f \rangle(\mathbf{z}) G(\mathbf{z})\, \mathrm{vol}(V) \cong \int_\Omega FG\, dv \tag{2.2}$$

holds for an arbitrary continuous V-macro function G defined on Ω. Generally speaking, V-macro functions describe the macroscopic behaviour of the body.

The second auxiliary concept we are to introduce is that of a micro-shape function. We assume that the phenomena related to heterogeneous micro-periodic material structure, from the qualitative point of view, will be described by means of independent continuous functions $h_a(\cdot)$, $a = 1, \ldots, n$, defined on $\mathbb{R}^3$ (which have physical sense only for $\mathbf{x} \in \Omega$), satisfying the conditions:

(i) $h_a(\mathbf{x}) = h_a(\mathbf{x} + l_i \mathbf{e}_i)$, $i = 1, 2, 3$ (no summation with respect to i) for every $\mathbf{x} \in \mathbb{R}^3$, where $\mathbf{e}_i$ is a versor of x_i-axis,

(ii) $\langle h_a \rangle(\mathbf{x}) = 0$, $\langle h_{a,i} \rangle(\mathbf{x}) = 0$, for every $\mathbf{x} \in \mathbb{R}^3$,

(iii) $\langle \varrho h_a \rangle(\mathbf{x}) = 0$ for every $\mathbf{x} \in \Omega_0$, where $\varrho(\cdot)$ is a mass density function defined almost everywhere on Ω.

Condition (i) is equivalent to the statement that $h_a(\cdot)$ are V-periodic. Functions $h_a(\cdot)$ will be called *micro-oscillatory shape functions*. The choice of these shape functions has to be postulated a priori in every special problem and depends on the character of micro-oscillations which we are going to analyze. As a simple example of functions $h_a(\cdot)$ we can take trigonometric functions $\sin(2\pi a x_i/l_i)$, $i = 1, 2, 3$, $a = 1, \ldots, n$ (no summation over i).

3. Basic assumptions

The proposed method of macro-modelling for micro-periodic composite body under consideration will be based on two fundamental assumptions. The first of them takes into account the physically reasonable hypotheses that the motion of micro-periodic structure can be obtained by a superimposition of micro-oscillations on a certain macro-motion. This macro-motion is represented by a V-macro field $U_i(\cdot, \tau)$. The micro-oscillations will be described by means of the micro-oscillatory shape functions $h_a(\cdot)$ and by certain arbitrary fields $Q_i^a(\cdot, \tau)$. Denoting by $u_i(\cdot, \tau)$ a displacement field at a time instant τ from the undeformed configuration of the body, the first of the basic assumptions will be referred to as:

Micro-macro localization hypothesis

Every motion of a micro-periodic body under consideration can be represented in the form

$$u_i(\mathbf{x}, \tau) = U_i(\mathbf{x}, \tau) + h_a(\mathbf{x}) Q_i^a(\mathbf{x}, \tau), \quad \mathbf{x} \in \Omega, \quad \tau \in [\tau_0, \tau_f], \tag{3.1}$$

where $U_i(\cdot, \tau)$, $Q_i^a(\cdot, \tau)$ are arbitrary continuous V-macro fields together with their first and second order space and time derivatives, and $h_a(\cdot)$ are postulated *a priori* linear independent micro-shape functions.

Fields $U_i(\cdot, \tau)$ and $Q_i^a(\cdot, \tau)$ will be called *macro-displacements* and *correctors*, respectively.

The second basic assumption takes into account the fact that in displacement gradients $u_{i,j}(\mathbf{x}, \tau)$ which can be obtained from Eq. (3.1) the terms involving $h_a(\mathbf{x})$ are small as compared to terms involving $h_{a,i}(\mathbf{x})$. Hence, denoting by $\varepsilon_{ij}(\mathbf{x}, \tau)$ the linearized strain tensor, we formulate the following assumption:

Micro-strain assumption

For the micro-periodic body under consideration the components $\varepsilon_{ij}(\mathbf{x}, \tau)$ of the linearized strain tensor will be assumed in the form

$$\varepsilon_{ij}(\mathbf{x}, \tau) = U_{(i,j)}(\mathbf{x}, \tau) + h_{a,(j}(\mathbf{x}) Q_{i)}^a(\mathbf{x}, \tau), \quad \mathbf{x} \in \Omega, \quad \tau \in [\tau_0, \tau_f]. \tag{3.2}$$

Denoting by $\sigma_{ij}(\mathbf{x}, \tau)$ components of the stress tensor and by $A_{ijkl}(\mathbf{x})$ components of the tensor of elastic moduli, the stress-strain relations for a micro-periodic body are:

$$\sigma_{ij}(\mathbf{x}, \tau) = A_{ijkl}(\mathbf{x})[U_{(k,l)}(\mathbf{x}, \tau) + h_{a,(k}(\mathbf{x}) Q_{l)}^a(\mathbf{x}, \tau)], \quad \mathbf{x} \in \Omega, \quad \tau \in [\tau_0, \tau_f]. \tag{3.3}$$

It has to be remembered that the tensor of elastic moduli $A_{ijkl}(\cdot)$ as well as the mass density $\varrho(\cdot)$ are V-periodic fields having the periods l_i along x_i-axes, respectively.

4. Analysis

The macro-modelling approach of the composite structure under consideration will be based on the assumptions formulated in Sec. 3, on the properties of V-macro functions (2.1), (2.2) and on the well-known principle of virtual work:

$$\int_{\Omega} \sigma_{ij}\delta\varepsilon_{ij}dv = \oint_{\partial\Omega} p_i\delta u_i da + \int_{\Omega} (\varrho b_i - \varrho\ddot{u}_i)\delta u_i dv\,, \tag{4.1}$$

where p_i, b_i are surface tractions and body forces, respectively. Condition (4.1) has to hold for every admissible virtual strain field $\delta\varepsilon_{ij}$ and every virtual displacement field δu_i. In the sequel we assume that body forces are constant and that $\delta u_i = 0$ on the boundary $\partial\Omega$. By means of Eqs. (3.1) and (3.2) we obtain $\delta u_i = \delta U_i + h_a\delta Q_i^a$, $\delta\varepsilon_{ij} = \delta U_{(i,j)} + h_{a,(j}\,\delta Q_{i)}^a$, where δU_i, δQ_i^a are arbitrary sufficiently regular V-macro functions defined on Ω, such that $\delta U_i = 0$, $\delta Q_i^a = 0$ on the boundary $\partial\Omega$. Using Eqs. (2.1) and (2.2), the left-hand side of Eq. (4.1) can be transformed as follows:

$$\int_{\Omega} \sigma_{ij}\delta\varepsilon_{ij}dv \cong \sum_{\mathbf{z}\in\Lambda}[\langle\sigma_{ij}\rangle(\mathbf{z},\tau)\delta U_{(i,j)}(\mathbf{z}) + \langle\sigma_{ij}h_{a,(j}\rangle(\mathbf{z},\tau)\delta Q_{i)}^a(\mathbf{z})]\,\mathrm{vol}(V)\,.$$

From Eq. (3.3) and (2.1) it follows that $\langle\sigma_{ij}\rangle(\cdot,\tau)$ and $\langle\sigma_{ij}h_{a,j}\rangle(\cdot,\tau)$ are V-macro fields defined on Ω_0. Hence, there exist V-macro fields $S_{ij}(\cdot,\tau)$, $H_{ai}(\cdot,\tau)$ defined on Ω such that

$$S_{ij}(\mathbf{x},\tau) \cong \langle\sigma_{ij}\rangle(\mathbf{x},\tau), \qquad H_{ai}(\mathbf{x},\tau) \cong \langle\sigma_{ij}h_{a,j}\rangle(\mathbf{x},\tau), \qquad \mathbf{x}\in\Omega_0\,. \tag{4.2}$$

By means of Eq. (2.2) we conclude that

$$\int_{\Omega} \sigma_{ij}\delta\varepsilon_{ij}dv \cong \int_{\Omega} (S_{ij}\delta U_{(i,j)} + H_{ai}\delta Q_i^a)dv\,. \tag{4.3}$$

For the right-hand side of Eq. (4.1), using Eqs. (2.1), (2.2) and bearing in mind that $\delta u_i = 0$ on $\partial\Omega$, we obtain

$$\int_{\partial\Omega} p_i\delta u_i da + \int_{\Omega} (\varrho b_i - \varrho\ddot{u}_i)\delta u_i dv \cong \int_{\Omega} [(\langle\varrho\rangle b_i - \langle\varrho\rangle\ddot{U}_i)\delta U_i - \langle\varrho h_a h_b\rangle\ddot{Q}_i^a\delta Q_i^b]dv\,. \tag{4.4}$$

V-macro fields $S_{ij}(\cdot,\tau)$ and $H_{ai}(\cdot,\tau)$ will be called *macro-stresses* and *structural forces*, respectively, and they represent an averaged state of stress in a micro-periodic body under consideration. On the basis of formulae (4.1), (4.3), (4.4) we shall postulate the following:

Macro-approximation of virtual work principle

Macro-stress field $S_{ij}(\cdot,\tau)$ and structural force fields $H_{ai}(\cdot,\tau)$ are interrelated with macro-displacement field $U_i(\cdot,\tau)$ and corrector fields $Q_i^a(\cdot,\tau)$ by means of condition

$$\int_{\Omega} (S_{ij}\delta U_{(i,j)} + H_{ai}\delta Q_i^a)dv = \int_{\Omega} [(\langle\varrho\rangle b_i - \langle\varrho\rangle\ddot{U}_i)\delta U_i - \langle\varrho h_a h_b\rangle\ddot{Q}_i^a\delta Q_i^b]dv\,, \tag{4.5}$$

which has to hold for arbitrary continuous (with their first derivatives) V-macro fields $\delta U_i(\cdot)$, $\delta Q_i^a(\cdot)$ defined on Ω, such that $\delta U_i = 0$, $\delta Q_i^a = 0$ on the boundary $\partial\Omega$.

Under the assumption that fields $\delta U_i(\cdot)$, $\delta Q_i^a(\cdot)$ are independent, from Eq. (4.5), we obtain

$$\int_\Omega (S_{ij,j} + \langle \varrho \rangle b_i - \langle \varrho \rangle \ddot{U}_i)\delta U_i dv = 0\,,$$
$$\int_\Omega (H_{ai} + \langle \varrho h_a h_b \rangle \ddot{Q}_i^b)\delta Q_i^a dv = 0\,, \tag{4.6}$$

for every (sufficiently regular) V-macro fields δU_i, δQ_i^a, which are equal to zero on the boundary $\partial\Omega$. It can be shown that the restriction of virtual fields δU_i, δQ_i^a to V-macro fields in Eqs. (4.6) is irrelevant. Hence, using the du Bois–Reymond lemma, conditions (4.6) are equivalent to

$$S_{ij,j}(\mathbf{x},\tau) + \langle \varrho \rangle b_i - \langle \varrho \rangle \ddot{U}_i(\mathbf{x},\tau) = 0\,,$$
$$H_{ai}(\mathbf{x},\tau) + \langle \varrho h_a h_b \rangle \ddot{Q}_i^b(\mathbf{x},\tau) = 0, \quad \mathbf{x} \in \Omega, \quad \tau \in [\tau_0, \tau_f]\,. \tag{4.7}$$

It can be easily observed that from Eqs. (4.2), (3.3) and using Eq. (2.1) we can assume fields $S_{ij}(\cdot,\tau)$, $H_{ai}(\cdot,\tau)$ in the form

$$S_{ij}(\mathbf{x},\tau) = \langle A_{ijkl} \rangle U_{(k,l)}(\mathbf{x},\tau) + \langle A_{ijkl} h_{a,k} \rangle Q_l^a(\mathbf{x},\tau)\,,$$
$$H_{ai}(\mathbf{x},\tau) = \langle A_{ijkl} h_{a,j} \rangle U_{(k,l)}(\mathbf{x},\tau) + \langle A_{ijkl} h_{a,j} h_{b,l} \rangle Q_k^b(\mathbf{x},\tau)\,, \tag{4.8}$$

where $\mathbf{x} \in \Omega$ and $\tau \in [\tau_0, \tau_f]$. Equations (4.7), (4.8) involve exclusively V-macro fields and constitute the final result of the foregoing analysis. The proposed method of macro-modelling in its part related to the modelling of the inertial properties of the body does not involve any asymptotic approximation. At the same time the material macro-properties of the body are obtained by using micro-strain assumption (3.2) where terms of order $O(l)$ were neglected, i.e. we have applied here an asymptotic approximation. That is why the proposed method can be referred to as a semi-asymptotic method of macro-modelling.

5. Conclusions

The macro-modelling approach introduced in Secs. 3 and 4 describes a certain generalized elastic continuum governed by the field equations (4.7) and constitutive relations (4.8). It is easy to see that the form of Eqs. (4.7) is independent of the material properties of the body under consideration, hence, Eqs. (4.7) will be called the equations of motion. Similarly, Eqs. (4.8) will be referred to as macro-constitutive equations. Combining Eqs. (4.7) and (4.8) we obtain the system of equations for macro-displacement and corrector fields

$$\langle A_{ijkl} \rangle U_{k,lj}(\mathbf{x},\tau) + \langle A_{ijkl} h_{a,l} \rangle Q_{k,j}^a(\mathbf{x},\tau) + \langle \varrho \rangle b_i = \langle \varrho \rangle \ddot{U}_i(\mathbf{x},\tau)\,,$$
$$\langle \varrho h_a h_b \rangle \ddot{Q}_i^b(\mathbf{x},\tau) + \langle A_{ijkl} h_{a,j} h_{b,l} \rangle Q_k^b(\mathbf{x},\tau) = -\langle A_{ijkl} h_{a,j} \rangle U_{k,l}(\mathbf{x},\tau)\,, \tag{5.1}$$
$$\mathbf{x} \in \Omega, \tau \in [\tau_0, \tau_f]\,.$$

These equations describe on the macro-level a micro-periodic material structure, the inertial properties of which are specified not only by the averaged mass density $\langle \varrho \rangle$ but also by *micro-inertial moduli* $\langle \varrho h_a h_b \rangle$. It has to be emphasized that the micro-inertial moduli $\langle \varrho h_a h_b \rangle$ involve the length dimension of the r.v.e. Hence, Eqs. (5.1) can be a basis for investigations of scale effects in micro-periodic composites. Moreover, the obtained equations allow to analyze micro-vibrations and propagation of micro-waves in composite

materials, what takes into account the dispersion effects due to the micro-heterogeneity of the medium. The aforementioned problems can not be investigated by means of the asymptotic homogenized continuum, cf. [2–8]; that is why we shall consider Eqs. (4.7), (4.8) and (5.1) as equations of the refined macro-dynamics of micro-periodic elastic composites.

The form of the derived equations depends on the choice and number of micro-shape functions $h_a(\cdot)$, $a = 1, \ldots, n$. Let us observe that on the micro-level (inside every cell $V(\mathbf{x})$, $\mathbf{x} \in \Omega_0$) the model is described by a proper choice of micro-shape functions; in such a manner we can obtain more or less exact descriptions of a problem. Let us also observe that the corrector fields Q_i^a are governed by the second of Eqs. (5.1) which are ordinary differential equations involving only time derivatives of correctors.

In every special problem governing equations (5.1) have to be considered together with the boundary conditions for macro-displacements U_i and with initial conditions for macro displacement U_i and correctors Q_i^a. The form of these conditions depends on the physical character of the problem and takes into account Eqs. (3.1) or (3.3); it has to be emphasized that the solutions to the boundary value problem for Eqs. (5.1) have a physical sense only if the obtained solutions U_i, Q_i^a, together with their first and second order derivatives, are continuous V-macro fields.

The refined macro-dynamics constitutes a certain generalization of the macro-modelling approach proposed in [5] and developed in [6–8], where the micro-inertial moduli were not taken into account. Disregarding in Eqs. (5.1) the terms involving $\langle \varrho h_a h_b \rangle$ as small of order $O(l^2)$, we obtain the asymptotic model of micro-periodic composites described in [5–8] and given by

$$\langle A_{ijkl} \rangle U_{k,lj}(\mathbf{x}, \tau) + \langle A_{ijkl} h_{a,l} \rangle Q^a_{k,j}(\mathbf{x}, \tau) + \langle \varrho \rangle b_i = \langle \varrho \rangle \ddot{U}_i(\mathbf{x}, \tau),$$

$$\langle A_{ijkl} h_{a,j} h_{b,l} \rangle Q^b_k(\mathbf{x}, \tau) = -\langle A_{ijkl} h_{a,j} \rangle U_{k,l}(\mathbf{x}, \tau), \mathbf{x} \in \Omega, \tau \in [\tau_0, \tau_f].$$

In this asymptotic case correctors Q_i^a are governed by a system of $3n$ linear algebraic equations and can be eliminated from the foregoing system of equations. Such situation does not hold for Eqs. (5.1) of refined macro-dynamics, where correctors are governed by a system of ordinary differential equations, involving second-order time derivatives of correctors.

It has to be emphasized that for the homogeneous material structures we obtain $\langle A_{ijkl} h_{a,l} \rangle = A_{ijkl} \langle h_{a,l} \rangle = 0$ and hence, Eqs. (5.1) yields two independent systems of equations

$$A_{ijkl} U_{k,lj}(\mathbf{x}, \tau) + \varrho b_i = \varrho \ddot{U}_i(\mathbf{x}, \tau),$$

$$\varrho \langle h_a h_b \rangle \ddot{Q}^b_i(\mathbf{x}, \tau) + A_{ijkl} \langle h_{a,j} h_{b,l} \rangle Q^b_k(\mathbf{x}, \tau) = 0.$$

Under initial conditions $Q_i^a(\mathbf{x}, \tau_0) = 0$, $\dot{Q}_i^a(\mathbf{x}, \tau_0) = 0$, $\mathbf{x} \in \Omega$, we obtain $Q_i^a(\mathbf{x}, \tau) = 0$ for $\tau \in [\tau_0, \tau_f]$, $\mathbf{x} \in \Omega$. Hence, we conclude that the oscillatory terms in Eqs. (3.1) are due to the micro-heterogeneity of a material composite structure.

An illustrative example of applications of the refined dynamics will be given in Sec. 6.

6. Applications

The proposed model of the refined macro-dynamics will now be applied to the problem of a straight micro-periodic linear-elastic bar treated as a uniaxial structure. The representative element is now given by the interval $(-l/2, l/2)$ of the x-axis, where $x \equiv x_1$. We

assume that the Young modulus $E(x)$ and the mass density $\varrho(x)$ are piecewise constant periodic functions in $(-a/2, a/2)$ are equal to E_1, ϱ_1, and in $(-l/2, l/2)/(-a/2, a/2)$ are equal to E_2, ϱ_2, respectively. For the sake of simplicity we introduce only one micro-shape function $h(x) \equiv h_1(x)$, $x \in \mathbb{R}$, which is continuous, linear in $(-l/2, -a/2)$, $(-a/2, a/2)$, $(a/2, l/2)$, and in $[-l/2, l/2]$ takes the values $h(-l/2) = h(0) = h(l/2) = 0, h(-a/2) = 1, h(a/2) = -1$, and has the period l. By $U(\cdot, \tau)$, $Q(\cdot, \tau)$ we denote macro-displacement and corrector fields, respectively, defined on a certain interval $(-L, L)$ of the x-axis, where $L \gg l$. The derivatives of $F(\cdot)$ with respect to $x \equiv x_1$ will be denoted by $F'(\cdot)$, and the averaging operator $\langle\cdot\rangle$ will now be related to a segment $(-l/2, l/2)$ of the x-axis. From Eqs. (5.1), neglecting the body forces, we obtain

$$
\begin{aligned}
\langle E\rangle U''(x,\tau) + \langle Eh'\rangle Q'(x,\tau) &= \langle\varrho\rangle \ddot{U}(x,\tau)\,,\\
\langle\varrho hh\rangle \ddot{Q}(x,\tau) + \langle Eh'h'\rangle Q(x,\tau) &= -\langle Eh'\rangle U'(x,\tau)\,.
\end{aligned}
\tag{6.1}
$$

Setting $\phi_1 \equiv a/l$, $\phi_2 \equiv (l-a)/l$ we obtain

$$
\begin{gathered}
\langle\varrho\rangle = \phi_1\varrho_1 + \phi_2\varrho_2, \quad \langle E\rangle = \phi_1 E_1 + \phi_2 E_2\,,\\
\langle Eh'\rangle = 2(E_1 - E_2), \quad \langle Eh'h'\rangle = 4(E_1/\phi_1 + E_2/\phi_2)\,,\\
\langle\varrho hh\rangle = l^2/3 \times \langle\varrho\rangle = l^2/3 \cdot (\phi_1\varrho_1 + \phi_2\varrho_2)\,.
\end{gathered}
$$

The aim of the following section is to analyze the problems of free vibrations and wave propagation related to the micro-periodic bar under consideration. The analysis will be based on Eqs. (6.1).

6.1. Vibrations

Let $x \in (-L/2, L/2)$ where $L \gg l$ and $\tau \in (-\infty, \infty)$. It can be observed that Eqs. (6.1) have a solution of the form

$$
U(x,\tau) = 0, \quad Q(x,\tau) = A\cos\mu\tau + B\sin\mu\tau\,,
$$

where A, B are arbitrary constants and $\mu^2 = \langle Eh'h'\rangle/\langle\varrho hh\rangle$. The positive constant μ will be referred to as the free micro-vibration frequency, and will play an important role in the subsequent analysis.

First we shall look for the solution of Eqs. (6.1) in the form $U(x,\tau) = U_0(x)\exp(i\omega\tau)$, $Q(x,\tau) = Q_0(x)\exp(i\omega\tau)$. Hence

$$
\begin{aligned}
\langle E\rangle U_0''(x) + \langle\varrho\rangle\omega^2 U_0(x) + \langle Eh'\rangle Q_0'(x) &= 0\,,\\
(\langle\varrho hh\rangle\omega^2 - \langle Eh'h'\rangle)Q_0(x) - \langle Eh'\rangle U_0'(x) &= 0\,;
\end{aligned}
\tag{6.2}
$$

after simple manipulations, defining $E^{\text{eff}} \equiv \langle E\rangle - \dfrac{\langle Eh'\rangle^2}{\langle Eh'h'\rangle} = (\phi_1/E_1 + \phi_2/E_2)^{-1}$ (cf. [1]) and introducing the micro-vibration frequency μ we obtain

$$
\left[\frac{E^{\text{eff}}}{\langle E\rangle} - \left(\frac{\omega}{\mu}\right)^2\right] U_0''(x) + \frac{\langle\varrho\rangle}{\langle E\rangle}\omega^2\left[1 - \left(\frac{\omega}{\mu}\right)^2\right] U_0(x) = 0\,;
\tag{6.3}
$$

a similar equation we obtain also for $Q_0(x)$.

REMARK. The effective modulus E^{eff} introduced above makes it possible in the asymptotic approximation ($l \to 0$), to represent the first equation of Eq. (4.1) in the form $E^{\text{eff}}U''(x,\tau) = \langle\varrho\rangle\ddot{U}(x,\tau)$. Let us also observe that $E^{\text{eff}}/\langle E\rangle < 1$.

From Eq. (6.3) it follows that (A, B are arbitrary constants):

(i) if $(\omega/\mu)^2 < E^{\text{eff}}/\langle E\rangle$ or $(\omega/\mu)^2 > 1$ then there exist sinusoidal vibrations $U_0(x) = A\cos kx$, $Q_0(x) = B\sin kx$;

(ii) if $E^{\text{eff}}/\langle E\rangle < (\omega/\mu)^2 < 1$ then there exist exponential vibrations $U_0(x) = A\cosh kx$, $Q_0(x) = B\sinh kx$;

(iii) if $(\omega/\mu)^2 = 1$ or $(\omega/\mu)^2 = E^{\text{eff}}/\langle E\rangle$ then we arrive at the degenerate or trivial case, respectively.

This classification holds if $\langle Eh'\rangle \neq 0$; if $\langle Eh'\rangle = 0$ then only sinusoidal vibrations are possible.

In the case (i) of sinusoidal waves, by substituting $U_0(x) = A\cos kx, Q_0(x) = B\sin kx$ into Eqs. (4.2) we obtain nontrivial solutions ($A \neq 0, B \neq 0$) only if

$$\begin{vmatrix} \omega^2\langle\varrho\rangle - k^2\langle E\rangle & k\langle Eh'\rangle \\ k\langle Eh'\rangle & \omega^2\langle\varrho hh\rangle - \langle Eh'h'\rangle \end{vmatrix} = 0\,.$$

Introducing the micro-vibration frequency μ defined above we obtain finally

$$\omega^2 = \frac{E^{\text{eff}}}{\langle\varrho\rangle}k^2 + \left(\omega^2 - \frac{\langle E\rangle}{\langle\varrho\rangle}k^2\right)\left(\frac{\omega}{\mu}\right)^2. \tag{6.4}$$

The second term on the RHS of Eq. (6.4) describes the dispersion effect due to the micro-heterogeneous structure of the bar, i.e. the nonlinear interrelation between ω and k. For a homogeneous bar $E^{\text{eff}} = \langle E\rangle$ and Eq. (6.4) yields $\omega^2\langle\varrho\rangle = \langle E\rangle k^2, \langle E\rangle = E$. If $l \to 0$ then $\mu \to \infty$ and the dispersion effect disappears. It has to be emphasized that $U_0(\cdot)$, $Q_0(\cdot)$ have to be V-macro fields (cf. Sec. 2), where now V reduces to the line segment $(-l/2, l/2)$. Hence, for sinusoidal waves the obtained results have a physical sense only if $lk \ll 1$. Treating lk as a small parameter we can derive from Eq. (6.4) the formula

$$\omega^2 = \frac{E^{\text{eff}}}{\langle\varrho\rangle}k^2\left[1 - \frac{1}{3}(lk)^2\frac{\langle Eh'\rangle^2}{\langle Eh'h'\rangle^2}\right] + o(k^2l^2)\,,$$

which represents ω^2 in the explicit form.

In the case (ii) of exponential waves, after substituting $U_0(x) = A\cosh kx$, $Q_0(x) = B\sinh kx$ into Eqs. (6.2), we obtain nontrivial solutions only if

$$\begin{vmatrix} \omega^2\langle\varrho\rangle + k^2\langle E\rangle & k\langle Eh'\rangle \\ -k\langle Eh'\rangle & \omega^2\langle\varrho hh\rangle - \langle Eh'h'\rangle \end{vmatrix} = 0\,.$$

Hence, after introducing μ, we obtain

$$\omega^2 = \frac{\langle E\rangle}{\langle\varrho\rangle}k^2\left[\left(\frac{\omega}{\mu}\right)^2 - \frac{E^{\text{eff}}}{\langle E\rangle}\right] + \omega^2\left(\frac{\omega}{\mu}\right)^2. \tag{6.5}$$

It has to be emphasized that Eq. (6.5) has a physical sense only for micro-heterogeneous bars, because in the case of homogeneity $E^{\text{eff}}/\langle E\rangle = 1$ and there are no exponential vibrations.

Summing up we conclude that the micro-periodic heterogeneity of bars leads to the dispersion effects and to the exponential vibrations which can not be treated on the basis of the asymptotic homogenized models of periodic structures.

6.2. Wave propagation

At the end of this example the wave propagation problem in a infinite micro-heterogeneous bar will be discussed. To this end we look for a solution of Eqs. (6.1) in the form $U(x,\tau) = F(x - c\tau)$, $Q(x,\tau) = G(x - c\tau)$, where c is the wave propagation velocity. Setting $\zeta \equiv x - c\tau$ we shall use the notation $F' = dF/d\zeta$, $G' \equiv dG/d\zeta$. Then from Eqs. (6.1) after manipulations, we obtain

$$(\langle E\rangle - \langle\rho\rangle c^2)\frac{c^2}{\mu^2}F'' + (E^{\text{eff}} - \langle\rho\rangle c^2)F = 0\,, \tag{6.6}$$

and a similar equation for G. Eq. (6.6) implies the following special cases of wave propagation in a micro-periodic bar:

(i) sinusoidal waves if $c^2 < (E^{\text{eff}}/\langle\varrho\rangle)$ or $c^2 > (\langle E\rangle/\langle\varrho\rangle)$;

(ii) exponential waves if $E^{\text{eff}}/\langle\varrho > c^2 < \langle E\rangle/\langle\varrho\rangle$;

(iii) degenerate case if $c^2 = E^{\text{eff}}/\langle\varrho\rangle$ or $c^2 = \langle E\rangle/\langle\varrho\rangle$.

Conditions (i)–(iii) hold if $\langle Eh'\rangle \neq 0$; for $\langle Eh'\rangle = 0$, $c^2 = \langle E\rangle/\langle\varrho\rangle$. In the case (i), substituting $U(x,\tau) = A\sin k(x - c\tau)$, $Q(x,\tau) = B\cos k(x - c\tau)$, $k = 2\pi/L$, where L is the wavelength and A, B are arbitrary constants, into Eqs. (6.1) we obtain nontrivial solutions only if

$$c^2\langle\varrho\rangle = E^{\text{eff}} + (\langle\varrho\rangle c^2 - \langle E\rangle)\frac{c^2k^2}{\mu^2}\,. \tag{6.7}$$

The second term on the RHS (6.7) describes the effect of dispersion. The obtained result has a physical sense only if $lk = 2\pi l/L \ll 1$ because functions $U(\cdot,\tau)$, $Q(\cdot,\tau)$ have to be V-macro functions, cf. Sec. 2. Treating kl as a small parameter and bearing in mind that $\mu^2 = \langle Eh'h'\rangle/\langle\varrho hh\rangle = 3\langle Eh'h'\rangle/l^2\langle\varrho\rangle$ from Eq. (6.7) we obtain

$$c^2 = \frac{E^{\text{eff}}}{\langle\varrho\rangle}\left[1 - \frac{1}{3}(kl)^2\frac{\langle Eh'\rangle^2}{\langle Eh'h'\rangle^2}\right] + o(k^2l^2)\,.$$

It can be seen that for $kl \to 0$ the dispersion effect disappears.

Summing up we see that the micro-heterogeneity of a bar implies the existence of exponential waves and dispersion. These effects can not be investigated on a basis of asymptotic homogenization equations (for $l \to 0$), i.e. for the known homogenized models of a bar.

7. Final remarks

The example given in Sec. 6 illustrates the fact that the refined macro-dynamics can be successfully applied to dynamic problems for micro-periodic structures. Among the advantages of the proposed macro-modelling approach we can mention the relatively simple form of the resulting equations (5.1). Moreover, the governing equations of refined macro-dynamics (4.7) and (4.8) can be obtained without any solution to a boundary value problem on the r.v.e. The main drawback lies in an imprecise choice of the micro-shape functions $h_\alpha(\cdot)$ based often on the intuition of the researcher. Different applications of the proposed method of modelling as well as the possible generalization of this approach are now under consideration and will be presented separately.

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POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH

Received April 21, 1993.

Arch. Mech., 45, 3, pp. 305–331, Warszawa 1993

Virial coefficients, collective modes and problems with the Galerkin procedure

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DISCUSSED ARE SOME PROBLEMS in collective dynamics of discrete and continuous systems, and discretization procedure based on imposing appropriate holonomic constraints and truncation of certain hierarchy of moment equations. The relationships with the Galerkin procedure is analyzed with special emphasis on certain relatively popular misuses of this method. As a special example, very instructive and geometrically privileged, the model of affinely-rigid body is reviewed in some geometric details. In this paper we deal with purely mechanical problems. Thermal phenomena and constitutive assumptions concerning reactions of thermal constraints will be discussed in a forthcoming paper.

1. D'Alembert principle, Galerkin techniques and truncation of the hierarchy of moments

LET US CONSIDER an arbitrary system of material points, without precising at this stage its being discrete or continuous. Position of the a-th material point at the time instant t will be denoted by $x(t, a)$. Newton's equations of motion may be written in the following form:

$$\frac{\partial^2 x}{\partial t^2}(t, a) = \Phi\left[x(\cdot, \cdot), \frac{\partial x}{\partial t}(\cdot, \cdot);\ t, a\right], \tag{1.1}$$

Φ denoting the density of forces per unit mass. This is a system of ordinary differential equations labelled by the "index" a; the latter may have a finite, denumerable, or continuous range. The density Φ may depend on the evolution $(t, a) \to x(t, a)$ in a functional way; we symbolize this dependence by using the square brackets. If there is no retardation or memory, Eq. (1.1) simplifies to

$$\frac{\partial^2 x}{\partial t^2}(t, a) = \Phi\left[x(t, \cdot), \frac{\partial x}{\partial t}(t, \cdot); t, a\right]. \tag{1.2}$$

If the system is finite, it is customary to write a as a capital subscript $A = 1, \ldots, N$, thus, Eq. (1.2) becomes

$$m_A \frac{d^2 x_A}{dt^2}(t) = F_A\left(x_1(t), \ldots, x_N(t); \frac{dx_1}{dt}(t), \ldots, \frac{dx_N}{dt}(t)\right), \tag{1.3}$$

m_A denotes the mass of the A-th material point, and $F_A = m_A \Phi_A$ — the force affecting this point. This is the usual system of Newton equations for N-particle systems.

If the system is continuous, the label a becomes Lagrangian radius-vector (material variables). In the special case of simple elastic bodies free of external interactions (dynamically homogeneous physical space), the density Φ in Eq. (1.2) is a local algebraic function of the placement, i.e., of the first derivative of the Eulerian radius-vector x with respect to the Lagrangian radius-vector a,

$$\Phi\left[x(t, \cdot), \frac{\partial x}{\partial t}(t, \cdot);\ t, a\right] = \Phi(\nabla_a x(t, .);\ t, a). \tag{1.4}$$

The system (1.1) then becomes a partial differential equation for the time-dependent vector field x:

$$\frac{\partial^2 x}{\partial t^2} = \Phi(\nabla_a x(t,\cdot);\ t, a). \tag{1.5}$$

When the body is materially homogeneous, there is no dependence of Φ on the last argument a.

By the analogy with Eq. (1.3), one writes equations of motion of a continuous body in the form

$$\varrho_0(a)\frac{\partial^2 x}{\partial t^2}(t,a) = \mathcal{F}\left[x(t,.), \frac{\partial x}{\partial t}(t,.); t, a\right], \tag{1.6}$$

where ϱ_0 denotes the Lagrangian density of mass, and $\mathcal{F} = \varrho_0\Phi$ — the Lagrangian density of forces per unit non-deformed volume. When written in Eulerian terms, Eq. (1.2) becomes

$$\varrho(t,x)\frac{Dv}{Dt}(t,x) = f[a(t,.), v(t,.); t, x], \tag{1.7}$$

where $a(t,\cdot)$ denotes the inverse mapping of $x(t,\cdot)$, i.e., $x(t, a(t,u)) = u$ for any arguments t and u; $v(t,\cdot)$ denotes the Eulerian velocity field, i.e., $v(u) = \dfrac{\partial x}{\partial t}(t, a(t,u))$ ($\dfrac{\partial}{\partial t}$ acting only on the first argument), and f denotes the Eulerian density of forces per unit deformed volume. The Euler velocity field may be expressed through the fields $a(\cdot,\cdot)$ as follows:

$$v^i(t,x) = -\frac{\partial x^i}{\partial a^j}(t, a(t,x))\frac{\partial a^j}{\partial t}(t,x). \tag{1.8}$$

D/Dt denotes the substantial derivative, thus

$$\frac{Dv^i}{Dt} = \frac{\partial v^i}{\partial t} + v^j\frac{\partial v^i}{\partial x^j}.$$

In the special case of fluids f does not depend on the $a(t,\cdot)$ — variable.

To become a closed system of equations, Eq. (1.7) must be completed by the continuity equation

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho v) = 0, \tag{1.9}$$

and — if we consider the full thermomechanical theory — by an appropriate equation describing the dynamics of the temperature field.

In a wide range of problems, including certain general considerations, one does not need to distinguish between continuous and discrete case. We shall use then the general form (1.1), (1.2) and represent the Lagrangian mass distribution by a positive measure μ on the set of labels a, i.e., on the material body. The Euler mass distribution in a given configuration φ (φ is a mapping from the material space to the physical space, e.g., $a \mapsto x(t,a)$) is represented by the measure $\nu = \mu_\varphi$ obtained from μ by the φ-transport; thus,

$$\int (f \circ \varphi)(a)\, d\mu(a) = \int f(x)\, d\mu_\varphi(x)$$

for any function f on the physical space.

The unique treatment of discrete and continuous systems is effectively applied in the theory of collective modes and in moment methods; we shall also follow it. Nevertheless,

it must be stressed, that there is a deep qualitative gap between both cases, and one must be very careful when considering them on the same footing. The most spectacular example we know is the theorem proved by H. ZORSKI [27] about the impossibility of dynamical injection of discrete systems into continua. Fortunately, the matter considered below is neutral with respect to those delicate and troublesome problems.

The material measure μ gives rise to the following scalar product of functions on the set of material points:

$$\langle f|g\rangle := \int \overline{f}(a)g(a)\,d\mu(a). \tag{1.10}$$

The symbol $L^2(\mu)$ will denote the space of functions square-integrable in the μ-sense, i.e., such ones that $\langle f|f\rangle < \infty$. Similarly, for the vector-valued material functions, i.e., functions assigning spatial vectors to material points, we define the scalar product:

$$G[u,w] := g_{ij}\langle u^i|w^j\rangle = g_{ij}\int u^i(a)w^j(a)\,d\mu(a), \tag{1.11}$$

where g denotes the spatial metric tensor (in Cartesian coordinates $g_{ij} = \delta_{ij}$).

Kinetic energy and power of forces Φ are given by the following functionals:

$$T\left[\frac{\partial x}{\partial t}(t,\cdot)\right] = \frac{1}{2}G\left[\frac{\partial x}{\partial t}(t,\cdot),\frac{\partial x}{\partial t}(t,\cdot)\right], \tag{1.12}$$

$$\mathcal{P}\left[x(t,\cdot),\frac{\partial x}{\partial t}(t,\cdot)\right] = G\left[\frac{\partial x}{\partial t}(t,\cdot),\Phi\left[x(t,\cdot),\frac{\partial x}{\partial t}(t,\cdot);t,\cdot\right]\right]. \tag{1.13}$$

The Hilbert space $L^2(\mu)$ is separable; let us choose a complete system of real-valued functions H^r, $r = 0,1,2,\ldots$. For certain reasons, which become clear later, it is convenient to include here a constant function; we shall put $H^0 = 1$ and denote the remaining H^r by H^ϱ, $\varrho = 1,2,\ldots$.

Let us calculate the moments of equations of motion (1.2) with respect to the complete system H^r, i.e., take the scalar products (1.10) of equations of motion and functions H^r. The resulting equations have the form of balance laws

$$\frac{d}{dt}M^{ri} = N^{ri}, \quad r = 0,1,2,\ldots, \quad i = 1,2,3, \tag{1.14}$$

where

$$M^{ri} = \left\langle H^r \middle| \frac{\partial x^i}{\partial t}\right\rangle, \quad N^{ri} = \left\langle H^r \middle| \Phi^i\right\rangle. \tag{1.15}$$

The quantities M^{ri}, N^{ri} are, respectively, H^r-moments of the distribution of linear momentum and the distribution of forces within the body. They provide a global, collective representation of those distributions. In many problems one is interested rather in averaged, materially smeared out moments (1.15) than in one-particle quantities like

$$\pi_a\left[\frac{\partial x}{\partial t}(t,\cdot)\right] = \pi\left(\frac{\partial x}{\partial t}(t,a)\right) = \varrho_0(a)\frac{\partial x}{\partial t}(t,a),$$

etc. It is clear that M^{0i}, N^{0i} equal, respectively, the total linear momentum of the body and the total force acting upon it,

$$M^{0i} = P^i, \quad N^{0i} = F^i, \tag{1.16}$$

and this is one of reasons for using $H^0 = 1$ as an element of the complete system H^r.

In principle, the system of moments (1.15) is equivalent to the one-particle distribution of linear momentum and forces, i.e., these distributions may be reconstructed from Eqs. (1.15). Thus, in the case of continuous medium, Eqs. (1.14) provide some kind of infinite discretization, because the system (1.1), labelled by the continuous variable a, is replaced by a countable system labelled by the discrete index r. However, in the case of solids, the dynamical moments N^{ri} are not functions of the kinematical moments M^{ri} alone, thus, Eq. (1.14) is not an effective dynamical system for M^{ri}. Lagrangian representation (1.1), (1.2) is adequate for solids, but non-effective for fluids, thus, when applying the moment techniques to fluids and to physical fields interacting with continua, one has to use the Eulerian form (1.7) of equations of motion. Instead of the material functions one uses then functions defined on the physical Euclidean space E^3. Projecting equations (1.7) onto elements of an appropriately chosen complete system of spatial functions, one obtains a discrete system of equations; in certain problems of fluid dynamics such a system provides an effective tool of integration or qualitative discussion. For example, calculating multipole moments of Eqs. (1.7) with respect to Eulerian coordinates, one obtains the hierarchy of so-called virial equations used for a long time in hydrodynamical problems of astrophysics and in the theory of the shape of Earth. Nevertheless, even in the case of fluids, described in Eulerian terms, the direct calculation of moments does not lead automatically to a closed dynamical system.

To transform Eqs. (1.14), at least formally, in a dynamical system, one has to introduce explicitly the H-moments of configurations, i.e., the quantities $\langle H^r | x^i \rangle$. Thus, we expand configurations $x(t, \cdot)$ with respect to the mode functions H^r,

$$x^i(t, a) = \sum_r q^i{}_r(t) H^r(a), \tag{1.17}$$

and substitute this expansion to Eqs. (1.14). We obtain the following system of ordinary differential equations for expansion coefficients $q^i{}_r$:

$$\sum_s Q^{rs} \frac{d^2}{dt^2} q^i{}_s = N^{ri}\left(q, \frac{dq}{dt}, t\right), \quad r = 0, 1, 2, \ldots, \quad i = 1, 2, 3, \tag{1.18}$$

where

$$Q^{rs} = \langle H^r | H^s \rangle = \int \overline{H^r(a)} H^s(a)\, d\mu(a). \tag{1.19}$$

If we consider a fluid, described by equations

$$\varrho(t, x) \frac{Dv}{Dt}(t, x) = f[v(t, \cdot); t, x], \tag{1.20}$$

and calculate the Eulerian moments of Eq. (1.20) with respect to some complete system F^r of functions on E^3, then the most natural way of transforming the resulting system of moment equations in a closed dynamical system consists in expanding the Eulerian velocity field with respect to F^r,

$$v(t, x) = \sum_r \nu_r(t) F^r(x). \tag{1.21}$$

Substituting Eq. (1.21) to Eq. (1.20) one obtains in principle a countable dynamical system for coefficients ν_r.

In this paper we concentrate mainly on the Lagrangian description and Eqs. (1.18). The mode functions H^r need not be mutually orthogonal and normalized; there are

numerous applications where non-orthonormal systems are more convenient. Thus, we do not assume that $Q^{rs} = \delta^{rs}$. It is natural to use real functions H^r and real coefficients q^i_r. If they were complex, we would have to use additional conditions for coefficients $q^i{}_r$ to ensure the reality of Eulerian coordinates x^i.

Equations (1.2), (1.18) are in principle equivalent. If the system is finite or discrete, then the transition from Eq. (1.2) to Eq. (1.18) is a change of coordinates — the natural variables x^i_A are replaced by generalized coordinates $q^i{}_r$. Any $q^i{}_r$ depends in general on all radius-vectors x_A, i.e., q are collective coordinates, parametrizing coherent multiparticle motions. The same is true in the continuous case, and, by abuse of language, we can say that the transition from the field variables $x(t,\cdot)$ to the discrete parameters $q^i{}_r$ is a change of representation, replacing the one-particle coordinates $x(t,a)$ by the collective variables $q^i{}_r$. As seen from the formula (1.17), the quantity $q^i{}_r$ is an amplitude of the intensity with which the collective mode H^r occurs in a given configuration x.

Q^{rs} are quadrupole moments of the Lagrangian mass distribution. They are H-collective coefficients of inertia. In the sequel the Einstein summation convention will be applied not only to the spatial indices, but also to the Hilbert-space labels, thus we write

$$x^i(t,a) = q^i{}_r(t)H^r(a), \quad Q^{rs}\frac{d^2}{dt^2}q^i{}_s = N^{ri}\left(q, \frac{dq}{dt}, t\right). \tag{1.22}$$

In equations of motion (1.18) the centre of mass motion and the relative motion of constituents are mixed in a non-physical way. However, if the system of mode functions H^r includes the constant function $H^0 = 1$, which we always assume, it is relatively easy to separate these two kinds of degrees of freedom. The parameters $q^i{}_\varrho$, $\varrho = 1, 2, \ldots$ refer to the relative motion, and the radius-vector of the centre of mass is expressed by generalized coordinates as follows:

$$q^i = \frac{1}{M}Q^r q^i{}_r = q^i{}_0 + \frac{1}{M}Q^\varrho q^i{}_\varrho, \tag{1.23}$$

where $M = \int d\mu$ is the total mass of the body, and

$$Q^r = H^r(a)\, d\mu(a) \tag{1.24}$$

is the H^r-dipole moment of the mass distribution; obviously,

$$Q^0 = M, \quad Q^{00} = M, \quad Q^{\varrho 0} = Q^\varrho. \tag{1.25}$$

Inertial properties of the body in relative motion are described by coefficients

$$Q^{\varrho\sigma}_{\text{int}} := Q^{\varrho\sigma} - \frac{1}{M}Q^\varrho Q^\sigma; \tag{1.26}$$

the subscript int refers to the internal motion, i.e., motion with respect to the centre of mass reference frame.

As stated above, our considerations are based on the assumption that all multipole moments Q are finite; in particular, the total mass M is finite. In foundations of statistical mechanics and in general theory of dynamical systems one considers also infinite systems of particles with the infinite total mass. The scheme presented here does not work for such systems because, as a rule, the inertial parameters Q will be infinite. Moreover, the very splitting of motion into translational and internal parts breaks down, because the centre of mass is no longer well-defined. Only in certain exceptional situations, namely, when the total linear momentum is finite, the centre of mass reference frame may be defined. We are indebted for this observation to S. PIEKARSKI.

Equations of motion (1.18) may be rewritten in the following form:

$$(1.27)\quad M\frac{d^2q^i}{dt^2} = F^i, \quad Q^{\varrho\sigma}_{\text{int}}\frac{d^2}{dt^2}q^i{}_\sigma = N^{\varrho i} - \frac{1}{M}Q^\varrho F^i, \quad i = 1,2,3, \quad \varrho = 1,2\ldots,$$

i.e., they consist of two systems of balance laws

$$(1.28)\qquad \frac{dp^i}{dt} = F^i, \qquad \frac{d}{dt}M^{\varrho i}_{\text{int}} = N^{\varrho i}_{\text{int}},$$

where $M_{\text{int}}, N_{\text{int}}$ denote the relative-motion-parts of M, N,

$$(1.29)\qquad M^{\varrho i}_{\text{int}} = Q^{\varrho\sigma}_{\text{int}}\frac{d}{dt}q^i_\sigma, \quad N^{\varrho i}_{\text{int}} = N^{\varrho i} - \frac{1}{M}Q^\varrho F^i.$$

In general, both F^i and $N^{\sigma i}_{\text{int}}$ depend on all arguments q^i, $\frac{dq^i}{dt}$, q^i_σ, $\frac{dq^i{}_\sigma}{dt}$, t; there is a coupling between translational and internal dynamics. However, in the theory of multi-particle systems and in the continuum theory, we are usually interested in a pure relative (internal) motion, where $F^i = 0$ and $N^{\varrho i}_{\text{int}}$ depend only on $q^i{}_\sigma$, $\frac{dq^i{}_\sigma}{dt}$, t. Taking into account the expansion (1.17) and the separation of translational and internal motion, we can rewrite the formulas for kinetic energy and power as follows:

$$(1.30)\qquad \begin{aligned} T &= \frac{1}{2}g_{ij}\frac{dq^i{}_r}{dt}\frac{dq^j{}_s}{dt}Q^{rs} = \frac{M}{2}g_{ij}\frac{dq^i}{dt}\frac{dq^j}{dt} + \frac{1}{2}g_{ij}\frac{dq^i{}_\varrho}{dt}\frac{dq^j{}_\sigma}{dt}Q^{\varrho\sigma}_{\text{int}}, \\ \mathcal{P} &= g_{ij}\frac{dq^i{}_r}{dt}N^{jr} = g_{ij}\frac{dq^i}{dt}F^j + g_{ij}\frac{dq^i{}_\varrho}{dt}N^{\varrho j}_{\text{int}}. \end{aligned}$$

In this way the system of one-particle variables $x(a)$ has been systematically replaced by the collective generalized coordinates q^i. In particular, if we deal with a continuous medium, i.e., if the label a is a threedimensional radius-vector running over the material manifold, then the above procedure results in an infinite discretization. This means that continuous fields $a \mapsto x(a)$ describing configurations are replaced by elements of $\mathbb{R}^\infty$, i.e., by infinite arrays of numbers $q^i{}_1, q^i{}_2, \ldots$. The functions H^r represent the basic collective modes; their choice depends on the particular physical nature of the problem. Reasonable choices are somehow suited to the structure of interactions. By this we mean, e.g., that for any r the degrees of freedom corresponding to generalized coordinates $q^i{}_1, \ldots, q^i{}_r$ are relatively autonomous, i.e., weakly affected by the degrees of freedom described by parameters $q^i{}_p$, $p > r$. In a wide range of problems it is convenient to choose and order the modes H^r in such a way that for increasing r the oscillations of functions H^r become faster (starting from the non-oscillating $H^0 = 1$). This corresponds to the hierarchy of increasing lengths of the excited waves. In many problems it is only a few long-wave modes that is relevant for the considered phenomena and decides upon their qualitative features. It is so not only in microphysical problems like the dynamics of suspensions, but also in macroscopic elastic problems, including technical applications.

If the system is continuous, its equations of motion (1.1), (1.2) are equivalent to an infinite system of ordinary differential equations of the form (1.18). Such infinite systems are computationally non-effective and do not provide any real simplification of models based on finite systems of partial differential equations. The same concerns finite but very large systems (1.3). However, in situations where it is only a few approximately autonomous long-wave modes that decides about the qualitative behaviour of the system,

we can perform an effective finite discretization of Eqs. (1.1), (1.2). This is achieved by imposing the discretization constraints.

Let us assume that it is known, from some theoretical considerations or from experimental data, that in a given range of phenomena described by Eqs. (1.2), motion of the system holds approximately in a finite-dimensional linear subspace Δ of $L^2(\mu)$, e.g., in the linear span of modes $H^0, H^1, \ldots, H^N$,

$$\Delta = \Big\{ \sum_{r=0}^{N} c_r H^r, c_r \in \mathbb{R} \Big\}. \tag{1.31}$$

By "approximately holds" we mean that, for any $\varepsilon > 0$, there exists an open range of initial conditions for which all trajectories do not depart from Δ by the Hilbert distance larger than ε, uniformy all over the time axis. Usually this follows from the fact that, due to a special organization of internal interactions, the subspace Δ has attractive properties, and for sufficiently small open domains of initial states, the system performs small vibrations about Δ. This is the typical mechanism of holonomic constraints in mechanics. Performing orthogonal projections of those oscillating trajectories onto Δ one obtains constrained trajectories, as seen by observers who do not notice oscillations or do not take them into account. Such constrained trajectories satisfy equations of motion following from the d'Alembert principle.

Our constraints equations have the form

$$q^i{}_p = 0, \quad p > N. \tag{1.32}$$

According to the d'Alembert principle, it would be wrong to substitute algebraically equations (1.32) to the primary equations of motion (1.2). Instead, equations of constraints should be substituted to modified equations of motion, involving some *a priori* non-specified reaction forces,

$$\frac{\partial^2 x}{\partial t^2} = \Phi + \Phi_R, \tag{1.33}$$

where the power of reactions Φ_R is assumed to vanish on any virtual velocity field V compatible with Eq. (1.32), i.e.,

$$G[V, \Phi_R] = 0 \tag{1.34}$$

if

$$V = \sum_{r=0}^{N} V_r H^r. \tag{1.35}$$

Thus, using Eqs. (1.18), we have

$$\frac{d}{dt} M^{ri} = Q^{rs} \frac{d^2}{dt^2} q^i{}_s = N^{ri} + N_R^{ri}, \quad r = 0, 1, 2, \ldots, \tag{1.36}$$

where

$$g_{ij} \sum_{r=0}^{N} N_R^{ri} V^j{}_r = 0 \tag{1.37}$$

for any system $V^k{}_1, \ldots, V^k{}_N$, $k = 1, 2, 3$. Expression N_R^{ri} in Eq. (1.36) denotes the dynamical H^r-moment built of the reaction forces,

$$N_R^{ri} = \langle H^r | \Phi_R^i \rangle. \tag{1.38}$$

Condition (1.37) means that all moments of reaction forces with $r \leq N$ vanish,

$$N_R^{ri} = 0, \quad \text{i.e.,} \quad F_R = 0, \quad N_R^{\varrho i} = 0, \quad r = 0, 1, 2, \ldots, N, \quad \varrho = 1, 2, \ldots, N. \tag{1.39}$$

Therefore, the first $(N + 1)$-tuple of the moment equations (1.36) is free of reaction forces, although the forces Φ_R themselves do not vanish. All motions compatible with holonomic constraints (1.32) satisfy the subsystem $r = 0, 1, \ldots, N$ with algebraically substituted conditions (1.32),

$$\frac{d}{dt} M^{ri} | \Delta = \sum_{s=0}^{N} Q^{rs} \frac{d^2}{dt^2} q^i{}_s = N^{ri}\left(q, \frac{dq}{dt}, t\right), \quad r = 0, 1, 2, \ldots. \tag{1.40}$$

Separating the translational and internal motion we obtain

$$M \frac{d^2 q^i}{dt^2} = F^i, \quad \frac{d}{dt} M_{\text{int}}^{\varrho i} = \sum_{\sigma=1}^{N} Q_{\text{int}}^{\varrho\sigma} \frac{d^2}{dt^2} q^i{}_\sigma = N_{\text{int}}^{\varrho i}(q, \frac{dq}{dt}, t). \tag{1.41}$$

When restricted to the range $0, 1, \ldots, N$, the index r will be denoted by capital Latin letters; similarly, for $\varrho = 1, \ldots, N$, we shall use capital Greek letters. It is a generic situation that the $(N + 1) \times (N + 1)$ matrix $[Q^{AB}]$ is non-degenerate; obviously, the $N \times N$ matrix $[Q^{\Sigma\Omega}]$ is then also non-singular. Therefore, Eqs. (1.41) with $\varrho = 1, \ldots, N$ is an effective system of equations of motion for the Δ-constrained system. All higher modes H^r, $r > N$, are then non-excited; $q^i{}_r = 0$ for $r > N$. Equations (1.41) with $\varrho = N + 1, N + 2, \ldots$ are superfluous. They may be used, however, for determining the reaction forces Φ_R.

Constraints (1.32) reduce the kinetic energy expression to

$$T = \frac{1}{2} g_{ij} \frac{dq^i{}_A}{dt} \frac{dq^j{}_B}{dt} Q^{AB} = \frac{M}{2} g_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt} + \frac{1}{2} g_{ij} \frac{dq^i{}_\Sigma}{dt} \frac{dq^j{}_\Omega}{dt} Q_{\text{int}}^{\Sigma\Omega}. \tag{1.42}$$

In this way we obtain some problem from the realm of analytical mechanics of systems with a finite number of degrees of freedom. Coefficients q^i_Σ are generalized coordinates, M and $Q^{\Sigma\Omega}$ are inertial parameters, and F^i, $N^{\Sigma i}$ are generalized forces. The assumed nonsingularity of $[Q_{\text{int}}^{\Sigma\Omega}]$ implies that the system (1.41) is regular, i.e., solvable with respect to generalized accelerations,

$$\frac{d^2 q^i{}_\Sigma}{dt^2} = Q_{\text{int}\Sigma\Omega} N_{\text{int}}^{\Omega i}, \tag{1.43}$$

where the matrix $[Q_{\text{int}\Sigma\Omega}]$ is reciprocal of $[Q_{\text{int}}^{\Sigma\Omega}]$, i.e.,

$$Q_{\text{int}\Sigma\Omega} Q_{\text{int}}^{\Omega\Delta} = \delta_\Sigma^\Delta. \tag{1.44}$$

This method provides an effective discretization of continuous systems and computational simplification of systems with a finite but large number of degrees of freedom.

As mentioned, the moment procedure is applicable also to field-theoretical problems in the physical space E^3, in particular, to the Eulerian formulation of the theory of continua. Lagrangian coordinates a are then expressed as functions of Eulerian variables x, i.e., as fields on E^3. We expand them with respect to the mode functions F^r used for calculating

the moments of Eqs. (1.7),

$$a(t,x) = \sum_r \alpha_r(t) F^r(x), \tag{1.45}$$

substitute this expansion to Eq. (1.7), and obtain a denumerable-infinite system of ordinary differential equations for α_r. There are some problems, however, with the truncation procedure. Namely, when the fields $x \mapsto a(t,x)$ expressing material coordinates through spatial ones are confined to some proper functional subspace, e.g., to a finite-dimensional space Γ spanned by functions $F^0, F^1, \ldots, F^N$, then, except for the very special situations (very special choices of functions F^r), the corresponding Eulerian velocity fields $x \mapsto v(t,x)$ fail to be elements of the same subspace, i.e., $v(t,\cdot)$ cannot be expressed as a linear combination of F^r. Thus, the F-moments n_R of reaction forces f_R need not vanish when contracted with virtual velocities, and it is impossible to obtain an effective system of differential equations for $\alpha_R(t)$, $R = 0, 1, \ldots, N$, by taking the F^R-moments of Eq. (1.7) ($R = 0, 1, \ldots, N$) and substituting $a(t,x) = \alpha_R(t) F^R(x)$. For solids the quantity f in Eq. (1.7) depends on $a(t,\cdot)$, thus, the method of moments and holonomic constraints is non-adequate as a tool of finite discretization. For fluids the situation is somewhat better, because f does not depend on $a(t,.)$, and the Eulerian velocity field $v(t,.)$ may be considered as a primary field. Thus, one could try to introduce constraints of finite discretization by truncation of Eq. (1.21),

$$v(t,x) = \sum_{r=0}^{N} \nu_r(t) F^r(x) = \nu_R(t) F^R(x), \tag{1.46}$$

i.e., by the direct assumption that it is the Eulerian velocity field which is confined to a finite-dimensional subspace Γ. This procedure is correct and the d'Alembert principle enables one to eliminate reactions f_R and obtain an effective system of $N+1$ ordinary differential equations for functions ν_R. There is one delicate point however, namely, the constraints (1.46) are in general non-holonomic. Thus, the problem becomes very complicated on the level of the field $a(t,\cdot)$, i.e., when in addition to the knowledge of $v(t,\cdot)$ we wish to follow the motion of fluid particles.

In spite of the very widespread use of moment techniques and discretization procedures, the understanding of the above-quoted subtle points is rather poor in the literature. It is typical that one does not notice the problem at all, and, for example, substitutes algebraically the finitely truncated expansion (1.45) to the $(N+1)$-tuple of equations obtained as F^R-moments of Eq. (1.7). One obtains in this way a closed dynamical system for the parameters α_R, $R = 0, 1, \ldots, N$, however, in spite of its formal consistency, this system is physically wrong, or at least non-justified, with an exception of some special cases. Fortunately, usually it is just those special cases that is practically used, thus, the final results are often correct, although incorrectly derived.

Those misunderstandings have to do with certain misuses of the Galerkin discretization procedure and other related methods. The Galerkin method of finding approximate solutions of differential equations like (1.5) or (1.7) consists in what follows [12]. The first step is to choose a finite number of mode functions $\mathcal{H}^R$, $R = 0, 1, \ldots, N$, satisfying appropriate boundary or asymptotic conditions. One expands then the unknown field functions Ψ with respect to $\mathcal{H}^R$,

$$\Psi = c_R \mathcal{H}^R, \tag{1.47}$$

c_R being yet non-specified parameters. Let us write differential equation to be solved in a symbolic operator form:

$$\mathcal{L}[\Psi] = 0. \tag{1.48}$$

For example, if we consider equations (1.2), and $\Psi = x(\cdot, \cdot)$, then

$$\mathcal{L}[x] = \frac{\partial^2 x}{\partial t^2} - \Phi\left[x, \frac{\partial x}{\partial t}; t, a\right]. \tag{1.49}$$

If we use the Euler form (1.7) and the field variables $\Psi = a(\cdot, \cdot)$, then

$$\mathcal{L}[a] = \varrho \frac{Dv}{Dt} - f[a, v; t, x], \tag{1.50}$$

where the Euler velocity field v is expressed through a as in Eq. (1.8). Substituting expressions (1.47) to $\mathcal{L}$ we obtain the following $(N + 1)$-parameter family of functions

$$E[c_0, c_1, \ldots, c_N] := \mathcal{L}[c_R \mathcal{H}^R]. \tag{1.51}$$

In general, functions (1.47) do not satisfy Eq. (1.48), thus $E[c_0, c_1, \ldots, c_N] \neq 0$. The next step of obtaining an approximate solution is to minimize the distance between zero and expression (1.51) with respect to the parameters c_R. The function Ψ (1.47) corresponding to optimal values of c_R, minimizing E, is an approximate solution of Eq. (1.48). The crucial point is to define a convenient optimization criterion for Eq. (1.51). To some extent, the proper choice is a matter of intuition and depends on our purposes. In some problems it is convenient to base the concept of "smallness" of $E[c_0, c_1, \ldots, c_N]$ on the L^2-norm or supremum norm in the space of Ψ. The Galerkin criterion consists in vanishing of the Hilbert-space projection of E onto the appropriately chosen finite-dimensional subspace Λ spanned by functions $\mathcal{K}^0, \mathcal{K}^1, \ldots, \mathcal{K}^M$. In other words, one substitutes the assumed form (1.47) to the considered equations $\mathcal{L}[\Psi] = 0$, and then one calculates the $\mathcal{K}$-moments of those equations. This results in $(M + 1)$ algebraic equations for determining the optimal $(N + 1)$-tuple of parameters c_R. Usually one puts $M = N$ and $\mathcal{K}^R = \mathcal{H}^R$, i.e., the same functions are used as basic modes and as moment weights, just as in the above-quoted derivation of collective dynamics and discretization constraints. Obviously, the Galerkin method may be modified so as to discretize the dependence of field functions on some subset of independent variables, but without imposing *a priori* restrictions like Eq. (1.47) on the dependence upon other coordinates. In particular, in mechanics of continuous media it is natural to exclude the time variable and, subject to the discretization procedure, the dependence of Eulerian coordinates on the Lagrangian ones or conversely. The only difference is that c_R are no longer constants, but become functions of time, and the above optimization procedure leads to ordinary differential equations for $c_R(t)$, $R = 0, 1, \ldots, N$. Applying this modified procedure to Eqs. (1.2), identifying $\mathcal{H}^R$ and $\mathcal{K}^R$ with $\mathcal{H}^R$, and c_R with q^i_R, we obtain exactly Eqs. (1.41), (1.43). Thus, in this case the Galerkin method and the constraints-based discretization are essentially identical. The reason is that, when applied to the Lagrangian representation of continuum equations, the Galerkin procedure is equivalent to the d'Alembert principle. Thus, there is nothing wrong in algebraic substitution of constraints to the moment equations, without the explicit introduction of reaction forces. When working in the Eulerian representation we can also formally apply the Galerkin method, however, this would lead to equations we have rejected above as inconsistent with the d'Alembert principle. Thus, the Galerkin procedure must not be automatically

applied to mechanical problems, without taking any care of the physical adequacy of the criterion it uses. The point is that this method answers correctly the question: what expressions within a given class do optimally approximate rigorous solutions in the sense of a given minimization criterion. However, in physical problems the terms like "a given class" and "a given criterion" must be not only rigorously defined, as they are in the Galerkin procedure, but also justified on the basis of some physical arguments. If it is known from experiment, or from some, even rough, physical considerations that (in a good approximation) the system moves in a neighbourhood of some stable submanifold in its configuration space and that this confinement is due to some internal elastic-like forces responsible for the small oscillations about the stable surface, then the proper criterion is given by the d'Alembert principle. This fact is not only deduced from experimental data, but may also be formally derived from the "micromodel" of constraints; cf. in this respect the reasoning presented in the books by LANCZOS and ARNOLD [1, 10]. It means that there is no exchange of energy between the "along constraints" and "across constraints" motion, thus, no pumping of energy from outside is necessary to maintain the constraints. If in addition to mechanical fields also other fields are included into the treatment, e.g., temperature or electromagnetic field, the situation becomes more complicated, because there are no traditional and well-established hints like the d'Alembert principle. In our opinion, there is no general, universal algorithm and any problem of this kind must be separately analyzed on the basis of some, even rough, micromodel. In thermodynamics and electrodynamics of continua, the best way is to maintain the constraints with the help of the formally introduced heat sources [9] and charges; they will play a similar role as reaction forces in mechanics. The problem of constitutive relations for constraints-maintaining forces was analyzed by TRUESDELL and NOLL [20]. More recently, the very detailed analysis was given by WOŹNIAK [21–26].

If one deals with a purely mechanical self-adjoint model, i.e., with a non-dissipative problem described by equations of motion derivable from a variational principle, then the d'Alembert principle for holonomic constraints follows directly from the substitution of constraints equations to the action functional. Discretization of variational problems is much simpler; there are traditional methods developed by Ritz, Tshebyshev and others.

The characteristic feature of the above-described discretization procedure based on functional series is that the discretization mapping is linear in generalized coordinates q,

$$f^i(\ldots, q^i{}_r, \ldots; a) = \sum_{r=0}^{N} q^i{}_r H^r(a). \tag{1.52}$$

By discretization mapping we mean an injection of a finite-dimensional configuration space Q into a functional space of all *a priori* possible configurations. This restriction is not necessary; it is possible to use discretizations depending in a general nonlinear way on generalized coordinates, $x = f(q, a)$. The range of the argument q may be a differential manifold, not necessarily a linear space. The expression (1.13) for power takes the following form on motions $t \mapsto x_t = f(q(t), \cdot)$ compatible with discretization constraints:

$$\mathcal{P} = g_{ij} \int \frac{\partial f^i(q, a)}{\partial q^B} \frac{dq^B}{dt} \Phi^j \left[f(q(t), a), \frac{\partial}{\partial t} f(q(t), a); t, a \right] d\mu(a). \tag{1.53}$$

Therefore, the effective, reaction-free equations of motion may be obtained by calculating

moments of the original equations (1.2) with respect to functions

$$a \mapsto \frac{\partial f^i}{\partial q^B}(q, a); \tag{1.54}$$

these functions will be called discretization gradients.

Just as in the case of linear constraints, moments are meant in the sense of the scalar product (1.10). The resulting equations read:

$$Q_{AB}\frac{d^2q^B}{dt^2} + [ABC]\frac{dq^B}{dt}\frac{dq^C}{dt} = N_A, \tag{1.55}$$

i.e.,

$$\frac{d^2q^A}{dt^2} + \left\{ \begin{matrix} A \\ BC \end{matrix} \right\}\frac{dq^B}{dt}\frac{dq^C}{dt} = Q^{AB}N_B, \tag{1.56}$$

where

$$Q_{AB} := G\left[\frac{\partial f}{\partial q^A}, \frac{\partial f}{\partial q^B}\right] \tag{1.57}$$

denotes the metric tensor induced by f from the Hilbert space $L^2(\mu)$ in the manifold Q, and $\left\{ \begin{matrix} A \\ BC \end{matrix} \right\}$ are the corresponding Christoffel symbols,

$$\left\{ \begin{matrix} A \\ BC \end{matrix} \right\} = \frac{1}{2}Q^{AD}(Q_{DB,C} + Q_{DC,B} - Q_{BC,D}) = Q^{AD}[DBC], \tag{1.58}$$

the commas denoting partial derivatives. Generalized forces N_A are given by

$$N_A = G\left[\frac{\partial f}{\partial q^A}, \Phi\right]. \tag{1.59}$$

Using the Q-covariant differentiation D/Dt along curves in Q, we can write Eqs. (1.56) as

$$\frac{D^2q^A}{Dt^2} = Q^{AB}N_B. \tag{1.60}$$

It may be convenient to use the balance form, analogous to Eq. (1.28),

$$\frac{DM_A}{Dt} = N_A, \tag{1.61}$$

where M_A are moments of the distribution of linear momentum with respect to the components of discretization gradients,

$$M_A = G\left[\frac{\partial f}{\partial q^A}, \frac{\partial x}{\partial t}\right] = Q_{AB}\frac{dq^B}{dt}. \tag{1.62}$$

If the function f is linear in variables q^A, $f^i(q, \cdot) = q^i{}_\alpha H^\alpha$, A becoming the multi-index $({}^i{}_\alpha)$, then the formulas above reduce to those derived previously for the discretization procedure based on moments. Nonlinear discretization is useful in certain applications like deformations of rods [11]. Besides, it may lead to interesting models in micromechanics.

2. Multipole moments, virial coefficients

In a wide class of problems it is polynomials and trigonometric functions of material coordinates that provides the most convenient and intuitive model of collective modes. From the purely mathematical point of view this model is satisfactory, because it is possible to approximate all sufficiently regular functions in a compact domain by polynomials and Fourier series. At the same time, both models are well-suited to the hierarchy of decreasing lengths of excited waves. The moments of fields with respect to homogeneous polynomials, first of all the moments of densities of extensive physical quantities, are known in physics as multipole moments and virial coeffiecients [2]. One is faced with them even in the elementary course of physics, e.g., in electrostatics. Usually a few lowest-order multipoles provide a satisfactory description of the spatial distribution of physical quantities within bounded domains; let us mention, e.g., electrostatic multipoles. If we use homogeneous polynomials as mode functions, then the expansion (1.17) becomes something very peculiar from the mathematical point of view, namely, the Taylor expansion of analytic functions. This is an additional distinguishing feature of polynomial discretization and polynomial collective modes.

The method of multipole moments is widely used in continuum theory, including the mechanics of generalized media with internal degrees of freedom — micromorphic continua [4]–[8]. Unfortunately, in literature there is plenty of misunderstandings following from the above-mentioned physical misuse of the Galerkin procedure. The point is that, in the case of multipoles and polynomial expansions it is especially easy to overlook the problem and confuse the Lagrangian and Eulerian moments.

Functions H^r used in polynomial method are homogeneous polynomials of Cartesian material coordinates. Thus, r is a multi-index of the form $r = (l, A_1, \ldots, A_l)$, where $l = 0, 1, 2, \ldots$ and $A_i = 1, 2, 3$

$$\{H^r\} = \{1, a^K, a^K a^L, \ldots, a^{K_1} \ldots a^{K_p}, \ldots\}. \tag{2.1}$$

In spite of their non-bounded character they belong to $L^2(\mu)$, because in realistic problems the measure μ is compactly-supported. The moments M^{ri}, N^{ri} become the following tensorial objects:

$${}_kM^{A_1 \ldots A_k i} = \int a^{A_1} \ldots a^{A_k} \frac{\partial x^i}{\partial t}(t, a) d\mu(a), \tag{2.2}$$

$${}_kN^{A_1 \ldots A_k i} = \int a^{A_1} \ldots a^{A_k} \Phi^i(t, a) d\mu(a). \tag{2.3}$$

They are mixed material-spatial quantities, because the indices $A_1, \ldots, A_k$ refer to the material space, and i is the usual spatial index. In other words, ${}_kM$, ${}_kN$ are partially Lagrangian and partially Eulerian objects. They are k-th order multipole moments of the material distribution of linear momentum and forces. We can define them both for continuous and discrete systems; in the latter case the integration in Eqs. (2.2), (2.3) becomes the summation over reference positions of material points. In continuous medium with internal contact forces and without boundary loads, ${}_kN$ may be expressed through the multipole moment of the Piola–Kirchhoff stress tensorial density

$${}_kN^{A_1 \ldots A_k i} = -\sum_{p=1}^{k} a^{A_1} \ldots a^{A_{p-1}} T^{A_p i} a^{A_{p+1}} \ldots a^{A_k}. \tag{2.4}$$

Inertial moments are given by Lagrangian multipoles of the mass distribution,

$$ {}_kQ^{A_1\ldots A_k} = \int a^{A_1}\ldots a^{A_k}\, d\mu(a). \tag{2.5}$$

They are completely symmetric tensors in the material space. Their internal parts are given by

$$ Q^{rs}_{\text{int}} = \underset{m,k}{Q}{}^{A_1\ldots A_m B_1\ldots B_k} = \underset{m+k}{Q}{}^{A_1\ldots A_m B_1\ldots B_k} - \frac{1}{M}\underset{m}{Q}{}^{A_1\ldots A_m}\underset{k}{Q}{}^{B_1\ldots B_k}, \tag{2.6}$$

where $r = (m; A_1, \ldots, A_m)$, $s = (k; B_1, \ldots, B_k)$.

In particular, $\underset{1}{Q}{}^{A} = \int a^A\, d\mu(a) = Mq^A$, where q^A denotes the radius-vector of the centre of mass in reference configuration. If we choose material coordinates a^K in such a way that the centre of mass of the reference state is placed at zero, then $\underset{1}{Q} = 0$. The quadrupole moment of the mass distribution $\underset{2}{Q}{}^{AB}_{\text{int}}$ is algebraically equivalent to the co-moving components of the tensor of inertia I^{AB} used, e.g., in rigid body mechanics,

$$ I^{AB} = \eta_{CD}\underset{2}{Q}{}^{CD}\eta^{AB} - \underset{2}{Q}{}^{AB}, \tag{2.7}$$

where η denotes the material metric tensor. In Cartesian coordinates $\eta_{AB} = \delta_{AB}$. If we choose those coordinates in such a way that $\underset{1}{Q} = 0$, then

$$ \underset{2}{Q}{}^{AB} = \underset{1,1}{Q}{}^{AB} = \underset{2}{Q}{}^{AB}_{\text{int}}. \tag{2.8}$$

Expression of instantaneous configurations $x(t, .)$ through the collective generalized coordinates q has the form:

$$ x^i(t,a) = {}_0q^i(t) + {}_1q^i{}_K(t)a^K + {}_2q^i{}_{KL}(t)a^K a^L + \ldots + \ldots + {}_pq^i{}_{K_1\ldots K_p}(t)a^{K_1}\ldots a^{K_p} + \ldots. \tag{2.9}$$

In certain applications it may be convenient to use another form of this expansion, expressed through spherical functions of angular variables,

$$ x^i(t,a) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} q^i{}_{lm}(t)|a|^l Y^{lm}\left(\frac{a}{|a|}\right), \tag{2.10}$$

where $|a|$ denotes the length of the vector a. Obviously, x is a real quantity, thus the coefficients $q^i{}_{lm}$ must be subject to the following condition:

$$ \overline{q^i{}_{lm}} = q^i{}_{l-m}. \tag{2.11}$$

The obvious advantage of Eq. (2.10) is that there are fewer redundant variables among $q^i{}_{lm}$. The coefficients ${}_pq^i{}_{K_1\ldots K_p}$ are fully symmetric in material indices, thus, only those corresponding to $K_1 \leq \ldots \leq K_p$ are independent generalized coordinates.

It must be stressed that, in general, ${}_0q^i$ do not coincide with coordinates of the spatial position of the centre of mass q^i. Namely,

$$ q^i - {}_0q^i = \frac{1}{M}\sum_{m=1}^{\infty} {}_mq^i{}_{A_1\ldots A_m}\underset{m}{Q}{}^{A_1\ldots A_m}. \tag{2.12}$$

Thus,

$$x^i(t,a) - q^i(t) = \sum_{m=1}^{\infty} {}_m q^i{}_{A_1\ldots A_m}(t)\left(a^{A_1}\ldots a^{A_m} - \frac{1}{M}Q_m^{A_1\ldots A_m}\right). \tag{2.13}$$

Kinetic moments ${}_lM$ have the form

$${}_lM^{A_1\ldots A_l i} = \sum_{m=0}^{\infty} Q_{m+l}^{A_1\ldots A_l B_1\ldots B_m} \frac{d}{dt}{}_m q^i{}_{B_1\ldots B_m}. \tag{2.14}$$

Decomposing them into translational and internal parts, we obtain

$${}_lM = {}_lM_{\text{tr}} + {}_lM_{\text{int}}, \tag{2.15}$$

where

$${}_lM_{\text{tr}}^{A_1\ldots A_l i} = Q_l^{A_1\ldots A_l}\frac{dq^i}{dt}, \tag{2.16}$$

$${}_lM_{\text{int}}^{A_1\ldots A_l i} = \sum_{m=1}^{\infty} Q_{l,m}^{A_1\ldots A_l B_1\ldots B_m}\frac{d}{dt}{}_m q^i_{B_1\ldots B_m}. \tag{2.17}$$

Similarly, for dynamical moments ${}_kN$ we have

$${}_lN = {}_lN_{\text{tr}} + {}_lN_{\text{int}}, \tag{2.18}$$

where

$$\begin{aligned} {}_lN_{\text{int}}^{A_1\ldots A_l i} &= {}_lN^{A_1\ldots A_l i} - \frac{1}{M}Q_l^{A_1\ldots A_l}F^i, \\ {}_lN_{\text{tr}}^{A_1\ldots A_l i} &= \frac{1}{M}Q_l^{A_1\ldots A_l}F^i. \end{aligned} \tag{2.19}$$

Equations of internal (relative) motion are:

$$\frac{d}{dt}{}_lM^{A_1\ldots A_l i} = \sum_{m=1}^{\infty} Q_{l,m}^{A_1\ldots A_l B_1\ldots B_m}\frac{d^2}{dt^2}{}_m q^i{}_{B_1\ldots B_m} = {}_lN^{A_1\ldots A_l i}, \tag{2.20}$$

$$l = 1, 2, \ldots.$$

The splitting of kinetic energy has the form

$$T = \frac{1}{2}g_{ij}\frac{dq^i}{dt}\frac{dq^j}{dt} + \frac{1}{2}g_{ij}\sum_{m,l=1}^{\infty}\frac{d}{dt}{}_m q^i{}_{A_1\ldots A_m} Q_{m,l}^{A_1\ldots A_m;B_1\ldots B_l}\frac{d}{dt}{}_l q^j{}_{B_1\ldots B_l}. \tag{2.21}$$

Polynomial constraints consist in assuming that configurations are described by polynomials of a given finite degree k, i.e., ${}_pq = 0$ if $p > k$. All infinite series (2.9), (2.14), (2.17), (2.20), (2.21) are then truncated at the step $m = k$ and become finite sums. The effective system of equations of motion has the form

$$M\frac{d^2q^i}{dt^2} = F^i, \quad \frac{d}{dt}{}_lM^{A_1\ldots A_l i} = {}_lN^{A_1\ldots A_l i}, \quad l = 1, \ldots, k, \tag{2.22}$$

with directly substituted constraints ${}_pq = 0$, $p > k$.

Let us notice that in spite of our use of terms like discretization, the above derivation is general and works also for discrete systems with collective modes. For example, for a

system of $N < \infty$ material points we have

$$(2.23) \qquad {}_lM^{A_1\dots A_l i} = \sum_{p=1}^{N} M_p v_p^i X_p^{A_1} \dots X_p^{A_l},$$

$$(2.24) \qquad {}_lN^{A_1\dots A_l i} = \sum_{p=1}^{N} X_p^{A_1} \dots X_p^{A_l} F_p^i,$$

where v_p, F_p denote, respectively, the velocity of the p-th material point and the force influencing it; X_p^A are the reference coordinates of the p-th particle. Similarly,

$$(2.25) \qquad {}_mQ^{A_1\dots A_m} = \sum_{p=1}^{N} X_p^{A_1} \dots X_p^{A_m} M_p.$$

Obviously, for any finite system there exists a certain critical number k of polynomial collective modes which cannot be exceeded without introducing fictitious degrees of freedom.

The above derivation of collective dynamics is based on multipoles in the material space, thus, it is automatically compatible with the d'Alembert principle. However, Lagrangian multipoles are not very intuitive quantities. It seems more natural to work with Eulerian multipoles, i.e., multipole moments with respect to spatial coordinates. In any case, spatial multipoles, having all tensorial indices in the physical space, are more operational quantities, and certainly are better suited to intuition developed in electrostatics. And in fact, many authors use them, and, unfortunately, combining them with the Galerkin procedure, commit the mistake described above and consisting in the violation of the d'Alembert principle. Let us discuss these problems in some detail.

The Eulerian multipole moments of linear momentum and forces are given by [2]:

$$(2.26) \qquad {}_lm^{i_1\dots i_l j} = \int x^{i_1}(t,a)\dots x^{i_l}(t,a)\frac{\partial x^j}{\partial t}(t,a)\,d\mu(a),$$

$$(2.27) \qquad {}_ln^{i_1\dots i_l j} = \int x^{i_1}(t,a)\dots x^{i_l}(t,a)\Phi^j(t,a)\,d\mu(a).$$

Both terms are symmetric in the first l-tuple of indices. In continuous medium with internal contact forces we have

$$(2.28) \qquad {}_ln^{i_1\dots i_l j} = -\sum_{p=1}^{l} \int x^{i_1}\dots x^{i_{p-1}}\sigma^{i_p j}x^{i_{p+1}}\dots x^{i_l},$$

where σ denotes the Cauchy stress tensor.

It is possible to express Eulerian multipoles through Lagrangian ones and generalized coordinates q. Namely, in polynomial theory of degree k we have

$$(2.29) \qquad m^{a\dots f_j} = \sum_{\alpha\dots\varphi=0}^{k} {}_\alpha q^a{}_{A_1\dots A_\alpha}\cdots{}_\varphi q^f{}_{F_1\dots F_\varphi(\alpha+\dots+\varphi)} M^{A_1\dots F_\varphi j},$$

and analogously for dynamical multipoles:

$$(2.30) \qquad n^{a\dots f_j} = \sum_{\alpha\dots\varphi=0}^{k} {}_\alpha q^a{}_{A_1\dots A_\alpha}\cdots{}_\varphi q^f{}_{F_1\dots F_\varphi(\alpha+\dots+\varphi)} N^{A_1\dots F_\varphi j}.$$

If we do not perform polynomial truncation and base on analytic functions, the same formulas remain valid after putting $k = \infty$; the sums (2.29), (2.30) become then infinite series.

The formulas (2.29), (2.30) tell us that Eulerian multipoles of degree l depend linearly (with functional coefficients) on Lagrangian multipoles of degrees $p = 0, 1, \ldots, kl$. Thus, we see again that the Eulerian multipoles of degrees $r = 0, 1, \ldots, k$ calculated for Eqs. (1.7), (1.20) do not lead to effective equations of motion of polynomially constrained body. The reason is that on the right-hand side there occur also Lagrangian dynamical multipoles ${}_sN$, $s = k + 1, \ldots, kl$. These multipoles are calculated for the total forces, involving both the given forces and the reactions maintaining constraints. However, d'Alembert principle implies only the vanishing of reaction multipoles ${}_rN_R$, $r = 0, 1, \ldots, k$, but not ${}_sN_R$, $s = k + 1, \ldots, kl$. Nevertheless, it is instructive to write down explicitly the system of Eulerian moment equations. For non-constrained systems, with configurations described by analytic functions, we have:

$$\frac{d}{dt}\,{}_lm = {}_ln + \sum_{p=1}^{k} \sum_{r_1 \ldots r_l s=0}^{\infty} \left({}_{r_1}q \otimes \ldots \otimes \frac{d}{dt}\,{}_{r_k}q \otimes \ldots \otimes {}_{r_l}q \otimes \frac{d}{dt}\,{}_s q\right) {}_{r_1+\ldots+r_m+s}Q \tag{2.31}$$

i.e., equivalently,

$$\sum_{r_1 \ldots r_m s=0}^{\infty} \left({}_{r_1}q \otimes \ldots \otimes {}_{r_l}q \otimes \frac{d^2}{dt^2}\,{}_s q\right) {}_{r_1+\ldots+r_l+s}Q = {}_ln. \tag{2.32}$$

It is clear that the balance law for ${}_lm$ does not reduce to any conservation principle even in the interaction-free case ${}_ln = 0$. This is prevented by the strongly nonlinear and purely kinematical term on the right-hand side of Eq. (2.31). When we impose polynomial constraints of a finite degree k, then on the right-hand side of all Eqs. (2.31), (2.32) with $l > 1$ the moments ${}_ln$ involve non-specified reaction terms ${}_ln_R$, thus, any $(k + 1)$-tuple of Eqs. (2.31), (2.32), in particular that one coresponding to $l = 0, 1, \ldots, k$ is either non-effective or simply wrong if one neglects the reaction moment.

Another important difference between Eulerian and Lagrangian moments is that, in contrast to ${}_lM$ and ${}_lN$, the multipoles ${}_lm$ and ${}_ln$ with $l > 1$ do not split into translational and orbital parts. Instead, they are very complicated superpositions of terms involving mutually mixed translational and internal coordinates.

In all these respects the dipole multipoles ${}_1m$ and ${}_1n$ provide a very interesting exception. This is important for physical applications and also has deep geometric reasons.

Equations (2.29), (2.30) imply that in the k-polynomial theory the dipole moments may be expressed as follows:

$$m^{ij} = \sum_{r=0}^{k} {}_rq^i{}_{A_1 \ldots A_r r}M^{A_1 \ldots A_r j}, \tag{2.33}$$

$$n^{ij} = \sum_{r=0}^{k} {}_rq^i{}_{A_1 \ldots A_r r}N^{A_1 \ldots A_r j}; \tag{2.34}$$

for non-constrained systems described in terms of analytic functions, $k = \infty$, i.e., the sums in Eqs. (2.33), (2.34) become infinite series. Expression (2.33) may be explicitly

written as

$$m = \sum_{r,s=0}^{k} \left({}_{r}q \otimes \frac{d_s q}{dt} \right) Q_{r+s} . \tag{2.35}$$

It is seen from the formula (2.34) that the dynamical dipole ${}_1n$ does not involve Lagrangian multipoles of any degree $s > k$. Thus, when the system is subject to k-polynomial constraints, the quantity ${}_1n$ is built only on given forces Φ but not on reactions Φ_R; ${}_1n_R = 0$. This implies that if $k \geq 1$, then besides of the monopole equation describing the centre of mass motion,

$$\frac{dP^i}{dt} = M \frac{d^2 q^i}{dt^2} = F^i , \tag{2.36}$$

also the dipole equation, i.e. Eqs. (2.31), (2.32) with $l = 1$, may be used as a subsystem of the effective system of equations of motion. This subsystem may be written as

$$\frac{d_1 m}{dt} = {}_1n + \sum_{r,s=0}^{k} \left(\frac{d_r q}{dt} \otimes \frac{d_s q}{dt} \right) Q_{r+s} , \tag{2.37}$$

i.e.,

$$\sum_{r,s=0}^{k} \left({}_{r}q \otimes \frac{d_s^2 q}{dt^2} \right) Q_{r+s} = {}_1n . \tag{2.38}$$

In analytic terms:

$$\frac{dm^{ij}}{dt} = n^{ij} + \sum_{r,s=0}^{k} \frac{d}{dt} {}_{r}q^{i}{}_{A_1 \ldots A_r} \frac{d}{dt} {}_{s}q^{j}{}_{B_1 \ldots B_s} Q_{r+s}^{\ A_1 \ldots A_r B_1 \ldots B_s} , \tag{2.39}$$

$$\sum_{r,s=0}^{k} {}_{r}q^{i}{}_{A_1 \ldots A_r} \frac{d^2}{dt^2} {}_{s}q^{j}{}_{B_1 \ldots B_s} Q_{r+s}^{\ A_1 \ldots A_r B_1 \ldots B_s} = n^{ij} . \tag{2.40}$$

In spite of its exceptional properties, the quantity ${}_1m$ is not a constant of motion even in the interaction-free case. This non-conservation of m^{ij} may be interpreted as a consequence of the parametric dependence of the kinetic energy (2.21) on the metric tensor; namely, we can rewrite Eqs. (2.39) as

$$\frac{dm^{ij}}{dt} = n^{ij} + 2 \frac{\partial T}{\partial g_{ij}} . \tag{2.41}$$

It is seen that the kinetic term preventing the conservation of ${}_1m$ is a symmetric tensor, thus, there are no kinetic obstacles against the conservation of $m^{[ij]}$, the skew-symmetric part of m^{ij},

$$\frac{d}{dt}(m^{ij} - m^{ji}) = n^{ij} - n^{ji} . \tag{2.42}$$

This is physically obvious, because the quantity $J^{ij} := m^{ij} - m^{ji}$ is nothing else but the total angular momentum of our system, related to the origin of coordinates x^i. Similarly, $D^{ij} := n^{ij} - n^{ji}$ is the total moment of forces with respect to the same origin. To be more in accord with traditional language: the angular momentum and the moment of

forces are axial vectors J^i, D^i, related to the skew-symmetric tensors J^{ij}, D^{ij} through the totally antisymmetric symbol ε_{ijk},

$$J^{ij} = \varepsilon^{ijk} J_k, \quad D^{ij} = \varepsilon^{ijk} D_k. \tag{2.43}$$

The next distinguishing feature of the moments ${}_1m$, ${}_1n$ among all other multipoles is their natural splitting into translational and internal parts,

$${}_1m = {}_1m_{\text{tr}} + {}_1m_{\text{int}}, \quad {}_1n = {}_1n_{\text{tr}} + {}_1n_{\text{int}}. \tag{2.44}$$

The internal moments ${}_1m_{\text{int}}$, ${}_1n_{\text{int}}$ are defined with the help of the formulas (2.26), (2.27) modified by substituting the quantities $x^i - q^i$ instead of x^i. In other words, they are dipole moments with respect to the instantaneous position of the centre of mass, whereas the total moments ${}_1m$, ${}_1n$ are related to a fixed origin in space (origin of coordinates x^i). Thus, ${}_1m_{\text{int}}$, ${}_1n_{\text{int}}$ are unambiguously defined tensorial quantities. One can easily show that

$${}_1m^{ij}_{\text{tr}} = q^i P^j = Mq^i \frac{dq^j}{dt}, \quad {}_1n^{ij}_{\text{tr}} = q^i F^j, \tag{2.45}$$

$${}_1m^{ij}_{\text{int}} = \sum_{\varrho=1}^{k} {}_\varrho q^i{}_{A_1 \dots A_\varrho} M^{A_1 \dots A_\varrho j}_{\text{int}} = \sum_{\varrho,\sigma=1}^{k} {}_\varrho q^i{}_{A_1 \dots A_\varrho} Q^{\ A_1 \dots A_\varrho B_1 \dots B_\sigma}_{\varrho,\sigma} \frac{d}{dt}{}_\sigma q^j_{B_1 \dots B_\sigma}, \tag{2.46}$$

$${}_1n^{ij}_{\text{int}} = \sum_{\varrho=1}^{k} {}_\varrho q^i_{A_1 \dots A_\varrho} N^{A_1 \dots A_\varrho j}_{\text{int}}. \tag{2.47}$$

It is seen that ${}_1m_{\text{int}}$ depends only on internal coordinates and velocities, whereas ${}_1m_{\text{tr}}$ involves only the centre of mass position and velocity.

Obviously, the same splitting is true for the antisymmetric parts of dipole moments.

$$J = L + S, \quad D = D_{\text{tr}} + D_{\text{int}}, \tag{2.48}$$

where L, S denote, respectively, the angular momentum of the centre of mass with respect to a fixed origin, and the spin, i.e., the angular momentum in the centre of mass reference frame. It is clear that

$$\begin{aligned} L^{ij} &= {}_1m^{ij}_{\text{tr}} - {}_1m^{ji}_{\text{tr}} = q^i P^j - q^j P^i = Mq^i \frac{dq^j}{dt} - Mq^j \frac{dq^i}{dt}, \\ S^{ij} &= {}_1m^{ij}_{\text{int}} - {}_1m^{ji}_{\text{int}}, \\ D^{ij}_{\text{tr}} &= N^{ij}_{\text{tr}} - N^{ji}_{\text{tr}} = q^i F^j - q^j F^i, \\ D^{ij}_{\text{int}} &= N^{ij}_{\text{int}} - N^{ji}_{\text{int}}. \end{aligned} \tag{2.49}$$

The balance law for the internal part of ${}_1m$ has the form

$$\frac{d}{dt}{}_1m_{\text{int}} = {}_1n_{\text{int}} + \sum_{r,s=1}^{k} \left(\frac{d_r q}{dt} \otimes \frac{d_s q}{dt} \right) Q_{\text{int}}{}_{r,s}, \tag{2.50}$$

i.e.,

$$\sum_{r,s=1}^{k} \left({}_r q \otimes \frac{d^2}{dt^2}{}_s q \right) Q_{r,s} = {}_1n_{\text{int}}. \tag{2.51}$$

The subsystem of Eq. (2.31) consisting of Eqs. (2.36) and (2.37) is equivalent to the system composed of Eqs. (2.36) and (2.50).

The exceptional properties of the Eulerian dipole moments ${}_1m$, ${}_1n$, and the operational status of Eulerian multipoles suggest us to rewrite the system (2.22) in the following, equivalent form of the balance laws:

$$
\begin{aligned}
&\frac{dP^i}{dt} = F^i,\\
&\frac{d}{dt}m^{ij}_{\text{int}} = n^{ij}_{\text{int}} + \sum_{r,s=1}^{k} \frac{d}{dt}{}_r q^i{}_{A_1\ldots A_r} Q^{A_1\ldots A_r B_1\ldots B_s}_{r,s} \frac{d}{dt}{}_s q^j{}_{B_1\ldots B_s},\\
&\frac{d}{dt}{}_l M^{A_1\ldots A_l i}_{\text{int}} = {}_l N^{A_1\ldots A_l i}_{\text{int}}, \quad l = 2,\ldots,k,
\end{aligned}
\tag{2.52}
$$

or, in the explicit terms of generalized coordinates

$$
\begin{aligned}
&M\frac{d^2q^i}{dt^2} = F^i,\\
&\sum_{m,l=1}^{k} {}_m q^i{}_{A_1\ldots A_m} Q^{A_1\ldots A_m B_1\ldots B_l}_{m,l} \frac{d^2}{dt^2}{}_l q^j{}_{B_1\ldots B_l} = n^{ij}_{\text{int}},\\
&\sum_{m=1}^{k} Q^{A_1\ldots A_l B_1\ldots B_m}_{l,m} \frac{d^2}{dt^2}{}_m q^i{}_{B_1\ldots B_m} = {}_l N^{A_1\ldots A_l i}_{\text{int}}, \quad l = 2,\ldots,k.
\end{aligned}
\tag{2.53}
$$

In many realistic problems the hierarchy of Eqs. (2.53) is suited to the ordering of collective modes which corresponds to their decreasing relevance for the phenomena considered. For example, one is often dealing with situation where the leading internal modes correspond to rigid rotations, finite homogeneous deformations, and higher-order polynomial excitations superposed over them, for example, as small corrections. To single out the rotational dynamics, it is convenient to split the second equation (2.53) into the skew-symmetric and symmetric parts.

Let us stress that in the special case of infinitesimal deformations and rotationless motion, the Eulerian equations (2.31) become effective and essentially coincide with Eq. (2.22), because then the spatial and material multipoles differ by higher-order terms.

The exceptional properties and the special role played by the dipole moment m^{ij} have very profound geometrical and group-theoretical roots. To explain this point we shall consider variational dynamical models with Lagrangians of the form $L = T - V$, T being the kinetic energy as above, and V — a potential energy depending only on generalized coordinates q. The corresponding Legendre transformation has the form

$$
\begin{aligned}
p_i &= \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial T}{\partial \dot{q}^i} = M g_{ij}\frac{dq^j}{dt} = g_{ij}P^j,\\
{}_l p^{A_1\ldots A_l}{}_i &= \frac{\partial L}{\partial {}_l\dot{q}^i{}_{A_1\ldots A_l}} = \frac{\partial T}{\partial {}_l\dot{q}^i{}_{A_1\ldots A_l}} = g_{ij}{}_l M^{A_1\ldots A_l j}_{\text{int}},
\end{aligned}
\tag{2.54}
$$

where p_i, ${}_l p^{A_1\ldots A_l}{}_i$ denote canonical momenta, conjugate, respectively, to coordinates q^i, ${}_l q^i{}_{A_1\ldots A_l}$. Obviously, just as ${}_l q^i{}_{A_1\ldots A_l}$, the quantities ${}_l p^{A_1\ldots A_l}{}_i$ are symmetric in indices $A_1,\ldots,A_l$. Independent phase space coordinates are given by q^i, ${}_l q^i{}_{A_1\ldots A_l}$, p_i, ${}_l p^{A_1\ldots A_l}{}_i$

with $i = 1, 2, 3$, $A_1 \leq A_2 \leq \ldots \leq A_l$, $l = 1, \ldots, k$; nevertheless, in many problems it is more convenient to use the symmetric redundant representation, formally admitting all possible orderings of indices.

It is seen that the internal parts of material multipoles are closely related to canonical momenta conjugate with internal coordinates; the relationship is given by the g-lowering of spatial indices. Thus, roughly speaking, the Lagrangian multipoles are Hamiltonian generators of translations in the configuration space of our problem,

$$(2.55) \qquad {}'q^i = q^i + a^i, \quad {}'_l q^i{}_{A_1 \ldots A_l} = {}_l q^i{}_{A_1 \ldots A_l} + {}_l a^i{}_{A_1 \ldots A_l},$$

${}_l a$ denoting constant tensors.

Expressing the spatial dipole moments m^{ij} through canonical variables q^i, ${}_l q^i{}_{A_1 \ldots A_l}$, p_i, ${}_l p^{A_1 \ldots A_l}{}_i$, we obtain in virtue of Eq. (2.54):

$$(2.56) \qquad m^i_{\mathrm{tr}\,j} = q^i p_j,$$

$$(2.57) \qquad m_{\mathrm{int}}{}^i{}_j = \sum_{l=1}^{k} {}_l q^i{}_{A_1 \ldots A_l} {}_l p^{A_1 \ldots A_l}{}_j,$$

where, obviously, $m_{\mathrm{tr}}{}^i{}_j := m_{\mathrm{tr}}{}^{ih} g_{hj}$, $m_{\mathrm{int}}{}^i{}_j := m_{\mathrm{int}}{}^{ih} g_{hj}$. Thus, for the total dipole m^{ij} we have

$$(2.58) \qquad m^i{}_j = m^{ih} g_{hj} = q^i p_j + \sum_{l=1}^{k} {}_l q^i{}_{A_1 \ldots A_l} {}_l p^{A_1 \ldots A_l}{}_j.$$

It may easily be shown that the quantities on the right-hand side are Hamiltonian generators of the spatial linear group $GL(3, \mathbb{R})$. This group acts on the Eulerian variables x^i according to the usual formula, $[x^i] \mapsto [U^i_j x^j]$, $U \in GL(3, \mathbb{R})$. The resulting transformation rule of generalized coordinates reads

$$(2.59) \qquad {}'q^i = U^i{}_j q^j, \quad {}'_l q^i{}_{A_1 \ldots A_l} = U^i{}_j {}_l q^j{}_{A_1 \ldots A_l}.$$

These formulas describe rigid rotations and homogeneous deformations of the body in the physical space. The corresponding transformations of canonical momenta have the form

$$(2.60) \qquad {}'p_i = p_j U^{-1j}{}_i, \quad {}'_l p^{A_1 \ldots A_l}{}_i = {}_l p^{A_1 \ldots A_l}{}_j U^{-1j}{}_i.$$

Let us consider an infinitesimal transformations $U = I + \varepsilon$, I denoting the identity matrix and ε — an arbitrary „small" matrix. For any function F of the phase-space variables q^i, ${}_l q^i{}_{A_1 \ldots A_l}$, p_i, ${}_l p^{A_1 \ldots A_l}{}_i$, we have the following Poisson-bracket expression for the increment of F under $U = I + \varepsilon$:

$$(2.61) \qquad \delta F = \{F, m^i{}_j\} \varepsilon^j{}_i,$$

where

$$\delta F(q, {}_l q, p, {}_l p) = F({}'q, {}_l{}'q, {}'p, {}_l{}'p) - F(q, {}_l q, p, {}_l p),$$

and the formula (2.61) is valid up to second-order terms in ε. Thus, in fact, the quantities $m^i{}_j$ are Hamiltonian generators of $GL(3, \mathbb{R})$ acting as in Eqs. (2.59). As it is seen from Eqs. (2.54), (2.55), the total linear momentum p_i generates spatial translations, i.e., transformations of the Eulerian variables x^i given by $x^i \mapsto x^i + a^i$, a^i being constants. In terms of canonical coordinates:

$$(2.62) \qquad {}'q^i = q^i + a^i, \quad {}_l{}'q = {}_l q, \quad {}'p_i = p_i, \quad {}_l{}'p = {}_l p.$$

Thus, the system of functions ${m^i}_j$, p_i generates the action of the spatial affine group $GAf(3,\mathbb{R}) = GL(3,\mathbb{R}) \times \mathbb{R}^3$.

We have the following system of Poisson brackets, corresponding to the structure constants of $GAf(3,\mathbb{R})$:

$$\begin{aligned} \{{m^i}_j, m^a_b\} &= {\delta^i}_b {m^a}_j - {\delta^a}_j {m^i}_b, \\ \{{m^i}_j, p_a\} &= {\delta^i}_a p_j, \\ \{p_i, p_j\} &= 0. \end{aligned} \tag{2.63}$$

We conclude that the crucial role of the monopole and dipole moments of linear momentum is based on the fact that they generate the spatial affine group. They are distinguished among all multipole moments by the very affine geometry of the physical space. Their balance laws give an account of the relationship between this geometry and the structure of internal interactions. Expressing the angular momentum J through canonical momenta we obtain Hamiltonian generators of spatial rotations,

$${J^i}_j = {m^i}_j - {m_j}^i = {m^i}_j - g_{jk} g^{il} {m^k}_l. \tag{2.64}$$

The system ${J^i}_j$, p_a generates the spatial Euclidean group. If there are no external forces, then for the usual non-polar continuum, the balance of ${J^i}_j$ and p_a becomes the system of conservation laws of angular and linear momentum. The quantities ${S^i}_j$ generate rotations about the centre of mass; the orbital angular momentum ${L^i}_j$ generates rotations of the centre of mass about the origin of spatial coordinates, without affecting the internal variables. Analogous statements may be formulated about ${m_{\text{int}}}^i{}_j$ and ${m_{\text{tr}}}^i{}_j$. In particular, the Poisson brackets for internal parts ${m_{\text{int}}}^i{}_j$, ${S^i}_j$ have the same form as those for the total quantities ${m^i}_j$, ${J^i}_j$. The same is true for translational parts ${m_{\text{tr}}}^i{}_j$, ${L^i}_j$. Obviously, internal quantities are in involution with translational ones, i.e., the mixed Poisson brackets vanish,

$$\{{m_{\text{tr}}}^i{}_j, {m_{\text{int}}}^a{}_b\} = \{p_a, {m_{\text{int}}}^i{}_j\} = \{{L^a}_b, {S^i}_j\} = \{p_a, {S^i}_j\} = 0. \tag{2.65}$$

Transformations generated by higher-order multipoles are not geometrically interesting. Besides, their Poisson brackets do not close to a Lie algebra. If $k > 1$, then the system of Eulerian multipoles ${}_l m$, $l = 0, 1, \ldots, k$ generates an infinite-dimensional Poisson–Lie algebra of phase-space functions.

Besides of spatial transformations one also considers material ones, acting on configurations through an appropriate action on Lagrangian variables a. For example, the material group $GL(3,\mathbb{R})$, i.e., the group of material rotations and homogeneous deformations transforms configurations $a \mapsto x(t,a)$ into configurations $a \mapsto (xU)(t,a)$, $U \in GL(3,\mathbb{R})$, where

$$(xU)(t,a) := x(t, Ua). \tag{2.66}$$

In terms of generalized coordinates q^i, ${}_l q^i{}_{A_1 \ldots A_l}$:

$${}'q^i = q^i, \quad {}'_l q^i{}_{A_1 \ldots A_l} = {}_l q^i{}_{B_1 \ldots B_l} {U^{B_1}}_{A_1} \ldots {U^{B_l}}_{A_l}. \tag{2.67}$$

The first two subsystems in Eqs. (2.52), (2.53) give an account of the geometric part of the dynamics — they are balance laws of the Hamiltonian generators of spatial (Eulerian) affine transformations (2.60), (2.62). The remaining equations (2.52), (2.54), corresponding to $l > 1$, have no natural geometric status.

3. Affinely rigid body and additional constraints

It seems to follow from the above geometric discussion that, at least from the academic point of view, particularly interesting should be such systems whose equations of motion are equivalent to the balance laws of the first two Eulerian multipole [13–19]. For such systems $k = 1$, i.e., the polynomial discretization is truncated at the step of affine functions, or, using a more traditional language, inhomogeneous linear functions,

$$x^i(t,a) = q^i(t) + q^i{}_A(t)a^A. \tag{3.1}$$

In this special case the zeroth-order term coincides with the centre of mass position in space, provided the origin of material radius-vectors is placed at the reference position of the centre of mass. The matrix $[q^i{}_A]$ is non-singular for any time instant t. Thus, at any fixed $t \in \mathbb{R}$, transformations (3.1) are elements of the spatial affine group. In this way, the configuration space [3] becomes identical with the group space of the three-dimensional affine group, i.e., with the homogeneous space of this group, with trivial isotropy groups. The considered system becomes an affinely rigid body. By this statement we mean that during any admissible motion all affine relations between constituents of the body are invariant, in particular, that material straight lines remain straight lines, their parallelism is conserved, and all mutual ratios of segments placed on the same straight lines are constant. In the special case of continuous medium the configuration space becomes the proper affine group $GAf^+(3,\mathbb{R}) = \mathbb{R}^3 \times GL^+(3,\mathbb{R})$, i.e., matrices $[q^i{}_A]$ have positive determinants. For discrete systems the total $GAf(3,\mathbb{R}) = \mathbb{R}^3 \times GL(3,\mathbb{R})$ is in principle admissible. Among all polynomially-discretized models, the affinely-rigid body is peculiar in that its configuration space is a Lie group, or, to be more precise, a group space. Transformations described by polynomials of a finite degree $k > 1$ do not form a group; their composition results in raising the degree.

Equations of motion of affinely constrained body result from the system (2.52) or (2.53) by the direct algebraic substitution ${}_lq = 0$ for $l > 1$ and the neglect of all multipole equations of degree $l > 1$; thus, they have the form:

$$\frac{dp^i}{dt} = F^i, \quad \frac{d}{dt}m^{ij}_{\text{int}} = n^{ij}_{\text{int}} + Q^{AB}\frac{dq^i{}_A}{dt}\frac{dq^i{}_B}{dt}, \tag{3.2}$$

i.e., explicitly,

$$m\frac{d^2q^i}{dt^2} = F^i, \quad q^i{}_A\frac{d^2q^j{}_B}{dt^2}Q^{AB} = n^{ij}_{\text{int}}, \tag{3.3}$$

where $[Q^{AB}]$ denotes the matrix of the quadrupole moment of the mass distribution,

$$Q^{AB} = \underset{2}{Q^{AB}_{\text{int}}} = \underset{1,1}{Q^{AB}} = \int a^A a^B \, d\mu(a). \tag{3.4}$$

As mentioned, $[Q^{AB}]$ is algebraically equivalent to the matrix of co-moving components of the inertial tensor I,

$$I^{AB} = \eta_{CD}Q^{CD}\eta^{AB} - Q^{AB}. \tag{3.5}$$

Let us introduce the quantities

$$\begin{aligned} m^A{}_B &= q^{-1A}{}_i m_{\text{int}}{}^i{}_j q^j{}_B, \quad n^A{}_B = q^{-1A}{}_i n_{\text{int}}{}^i{}_j q^j{}_B, \\ m^{AB} &= q^{-1A}{}_i q^{-1B}{}_j m^{ij}_{\text{int}}, \quad n^{AB} = q^{-1A}{}_i q^{-1B}{}_j n^{ij}_{\text{int}}. \end{aligned} \tag{3.6}$$

It is obvious that they may be interpreted as co-moving components of the tensors $m^i{}_j$, $n^i{}_j$, m^{ij}, n^{ij}, i.e., their components with respect to a reference frame affinely frozen into the body. A certain delicate point must be stressed here. Namely, the distinction between mixed and contravariant Eulerian tensors is not very essential, because they are related through the spatial metric tensor, i.e.,

$$m^i{}_j = m^{ik}g_{kj}, \quad n^i{}_j = n^{ik}g_{kj}. \tag{3.7}$$

In affine coordinates the metric components g_{kj} are constant, in particular, in orthonormal frames $g_{kj} = \delta_{kj}$. Unlike this, the material tensors $m^A{}_B$, m^{AB} and $n^A{}_B$, n^{AB} are not interrelated through the constant reference metric η_{AB}. Instead, we have

$$m^A{}_B = m^{AC}G_{CB}, \quad n^A_B = n^{AC}G_{CB}, \tag{3.8}$$

where G denotes the Green deformation tensor,

$$G_{AB} := g_{ij}q^i{}_A q^j{}_B. \tag{3.9}$$

Only in non-deformed configurations, when $G_{AB} = \eta_{AB}$, mixed tensors are related to the contravariant ones through linear expressions with constant coefficients η_{AB}.

It is not m^{AB}, but m^A_B that is geometrically interpretable in a similar fashion as $m^i{}_j$. Namely, the quantities $m^A{}_B$ are Hamiltonian generators of the material action of $GL(3,\mathbf{R})$. Let us consider the action of $U \in GL(3,\mathbb{R})$ on the phase-space variables q^i, $q^i{}_A$; p_i, $p^A{}_i$:

$$({}'q^i, {}'q^i{}_A; {}'p_i, {}'p^A{}_i) = (q^i, q^i{}_B U^B{}_A; p_i, U^{-1A}{}_B p^B{}_i). \tag{3.10}$$

If $U = I + \varepsilon$ is infinitesimal, i.e., $\varepsilon \approx 0$, then, up to higher-order terms,

$$\delta F = \{F, m^A_B\}\varepsilon^B{}_A, \tag{3.11}$$

where

$$\delta F(q^i, a^i{}_A; p_i, p^A{}_i) = F({}'q^i, {}'q^i{}_A; {}'p_i, {}'p^A{}_i) - F(q^i, q^i{}_A; p_i, p^A{}_i).$$

Poisson brackets of $m^A{}_B$ have the form

$$\{m^A{}_B, m^C{}_D\} = \delta^C{}_B m^A{}_D - \delta^A{}_D m^C{}_B. \tag{3.12}$$

Besides, we have

$$\{m^i_j, m^A{}_B\} = 0, \quad \{p_i, m^A{}_B\} = 0, \tag{3.13}$$

because spatial and material transformations mutually commute.

In the special case of affinely-rigid body the expression of quantities $m^i{}_j$, $m^A{}_B$ through the phase-space coordinates takes on the following simple form:

$$m^i{}_j = q^i p_j + q^i{}_A p^A{}_j = m^i_{\mathrm{tr}j} + m^i_{\mathrm{int}j}, \quad m^A{}_B = q^A{}_i p^i{}_B, \tag{3.14}$$

where $q^A{}_i$ is an abbreviation for $q^{-1A}{}_i$, i.e.,

$$q^i{}_A q^A{}_j = \delta^i{}_j.$$

If the Lagrangian has the form $L = T - V$, V being independent of generalized velocities, then the corresponding Legendre transformation is given by

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = m g_{ij}\frac{dq^j}{dt}, \quad p^A{}_i = \frac{\partial L}{\partial \dot{q}^i{}_A} = g_{ij}\frac{dq^j{}_B}{dt}Q^{BA}. \tag{3.15}$$

Kinematical quantities p^i, m^{ij} are given by

$$p^i = m\frac{dq^i}{dt}, \quad m^{ij} = q^i{}_A\frac{dq^j{}_B}{dt}Q^{AB}. \tag{3.16}$$

The form (2.51), (2.52) of equations of motion is very convenient when we aim at deriving equations of motion of additionally constrained systems. For example, d'Alembert principle implies that equations of motion of the usual rigid body are given by the balance laws of linear momentum and angular momentum, i.e., by Eq. $(2.53)_1$ and the skew-symmetric part of Eq. $(2.53)_2$,

$$m\frac{d^2}{dt^2}q^i = F^i, \quad q^{[i}{}_A\frac{d^2}{dt^2}q^{j]}{}_BQ^{AB} = n^{[ij]}_{\text{int}}. \tag{3.17}$$

By the same argument, based on the d'Alembert principle, we can show that equations of motion of an incompressible affinely-rigid body are given by Eq. $(2.53)_1$ and the g-traceless part of Eq. $(2.53)_2$,

$$q^i{}_A\frac{d^2q^j{}_B}{dt^2}Q^{AB} - \frac{1}{3}g_{ab}q^a{}_A\frac{d^2q^b{}_B}{dt^2}Q^{AB}g^{ij} = n^{ij} - \frac{1}{3}g_{ab}n^{ab}g^{ij}. \tag{3.18}$$

(The $\frac{1}{3}$-factor comes from the dimension of space; in an m-dimensional space it would be replaced by $1/m$).

If the only admissible modes of internal motion are dilatations, then the internal part of dynamics is described by a single scalar equation — the g-trace of $(2.53)_2$,

$$g_{ij}q^i{}_A\frac{d^2}{dt^2}q^j{}_BQ^{AB} = g_{ij}n^{ij}. \tag{3.19}$$

If the body undergoes only rigid rotations and dilatations, then the internal part of dynamics is given by the system composed of Eqs. $(3.17)_2$ and (3.19), i.e., skew-symmetric part and g-trace of Eq. $(2.53)_2$.

There is an interesting example of non-holonomic constraints imposed on affine motion. To describe them we must introduce additional concept, namely that of affine quasivelocity. It is defined by the formula

$$e^i{}_j = \frac{dq^i{}_A}{dt}q^A{}_j. \tag{3.20}$$

The object $e^i{}_j$ is non-holonomic in the sense that there are no coordinates $x^i{}_j$ satisfying $e^i{}_j = \frac{d}{dt}x^i{}_j$. Its kinematical meaning is that it defines the Eulerian velocity field of affinely constrained continuum,

$$v^i(t,x) = \frac{\partial x^i}{\partial t}(t,a(t,x)) = \frac{dq^i}{dt} + e^i{}_j(x^j - q^j). \tag{3.21}$$

One also uses the quantities

$$e^A{}_B = q^A{}_i\frac{dq^i{}_B}{dt}, \tag{3.22}$$

which are exactly the co-moving components of the tensor $e^i{}_j$,

$$e^i{}_j = q^i{}_Ae^A{}_Bq^B{}_j. \tag{3.23}$$

From the geometrical point of view the quantities $e^i{}_j$, $e^A{}_B$ are Lie-algebraic objects, corresponding, respectively, to the right-invariant and left-invariant vector fields on the group $GL(3,\mathbb{R})$. Under the spatial action of $U \in GL(3,\mathbb{R})$, $[x^i] \mapsto [U^i{}_j x^j]$, the object $e^i{}_j$ transforms as:

$$[e^i{}_j] \mapsto [U^i{}_k e^k{}_l U^{-1l}{}_j]. \tag{3.24}$$

Obviously, $e^i{}_j$ is non-sensitive with respect to the material action of $U \in GL(3,\mathbb{R})$, $[a^A] \mapsto [U^A{}_B a^B]$. The material object $e^A{}_B$ has reciprocal transformation properties. It is easy to see that the internal affine moments may be interpreted as non-holonomic canonical momenta conjugate to affine velocities, because

$$p^A{}_i \frac{dq^i{}_A}{dt} = m^i{}_j e^j{}_i = m^A{}_B e^B{}_A. \tag{3.25}$$

The quantity

$$\omega^i{}_j := \frac{1}{2}(e^i{}_j - e_j{}^i) = e^i{}_j - g_{jk} g^{il} e^k{}_l \tag{3.26}$$

is interpreted as the angular velocity of affine motion.

The deformation rate is given by

$$d^i{}_j := e^i{}_j + e^j{}_i = e^i{}_j + g^j{}_k g^{il} e^k{}_l. \tag{3.27}$$

The co-moving components of $d_{ij} = g_{ik} d^k{}_j = d_{ji}$, i.e., $d_{AB} = d_{ij} q^i{}_A q^j{}_B$, represent the strain rate,

$$d_{AB} = \frac{1}{2} \frac{d}{dt} G_{AB}, \tag{3.28}$$

where $G_{AB} = g_{ij} q^i{}_A q^j{}_B$ are components of the Green deformation tensor.

Rigid-body constraints may be alternatively described by the condition $d^i{}_j = 0$. By analogy, the constraints of rotation-less motion have the form $\omega^i{}_j = 0$. However, there is a deep geometric novelty, namely, they are non-holonomic and do not impose any restriction on the attainability of affine configurations. Let us remind for comparison that, in spite of its anholonomic form, the equation $d^i{}_j = 0$ may be integrated to the finite condition

$$G_{AB} = g_{ij} q^i{}_A q^j{}_B = \eta_{AB}, \tag{3.29}$$

defining the rigid-body configurations.

Using the d'Alembert principle, we can show that equations of rotation-less motion are given by the symmetric part of Eq. $(2.53)_2$, i.e.,

$$q^{(i}{}_A \frac{d^2}{dt^2} q^{j)}{}_B Q^{AB} = n^{(ij)}_{\text{int}}. \tag{3.30}$$

Approximation of rotation-less motion may provide a reasonable model of the dynamics of small inclusions in very viscous fluids.

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POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received May 12, 1993.

Arch. Mech., 45, 3, pp. 333–359, Warszawa 1993

Deduction of thermodynamic balance laws for bidimensional nonmaterial directed continua modelling interphase layers

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Ϗ Ἄφρων δ', ϲ ὅς κ' Ϗ εὐθ ἔλῃ πρὸς κρείσσονας Ϗ αὐτιφερίζειν
ὑίκης τε στέρεται πρὸς τ' Ϗ αἴσχεσιν Ϗ ἄλγεα πάσχει
ϲ Ὡς Ϗ ἔφατ' Ϗ ὠκυπέτης Ϗ ἶρηξ, ταυσίπτερος Ϗ ὄρνις
Hesiod, Op vv. 210–212

AN HEURISTIC METHOD for the solution of some contradictions arising in the formulation of direct models for nonmaterial interphase layers is proposed. The novelty of the method consists in its use as a guide for the formulation of the right balance laws for a pretty new object: nonmaterial bidimensional continua. By means of an integration across the thickness of the layer starting from balance laws which are valid for $3D$ nonpolar continua, new $2D$ balance laws are derived. In modelling surface phenomena associated with interfaces between phases a notion of "interfacial thickness" that could vary both with respect to time and along the interface itself, is recognized, even when the interphase layer is modeled as a $2D$ continuum. The quantities — together with their evolution equations — necessary to describe surface phenomena, heuristically assuming that the $2D$ continuum actually models a three-dimensional interfacial continuum, are indicated. Surface balance laws for all h-moments with respect to the thickness of the interface, where suitable surface extra-fluxes naturally appear, are deduced with quoted heuristic method and therefore postulated. Preliminary considerations aiming to prove that Tolmann's formula cannot be improved — in order to match all experimental evidence — without introducing at least a first order model for the interface between two phases, are developed.

1. Introduction

THE SURFACE phenomena play an essential role in the border land between the chemistry, physics and mechanics of fluids and solids. Since the days of Young [51] mechanical phenomena associated with fluid interface regions in equilibrium are well-described in terms of a surface tension: indeed, the interface between two fluids — say, a liquid and its vapor — has been considered from the mechanical point of view as if it were a uniformly stretched mass-less membrane of zero thickness.

When a system in equilibrium is composed of two or more phases, the interface region between any two phases has a small but perceptible contribution to the mechanical and thermodynamic behavior of the system. An extensive description of thermostatic behavior of multi-component interfaces was established by GIBBS [20]; it used the method of the dividing surface. However, non-equilibrium situations are more complex. Any comprehensive theory must accommodate the possibility of transport phenomena both within and across the interface; bulk motions may be induced by inhomogeneities of fields and of the matter in the interfacial region (cf. BUFF [9], LEVINE [33], LEVICH and KRYLOV [32], MURREL and BOUCHER [37], ONO and KONDO [41], OŚCIK [42], SCRIVEN [44]) and moreover, physical adsorption and evaporation can occur there.

The use of the classical method of the dividing surface of Gibbs has been preferred by many authors (cf. ONO and KONDO [41]) to avoid the necessity of assigning

some thickness to the interface zone. In that method the surface tension is not only a quantity strictly defined with reference to the dividing surface since the excess contributions to the densities and fluxes (currents) appear there in the form of superficial quantities.

The above method for treating surface tension and description of the excess contributions to the densities and fluxes (currents) at the dividing surface in the form of superficial quantities, is simple, intuitive and often useful but:

i) it is of approximate nature from the molecular point of view (cf. ONO and KONDO [41]), for the structure of the fluid undergoes not a discontinuous but progressive modification across the actual interface.

ii) it does not seem to be easily extendible to describe non-equilibrium phenomena (for a full discussion of this point see DELL'ISOLA and KOSIŃSKI [11]).

In the majority of known models of interfaces a surface or a layer of singularity of bulk fields appear, which in the case of models dealing with a $3D$ layer become additionally carriers of interfacial fields; the latter being defined as mean quantities of either true or excess bulk localized fields. In the both approaches: the singular dividing surface and the interfacial layer of finite thickness, the resulting systems of universal balance laws of mechanics and thermodynamics have in principle similar forms. The differences concern interpretations of the terms appearing in the resulting balance equations ([1]).

At this point one should underline the difference appearing between two models based on the concept of the interface $3D$ layer: the first referring to the excess quantities and the second referring to the true ones. In the first model one introduces a dividing surface located somewhere in the transition (interface) zone, and then the bulk quantities are extrapolated up to this surface by stipulating (cf. DREEMER and SLATTERY [15], DUMAIS [16], ALTS and HUTTER [5], GOGOSOV *et al.* [21]) that they must satisfy the typical $3D$ balance equations and the bulk constitutive relations (whatever these may be). The main problem of this model consists in introducing surface excess densities (quantities) to compensate the error introduced by replacing the exact (true) quantities by the extrapolated quantities in the transition zone.

In the second model no extrapolation is made ([2]), instead two dividing surfaces are introduced (cf. GATIGNOL [17]), which make the boundary between the single phase bulk media and the interface zone; in the latter multi-phase behavior is observed, in which the confining matter possess constitutive properties different from the surrounding bulk phases.

In both models the averaging procedure is applied in which the integration along the thickness is performed thus yielding mean quantities defined as surface fields. In the first model one relates those quantities to the deviations between exact and extrapolated quantities in the layer, in the second one the mean quantities are defined as the line integrals of the exact fields on some reference (e.g. mean) surface located between the previous two ones. Here no physical meaning is ascribed to that surface as it simply models the "geometry" of the interface layer: however, for the convenience one can call it the dividing surface (like in the first model).

([1]) A slightly different point of view represents BLINOWSKI [7].

([2]) For seek of selfconsistence we describe here the deduction procedure formally introduced in DELL'ISOLA and KOSIŃSKI [11], and KOSIŃSKI and ROMANO [29].

In the present paper as well as in the second approach the interface is modeled as a shell-like Eulerian region composed in principle of different material points at different time instants. ([3])

Having, without any extrapolations, the exact integral relations for the surface true fields in terms of the bulk quantities of the layer, one tries to make constitutive description, that takes into account the interface and its interaction with the bulk phases as a whole, without retaining those "microscopical" details of its structure which could be regarded (for a certain class of phenomena) as irrelevant. A more detailed description ([4]) for the interface can be given by developing theories in which higher order moments of the true fields appear (cf. DUMAIS [16] and DELL'ISOLA and KOSIŃSKI [12]) as proposed in the last section of this paper.

In the phenomenological approach we are presenting the interface is modelled as a finite slab, and more detailed information about the structure of the dividing surface is introduced by relating the interfacial quantities to their $3D$ counterparts.

It should be pointed out at this place that both the approaches lead finally to equations which are similar to those of the thermodynamics with surface field singularities. In this way one can find a common point with the singular surface approach well developed in the literature (cf. SCRIVEN [44], GHEZ [19], MOECKEL [35], ROMANO [43], DELL'ISOLA and IANNECE [10], ALBANO, BEDEAUX and VLIEGER [2], KOSIŃSKI [24]).

It is one of our aims to draw attention that by localizing the surface phenomena to their carrier, namely to the moving surfaces, we are loosing some information necessary in the constitutive modelling (cf. MURDOCH [36].) To get it back we can explore the results of our exact derivation and the formula in which the interfacial quantities appearing in the interfacial balance laws are defined in terms of the corresponding $3D$ quantities. In this way we get some "hints" and "compatibility" conditions to be taken into account even when $2D$ balance laws are to be postulated, in that which is usually called the *direct* approach for the development of the theory of $2D$ nonmaterial continua. The reasoning developed in this paper could be considered in such a direct approach as "heuristic" in nature.

2. Moving shell-like Eulerian region in a continuum

Let us now assume that the effect of the interface in a continuous material system $\mathcal{B}$ occupying at time t in a motion χ a simply-connected region $\mathcal{B}_t$ in $3D$ space E^3 may be localized in a three-dimensional *moving region* $\mathcal{Z}_t$ of finite (and in a sense to be made precise "small" when compared with its "area") thickness. The region $\mathcal{B}_t$ consists of two phases $\mathcal{B}_t^+$ and $\mathcal{B}_t^-$. In addition there exists a narrow layer $\mathcal{Z}_t$ which divides the volume phases $\mathcal{B}_t^{\pm}$.

The boundaries between $\mathcal{Z}_t$ and both $\mathcal{B}_t^{\pm}$ are regular surfaces Σ_t^+ and Σ_t^-; between them a reference (mean) surface Σ_t is located to which the mean, interfacial, fields will be referred. (Note the subscript t appearing in the sets introduced, which reflects the fact

([3]) This is unavoidable in the case of phase transition, while the case of adsorption may be also modeled by a material region with extra mass supply sources, cf. OŚCIK [42].

([4]) Another approach can be developed by introducing two different scales: micro- and macro-coordinates, the former responsible for the inner structure of the layer. The nonstandard analysis tool could be helpful here.

that the sets are not fixed in the time, they change ([5]) with time.) In the case when the excess quantities are used the reference surface Σ_t plays the role of the dividing surface (cf. DEEMER and SLATTERY [15] or ALTS and HUTTER [4–6]).

The model with a shell-like interfacial region, whose geometry was described above, is aimed to formulate an initial boundary-value problem in terms of the bulk field equations valid in the regions $\mathcal{B}_t^{\pm}$ and the interfacial field equations valid on Σ_t: indeed in the present formulation of the problem the motion of the region $\mathcal{Z}_t$ will be determined by means of (free-moving) boundary conditions, since the conditions primitively (and physically) formulated on the lateral boundary of $\mathcal{Z}_t$, i.e on $\Omega_t := \partial\mathcal{Z}_t \setminus \Sigma_t^{\pm}$, once recalculated (read: integrated along the thickness) will lead to boundary conditions for the interface equations.

2.1. Geometry and kinematics of the interfacial layer

2.1.1. Normal coordinate system. In the case of layers of constant thickness $z = z^+ - z^-$ and the boundary surfaces $\Sigma_t^{\pm}$ are equidistant (parallel) and the parallel surface coordinate system (NAGHDI [38], NAPOLITANO [40]) is most convenient for the description of an arbitrary point in the layer $\mathcal{Z}_t$. If the position of the reference surface at time t is given by

$$y = \mathbf{r}(l^1, l^2, t), \tag{2.1}$$

where l^1 and l^2 are Gauss parameters of the surface, then an arbitrary point $\mathbf{x}$ in $\mathcal{Z}_t$ can be represented as

$$\mathbf{x} = \mathbf{r}(l^1, l^2, t) + l\mathbf{n}(\mathbf{r}(l^1, l^2, t)), \tag{2.2}$$

where $l \in [z^-, z^+]$ is the third coordinate, measuring the distance of the point x from Σ_t along the unit normal $\mathbf{n}$. Here points of the region $\mathcal{Z}_t$ are referred to a fixed rectangular Cartesian coordinate system.

The representation (2.2) means that the zone $\mathcal{Z}_t$ is *delineated* by the surfaces Σ_t^l, at the distance of l from Σ_t and represented by Eq. (2.2) with fixed l, and is delimited by the surfaces Σ_t^+ and Σ_t^-, at the distance of $l = z^+$ and $l = z^-$ from Σ_t, respectively, where to the surfaces lying between Σ_t and $\mathcal{B}_t^-$ the negative values of the coordinate l are attributed. We will see in the next subsections that Eq. (2.2) allows us to "delineate" the interfacial layer also when its "thickness" is variable.

2.1.2. Geometrical properties of parallel surfaces. Let

$$\frac{\partial \mathbf{r}(l^\alpha, t)}{\partial l^1} =: \mathbf{a}_1, \qquad \frac{\partial \mathbf{r}(l^\alpha, t)}{\partial l^2} =: \mathbf{a}_2 \tag{2.3}$$

denote two linearly independent vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ *tangent* to Σ_t at $\mathbf{r}$, then $\mathbf{a}^\alpha$, where $\alpha = 1, 2$, denote the co-tangent vectors of the surface Σ_t defined by the relation $\mathbf{a}_\beta \cdot \mathbf{a}^\alpha = \delta_\beta^\alpha$, where $\beta = 1, 2$ and $\mathbf{a}^\alpha \cdot \mathbf{n} = 0$. If we define the *surface gradient* $\operatorname{grad}_s(\cdot)$ by the formula

$$\operatorname{grad}_s \mathbf{u}(l^\beta, t) := \frac{\partial \mathbf{u}(l^\beta, t)}{\partial l^\alpha} \otimes \mathbf{a}^\alpha, \tag{2.4}$$

([5]) Only Eulerean formalism is possible in modeling surface phenomena at phase interfaces in phase transition problems (c.f. DELL'ISOLA and ROMANO [13, 14].

for an arbitrary vector (or tensor)-valued smooth field **u**, then Eq. (2.4) allows us to define the tenso field on (the tangent space of) the surface Σ_t, called the *curvature or second fundamental tensor*, by

$$\mathbf{b} := -\operatorname{grad}_s \mathbf{n}\,, \tag{2.5}$$

for which $b_{\alpha\beta}$ are its components in the tensor basis $\{\mathbf{a}^\alpha \otimes \mathbf{a}^\beta\}$. The components of the curvature tensor in the basis $\{\mathbf{a}_\alpha \otimes \mathbf{a}_\beta\}$ or in the mixed basis $\{\mathbf{a}_\alpha \otimes \mathbf{a}^\beta\}$, are $b^{\alpha\beta}$ and b^α_β, respectively. As we will need it when manipulating the expressions where **b** appears, we recall here the *Cayley–Hamilton theorem* for 2×2 matrices, from which the identity follows

$$\mathbf{d}^2 - \mathbf{d}\operatorname{tr}\mathbf{d} + \mathbf{1}_s \det\mathbf{d} = \mathbf{0} \tag{2.6}$$

for any surface tensor **d**, where $\mathbf{1}_s$ denotes the unit (metric) tensor of Σ_t.

Obviously Eq. (2.2) allows us to use for the parallel surface Σ_t^l the same Gauss parameters used for Σ_t: in this way all geometric differential objects we have introduced for Σ_t are naturally inherited by all Σ_t^l. To underline that except for the field of the normal vector **n**, the intrinsic as well as the embedded geometries of the surfaces Σ_t^l can be different, we shall explicitly consider the dependence of the objects $\mathbf{a}^\alpha$, $\mathbf{a}_\beta$ and **b** on the third coordinate l (if l differs from zero).

Using the representation (2.2), together with Eq. (2.3), we have, after performing the necessary differentiation, the relation

$$\mathbf{a}_\alpha(l) = (\mathbf{1}_s - l\mathbf{b})\mathbf{a}_\alpha\,. \tag{2.7}$$

In what follows we shall need the expansion formula for the second invariant of the surface tensor $(\mathbf{1}_s - l\mathbf{b})$, denoted by $j(l)$, i.e. for the determinant of the matrix $[\delta^\beta_\alpha - lb^\beta_\alpha]$,

$$j(l) := \det[\delta^\beta_\alpha - lb^\beta_\alpha] = 1 - 2Hl + Kl^2\,, \tag{2.8}$$

where H and K are invariants of **b**, i.e. the mean and Gauss curvatures, respectively. If we put $k_1(\mathbf{r})$ and $k_2(\mathbf{r})$, for principal curvatures at $\mathbf{r} \in \Sigma_t$, then to avoid loss of regularity of the representation of the layer, we have to restrict the thickness z of the layer to the value

$$z < \inf\{\min(|k_1(\mathbf{r})^{-1}|, |k_2(\mathbf{r})^{-1}|) : \mathbf{r} \in \Sigma_t\}\,.$$

The relationships (2.7) between the basis of a typical surface Σ_t^l and Σ_t allows to calculate its surface metric tensor components $a_{\alpha\beta}(l)$, and the area element $\sqrt{a(l)}$; we obtain

$$a_{\alpha\beta}(l) := \mathbf{a}_\alpha(l) \cdot \mathbf{a}_\beta(l) = \mathbf{a}_\alpha \cdot (\mathbf{1}_s - l\mathbf{b})^2\mathbf{a}_\beta\,.$$

Hence $a(l) := \det[a_{\alpha\beta}(l)] = \det[a_{\alpha\beta}]\det(\mathbf{1}_s - l\mathbf{b})^2 = aj(l)^2$.

It means that the ratio of the surface area elements of Σ_t^l to Σ_t is given by

$$j(l) = \sqrt{a(l)a^{-1}}\,. \tag{2.9}$$

The last formula is particularly useful in splitting the volume measure dv in the layer $\mathcal{Z}_t$ into the product of two measures: dl and da_l. Here dl represents the line measure (element) of a typical segment $\{\mathbf{r} + l\mathbf{n}(\mathbf{r}) : l \in [z^-, z^+],\ \mathbf{r} \in \Sigma_t\}$ in the layer, orthogonal to each surface Σ_t^l, while da_l is the surface measure (element) of Σ_t^l. The last measure can be written as $da_l = \sqrt{a(l)}dl^1dl^2$. From the orthogonality of segments and

the surfaces Σ_t^l, and the formula for $j(l)$, follows

$$dv = dl\,da_l = j(l)dl\,da\,, \tag{2.10}$$

where da is the the surface measure (element) of Σ_t. We can proceed further to get relations for other geometrical objects in the layer. Due to Eqs (2.2), (2.5) and (2.8) we obtain

$$\mathbf{b}(l) = j(l)^{-1}(\mathbf{b} - lK\mathbf{1}_s) = j(l)^{-1}\mathbf{b}(\mathbf{1}_s + l(\mathbf{b} - 2H\mathbf{1}_s)\,. \tag{2.11}$$

The surface tensors appearing under the sign of the last bracket, will play an important role in the further derivation, they satisfy the relation

$$j(l)\mathbf{1}_s = (\mathbf{1}_s - l\mathbf{b})\mathbf{A}_s(l)\,, \tag{2.12}$$

where

$$\mathbf{A}(l) := \mathbf{1}_s + l\tilde{\mathbf{b}}, \quad \tilde{\mathbf{b}} := \mathbf{b} - 2H\mathbf{1}_s\,. \tag{2.13}$$

Note that at a spherical point of the surface Σ_t, where $\mathbf{b} = H\mathbf{1}_s$, the tensor $\tilde{\mathbf{b}}$ is of opposite sign to $\mathbf{b}$, i.e. $\tilde{\mathbf{b}} = -\mathbf{b}$. To determine $\mathbf{a}^\beta(l)$ we use the relation $\mathbf{a}(l)\cdot\mathbf{a}^\beta(l) = \delta_\alpha^\beta$, valid for any l, to get

$$\mathbf{a}^\beta(l) = j(l)^{-1}(\mathbf{1}_s + l\tilde{\mathbf{b}})\mathbf{a}^\beta = j(l)^{-1}\mathbf{A}_s(l)\mathbf{a}^\beta\,. \tag{2.14}$$

The expressions (2.7), (2.11), (2.14) relate the geometry of a typical surface Σ_t^l to the geometry of Σ_t, quite similarly to the thin shell theory. What differs this derivation from that in the shell theory is the fact that here the region $\mathcal{Z}_t$ is not material, in general. To finish the derivation of geometrical relationships in the layer let us transform the oriented surface element $\mathbf{N}(l)da$ of the ruled surface Ω_t formed of segments

$$\{\mathbf{r} + l\mathbf{n}(\mathbf{r}) : l \in [z^-, z^+],\ \mathbf{r} \in \mathcal{C}_t \subset \Sigma_t\}\,,$$

where $\mathcal{C}_t$ is a curve ([6]) on Σ_t. Here $\mathbf{N}(l)$ is the outward unit normal to Ω_t, given by

$$\mathbf{N}(l) := \mathbf{t}_l \times \mathbf{n}\|\mathbf{t}_l \times \mathbf{n}\|^{-1}\,, \tag{2.15}$$

where $\mathbf{t}_l$ is a tangent vector to the curve $\mathcal{C}_t^l$ which is the lifting of the curve $\mathcal{C}_t$ to Σ_t^l; the vector tangent to the latter we shall denote by $\mathbf{t}_0$. Due to (2.2) and the fact that each of $\mathbf{t}_l$ and $\mathbf{t}_0$ are orthogonal to $\mathbf{n}$, and $\mathbf{n}$ has the unit length, we get (cf. Appendix)

$$\mathbf{N}(l)da = j(l)(\mathbf{a}^\gamma(l)\otimes\mathbf{a}_\gamma)\tilde{\mathbf{n}}dl\,ds\,, \tag{2.16}$$

where we have put $\tilde{\mathbf{n}} := \mathbf{t}_0 \times \mathbf{n}/\|\mathbf{t}_0\|$ for the unit normal to the curve $\mathcal{C}_t$ that is both tangent and outwardly directed with respect to Σ_t, and by ds we denote the line element of the curve $\mathcal{C}_0$.

2.1.3. Kinematics of a family of parallel surfaces. To describe the kinematics of $\mathcal{Z}_t$ we employ the vector

$$\mathbf{c} := \frac{\partial\mathbf{r}(l^\alpha, t)}{\partial t} \tag{2.17}$$

which represents the *velocity of the displacement* of the point

$$l^\alpha = \text{const}\,.$$

([6]) If $\mathcal{C}_t = \Sigma_t \cap \overline{\mathcal{Z}}_t$, then the ruled surface will be the lateral boundary of $\mathcal{Z}_t$, i.e. $\partial\mathcal{Z}_t\setminus(\Sigma_t^- \cup \Sigma_t^+)$.

It is obvious that in another parametric representation of the moving surface, say $x = \mathbf{r}'(k^\alpha, t)$, the calculated time derivative will lead in general to another value

$$\mathbf{c}' = \frac{\partial \mathbf{r}(k^\alpha, t)}{\partial t}.$$

However, the normal component of both the derivatives $\mathbf{c}$ and $\mathbf{c}'$,

$$\mathbf{n} \cdot \mathbf{c} = \mathbf{n} \cdot \mathbf{c}' = c_n \tag{2.18}$$

are equal. That is easy to see, if one starts with the implicit representation of the hypersurface $S := \bigcup \Sigma_t \times \{t\}$ in terms of some nontrivial differentiable function g by the equation

$$S \subseteq \{(x, t) : g(x, t) = 0\}.$$

Then substituting $\mathbf{r}$ or $\mathbf{r}'$ into the equation $g(x, t) = 0$ and performing the differentiation under the identity sign, we get

$$\frac{\partial g(\mathbf{r}, t)}{\partial t} / \| \operatorname{grad} g \| = -c_n, \tag{2.19}$$

due to the expression for the normal vector $\mathbf{n}$. The normal component c_n of the velocity of the displacement of the point $l^\alpha = \text{const}$ is called the *normal speed of displacement of the surface*, since it is independent of the parameterization. On the contrary, the tangential component $c^\alpha := \mathbf{c} \cdot \mathbf{a}^\alpha$ is strictly related to the parameterization (l^α), for in another parameterization $c'^\alpha = \mathbf{c}' \cdot \mathbf{a}^\alpha$. Let us notice that if the both are related by a time-dependent transformation

$$k^\alpha = \hat{k}^\alpha(l^\beta, t) \quad \text{then} \quad c'^\alpha = c^\alpha + \frac{\partial \hat{k}^\alpha(l^\beta, t)}{\partial t}.$$

Now, if we choose the transformation $\hat{k}^\alpha(l^\beta, t)$ such that its time derivative is equal to $-c^\alpha$, then in the parameterization k^α the tangential velocity c'^α vanishes. That particular parameterization of the moving surface is called the *convected parameterization* (BOWEN and WANG [8], KOSIŃSKI [27]), in that parameterization $\mathbf{c}'$ is equal to $c_n \mathbf{n}$. The integral curve of the field $\mathbf{u} = c_n \mathbf{n}$, i.e. the spatial projection of a solution of the vector differential equation

$$\frac{dx}{ds} = c_n \mathbf{n}(x), \qquad \frac{dt}{ds} = 1,$$

is called the *normal trajectory* of the moving surface $\{S_t\}$ if at $t_0 = t(0)$ it begins at a certain point of the initial surface Σ_{t_0}; each point of Σ_{t_0} is the starting point for a certain normal trajectory; moreover, through different points of Σ_t different normal trajectories are passing. In the convected parameterization the normal trajectory is a locus of the surface point $k^\alpha = \text{const}$. The choice of the convected parameterization in the description is of particular convenience in the derivation of any formula of a general nature.

Moreover, no final formula should depend on the particular choice of the parameterization of the moving surface, consequently it should be independent of the tangential component of the velocity field $\mathbf{c}$, only its normal component c_n has the geometrical meaning.

In the literature, however, one can find a discussion concerning the form of the $c^{\alpha}\mathbf{a}_{\alpha}$. In our opinion this velocity has no physical meaning, unless a fictitious point $l^{\alpha} = \text{const}$ will be equipped with an additional structure.

2.2. Displacement and other time derivatives

In the derivation of the local balance laws in the interfacial layer, an invariant time derivative has to appear, which is independent of the chosen parameterization of the moving surface Σ_t. Introducing the velocity of displacement $\mathbf{c}$ of the moving surface in a particular parameterization, namely in the convected parameterization, and following THOMAS [47, 48] and HAYES [22], the so-called

$$\textit{displacement time derivative } \frac{\delta}{\delta t}\mathbf{u}$$

of a quantity (a C^1-smooth field) $\mathbf{u}$ can be defined on the hypersurface $\mathcal{S}$ as the time derivative of $\mathbf{u}$ at fixed convected parameters of the moving surface $\{\Sigma_t\}$, i.e. by the formula

$$\frac{\delta}{\delta t}\mathbf{u} := \frac{\partial \mathbf{u}(k^{\alpha}, t)}{\partial t}, \tag{2.20}$$

where (k^{α}, t) is the convected parameterization of $\mathcal{S}$. It is a direct consequence of Eq. (2.20) that in any (not necessarily convected) parameterization (l^{α}, t) the operation formula for this derivative is

$$\frac{\delta}{\delta t}\mathbf{u} = \frac{\partial \mathbf{u}(l^{\alpha}, t)}{\partial t} - \operatorname{grad}_s \mathbf{u}\,\mathbf{c}. \tag{2.21}$$

Together with displacement (or Thomas) derivative we need to introduce other time derivatives of fields defined on a moving surface . To this aim we begin to apply the representation (2.2) of the surface Σ_t^l so that the velocity of displacement of Σ_t^l can be calculated as

$$\mathbf{c}(l) = \mathbf{c} - l(\operatorname{grad}_s c_n + \mathbf{bc}). \tag{2.22}$$

Here the explicit dependence of the fields on the coordinate l is written only, neither dependence on l^{α} nor on t is represented. To get Eq. (2.22) we have to differentiate (2.2) with respect to time t keeping l^{α} and l constant. Hence we need the time derivative of the normal vector field $\mathbf{n}$ (note that here $\mathbf{n}$ is the same for each l in the zone $\mathcal{Z}_t$). Due to the fact that $\mathbf{n} \cdot \mathbf{a}_{\alpha} = 0$, where the dot $\cdot$ denotes the inner product in E^3 and $\alpha = 1, 2$, we get

$$\frac{\partial \mathbf{n}}{\partial t} = -\mathbf{n} \cdot \frac{\partial \mathbf{c}(l^{\alpha}, t)}{\partial l^{\beta}} \otimes \mathbf{a}^{\beta} = -\mathbf{n} \operatorname{grad}_s \mathbf{c} = -(\operatorname{grad}_s c_n + \mathbf{bc}). \tag{2.23}$$

The derived formula (2.22) for the velocity is exact and can be compared with that proposed by GATIGNOL and SEPPECHER [18] and others. Additionally to the derivative related to the motion of the reference surface Σ_t and due to the derived representation for $\mathbf{c}(l)$ in Eq. (2.22), we may introduce the time derivative following the displacement of the whole region (layer) $\mathcal{Z}_t$. We denote this derivative by d_c/dt and define it by

$$\frac{d_c}{dt}\psi := \frac{\partial \psi(x, t)}{\partial t} + \operatorname{grad} \psi(x, t)\mathbf{c}(x), \tag{2.24}$$

where, according to Eq. (2.2), we put $x = \mathbf{r}(l^1, l^2, t) + l\mathbf{n}(\mathbf{r}(l^1, l^2, t))$ and ψ is an arbitrary (C^1-smooth field) defined (at least) on $\mathcal{S}$

$$\mathcal{S} := \cup\{\mathcal{Z}_t \times \{t\} : t \in I\},$$

2.3. Moving regions of variable thickness

The assumption about the constant thickness of the layer is rather reasonable when very thin layer of the interfacial medium is modeled. However, the constant in time (and in the surface coordinates) interfacial layer restricts the class of physical problems successfully treated by both the models based upon Gibbs' excess or true quantities. Consequently in this section to overcome this drawback we shall try to drop this assumption and to check its consequences on the formulae derived till now.

We consider a narrow layer $\mathcal{Z}_t$ dividing the volume phases $\mathcal{B}_t^{\pm}$: the boundaries between $\mathcal{Z}_t$ and both $\mathcal{B}_t^{\pm}$ are regular surfaces Σ_t^+ and Σ_t^-, which, however, are not in general any more equidistant (parallel). As before we use the parallel surface coordinate system describing an arbitrary point in the layer $\mathcal{Z}_t$: if the position of the reference surface is given by Eq. (2.1), then an arbitrary point x in $\mathcal{Z}_t$ can be represented as in Eq. (2.2), where the third coordinate l measuring the distance of the point x from Σ_t does not run over a fixed interval. Indeed, two scalar fields ζ^- and ζ^+ on the hypersurface $\mathcal{S}$ are defined such that their values give the distance of the boundary surfaces $\Sigma_t^{\pm}$ from Σ_t. The thickness z of the layer can then change according to the difference of both the functions, depending on the point and time, i.e.

$$z(\mathbf{r}, t) := \zeta^+(\mathbf{r}, t) - \zeta^-(\mathbf{r}, t). \tag{2.25}$$

The interfacial zone $\mathcal{Z}_t$ is therefore *delineated* by the surfaces Σ_t^l, at distance l from Σ_t and represented by Eq. (2.2) with fixed l, and is *delimited* by the surfaces Σ_t^+ and Σ_t^- given by

$$\Sigma_t^{\pm} \equiv \{y \in \mathcal{Z}_t : y = \mathbf{r} + \zeta^{\pm}(\mathbf{r}, t)\mathbf{n}(\mathbf{r}),\ \mathbf{r} \in \Sigma_t\} \tag{2.26}$$

which are not parallel to Σ_t, unless $\zeta^{\pm}(\mathbf{r}, t)$ is independent of the position $\mathbf{r}$. Let us notice that under the present weaker assumptions the layer can shrink locally to a surface if $\zeta^{\pm} = 0$. Moreover, it is possible now to describe the situation when the lateral boundary of the whole layer is not a ruled surface.

To avoid any singularities in the representation of the layer described in terms of the parallel surface coordinates, the assumption similar to that made in Sec. 2.2 and concerning the maximal thickness of the layer

$$\zeta := \sup\{\max(|\zeta^-(\mathbf{r}, t)|, |\zeta^+(\mathbf{r}, t)|) : (\mathbf{r}, t) \in \mathcal{S}\}$$

should be done. The geometry of the boundary surfaces $\Sigma_t^{\pm}$ will be related to that of Σ_t as follows. If $\mathbf{a}_{\alpha}^{\pm}$, $\alpha = 1, 2$, denote the natural base vectors of either surface, then due to Eq. (2.26), we get

$$\mathbf{a}_{\alpha}^{\pm} = (\mathbf{1}_s - \zeta^{\pm}\mathbf{b})\mathbf{a}_{\alpha} + \frac{\partial \zeta^{\pm}(\mathbf{r}, t)}{\partial l^{\alpha}}\mathbf{n}. \tag{2.27}$$

For the components of the metric tensor we then obtain

$$a_{\alpha\beta}^{\pm} := \mathbf{a}_{\alpha}^{\pm} \cdot \mathbf{a}_{\beta}^{\pm} = \mathbf{a}_{\alpha} \cdot (\mathbf{1}_s - \zeta^{\pm}\mathbf{b})^2\mathbf{a}_{\beta} + \frac{\partial \zeta^{\pm}}{\partial l^{\alpha}}\frac{\partial \zeta^{\pm}}{\partial l^{\beta}}. \tag{2.28}$$

The directed surface element of $\Sigma^{\pm}$ is given by (cf. Appendix)

$$\mathbf{n}^{\pm}(\mathbf{r})da^{\pm} := \mathbf{a}_1^{\pm} \times \mathbf{a}_2^{\pm} dl^1 dl^2 = (j(\zeta^{\pm}, \mathbf{r})\mathbf{n}(\mathbf{r}) - \mathbf{A}(\zeta^{\pm}) \operatorname{grad}_s \zeta^{\pm}) da\,, \tag{2.29}$$

where on the RHS the surface element $da = \sqrt{a} dl^1 dl^2$. In what follows the ratio $da^{\pm}/da$ we denote by $j^{\pm}$; it is a function of $\mathbf{r}$ and t.

The boundary surfaces $\Sigma_t^{\pm}$ move now with a velocity different than that calculated by Eq. (2.22). In fact, performing the time differentiation in Eq. (2.26) we obtain the following velocities $\mathbf{c}^{\pm}$ of displacement of $\Sigma^{\pm}$:

$$\mathbf{c}^{\pm} = \mathbf{c} - \zeta^{\pm}(\operatorname{grad}_s c_n + \mathbf{bc}) + \frac{\partial \zeta^{\pm}}{\partial t}\mathbf{n}\,. \tag{2.30}$$

To finish the geometrical preparation to the next section concerning the general balance law for $3D$ fields we write the product of $j^{\pm}$ and the normal speed of displacement of either boundary surface $\Sigma^{\pm}$ (cf. Appendix)

$$j^{\pm}\mathbf{c}^{\pm} \cdot \mathbf{n}^{\pm}(\mathbf{r}) = j(\zeta^{\pm}, \mathbf{r})\left(c_n + \frac{\delta \zeta^{\pm}}{\delta t}\right) + \zeta^{\pm} \operatorname{grad}_s \zeta^{\pm} \mathbf{A}_s(\zeta^{\pm}) \cdot \operatorname{grad}_s c_n\,. \tag{2.31}$$

3. Balance laws for a moving non-material shell-like region

The tools we developed in the previous sections will be now applied to derive the general and particular balance laws of thermomechanics. It will be done for the case of interfacial layer with nonvanishing thickness and for true, not excess, quantities.

3.1. General balance law

In one of previous papers (cf. DELL'ISOLA and ROMANO [14] or KOSIŃSKI and ROMANO [2]) it was assumed that the lateral boundary of the whole interface (transition) region, i.e.

$$\mathcal{Z}_t \setminus \Sigma_t^+ \cup \Sigma_t^-$$

is a ruled "lifted from a curve" surface .

It turns out that this assumption can be disregarded out on the global level keeping, however, this assumption, on the local level, i.e. during the passage from global to local forms of balance laws. It can be done assuming the integral form of the laws to be valid for any sub-layer, which is a proper subset of the whole layer bounded by subsurfaces of $\Sigma_t^{\pm}$ and a lateral boundary which is a ruled surface. In such a case the natural boundary conditions given on the lateral boundary of $\mathcal{Z}_t$ need to be recalculated in an appropriate way, also by introducing lines with material properties.

According to our notation the normal unit vector of $\Sigma_t^{\pm}$ will point from the region "$-$" to "$+$". The question how to choose the surfaces $\Sigma_t^{\pm}$ and the reference surface Σ_t lying between $\Sigma_t^{\pm}$ will not be discussed here (cf. DELL'ISOLA and KOSIŃSKI [11]). The choice of its position should be based on the mathematical convenience in the process of modelling and solving; this convenience is especially evident in the case of variable thickness of the layer.

The surface field ρ^s defined by

$$\rho^s(\mathbf{r}(l^1, l^2, t), t) = \int_{\zeta^-}^{\zeta^+} \rho(\mathbf{r}(l^\alpha, t) + l\mathbf{n}(\mathbf{r}), t) j(l, \mathbf{r}(l^\alpha, t)) dl \tag{3.1}$$

with $j(l, \mathbf{r}(l^\alpha, t))$, $\alpha = 1, 2$, given by Eq. (2.8), will be called the *surface mass density*. Here and in what follows both the cases of constant and variable thickness are discussed, for the former $\zeta^\pm$ should be identified with $z^\pm$.

Notice that in terms of the surface mass density ρ^s the mass of the matter contained at instant t in the zone $\mathcal{Z}_t$ is

$$\mathcal{M}(\mathcal{Z}_t) := \int_{\mathcal{Z}_t} \rho(x, t) d\nu = \int_{\Sigma_t} \rho^s da .$$

Therefore our definition of "surface" density is such that introduction of the $2D$ continuum Σ_t as a model for the interface $\mathcal{Z}_t$ does not cause a *loss* of the quantity $\mathcal{M}$. On the other hand, let us notice, that the definition of the surface mass does not take into account the type and the form of the $3D$ motion governed by the particle velocity field $\mathbf{v}$ in the layer (especially its tangential components): as a consequence we shall get an extra term in the flux of the mass in the $2D$ continuity equation (3.21). However, confining the velocity field to a particular form we could define the surface mass density in a different way appropriate to this form. Particular examples of $3D$ motions in a material zone are discussed in KOSIŃSKI and WĄSOWSKI [30].

After choosing the moving reference surface Σ_t as a geometrical object the remaining material structure of $\mathcal{Z}_t$ is preserved by equipping the surface Σ_t with a structure of $2D$ continuum. It is done by defining, in addition to ρ^s, next surface densities and fluxes of physical quantities as suitable integrals of the corresponding volume ones along the thickness of $\mathcal{Z}_t$. Hence, following the definition of the mass density, we can define for f, representing the density of a bulk quantity (i.e. a $3D$ density field) in the layer $\mathcal{Z}_t$, the corresponding surface field f^s as

$$f^s = \int_{\zeta^-}^{\zeta^+} j(l, \mathbf{r}) f(\mathbf{r} + l\mathbf{n}(\mathbf{r}), t) dl =: \langle jf \rangle . \tag{3.2}$$

The procedure makes possible the identification of interfacial quantities which appear in surface balance laws, even when we are dealing with non-material continua. Moreover, a more careful discussion of the *Galilean invariance* of the derived interfacial balance laws is possible when dealing with the explicit representations of the interfacial densities and fluxes (cf. Eq. (3.2) and (3.11)), respectively. This was the subject of the previous papers of DELL'ISOLA and ROMANO [14], KOSIŃSKI and ROMANO [29], DELL'ISOLA and KOSIŃSKI [11].

Referring to this last paper for a more detailed discussion, we remark here that, in our opinion, to the *geometrical* surface speed field c_n, describing how $\mathcal{Z}_t$ moves in E^3, no physically meaningful tangential component can be added. In the literature, however, such a component is searched for, which is "reasonable" from the physical point of view, thus getting a "complete" velocity field to be used in the balance of linear momentum (cf. ISHII [23], GATIGNOL and SEPPECHER [18], MOECKEL [35]): in DELL'ISOLA and

ROMANO [14] it is proved that these reasonable reasonings are valid only in the case of perfectly viscous interfaces as defined by Ishii.

On the other hand, in phase transition problems the material particles constituting the interfacial matter at instant t differ from those at another instant t'. Hence, together with the field $c_n\mathbf{n}$, we have to introduce an average velocity $\mathbf{V}^s$ of particles belonging to the layer. In terms of the $3D$ material (particle) velocity field $\mathbf{v}$, the densities ρ and ρ^s, the *"surface" material point velocity* $\mathbf{V}^s$ is given by the relation for the *surface momentum density*

$$\rho^s\mathbf{V}^s = \int_{\zeta^-}^{\zeta^+} \rho\mathbf{v}j(l,\mathbf{r})dl\,, \tag{3.3}$$

together with Eq. (3.1). A continuous $2D$ system modelled by Σ_t will be called *non-material* if ([7])

$$c_n \equiv \mathbf{c}\cdot\mathbf{n} \neq \mathbf{V}^s\cdot\mathbf{n}\,,$$

which means that in the mean the material points (particles) occupying the interface layer will not stay in it all the time. The difference

$$d_n := (\mathbf{c} - \mathbf{V}^s)\cdot\mathbf{n} \tag{3.4}$$

is a Galilean invariant and it is relevant to phase transition and adsorption processes if it does not vanish. It can be regarded as a quantity which needs to be determined by a constitutive equation (cf. DELL'ISOLA and ROMANO [11] and KOSIŃSKI [28]).

The classical balance law for the quantity ψ with its Galilean invariant flux (current) $\mathbf{w}$ and the source (supply + production) term p in the material volume $\mathcal{P}_t \subseteq \mathcal{B}_t$ is of the form

$$\frac{d}{dt}\int_{\mathcal{P}_t}\psi d\nu = -\int_{\partial\mathcal{P}_t}\mathbf{w}\cdot\mathbf{N}da + \int_{\mathcal{P}_t} p d\nu\,, \tag{3.5}$$

where $\mathbf{N}$ is the outward unit normal to $\partial\mathcal{P}_t$. Using the derivative d_c/dt , we get the following integral balance law for ψ in the non-material, in general, region $\mathcal{Z}_t$ moving with the velocity $\mathbf{c}$,

$$\frac{d_c}{dt}\int_{\mathcal{Z}_t}\psi dv = -\int_{\partial\mathcal{Z}_t}(\psi(\mathbf{v}-\mathbf{c})+\mathbf{w})\cdot\mathbf{N}da + \int_{\mathcal{Z}_t} p dv\,. \tag{3.6}$$

After the partition of both volumetric and surface measures into the product measures (cf. Eqs. (2.10) and (2.16)) we get ([8])

$$\frac{d_c}{dt}\int_{\Sigma}\psi^s da = -\int_{\mathcal{C}_t}(\mathbf{W}^{s'}\{\psi\} - \psi^s\mathbf{P}\mathbf{c})\tilde{\mathbf{n}}ds - \int_{\Sigma_t}((\tilde{\mathbf{w}}j\mathbf{n})^- + (\tilde{\mathbf{w}}j\mathbf{n})^+)da + \int_{\Sigma_t} p^s da\,. \tag{3.7}$$

The "weighted" limiting bulk-field values $(\tilde{\mathbf{w}}j\mathbf{n})^{\pm}$ are

$$(\tilde{\mathbf{w}}j\mathbf{n})^{\pm} := \tilde{\mathbf{w}}(\mathbf{r} + \zeta^{\pm}\mathbf{n}(\mathbf{r}), t)(j\mathbf{n})^{\pm}\,, \tag{3.8}$$

([7]) The surface is p-material if the equality holds (KOSIŃSKI [27]).

([8]) Here $\mathbf{P}$ denotes the tangent projection operator, i.e. $\mathbf{Pc} = \mathbf{c}_T$.

where

$$\tilde{\mathbf{w}} := \psi \otimes (\mathbf{v} - \mathbf{c}) + \mathbf{w} \tag{3.9}$$

is the new flux in Eq. (3.6) appearing under the surface integral over $\partial \mathcal{Z}_t$, and we have used the fact that $\mathbf{N}da = (j\mathbf{n})^{\pm}da$ on $\Sigma_t^{\pm}$. Let us notice that in the case of a constant thickness $\zeta^{\pm} = z^{\pm}$ is independent of $\mathbf{r}$ and t, and

$$(j\mathbf{n})^{\pm} = \pm j(z^{\pm}, \mathbf{r})\mathbf{n}\,, \tag{3.10}$$

while the case of a variable thickness is governed by Eq. (2.29) with $\zeta^{\pm}$ depending on $\mathbf{r}$ and t, in general. Since the definition of a general surface density has been given by Eq. (3.2) we have the definition of the surface flux in place of $\tilde{\mathbf{w}}$. To the flux $\tilde{\mathbf{w}}$ in Eq. (3.9) the corresponding surface flux $\mathbf{W}^{s'}\{\psi\}$ (note the prime over s) is defined as

$$\mathbf{W}^{s'}\{\psi\} \equiv \langle \psi \otimes (\mathbf{v} + l \operatorname{grad}_s c_n)\mathbf{A}_s(l)\rangle + \langle \mathbf{w}\mathbf{A}_s(l)\rangle\,. \tag{3.11}$$

The above definition together with Eq. (3.2) give at the same time the only possible relationships between the surface quantities and their bulk counterparts (better to say — their primitives), in order to make the interfacial balance law localized on the surface Σ_t compatible and derivable from the $3D$ law. The latter is postulated for ψ in the integral form Eq. (3.5). Let us notice that the surface flux $\mathbf{W}^{s'}\{\psi\}$ is Galilean invariant (due to the Cayley–Hamilton identity (2.6)). Now, we can get the final form of the integral balance law for layer $\mathcal{Z}_t$

$$\int_{\Sigma_t} \left(\frac{d_c}{dt}\psi^s + \psi^s \operatorname{div}_s \mathbf{c}_\tau\right) da + \int_{\mathcal{C}_t} (\mathbf{W}^{s'}\{\psi\} - \psi^s \otimes \mathbf{c}_\tau)\tilde{\mathbf{n}}\, ds$$
$$= -\int_{\Sigma_t} ((\tilde{\mathbf{w}}j\mathbf{n})^- + (\tilde{\mathbf{w}}j\mathbf{n})^+) da + \int_{\Sigma_t} p^s da\,.$$

Here we have used $\mathbf{c}_\tau$ to denote the tangential part of $\mathbf{c}$. However, in order to obtain the local, differential form of the law we have to perform the localization procedure by applying the integral law to an arbitrary *subzone* $\mathcal{Z}_t'$ of $\mathcal{Z}_t$. Here by *a subzone* we mean an arbitrary (shell-like) subregion $\mathcal{Z}_t'$ of $\mathcal{Z}_t \subset \mathcal{P}_t$ bounded by subsurfaces $\Sigma_t'^{\pm}$ of $\Sigma_t^{\pm}$ with a subsurface Σ_t' of Σ_t and with a nonvanishing lateral boundary being a ruled surface, for which the Stokes and Green-Gauss theorems can be applied.

After calculating the time derivative of the first integral and applying the obtained integral law to the arbitrary Σ_t' (supporting, by means the lifting along its normal field the subzone $\mathcal{Z}_t'$ of the layer $\mathcal{Z}_t$), we get under the continuity of the integrand ([9]), the following local equations:

$$\frac{d_c}{dt}\psi^s + \psi^s \operatorname{div}_s \mathbf{c} + \operatorname{div}_s(\mathbf{W}^{s'}\{\psi\} - \psi^s \otimes \mathbf{c}_\tau)$$
$$= -(\{\psi \otimes (\mathbf{v} - \mathbf{c}) + \mathbf{w}\}j\mathbf{n})^- + (\{\psi \otimes (\mathbf{v} - \mathbf{c}) + \mathbf{w}\}j\mathbf{n})^+\} + p^s\,. \tag{3.12}$$

([9]) If weaker conditions were assumed, like measurability of the integrand, the derived equation (3.12) should hold except for a set of H^2-measure zero, (i.e. at most on curves) where H^2-measure represents the $2D$ Hausdorff measure. This more general case leads to additional equations responsible for contact line effects.

Using the Thomas displacement derivative $\delta_n/\delta t$ (cf. (2.20)), together with the relationship

$$\frac{d_c}{dt}\psi^s + \psi^s \operatorname{div}_s \mathbf{c} - \operatorname{div}_s(\psi^s \mathbf{c}_\tau) = \frac{\delta_n}{\delta t}\psi^s - 2Hc_n\psi^s,$$

we arrive at

$$\frac{\delta_n}{\delta t}\psi^s - 2Hc_n\psi^s + \operatorname{div}_s \mathbf{W}^{s'}\{\psi\} = -\{(\{\psi \otimes (\mathbf{v}+\mathbf{c}) + \mathbf{w}\}j\mathbf{n})^- + (\{\psi \otimes (\mathbf{v}-\mathbf{c}) + \mathbf{w}\}j\mathbf{n})^+\} + p^s. \tag{3.13}$$

In *the constant thickness case* we obtain the following formula, which we quote here for the sake of completeness:

$$\frac{\delta_n}{\delta t}\psi^s - 2Hc_n\psi^s + \operatorname{div}_s \mathbf{W}^{s'}\{\psi\} = [\![\psi(\mathbf{v}-\mathbf{c}) + \mathbf{w}]\!]\cdot\mathbf{n} + \tilde{p}^s, \tag{3.14}$$

where

$$[\![g]\!] := g(\mathbf{r} + \zeta^-\mathbf{n}(\mathbf{r}), t) - g(\mathbf{r} + \zeta^+\mathbf{n}(\mathbf{r}), t) \tag{3.15}$$

for an arbitrary field g defined on $\mathcal{B}_t$.

The last equation is very well known in thermodynamics with surface singularities (called also: thermodynamics with singular surfaces, cf. MOECKEL [35], ISHII [23], KOSIŃSKI [24, 26, 27], ROMANO [43], ALTS [3], DELL'ISOLA and ROMANO [13]). The surface term $\tilde{p}^s$, called there a *surface supply*, is equal here to

$$\tilde{p}^s = p^s + [\![h\{\psi(\mathbf{v}-\mathbf{c}) + \mathbf{w}\}]\!]\cdot\mathbf{n}, \tag{3.16}$$

where $h(z^\pm, \mathbf{r}) := (K(\mathbf{r})z^\pm - 2H(\mathbf{r}))z^\pm$, and as previously $\mathbf{r} \in \Sigma_t$.

ALTS [3], ALTS and HUTTER [5, 6] gave a boundary layer model for curved phase boundaries and compared it with the model employing a singular surface. In their derivation, however, the surface quantities are identified with the so-called *excess* interfacial quantities in contrast with the definition given in the present paper. Moreover, they made use of balance laws for fields which are "extensions" of the bulk-field values at the boundary layer edges, i.e. at $\Sigma_t^\pm$, in our notation. In this way their definition of the surface quantities depends on the method of continuation of the bulk fields into the layer. The same definition of surface quantities has been used by DUMAIS [16], who performed the derivation for the case of a fixed material volume, not taking into account the diffusion terms. Partial results under the similar definition was obtained by Deemer and SLATTERY [18] together with the structural models for interface employing local area averages dealing with excess quantities. The present results could be compared with those of GATIGNOL and SEPPECHER [18], where dimensional and quantitive analyses of an approximation were performed. The present derivations are rather close to the results of the $2D$ approximation theory of shells. A comparison will be done in the other paper by KOSIŃSKI and WĄSOWSKI [30].

The variable thickness case ends with the law similar to Eq. (3.14), in which however the surface supply term $\tilde{p}^s$ is different and if we denote it by $\tilde{p}^s_v$, then due to Eqs. (2.29) and (2.31) it is equal to

$$\tilde{p}^s_v = p^s + [\![h\{\psi(\mathbf{v}-\mathbf{c}) + \mathbf{w}\}]\!]\cdot\mathbf{n} - [\![\psi \otimes (\mathbf{v} + \zeta^\pm \operatorname{grad}_s c_n)\mathbf{A}_s(\zeta^\pm)\operatorname{grad}_s \zeta^\pm]\!] - [\![\mathbf{w}\mathbf{A}_s(\zeta^\pm)\operatorname{grad}_s \zeta^\pm]\!] - \left[\!\left[\frac{\delta_n}{\delta t}\zeta^\pm \psi j\right]\!\right]. \tag{3.17}$$

Brief inspection of this term (in view of the previous one) shows the contribution of a new tangential part. This can be particularly important even in the case of the equilibrium equation for the interfacial stress tensor. The change of the thickness of the soap film on the bubble could be explained by means of those new terms.

3.2. Particular balance laws

Let us consider the particular quantities to be balanced by Eq. (3.14).

a. *Mass balance equation*: ψ is equal to ρ, and if the mass is conserved in the bulk medium, the flux of mass $\mathbf{w}$ and supply p of ρ (compare notation in (3.11)) are zero. The surface flux $\mathbf{W}^{s'}\{\rho\}$ is given by

$$\mathbf{W}^{s'}\{\rho\} = \langle \rho(\mathbf{v} + l \operatorname{grad}_s c_n)\mathbf{A}_s(l)\rangle =: \langle \mathbf{m}(l)\rangle , \tag{3.18}$$

which can be split into two parts

$$\mathbf{W}^{s'}\{\rho\} = \rho^s \mathbf{V}_\tau^s + \mathbf{W}_\rho = \rho^s \mathbf{V}_\tau^s + \langle \rho(\mathbf{A}_s(l) - j(l)\mathbf{1}_s)\mathbf{v} + l\rho \operatorname{grad}_s c_n \mathbf{A}_s(l)\rangle . \tag{3.19}$$

In the obvious way this equation leads to the definition of the extra mass flux $\mathbf{W}_\rho$. Hence the local balance equation for the mass is

$$\frac{\delta_n}{\delta t}\rho^s - 2H c_n \rho^s + \operatorname{div}_s(\rho^s \mathbf{V}_\tau^s) + \operatorname{div}_s \mathbf{W}_\rho = [\![j\rho(\mathbf{v} - \mathbf{c})]\!] \cdot \mathbf{n} + [\![\rho(\mathbf{v} + \zeta^{\pm} \operatorname{grad}_s c_n)\mathbf{A}_s \ (\zeta^{\pm}) \cdot \operatorname{grad}_s \zeta^{\pm}]\!] - \left[\!\!\left[\frac{\delta_n}{\delta t}\zeta^{\pm} \rho j \right]\!\!\right] , \tag{3.20}$$

where from (3.18) and (3.19) follows the explicit form of the surface extra mass flux $\mathbf{W}_\rho$

$$\mathbf{W}_\rho := \langle l\rho\mathbf{v}\rangle \mathbf{b} - K\langle l^2 \rho \mathbf{P}\mathbf{v}\rangle + \{\langle l\rho\rangle \mathbf{1}_s + \langle l^2\rho\rangle \tilde{\mathbf{b}}\} \operatorname{grad}_s c_n . \tag{3.21}$$

The last two terms on the RHS of Eq. (3.20) disappear in the *constant thickness case*. The simple inspection of Eq. (3.21) shows that the first two moments of the mass (i.e. $\langle l\rho\rangle$ and $\langle l^2\rho\rangle$) and of the momenta (i.e. $\langle l\rho\mathbf{v}\rangle$ and $\langle l^2\rho\mathbf{v}\rangle$) lead to the nonvanishing, in general, extra flux of the mass. Dealing with a p-material interface and the excess mass density field $\rho - \rho^{\pm}$, in the constant thickness case, ALTS [3] and then ALTS and HUTTER [4, 5] put the term corresponding to our $\mathbf{W}_\rho$ equal to zero. They chose, however, the surface coordinates as lines that are frozen to the motion of the surface "particles" (where the particles are defined in terms of the excess mass density). They should however distinguish between the density of surface particles defined in terms of the true mass distribution and that defined in terms of the excess fields, since their mass density can be negative. Moreover, their choice is local in l^β, and the disappearing of $\mathbf{W}_\rho$ cannot be interpreted as a constraint on the thickness of the layer which, in the constant thickness case, is a material intrinsic quantity, independent of l^β.

On the other hand, assuming that a reference surface should be chosen in the layer so that the quantity on RHS in Eq. (3.3) could represent the flux of mass, one cannot regard the same quantity as the surface linear momentum density, independently of the kinematics and the geometry of the surface, due to the K and $\operatorname{grad}_s c_n$ appearing in it. It follows that in the exact theory of interfaces one should expect the additional term $\mathbf{W}_\rho$ in the balance of mass, as compared with the form of this law given by the singular surface approach.

Let us notice that in the present model of an interface layer, the appearance of the non-vanishing extra flux of the mass $\mathbf{W}_\rho$ means that the flux of the surface mass is different from the density of interfacial linear momentum density; this observation has been already made by DELL'ISOLA and ROMANO [14] and SEPPECHER [45], using however an approximate theory.

b. *Linear momentum balance equation*: $\psi = \rho\mathbf{v}$ and the Cauchy stress $\mathbf{T}$ with the minus sign serves as the flux of linear momentum, and the body force $\rho\mathbf{f}$ is the supply term (no production in the bulk medium is admitted). For the surface flux $\mathbf{W}^{s'}\{\rho\mathbf{v}\}$ we have

$$\mathbf{W}^{s'}\{\rho\mathbf{v}\} \equiv \langle \rho\mathbf{v} \otimes (\mathbf{v} + l\,\mathrm{grad}_s\, c_n)\mathbf{A}_s(l)\rangle - \langle \mathbf{T}\mathbf{A}_s(l)\rangle\,, \tag{3.22}$$

which can be split into two parts

$$\mathbf{W}^{s'}\{\rho\mathbf{v}\} = \mathbf{V}^s \otimes \mathbf{W}^{s'}\{\rho\} + \mathbf{T}_s\,. \tag{3.22'}$$

The Galilean invariant *interfacial surface stress tensor* $\mathbf{T}_s$ can be written as the sum of two invariant parts $\mathbf{S}$ and $\mathbf{W}_{\rho v}$, where the definition of the first part imitates exactly that known in the continuum mixture theory (with the integration across the thickness instead of summing up over mixture constituents), and the second part can be called the extra surface linear momentum flux, i.e.

$$\begin{aligned} -\mathbf{S}(\mathbf{r},t) &:= \langle \mathbf{T}\mathbf{A}_s(l)\rangle - \langle \rho(\mathbf{v}-\mathbf{V}^s)\otimes(\mathbf{v}-\mathbf{V}^s)\mathbf{A}_s(l)\rangle\,,\\ \mathbf{W}_{\rho v}(\mathbf{r},t) &:= \langle \rho(\mathbf{v}-\mathbf{V}^s)\otimes((\mathbf{A}_s(l) - j(l)\mathbf{1}_s)\mathbf{V}^s + l\rho\,\mathrm{grad}_s\, c_n\mathbf{A}_s(l))\rangle\,. \end{aligned} \tag{3.23}$$

The other splitting can be made by defining two new components contributing to $\mathbf{T}_s$ as $\mathbf{S}^1$ and $\mathbf{S}^2$, where we put

$$\begin{aligned} \mathbf{S}^1(\mathbf{r},t) &:= -\langle \mathbf{T}\mathbf{A}_s(l)\rangle\,,\\ \mathbf{S}^2(\mathbf{r},t) &:= \langle(\mathbf{v}-\mathbf{V}^s)\otimes\mathbf{m}(l)\rangle\,,\\ \mathbf{T}_s &= \mathbf{S}^1 + \mathbf{S}^2\,. \end{aligned} \tag{3.24}$$

Using the last formula we can write the local balance equation for the linear momentum as below, where the superscript $\pm$ is omitted,

$$\begin{aligned} \frac{\delta_n}{\delta t}(\rho^s\mathbf{V}^s) - 2Hc_n\rho^s\mathbf{V}^s + \mathrm{div}_s(\mathbf{V}^s\otimes(\rho^s\mathbf{V}^s_\tau + \mathbf{W}_\rho) + \mathbf{T}_s) &= \rho^s\mathbf{f}^s\\ +[\![(\rho\mathbf{v}\otimes(\mathbf{v}\cdot\mathbf{n} - c_n) - \mathbf{T}\mathbf{n})j]\!] - [\![\rho\mathbf{v}\otimes(\mathbf{v} + \zeta\,\mathrm{grad}_s\, c_n)\mathbf{A}_s(\zeta)\,\mathrm{grad}_s\,\zeta]\!]&\\ +[\![\mathbf{T}\mathbf{A}_s(\zeta)\,\mathrm{grad}_s\,\zeta]\!] - \left[\!\!\left[\frac{\delta_n}{\delta t}\zeta\rho\mathbf{v}j\right]\!\!\right]\,.& \end{aligned} \tag{3.25}$$

The last three terms on the RHS of Eq. (3.25) disappear in the *constant thickness case*. The above expressions for the interfacial stress tensor $\mathbf{T}_s$ show that even in the equilibrium case, when the diffusion terms $\mathbf{S}^2$ and $\mathbf{W}_{\rho v}$ are put equal to zero, the symmetry of the tangential components of $\mathbf{S}$ cannot hold automatically. Moreover, the normal component $\mathbf{n}\mathbf{T}_s$ of the surface stress tensor $\mathbf{T}_s$ contains a contribution from the diffusion terms unless

the tangential component $\mathbf{v}_\tau$ of the velocity field $\mathbf{v}$ is constant along each segment of the layer ([10]), i.e. is independent of l.

The last but not the least important proposition concerning the interfacial stress tensor concerns the explicit dependence of $\mathbf{S}$ on the curvature of the interface Σ_l. The dependence will be present even in the case when the contribution due to the diffusion is neglected. Consequently, in formulating a constitutive equation for the surface stress tensor one should not forget that $\mathbf{S}$ is a function of $\mathbf{b}$ as well. In particular, if the interface is modelled as a two-phase Korteweg fluid, then

$$-\mathbf{T} = p\mathbf{1} + \alpha \mathbf{n} \otimes \mathbf{n}, \quad ([11])$$

with a scalar p as the pressure, then even in equilibrium (i.e. when $\mathbf{v} - \mathbf{V}^s = \mathbf{0}$), the surface stress will be different from the spherical tensor because of the term $-\langle lp \rangle \tilde{\mathbf{b}}$. Note that in the case of a spherical interface with non-vanishing thickness (e.g. a soap bubble) under equilibrium conditions $\mathbf{v} - \mathbf{V}^s = \mathbf{0}$, the surface stress will be

$$\mathbf{T}_s = \langle \mathbf{p}(1 + l/r) \rangle \mathbf{1}_s, \tag{3.26}$$

where r is a radius of the bubble. We can interpret the term $\langle p \rangle$ as the classical surface tension, here the additional part $\langle pl/r \rangle$ appears, which is normally very small, unless the thickness of the bubble is comparable with the radius r. This will be the case of very small bubbles. In our opinion, a deeper investigation of the consequences of (3.26) will lead to the improvement of Tolmann's formula which is considered necessary by ADAMSON [17] to match the experimental evidence. The quoted investigation will be made possible by means of the introduction of a first order model for the interface (cf. the following section).

It is seen from the above derivation that the local balance equations known in the singular surface approach are limited to very particular cases of the present approach and are valid under a particular set of assumptions.

c. *Angular momentum balance equation*: $\psi = \mathbf{x} \times \rho \mathbf{v}$. We restrict ourselves only to nonpolar continua. The master angular momentum balance law is well known in the $3D$ theory; its interfacial counterpart requires to define two quantities, namely

$$\psi^s := \langle j(\mathbf{r} + l\mathbf{n}) \times \rho \mathbf{v} \rangle, \quad \psi_2^s := \langle jl\mathbf{n} \times \rho \mathbf{v} \rangle \tag{3.27}$$

and the corresponding flux and production terms

$$\begin{aligned} \mathbf{w}(l) &= -(\mathbf{r} + l\mathbf{n}) \times \mathbf{T}, \quad \mathbf{w}_2(l) = -l\mathbf{n} \times \mathbf{T}, \\ \mathbf{p}^s &= \langle j(\mathbf{r} + l\mathbf{n}) \times \rho \mathbf{f} \rangle, \quad \mathbf{p}_2^s = \langle jl\mathbf{n} \times \rho \mathbf{f} \rangle - \langle j\mathbf{F} \rangle, \end{aligned} \tag{3.28}$$

where

$$j\mathbf{F} := (j\mathbf{1} + l(\mathbf{b} - Kl\mathbf{1}_s) \times \mathbf{T}) - j\mathbf{n} \times \mathbf{T} \cdot \mathbf{n} - \mathbf{m}(l) \times (\mathbf{v} \cdot \mathbf{n})\mathbf{n} + j\rho \mathbf{v} \times c_n \mathbf{n}.$$

Making use of the balance law (3.14), replacing ψ^s with those defined in Eqs. (3.27) together with the corresponding terms from Eqs. (3.28), we arrive at a pair of equations.

([10]) In that case $\langle f\mathbf{v} \rangle_\tau = \langle f \rangle \mathbf{v}_\tau$ for an arbitrary field f, and $\mathbf{v}_\tau = \mathbf{v}_\tau^\pm = \mathbf{V}_\tau^s$. Such condition has been admitted by DELL'ISOLA and ROMANO [14] and interpreted as the perfect viscosity consequence of the viscosity of the $3D$ matter contained in the layer.

([11]) We are assuming here that inside the layer i) $\nabla \rho$ is parallel to the field $\mathbf{n}$ inside the layer, ii) α is a nonvanishing scalar field.

Subtracting the second from the first one and using the cross product of $\mathbf{r}$ with the linear momentum equation (3.25), we end up with

$$\langle j(\mathbf{1} \times \mathbf{T}) \rangle = \mathbf{0} \tag{3.29}_1$$

which is automatically satisfied if $\mathbf{T}$ is symmetric. On the other hand, the last equation, which represents the *zero* order condition of non-polarity of the layer, leads to the following one:

$$\mathbf{l}_0 + \mathbf{l}_1 + (\mathbf{1} - \mathbf{n} \otimes \mathbf{n}) \underset{1,3}{\times} \mathbf{T}_s = \mathbf{b} \underset{1,3}{\times} {}^1\mathbf{T}_\sigma \tag{3.29}_2$$

where:

i. The product

$$\mathbf{N} \underset{1,3}{\times} \mathbf{M}$$

between 3-3 double tensors $\mathbf{N}$ and $\mathbf{M}$, whose components in a given basis are $N^j{}_m$ and M^{km}, respectively, is a 3-vector whose i-th component in the same basis is:

$$\varepsilon_{ijk} N^j{}_m M^{km},$$

with ε_{ijk} being the components of the three-dimensional Levi–Civita tensor.

Similarly the $\underset{1,3}{\times}$ product between 3-2 tensors is introduced; if $\mathbf{N}$ and $\mathbf{M}$ are 3-2 tensors whose components are $N^j{}_\alpha$ and $M^{k\alpha}$, respectively, with a fixed basis $\mathbf{e}_i$ for E^3 and another one $\mathbf{a}_\alpha$ for the tangent plane to Σ_t, then we denote with the symbol $\mathbf{N} \underset{1,3}{\times} \mathbf{M}$ a three-dimensional vector whose i-th component in the basis $\mathbf{e}_i$ is given by:

$$\varepsilon_{ijk} N^j{}_\alpha M^{k\alpha}.$$

ii. The quantities $\mathbf{l}_0$, $\mathbf{l}_1$ and ${}^1\mathbf{T}_\sigma$ have to be determined by means of constitutive equations. The quantities $\mathbf{l}_0$ and $\mathbf{l}_1$ represent, respectively, the *zero* and *higher order sources of kinetical couples*, and are introduced in order to describe those features of the kinematics of the layer which cannot be completely neglected even in the case of very thin layers. Indeed they can be represented, in terms of $3D$ fields, as follows:

$$\mathbf{l}_0 := \langle \rho(v_n - c_n)\mathbf{n} \times (\mathbf{v}_\top - \mathbf{V}^s_\top) \rangle,$$

$$\mathbf{l}_1 := \langle (1 - j)\rho(v_n - c_n)\mathbf{n} \times (\mathbf{v}_\top - \mathbf{V}^s_\top) \rangle + \langle \mathbf{w}_\rho \times (\mathbf{V} \cdot \mathbf{n})\mathbf{n} \rangle + \mathbf{b}\, {}^1\mathbf{M}^s \times {}^1\mathbf{V}^s - \langle \operatorname{grad}_s c_n \times l\rho j\mathbf{v} \rangle,$$

where

$${}^1\mathbf{M}^s := \langle {}^1\mathbf{m}(l) \rangle \equiv \langle jl\rho\mathbf{v}\mathbf{1}_s \rangle + {}^1\mathbf{w}_\rho$$

is the flux of the first moment of mass, ${}^1\mathbf{V}^s$ is defined by

$${}^1\rho^s\, {}^1\mathbf{V}^s = \langle \rho l j \mathbf{v} \rangle, \qquad {}^1\rho^s := \langle \rho j l \rangle,$$

and

$${}^1\mathbf{w}_\rho := \langle l\rho(\mathbf{A}_s(l) - j\mathbf{1}_s)\mathbf{v} + l^2\rho \operatorname{grad}_s c_n \mathbf{A}_s(l) \rangle.$$

Finally the quantity

$${}^1\mathbf{T}_\sigma := {}^1\mathbf{S}_1 + {}^1\mathbf{S}_2$$

represents the *interfacial first moment of stress tensor*, where

$${}^1\mathbf{S}_1\{\rho\mathbf{v}\} := \langle (\mathbf{v} - {}^1\mathbf{V}^s) \otimes {}^1\mathbf{m}(l) \rangle, \qquad {}^1\mathbf{S}_2\{\rho\mathbf{v}\} := -\langle l\mathbf{T}\mathbf{A}_s(l) \rangle.$$

Before closing the discussion of the equation of motion we shall write the explicit relation for the normal and antisymmetric parts of the surface stress tensor $\mathbf{T}_s$ in the natural basis $\{\mathbf{a}_\alpha, \mathbf{n}\}$. They are

$$T_s^{n\alpha} = -\langle jT^{n\alpha}\rangle + \langle j\rho(\mathbf{v}\cdot\mathbf{n} - c_n)(v^\alpha - V^{s\alpha})\rangle + \langle(\mathbf{v}-\mathbf{V}^s)\cdot\mathbf{n}w_\rho^\alpha\rangle - \langle jlb_\delta^\alpha T^{n\delta}\rangle\rangle\,,$$

(3.30)

$$T_s^{12} - T_s^{21} = -\langle jl(b_2^1(T^{22} - T^{11}) + (b_1^1 - b_2^2)T^{12})\rangle\,,$$

where

$$T_s^{n\alpha} := \mathbf{n}\cdot\mathbf{T}_s\mathbf{a}^\alpha, \quad T^{12} := \mathbf{a}^1\cdot\mathbf{T}\mathbf{a}^2, \quad \text{etc.}$$

d. *Energy balance equation*: $\psi = \rho(e + 0.5\mathbf{v}\cdot\mathbf{v}) =: \rho E$, where e represents the specific internal energy, the sum $-\mathbf{v}\mathbf{T} + \mathbf{q}$ serves as the flux of the total energy, where $\mathbf{q}$ is the heat flux vector, the sum $\rho(\mathbf{f}\cdot\mathbf{v} + r)$ is the supply term, where r represents the body heat supply density. For the surface flux $\mathbf{W}^{s'}\{\rho E\}$ we have

$$\mathbf{W}^{s'}\{\rho E\} \equiv \langle E\mathbf{m}(l)\rangle - \langle(\mathbf{v}\mathbf{T} - \mathbf{q})\mathbf{A}_s(l)\rangle\,. \tag{3.31}$$

If we define

$$\rho^s e^s := \langle j\rho\tilde{e}\rangle, \quad \tilde{e} := e + 0.5(\mathbf{v}-\mathbf{V}^s)^2, \quad \rho^s\tilde{r}^s := \langle j\rho(r + \mathbf{f}\cdot(\mathbf{v}-\mathbf{V}^s))\rangle\,,$$

$$\mathbf{q}^s := \langle(\rho\tilde{e}(\mathbf{v}-\mathbf{V}^s) + \mathbf{q} - (\mathbf{v}-\mathbf{V}^s)\mathbf{T})\mathbf{A}_s(l)\rangle\,, \tag{3.32}$$

$$\mathbf{W}_E := \langle\rho\tilde{e}(\mathbf{A}_s(l) - j(l)\mathbf{1}_s)\mathbf{V}^s + l\rho\tilde{e}\,\mathrm{grad}_s\, c_n\mathbf{A}_s(l)\rangle\,,$$

then the local energy balance equation will be

$$\begin{aligned}\frac{\delta_n}{\delta t}(\rho^s(e^s + 0.5\mathbf{V}^s\cdot\mathbf{V}^s)) - 2Hc_n\rho^s(e^s + 0.5\mathbf{V}^s\cdot\mathbf{V}^s)\\ + \mathrm{div}_s(\rho^s(e^s + 0.5\mathbf{V}^s\cdot\mathbf{V}^s)\mathbf{V}_\tau^s + \mathbf{q}^s + \mathbf{W}_E + \mathbf{V}^s\mathbf{T}_s + 0.5\mathbf{V}^s\cdot\mathbf{V}^s\mathbf{W}_\rho)\\ = \rho^s\mathbf{f}^s\cdot\mathbf{V}^s + \rho^s\tilde{r}^s + [\![(\rho(e + 0.5\mathbf{v}\cdot\mathbf{v})(\mathbf{v}-\mathbf{c}) + \mathbf{q} - \mathbf{v}\mathbf{T})j]\!]\cdot\mathbf{n}\\ -[\![\rho(e + 0.5\mathbf{v}\cdot\mathbf{v})(\mathbf{v} + \zeta\,\mathrm{grad}_s\, c_n)\mathbf{A}_s(\zeta)\,\mathrm{grad}_s\,\zeta]\!] + [\![(\mathbf{v}\mathbf{T} - \mathbf{q})\mathbf{A}_s(\zeta)\,\mathrm{grad}_s\,\zeta]\!]\\ -\left[\!\!\left[\frac{\delta_n}{\delta t}\zeta\rho(e + 0.5\mathbf{v}\cdot\mathbf{v})j\right]\!\!\right].\end{aligned} \tag{3.33}$$

The last three terms on the RHS of Eq. (3.33) disappear in the *constant thickness case*. The above expressions for the interfacial heat flux $\mathbf{Q}^s$ and the supply terms $\rho^s\tilde{r}^s$ lead to the following relations:

$$\begin{aligned}\mathbf{Q}^s &= \mathbf{q}^s + \mathbf{W}_E\,,\\ \rho^s r^s &\neq \rho^s\tilde{r}^s\,,\end{aligned} \tag{3.34}$$

which mean that even the case of a nonconductor of heat at the $3D$ level leads to the nonvanishing interfacial heat flux, and the vanishing heat supply term ρr at the $3D$ level leads to the interfacial heat supply $\rho^s\tilde{r}^s$ equal to $\langle j\rho\mathbf{f}\cdot(\mathbf{v}-\mathbf{V}^s)\rangle$, which does not need to vanish if $\mathbf{f}$ is different from zero.

f. *Thermodynamic inequality*

The second law of thermodynamics for the $3D$ material continuum is assumed in the form of the entropy production inequality

$$\frac{d}{dt}\int_{\mathcal{P}_t}\rho\eta d\nu \geq -\int_{\partial\mathcal{P}_t}(\mathbf{q}/\vartheta + \mathbf{k})\cdot\mathbf{N}da + \int_{\mathcal{P}_t}\rho r/\vartheta d\nu\,, \tag{3.35}$$

where η and ϑ represent the specific entropy and the absolute temperature, respectively, while $\mathbf{k}$ is the so-called extra entropy flux. Performing the usual localization for the interfacial layer we get the inequality

$$\begin{aligned}(3.36)\quad &\frac{\delta_n}{\delta t}\rho^s\eta^s - 2Hc_n\rho^s\eta^s + \operatorname{div}_s(\rho^s\eta^s\mathbf{V}_\tau^s + \tilde{\mathbf{k}}^s + \mathbf{W}_\eta)\\ &\geq \rho^s r_\eta^s + [\![(\rho\eta(\mathbf{v}-\mathbf{c}) + \mathbf{q}/\vartheta + \mathbf{k})j]\!]\cdot\mathbf{n}\\ &-[\![\rho\eta(\mathbf{v}+\zeta\operatorname{grad}_s c_n)\mathbf{A}_s(\zeta)\operatorname{grad}_s\zeta]\!] - [\![(\mathbf{q}/\vartheta+\mathbf{k})\mathbf{A}_s(\zeta)\operatorname{grad}_s\zeta]\!] - \left[\!\!\left[\frac{\delta_n}{\delta t}\zeta\rho\eta j\right]\!\!\right],\end{aligned}$$

where

$$\mathbf{W}^{s'}\{\rho\eta\} \equiv \langle\rho\eta\mathbf{m}(l)\rangle + \langle(\mathbf{q}/\vartheta+\mathbf{k})\mathbf{A}_s(l)\rangle,$$

and we have defined

$$\begin{aligned}&\rho^s\eta^s := \langle j\rho\eta\rangle, \quad \rho^s r_\eta^s := \langle j\rho(r/\vartheta)\rangle,\\ (3.37)\qquad &\tilde{\mathbf{k}}^s := \langle(\rho(\mathbf{q}/\vartheta+\mathbf{k})\mathbf{A}_s(l) + (\rho\eta(\mathbf{v}-\mathbf{V}^s))\mathbf{A}_s(l)\rangle,\\ &\mathbf{W}_\eta := \langle\rho\eta(\mathbf{A}_s(l) - j(l)\mathbf{1}_s)\mathbf{V}^s + l\rho\eta\operatorname{grad}_s c_n\mathbf{A}_s(l)\rangle.\end{aligned}$$

The last three terms on the RHS of (3.36) disappear in the *constant thickness case*. The above expressions for the interfacial entropy flux cannot be simple related to the interfacial heat flux $\mathbf{Q}^s$, even in the case of the vanishing extra-term $\mathbf{k}$. However, under particular set of assumptions concerning the kinematics and the constitutive properties of the matter in the layer, some simplification can be made in order to derive such a relation. This will be the subject of a future paper.

4. H-order models for nonmaterial twodimensional continua

Using the model developed in the previous sections one could not completely take into account the influence of the thickness of the layer on the thermomechanical behaviour of phase interfaces. The quoted model will be called a *0-th order model*.

On the other hand the considerations preceding and following formula for (3.26) clearly point out the following circumstance: if one wants to account the influence of surface effects in the formation of small drops, one has to consider (at least) the first moment of the interfacial tension field.

In order to provide a guide to the introduction of a more complex structure to nonmaterial bidimensional continua, an H-th order model can be developed (cf. DELL'ISOLA and KOSIŃSKI [12] for the case of the constant thickness) the idea of which comes from DUMAIS [16]. Similarly to the previous sections we will assume that the interfacial layer has a variable thickness, in the sense made precise in Sec. 2.

4.1. H-th order model

One introduces the k-th moment field ($k \leq H$) of a typical interfacial quantity f by

$$(4.1)\qquad {}^k f \equiv l^k(x,t^*)f \quad \text{when } x \in \mathcal{Z}_t.$$

Here l^k means the k-th power of l.

Then for every k the following k-th local balance equation for the physical quantity represented by the field ψ can be easily derived by evaluating partial time derivative of

the k-th moment ${}^k\psi$, using the properties of the function $l(x, t^*)$ and using the local balance equation for the field ψ (see Eqs. (3.17)–(3.20) in KOSIŃSKI [31]):

$$\frac{\partial}{\partial t}\,{}^k\psi + \operatorname{div}({}^k\psi \otimes \mathbf{v} + {}^k\mathbf{w}) = {}^kp + k({}^{k-1}\psi \otimes (\mathbf{v}\cdot\mathbf{n} - c_n) + {}^{k-1}\mathbf{w}\cdot\mathbf{n}). \tag{4.2}$$

Regarding Eq. (4.2) as the local form of a particular case of Eq. (3.5) and recalling the form (3.14) we get the following surface balance equation:

$$\begin{aligned}\frac{\delta_n}{\delta t}\,{}^k\psi^s - 2Hc_n\,{}^k\psi^s + \operatorname{div}_s(\langle {}^k\psi \otimes (\mathbf{v} + l\operatorname{grad}_s c_n)\mathbf{A}_s(l)\rangle + \langle {}^k\mathbf{w}\mathbf{A}_s(l)\rangle) \\ = [\![({}^k\psi \otimes (\mathbf{v} - \mathbf{c}) + {}^k\mathbf{w})]\!]\cdot\mathbf{n} + {}^k\tilde{p}_v^s + \langle jk({}^{k-1}\psi \otimes (\mathbf{v}\cdot\mathbf{n} - c_n) + {}^{k-1}\mathbf{w}\cdot\mathbf{n})\rangle,\end{aligned} \tag{4.3}$$

where the following definition has been used:

$$\begin{aligned}{}^k\tilde{p}_v^s := {}^kp^s + [\![h\{{}^k\psi \otimes (\mathbf{v} - \mathbf{c}) + {}^k\mathbf{w}\}]\!]\cdot\mathbf{n} \\ -[\![{}^k\psi \otimes (\mathbf{v} + \zeta^{\pm}\operatorname{grad}_s c_n)\mathbf{A}_s(\zeta^{\pm})\operatorname{grad}_s\zeta^{\pm}]\!] - [\![{}^k\mathbf{w}\mathbf{A}_s(\zeta^{\pm})\operatorname{grad}_s\zeta^{\pm}]\!] - \left[\!\!\left[\frac{\delta_n}{\delta t}\zeta^{\pm}\,{}^k\psi j\right]\!\!\right].\end{aligned} \tag{4.4}$$

Equation (4.3) is valid for any $k \geq 1$.

We remark that when $k = 0$ the primitive interfacial balance law (3.14) with (3.17) yields Eq. (4.3) as a particular case.

However, a question may arise concerning the completeness of a k-th order model, formulated by postulating, for the interface, the first $k + 1$ surface balance equations of the type (4.3) and assuming that all the higher order quantities appearing in them are determined by means of constitutive equations.

To answer this question one should first notice that the k-th moment of a typical function f in Eq. (4.1) (regarded as a function of l only) defines the projection of f on the polynomial l^k belonging to the basis formed by all polynomials of the function space

$$L^2([z^-, z^+], d\mu = j\,dl).$$

The measure μ is positive and absolutely continuous with respect to the Lebesgue measure as long as j is positive and H and K are finite; this corresponds to the assumed hypothesis on the thickness of the layer $\mathcal{Z}_t$. Therefore the H-order theory deals with truncated expansions across the thickness of the layer of the physical quantities to be balanced.

4.2. H-th order thermomechanical balance equations

We present here the balance equations of k-th moments of mass, linear momentum, energy and entropy.

We explicitly remark that balance of angular momentum leads to some surface equations which simply generalize either Eq. $(3.29)_1$ or its different formulation $(3.29)_2$.

a. *Balance of mass*

$$\begin{aligned}\frac{\delta_n}{\delta t}\,{}^k\rho^s - 2Hc_n\,{}^k\rho^s + \operatorname{div}_s({}^k\rho^s\,{}^k\mathbf{V}_\tau^s) + \operatorname{div}_s\,{}^k\mathbf{W}_\rho \\ = \langle jk\,{}^{k-1}\rho(\mathbf{v} - \mathbf{c})\rangle\cdot\mathbf{n} + [\![j\,{}^k\rho(\mathbf{v} - \mathbf{c})]\!]\cdot\mathbf{n} \\ -[\![{}^k\rho(\mathbf{v} + \zeta^{\pm}\operatorname{grad}_s c_n)\mathbf{A}_s(\zeta^{\pm})\cdot\operatorname{grad}_s\zeta^{\pm}]\!] - \left[\!\!\left[\frac{\delta_n}{\delta t}\zeta^{\pm}\,{}^k\rho j\right]\!\!\right],\end{aligned} \tag{4.5}$$

where we have used the following notation:

$$ {}^k\rho^s\ {}^k\mathbf{V}^s := \langle j\ {}^k\rho\mathbf{v}\rangle . $$

b. *Balance of linear momentum*

$$ \begin{aligned} (4.6)\quad & \frac{\delta_n}{\delta t}({}^k\rho^s\ {}^k\mathbf{V}^s) - 2Hc_n\ {}^k\rho^s\ {}^k\mathbf{V}^s + \operatorname{div}_s({}^k\mathbf{V}^s \otimes {}^k\mathbf{M}^s + {}^k\mathbf{S}^1 + {}^k\mathbf{S}^1) \\ &= {}^k(\rho\mathbf{f})^s + [\![({}^k\rho\mathbf{v} \otimes (\mathbf{v}\cdot\mathbf{n} - c_n) - {}^k\mathbf{T}\mathbf{n})j]\!] + \langle k({}^{k-1}\rho\mathbf{v} \otimes (\mathbf{v}\cdot\mathbf{n} - c_n) - {}^{k-1}\mathbf{T}\mathbf{n})j\rangle \\ &\quad -[\![\,{}^k\rho\mathbf{v} \otimes (\mathbf{v} + \zeta\,\operatorname{grad}_s c_n)\mathbf{A}_s(\zeta)\operatorname{grad}_s\zeta]\!] + [\![\,{}^k\mathbf{T}\mathbf{A}_s(\zeta)\operatorname{grad}_s\zeta]\!] - \left[\!\!\left[\frac{\delta_n}{\delta t}\zeta\ {}^k\rho\mathbf{v}j\right]\!\!\right], \end{aligned} $$

where we have omitted the superscript $\pm$ and used the following notations:

$$ \begin{aligned} \langle\ {}^k\rho(\mathbf{v} + l\,\operatorname{grad}_s c_n)\mathbf{A}_s(l)\rangle &=: \langle\ {}^k\mathbf{m}(l)\rangle \equiv\ {}^k\mathbf{M}^s , \\ {}^k\mathbf{S}^1(\mathbf{r},t) &:= -\langle\ {}^k\mathbf{T}\mathbf{A}_s(l)\rangle , \\ {}^k\mathbf{S}^2(\mathbf{r},t) &:= \langle(\mathbf{v} -\ {}^k\mathbf{V}^s) \otimes\ {}^k\mathbf{m}(l)\rangle . \end{aligned} $$

c. *Balance of energy*

$$ \begin{aligned} (4.7)\quad & \frac{\delta_n}{\delta t}({}^k\rho^s({}^k e^s + 0.5\ {}^k\mathbf{V}^s\cdot\ {}^k\mathbf{V}^s)) - 2Hc_n\ {}^k\rho^s({}^k e^s + 0.5\ {}^k\mathbf{V}^s\cdot\ {}^k\mathbf{V}^s) \\ &\quad + \operatorname{div}_s({}^k\rho^s({}^k e^s + 0.5\ {}^k\mathbf{V}^s\cdot\mathbf{V}^s)\ {}^k\mathbf{V}^s_\tau + {}^k\mathbf{q}^s + {}^k\mathbf{W}_E \\ &\quad + {}^k\mathbf{V}^s\ {}^k\mathbf{T}_s + 0.5\ {}^k\mathbf{V}^s\cdot\ {}^k\mathbf{V}^s\ {}^k\mathbf{W}_\rho) = {}^k(\rho\mathbf{f})^s\cdot\ {}^k\mathbf{V}^s + {}^k\rho^s\ {}^k\tilde{r}^s \\ &\quad + [\![({}^k\rho(e + 0.5\mathbf{v}\cdot\mathbf{v})(\mathbf{v} - \mathbf{c}) + {}^k\mathbf{q} - {}^k\mathbf{v}\mathbf{T})j]\!]\cdot\mathbf{n} \\ &\quad + \langle k({}^{k-1}\rho(e + 0.5\mathbf{v}\cdot\mathbf{v})(\mathbf{v} - \mathbf{c}) + {}^{k-1}\mathbf{q} - {}^{k-1}\mathbf{v}\mathbf{T})j\rangle\cdot\mathbf{n} \\ &\quad -[\![\,{}^k\rho(e + 0.5\mathbf{v}\cdot\mathbf{v}) \otimes (\mathbf{v} + \zeta\,\operatorname{grad}_s c_n)\mathbf{A}_s(\zeta)\operatorname{grad}_s\zeta]\!] \\ &\quad [\![({}^k\mathbf{v}\mathbf{T} - {}^k\mathbf{q})\mathbf{A}_s(\zeta)\operatorname{grad}_s\zeta]\!] - \left[\!\!\left[\frac{\delta_n}{\delta t}\zeta\ {}^k\rho(e + 0.5\mathbf{v}\cdot\mathbf{v})j\right]\!\!\right], \end{aligned} $$

where we have used the notations:

$$ \begin{aligned} {}^k\rho^s\ {}^k e^s &:= \langle j\rho\ {}^k\tilde{e}\rangle, \qquad \rho^s\tilde{r}^s := \langle j^k\rho(r + \mathbf{f}\cdot(\mathbf{v} - \mathbf{V}^s))\rangle , \\ {}^k\mathbf{T}_s &:=\ {}^k\mathbf{S}_1 +\ {}^k\mathbf{S}_2 . \end{aligned} $$

d. *Thermodynamic inequality*

$$ \begin{aligned} (4.8)\quad & \frac{\delta_n}{\delta t}\ {}^k\rho^s\ {}^k\eta^s - 2Hc_n\ {}^k\rho^s\ {}^k\eta^s + \operatorname{div}_s({}^k\rho^s\ {}^k\eta^s\ {}^k\mathbf{V}^s_\tau + {}^k\tilde{\mathbf{k}}^s + {}^k\mathbf{W}_\eta) \\ &\geq {}^k\rho^s\ {}^k r^s_\eta + [\![({}^k\rho\eta(\mathbf{v} - \mathbf{c}) + {}^k\mathbf{q}/\vartheta + {}^k\mathbf{k})j]\!]\cdot\mathbf{n} \\ &\quad -[\![\,{}^k\rho\eta(\mathbf{v} + \zeta\,\operatorname{grad}_s c_n)\mathbf{A}_s(\zeta)\operatorname{grad}_s\zeta]\!] - [\![({}^k\mathbf{q}/\vartheta + {}^k\mathbf{k})\mathbf{A}_s(\zeta)\operatorname{grad}_s\zeta]\!] \\ &\quad -\left[\!\!\left[\frac{\delta_n}{\delta t}\zeta\ {}^k\rho\eta j\right]\!\!\right] + \langle k({}^k\rho\eta(\mathbf{v} - \mathbf{c}) + {}^k\mathbf{q}/\vartheta + {}^k\mathbf{k})j\rangle\cdot\mathbf{n} , \end{aligned} $$

where we have used the following notations:

$$ {}^k\rho^s\ {}^k\eta^s := \langle j^k\rho\eta\rangle, \qquad {}^k\rho^{sk}r^s_\eta := \langle j^k\rho(r/\vartheta)\rangle , $$

$$
{}^{k}\widetilde{\mathbf{k}}^{s} := \langle ({}^{k}\rho(\mathbf{q}/\vartheta + \mathbf{k})\mathbf{A}_s(l) + ({}^{k}\rho\eta(\mathbf{v} - \mathbf{V}^s))\mathbf{A}_s(l)\rangle ,
$$

$$
\mathbf{W}_\eta := \langle {}^{k}\rho\eta(\mathbf{A}_s(l) - j(l)\mathbf{1}_s)\mathbf{V}^s + l\,{}^{k}\rho\eta\,\mathrm{grad}_s\, c_n\mathbf{A}_s(l)\rangle .
$$

4.3. Epilogue: future developments towards a constitutive theory for nonmaterial twodimensional continua

The problem which has to be faced at this stage of the development of our model concerns Constitutive Modelling of the interfacial layers.

Self-consistence arguments will lead us to try to deduce, exactly so as it was done for balance equations, the constitutive relations for nonmaterial bidimensional continua from the corresponding threedimensional ones.

One observation is immediately possible: let us look, for instance, at the previously obtained expression for surface stress tensor: the deduction of the surface constitutive relations is not straightforward from the threedimensional ones. Indeed, in the expression for $2D$ quantities the $3D$ velocity field $\mathbf{v}$ appears: this means that the particular kinematical situations inside the layer affect the behaviour of the twodimensional continuum modelling it.

The last circumstance, while giving reasonable chances to yield the right framework for modelling Marangoni effects, renders the constitutive description of bidimensional nonmaterial continua more complex.

A possible way of avoiding such a difficulty is suggested in the classical "*Theory of Shells*" (material bidimensional continua) by LOVE [34]: some physically reasonable assumptions about the kinematics inside the thin continua are made, what leads to sufficiently accurate bidimensional models.

This procedure proved itself to be fruitful also in the theory of nonmaterial bidimensional continua: indeed, for the so-called *perfectly viscous interfacial layers* (introduced by ISHII [23]) even the deduction of surface balance laws is greatly simplified (for more details see DELL'ISOLA and ROMANO [13, 14] or SEPPECHER [45]).

For this reasons we are led to define the *Kirchhoff–Love type interfaces*.

We call the Kirchhoff–Love type interfaces such interfaces $\mathcal{Z}_t$ that:

i) $\mathcal{Z}_t$ is delineated (in the sense of our Sec. 2) by a surface Σ_t and surface fields $\zeta^{\pm}$ (or $z^{\pm}$),

ii) the material $3D$ field $\mathbf{v}$ has the following particular form:

$$
\mathbf{v}(l; \mathbf{r}, t) = \mathbf{v}_0(\mathbf{r}, t) - l(\mathrm{grad}_s\, w_n + \mathbf{b}_0\mathbf{v}_\tau) , \tag{4.9}
$$

where $\mathbf{v}_0$, $\mathbf{v}_\tau$ and w_n are the suitable vectors, tangent (to Σ_t) and scalar fields defined on Σ_t and on every Σ_t^l, respectively.

Previous definition can be interpreted as follows:

a) all the particles belonging to the line

$$
\mathbf{l}_t := \{\mathbf{z} \in \mathcal{Z}_t / \exists l \in [\zeta^-, \zeta^+] \quad \mathbf{z} = \mathbf{r} + l\mathbf{n}\}
$$

move with the same normal (i.e. along the vector field $\mathbf{n}$) speed: $\mathbf{v}_0 \cdot \mathbf{n}$;

b) when the fields w_n and $\mathbf{v}_\tau$ are independent of the variable l, then the tangential velocity field along the same $\mathbf{l}_t$ depends linearly upon the variable l;

c) when the following equalities

$$\forall l \in [\zeta^-, \zeta^+] \quad w_n = c_n, \quad (\mathbf{1} - \mathbf{n} \otimes \mathbf{n})\mathbf{v}_0 = \mathbf{v}_\tau \tag{4.10}$$

hold, then all the particles belonging to the same $\mathbf{l}_t$ at a given instant t will belong (if they still belong to $\mathcal{Z}_\tau$) to the same line $\mathbf{l}_\tau$, for every different instant τ,

d) when, togheter with the relations (4.10), also the following equality

$$\mathbf{v}_0 \cdot \mathbf{n} = c_n ,$$

holds, then the layer $\mathcal{Z}_t$ is material,

The case d) was considered by Love, in his classical treatment of the theory of material shells.

Appendix

Here the derivation of the transformation of the oriented surface element $\mathbf{N}(l)da$ of the ruled surface Ω_t formed by segments $\{\mathbf{r} + l\mathbf{n}(\mathbf{r}) : l \in [z^-, z^+], \mathbf{r} \in \mathcal{C}_t \subset \Sigma_t\}$, where $\mathcal{C}_t$ is a curve ([12]) on Σ_t, will be given. Here $\mathbf{N}(l)$ is the outward unit normal to Ω_t, given by

$$\mathbf{N}(l) := \mathbf{t}_l \times \mathbf{n} \| \mathbf{t}_l \times \mathbf{n} \|^{-1} , \tag{A.1}$$

where $\mathbf{t}_l$ is a vector tangent to the curve $\mathcal{C}_t^l$ which is lifting of the curve $\mathcal{C}_t$ to Σ_t^l; the tangent vector to the latter we shall denote by $\mathbf{t}_0$. Due to Eq. (2.2) and to the fact that each of $\mathbf{t}_l$ and $\mathbf{t}_0$ is orthogonal to $\mathbf{n}$, and $\mathbf{n}$ has a unit length, we get

$$\mathbf{t}_l = (\mathbf{1}_s - l\mathbf{b})\mathbf{t}_0 \quad \text{and} \quad \mathbf{t}_l \times \mathbf{n} = (\mathbf{1}_s - l\mathbf{b})\mathbf{t}_0 \times \mathbf{n}$$

and

$$\|\mathbf{t}_l \times \mathbf{n}\| = \|\mathbf{t}_l\| = j(l)\|\mathbf{t}_0\| . \tag{A.2}$$

If by ds_l and ds we denote the line element of the curves $\mathcal{C}_t^l$ and $\mathcal{C}_t$, respectively, then from Eq. (A.2) it follows that its ratio $ds_l/ds := \|\mathbf{t}_l\|/\|\mathbf{t}_0\|$ is equal to $j(l)$. If the tangent vector $\mathbf{t}_0$ has ([13]) the splitting $d^\alpha \mathbf{a}_\alpha$, then $\mathbf{t}_l = d^\alpha \mathbf{a}_\alpha(l)$, and denoting components of the alternation tensor $\varepsilon(l)$ of the surface Σ_t^l by $\varepsilon_{\alpha\beta}(l) = j(l)\varepsilon_{\gamma\alpha}$ (cf. one formula above Eq. (2.9) and the next one, i.e. (2.10)), we obtain

$$\begin{aligned}\mathbf{t}_l \times \mathbf{n} = d^\alpha \mathbf{a}_\alpha(l) \times \mathbf{n} &= d^\alpha \varepsilon_{\gamma\alpha}(l)\mathbf{a}^\gamma(l) = d^\alpha j(l)\varepsilon_{\gamma\alpha}\mathbf{a}^\gamma(l) \\ &= d^\alpha j(l)\varepsilon_{\beta\alpha}\delta^\beta_\gamma \mathbf{a}^\gamma(l) = d^\alpha j(l)\varepsilon_{\beta\alpha}(\mathbf{a}^\beta \cdot \mathbf{a}_\gamma)\mathbf{a}^\gamma(l) \\ &= d^\alpha j(l)(\mathbf{a}_\alpha \times \mathbf{n}) \cdot \mathbf{a}_\gamma \mathbf{a}^\gamma(l) = j(l)(\mathbf{a}^\gamma(l) \otimes \mathbf{a}_\gamma)\mathbf{t}_0 \times \mathbf{n} .\end{aligned}$$

If we put $\tilde{\mathbf{n}} := \mathbf{t}_0 \times \mathbf{n}/\|\mathbf{t}_0\|$ for the unit normal to the curve $\mathcal{C}_t$ that is both tangent and outwardly directed with respect to Σ_t, then the last expression, together with Eqs. (A.1) and (A.2), will lead to relation

$$\mathbf{N}(l)da = \mathbf{N}(l)dl\, ds_l = j(l)(\mathbf{a}^\gamma(l) \otimes \mathbf{a}_\gamma)\tilde{\mathbf{n}} dl\, ds . \tag{A.3}$$

The componentwise derivation of the last formula can be found in ALT and HUTTER [4].

([12]) If $\mathcal{C}_t = \Sigma_t \cap \overline{\mathcal{Z}}_t$, then the ruled surface will be the lateral boundary of $\mathcal{Z}_t$, i.e. $\partial\mathcal{Z}_t \backslash (\Sigma_t^- \cup \Sigma_t^+)$.

([13]) If λ, running along some interval of the real line, is a parameter of the curve $\mathcal{C}_t$ and each of $\mathcal{C}_t^l$, then $d^\alpha := dl^\alpha/d\lambda$.

The normal (not unit) vector to either surface $\Sigma_t^{\pm}$ in Eq. (2.26) will be given by

$$\begin{aligned}(A.4)\quad \mathbf{a}_1^{\pm}\times\mathbf{a}_2^{\pm} &= \mathbf{a}_1\times\mathbf{a}_2-\zeta^{\pm}(\mathbf{b}\mathbf{a}_1\times\mathbf{a}_2+\mathbf{a}_1\times\mathbf{b}\mathbf{a}_2)+(\zeta^{\pm})^2\mathbf{b}\mathbf{a}_1\times\mathbf{b}\mathbf{a}_2\\ &+\mathbf{a}_1\times\mathbf{n}\frac{\partial\zeta^{\pm}}{\partial l^2}+\mathbf{n}\times\mathbf{a}_2\frac{\partial\zeta^{\pm}}{\partial l^1}-\zeta^{\pm}\left(\mathbf{b}\mathbf{a}_1\times\mathbf{n}\frac{\partial\zeta^{\pm}}{\partial l^2}+\mathbf{n}\times\mathbf{b}\mathbf{a}_2\frac{\partial\zeta^{\pm}}{\partial l^1}\right)\\ &=\varepsilon_{12}(1-\zeta^{\pm}\operatorname{tr}\mathbf{b}+(\zeta^{\pm})^2\det\mathbf{b})\mathbf{n}+\varepsilon_{12}\zeta^{\pm}\left(b_1^1\frac{\partial\zeta^{\pm}}{\partial l^2}-b_2^1\frac{\partial\zeta^{\pm}}{\partial l^1}\right)\mathbf{a}^2\\ &-\varepsilon_{12}\zeta^{\pm}\left(b_1^2\frac{\partial\zeta^{\pm}}{\partial l^2}-b_2^2\frac{\partial\zeta^{\pm}}{\partial l^1}\right)\mathbf{a}^1-\varepsilon_{12}\operatorname{grad}_s\zeta^{\pm}.\end{aligned}$$

Inspecting the contents of the first bracket we can see the value of the Jacobian j at $l=\zeta^{\pm}$, while the next two brackets give minus $\zeta^{\pm}(\mathbf{b}-2H\mathbf{1}_s)\operatorname{grad}_s\zeta^{\pm}$ equal (cf. Eq. (2.12)) to $-\zeta^{\pm}\tilde{\mathbf{b}}\operatorname{grad}_s\zeta^{\pm}$. With this at hand and remembering the tensor $\mathbf{A}_s(l)$ from Eq. (2.13), we write the final relationship

$$(A.5)\qquad \mathbf{n}^{\pm}(\mathbf{r})da^{\pm}:=\mathbf{a}_1^{\pm}\times\mathbf{a}_2^{\pm}dl^1dl^2=(j(\zeta^{\pm},\mathbf{r})\mathbf{n}(\mathbf{r})-\mathbf{A}(\zeta^{\pm})\operatorname{grad}_s\zeta^{\pm})da\,,$$

where on RHS the surface element $da=\sqrt{a}dl^1dl^2$. In what follows the ratio $da^{\pm}/da$ is denoted by $j^{\pm}$; it is a function of $\mathbf{r}$ and t.

The boundary surfaces $\Sigma_t^{\pm}$ move now with a velocity different than that calculated by Eq. (2.22). In fact, performing the time differentiation in Eq. (2.26) we obtain the following velocities $\mathbf{c}^{\pm}$ of displacement of $\Sigma_t^{\pm}$:

$$(A.6)\qquad \mathbf{c}^{\pm}=\mathbf{c}-\zeta^{\pm}(\operatorname{grad}_s c_n+\mathbf{b}\mathbf{c})+\frac{\partial\zeta^{\pm}}{\partial t}\mathbf{n}\,.$$

Now we are making the final calculation of the product of $j^{\pm}$ and the normal speed of displacement of either boundary surface $\Sigma_t^{\pm}$. It will proceed as follows

$$\begin{aligned}(A.7)\quad & j^{\pm}\mathbf{c}^{\pm}\cdot\mathbf{n}^{\pm}(\mathbf{r})\\ &=\left(\mathbf{c}-\zeta^{\pm}(\operatorname{grad}_s c_n+\mathbf{b}\mathbf{c})+\frac{\partial\zeta^{\pm}}{\partial t}\mathbf{n}\right)\cdot(j(\zeta^{\pm},\mathbf{r})\mathbf{n}(\mathbf{r})-\mathbf{A}(\zeta^{\pm})\operatorname{grad}_s\zeta^{\pm})\\ &=j(\zeta^{\pm},\mathbf{r})\left(c_n+\frac{\partial\zeta^{\pm}}{\partial t}-\operatorname{grad}_s\zeta^{\pm}\mathbf{c}\right)+\zeta^{\pm}\operatorname{grad}_s\zeta^{\pm}\mathbf{A}_s(\zeta^{\pm})\cdot\operatorname{grad}_s c_n\,.\end{aligned}$$

Using the formula (2.21) for the displacement derivative we can rewrite Eq. (A.7) to get

$$(A.8)\qquad j^{\pm}\mathbf{c}^{\pm}\cdot\mathbf{n}^{\pm}(\mathbf{r})=j(\zeta^{\pm},\mathbf{r})\left(c_n+\frac{\delta\zeta^{\pm}}{\delta t}\right)+\zeta^{\pm}\operatorname{grad}_s\zeta^{\pm}\mathbf{A}_s(\zeta^{\pm})\cdot\operatorname{grad}_s c_n\,.$$

Hence follows the expression for $j^{\pm}$

$$j^{\pm}:=\|\mathbf{a}_1^{\pm}\times\mathbf{a}_2^{\pm}\|=\sqrt{a}(j^2((\zeta^{\pm},\mathbf{r})+\operatorname{grad}_s\zeta^{\pm}\cdot\mathbf{A}_s^2(\zeta^{\pm})\operatorname{grad}_s\zeta^{\pm})^{0.5}\,,$$

which can be useful in some derivations.

Acknowledgements

This research was sponsored by the Polish Academy of Sciences (Research Project CPBP 02.01) and Italian CNR, GNFM.

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Received May 28, 1993.

Arch. Mech., 45, 3, pp. 361–369, Warszawa 1993

On transient development of waves generated by a porous wave maker

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THE INITIAL VALUE problem of surface waves generated by a harmonically oscillating porous wave maker immersed vertically in incompressible homogeneous liquid is considered. The resulting motion is investigated by using the Laplace transform in time and the Fourier transform of special type in space. An asymptotic analysis for large time and large distance is given for the free surface elevation.

1. Introduction

THE LINEAR theory of surface waves generated due to harmonic oscillations of a wave maker was initially developed by HAVELOCK [1]. During the last two decades this theory was studied and developed by many investigators (cf. URSELL, DEAN [2], KENNARD [3]) for vertical and nearly vertical wave makers. The influence of leakage around a wave maker on the wave amplitude is an important effect and an interesting phenomenon to study as it may have application in the study of surface waves in reservoirs or lakes caused by landslides during earthquakes, and in designing the construction of dams.

In the present paper we consider the transient development of waves at the free surface of the fluid produced by harmonically oscillating porous wave maker immersed vertically in a homogeneous incompressible fluid of finite depth. Following TAYLOR [4], regarding the boundary condition on the porous wave maker it is assumed that the velocity perpendicular to the porous plane is linearly proportional to the difference in pressure between the two sides of the wave maker. The integral representation of the free surface elevation is obtained by application of the Laplace transform in time and the Fourier transform of special type in space coordinate (following SNEDDON [5]). Then the solutions for asymptotic waves for large time and distance are obtained by the application of stationary phase method. It may be mentioned that CHWANG [6] solved a porous wave maker problem by the eigenfunction expansion procedure. In the present paper, an initial value formulation of the generation of surface waves due to oscillatory wave maker is presented and solved by means of the integral transform technique to obtain the solutions representing progressive transient wave and ultimate steady wave as $t \to \infty$.

2. Statement and formulation

We consider the two-dimensional motion in a liquid of finite depth h and volume density ϱ in the gravitational field, neglecting the effect of surface tension at the free surface. Let the $y = 0$ plane be chosen as the undisturbed free surface with the y-axis taken vertically downwards. The motion is generated due to a prescribed time-harmonic normal velocity

$$U(y,t) = u(y)e^{i\omega t}H(t) \tag{2.1}$$

on a porous wave maker occupying the position $x = 0$, $0 \leq y \leq h$, where $u(y)$ is an arbitrary function of y, ω is the frequency and $H(t)$ is the unit step function.

The velocity potential $\varphi(x, y, t)$ satisfies an initial boundary value problem in which

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0, \quad 0 \leq x \leq \infty, \quad 0 \leq y \leq h, \quad t > 0. \tag{2.2}$$

The bottom condition is

$$\frac{\partial \varphi}{\partial y} = 0 \quad \text{on} \quad y = h, \quad t > 0. \tag{2.3}$$

The linearized dynamical and kinematical boundary conditions are, respectively,

$$\frac{\partial \varphi}{\partial t} = g\eta(x, t) \quad \text{on} \quad y = 0, \quad t > 0, \tag{2.4}$$

$$\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial t}\eta(x, t) \quad \text{on} \quad y = 0, \quad t > 0, \tag{2.5}$$

where $\eta(x, t)$ is the elevation of the free surface above its mean level, g being the acceleration due to gravity.

Following CHWANG [6] and basing on TAYLOR'S [4] idea, the hydrodynamic pressure on the positive side of the wave maker surface $p^+(y, t)$ is related to that on the negative side of the plate by

$$P(0, y, t) = p^+(y, t) = -P^-(y, t),$$

where $P(x, y, t)$ is the hydrodynamic pressure. As the wave maker is made of material with very fine pores, the normal velocity of the fluid passing through the porous plate is thus linearly proportional to the pressure difference between two sides of the wave maker plate, and is given by

$$W(y, t) = \frac{b}{\mu}(p^+ - p^-) = \frac{2b}{\mu}P(0, y, t),$$

where μ is the dynamic viscosity and b is a coefficient which has the dimension of length. But the hydrodynamic pressure is related to the velocity potential $\varphi(x, y, t)$ by the Bernoulli equation as

$$P = -\varrho\varphi_t.$$

Therefore

$$W(y, t) = -\frac{2b\varrho}{\mu}\varphi_t \quad \text{at} \quad x = 0.$$

The boundary condition on the wave maker surface is the difference of the horizontal velocity and normal velocity

$$\frac{\partial \varphi}{\partial x} = U(y, t) - W(y, t) \quad \text{at} \quad x = 0,$$

or

$$\frac{\partial \varphi}{\partial x} - c\frac{\partial \varphi}{\partial t} = U(y, t) \quad \text{at} \quad x = 0, \tag{2.6}$$

where $c = 2b/\mu$, μ being the dynamical viscosity, and b — the constant having the dimension of length.

The initial conditions are

$$\varphi = \eta = 0 \quad \text{when} \quad t = 0 . \tag{2.7}$$

3. Method of solution

We define the Laplace transform in time of the function $F(x, y, t)$ and denote it by $\overline{F}(x, y, p)$,

$$\overline{F}(x, y, p) = \int_0^\infty e^{-pt} F(x, y, t)\, dt . \tag{3.1}$$

We also introduce the Fourier transform of a special type in the horizontal space coordinate x as

$$\Psi(\xi, y, p) = \int_0^\infty (\xi \cos \xi x + cp \sin \xi x)\overline{\varphi}(x, y, p)\, dx , \tag{3.2}$$

where $\overline{\varphi}(x, y, p)$ is the Laplace transform of $\varphi(x, y, t)$.

The corresponding inverse transform of Eq. (3.2) is given by

$$\overline{\varphi}(x, y, p) = \frac{2}{\pi} \int_0^\infty \frac{(\xi \cos \xi x + cp \sin \xi x)}{\xi^2 + c^2 p^2} \Psi(\xi, y, p)\, d\xi . \tag{3.3}$$

Hence application of the transform (3.1) and (3.2) to the system of Eqs. (2.2)–(2.6) gives a set of equations to be satisfied by $\Psi(\xi, y, p)$ in the form

$$\frac{d^2\Psi}{dy^2} - \xi^2 \Psi = \xi \overline{U}(y, p) , \tag{3.4}$$

$$\frac{d}{dy}\Psi = 0 \quad \text{on} \quad y = h , \tag{3.5}$$

$$\Psi = \frac{g}{p}\Omega \quad \text{on} \quad y = 0 , \tag{3.6}$$

and

$$\frac{d}{dy}\Psi = p\Omega(\xi, p) \quad \text{on} \quad y = 0 , \tag{3.7}$$

where $\Omega(\xi, p)$ is the Fourier transform (defined in Eq. (3.2)) of the function $\overline{\eta}(x, p)$, the Laplace transform of $\eta(x, t)$.

The solution of Eq. (3.4) is given by

$$\Psi(\xi, y, p) = A(\xi, p)e^{\xi y} + B(\xi, p)e^{-\xi y} + \int_0^y \sinh \xi(y - z)\overline{U}(z, p)\, dz , \tag{3.8}$$

where $A(\xi, p)$ and $B(\xi, p)$ are functions to be determined from Eqs. (3.5)–(3.7).

Using Eqs. (3.6) and (3.7), we get

$$\begin{aligned} A(\xi, p) &= \left(\frac{g\xi + p^2}{2p\xi}\right)\Omega(\xi, p) , \\ B(\xi, p) &= \left(\frac{g\xi - p^2}{2p\xi}\right)\Omega(\xi, p) . \end{aligned} \tag{3.9}$$

Hence, by using relations (3.9) in Eq. (3.8) we finally get $\Psi(\xi, y, p)$ in the form

$$\Psi(\xi, y, p) = \left[\frac{g}{p}\cosh \xi y + \frac{p}{\xi}\sinh \xi y\right]\Omega(\xi, p) + \int_0^y \sinh \xi(y - z)\overline{U}(z, p)\, dz\,. \tag{3.10}$$

Again, to get $\Omega(\xi, p)$ we substitute Eq. (3.10) in Eq. (3.5) so that

$$\Omega(\xi, p) = -\frac{p}{p^2 + \alpha^2}\int_0^h \frac{\xi \cosh \xi(h - z)\overline{U}(z, p)}{\cosh \xi h}\, dz\,, \tag{3.11}$$

where $\alpha^2(\xi) = g\xi \tanh \xi h$.

Substitution of Eq. (3.11) in Eq. (3.10) gives $\Psi(\xi, y, p)$ in the form

$$\Psi(\xi, y, p) = \int_0^y \sinh \xi(y - z)\overline{U}(z, p)\, dz - \sinh \xi y \int_0^h \frac{\cosh \xi(h - z)\overline{U}(z, p)}{\cosh \xi h}\, dz - \frac{g\xi \cosh \xi(h - y)}{(p^2 + \alpha^2)\cosh \xi h}\int_0^h \frac{\cosh \xi(h - z)\overline{U}(z, p)}{\cosh \xi h}\, dz\,. \tag{3.12}$$

Taking the inverse Laplace and Fourier transforms and applying the convolution theorem for the Laplace transform in Eqs. (3.11) and (3.12) together with Eq. (2.1), we get

$$\eta(x, t) = -\frac{2}{\pi}\int_0^\infty sM(\xi)\Big[\int_0^t e^{i\omega\tau}\Big\{\int_0^{t-\tau} \cos \alpha(t - \tau - \lambda)\sin(s\lambda + x\xi)\, d\lambda\Big\}d\tau\Big]\, d\xi\,, \tag{3.13}$$

$$\varphi(x, y, t) = \frac{2}{\pi c}\int_0^\infty \{K(\xi, y) - M(\xi)\sinh \xi y\}\Big\{\int_0^t e^{i\omega\tau}[\sin s(t - \tau)\cos \xi x + \cos s(t - \tau)\sin \xi x]\, d\tau\Big\}d\xi - \frac{2g}{\pi c}\int_0^\infty \frac{\xi M(\xi)\cosh \xi(h - y)}{\alpha \cosh \xi h}\Big\{\int_0^t e^{i\omega\tau}\int_0^{t-\tau}[\sin s(t - \tau - \lambda) \times \sin \alpha\lambda \cos \xi x + \cos s(t - \tau - \lambda)\sin \alpha\lambda \sin \xi x]\, d\lambda\, d\tau\Big\}d\xi\,, \tag{3.14}$$

where

$$K(\xi, y) = \int_0^y \sinh \xi(y - z)u(z)\, dz\,,$$

$$M(\xi) = \int_0^h \frac{\cosh \xi(h - z)u(z)}{\cosh \xi h}\, dz\,,$$

$$s(\xi) = \xi/c\,.$$

Again, Eqs. (3.13) and (3.14) can be expressed as

$$\eta(x, t) = \int_0^\infty \frac{sM(\xi)}{\alpha^2 - s^2}\left[\frac{s}{s^2 - \omega^2}\{e^{i\omega t}(s \sin x\xi - i\omega \cos x\xi) + i\omega \cos(st + x\xi)\right. \tag{3.15}$$

(3.15) [cont.)
$$-s\sin(st+x\xi)\}+\frac{1}{\alpha^2-\omega^2}\{e^{i\omega t}(i\omega s\cos x\xi-\alpha^2\sin x\xi)+s\alpha\sin\alpha t\cos x\xi$$
$$-i\omega s\cos st\cos x\xi+\alpha^2\cos\alpha t\sin x\xi+i\omega\sin\alpha t\sin x\xi\}\Big]\,d\xi$$

and

(3.16)
$$\varphi(x,y,t)=\frac{2}{\pi c}\int_0^\infty\left\{\frac{K(\xi,y)-M(\xi)\sinh\xi y}{s^2-\omega^2}\right\}[e^{i\omega t}(s\cos x\xi+i\omega\sin x\xi)$$
$$-s\cos(st+x\xi)-i\omega\sin(st+x\xi)]\,d\xi-\frac{2g}{\pi c}\int_0^\infty\frac{\xi M(\xi)\cosh\xi(h-y)}{(\alpha^2-s^2)\cosh\xi h}$$
$$\times\Big[\frac{\alpha}{s^2-\omega^2}\{e^{i\omega t}(s\cos x\xi+i\omega\sin x\xi)-s\cos(st+x\xi)-i\omega\sin(st+x\xi)\}$$
$$-\frac{1}{\alpha^2-\omega^2}\{\alpha e^{i\omega t}(s\cos x\xi+i\omega\sin x\xi)$$
$$-\alpha s\cos\alpha t\cos x\xi-i\omega s\sin\alpha t\cos x\xi+\alpha^2\sin\alpha t\sin x\xi-i\omega\alpha\cos\alpha t\sin x\xi\}\Big]\,d\xi\,.$$

4. Asymptotic analysis of solution

We are interested to investigate the principal properties of the wave motion after a long time and a large distance; the free surface elevation $\eta(x,t)$ is obtained in integral form given by (3.15). We write

(4.1)
$$\eta(x,t)=\frac{2}{\pi}\sum_{n=1}^{4}\eta_n\,,$$

where

$$\eta_1=\int_0^\infty\frac{s^2(\xi)M(\xi)}{(\alpha^2-s^2)(\alpha^2-\omega^2)}(\alpha\sin\alpha t-i\omega\cos\alpha t+i\omega e^{i\omega t})\cos x\xi\,d\xi\,,$$

$$\eta_2=-\int_0^\infty\frac{s^2(\xi)M(\xi)}{(\alpha^2-s^2)(s^2-\omega^2)}(s\sin st-i\omega\cos st+i\omega e^{i\omega t})\cos x\xi\,d\xi\,,$$

$$\eta_3=\int_0^\infty\frac{s^2(\xi)M(\xi)}{(\alpha^2-s^2)(s^2-\omega^2)}(se^{i\omega t}-s\cos st-i\omega\sin st)\sin x\xi\,d\xi\,,$$

$$\eta_4=-\int_0^\infty\frac{s(\xi)\alpha(\xi)M(\xi)}{(\alpha^2-s^2)(\alpha^2-\omega^2)}(\alpha e^{i\omega t}-\alpha\cos\alpha t-i\omega\sin\alpha t)\sin x\xi\,d\xi\,.$$

Again, we can write

(4.2)
$$\eta_j=I_j+J_j\qquad(j=1,2,3,4)\,,$$

where

$$I_1=\frac{i\omega e^{i\omega t}}{2}\int_0^\infty\frac{s^2(\xi)M(\xi)}{(\alpha^2-s^2)(\alpha^2-\omega^2)}(e^{ix\xi}+e^{-ix\xi})\,d\xi\,,$$

$$I_2 = -\frac{i\omega e^{i\omega t}}{2} \int_0^\infty \frac{s^2(\xi)M(\xi)}{(\alpha^2 - s^2)(s^2 - \omega^2)}(e^{ix\xi} + e^{-ix\xi})\,d\xi\,,$$

$$I_3 = \frac{e^{i\omega t}}{2i} \int_0^\infty \frac{s^3(\xi)M(\xi)}{(\alpha^2 - s^2)(s^2 - \omega^2)}(e^{ix\xi} - e^{-ix\xi})\,d\xi\,,$$

$$I_4 = -\frac{e^{i\omega t}}{2i} \int_0^\infty \frac{s(\xi)\alpha^2(\xi)M(\xi)}{(\alpha^2 - s^2)(\alpha^2 - \omega^2)}(e^{ix\xi} - e^{-ix\xi})\,d\xi\,.$$

It is also found that

(4.3)
$$J_j = \frac{1}{4i}\sum_{l=1}^{4} J_{jl} \qquad (l = 1,2,3,4),$$

$$J_{11}, J_{12} = \int_0^\infty \frac{s^2(\xi)M(\xi)}{(\alpha^2 - s^2)(\alpha - \omega)} e^{i(\alpha t \pm x\xi)}\,d\xi\,,$$

$$J_{13}, J_{14} = -\int_0^\infty \frac{s^2(\xi)M(\xi)}{(\alpha^2 - s^2)(\alpha + \omega)} e^{-i(\alpha t \mp x\xi)}\,d\xi\,,$$

$$J_{21}, J_{22} = -\int_0^\infty \frac{s^2(\xi)M(\xi)}{(\alpha^2 - s^2)(s - \omega)} e^{i(st \pm x\xi)}\,d\xi\,,$$

$$J_{23}, J_{24} = \int_0^\infty \frac{s^2(\xi)M(\xi)}{(\alpha^2 - s^2)(s + \omega)} e^{-i(st \mp x\xi)}\,d\xi\,,$$

$$J_{31}, J_{32} = \mp\int_0^\infty \frac{s^2(\xi)M(\xi)}{(\alpha^2 - s^2)(s - \omega)} e^{i(st \pm x\xi)}\,d\xi\,,$$

$$J_{33}, J_{34} = \mp\int_0^\infty \frac{s^2(\xi)M(\xi)}{(\alpha^2 - s^2)(s + \omega)} e^{-i(st \mp x\xi)}\,d\xi\,,$$

$$J_{41}, J_{42} = \pm\int_0^\infty \frac{s(\xi)\alpha(\xi)M(\xi)}{(\alpha^2 - s^2)(\alpha - \omega)} e^{i(\alpha t \pm x\xi)}\,d\xi\,,$$

$$J_{43}, J_{44} = \pm\int_0^\infty \frac{s(\xi)\alpha(\xi)M(\xi)}{(\alpha^2 - s^2)(\alpha + \omega)} e^{-(\alpha t \mp x\xi)}\,d\xi\,.$$

If $f(\xi)$ has a simple pole at $\xi = \xi_0$ in $a < \xi_0 < b$, then $|x| \to \infty$. We get (cf. LIGHTHILL [7])

$$\int_a^b f(\xi)e^{i\xi x}d\xi \sim i\pi\,\mathrm{sgn}(x)e^{i\xi_0 x}\cdot\left[\text{residue of } f(\xi) \text{ at } \xi = \xi_0 + O\left(\frac{1}{|x|}\right)\right].$$

For large t and x, the pole and stationary point of the integral give the main contribution to asymptotic value of the above integrals. It is noted that I_1, I_4, J_{11}, J_{12}, J_{41}, J_{42} contain two simple poles at $\xi = \xi_1$ and $\xi = \xi_2$; I_2, I_3, J_{21}, J_{22}, J_{31}, J_{32} have two simple poles at $\xi = \xi_1$ and $\xi = \xi_3$. J_{13}, J_{14}, J_{23}, J_{24}, J_{33}, J_{34}, J_{43}, J_{44} have a simple pole at $\xi = \xi_1$

where ξ_1, ξ_2, ξ_3 are only real roots, and this results from the equations given below

$$\begin{aligned} \alpha(\xi) &= s(\xi)\,, \\ \alpha(\xi) &= \omega\,, \\ s(\xi) &= \omega\,. \end{aligned} \tag{4.4}$$

Equation $(4.4)_1$ gives an infinite number of imaginary roots $\xi = i\xi_k$, $k = 1, 2, \ldots, \infty$, but as $|x| \to \infty$, by choosing the contour suitably, the polar contributions due to imaginary roots vanish.

Therefore, using Eqs. (4.4) and adding them we get

$$\eta_{\text{polar}} \sim -\frac{s(\xi_2)M(\xi_2)}{\alpha'(\xi_2)\{\omega + s(\xi_2)\}} e^{i(\omega t - x\xi_2)} + O\left(\frac{1}{x}\right). \tag{4.5}$$

None of I_j, J_{2j}, J_{3j} $(j = 1, 2, 3, 4)$ and J_{11}, J_{14}, J_{41}, J_{44} has a stationary point. Only J_{12}, J_{13}, J_{42}, J_{43} have a stationary point at $\xi = \xi_0$, which is found from the equation

$$\frac{d\alpha}{d\xi} = \frac{x}{t}\,. \tag{4.6}$$

If $f(\lambda) = \int_a^b g(t)e^{i\lambda h(t)}\,dt$ has a stationary point at $t = t_0$ then, following COPSON [8],

$$\int_a^b g(t)e^{i\lambda h(t)}dt \sim g(t_0)\left\{\frac{2\pi}{\lambda|h''(t_0)|}\right\}^{1/2} e^{i\{\lambda h(t) \pm \pi/4\}} + O\left(\frac{1}{\lambda}\right). \tag{4.7}$$

Using formula (4.7) we get the contribution for $\eta_{\text{transient}}$,

$$\eta_{\text{tr}} \sim \frac{i}{\sqrt{2\pi}}\left\{\frac{1}{t|\alpha''(\xi_0)|}\right\}^{1/2} \frac{s(\xi_0)M(\xi_0)}{\{\alpha(\xi_0) + s(\xi_0)\}}\left[\frac{e^{i\{\alpha(\xi_0)t - x\xi_0 - \pi/4\}}}{\alpha(\xi_0) - \omega} - \frac{e^{-i\{\alpha(\xi_0)t - x\xi_0 - \pi/4\}}}{\alpha(\xi_0) + \omega}\right] + O\left(\frac{1}{t}\right). \tag{4.8}$$

$\eta(x, t) = \eta_{\text{polar}} + \eta_{\text{transient}}$, where η_{polar} and $\eta_{\text{transient}}$ is given by Eqs. (4.5) and (4.8), respectively.

5. Asymptotic solution in the case of infinite depth

If the fluid is of infinite depth, that is when $h \to \infty$, the functions $M(\xi)$, $\alpha(\xi)$, $s(\xi)$, the poles, ξ_1, ξ_2, ξ_3 and stationary point ξ_0 are all simpler in the form and they are given by

$$M(\xi) = \int_0^\infty u(z)e^{-\xi z}\,dz\,, \qquad \alpha(\xi) = \sqrt{g\xi}\,,$$

$$s(\xi) = \xi/c\,, \qquad \xi_1 = c^2 g\,, \qquad \xi_2 = \omega^2/g\,, \qquad \xi_3 = c\omega\,, \qquad \xi_0 = gt^2/4x^2\,.$$

Therefore in this case, the asymptotic solutions for $\eta(x, t)$ can be obtained independently, or from Eqs. (4.5) and (4.8) by letting formally $h \to \infty$

$$\eta_{\text{st}} \sim -\frac{2\omega}{g}\frac{(\omega^2/g)M(\omega^2/g)}{\{c\omega + (\omega^2/g)\}} e^{i\{\omega t - (\omega^2/g)x\}} + O\left(\frac{1}{|x|}\right), \tag{5.1}$$

$$(5.2)\qquad \eta_{\mathrm{tr}} \sim \frac{i}{2}\sqrt{\frac{g}{\pi}}\frac{t}{x^{3/2}}\frac{(gt^2/4x^2)M(gt^2/4x^2)}{\{c(gt/2x)+(gt^2/4x^2)\}}\left[\frac{e^{i\{(gt^2/4x)-\pi/4\}}}{\{(gt/2x)-\omega\}}\right.$$
$$\left.-\frac{e^{-i\{(gt^2/4x)-\pi/4\}}}{\{(gt/2x)+\omega\}}\right]+O\left(\frac{1}{t}\right).$$

6. Conclusions

For a deeper understanding of the wave motion and the behaviour of wave formation at all times it is necessary to study the steady state and transient solution explicitly. The surface wave generation by a harmonically oscillating vertical porous wave maker immersed in an incompressible fluid of finite depth is investigated following the method of generalised function (cf. DEBNATH [9]), and an asymptotic analysis for large times and distances is given for the free surface elevation. For impermeable wave maker, taking $c = 0$, the results of FALTAS [10] can be recovered. The analysis presented here reveals the fact that the transient solution η_{tr} decays rapidly to zero as $t \to \infty$. Thus the ultimate steady state solution in Eqs. (4.5) and (5.1) represents the outgoing progressive wave propagating with the phase velocity ω/ξ_2 and g/ω for finite and infinite depth of liquid, respectively.

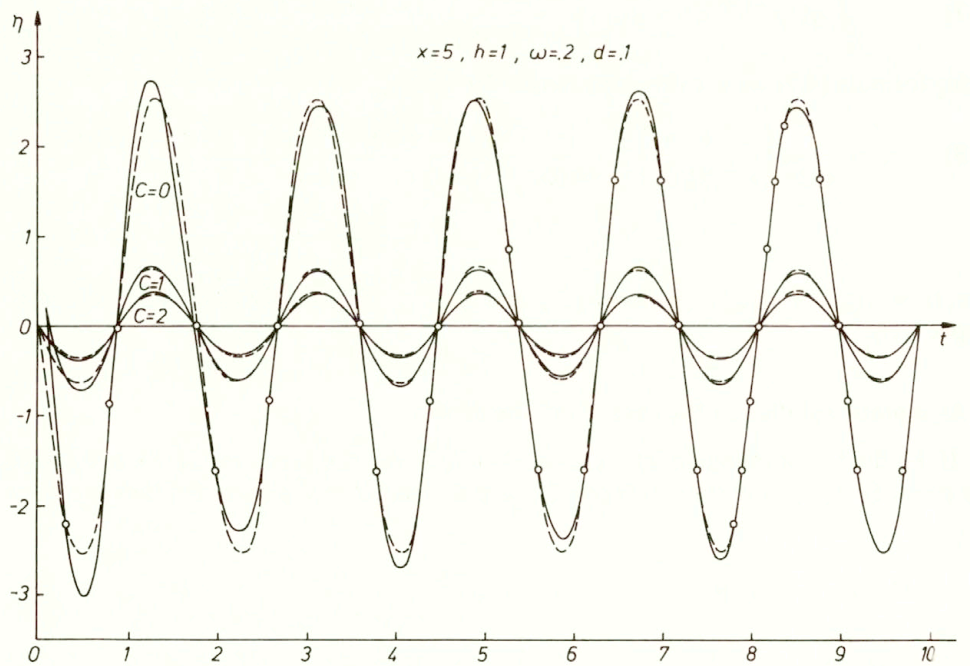

FIG. 1. Surface wave profile for different values of c at $x = 5$, $\omega = 2$, $h = 1$, $d = 1$.

The graphical representation of wave profile have been presented for CHWANG'S [6] results by a dotted line, and for our results — by a continuous line in Fig. 1. The surface wave profiles are plotted against time for fixed space variable x, frequency ω, depth h

and parameter $c(= 0, 1, 2)$ in Fig. 1. It is interesting to note that, as time increases, the ultimate steady state solutions for both the cases coincide exactly, although the initial development of wave amplitudes differ which is an expected and very interesting feature to note. The graphical pictures confirm that the transient part η_{tr} of wave amplitude rapidly decays to zero. By plotting the curve 1, 2, 3 for different values of c, the important result should be noted that the wave amplitude decreases with the increasing value of c. If we take $c = 0$, then the permeability character of wave maker can be eliminated, and the surface waves have maximum amplitudes, as it also has been noted by CHWANG [6]. Again it can be stated that if c becomes very large, the wave amplitude reduces to zero, i.e. the wave maker behaves as a transparent object and this result was also explained by CHWANG [6]. Due to the effect of permeability on the wave maker, the amplitude of a progressive wave is changed quantitatively, but the qualitative nature of the wave remains unchanged.

Acknowledgement

The paper has been revised incorporating all the suggestions and comments of the referee for which we are very thankful. The present work is partially supported by UGC-965/GRW and Calcutta Mathematical Society.

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Received June 3, 1993.

Arch. Mech., 45, 3, pp. 371–379, Warszawa 1993

Geometrical aspect of symmetrization of quasi-linear systems of the first order partial differential equations

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IN CONTINUUM MECHANICS, one often discusses symmetric and symmetric hyperbolic systems. Such systems are of interest for phenomenological thermomechanics where the additional balance law implied by a symmetric hyperbolic system of field equation is often interpreted as the entropy balance. The aim of this paper is to discuss the structure of the process of symmetrization of a system of partial differential equations. The approach to symmetrization developed by Boillat (G. Boillat, Sur l'existence et la recherche d'equations de conservation supplementaires pour les systems hyperboliques, C. R. Acad. Sc. Paris 278A,909,1974) is formulated in an invariant manner and its effectiveness is discussed. The geometrical approach naturally leads to a distinction between the trivial and nontrivial equivalencies of symmetric systems.

1. Introduction

IN CONTINUUM MECHANICS, one often discusses symmetric and symmetric hyperbolic systems. Such systems are of interest for phenomenological thermomechanics where the additional balance law implied by a symmetric hyperbolic system of field equation is often interpreted as the entropy balance [1–7].

One interesting problem concerns the conditions which are necessary or sufficient in order to ensure that a given system of field equations can be written in a symmetric form; the other problem is how to construct such a transformation explicitly. For the case of hyperbolic systems, this problem was discussed by BOILLAT [12] who expressed the symmetrizing matrix in terms of the eigenvectors occurring in the definition of hyperbolicity. In this paper, we apply the invariant definition of a symmetric system introduced in [10] in order to discuss the process of symmetrization. In Section 2 we show that Boillat's result is valid only for a certain fixed normalization of eigenvectors; however, in order to determine this normalization we first have to know the symmetrizing matrix and therefore Boillat's approach is not directly effective. The improved procedure leads to the system of partial differential equations (Eqs. (2.41), (2.42)) which, in principle, allows us to determine all symmetrizing matrices. However, the point which seems to be the most important is the definition of a symmetric system. In our approach the symmetric system is defined invariantly, that is, in the manner which does not depend on the choice of a coordinate system in the space of the dependent variables. From geometrical point of view, our approach is equivalent to Cartan's invariant description of partial differential equations and our invariant definition of a symmetric system can be easily written, in Cartan's language, in terms of differential forms. Of course, every differential form representing the symmetric system can be explicitly written in different coordinate systems what leads to the symmetric systems which are, from „analytic" point of view, different. In fact, they are not different since they are represented by the same differential form. In turn, sometimes a given symmetric system of partial differential equations can be geometrically represented by the different differential forms which cannot be transformed one into another after multiplication by a scalar function of the dependent variables; in such

a case, we say that the discussed systems possess nontrivially equivalent symmetric forms. Therefore, a distinction between the trivial and nontrivial equivalencies of the symmetric systems seems to be important for the purpose of formulating the theory of transforming the systems of partial differential equations into the symmetric form. In Section 2 we discuss these problems in more detail and we use the approach which, being equivalent to that of Cartan, is simpler from the computational point of view. In our approach, it is possible to describe the process of symmetrization of a system of partial differential equations in a manner which does not depend on the choice of „independent" equations and on the coordinate system.

By means of the geometrical formulation introduced by PERADZYŃSKI [8, 9], the system before symmetrization can be written in the „abstract Cauchy form" which has the above mentioned invariance properties. Then, exploiting the geometrical definition of symmetric system given in [10, 11], the approach to symmetrization developed by BOILLAT [12] is formulated in the invariant manner and its effectiveness is discussed. In our approach, the symmetric system is represented as a cross-section of the vector bundle $T^*(Q) \otimes T^*(Q) \otimes T_A$ (Q is the manifold of dependent variables, $T^*(Q)$ is its cotangent bundle, and T_A is the translation space corresponding to the affine space A of independent variables). Of course, the same cross-section of $T^*(Q) \otimes T^*(Q) \otimes T_A$ can be multiplied by non-vanishing real functions defined on Q and, written in different coordinates, gives rise to the infinite number of "symmetric forms" of a discussed system of partial differential equations. All such symmetric forms correspond to the same geometrical object and are in this sense indistinguishable. However, the existence of the systems of P.D.E. possessing "nontrivially different" symmetric forms can be demonstrated. For simplicity, we discuss systems of P.D.E. with vanishing source terms. The presence of non-vanishing sources does nor modify our results. Throughout this paper, we apply Einstein's convention assuming summation with respect to repeated indices. In order to maintain the self-consistency of presentation, all necessary definitions are presented at the beginning of Sec. 2.

2. Geometrical aspect of symmetrization of quasi-linear systems of the first order partial differential equations

In this paper, we shall apply the notation used in [7]. In the system

$$w^I{}_j{}^i(y_{j'}) \frac{\partial y_j}{\partial x_i} = 0, \quad I = 1, \ldots, m, \; j = 1, \ldots, m, \; i = 1, \ldots, n, \tag{2.1}$$

the independent variables (x_i), $i = 1, \ldots, n$ shall be interpreted as a coordinate system on affine space A

$$\mathbf{R} \ni (x_i) \to \vartheta + x_i \mathbf{e}_i \in A, \tag{2.2}$$

where $\mathbf{e}_i$, $i = 1, \ldots, n$ is a basis in the translation space T_A of A and ϑ is an arbitrary point of A. In turn, the dependent variables (y_j), $j = 1, \ldots, m$ shall be interpreted as a local coordinate system on the manifold Q. The solution of Eq. (2.1) which in coordinates is given by the functions

$$y_j = y_j(x_i) \tag{2.3}$$

attains the geometrical meaning of the function f acting from the subset of the affine space A into the manifold Q (in order to simplify the notation, in Eq. (2.3) we do not

distinguish between the function $y_j(x_i)$ and its value y_j). The natural base vectors ∂_{x_i}, $i = 1, \ldots, m$ of the coordinate system (2.2) can be identified with the corresponding vectors from the basis $\mathbf{e}_i$, $i = 1, \ldots, n$, whereas the natural base forms dx^i, $i = 1, \ldots, n$ of the coordinate system (2.2) can be identified with the forms from the dual basis F^i, $i = 1, \ldots, n$ of the basis $\mathbf{e}_i$, $i = 1, \ldots, n$.

The equations of the system (2.1) can be written in terms of contractions of the vector-valued 1-forms w^I,

$$w^I := w^I{}_j{}^i dy^j \otimes \mathbf{e}_i, \qquad I = 1, \ldots, m, \tag{2.4}$$

with the derivative f_* of f

$$\begin{aligned} f_* &= \partial y_j \frac{\partial y_j}{\partial x_i} \otimes dx^i = \partial y_j \frac{\partial y_j}{\partial x_i} \otimes F^i, \\ \langle w^I, f_* \rangle &= \left\langle w^I{}_j{}^i dy^j \otimes \mathbf{e}_i, \partial y_{j'} \frac{\partial y_{j'}}{\partial x_{i'}} \otimes F^{i'} \right\rangle \\ &= w^I{}_j{}^i \delta^j_{j'} \delta^{i'}_i \frac{\partial y_{j'}}{\partial x_{i'}} = w^I{}_j{}^i \frac{\partial y_j}{\partial x_i} = 0 \end{aligned} \tag{2.5}$$

(in Eqs. (2.5) $\delta^j_{j'}$ and $\delta^{i'}_i$ denote Kronecker's symbols).

Let us assume that the system (2.1) is determined. Then the set of equations which are algebraically implied by the system (2.1) can be identified with the set of cross-sections of the m-dimensional vector subbundle S of the vector bundle $T^*(Q) \otimes T_A$ which is spanned by $w^1, \ldots, w^m$ [8, 9].

Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be the vector fields on Q which are such that for each $q \in Q$ the vectors $\mathbf{v}_1(q), \ldots, \mathbf{v}_m(q)$ are linearly independent.

Let us define the field of the two-point tensors κ

$$\kappa := \sum_{I=1}^{m} \mathbf{v}_I \otimes w^I. \tag{2.6}$$

The system (2.1) can be alternatively written in the form

$$\operatorname*{Tr}_{(2,4)} \operatorname*{Tr}_{(3,5)} \kappa \otimes f_* = \sum_{I=1}^{m} \mathbf{v}_I \langle w^I, f_* \rangle = 0, \tag{2.7}$$

where $\operatorname*{Tr}_{(\alpha,\beta)}$ means the trace operation taken with respect to the indices (α, β). In order to simplify the notation, the trace operation on $A \otimes B$ in which the 2-nd index is contracted with the 4-th, the 3-rd index is contracted with the 5-th, A is a tensor field of the order 3 and B is a tensor field of the order 2, will be denoted $\langle A, B \rangle_2$. If, instead of $w^1, \ldots, w^m$, we take any other set of independent equations $\Omega^1, \ldots, \Omega^m$ (corresponding to another maximal set of linearly independent cross-section of S) then

$$w^I = A^I_J \Omega^J, \tag{2.8}$$

where A^I_J is a field of nondegenerate real matrices on Q, and the object κ defined in Eq. (2.6) transforms according to the following rule

$$\kappa := \sum_I \mathbf{v}_I \otimes w^I = \sum_I \mathbf{v}_I \otimes A^I_J \Omega^J = \sum_J (\mathbf{v}_I A^I_J) \otimes \Omega^J. \tag{2.9}$$

It should be stressed that the geometrical representation of the system (2.1) by the field of the two-point tensors κ is not defined uniquely since instead of $\mathbf{v}_1, \ldots, \mathbf{v}_m$ we can take any other set of vectors fields on Q which form a moving frame on Q. The expression (2.7) gives us the geometrical representation of the discussed system as the cross-section of the vector bundle $T(Q) \otimes T^*(Q) \otimes T_A$ and the important fact is that such a representation is not uniquely defined. Instead of the vector fields, we can use alternatively the fields of forms; for example, let us multiply the I-th equation of the system (2.1) by the form dy^I and let us add the results. The expression

$$w^I{}_j{}^i \frac{\partial y_j}{\partial x_i} dy^I = 0 \tag{2.10}$$

obtained in that way can be equivalently written in the form

$$\langle w^I{}_j{}^i dy^I \otimes dy^j \otimes \mathbf{e}_i, f_* \rangle_2 = 0 \tag{2.11}$$

and therefore the discussed system can be equivalently written as the contraction of f_* with the field of the two-point tensors G defined by

$$G := w^I{}_j{}^i dy^I \otimes dy^j \otimes \mathbf{e}_i. \tag{2.12}$$

Formula (2.11) defines the geometrical representation of the discussed P.D.E. as a cross-section of the vector bundle $T^*(Q) \otimes T^*(Q) \otimes T_A$ and is convenient for a discussion of symmetric systems; as it can be easily checked, the symmetric systems can be invariantly described by a such fields of the two-point tensors of the kind (2.12) which are symmetric in the first two indices. The invariant definition of symmetric systems based on this geometrization was introduced in [10] an exploited in [11].

As it has been already mentioned, the geometrical representation of the system of P.D.E. as the cross-section of $T(Q) \otimes T^*(Q) \otimes T_A$ is not uniquely defined. In order to eliminate this ambiguity, we shall restrict our considerations to the case of such systems which can be transformed to the Cauchy form. The convenient manner of discussing such systems is to apply the notion of a chronological structure [7]. Such a chronological structure is similar to that known from the theory of Galilean spacetime and is described by the so-called chronological form Ψ, $0 \neq \Psi \in T_A^*$ (by T_A^* we mean here the dual space of T_A). The form Ψ distinguishes the class of bases in T_A; we say that a basis $\mathbf{E}_1, \ldots, \mathbf{E}_n$ of T_A is consistent with Ψ if and only if

$$\begin{aligned} &\langle \Psi, \mathbf{E}_1 \rangle = 1, \\ &\langle \Psi, \mathbf{E}_l \rangle = 0, \quad l = 2, \ldots, n. \end{aligned} \tag{2.13}$$

In the coordinate system on A corresponding to the basis $\mathbf{E}_1, \ldots, \mathbf{E}_n$

$$\mathbf{R}^n \ni (t, z^l) \to \vartheta + t\mathbf{E}_1 + z^l \mathbf{E}_l \in A, \quad l = 2, \ldots, n. \tag{2.14}$$

The coordinate multiplying the vector $\mathbf{E}_1$ can be interpreted as the time coordinate, whereas $z^2, \ldots, z^n$ have a meaning of "spatial" coordinates.

Let the system (2.1) be in a Cauchy form with respect to the time coordinate

$$\frac{\partial y^I}{\partial t} + \hat{w}^I{}_j{}^i \frac{\partial y_j}{\partial z_l} = 0, \quad I = 1, \ldots, m, \quad j = 1, \ldots, m, \quad l = 2, \ldots, n. \tag{2.15}$$

After multiplying equations of the system (2.15) by the corresponding natural base vectors ∂y_I of the coordinate system (y_I), $I = 1, \ldots, m$ and adding the results, we arrive at

the following geometrical representation of the system (2.15) as the cross-section G' of $T(Q) \otimes T^*(Q) \otimes T_A$:

$$G' = \partial y_I \otimes dy^I \odot \mathbf{E}_1 + \partial y_I \otimes \widehat{w}^I{}_j{}^l dy^j \otimes \mathbf{E}_l, \tag{2.16}$$

$$\langle G', f_* \rangle_2 = \left(\frac{\partial y_I}{\partial t} + \widehat{w}^I{}_j{}^l \frac{\partial y_j}{\partial z_l} \right) \partial y_I = 0 \tag{2.17}$$

and the equivalence of Eqs. (2.17) and (2.15) follows from the linear independence of the vector fields ∂y_I, $I = 1, \ldots, m$. Let us take a contraction of the 3-rd index of G' with a chronological form Ψ; in order to simplify the notation, the result of such a contraction will be denoted by $G' \odot \Psi$ (such a simplified notation for the operation of contraction will be applied also in the rest of this paper). From the relation (2.13), we immediately obtain

$$G' \odot \Psi = \partial y_I \otimes dy^I < \Psi, E_1 >= \partial y_I \otimes dy^I. \tag{2.18}$$

The expression (2.18) defines the identity operator $\mathrm{Id}_{T(Q)}$ acting on vectors tangent to Q. Of course the operator $\mathrm{Id}_{T(Q)}$ does not depend on the choice of the coordinate system. This suggests the following definition. We shall say that the cross-section G' of $T(Q) \otimes T^*(Q) \otimes T_A$ defines the system P.D.E. in the abstract Cauchy form in the direction Ψ if and only if

$$G' \odot \Psi = \mathrm{Id}_{T(Q)}. \tag{2.19}$$

If the property (2.19) is satisfied, then the discussed system is in Cauchy form with respect to the time coordinate in all coordinate systems corresponding to bases consistent with Ψ. Let $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n$ and $\widetilde{\mathbf{E}}_1, \ldots, \widetilde{\mathbf{E}}_n$ be two bases consistent with Ψ. The transformation rule between such bases is

$$\begin{aligned} \mathbf{E}_1 &= \widetilde{\mathbf{E}}_1 + \beta_l \widetilde{\mathbf{E}}_l, \\ \mathbf{E}_l &= B_l^{l'} \widetilde{\mathbf{E}}_{l'}, \quad l, l' = 2, \ldots, n, \end{aligned} \tag{2.20}$$

where $B_l^{l'}$ is a nondegenerate $(n-1) \times (n-1)$ matrix and β_l, $l = 2, \ldots, n$ are the arbitrary real constants [7]. After inserting Eqs. (2.20) into Eq. (2.16) we arrive at the following transformation rule for G'

$$\begin{aligned} G' &= \mathrm{Id}_{T(Q)} \otimes \mathbf{E}_1 + \partial y_I \otimes \widehat{w}^I{}_j{}^l dy^j \otimes \mathbf{E}_l \\ &= \mathrm{Id}_{T(Q)} \otimes (\widetilde{\mathbf{E}}_1 + \beta_{l'} \widetilde{\mathbf{E}}_{l'}) + \partial y_I \otimes \widehat{w}^I{}_j{}^l dy^j \otimes B_l^{l'} \widetilde{\mathbf{E}}_{l'} \\ &= \mathrm{Id}_{T(Q)} \otimes \widetilde{\mathbf{E}}_1 + [\delta_j^I \beta_{l'} + \widehat{w}^I{}_j{}^l B_l^{l'}] \partial y_I \otimes dy^j \otimes \widetilde{\mathbf{E}}_{l'}. \end{aligned} \tag{2.21}$$

For simplicity, the part of Eq. (2.16) describing "spatial" derivatives will by denoted by K,

$$\mathbf{K} := \partial y_I \otimes w^I{}_j{}^l dy^j \otimes \mathbf{E}_l. \tag{2.22}$$

Let $E^1, E^2, \ldots, E^n$ denote the basis in T_A^* dual to the basis $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n$ in T_A. Let L denote the vector space spanned by the linear combinations of the forms $E^2, \ldots, E^n$ (of course, L is a vector subspace of T_A^*). The system (2.16) is hyperbolic if and only if for each form Φ from L the linear operator $\mathbf{K}(q) \odot \Phi$ defines the eigenproblem

$$(\mathbf{K}(q) \odot \Phi) \gamma_\alpha(q, \Phi) = c_\alpha(q, \Phi) \gamma_\alpha(q, \Phi) \tag{2.23}$$

with real eigenvalues $c_\alpha(q, \Phi)$ and eigenvectors $\gamma_\alpha(q, \Phi)$ which, for each $q \in Q$, span the m-dimensional vector space. Of course, the eigenvectors in Eq. (2.23) are defined up to

multiplicative constants. It can be easily checked that the transformations (2.20), (2.21) preserve the hyperbolicity. For simplicity, we shall assume that the eigenvalues $c_\alpha(q, \Phi)$ are nondegenerate. We shall also make use of the eigenvalue problem dual to Eq. (2.23). In order to describe it in a more detail, we shall write the eigenvalue problem for $\mathbf{K} \odot \Phi$ in slightly modified form

$$(\lambda Id_{T(Q)} + \mathbf{K} \odot \Phi)\mathbf{V} = 0, \tag{2.24}$$

where λ is a real parameter and $\mathbf{V}$ denotes the vector field on Q. Let $\gamma^{\alpha'}(q, \Phi), \alpha' = 1, \ldots, n$ denote the forms dual to the vectors $\gamma_\alpha(q, \Phi), \alpha = 1, \ldots, n$. Equation (2.24) can be alternatively written as

$$\begin{aligned} &\Big\{\lambda\Big[\sum_\alpha \gamma_\alpha(q, \Phi) \otimes \gamma^\alpha(q, \Phi)\Big] + \Big[\sum_\alpha c_\alpha(q, \Phi) \otimes \gamma_\alpha \otimes \gamma^a\Big]\Big\} \odot \mathbf{V} \\ &\qquad = \Big\{\sum_\alpha (\lambda + c_\alpha(q, \Phi))\gamma_\alpha(q, \Phi) \otimes \gamma^\alpha(q, \Phi)\Big\} \odot \mathbf{V} = 0 \end{aligned} \tag{2.25}$$

which has a nontrivial solution if and only if

$$\lambda = -c_\alpha(q, \Phi), \tag{2.26}$$

and the vector $\mathbf{V}$ is proportional to $\gamma_\alpha(q, \Phi)$. In turn, we can also consider the equation

$$w \odot (\lambda \operatorname{Id}_{T(Q)} + \mathbf{K} \odot \Phi) = 0 \tag{2.27}$$

which is solved for

$$\lambda = -c_\alpha(q, \Phi) \tag{2.28}$$

by the forms w being proportional to $\gamma^\alpha(q, \Phi)$. Let

$$H = H_{jj'} dy^j \otimes dy^{j'} \tag{2.29}$$

be a field of tensors on Q which is covariant in both indices and which symmetrizes the discussed system of P.D.E. in the sense that

$$\det H_{jj'} \neq 0 \tag{2.30}$$

and the contraction of H with $\operatorname{Id}_{T(Q)} \otimes \mathbf{E}_1 + \mathbf{K}$ given by

$$H \odot (\operatorname{Id}_{T(Q)} \otimes \mathbf{E}_1 + \mathbf{K}) = H \otimes \mathbf{E}_1 + H \odot \mathbf{K} \tag{2.31}$$

is symmetric in the first two indices. Of course, in coordinates this is equivalent to the symmetrization of a quasi-linear system by means of multiplying it by a nondegenerate matrix $H_{jj'}$; it can be easily checked that after writing the explicit form of Eq. (2.31)

$$\begin{aligned} &H \otimes \mathbf{E}_1 + H \odot \mathbf{K} \\ &\qquad = H_{jj'} dy^j \otimes dy^{j'} \otimes \mathbf{E}_1 + (H_{jj'} dy^j \otimes dy^{j'}) \odot (\partial y_I \otimes \widehat{w}^I{}_{j''}{}^l dy^{j''} \otimes \mathbf{E}_l) \\ &\qquad = H_{jj'} dy^j \otimes dy^{j'} \otimes \mathbf{E}_1 + H_{jj''} w^{j''}{}_{j'}{}^l dy^j \otimes dy^{j'} \otimes \mathbf{E}_l, \quad j, j', j'' = 1, \ldots, m \end{aligned} \tag{2.32}$$

and after taking the contradiction of Eq. (2.32) with f_* according to the rule (2.11), we obtain the symmetric system. Let us discuss the following equation

$$(H\lambda + H \odot \mathbf{K} \odot \Phi)\xi = 0, \tag{2.33}$$

where the field of symmetric twice covariant tensors standing in parentheses is interpreted as the linear mapping assigning to vectors tangent to Q the corresponding covectors. It

can be easily checked that the nontrivial solution of Eq. (2.32) exists only for

$$\lambda = -c_\alpha(q, \Phi) \tag{2.34}$$

and requires the vector ξ to be proportional to $\gamma_\alpha(q, \Phi)$. From the symmetry of the tensor field standing in Eq. (2.33) in parentheses it immediately follows that the solutions of Eq. (2.33) satisfy also the equivalent equation

$$\xi \odot (\lambda H + H \odot \mathbf{K} \odot \Phi) = 0. \tag{2.35}$$

In turn Eq. (2.35) can be written as

$$\xi H \odot (\lambda \, \mathrm{Id}_{T(Q)} + \mathbf{K} \odot \Phi) = 0 \tag{2.36}$$

which, compared with Eq. (2.27), shows that ξH must be proportional to $\gamma^\alpha(q, \Phi)$ for certain α, $1 \leq \alpha \leq m$. If we denote the corresponding proportionality coefficient by $Z_\alpha(q, \Phi)$ then we arrive at the identity

$$\gamma_\alpha(q, \Phi) H = Z_\alpha(q, \Phi) \gamma^\alpha(q, \Phi) \tag{2.37}$$

(in Eq. (2.37), we do not apply Einsteins convention!).

For a fixed value of the form Φ we can express the tensor field H by means of the fields of forms $\gamma^\alpha(q, \Phi)$, $\alpha = 1, \ldots, m$

$$H = \sum_{\alpha, \alpha'} H_{\alpha, \alpha'} \gamma^\alpha(q, \Phi) \otimes \gamma^{\alpha'}(q, \Phi). \tag{2.38}$$

The elements of the matrix $H_{\alpha, \alpha'}$, can be determined by taking the double contraction of H with the vector fields $\gamma_\alpha(q, \Phi)$, $\gamma_{\alpha'}(q, \Phi)$, $\alpha, \alpha' = 1, \ldots, m$. After taking into account (2.37), we obtain the following explicit form of H:

$$H = \sum_\alpha Z_\alpha(q, \Phi) \gamma^\alpha(q, \Phi) \otimes \gamma^\alpha(q, \Phi). \tag{2.39}$$

The important fact is that the expression (2.39) does not depend on the form Φ since the tensor field H does not depend on Φ. Therefore, the identity (2.39) shows what constraints are imposed on a distribution of eigenvectors for a hyperbolic system by the requirement that the discussed system can be transformed to the equivalent symmetric system. If the functions $Z_\alpha(q, \Phi)$ are positive, then one can change the normalization of $\gamma_\alpha, \gamma^{\alpha'}$ according to

$$\gamma^\alpha \to \tilde{\gamma}^\alpha := \frac{1}{\sqrt{Z_\alpha(q, \Phi)}} \gamma^\alpha,$$

$$\gamma_\alpha \to \tilde{\gamma}_\alpha := \sqrt{Z_\alpha(q, \Phi)} \gamma_\alpha,$$

and the identity (2.39) takes then the form

$$H = \sum_\alpha \tilde{\gamma}^\alpha \otimes \tilde{\gamma}^\alpha. \tag{2.40}$$

In principle, the formula (2.40) (which was obtained by Boillat) expresses the "symmetrizing matrix" H in terms of the eigenvectors $\tilde{\gamma}^\alpha$, $\alpha = 1, \ldots, m$. However, it is not directly effective for the purpose of obtaining H since one has first to find a particular normalization which is needed in order to determine $\tilde{\gamma}^\alpha$, $\alpha = 1, \ldots, m$ (see Eq. (2.40)).

In order to symmetrize the discussed system we can proceed in the following way. If we determine the distribution of eigenvectors γ_α, $\alpha = 1, \ldots, m$ then we can form the expression

$$H' := \sum_\alpha Z'_\alpha(q, \Phi)\gamma^\alpha(q, \Phi) \otimes \gamma^\alpha(q, \Phi), \tag{2.41}$$

where Z'_α are treated as unknown functions. If we impose the condition that

$$\frac{dH'}{d\Phi} = 0 \tag{2.42}$$

then from the system of P.D.E. for Z'_α obtained in that manner we can try to determine all the symmetrizing matrices. It should be stressed that the functions Z_α in Eq. (2.39) can be sometimes negative; in such a case the discussed system of P.D.E. can be hyperbolic and symmetric but not symmetric hyperbolic. It can be easily checked that there are examples for such systems. If a system of P.D.E. can be geometrically represented by a field of two-point tensors of the form (2.12) which is symmetric in the first two indices, then after multiplying it by a non-vanishing real function on Q, we obtain the equivalent symmetric system. Moreover, such systems can be explicitly written in different coordinate systems (the transformation rule between different coordinates follows immediately from the fact that the field of the two-point tensors of the form (2.12) are twice covariant in the first two indices). The equivalencies between the symmetric systems obtained in that manner will be called trivial (they are a direct consequence of the invariant definition of a symmetric system). The interesting question occurs whether there exist such symmetric systems which are nontrivially equivalent. Below we shall give a simple example of such systems. Let ν^α, $\alpha = 1, \ldots, m$ denote fields of forms on Q which are such that for each $q \in Q$ the forms $\nu^1(q), \ldots, \nu^m(q)$ are linearly independent. Let $\mathbf{U}_\alpha$, $\alpha = 1, \ldots, m$ denote the nowhere vanishing mappings from Q into the translation space T_A of the affine space A. We can consider the fields of the two-point tensors $w^1, \ldots, w^m$ defined by the following rule:

$$w^\alpha := \nu^\alpha \otimes \mathbf{U}_\alpha, \quad \alpha = 1, \ldots, m. \tag{2.43}$$

The two-point tensors (2.43) define, via the contraction given by Eq. $(2.5)_2$, the determined system of quasi-linear partial differential equations. Let ν_α, $\alpha = 1, \ldots, m$ denote the vector fields on Q dual to the form fields ν^α, $\alpha = 1, \ldots, m$. Let us define the field of the two-point tensors κ',

$$\kappa' := \sum_\alpha \nu_\alpha \otimes \nu^\alpha \otimes \mathbf{U}_\alpha. \tag{2.44}$$

The system of P.D.E. corresponding to (2.43) can be alternatively written by means of the following contraction of κ' with f_*:

$$\langle \kappa, f_* \rangle_2 = \left\langle \sum_\alpha \nu_\alpha \otimes \nu^\alpha \otimes \mathbf{U}_\alpha, f_* \right\rangle = \sum_\alpha \nu_\alpha \langle \nu^\alpha \otimes \mathbf{U}_\alpha, f_* \rangle = 0. \tag{2.45}$$

Let ξ_α, $\alpha = 1, \ldots, m$ be the non-vanishing real functions on Q. Let us define the twice covariant tensor field H' on Q by the rule

$$H' := \sum_\alpha \xi_\alpha \nu^\alpha \otimes \nu^\alpha. \tag{2.46}$$

As it can be easily checked, (2.46) is a "symmetrizing matrix" for the system of P.D.E. defined by Eq. (2.43):

$$(2.47)\quad H' \odot \kappa' = \Big(\sum_{\alpha'} \xi_{\alpha'} \nu^{\alpha'} \otimes \nu^{\alpha'}\Big) \odot \Big(\sum_{\alpha} \nu_{\alpha} \otimes \nu^{\alpha} \otimes \mathbf{U}_{\alpha}\Big)$$

$$= \sum_{\alpha',\alpha} \xi_{\alpha'} \delta^{\alpha'}_{\alpha} \nu^{\alpha'} \otimes \nu^{\alpha} \otimes \mathbf{U}_{\alpha} = \sum_{\alpha} \xi_{\alpha} \nu^{\alpha} \otimes \nu^{\alpha} \otimes \mathbf{U}_{\alpha},$$

and symmetric systems defined via the contractions of (2.47) with f_* are, in general (that is, without imposing additional constraints on ξ_α), nontrivially equivalent (what means that they cannot be transformed one into another by means of trivial equivalencies, corresponding to multiplications of Eq. (2.47) by smooth real functions combined with changes of coordinates).

Acknowledgements

I am grateful to Prof. W. KOSIŃSKI and Dr. E. MAŁYSZKO for helpful discussions.

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Received June 18, 1993.

BOOK REVIEWS

Mechanics of continuous media by Cz. RYMARZ, Polish Scientific Publishers, Warszawa 1993, 513 pages.

THE BOOK REVIEWED is a fully mature monograph concerning the mechanics of material continua. It contains a complete description of basic problems of mechanics of continuous media, starting from its general foundations and concluding with the analysis of most of the mathematical models used in the theory, both the classical and nonclassical ones, such as e.g. the microstructural media.

The book consists of two parts and three appendices; both parts of the book are divided into six chapters, each of them ending up with a suitable set of problems and questions addressed to the readers.

Part 1 deals with the general laws and relations which are valid for all kinds of continuous media. The considerations concern the description of motion and deformation of material continua, the conservation laws, foundation of thermodynamics, theory of the constitutive equations, and some elements of the dimensional analysis and the similarity theory.

In Part 2 the author presents and reviews some of the models used in the theory of continuous media. In addition to the classical models such as liquids or elastic, viscoelastic, elastic-plastic and elastic-viscoplastic media, also certain nonclassical models are discussed, and namely the microstructural, micropolar as well as liquid crystalline media.

The last chapter of the book is devoted to the problem of interaction of mechanical and electromagnetic fields. The three appendices concluding the monograph are aimed at familiarizing the readers with certain mathematical notions appearing in the text; they contain concise exposition of the foundations of the tensor calculus, group theory and differential geometry. The extensive bibliography grouped according to the consecutive chapters includes 246 positions.

The book has been very carefully edited and printed on paper of good quality; the material is presented on a rather high level of mathematical abstraction, modern mathematical apparatus being extensively used; this, together with the indicial notation written mainly in curvilinear coordinates, will markedly reduce the readership of the book. The book, however, is intended for the readers of a good mathematical background; it will be a useful tool for the students and research associates in the fields of applied mathematics and theoretical physics.

A very wide range of models used in mechanics of continua makes it virtually impossible to apply clear and uniform system of notations throughout the monograph.

The book is definitely a very interesting and valuable contribution to the literature concerning the mechanics of continuous media.

J. Ostrowska-Maciejewska

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