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Variational formulations for the vibration of a piezoelectric plate

J. S. YANG (PRINCETON)

THIS PAPER PRESENTS a systematic discussion on the variational principles for the vibration of a piezoelectric plate. It is shown that there exist four types of variation formulations depending on the internal energy, electric enthalpy, mechanical enthalpy, and enthalpy, respectively. The one depending on the internal energy is in a positive definite form which leads to a few properties of the resonance frequency.

1. Introduction

IN CLASSICAL ELASTICITY and in the corresponding plate theory, there are two types of variational principles for the free vibration of an elastic body or plate. One is associated with the potential energy, the other with the complementary energy [1–4]. Besides their theoretical significance, these variational principles are the foundations of various approximate methods, especially the finite element method.

For the free vibration of a three-dimensional piezoelectric body, it has been shown [5] that there exist four types of variational formulations, depending on internal energy, electric enthalpy, mechanical enthalpy, and enthalpy, respectively, which is a generalization of the case of classical elasticity.

This paper is a continuation of [5] into the two-dimensional piezoelectric plate theory which is important in many applications. Four variational formulations for the two-dimensional piezoelectric plate vibration theory are presented. They are related to the two-dimensional internal energy, electric enthalpy, mechanical enthalpy, and enthalpy, respectively. These variational principles can be considered as generalizations of the corresponding variational formulations in classical elastic plate theory. Because of the presence of the electric fields, there can be four generalizations for the two formulations in classical elastic plate theory. They all have a different set of independent arguments, which allow for different but equivalent formulations of the same eigenvalue problem. The variational principles are given without constraints, with all the physical quantities, as independent variables and, therefore, they can be called mixed variational principles. They can be reduced to various constraint variational principles or variational principles with fewer independent variables. The constraint internal energy formulation is in a positive definite form which can be used to show a few properties of the lowest resonant frequency of a piezoelectric plate.

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2. The eigenvalue problem

Let the two-dimensional region occupied by the piezoelectric plate be A, the boundary curve of A be C, the unit outward normal of C be n_i (with $n_2 = 0$), and C be partitioned in the following way:

$$C_u \cup C_T = C_\phi \cup C_D = C, \qquad C_u \cap C_T = C_\phi \cap C_D = \emptyset;$$

then the eigenvalue problem for the free vibration of a linear piezoelectric plate is [6]

$$
\begin{aligned}
&-T^{(0)}_{ji,j} = \omega^2 2b\rho u^{(0)}_i, \qquad -T^{(1)}_{\beta\alpha,\beta} + T^{(0)}_{2\alpha} = \omega^2 \frac{2}{3} b^3 \rho u^{(1)}_\alpha \qquad \text{in} \quad A,\\
&-D^{(0)}_{i,i} = 0, \qquad -D^{(1)}_{\alpha,\alpha} + D^{(0)}_2 = 0 \qquad \text{in} \quad A,\\
&-S^{(0)}_{ij} + \frac{1}{2}(u^{(0)}_{i,j} + u^{(0)}_{j,i} + \delta_{2i} u^{(1)}_j + \delta_{2j} u^{(1)}_i) = 0 \qquad \text{in} \quad A,\\
&-S^{(1)}_{\alpha\beta} + \frac{1}{2}(u^{(1)}_{\alpha,\beta} + u^{(1)}_{\beta,\alpha}) = 0 \qquad \text{in} \quad A,
\end{aligned}
$$

$$
\begin{aligned}
E^{(0)}_i + \phi^{(0)}_{,i} + \delta_{2i}\phi^{(1)} &= 0, & E^{(1)}_\alpha + \phi^{(1)}_{,\alpha} &= 0 \quad \text{in} \quad A,\\
-T^{(0)}_{ij} + \frac{\partial H}{\partial S^{(0)}_{ij}} &= 0, & -T^{(1)}_{\alpha\beta} + \frac{\partial H}{\partial S^{(1)}_{\alpha\beta}} &= 0 \quad \text{in} \quad A,\\
D^{(0)}_i + \frac{\partial H}{\partial E^{(0)}_i} &= 0, & D^{(1)}_\alpha + \frac{\partial H}{\partial E^{(1)}_\alpha} &= 0 \quad \text{in} \quad A,\\
-u^{(0)}_i &= 0, & -u^{(1)}_\alpha &= 0 \quad \text{on} \quad C_u,\\
n_j T^{(0)}_{ji} &= 0, & n_\beta T^{(1)}_{\beta\alpha} &= 0 \quad \text{on} \quad C_T,\\
-\phi^{(0)} &= 0, & -\phi^{(1)} &= 0 \quad \text{on} \quad C_\phi,\\
n_i D^{(0)}_i &= 0, & n_\alpha D^{(1)}_\alpha &= 0 \quad \text{on} \quad C_D
\end{aligned}
\tag{2.1}
$$

where $u^{(0)}_i$ and $u^{(1)}_\alpha$ are the zeroth and first order displacements, $S^{(0)}_{ij}$ and $S^{(1)}_{\alpha\beta}$ the zeroth and first order strains, $T^{(0)}_{ij}$ and $T^{(1)}_{\alpha\beta}$ the zeroth and first order stresses, $\phi^{(0)}$ and $\phi^{(1)}$ the zeroth and first order electric potentials, $E^{(0)}_i$ and $E^{(1)}_\alpha$ the zeroth and first order electric fields, $D^{(0)}_i$ and $D^{(1)}_\alpha$ the zeroth and first order electric displacements, $2b$ the thickness of the plate, ρ the mass density, and ω the resonant frequency. We note that Latin subscripts i, j, k range from 1 to 3, Greek subscripts α, β do not assume the value 2, and $()_{,2} = 0$. $H = H(\mathbf{S}^{(0)}, \mathbf{S}^{(1)}, \mathbf{E}^{(0)}, \mathbf{E}^{(1)})$ is the electric enthalpy function. For a linear piezoelectric plate, H and the corresponding linear constitutive relations assume the following

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form [6]:

$$H = b(c^{(0)}_{ijkl}S^{(0)}_{ij}S^{(0)}_{kl} - \varepsilon_{ij}E^{(0)}_i E^{(0)}_j - 2e^{(0)}_{ijk}E^{(0)}_i S^{(0)}_{jk}) + \frac{1}{3}b^3(c^{(1)}_{\alpha\beta\gamma\delta}S^{(1)}_{\alpha\beta}S^{(1)}_{\gamma\delta} - \varepsilon_{\alpha\beta}E^{(1)}_\alpha E^{(1)}_\beta - 2e^{(1)}_{\alpha\beta\gamma}E^{(1)}_\alpha S^{(1)}_{\beta\gamma}), \tag{2.2}$$

$$\begin{aligned} T^{(0)}_{ij} &= 2b(c^{(0)}_{ijkl}S^{(0)}_{kl} - e^{(0)}_{kij}E^{(0)}_k), & T^{(1)}_{\alpha\beta} &= \frac{2}{3}b^3(c^{(1)}_{\alpha\beta\gamma\delta}S^{(1)}_{\gamma\delta} - e^{(1)}_{\gamma\alpha\beta}E^{(1)}_\gamma), \\ D^{(0)}_i &= 2b(\varepsilon_{ij}E^{(0)}_j + e^{(0)}_{ijk}S^{(0)}_{jk}), & D^{(1)}_\alpha &= \frac{2}{3}b^3(\varepsilon_{\alpha\beta}E^{(1)}_\beta + e^{(1)}_{\alpha\beta\gamma}S^{(1)}_{\beta\gamma}), \end{aligned} \tag{2.3}$$

where $c^{(0)}_{ijkl}$, $c^{(1)}_{\alpha\beta\gamma\delta}$, ε_{ij}, $e^{(0)}_{ijk}$ $e^{(1)}_{\alpha\beta\gamma}$ are material properties.

Given ρ, b, and H, values of ω^2 are sought corresponding to which nontrivial solutions of $u^{(0)}_i$, $u^{(1)}_\alpha$, $S^{(0)}_{ij}$, $S^{(1)}_{\alpha\beta}$, $T^{(0)}_{ij}$, $T^{(1)}_{\alpha\beta}$, $\phi^{(0)}$, $\phi^{(1)}$, $E^{(0)}_i$, $E^{(1)}_\alpha$, $D^{(0)}_i$ and $D^{(1)}_\alpha$ exist.

3. The electric enthalpy $H(\mathbf{S}^{(0)}, \mathbf{S}^{(1)}, \mathbf{E}^{(0)}, \mathbf{E}^{(1)})$ formulation

The eigenvalue problem (2.1) can be written in an abstract self-adjoint form and then a variational formulation can be obtained which can be further generalized by Lagrange multipliers. The procedure is similar to that in an earlier paper [7]. We define

$$\begin{aligned} &\Lambda_1(\mathbf{u}^{(0)}, \mathbf{u}^{(1)}, \mathbf{S}^{(0)}, \mathbf{S}^{(1)}, \mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \phi^{(0)}, \phi^{(1)}, \mathbf{E}^{(0)}, \mathbf{E}^{(1)}, \mathbf{D}^{(0)}, \mathbf{D}^{(1)}) \\ &= \int_A \Big[T^{(0)}_{ji}u^{(0)}_{i,j} + T^{(1)}_{\beta\alpha}u^{(1)}_{\alpha,\beta} + T^{(0)}_{2\alpha}u^{(1)}_\alpha + D^{(0)}_i\phi^{(0)}_{,i} + D^{(1)}_\alpha\phi^{(1)}_{,\alpha} + D^{(0)}_2\phi^{(1)} \\ &\quad + H(\mathbf{S}^{(0)}, \mathbf{S}^{(1)}, \mathbf{E}^{(0)}, \mathbf{E}^{(1)}) + E^{(0)}_i D^{(0)}_i + E^{(1)}_\alpha D^{(1)}_\alpha - T^{(0)}_{ij}S^{(0)}_{ij} - T^{(1)}_{\alpha\beta}S^{(1)}_{\alpha\beta}\Big]\, dS \\ &\quad - \int_{C_u}\left(n_j T^{(0)}_{ji}u^{(0)}_i + n_\beta T^{(1)}_{\beta\alpha}u^{(1)}_\alpha\right) ds - \int_{C_\phi}\left(n_i D^{(0)}_i\phi^{(0)} + n_\alpha D^{(1)}_\alpha\phi^{(1)}\right) ds, \end{aligned}$$

$$\Gamma_1\left(\mathbf{u}^{(0)}, \mathbf{u}^{(1)}\right) = \int_A \frac{1}{2}\left(2b\rho u^{(0)}_i u^{(0)}_i + \frac{2}{3}b^3\rho u^{(1)}_\alpha u^{(1)}_\alpha\right) dS,$$

$$\begin{aligned} &\Pi_1(\mathbf{u}^{(0)}, \mathbf{u}^{(1)}, \mathbf{S}^{(0)}, \mathbf{S}^{(1)}, \mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \phi^{(0)}, \phi^{(1)}, \mathbf{E}^{(0)}, \mathbf{E}^{(1)}, \mathbf{D}^{(0)}, \mathbf{D}^{(1)}) \\ &= \frac{\Lambda_1(\mathbf{u}^{(0)}, \mathbf{u}^{(1)}, \mathbf{S}^{(0)}, \mathbf{S}^{(1)}, \mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \phi^{(0)}, \phi^{(1)}, \mathbf{E}^{(0)}, \mathbf{E}^{(1)}, \mathbf{D}^{(0)}, \mathbf{D}^{(1)})}{\Gamma_1(\mathbf{u}^{(0)}, \mathbf{u}^{(1)})}. \end{aligned}$$

To obtain the stationary conditions of Π_1 with all its arguments as independent variables, we begin with

$$\delta\Pi_1 = \frac{1}{\Gamma_1^2}(\Gamma_1\delta\Lambda_1 - \Lambda_1\delta\Gamma_1) = \frac{1}{\Gamma_1}(\delta\Lambda_1 - \frac{\Lambda_1}{\Gamma_1}\delta\Gamma_1).$$

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Therefore $\delta \Pi_1 = 0$ implies

$$\delta \Lambda_1 - \frac{\Lambda_1}{\Gamma_1}\delta \Gamma_1 = 0.$$

With integration by parts, we can obtain

$$\begin{aligned}\delta \Lambda_1 = \int_A \Big\{ &- T^{(0)}_{ji,j}\delta u^{(0)}_i + \left(-T^{(1)}_{\beta\alpha,\beta} + T^{(0)}_{2\alpha}\right)\delta u^{(1)}_\alpha \\ &- D^{(0)}_{i,i}\delta\phi^{(0)} + \left(-D^{(1)}_{\alpha,\alpha} + D^{(0)}_2\right)\delta\phi^{(1)} \\ &+ \left(\frac{\partial H}{\partial S^{(0)}_{ij}} - T^{(0)}_{ij}\right)\delta S^{(0)}_{ij} + \left(\frac{\partial H}{\partial S^{(1)}_{\alpha\beta}} - T^{(1)}_{\alpha\beta}\right)\delta S^{(1)}_{\alpha\beta} \\ &+ \left(\frac{\partial H}{\partial E^{(0)}_i} + D^{(0)}_i\right)\delta E^{(0)}_i + \left(\frac{\partial H}{\partial E^{(1)}_\alpha} + D^{(1)}_\alpha\right)\delta E^{(1)}_\alpha \\ &+ \left[\frac{1}{2}\left(u^{(0)}_{i,j} + u^{(0)}_{j,i} + \delta_{2i}u^{(1)}_j + \delta_{2j}u^{(1)}_i\right) - S^{(0)}_{ij}\right]\delta T^{(0)}_{ij} \\ &+ \left[\frac{1}{2}\left(u^{(1)}_{\alpha,\beta} + u^{(1)}_{\beta,\alpha}\right) - S^{(1)}_{\alpha\beta}\right]\delta T^{(1)}_{\alpha\beta} \\ &+ \left(E^{(0)}_i + \phi^{(0)}_{,i} + \delta_{2i}\phi^{(1)}\right)\delta D^{(0)}_i + \left(E^{(1)}_\alpha + \phi^{(1)}_{,\alpha}\right)\delta D^{(1)}_\alpha\Big\} dS \\ &- \int_{C_u}\left(u^{(0)}_i\delta n_j T^{(0)}_{ji} + u^{(1)}_\alpha \delta n_\beta T^{(1)}_{\beta\alpha}\right) ds + \int_{C_T}\left(n_j T^{(0)}_{ji}\delta u^{(0)}_i + n_\beta T^{(1)}_{\beta\alpha}\delta u^{(1)}_\alpha\right) ds \\ &- \int_{C_\phi}\left(\phi^{(0)}\delta n_i D^{(0)}_i + \phi^{(1)}\delta n_\alpha D^{(1)}_\alpha\right) ds + \int_{C_D}\left(n_i D^{(0)}_i\delta\phi^{(0)} + n_\alpha D^{(1)}_\alpha\delta\phi^{(1)}\right) ds,\end{aligned}$$

$$\delta \Gamma_1 = \int_A \left(2b\rho u^{(0)}_i \delta u^{(0)}_i + \frac{2}{3}b^3 u^{(1)}_\alpha \delta u^{(1)}_\alpha\right) dS.$$

Since all the variations of $\delta u^{(0)}_i$, $\delta u^{(1)}_\alpha$, $\delta S^{(0)}_{ij}$, $\delta S^{(1)}_{\alpha\beta}$, $\delta T^{(0)}_{ij}$, $\delta T^{(1)}_{\alpha\beta}$, $\delta\phi^{(0)}$, $\delta\phi^{(1)}$, $\delta E^{(0)}_i$, $\delta E^{(1)}_\alpha$, $\delta D^{(0)}_i$ and $\delta D^{(1)}_\alpha$ are independent, $\delta \Pi_1 = 0$ implies

$$\begin{aligned}&-T^{(0)}_{ji,j} = \frac{\Lambda_1}{\Gamma_1}2b\rho u^{(0)}_i, \qquad -T^{(1)}_{\beta\alpha,\beta} + T^{(0)}_{2\alpha} = \frac{\Lambda_1}{\Gamma_1}\frac{2}{3}b^3\rho u^{(1)}_\alpha \quad \text{in} \quad A, \\ &-D^{(0)}_{i,i} = 0, \qquad -D^{(1)}_{\alpha,\alpha} + D^{(0)}_2 = 0 \quad \text{in} \quad A, \\ &-S^{(0)}_{ij} + \frac{1}{2}(u^{(0)}_{i,j} + u^{(0)}_{j,i} + \delta_{2i}u^{(1)}_j + \delta_{2j}u^{(1)}_i) = 0 \quad \text{in} \quad A, \\ &-S^{(1)}_{\alpha\beta} + \frac{1}{2}(u^{(1)}_{\alpha,\beta} + u^{(1)}_{\beta,\alpha}) = 0 \quad \text{in} \quad A,\end{aligned} \tag{3.1}$$

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(3.1) [cont.]
$$
\begin{aligned}
&E_i^{(0)} + \phi_{,i}^{(0)} + \delta_{2i}\phi^{(1)} = 0, && E_\alpha^{(1)} + \phi_{,\alpha}^{(1)} = 0 \quad \text{in} \quad A,\\
&-T_{ij}^{(0)} + \frac{\partial H}{\partial S_{ij}^{(0)}} = 0, && -T_{\alpha\beta}^{(1)} + \frac{\partial H}{\partial S_{\alpha\beta}^{(1)}} = 0 \quad \text{in} \quad A,\\
&D_i^{(0)} + \frac{\partial H}{\partial E_i^{(0)}} = 0, && D_\alpha^{(1)} + \frac{\partial H}{\partial E_\alpha^{(1)}} = 0 \quad \text{in} \quad A,\\
&-u_i^{(0)} = 0, && -u_\alpha^{(1)} = 0 \quad \text{on} \quad C_u,\\
&n_j T_{ji}^{(0)} = 0, && n_\beta T_{\beta\alpha}^{(1)} = 0 \quad \text{on} \quad C_T,\\
&-\phi^{(0)} = 0, && -\phi^{(1)} = 0 \quad \text{on} \quad C_\phi,\\
&n_i D_i^{(0)} = 0, && n_\alpha D_\alpha^{(1)} = 0 \quad \text{on} \quad C_D.
\end{aligned}
$$

Comparing (3.1) to (2.1), we have the following variational principle: the stationary condition of Π_1 gives the eigenvalue problem (2.1), with the stationary value of Π_1 as ω^2.

The above variational principle has no constraints. If we choose our admissible functions to satisfy the conditions

$$
\begin{aligned}
&-S_{ij}^{(0)} + \frac{1}{2}(u_{i,j}^{(0)} + u_{j,i}^{(0)} + \delta_{2i}u_j^{(1)} + \delta_{2j}u_i^{(1)}) = 0 \qquad \text{in} \quad A,\\
&-S_{\alpha\beta}^{(1)} + \frac{1}{2}(u_{\alpha,\beta}^{(1)} + u_{\beta,\alpha}^{(1)}) = 0 \qquad \text{in} \quad A,
\end{aligned}
$$

(3.2)
$$
\begin{aligned}
&E_i^{(0)} + \phi_{,i}^{(0)} + \delta_{2i}\phi^{(1)} = 0, && E_\alpha^{(1)} + \phi_{,\alpha}^{(1)} = 0 \quad \text{in} \quad A,\\
&-T_{ij}^{(0)} + \frac{\partial H}{\partial S_{ij}^{(0)}} = 0, && -T_{\alpha\beta}^{(1)} + \frac{\partial H}{\partial S_{\alpha\beta}^{(1)}} = 0 \quad \text{in} \quad A,\\
&D_i^{(0)} + \frac{\partial H}{\partial E_i^{(0)}} = 0, && D_\alpha^{(1)} + \frac{\partial H}{\partial E_\alpha^{(1)}} = 0 \quad \text{in} \quad A,\\
&-u_i^{(0)} = 0, && -u_\alpha^{(1)} = 0 \quad \text{on} \quad C_u,\\
&-\phi^{(0)} = 0, && -\phi^{(1)} = 0 \quad \text{on} \quad C_\phi,
\end{aligned}
$$

then Π_1 reduces to

$$
\Pi_1 = \frac{\int\limits_A H\, dS}{\int\limits_A \frac{1}{2}\left(2b\rho u_i^{(0)}u_i^{(0)} + \frac{2}{3}b^3\rho u_\alpha^{(1)}u_\alpha^{(1)}\right) dS} \tag{3.3}
$$

and the stationary conditions of Π_1 become

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$$
(3.4)\quad
\begin{aligned}
-T^{(0)}_{ji,j} &= \frac{\Lambda_1}{\Gamma_1}2b\rho u^{(0)}_i, & -T^{(1)}_{\beta\alpha,\beta} + T^{(0)}_{2\alpha} &= \frac{\Lambda_1}{\Gamma_1}\frac{2}{3}b^3\rho u^{(1)}_\alpha \quad \text{in} \quad A,\\
-D^{(0)}_{i,i} &= 0, & -D^{(1)}_{\alpha,\alpha} + D^{(0)}_2 &= 0 \quad \text{in} \quad A,\\
n_j T^{(0)}_{ji} &= 0, & n_\beta T^{(1)}_{\beta\alpha} &= 0 \quad \text{on} \quad C_T,\\
n_i D^{(0)}_i &= 0, & n_\alpha D^{(1)}_\alpha &= 0 \quad \text{on} \quad C_D.
\end{aligned}
$$

In fact, we can substitute Eqs. $(3.2)_{1-5}$ into Eq. (3.3) and express everything in terms of $u^{(0)}_i$, $u^{(1)}_\alpha$, $\phi^{(0)}$, and $\phi^{(1)}$. Then we will have a variational principle with constraints $(3.2)_{6,7}$ and independent variables $u^{(0)}_i$, $u^{(1)}_\alpha$, $\phi^{(0)}$, and $\phi^{(1)}$. The stationary conditions will be Eqs. (3.4) in terms of $u^{(0)}_i$, $u^{(1)}_\alpha$, $\phi^{(0)}$ and $\phi^{(1)}$.

4. The internal energy $U(\mathbf{S}^{(0)}, \mathbf{S}^{(1)}, \mathbf{D}^{(0)}, \mathbf{D}^{(1)})$ formulation

The internal energy U can be as introduced from H through Legendre transform as

$$
(4.1)\qquad U = U(\mathbf{S}^{(0)}, \mathbf{S}^{(1)}, \mathbf{D}^{(0)}, \mathbf{D}^{(1)}) = H + E^{(0)}_i D^{(0)}_i + E^{(1)}_\alpha D^{(1)}_\alpha,
$$

which generates the following constitutive relations:

$$
\begin{aligned}
T^{(0)}_{ij} &= \frac{\partial U}{\partial S^{(0)}_{ij}}, & T^{(1)}_{\alpha\beta} &= \frac{\partial U}{\partial S^{(1)}_{\alpha\beta}},\\
E^{(0)}_i &= \frac{\partial U}{\partial D^{(0)}_i}, & E^{(1)}_\alpha &= \frac{\partial U}{\partial D^{(1)}_\alpha}.
\end{aligned}
$$

Let

$$
\begin{aligned}
&\Lambda_2(\mathbf{u}^{(0)}, \mathbf{u}^{(1)}, \mathbf{S}^{(0)}, \mathbf{S}^{(1)}, \mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \phi^{(0)}, \phi^{(1)}, \mathbf{D}^{(0)}, \mathbf{D}^{(1)})\\
&\quad = \int_A \Big[T^{(0)}_{ji} u^{(0)}_{i,j} + T^{(1)}_{\beta\alpha} u^{(1)}_{\alpha,\beta} + T^{(0)}_{2\alpha} u^{(1)}_\alpha - D^{(0)}_{i,i}\phi^{(0)} - D^{(1)}_{\alpha,\alpha}\phi^{(1)} + D^{(0)}_2\phi^{(1)}\\
&\qquad\qquad + U(\mathbf{S}^{(0)}, \mathbf{S}^{(1)}, \mathbf{D}^{(0)}, \mathbf{D}^{(1)}) - T^{(0)}_{ij}S^{(0)}_{ij} - T^{(1)}_{\alpha\beta}S^{(1)}_{\alpha\beta}\Big]\, dS\\
&\quad - \int_{C_u} \left(n_j T^{(0)}_{ji} u^{(0)}_i + n_\beta T^{(1)}_{\beta\alpha} u^{(1)}_\alpha\right) ds + \int_{C_D} \left(n_i D^{(0)}_i \phi^{(0)} + n_\alpha D^{(1)}_\alpha \phi^{(1)}\right) ds,\\
&\Gamma_2(\mathbf{u}^{(0)}, \mathbf{u}^{(1)}) = \int_A \frac{1}{2}\left(2b\rho u^{(0)}_i u^{(0)}_i + \frac{2}{3}b^3\rho u^{(1)}_\alpha u^{(1)}_\alpha\right) dS,\\
&\Pi_2(\mathbf{u}^{(0)}, \mathbf{u}^{(1)}, \mathbf{S}^{(0)}, \mathbf{S}^{(1)}, \mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \phi^{(0)}, \phi^{(1)}, \mathbf{D}^{(0)}, \mathbf{D}^{(1)})\\
&\qquad = \frac{\Lambda_2(\mathbf{u}^{(0)}, \mathbf{u}^{(1)}, \mathbf{S}^{(0)}, \mathbf{S}^{(1)}, \mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \phi^{(0)}, \phi^{(1)}, \mathbf{D}^{(0)}, \mathbf{D}^{(1)})}{\Gamma_2(\mathbf{u}^{(0)}, \mathbf{u}^{(1)})}.
\end{aligned}
$$

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Then

$$\delta\Lambda_2 = \int_A \Big\{ - T^{(0)}_{ji,j}\delta u^{(0)}_i + \left(-T^{(1)}_{\beta\alpha,\beta} + T^{(0)}_{2\alpha}\right)\delta u^{(1)}_\alpha - D^{(0)}_{i,i}\delta\phi^{(0)} + \left(-D^{(1)}_{\alpha,\alpha} + D^{(0)}_2\right)\delta\phi^{(1)} + \left(\frac{\partial U}{\partial S^{(0)}_{ij}} - T^{(0)}_{ij}\right)\delta S^{(0)}_{ij}$$

$$+ \left(\frac{\partial U}{\partial S^{(1)}_{\alpha\beta}} - T^{(1)}_{\alpha\beta}\right)\delta S^{(1)}_{\alpha\beta} + \left(\frac{\partial U}{\partial D^{(0)}_i} + \phi^{(0)}_{,i} + \delta_{2i}\phi^{(1)}\right)\delta D^{(0)}_i$$

$$+ \left(\frac{\partial U}{\partial D^{(1)}_\alpha} + \phi^{(1)}_{,\alpha}\right)\delta D^{(1)}_\alpha + \left[\frac{1}{2}\left(u^{(0)}_{i,j} + u^{(0)}_{j,i} + \delta_{2i}u^{(1)}_j + \delta_{2j}u^{(1)}_i\right) - S^{(0)}_{ij}\right]\delta T^{(0)}_{ij}$$

$$+ \left[\frac{1}{2}\left(u^{(1)}_{\alpha,\beta} + u^{(1)}_{\beta,\alpha}\right) - S^{(1)}_{\alpha\beta}\right]\delta T^{(1)}_{\alpha\beta}\Big\} dS$$

$$- \int_{C_u} \left(u^{(0)}_i \delta n_j T^{(0)}_{ji} + u^{(1)}_\alpha \delta n_\beta T^{(1)}_{\beta\alpha}\right) ds + \int_{C_T} \left(n_j T^{(0)}_{ji}\delta u^{(0)}_i + n_\beta T^{(1)}_{\beta\alpha}\delta u^{(1)}_\alpha\right) ds$$

$$- \int_{C_\phi} \left(\phi^{(0)}\delta n_i D^{(0)}_i + \phi^{(1)}\delta n_\alpha D^{(1)}_\alpha\right) ds + \int_{C_D} \left(n_i D^{(0)}_i \delta\phi^{(0)} + n_\alpha D^{(1)}_\alpha \delta\phi^{(1)}\right) ds,$$

$$\delta\Gamma_2 = \int_A \left(2b\rho u^{(0)}_i \delta u^{(0)}_i + \frac{2}{3}b^3 u^{(1)}_\alpha \delta u^{(1)}_\alpha\right) dS;$$

hence $\delta\Pi_2 = 0$ implies

$$-T^{(0)}_{ji,j} = \frac{\Lambda_2}{\Gamma_2}2b\rho u^{(0)}_i, \qquad -T^{(1)}_{\beta\alpha,\beta} + T^{(0)}_{2\alpha} = \frac{\Lambda_2}{\Gamma_2}\frac{2}{3}b^3\rho u^{(1)}_\alpha \qquad \text{in} \quad A,$$

$$-D^{(0)}_{i,i} = 0, \qquad -D^{(1)}_{\alpha,\alpha} + D^{(0)}_2 = 0 \qquad \text{in} \quad A,$$

$$-S^{(0)}_{ij} + \frac{1}{2}\left(u^{(0)}_{i,j} + u^{(0)}_{j,i} + \delta_{2i}u^{(1)}_j + \delta_{2j}u^{(1)}_i\right) = 0 \qquad \text{in} \quad A,$$

$$-S^{(1)}_{\alpha\beta} + \frac{1}{2}\left(u^{(1)}_{\alpha,\beta} + u^{(1)}_{\beta,\alpha}\right) = 0 \qquad \text{in} \quad A,$$

$$\text{(4.2)} \qquad -T^{(0)}_{ij} + \frac{\partial U}{\partial S^{(0)}_{ij}} = 0, \qquad -T^{(1)}_{\alpha\beta} + \frac{\partial U}{\partial S^{(1)}_{\alpha\beta}} = 0 \qquad \text{in} \quad A,$$

$$\phi^{(0)}_{,i} + \delta_{2i}\phi^{(1)} + \frac{\partial U}{\partial D^{(0)}_i} = 0, \qquad \phi^{(1)}_{,\alpha} + \frac{\partial U}{\partial D^{(1)}_\alpha} = 0 \qquad \text{in} \quad A,$$

$$\begin{aligned} -u^{(0)}_i &= 0, & -u^{(1)}_\alpha &= 0 & \text{on} \quad & C_u, \\ n_j T^{(0)}_{ji} &= 0, & n_\beta T^{(1)}_{\beta\alpha} &= 0 & \text{on} \quad & C_T, \\ -\phi^{(0)} &= 0, & -\phi^{(1)} &= 0 & \text{on} \quad & C_\phi, \\ n_i D^{(0)}_i &= 0, & n_\alpha D^{(1)}_\alpha &= 0 & \text{on} \quad & C_D, \end{aligned}$$

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which is an equivalent form of the original eigenvalue problem (2.1). (4.2) can be obtained from (2.1) by eliminating $E_i^{(0)}$ and $E_\alpha^{(1)}$. The stationary condition of Π_2 gives the eigenvalue problem (4.2), with the stationary value of Π_2 as ω^2. If we choose admissible functions to satisfy

$$
\begin{aligned}
&-D_{i,i}^{(0)} = 0, \qquad -D_{\alpha,\alpha}^{(1)} + D_2^{(0)} = 0 \quad \text{in} \quad A,\\
&-S_{ij}^{(0)} + \frac{1}{2}\left(u_{i,j}^{(0)} + u_{j,i}^{(0)} + \delta_{2i}u_j^{(1)} + \delta_{2j}u_i^{(1)}\right) = 0 \quad \text{in} \quad A,\\
&-S_{\alpha\beta}^{(1)} + \frac{1}{2}\left(u_{\alpha,\beta}^{(1)} + u_{\beta,\alpha}^{(1)}\right) = 0 \quad \text{in} \quad A,\\
&-u_i^{(0)} = 0, \qquad -u_\alpha^{(1)} = 0 \quad \text{on} \quad C_u,\\
&n_i D_i^{(0)} = 0, \qquad n_\alpha D_\alpha^{(1)} = 0 \quad \text{on} \quad C_D
\end{aligned}
$$

then Π_2 reduces to

$$
\Pi_2 = \frac{\int_A U\,dS}{\int_A \frac{1}{2}\left(2b\rho u_i^{(0)}u_i^{(0)} + \frac{2}{3}b^3\rho u_\alpha^{(1)}u_\alpha^{(1)}\right)dS}. \tag{4.3}
$$

We note that from Eqs. (4.1), (2.2) and (2.3) the internal energy can be expressed in terms of $S_{ij}^{(0)}$, $S_{\alpha\beta}^{(1)}$, $E_i^{(0)}$, $E_\alpha^{(1)}$ in the form

$$
\begin{aligned}
U = H + E_i^{(0)}D_i^{(0)} + E_\alpha^{(1)}D_\alpha^{(1)} &= b\left(c_{ijkl}^{(0)}S_{ij}^{(0)}S_{kl}^{(0)} - \varepsilon_{ij}E_i^{(0)}E_j^{(0)} - 2e_{ijk}^{(0)}E_i^{(0)}S_{jk}^{(0)}\right)\\
&+\frac{1}{3}b^3\left(c_{\alpha\beta\gamma\delta}^{(1)}S_{\alpha\beta}^{(1)}S_{\gamma\delta}^{(1)} - \varepsilon_{\alpha\beta}E_\alpha^{(1)}E_\beta^{(1)} - 2e_{\alpha\beta\gamma}^{(1)}E_\alpha^{(1)}S_{\beta\gamma}^{(1)}\right)\\
&+E_i^{(0)}2b\left(\varepsilon_{ij}E_j^{(0)} + e_{ijk}^{(0)}S_{jk}^{(0)}\right) + E_\alpha^{(0)}\frac{2}{3}b^3\left(\varepsilon_{\alpha\beta}E_\beta^{(1)} + e_{\alpha\beta\gamma}^{(1)}S_{\beta\gamma}^{(1)}\right)\\
&= b\left(c_{ijkl}^{(0)}S_{ij}^{(0)}S_{kl}^{(0)} + \varepsilon_{ij}E_i^{(0)}E_j^{(0)}\right) + \frac{1}{3}b^3\left(c_{\alpha\beta\gamma\delta}^{(1)}S_{\alpha\beta}^{(1)}S_{\gamma\delta}^{(1)} + \varepsilon_{\alpha\beta}E_\alpha^{(1)}E_\beta^{(1)}\right).
\end{aligned}
$$

The positive definiteness of the internal energy function U (or $c_{ijkl}^{(0)}$, $c_{\alpha\beta\gamma\delta}^{(1)}$, ε_{ij}) is usually assumed for stability considerations [8], then the constraint Π_2 is positive and hence bounded from below. Therefore the lowest resonant frequency must be a minimum. Following some standard arguments in variational analysis [9], we have the following immediate properties.

The lowest resonant frequency will increase if any of the following happens:

i. C_u increases, or C_D increases;

ii. ρ decreases;

iii. $c_{ijkl}^{(0)}$ increases to $\bar{c}_{ijkl}^{(0)}$ such that $(\bar{c}_{ijkl}^{(0)} - c_{ijkl}^{(0)})a_{ij}a_{kl} > 0$ for any nonzero symmetric a_{ij}, or $c_{\alpha\beta\gamma\delta}$ increases to $\bar{c}_{\alpha\beta\gamma\delta}^{(1)}$ such that $(\bar{c}_{\alpha\beta\gamma\delta}^{(1)} - c_{\alpha\beta\gamma\delta}^{(1)})a_{\alpha\beta}a_{\gamma\delta} > 0$ for any nonzero symmetric $a_{\alpha\beta}$;

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iv. ε_{ij} increases to $\bar{\varepsilon}_{ij}$ such that $(\bar{\varepsilon}_{ij} - \varepsilon_{ij})b_i b_j > 0$ for any nonzero b_i.

These properties are consistent with similar properties obtained from the three-dimensional theory in [5]. Some of them may be considered as generalizations of the corresponding properties in classical elastic plate theory. In fact, it can be expected that with the orthogonality conditions with lower modes as variational constraints of Eqs. (4.3), some of the above properties may be proved for higher resonant frequencies.

The above two formulations in terms of H and U can be considered as generalizations of the potential energy formulation for the vibration problem in elastic plate theory. In the following, we will discuss two other formulations which are generalizations of the complementary energy formulation in elastic plate theory.

5. The mechanical enthalpy $M(\mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \mathbf{D}^{(0)}, \mathbf{D}^{(1)})$ formulation

The mechanical enthalpy M can be introduced through Legendre transform from H as

$$M = M(\mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \mathbf{D}^{(0)}, \mathbf{D}^{(1)}) = H + E_i^{(0)} D_i^{(0)} + E_\alpha^{(1)} D_\alpha^{(1)} - T_{ij}^{(0)} S_{ij}^{(0)} - T_{\alpha\beta}^{(1)} S_{\alpha\beta}^{(1)} .$$

which generates the following constitutive relations

$$\begin{aligned} S_{ij}^{(0)} &= -\frac{\partial M}{\partial T_{ij}^{(0)}}, \qquad & S_{\alpha\beta}^{(1)} &= -\frac{\partial M}{\partial T_{\alpha\beta}^{(1)}}, \\ E_i^{(0)} &= \frac{\partial M}{\partial D_i^{(0)}}, \qquad & E_\alpha^{(1)} &= \frac{\partial M}{\partial D_\alpha^{(1)}} . \end{aligned} \tag{5.1}$$

For this formulation, we need to introduce (when $\omega \neq 0$)

$$\begin{aligned} a_i^{(0)} &= -\omega^2 u_i^{(0)}, \qquad & a_\alpha^{(1)} &= -\omega^2 u_\alpha^{(1)}, \\ \psi^{(0)} &= -\omega^2 \phi^{(0)}, \qquad & \psi^{(1)} &= -\omega^2 \phi^{(1)} . \end{aligned} \tag{5.2}$$

We note that the physical meaning of $a_i^{(0)}$ and $a_\alpha^{(1)}$ is related to accelerations. Let

$$\begin{aligned} &\Lambda_3(\mathbf{a}^{(0)}, \mathbf{a}^{(1)}, \mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \psi^{(0)}, \psi^{(1)}, \mathbf{D}^{(0)}, \mathbf{D}^{(1)}) \\ &\quad = \int_A \Big[- T_{ji,j}^{(0)} a_i^{(0)} - T_{\beta\alpha,\beta}^{(1)} a_\alpha^{(1)} + T_{2\alpha}^{(0)} a_\alpha^{(1)} - D_{i,i}^{(0)} \psi^{(0)} - D_{\alpha,\alpha}^{(1)} \psi^{(1)} \\ &\qquad + D_2^{(0)} \psi^{(1)} + \frac{1}{2} \left(2b\rho a_i^{(0)} a_i^{(0)} + \frac{2}{3} b^3 \rho a_\alpha^{(1)} a_\alpha^{(1)} \right) \Big] dS \\ &\quad + \int_{C_T} \left(n_j T_{ji}^{(0)} a_i^{(0)} + n_\beta T_{\beta\alpha}^{(1)} a_\alpha^{(1)} \right) ds + \int_{C_D} \left(n_i D_i^{(0)} \psi^{(0)} + n_\alpha D_\alpha^{(1)} \psi^{(1)} \right) ds, \end{aligned}$$

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$$\Gamma_3\left(\mathbf{T}^{(0)},\mathbf{T}^{(1)},\mathbf{D}^{(0)},\mathbf{D}^{(1)}\right)=\int\limits_A M\left(\mathbf{T}^{(0)},\mathbf{T}^{(1)},\mathbf{D}^{(0)},\mathbf{D}^{(1)}\right)dS,$$

$$\Pi_3\left(\mathbf{a}^{(0)},\mathbf{a}^{(1)},\mathbf{T}^{(0)},\mathbf{T}^{(1)},\psi^{(0)},\psi^{(1)},\mathbf{D}^{(0)},\mathbf{D}^{(1)}\right)$$
$$=\frac{\Lambda_3\left(\mathbf{a}^{(0)},\mathbf{a}^{(1)},\psi^{(0)},\psi^{(1)},\mathbf{T}^{(0)},\mathbf{T}^{(1)},\mathbf{D}^{(0)},\mathbf{D}^{(1)}\right)}{\Gamma_3\left(\mathbf{T}^{(0)},\mathbf{T}^{(1)},\mathbf{D}^{(0)},\mathbf{D}^{(1)}\right)},$$

where independent arguments are now $a_i^{(0)}$, $a_\alpha^{(1)}$, $T_{ij}^{(0)}$, $T_{\alpha\beta}^{(1)}$, $\psi^{(0)}$, $\psi^{(1)}$, $D_i^{(0)}$ and $D_\alpha^{(1)}$. Since

$$\delta\Lambda_3=\int\limits_A\Big[\left(-T_{ji,j}^{(0)}+2b\rho a_i^{(0)}\right)\delta a_i^{(0)}+\left(-T_{\beta\alpha,\beta}^{(1)}+T_{2\alpha}^{(0)}+\frac{2}{3}b^3\rho a_\alpha^{(1)}\right)\delta a_\alpha^{(1)}$$
$$-D_{i,i}^{(0)}\delta\psi^{(0)}+\left(-D_{\alpha,\alpha}^{(1)}+D_2^{(0)}\right)\delta\psi^{(1)}$$
$$+\frac{1}{2}\left(a_{i,j}^{(0)}+a_{j,i}^{(0)}+\delta_{2i}a_j^{(1)}+\delta_{2j}a_i^{(1)}\right)\delta T_{ij}^{(0)}+\frac{1}{2}\left(a_{\alpha,\beta}^{(1)}+a_{\beta,\alpha}^{(1)}\right)\delta T_{\alpha\beta}^{(1)}$$
$$+\left(\psi_{,i}^{(0)}+\delta_{2i}\psi^{(1)}\right)\delta D_i^{(0)}+\psi_{,\alpha}^{(1)}\delta D_\alpha^{(1)}\Big]dS$$
$$-\int\limits_{C_u}\left(a_i^{(0)}\delta n_jT_{ji}^{(0)}+a_\alpha^{(1)}\delta n_\beta T_{\beta\alpha}^{(1)}\right)ds+\int\limits_{C_T}\left(n_jT_{ji}^{(0)}\delta a_i^{(0)}+n_\beta T_{\beta\alpha}^{(1)}\delta a_\beta^{(1)}\right)ds$$
$$-\int\limits_{C_\phi}\left(\psi^{(0)}\delta n_iD_i^{(0)}+\psi^{(1)}\delta n_\alpha D_\alpha^{(1)}\right)ds+\int\limits_{C_D}\left(n_iD_i^{(0)}\delta\psi^{(0)}+n_\alpha D_\alpha^{(1)}\delta\psi^{(1)}\right)ds,$$

$$\delta\Gamma_3=\int\limits_A\left(\frac{\partial M}{\partial T_{ij}^{(0)}}\delta T_{ij}^{(0)}+\frac{\partial M}{\partial T_{\alpha\beta}^{(1)}}\delta T_{\alpha\beta}^{(1)}+\frac{\partial M}{\partial D_i^{(0)}}\delta D_i^{(0)}+\frac{\partial M}{\partial D_\alpha^{(1)}}\delta D_\alpha^{(1)}\right)dS,$$

$\delta\Pi_3=0$ implies

$$-T_{ji,j}^{(0)}+2b\rho a_i^{(0)}=0,\qquad -T_{\beta\alpha,\beta}^{(1)}+T_{2\alpha}^{(0)}+\frac{2}{3}b^3\rho a_\alpha^{(1)}=0\qquad\text{in}\quad A,$$
$$-D_{i,i}^{(0)}=0,\qquad -D_{\alpha,\alpha}^{(1)}+D_2^{(0)}=0\qquad\text{in}\quad A,$$
$$(5.3)\qquad \frac{1}{2}\left(a_{i,j}^{(0)}+a_{j,i}^{(0)}+\delta_{2i}a_j^{(1)}+\delta_{2j}a_i^{(1)}\right)=\frac{\Lambda_3}{\Gamma_3}\frac{\partial M}{\partial T_{ij}^{(0)}}\qquad\text{in}\quad A,$$
$$\frac{1}{2}\left(a_{\alpha,\beta}^{(1)}+a_{\beta,\alpha}^{(1)}\right)=\frac{\Lambda_3}{\Gamma_3}\frac{\partial M}{\partial T_{\alpha\beta}^{(1)}}\qquad\text{in}\quad A,$$
$$\psi_{,i}^{(0)}+\delta_{2i}\psi^{(1)}=\frac{\Lambda_3}{\Gamma_3}\frac{\partial M}{\partial D_i^{(0)}},\qquad \psi_{,\alpha}^{(1)}=\frac{\Lambda_3}{\Gamma_3}\frac{\partial M}{\partial D_\alpha^{(1)}}\qquad\text{in}\quad A,$$

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(5.3) [cont.]

$$
\begin{aligned}
-a_i^{(0)} &= 0, & -a_\alpha^{(1)} &= 0 & \text{on} \quad & C_u,\\
n_j T_{ji}^{(0)} &= 0 & n_\beta T_{\beta\alpha}^{(1)} &= 0 & \text{on} \quad & C_T,\\
-\psi^{(0)} &= 0, & -\psi^{(1)} &= 0 & \text{on} \quad & C_\phi,\\
n_i D_i^{(0)} &= 0, & n_\alpha D_\alpha^{(1)} &= 0 & \text{on} \quad & C_D,
\end{aligned}
$$

which is an equivalent system of the original eigenvalue problem (2.1) (when $\omega \neq 0$). Equations $(5.3)_{3-5}$ can be obtained by multiplying both sides of Eqs. (5.1) by ω^2 and substitutions of Eqs. $(2.1)_{3-5}$ and (5.2). Hence, the stationary condition of the functional Π_3 gives the eigenvalue problem (5.3) with the stationary value of Π_3 as ω^2. If we choose the admissible functions to satisfy

$$
-T_{ji,j}^{(0)} + 2b\rho a_i^{(0)} = 0, \qquad -T_{\beta\alpha,\beta}^{(1)} + T_{2\alpha}^{(0)} + \frac{2}{3}b^3\rho a_\alpha^{(1)} = 0 \qquad \text{in} \quad A,
$$

$$
-D_{i,i}^{(0)} = 0, \qquad -D_{\alpha,\alpha}^{(1)} + D_2^{(0)} = 0 \qquad \text{in} \quad A,
$$

$$
\begin{aligned}
n_j T_{ji}^{(0)} &= 0, & n_\beta T_{\beta\alpha}^{(1)} &= 0 \ \text{on} \quad C_T,\\
n_i D_i^{(0)} &= 0, & n_\alpha D_\alpha^{(1)} &= 0 \ \text{on} \quad C_D,
\end{aligned}
$$

then Π_3 reduces to

$$
\Pi_3 = \frac{\displaystyle\int_A -\frac{1}{2}\left(2b\rho a_i^{(0)} a_i^{(0)} + \frac{2}{3}b^3\rho a_\alpha^{(1)} a_\alpha^{(1)}\right) dS}{\displaystyle\int_A M\, dS} .
$$

6. The enthalpy $G(\mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \mathbf{E}^{(0)}, \mathbf{E}^{(1)})$ formulation

The enthalpy G can be obtained from H as

$$
G = G(\mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \mathbf{E}^{(0)}, \mathbf{E}^{(1)}) = H - T_{ij}^{(0)} S_{ij}^{(0)} - T_{\alpha\beta}^{(1)} S_{\alpha\beta}^{(1)},
$$

which generates the following constitutive relations

$$
\begin{aligned}
S_{ij}^{(0)} &= -\frac{\partial G}{\partial T_{ij}^{(0)}}, & S_{\alpha\beta}^{(1)} &= -\frac{\partial G}{\partial T_{\alpha\beta}^{(1)}},\\
D_i^{(0)} &= -\frac{\partial G}{\partial E_i^{(0)}}, & D_\alpha^{(1)} &= -\frac{\partial G}{\partial E_\alpha^{(1)}}.
\end{aligned}
$$

We introduce

$$
\begin{aligned}
a_i^{(0)} &= -\omega^2 u_i^{(0)}, & a_\alpha^{(1)} &= -\omega^2 u_\alpha^{(1)},\\
Q_i^{(0)} &= -\omega^2 D_i^{(0)}, & Q_\alpha^{(1)} &= -\omega^2 D_\alpha^{(1)},
\end{aligned}
$$

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and let

$$\Lambda_4(\mathbf{a}^{(0)}, \mathbf{a}^{(1)}, \mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \phi^{(0)}, \phi^{(1)}, \mathbf{E}^{(0)}, \mathbf{E}^{(1)}, \mathbf{Q}^{(0)}, \mathbf{Q}^{(0)})$$
$$= \int_A \Big[-T^{(0)}_{ji,j}a^{(0)}_i - T^{(1)}_{\beta\alpha,\beta}a^{(1)}_\alpha + T^{(0)}_{2\alpha}a^{(1)}_\alpha + \frac{1}{2}\left(2b\rho a^{(0)}_i a^{(0)}_i + \frac{2}{3}b^3\rho a^{(1)}_\alpha a^{(1)}_\alpha\right)$$
$$+ Q^{(0)}_i\phi^{(0)}_{,i} + Q^{(0)}_2\phi^{(1)} + Q^{(1)}_\alpha\phi^{(1)}_{,\alpha} + Q^{(0)}_i E^{(0)}_i + Q^{(1)}_\alpha E^{(1)}_\alpha\Big] dS$$
$$+ \int_{C_T}\left(n_j T^{(0)}_{ji}a^{(0)}_i + n_\beta T^{(0)}_{\beta\alpha}a^{(1)}_\alpha\right) ds - \int_{C_\phi}\left(n_i Q^{(0)}_i\phi^{(0)} + n_\alpha Q^{(1)}_\alpha\phi^{(0)}\right) ds,$$

$$\Gamma_4\left(\mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \mathbf{E}^{(0)}, \mathbf{E}^{(1)}\right) = \int_A G\left(\mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \mathbf{E}^{(0)}, \mathbf{E}^{(1)}\right) dS,$$

$$\Pi_4\left(\mathbf{a}^{(0)}, \mathbf{a}^{(1)}, \mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \phi^{(0)}, \phi^{(1)}, \mathbf{E}^{(0)}, \mathbf{E}^{(1)}, \mathbf{Q}^{(0)}, \mathbf{Q}^{(1)}\right)$$
$$= \frac{\Lambda_4\left(\mathbf{a}^{(0)}, \mathbf{a}^{(1)}, \mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \phi^{(0)}, \phi^{(1)}, \mathbf{E}^{(0)}, \mathbf{E}^{(1)}, \mathbf{Q}^{(0)}, \mathbf{Q}^{(1)}\right)}{\Gamma_4\left(\mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \mathbf{E}^{(0)}, \mathbf{E}^{(1)}\right)}.$$

It can be verified that the stationary conditions of Π_4 are

$$-T^{(0)}_{ji,j} + 2b\rho a^{(0)}_i = 0, \qquad -T^{(1)}_{\beta\alpha,\beta} + T^{(0)}_{2\alpha} + \frac{2}{3}b^3\rho a^{(1)}_\alpha = 0 \qquad \text{in} \quad A,$$
$$-Q^{(0)}_{i,i} = 0, \qquad -Q^{(1)}_{\alpha,\alpha} + Q^{(0)}_2 = 0 \qquad \text{in} \quad A,$$
$$E^{(0)}_i + \phi^{(0)}_{,i} + \delta_{2i}\phi^{(1)} = 0, \qquad E^{(1)}_\alpha + \phi^{(1)}_{,\alpha} = 0 \qquad \text{in} \quad A,$$
$$\frac{1}{2}\left(a^{(0)}_{i,j} + a^{(0)}_{j,i} + \delta_{2i}a^{(1)}_j + \delta_{2j}a^{(1)}_i\right) = \frac{\Lambda_4}{\Gamma_4}\frac{\partial G}{\partial T^{(0)}_{ij}} \qquad \text{in} \quad A,$$
$$\frac{1}{2}(a^{(1)}_{\alpha,\beta} + a^{(1)}_{\beta,\alpha}) = \frac{\Lambda_4}{\Gamma_4}\frac{\partial G}{\partial T^{(1)}_{\alpha\beta}} \qquad \text{in} \quad A,$$
$$Q^{(0)}_i = \frac{\Lambda_4}{\Gamma_4}\frac{\partial G}{\partial E^{(0)}_i}, \qquad Q^{(1)}_\alpha = \frac{\Lambda_4}{\Gamma_4}\frac{\partial G}{\partial E^{(1)}_\alpha} \qquad \text{in} \quad A,$$
$$-a^{(0)}_i = 0, \qquad -a^{(1)}_\alpha = 0 \qquad \text{on} \quad C_u,$$
$$n_j T^{(0)}_{ji} = 0, \qquad n_\beta T^{(0)}_{\beta\alpha} = 0 \qquad \text{on} \quad C_T,$$
$$-\phi^{(0)} = 0, \qquad -\phi^{(1)} = 0 \qquad \text{on} \quad C_\phi,$$
$$n_i Q^{(0)}_i = 0, \qquad n_\alpha Q^{(1)}_\alpha = 0 \qquad \text{on} \quad C_D,$$

which is another equivalent system of the original eigenvalue problem (2.1). If we choose our admissible functions to satisfy

$$-T^{(0)}_{ji,j} + 2b\rho a^{(0)}_i = 0, \qquad -T^{(1)}_{\beta\alpha,\beta} + T^{(0)}_{2\alpha} + \frac{2}{3}b^3\rho a^{(1)}_\alpha = 0 \qquad \text{in} \quad A,$$

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$$E_i^{(0)} + \phi_{,i}^{(0)} + \delta_{2i}\phi^{(1)} = 0, \qquad E_\alpha^{(1)} + \phi_{,\alpha}^{(1)} = 0 \qquad \text{in} \quad A,$$
$$n_j T_{ji}^{(0)} = 0, \quad n_\beta T_{\beta\alpha}^{(0)} = 0 \quad \text{on} \quad C_T,$$
$$-\phi^{(0)} = 0, \quad -\phi^{(1)} = 0 \quad \text{on} \quad C_\phi,$$

then Π_4 reduces to

$$\Pi_4 = \frac{\displaystyle\int_A -\frac{1}{2}\left(2b\rho a_i^{(0)} a_i^{(0)} + \frac{2}{3} b^3 \rho a_\alpha^{(1)} a_\alpha^{(1)}\right) dS}{\displaystyle\int_A G\, dS}.$$

7. Conclusions

In summary, four variational formulations for the vibration of a piezoelectric plate are obtained. They are equivalent in the sense that the eigenvalue problems defined by the stationary conditions of the variational formulations are equivalent. Each variational formulation has a different set of independent variables. These variational formulations can reduce to variational formulations with fewer independent variables. They can also be used to construct various finite element formulations for the vibration of piezoelectric plates.

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Equations for a laminated piezoelectric plate

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TWO-DIMENSIONAL equations of motion for a composite piezoelectric crystal plate symmetrically laminated about the middle plane are obtained by retaining the initial terms of power series expansions of the mechanical displacement and electric potential of the three-dimensional equations of piezoelectricity.

1. Introduction

TWO-DIMENSIONAL equations for motion of piezoelectric crystal plates were derived, extended, and revised by MINDLIN and TIERSTEN [1–5] using power series expansions in the plate thickness direction. LEE [6] derived similar equations using trigonometric series expansions.

A few versions of the two-dimensional plate equations for laminated piezoelectric plates have been derived recently [7–10] because of the development of intelligent structures. These plate theories for the piezoelectric laminates are not true plate theories coupling the electric and mechanical fields. They usually either neglect the electric fields resulting from the variation in stress (the so-called direct piezoelectric effect) [8] or use some equivalent circuit model [7, 10] for the electric fields, so that only the plate equations for the mechanical fields are developed.

In this paper, two-dimensional equations for a composite piezoelectric plate symmetrically laminated about the middle plane are obtained by retaining the initial terms of power series expansions of the mechanical displacement and electric potential of the three-dimensional equations of piezoelectricity. These equations are true coupled plate equations for the laminated plate in the sense that plate equations for the electric fields are also derived and are coupled to the mechanical fields. The equations are derived for general anisotropic materials. They are then specialized to materials with monoclinic symmetry. Finally, thickness-shear vibrations, which are of interest in resonator industry, are discussed.

2. Three-dimensional equations

The three-dimensional equations of piezoelectricity are

$$
\begin{aligned}
T_{ij,i} &= \rho \ddot{u}_j, \qquad D_{i,i} = 0,\\
S_{ij} &= \frac{1}{2}(u_{j,i} + u_{i,j}), \qquad E_i = -\phi_{,i},\\
T_{ij} &= c_{ijkl} S_{kl} - e_{kij} E_k, \qquad D_i = e_{ijk} S_{jk} + \varepsilon_{ij} E_j,
\end{aligned}
\tag{2.1}
$$

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where T_{ij} is stress, u_i mechanical displacement, ρ mass density, D_i electric displacement, S_{ij} strain, E_i electric field, ϕ electric potential, c_{ijkl}, e_{ijk} and ε_{ij} material constants. The above equations can be combined to give equations governing u_i and ϕ:

$$c_{ijkl}u_{k,li} + e_{kij}\phi_{,ki} = \rho\ddot{u}_j\,, \qquad e_{kij}u_{i,jk} - \varepsilon_{ij}\phi_{,ji} = 0.$$

For a piezoelectric composite of N phases, the above equations must be true for each phase with the corresponding material constants c^I_{ijkl}, e^I_{ijk} and ε^I_{ij}, $I = 1, 2, \cdots, N$. At the interfaces of the composite, continuity of the mechanical displacement vector, electric potential, traction vector, and normal component of electric displacement is required.

3. Series of two-dimensional equations

We consider an N-layer laminated plate referred to rectangular coordinates x_i with plate faces at $x_2 = \pm b$, and with x_1 and x_3 being the axes in the the middle plane. The two plate faces and $N-1$ interfaces are sequentially determined by $-b = b_0, b_1, \cdots, b_{N-1}, b_N = b$.

First we expand the mechanical displacement and electric potential into power series in x_2

$$u_i = \sum_n x_2^n u_i^{(n)}\,, \qquad \phi = \sum_n x_2^n \phi^{(n)}\,,$$

where $u_i^{(n)}$ and $\phi^{(n)}$, $n = 0, 1, \cdots$, are functions of x_1, x_3 and t only.

Then, from Eq. (2.1)$_2$

$$S_{ij} = \sum_n x_2^n S_{ij}^{(n)}\,, \qquad E_i = \sum_n x_2^n E_i^{(n)}\,,$$

where

$$\begin{aligned} S_{ij}^{(n)} &= \frac{1}{2}\left[u_{j,i}^{(n)} + u_{i,j}^{(n)} + (n+1)\left(\delta_{i2}u_j^{(n+1)} + \delta_{2j}u_i^{(n+1)}\right)\right], \\ E_i^{(n)} &= -\phi_{,i}^{(n)} - (n+1)\delta_{2i}\phi^{(n+1)}. \end{aligned}$$

Next we multiply Eq. (2.1)$_1$ by x_2^n and integrate across the thickness of the I-th layer from b_{I-1} to b_I. With integration by parts in x_2, we obtain

$$\left(\int_{b_{I-1}}^{b_I} x_2^n T_{ij}dx_2\right)_{,i} - n\left(\int_{b_{I-1}}^{b_I} x_2^{n-1}T_{2j}dx_2\right) + \left[x_2^n T_{2j}\right]_{b_{I-1}}^{b_I} = \sum_m \left(\int_{b_{I-1}}^{b_I} x_2^n x_2^m dx_2\right)\rho^I \ddot{u}_j^{(m)}\,,$$

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$$\left(\int_{b_{I-1}}^{b_I} x_2^n D_i dx_2 \right)_{,i} - n \left(\int_{b_{I-1}}^{b_I} x_2^{n-1} D_2 dx_2 \right) + [x_2^n D_2]_{b_{I-1}}^{b_I} = 0.$$

Summing over I, we have

$$\left(\sum_{I=1}^{N} \int_{b_{I-1}}^{b_I} x_2^n T_{ij} dx_2 \right)_{,i} - n \left(\sum_{I=1}^{N} \int_{b_{I-1}}^{b_I} x_2^{n-1} T_{2j} dx_2 \right) + [x_2^n T_{2j}]_{-b}^{b}$$
$$= \sum_m \left[\sum_{I=1}^{N} \left(\int_{b_{I-1}}^{b_I} x_2^n x_2^m dx_2 \right) \rho^I \right] \ddot{u}_j^{(m)},$$
$$\left(\sum_{I=1}^{N} \int_{b_{I-1}}^{b_I} x_2^n D_i dx_2 \right)_{,i} - n \left(\sum_{I=1}^{N} \int_{b_{I-1}}^{b_I} x_2^{n-1} D_2 dx_2 \right) + [x_2^n D_2]_{-b}^{b} = 0.$$

Defining

$$\begin{aligned}
T_{ij}^{(n)} &= \int_{-b}^{b} x_2^n T_{ij} dx_2 = \sum_{I=1}^{N} \int_{b_{I-1}}^{b_I} x_2^n T_{ij} dx_2, \\
D_i^{(n)} &= \int_{-b}^{b} x_2^n D_i dx_2 = \sum_{I=1}^{N} \int_{b_{I-1}}^{b_I} x_2^n D_i dx_2, \\
T_j^{(n)} &= [x_2^n T_{2j}]_{-b}^{b}, \\
D^{(n)} &= [x_2^n D_2]_{-b}^{b}, \\
\rho^{(mn)} &= \sum_{I=1}^{N} \left(\int_{b_{I-1}}^{b_I} x_2^n x_2^m dx_2 \right) \rho^I,
\end{aligned}$$

we have the following n-th order field equations:

$$\begin{aligned}
T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + T_j^{(n)} &= \sum_m \rho^{(mn)} \ddot{u}_j^{(m)}, \\
D_{i,i}^{(n)} - nD_2^{(n-1)} + D^{(n)} &= 0.
\end{aligned} \tag{3.1}$$

We note that $\rho^{(mn)} = 0$ when $m + n$ is odd because of the symmetry of the lamination.

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The constitutive equations of order n are obtained as follows:

$$(3.2)\qquad T_{ij}^{(n)} = \int\limits_{-b}^{b} x_2^n T_{ij} dx_2 = \sum_{I=1}^{N} \int\limits_{b_{I-1}}^{b_I} x_2^n T_{ij} dx_2 = \sum_{I=1}^{N} \int\limits_{b_{I-1}}^{b_I} x_2^n (c_{ijkl}^I S_{kl} - e_{kij}^I E_k)\, dx_2$$

$$= \sum_{I=1}^{N} \int\limits_{b_{I-1}}^{b_I} x_2^n \sum_m x_2^m \left(c_{ijkl}^I S_{kl}^{(m)} - e_{kij}^I E_k^{(m)}\right) dx_2$$

$$= \sum_m \left\{ \left[\sum_{I=1}^{N} \left(\int\limits_{b_{I-1}}^{b_I} x_2^n x_2^m dx_2 \right) c_{ijkl}^I \right] S_{kl}^{(m)} - \left[\sum_{I=1}^{N} \left(\int\limits_{b_{I-1}}^{b_I} x_2^n x_2^m dx_2 \right) e_{kij}^I \right] E_k^{(m)} \right\}$$

$$= \sum_m \left(c_{ijkl}^{(mn)} S_{kl}^{(m)} - e_{kij}^{(mn)} E_k^{(m)} \right),$$

$$(3.3)\qquad D_i^{(n)} = \int\limits_{-b}^{b} x_2^n D_i dx_2 = \sum_{I=1}^{N} \int\limits_{b_{I-1}}^{b_I} x_2^n D_i dx_2 = \sum_{I=1}^{N} \int\limits_{b_{I-1}}^{b_I} x_2^n (e_{ijk}^I S_{jk} + \varepsilon_{ij}^I E_j)\, dx_2$$

$$= \sum_{I=1}^{N} \int\limits_{b_{I-1}}^{b_I} x_2^n \sum_m x_2^m \left(e_{ijk}^I S_{jk}^{(m)} + \varepsilon_{ij}^I E_j^{(m)}\right) dx_2$$

$$= \sum_m \left\{ \left[\sum_{I=1}^{N} \left(\int\limits_{b_{I-1}}^{b_I} x_2^n x_2^m dx_2 \right) e_{ijk}^I \right] S_{jk}^{(m)} + \left[\sum_{I=1}^{N} \left(\int\limits_{b_{I-1}}^{b_I} x_2^n x_2^m dx_2 \right) \varepsilon_{ij}^I \right] E_j^{(m)} \right\}$$

$$= \sum_m \left(e_{ijk}^{(mn)} S_{jk}^{(m)} + \varepsilon_{ij}^{(mn)} E_j^{(m)} \right),$$

where

$$c_{ijkl}^{(mn)} = \sum_{I=1}^{N} \left(\int\limits_{b_{I-1}}^{b_I} x_2^n x_2^m dx_2 \right) c_{ijkl}^I,$$

$$e_{kij}^{(mn)} = \sum_{I=1}^{N} \left(\int\limits_{b_{I-1}}^{b_I} x_2^n x_2^m dx_2 \right) e_{kij}^I,$$

$$\varepsilon_{ij}^{(mn)} = \sum_{I=1}^{N} \left(\int\limits_{b_{I-1}}^{b_I} x_2^n x_2^m dx_2 \right) \varepsilon_{ij}^I,$$

and $c_{ijkl}^{(mn)} = 0$, $e_{kij}^{(mn)} = 0$, $\varepsilon_{ij}^{(mn)} = 0$ when $m + n$ is odd.

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4. Truncation and adjustment

We begin the truncation of series and adjust the remaining terms, discarding the strains and electric fields of orders higher than the first, leaving $S_{ij}^{(0)}$, $S_{ij}^{(1)}$, $E_i^{(0)}$, $E_i^{(1)}$, which contain the zeroth, first and second order mechanical displacement $u_j^{(0)}$, $u_j^{(1)}$, $u_j^{(2)}$ and electric potential $\phi^{(0)}$, $\phi^{(1)}$, $\phi^{(2)}$, some of which will be eliminated subsequently. At this stage, the constitutive relation (3.2) and (3.3) reduce to

$$(4.1) \qquad T_{ij}^{(0)} = c_{ijkl}^{(00)} S_{kl}^{(0)} - e_{kij}^{(00)} E_k^{(0)}, \qquad D_i^{(0)} = e_{ijk}^{(00)} S_{jk}^{(0)} + \varepsilon_{ij}^{(00)} E_j^{(0)},$$

$$(4.2) \qquad T_{ij}^{(1)} = c_{ijkl}^{(11)} S_{kl}^{(1)} - e_{kij}^{(11)} E_k^{(1)}, \qquad D_i^{(1)} = e_{ijk}^{(11)} S_{jk}^{(1)} + \varepsilon_{ij}^{(11)} E_j^{(1)}.$$

Next, following MINDLIN [3], we neglect $\ddot{u}_2^{(1)}$ in the field equation $(3.1)_1$, and provide for free development of the strain $S_{22}^{(0)}$ $(= u_2^{(1)})$ by setting $T_{22}^{(0)} = 0$ in Eqs. (4.1). Thus

$$T_{22}^{(0)} = c_{22kl}^{(00)} S_{kl}^{(0)} - e_{k22}^{(00)} E_k^{(0)} = 0$$

which implies

$$(4.3) \qquad \left(c_{ij22}^{(00)} c_{22kl}^{(00)} / c_{2222}^{(00)}\right) S_{kl}^{(0)} = \left(e_{k22}^{(00)} c_{ij22}^{(00)} / c_{2222}^{(00)}\right) E_k^{(0)}.$$

With Eq. (4.3), one of Eqs. (4.1) becomes

$$\begin{aligned}
(4.4) \qquad T_{ij}^{(0)} &= c_{ijkl}^{(00)} S_{kl}^{(0)} - e_{kij}^{(00)} E_k^{(0)} \\
&= \left(c_{ijkl}^{(00)} - c_{ij22}^{(00)} c_{22kl}^{(00)} / c_{2222}^{(00)}\right) S_{kl}^{(0)} + \left(c_{ij22}^{(00)} c_{22kl}^{(00)} / c_{2222}^{(00)}\right) S_{kl}^{(0)} - e_{kij}^{(00)} E_k^{(0)} \\
&= \left(c_{ijkl}^{(00)} - c_{ij22}^{(00)} c_{22kl}^{(00)} / c_{2222}^{(00)}\right) S_{kl}^{(0)} + \left(e_{k22}^{(00)} c_{ij22}^{(00)} / c_{2222}^{(00)}\right) E_k^{(0)} - e_{kij}^{(00)} E_k^{(0)} \\
&= \left(c_{ijkl}^{(00)} - c_{ij22}^{(00)} c_{22kl}^{(00)} / c_{2222}^{(00)}\right) S_{kl}^{(0)} - \left(e_{kij}^{(00)} - e_{k22}^{(00)} c_{ij22}^{(00)} / c_{2222}^{(00)}\right) E_k^{(0)} \\
&= \bar{c}_{ijkl}^{(00)} S_{kl}^{(0)} - \bar{e}_{kij}^{(00)} E_k^{(0)},
\end{aligned}$$

and the other of Eqs. (4.1) is replaced by

$$(4.5) \qquad D_i^{(0)} = \bar{e}_{ijk}^{(00)} S_{jk}^{(0)} + \varepsilon_{ij}^{(00)} E_j^{(0)},$$

where

$$\bar{c}_{ijkl}^{(00)} = c_{ijkl}^{(00)} - c_{ij22}^{(00)} c_{22kl}^{(00)} / c_{2222}^{(00)}, \qquad \bar{e}_{kij}^{(00)} = e_{kij}^{(00)} - e_{k22}^{(00)} c_{ij22}^{(00)} / c_{2222}^{(00)}.$$

Note that in Eqs. (4.4) and (4.5) $T_{22}^{(0)}$ is now zero and $S_{22}^{(0)} = u_2^{(1)}$ is no longer present.

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The first order terms are treated similarly except that all three $\ddot{u}_j^{(2)}$ in the field equation $(3.1)_1$ are neglected and free development of the three strains $S_{2j}^{(1)}$ are accommodated by setting $T_{2j}^{(1)} = 0$ in Eqs. (4.2). Since many components of stress are to be set to zero, it is simpler to start with expressions for strains in terms of stresses. We define $s_{ijkl}^{(11)}$ and $d_{ijk}^{(11)}$ such that

$$s_{ijmn}^{(11)} c_{ijkl}^{(11)} = I_{mnkl}\,, \qquad d_{kmn}^{(11)} = s_{ijmn}^{(11)} e_{kij}^{(11)}\,,$$

where I_{mnkl} is the fourth rank unit tensor. Then multiply the formula for $T_{ij}^{(1)}$ in Eqs. (4.2) by $s_{ijmn}^{(11)}$ to obtain

$$s_{ijmn}^{(11)} T_{ij}^{(1)} = S_{mn}^{(1)} - d_{kmn}^{(11)} E_k^{(1)}\,. \tag{4.6}$$

In Eq. (4.6), set $T_{2j}^{(1)} = 0$, $\phi^{(2)} = 0$, so that

$$s_{\gamma\delta\alpha\beta}^{(11)} T_{\gamma\delta}^{(1)} = S_{\alpha\beta}^{(1)} - d_{\gamma\alpha\beta}^{(11)} E_\gamma^{(1)}\,, \tag{4.7}$$

where Greek subscripts range over 1 and 3 only. Now, solve the three equations (4.7) for the three independent $T_{\gamma\delta}^{(1)}$

$$T_{\alpha\beta}^{(1)} = c_{\alpha\beta\gamma\delta}^{(1)} S_{\gamma\delta}^{(1)} - e_{\gamma\alpha\beta}^{(1)} E_\gamma^{(1)}\,,$$

and the other equation in Eqs. (4.2) becomes

$$D_\alpha^{(1)} = e_{\alpha\beta\gamma}^{(1)} S_{\beta\gamma}^{(1)} + \varepsilon_{\alpha\beta}^{(1)} E_\beta^{(1)}\,,$$

where

$$c_{\alpha\beta\gamma\delta}^{(1)} = \left[s_{\alpha\beta\gamma\delta}^{(11)}\right]^{-1}, \qquad e_{\lambda\alpha\beta}^{(1)} = \left[s_{\alpha\beta\gamma\delta}^{(11)}\right]^{-1} d_{\lambda\gamma\delta}^{(11)}\,, \qquad \varepsilon_{\alpha\beta}^{(1)} = \varepsilon_{\alpha\beta}^{(11)}\,.$$

The final adjustment is made by introducing shear correction factors [3]. The thickness shear strains $S_{21}^{(0)}$ and $S_{23}^{(0)}$ are replaced by $\kappa_1 S_{21}^{(0)}$ and $\kappa_3 S_{23}^{(0)}$ in the electric enthalpy density [3], where κ_1 and κ_3 are correction factors whose values may be chosen in such a way that the important thickness shear frequencies have correct values, thus compensating, in part, for the omission of terms of higher orders in the series expansions. With the correction factors, Eqs. (4.4) and (4.5) assume the following form [3]:

$$T_{ij}^{(0)} = c_{ijkl}^{(0)} S_{kl}^{(0)} - e_{kij}^{(0)} E_k^{(0)}\,, \qquad D_i^{(0)} = e_{ijk}^{(0)} S_{jk}^{(0)} + \varepsilon_{ij}^{(0)} E_j^{(0)}\,,$$

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where

$$c^{(0)}_{ijkl} = \kappa^{\mu}_{i+j-2}\kappa^{\nu}_{k+l-2}\bar{c}^{(00)}_{ijkl} \qquad \text{(no sum)},$$
$$e^{(0)}_{kij} = \kappa^{\mu}_{i+j-2}\bar{e}^{(00)}_{kij} \qquad \text{(no sum)},$$
$$\varepsilon^{(0)}_{ij} = \varepsilon^{(00)}_{ij}$$

and μ and ν are

$$\mu = \cos^2(ij\pi/2), \qquad \nu = \cos^2(kl\pi/2).$$

Thus, κ^{μ}_{i+j-2} (or κ^{ν}_{k+l-2}) is equal to κ_1, κ_3 or unity according to whether $i+j$ (or $k+l$) is 3, 5 or neither, respectively.

5. Summary of equations

We summarize the equations below.

Field equations

$$\text{(5.1)} \quad \begin{aligned} T^{(0)}_{ij,i} + T^{(0)}_{j} &= \rho^{(0)}\ddot{u}^{(0)}_{j}, & D^{(0)}_{i,i} + D^{(0)} &= 0, \\ T^{(1)}_{\alpha\beta,\alpha} - T^{(0)}_{2\beta} + T^{(1)}_{\beta} &= \rho^{(1)}\ddot{u}^{(1)}_{\beta}, & D^{(1)}_{\alpha,\alpha} - D^{(0)}_{2} + D^{(1)} &= 0. \end{aligned}$$

Strain-displacement relation

$$\text{(5.2)} \quad S^{(0)}_{ij} = \frac{1}{2}\left(u^{(0)}_{i,j} + u^{(0)}_{j,i} + \delta_{2i}u^{(1)}_{j} + \delta_{2j}u^{(1)}_{i}\right), \qquad S^{(1)}_{\alpha\beta} = \frac{1}{2}\left(u^{(1)}_{\alpha,\beta} + u^{(1)}_{\beta,\alpha}\right).$$

Electric field-potential relation

$$\text{(5.3)} \quad E^{(0)}_{i} = -\phi^{(0)}_{,i} - \delta_{2i}\phi^{(1)}, \qquad E^{(1)}_{\alpha} = -\phi^{(1)}_{,\alpha}.$$

Constitutive equations

$$\text{(5.4)} \quad \begin{aligned} T^{(0)}_{ij} &= c^{(0)}_{ijkl}S^{(0)}_{kl} - e^{(0)}_{kij}E^{(0)}_{k}, & D^{(0)}_{i} &= e^{(0)}_{ijk}S^{(0)}_{jk} + \varepsilon^{(0)}_{ij}E^{(0)}_{j}, \\ T^{(1)}_{\alpha\beta} &= c^{(1)}_{\alpha\beta\gamma\delta}S^{(1)}_{\gamma\delta} - e^{(1)}_{\gamma\alpha\beta}E^{(1)}_{\gamma}, & D^{(1)}_{\alpha} &= e^{(1)}_{\alpha\beta\gamma}S^{(1)}_{\beta\gamma} + \varepsilon^{(1)}_{\alpha\beta}E^{(1)}_{\beta}. \end{aligned}$$

Edge conditions [3]

$$\begin{aligned} n_\alpha T^{(0)}_{\alpha j} &= \bar{t}^{(0)}_{j} & \text{or} \quad u^{(0)}_{j} &= \bar{u}^{(0)}_{j}, \\ n_\alpha D^{(0)}_{\alpha} &= \bar{d}^{(0)} & \text{or} \quad \phi^{(0)} &= \bar{\phi}^{(0)}, \\ n_\alpha T^{(1)}_{\alpha\beta} &= \bar{t}^{(1)}_{\beta} & \text{or} \quad u^{(1)}_{\beta} &= \bar{u}^{(1)}_{\beta}, \\ n_\alpha D^{(1)}_{\alpha} &= \bar{d}^{(1)} & \text{or} \quad \phi^{(1)} &= \bar{\phi}^{(1)}, \end{aligned}$$

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where

$$\begin{aligned}
\rho^{(0)} &= \rho^{(00)}, \\
c^{(0)}_{ijkl} &= \kappa^{\mu}_{i+j-2}\kappa^{\nu}_{k+l-2}\bar{c}^{(00)}_{ijkl} \qquad \text{(no sum)}, \\
e^{(0)}_{kij} &= \kappa^{\mu}_{i+j-2}\bar{e}^{(00)}_{kij} \qquad \text{(no sum)}, \\
\varepsilon^{(0)}_{ij} &= \varepsilon^{(00)}_{ij}, \\
\mu &= \cos^2(ij\pi/2), \qquad \nu = \cos^2(kl\pi/2), \\
\bar{c}^{(00)}_{ijkl} &= c^{(00)}_{ijkl} - c^{(00)}_{ij22}c^{(00)}_{22kl}/c^{(00)}_{2222}, \qquad \bar{e}^{(00)}_{kij} = e^{(00)}_{kij} - e^{(00)}_{k22}c^{(00)}_{ij22}/c^{(00)}_{2222}, \\
\rho^{(00)} &= \sum_{I=1}^{N}(b_I - b_{I-1})\rho^I, \qquad c^{(00)}_{ijkl} = \sum_{I=1}^{N}(b_I - b_{I-1})c^I_{ijkl}, \\
e^{(00)}_{kij} &= \sum_{I=1}^{N}(b_I - b_{I-1})e^I_{kij}, \qquad \varepsilon^{(00)}_{ij} = \sum_{I=1}^{N}(b_I - b_{I-1})\varepsilon^I_{ij}, \\
\rho^{(1)} &= \rho^{(11)}, \\
c^{(1)}_{\alpha\beta\gamma\delta} &= \left[s^{(11)}_{\alpha\beta\gamma\delta}\right]^{-1}, \qquad e^{(1)}_{\lambda\alpha\beta} = \left[s^{(11)}_{\alpha\beta\gamma\delta}\right]^{-1} d^{(11)}_{\lambda\gamma\delta}, \qquad \varepsilon^{(1)}_{\alpha\beta} = \varepsilon^{(11)}_{\alpha\beta}, \\
s^{(11)}_{ijmn}c^{(11)}_{ijkl} &= I_{mnkl}, \qquad d^{(11)}_{kmn} = s^{(11)}_{ijmn}e^{(11)}_{kij}, \\
c^{(11)}_{ijkl} &= \sum_{I=1}^{N}\frac{1}{3}\left[(b_I)^3 - (b_{I-1})^3\right] c^I_{ijkl}, \\
e^{(11)}_{kij} &= \sum_{I=1}^{N}\frac{1}{3}\left[(b_I)^3 - (b_{I-1})^3\right] e^I_{kij}, \\
\varepsilon^{(11)}_{ij} &= \sum_{I=1}^{N}\frac{1}{3}\left[(b_I)^3 - (b_{I-1})^3\right] \varepsilon^I_{ij}, \\
\rho^{(11)} &= \sum_{I=1}^{N}\frac{1}{3}\left[(b_I)^3 - (b_{I-1})^3\right] \rho^I,
\end{aligned}$$

Equations (5.1)–(5.4) can be combined to give the following equations on $u^{(0)}_j$, $u^{(1)}_\alpha$, $\phi^{(0)}$, $\phi^{(1)}$:

$$\begin{aligned}
&c^{(0)}_{ijkl}(u^{(0)}_{k,li} + \delta_{2l}u^{(1)}_{k,i}) + e^{(0)}_{kij}(\phi^{(0)}_{,ki} + \delta_{2k}\phi^{(1)}_{,i}) + T^{(0)}_j = \rho^{(0)}\ddot{u}^{(0)}_j, \\
&e^{(0)}_{ijk}(u^{(0)}_{k,ji} + \delta_{2j}u^{(1)}_{k,i}) - \varepsilon^{(0)}_{ij}(\phi^{(0)}_{,ji} + \delta_{2j}\phi^{(1)}_{,i}) + D^{(0)} = 0, \\
&c^{(1)}_{\alpha\beta\gamma\delta}u^{(1)}_{\delta,\gamma\alpha} + e^{(1)}_{\gamma\alpha\beta}\phi^{(1)}_{,\gamma\alpha} - c^{(0)}_{2\beta kl}(u^{(0)}_{l,k} + \delta_{2k}u^{(1)}_l) \\
&\qquad\qquad - e^{(0)}_{\alpha 2\beta}\phi^{(0)}_{,\alpha} - e^{(0)}_{22\beta}\phi^{(1)} + T^{(1)}_\beta = \rho^{(1)}\ddot{u}^{(1)}_\beta, \\
&e^{(1)}_{\alpha\beta\gamma}u^{(1)}_{\gamma,\beta\alpha} - \varepsilon^{(1)}_{\alpha\beta}\phi^{(1)}_{,\beta\alpha} - e^{(0)}_{2jk}(u^{(0)}_{k,j} + \delta_{2j}u^{(1)}_k) + \varepsilon^{(0)}_{2\alpha}\phi^{(0)}_{,\alpha} + \varepsilon^{(0)}_{22}\phi^{(1)} + D^{(1)} = 0.
\end{aligned}$$

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6. Equations for materials with monoclinic symmetry

In applications it is convenient to the employ the following notation in which a pair of small Latin subscripts ij or kl is replaced by a single capital Latin subscript P or Q ranging over 1 – 6 [11]

$$\begin{aligned} T_1 &= T_{11}, & T_2 &= T_{22}, & T_3 &= T_{33}, \\ T_4 &= T_{23}, & T_5 &= T_{31}, & T_6 &= T_{12}, \\ S_1 &= S_{11}, & S_2 &= S_{22}, & S_3 &= S_{33}, \\ S_4 &= 2S_{23}, & S_5 &= 2S_{31}, & S_6 &= 2S_{12}, \end{aligned}$$

and correspondingly

$$c_{ijkl} \to c_{PQ}, \qquad e_{kij} \to e_{kP}.$$

For materials with monoclinic symmetry, many constants are zero. If x_1 is the diagonal axis of the plate [3] and the layers are laminated in the same direction

$$\begin{aligned} c^I_{15} &= c^I_{16} = c^I_{25} = c^I_{26} = c^I_{35} = c^I_{36} = c^I_{45} = c^I_{46} = 0, \\ e^I_{21} &= e^I_{31} = e^I_{22} = e^I_{32} = e^I_{23} = e^I_{33} = e^I_{24} = e^I_{34} = e^I_{15} = e^I_{16} = 0, \\ \varepsilon^I_{12} &= \varepsilon^I_{13} = 0, \end{aligned}$$

which immediately implies

$$\begin{aligned} c^{(0)}_{15} &= c^{(0)}_{16} = c^{(0)}_{25} = c^{(0)}_{26} = c^{(0)}_{35} = c^{(0)}_{36} = c^{(0)}_{45} = c^{(0)}_{46} = 0, \\ c^{(0)}_{21} &= c^{(0)}_{22} = c^{(0)}_{23} = c^{(0)}_{24} = 0, \\ e^{(0)}_{21} &= e^{(0)}_{31} = e^{(0)}_{22} = e^{(0)}_{32} = e^{(0)}_{23} = e^{(0)}_{33} = e^{(0)}_{24} = e^{(0)}_{34} = e^{(0)}_{15} = e^{(0)}_{16} = 0, \\ e^{(0)}_{12} &= 0, \\ \varepsilon^{(0)}_{12} &= \varepsilon^{(0)}_{13} = 0, \\ c^{(1)}_{15} &= c^{(1)}_{35} = 0, \\ e^{(1)}_{21} &= e^{(1)}_{31} = e^{(1)}_{23} = e^{(1)}_{33} = e^{(1)}_{15} = 0, \\ \varepsilon^{(1)}_{12} &= \varepsilon^{(1)}_{13} = 0. \end{aligned}$$

The equations on $u^{(0)}_j$, $u^{(1)}_\alpha$, $\phi^{(0)}$, $\phi^{(1)}$ then reduce to

$$\begin{aligned} (6.1) \qquad & c^{(0)}_{11} u^{(0)}_{1,11} + c^{(0)}_{55} u^{(0)}_{1,33} + (c^{(0)}_{14} + c^{(0)}_{56}) u^{(0)}_{2,13} + (c^{(0)}_{13} + c^{(0)}_{55}) u^{(0)}_{3,13} \\ & + e^{(0)}_{11} \phi^{(0)}_{,11} + e^{(0)}_{35} \phi^{(0)}_{,33} + c^{(0)}_{14} u^{(1)}_{3,1} + c^{(0)}_{56} u^{(1)}_{1,3} + e^{(0)}_{25} \phi^{(1)}_{,3} + T^{(0)}_1 = \rho^{(0)} \ddot{u}^{(0)}_1, \end{aligned}$$

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(6.1) [cont.]

$$
\begin{aligned}
&(c_{56}^{(0)} + c_{14}^{(0)})u_{1,13}^{(0)} + c_{66}^{(0)}u_{2,11}^{(0)} + c_{44}^{(0)}u_{2,33}^{(0)} + c_{34}^{(0)}u_{3,33}^{(0)} + c_{56}^{(0)}u_{3,11}^{(0)} \\
&\qquad +(e_{14}^{(0)} + e_{36}^{(0)})\phi_{,13}^{(0)} + c_{66}^{(0)}u_{1,1}^{(1)} + c_{44}^{(0)}u_{3,3}^{(1)} + e_{26}^{(0)}\phi_{,1}^{(1)} + T_2^{(0)} = \rho^{(0)}\ddot{u}_2^{(0)}, \\
&(c_{13}^{(0)} + c_{55}^{(0)})u_{1,13}^{(0)} + c_{56}^{(0)}u_{2,11}^{(0)} + c_{34}^{(0)}u_{2,33}^{(0)} + c_{55}^{(0)}u_{3,11}^{(0)} + c_{33}^{(0)}u_{3,33}^{(0)} \\
&\qquad +(e_{13}^{(0)} + e_{35}^{(0)})\phi_{,13}^{(0)} + c_{56}^{(0)}u_{1,1}^{(1)} + c_{34}^{(0)}u_{3,3}^{(1)} + e_{25}^{(0)}\phi_{,1}^{(1)} + T_3^{(0)} = \rho^{(0)}\ddot{u}_3^{(0)}, \\
&e_{11}^{(0)}u_{1,11}^{(0)} + e_{35}^{(0)}u_{1,33}^{(0)} + (e_{14}^{(0)} + e_{36}^{(0)})u_{2,13}^{(0)} + (e_{13}^{(0)} + e_{35}^{(0)})u_{3,13}^{(0)} \\
&\qquad +e_{36}^{(0)}u_{1,3}^{(1)} + e_{14}^{(0)}u_{3,1}^{(1)} - \varepsilon_{11}^{(0)}\phi_{,11}^{(0)} - \varepsilon_{33}^{(0)}\phi_{,33}^{(0)} - \varepsilon_{32}^{(0)}\phi_{,3}^{(1)} + D^{(0)} = 0, \\
&c_{11}^{(1)}u_{1,11}^{(1)} + c_{55}^{(1)}u_{1,33}^{(1)} + (c_{13}^{(1)} + c_{55}^{(1)})u_{3,13}^{(1)} + e_{11}^{(1)}\phi_{,11}^{(1)} + e_{35}^{(1)}\phi_{,33}^{(1)} \\
&-c_{66}^{(0)}u_{2,1}^{(0)} - c_{56}^{(0)}(u_{1,3}^{(0)} + u_{3,1}^{(0)}) - e_{36}^{(0)}\phi_{,3}^{(0)} - c_{66}^{(0)}u_1^{(1)} - e_{26}^{(0)}\phi^{(1)} + T_1^{(1)} = \rho^{(1)}\ddot{u}_1^{(1)}, \\
&(c_{13}^{(1)} + c_{55}^{(1)})u_{1,13}^{(1)} + c_{55}^{(1)}u_{3,11}^{(1)} + c_{33}^{(1)}u_{3,33}^{(1)} + (e_{13}^{(1)} + e_{35}^{(1)})\phi_{,13}^{(1)} \\
&\qquad -c_{14}^{(0)}u_{1,1}^{(0)} - c_{44}^{(0)}u_{2,3}^{(0)} - c_{43}^{(0)}u_{3,3}^{(0)} - e_{14}^{(0)}\phi_{,1}^{(0)} - c_{44}^{(0)}u_3^{(1)} + T_3^{(1)} = \rho^{(1)}\ddot{u}_3^{(1)}, \\
&e_{11}^{(1)}u_{1,11}^{(1)} + e_{35}^{(1)}u_{1,33}^{(1)} + (e_{13}^{(1)} + e_{35}^{(1)})u_{3,13}^{(1)} - \varepsilon_{11}^{(1)}\phi_{,11}^{(1)} - \varepsilon_{33}^{(1)}\phi_{,33}^{(1)} \\
&\qquad -e_{26}^{(0)}u_{2,1}^{(0)} - e_{25}^{(0)}(u_{1,3}^{(0)} + u_{3,1}^{(0)}) + \varepsilon_{23}^{(0)}\phi_{,3}^{(0)} - e_{26}^{(0)}u_1^{(1)} + \varepsilon_{22}^{(0)}\phi^{(1)} + D^{(1)} = 0.
\end{aligned}
$$

7. Thickness shear vibrations

For thickness-shear modes, we drop all terms with spatial derivatives in Eqs. (6.1), and consider the case in which the faces of the plate are free of traction and charge, which implies $T_j^{(0)} = 0$, $D^{(0)} = 0$, $T_\beta^{(1)} = 0$, $D^{(1)} = 0$. We seek for solutions such that $u_j^{(0)} = 0$. We are then left with the last three equations of Eqs. (6.1) which now assume the following form:

(7.1)
$$
\begin{aligned}
&-c_{66}^{(0)}u_1^{(1)} - e_{26}^{(0)}\phi^{(1)} = \rho^{(1)}\ddot{u}_1^{(1)}, \\
&-c_{44}^{(0)}u_3^{(1)} = \rho^{(1)}\ddot{u}_3^{(1)}, \\
&-e_{26}^{(0)}u_1^{(1)} + \varepsilon_{22}^{(0)}\phi^{(1)} = 0.
\end{aligned}
$$

(i) Thickness-shear in x_1-direction

For thickness-shear in x_1 direction, we keep $u_1^{(1)}$, $\phi^{(1)}$ and Eqs. $(7.1)_{1,3}$

(7.2)
$$
\begin{aligned}
-c_{66}^{(0)}u_1^{(1)} - e_{26}^{(0)}\phi^{(1)} &= \rho^{(1)}\ddot{u}_1^{(1)}, \\
-e_{26}^{(0)}u_1^{(1)} + \varepsilon_{22}^{(0)}\phi^{(1)} &= 0.
\end{aligned}
$$

Now let

$$u_1^{(1)} = Ae^{i\omega t}, \qquad \phi^{(1)} = Be^{i\omega t}.$$

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Equation (7.2) becomes

$$(7.3)\qquad \begin{aligned} (\rho^{(1)}\omega^2 - c_{66}^{(0)})A - e_{26}^{(0)}B &= 0, \\ -e_{26}^{(0)}A + \varepsilon_{22}^{(0)}B &= 0. \end{aligned}$$

Vanishing of the determinant of the coefficient matrix in Eqs. (7.3) gives the frequency expression

$$\omega^2 = \left[\frac{(e_{26}^{(0)})^2}{\varepsilon_{22}^{(0)}} + c_{66}^{(0)}\right] / \rho^{(1)}.$$

(ii) Thickness-shear in x_3-direction

For thickness-shear in x_3 direction, we keep $u_3^{(1)}$ and Eq. $(7.1)_2$

$$(7.4)\qquad -c_{44}^{(0)}u_3^{(1)} = \rho^{(1)}\ddot{u}_3^{(1)},$$

$$(7.5)\qquad u_3^{(1)} = Ae^{i\omega t}.$$

Substitution of Eq. (7.5) into Eq. (7.4) gives the following frequency expression:

$$\omega^2 = c_{44}^{(0)}/\rho^{(1)},$$

where

$$c_{44}^{(0)} = \kappa_3\kappa_3\left\{\sum_{I=1}^{N}(b_I - b_{I-1})c_{44}^I - \left[\sum_{I=1}^{N}(b_I - b_{I-1})c_{24}^I\right]^2 \Big/ \left(\sum_{I=1}^{N}(b_I - b_{I-1})c_{22}^I\right)\right\},$$

$$c_{66}^{(0)} = \kappa_1\kappa_1\sum_{I=1}^{N}(b_I - b_{I-1})c_{66}^I,$$

$$e_{26}^{(0)} = \kappa_1\sum_{I=1}^{N}(b_I - b_{I-1})e_{26}^I,$$

$$\varepsilon_{22}^{(0)} = \sum_{I=1}^{N}(b_I - b_{I-1})\varepsilon_{22}^I,$$

$$\rho^{(1)} = \sum_{I=1}^{N}\frac{1}{3}\left[(b_I)^3 - (b_{I-1})^3\right]\rho^I.$$

It can be seen that only one correction factor κ_1 is involved in thickness-shear in the x_1-direction, the other correction factor κ_3 is involved in thickness-shear in the x_3-direction only. The values of κ_1 and κ_3 are determined in [3] for a plate

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of a single material as

$$\kappa_1^2 = \frac{\pi^2}{12},$$
$$\kappa_3^2 = \frac{\pi^2}{12}\frac{1}{2}\frac{\{c_{22} + c_{44} - [(c_{22} - c_{44})^2 + 4c_{24}^2]^{\frac{1}{2}}\}}{c_{44} - c_{24}^2/c_{22}} .$$

The above expressions can not be used directly in a laminated plate. The correct values of the correction factors for laminates should be determined for each specific laminated plate, by matching the solutions for the thickness shear vibrations obtained from the plate equations derived above, to the exact solutions obtained from the three-dimensional equations. In many cases only flexural motions are of interest; then these correction factors just assume the value 1.

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Many-sphere hydrodynamic interactions: first order Oseen's effects

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THE PAPER CONCERNS hydrodynamic interactions of a finite number N of rigid spheres, immersed in an incompressible, unbounded fluid. In particular, we are interested in the weak momentum convection effects, arising in the hydrodynamic interactions. Hence, the regime of flows, characterized by the condition $\mathrm{Re} < 1$, where Re is the characteristic Reynolds number, will be regarded. The hydrodynamic interactions are analysed in the framework of the Oseen equations of the fluid motion. The results obtained describe the relations between the forces $\mathbf{F}_i$, $j = 1, \ldots, N$, exerted by the fluid on the spheres, and the streaming velocity of the fluid at the infinity $\mathbf{U}$.

1. Introduction

THE PRESENT PAPER deals with hydrodynamic interactions of a finite number of rigid spheres immersed in an incompressible, unbounded fluid. We are interested in many-spheres hydrodynamic interactions, characterized by the small, but finite Reynolds number Re.

The current status of the problem of many-particles hydrodynamic interactions, in the regimes of Stokes, and transient Stokes flows, is discussed by KIM and KARRILA [1]. The regime of small Re for the particular case of the motion of a single body through a fluid has been recently considered by CHESTER [2]. The first results concerning the dependence of the hydrodynamic interactions on Re for two spheres have been obtained by OSEEN [3]. C.W. Oseen used his approximate form of the equations of the fluid motion. He considered the spheres, translating with parallel velocities in a fluid being at rest at infinity. According to these results, the forces exerted by the fluid on the leading and trailing spheres are different, in contrast to the equality of the respective forces in the regime of Stokes flows. In the subsequent papers, the hydrodynamic interactions of two spheres have been analysed, allowing for both translational, and rotational motions of the spheres, and starting from the Navier–Stokes equations. To take into account the nonlinear terms, the method of matched asymptotic expansions has been used. The results obtained indicate, that two different ranges of the hydrodynamic interactions should be distinguished, depending on the quotient $\mathrm{Re}/(a/R)$, where a is the radius of the spheres, R is the characteristic distance of the centres of two spheres, $\sigma = a/R$. KANEDA and ISHII [4] dealt with the case $\mathrm{Re}/\sigma < 1$; it means, they referred to the case, when the spheres are within their respective inner expansions. A part of the paper by VASSEUR and COX [5] concerns the case $\mathrm{Re}/\sigma = 0(1)$, describing the hydrodynamic interactions of the

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spheres, situated within their respective outer expansions. The former hydrodynamic interactions are, qualitatively speaking, stronger in comparison with the latter ones.

In this paper, we consider the hydrodynamic interactions of a finite number N of rigid spheres, using the classical Oseen equations, in the range of flows with small, but finite Re. The presence of the spheres in the flow is expressed by the so-called induced forces, distributed on the surfaces of the spheres. The integral equation approach is used. It involves the second order hydrodynamic interaction tensors, depending on Re, and on the spatial distribution of the spheres. The first-order Oseen effects are discussed. As an example, the first contributions of the order of Re to the lift forces, acting on two spheres, which are at rest in a fluid having the streaming velocity $\mathbf{U}$ at infinity, are calculated.

2. Basic equations

We consider N rigid spheres of radius a, being at rest in an incompressible, unbounded fluid. The streaming velocity of the fluid at infinity is given by $\mathbf{U}$. To describe the spatial distribution of the spheres we use the fixed Cartesian coordinate system $\mathbf{r}(x, y, z)$. The positions of the centres of the spheres are given by $\mathbf{R}_j^0$, $j = 1, \ldots, N$, and the positions of the surfaces of the spheres are given by $\mathbf{R}_j$. We introduce also the local coordinate system for each sphere:

$$\mathbf{r}_j = \mathbf{R}_j - \mathbf{R}_j^0, \; \mathbf{r}_j(a, \Omega_j).$$

The presence of spheres in the flow is replaced by the induced forces $\mathbf{f}_j(\mathbf{r}_j)$ [6], distributed on the surfaces of the spheres. These surfaces are described by the appriopriate δ-functions: $\delta(\mathbf{r} - \mathbf{R}_j)$. The forces $\mathbf{f}_j(\mathbf{r}_j)$ give rise to the source term in the Oseen equation of motion:

$$\begin{gathered} \rho \mathbf{U} \cdot \nabla \mathbf{v} - \mu \Delta \mathbf{v} + \nabla p = \sum_{j=1}^{N} \int d\Omega_j \delta[\mathbf{r} - \mathbf{R}_j(\Omega_j)] \mathbf{f}_j(\Omega_j), \\ \nabla \cdot \mathbf{v} = 0, \end{gathered} \tag{2.1}$$

where ρ is the density, μ – the dynamic viscosity, $\mathbf{v}(\mathbf{r})$ – the velocity, p – the pressure of the incompressible fluid. On the surfaces of the spheres, we assume the non-slip boundary conditions

$$\dot{\mathbf{R}}_j\,(\Omega_j) = \mathbf{v}(\mathbf{R}_j(\Omega_j)), \tag{2.2}$$

where $\dot{\mathbf{R}}_j$ denotes the velocity of the j-th sphere. The equations of motion can be used in the whole space if the divergence of the stress tensor $\mathbf{P}(\mathbf{r})$ is determined inside the volumes occupied by the immersed spheres:

$$\nabla \cdot \mathbf{P}(\mathbf{r}_j) = 0, \qquad |\mathbf{r}_j| < a. \tag{2.3}$$

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To examine the relations of the induced forces $\mathbf{f}_j$ to the fluid velocity $\mathbf{U}$, we introduce the respective integral equations. The velocity field of the fluid is expressed in terms of the Green tensor $\mathbf{T}(\mathbf{r})$, acting on the induced forces $\mathbf{f}_j$

(2.4)
$$\mathbf{v}(\mathbf{r}) = \mathbf{v}^0(\mathbf{r}) + \int d\mathbf{r}'\mathbf{T}(\mathbf{r}-\mathbf{r}') \cdot \sum_{j=1}^{N} \int d\Omega'_j \delta\left[\mathbf{r}' - R'_j(\Omega'_j)\right] \mathbf{f}_j{}'(\Omega'_j),$$
$$\mathbf{v}^0(\mathbf{r}) = \mathbf{U}.$$

$\mathbf{v}^0(\mathbf{r})$ denotes the fluid velocity in the absence of the spheres. The Green tensor may be presented by means of the space – Fourier representation [7]:

(2.5)
$$\mathbf{T}(\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\exp(i\mathbf{k}\cdot\mathbf{r})}{\mu(k^2 + i\nu^{-1}\mathbf{U}\cdot\mathbf{k})}(\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}),$$
$$\nu = \mu/\rho, \qquad \hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|, \qquad k = |\mathbf{k}|.$$

Using the expression for the velocity field and the boundary conditions on the surfaces of the spheres, we arrive at N coupled integral equations [11]:

(2.6)
$$\dot{\mathbf{R}}_j(\Omega_j) = \mathbf{v}^0(\mathbf{R}_j(\Omega_j)) + \int d\Omega'_j \mathbf{T}[\mathbf{R}_j(\Omega_j) - \mathbf{R}'_j(\Omega'_j)] \cdot \mathbf{f}_j{}'(\Omega'_j)$$
$$+ \sum_{k\neq j}^{N} \int d\Omega_k \mathbf{T}[\mathbf{R}_j(\Omega_j) - \mathbf{R}_k(\Omega_k)] \cdot \mathbf{f}_k(\Omega_k),$$
$$\mathbf{V}_j(\Omega_j) = \dot{\mathbf{R}}_j(\Omega_j) - \mathbf{v}^0(\mathbf{R}_j(\Omega_j)).$$

The first integral on the r.h.s. accounts for the interaction of the j-th sphere with the fluid, the second integral is due to the hydrodynamic interaction between the spheres. $\mathbf{V}_j$ is the relative velocity of the j-th sphere with respect to the fluid.

3. The set of algebraic equations

To transform the set of integral equations to the form of the algebraic equations, we expand $\mathbf{V}_j$ and $\mathbf{f}_j$ in terms of the normalized spherical harmonics [8, 9]:

(3.1)
$$\mathbf{V}_j(\Omega_j) = \sqrt{4\pi}\sum_{l\,m}^{\infty} \mathbf{V}_{j,lm} Y_l^m(\Omega_j),$$
$$\mathbf{f}_j(\Omega_j) = \frac{1}{\sqrt{4\pi}a^2}\sum_{l\,m}^{\infty} \mathbf{f}_{j,lm} Y_j^m(\Omega_j),$$
$$|m| \leq l, \qquad 0 \leq l < \infty.$$

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Integrating over the surfaces of the spheres, one obtains the following set of algebraic equations for the expansion coefficients:

(3.2) $$\mathbf{V}_{j,l_1m_1} = \sum_{l_2m_2} \mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{0}_j)\cdot \mathbf{f}_{j,l_2m_2} + \sum_{k\neq j}^{N}\sum_{l_2m_2} \mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{R}_{jk})\mathbf{f}_{k,l_2m_2},$$

where $\mathbf{R}_{jk} = \mathbf{R}^0_k - \mathbf{R}^0_j$ is the distance between the centres of two spheres, $\mathbf{T}^{l_2m_2}_{l_1m_1}$ are called, after YOSHIZAKI and YAMAKAWA [9], the hydrodynamic interaction tensors. The self-interaction tensors

$$\mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{0}_j)$$

describe the presence of the j-th sphere in the flow, whereas the mutual interaction tensors

$$\mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{R}_{jk})$$

account for the interactions between the j-th and the k-th sphere. The interaction tensors are given by the following expressions:

(i) the self-interaction tensors

(3.3) $$\mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{0}_j) = \frac{1}{2\pi^2\mu} i^{l_1-l_2}\int d\mathbf{k}\frac{(\mathbf{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})}{k^2+i\nu^{-1}\mathbf{U}\cdot\mathbf{k}} Y^{-m_1}_{l_1} Y^{m_2}_{l_2} j_{l_1}(ak) j_{l_2}(ak),$$

where j_{l_1}, j_{l_2} are the spherical Bessel functions;

(ii) the mutual-interaction tensors, using

(3.4) $$\exp(i\mathbf{k}\cdot\mathbf{r}_j) = 4\pi\sum_{lm} i^l j_l(kr_j) Y^m_l(\Omega_j) Y^{-m}_l(X,\eta), \qquad \mathbf{k}(k, X, \eta),$$

are presented in the following form:

$$\mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{R}_{jk}) = \sum_{l_3m_3} \mathbf{T}^{l_2m_2}_{l_1m_1,l_3m_3}(|\mathbf{R}_{jk}|) Y^{m_3}_{l_3}(\Omega_{jk}), \qquad R_{jk} = |\mathbf{R}_{jk}|,$$

$$\mathbf{T}^{l_2m_2}_{l_1m_1,l_3m_3} = \frac{2}{\pi\mu} i^{l_1-l_2-l_3}\int d\mathbf{k}\frac{(\mathbf{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})}{k^2+i\nu^{-1}\mathbf{U}\cdot\mathbf{k}} \times Y^{-m_1}_{l_1} Y^{m_2}_{l_2} Y^{-m_3}_{l_3} j_{l_1}(ak) j_{l_2}(ak) j_{l_3}(R_{jk}k).$$

The properties of the above tensors are directly related to the properties of the Green tensor involved (2.5). Here the tensors $\mathbf{T}^{l_2m_2}_{l_1m_1}$ are examined from the point of view of their dependence on Re, $\mathrm{Re} = a|\mathbf{U}|/\nu$, and on the spatial distribution of spheres. For that aim we use the double expansion: the expressions for $\mathbf{T}^{l_2m_2}_{l_1m_1}$ are expanded into the power series of Re; the resulting coefficients are, in turn, expanded into the power series of σ. Knowing the above properties of the hydrodynamic interaction tensors, one can use Eq. (3.2) to obtain the relations of the forces $\mathbf{F}_j$, exerted by the fluid on the spheres, to the fluid velocity $\mathbf{U}$, within the assumed approximation with respect to Re and σ.

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4. The properties of the hydrodynamic interaction tensors

4.1. Self-interaction tensors $\mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{0}_j)$

The dependence of the self-interaction tensors on Re can be expressed in terms of the functions $F_{l_1l_2}$:

$$\text{(4.1)}\quad \begin{aligned} \mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{0}_j) &= \frac{1}{4\pi\mu a} i^{l_1-l_2}\int d\mathbf{k}(\mathbf{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})Y^{-m_1}_{l_1}Y^{m_2}_{l_2}F_{l_1l_2}(\mathrm{Re},\xi),\\ F_{l_1l_2}(\mathrm{Re},\xi) &= \int dk\frac{k-i\alpha\xi}{k^2+\alpha^2\xi^2}J_{l_1+1/2}(ak)J_{l_2+1/2}(ak), \end{aligned}$$

where $\xi = \cos(\hat{\mathbf{U}},\hat{\mathbf{k}})$, $\alpha = |\mathbf{U}|/\nu$, $\mathrm{Re} = \alpha a$. For the cases $|l_1-l_2| = 2n$, $n = 0,1,2,...$

$$\text{(4.1}'\text{)}\quad \mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{0}_j) = \frac{1}{4a\mu\pi} i^{l_1-l_2+|l_1-l_2|}\int d\hat{\mathbf{k}}(\mathbf{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})Y^{-m_1}_{l_1}Y^{m_2}_{l_2}I_{\Lambda}(\mathrm{Re}\,\xi)K_{\lambda}(\mathrm{Re}\,\xi),$$

and respectively for the cases $|l_1-l_2| = 2n+1$,

$$\mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{0}_j) = \frac{1}{4\pi a\mu} i^{l_1-l_2+|l_1-l_2|+2}\int d\hat{\mathbf{k}}(\mathbf{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})Y^{-m_1}_{l_1}Y^{m_2}_{l_2}I_{\Lambda}(\mathrm{Re}\,\xi)K_{\lambda}(\mathrm{Re}\,\xi),$$

$$\Lambda = \max(l_1+1/2,\, l_2+1/2), \qquad \lambda = \min(l_1+1/2,\, l_2+1/2).$$

Λ and λ are the larger and smaller number, respectively, of $l_1+1/2$, and $l_2+1/2$, and $I_{l_1+1/2}$, $K_{l_2+1/2}$ denote the modified Bessel functions.

From the properties of the Bessel functions in the range $\mathrm{Re} < 1$ it follows that the functions $F_{l_1l_2}$ behave as

$$\text{(4.2)}\quad F_{l_1l_2} = A_{l_1l_2}(\mathrm{Re}\,\xi)^{|l_1-l_2|}, \qquad \mathrm{Re} < 1.$$

Hence, for the particular case of $l_1 = l_2$, the leading order terms of the self-interaction tensor are independent of Re. These terms refer to the regime of Stokes flows, which do not exhibit the effects of the inertia of the fluid. At zero Reynolds number the self-interaction tensors become equal to

$$\text{(4.3)}\quad \begin{aligned} \mathbf{T}^{l_1m_2}_{l_1m_1}(\mathbf{0}_j) &= \frac{1}{4\pi a\mu}A_{l_1l_1}\int d\hat{\mathbf{k}}\,(\mathbf{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})\,Y^{-m_1}_{l_1}Y^{m_2}_{l_1}\\ &= \frac{1}{4\sqrt{\pi}a\mu}\frac{1}{(l_1+1/2)}\mathbf{K}^{l_1m_2}_{l_1m_1,00}, \end{aligned}$$

where

$$\mathbf{K}^{l_2m_2}_{l_1m_1,l_3m_3} = i^{l_1-l_2-l_3}\int d\hat{\mathbf{k}}(\mathbf{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})Y^{-m_1}_{l_1}Y^{m_2}_{l_2}Y^{-m_3}_{l_3}.$$

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The constant tensors $\mathbf{K}^{l_2m_2}_{l_1m_1,l_3m_3}$ describe the dependence of the self-interaction tensors on the sets of indices (l_i, m_i). The second order tensors $\mathbf{K}$ have been introduced, in the context of the Stokes flows, by YOSHIZAKI and YAMAKAWA [9]. Here, as Re $\Rightarrow 0$, we recover the Stokes self-interaction tensors.

Examining weak inertia effects, we are interested in the linear in Re contributions to the relevant tensors. It follows from the properties of the functions $F_{l_1l_2}$ that there are two sources of such contributions. Firstly, we have a group of the self-interaction tensors, being of the leading order of Re (specified by the condition $|l_1 - l_2| = 1$). These self-interaction tensors are given by

$$(4.4) \qquad \mathbf{T}^{l_2m_2}_{l_2+1m_1}(\mathbf{0}_j) = \frac{\mathrm{Re}\,\widehat{U}}{8\mu a\sqrt{3\pi}(l_2+1/2)(l_2+3/2)}\mathbf{K}^{l_2m_2}_{l_2+1m_1,10} + \dots .$$

and, respectively, by

$$(4.4') \qquad \mathbf{T}^{l_1+1m_2}_{l_1m_1}(\mathbf{0}_j) = \frac{\mathrm{Re}\,\widehat{U}}{8\mu a\sqrt{3\pi}(l_1+1/2)(l_1+3/2)}\mathbf{K}^{l_1+1m_2}_{l_1m_1,10} + \dots .$$

The above expressions are obtained for the particular case of $\mathbf{U} = (0, 0, U)$. We see that here the tensors $\mathbf{K}^{l_2m_2}_{l_1m_1,l_3m_3}$ have higher values of the indices (l_3, m_3), in comparison with the tensors $\mathbf{K}$, referring to the Stokes flow regime. Secondly, the contributions linear in Re appear in a series expansion of the function F_{00}. Hence, the self-interaction tensor $\mathbf{T}^{00}_{00}$ containing that function, can be presented in the following form:

$$(4.5) \qquad \mathbf{T}^{00}_{00}(\mathbf{0}_j) = \frac{1}{6\pi\mu a}\left[\mathbf{1} - \frac{3}{8\pi}\mathrm{Re}\int d\hat{\mathbf{k}}(\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}})|\cos(\widehat{\mathbf{U}}, \hat{\mathbf{k}})|\right] + \dots .$$

Thus, we have collected all contributions, linear in Re, to the self-interaction tensors, needed to discuss the weak inertia effects of the interaction of a single sphere with the fluid.

4.2. Mutual-interaction tensors $\mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{R}_{jk}, \mathrm{Re})$

Using the Eq. (3.4), the mutual-interaction tensors are presented in the following form:

$$(4.6) \qquad \mathbf{T}^{l_2m_2}_{l_1m_1,l_3m_3} = \frac{1}{a\mu} i^{l_1-l_2-l_3}\sqrt{\frac{\pi}{2R_{jk}}}\int d\hat{\mathbf{k}}(\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}) \times Y^{-m_1}_{l_1} Y^{m_2}_{l_2} Y^{-m_3}_{l_3} F_{l_1l_2,l_3}(R_{jk}, \mathrm{Re}, \xi),$$

where the functions $F_{l_1l_2,l_3}$ describe the effects of Re, and of the spatial distribution of the spheres,

$$(4.7) \qquad F_{l_1l_2,l_3} = \int \frac{dk}{\sqrt{k}}\frac{(k - i\alpha\xi)}{k^2 + \alpha^2\xi^2} J_{l_1+1/2}(ak) J_{l_2+1/2}(ak) J_{l_3+1/2}(R_{jk}k).$$

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To consider the effects of Re, the tensors $\mathbf{T}^{l_2 m_2}_{l_1 m_1, l_3 m_3}$ are presented in terms of a series expansion, using the formula (7) from § 7.15 of [10]:

$$\mathbf{T}^{l_2 m_2}_{l_1 m_1, l_3 m_3} = \sum_{m=0}^{\infty} \mathbf{T}^{l_2 m_2, m}_{l_1 m_1, l_3 m_3} = \frac{\sqrt{\pi}}{2a\mu \Gamma(l_1 + 3/2)\Gamma(l_2 + 3/2)} i^{l_1 - l_2 - l_3}$$
$$\times \left(\frac{a}{R_{jk}}\right)^{l_1+l_2+1} \sum_{m=0}^{\infty} \frac{(l_1 + l_2 + 2m + 1/2)\Gamma(l_1 + l_2 + m + 1/2)}{m!}$$
$$\times F_4 \left[-m, l_1 + l_2 + m + 1/2; l_1 + 3/2, l_2 + 3/2; \left(\frac{a}{R_{jk}}\right)^2, \left(\frac{a}{R_{jk}}\right)^2 \right]$$
$$\times \int d\hat{\mathbf{k}} (\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}) Y^{-m_1}_{l_1} Y^{m_2}_{l_2} Y^{-m_3}_{l_3} \int d\kappa \frac{\kappa - iR_{jk}\alpha\xi}{\kappa^2 + (R_{jk}\alpha\xi)^2} J_{l_1+l_2+2m+1/2}(\kappa) J_{l_3+1/2}(\kappa),$$

where, for $|l_1 + l_2 + 2m - l_3| = 2n, \ n = 0, 1, 2, \ldots$

$$\text{(4.8)} \qquad \int d\kappa \frac{\kappa - iR_{jk}\alpha\xi}{\kappa^2 + (R_{jk}\alpha\xi)^2} J_{l_1+l_2+2m+1/2}(\kappa) J_{l_3+1/2}(\kappa)$$
$$= i^{|l_1+l_2+2m-l_3|} I_Z(R_{jk}\alpha\xi) K_\zeta(R_{jk}\alpha\xi),$$

and for $|l_1 + l_2 + 2m - l_3| = 2n + 1, \ n = 0, 1, 2, \ldots$

$$\int d\kappa \frac{\kappa - iR_{jk}\alpha\xi}{\kappa^2 + (R_{jk}\alpha\xi)^2} J_{l_1+l_2+2m+1/2}(\kappa) J_{l_3+1/2}(\kappa)$$
$$= -i^{|l_1+l_2+2m-l_3|} I_Z(R_{jk}\alpha\xi) K_\zeta(R_{jk}\alpha\xi),$$

$$Z = \max(l_1 + l_2 + 2m + 1/2, \ l_3 + 1/2),$$
$$\zeta = \min(l_1 + l_2 + 2m + 1/2, \ l_3 + 1/2);$$

F_4 is the hypergeometric series of two variables.

In that series the dependence on Re is expressed in the form of the product of two modified Bessel functions. The arguments of these functions contain the quotients $\mathrm{Re}_m = \mathrm{Re}/(a/R_{jk})$ in view of the fact that the mutual interactions take place at the distances R_{jk} between the centres of the two respective spheres.

In the considered range

$$\mathrm{Re}_m < 1,$$

the products of the modified Bessel functions behave as:

$$\text{(4.9)} \qquad I_Z(\mathrm{Re}_m\xi) K_\zeta(\mathrm{Re}_m\xi) = \left(\frac{1}{2}\right)^{Z-\zeta+1} \frac{\Gamma(Z)}{\Gamma(\zeta+1)} (\mathrm{Re}_m\xi)^{|l_1+l_2+2m-l_3|}.$$

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It can be seen that the Stokes regime is described by the following leading order contributions to the respective mutual-interaction tensors:

$$(4.10)\qquad \mathbf{T}^{l_2m_2}_{l_1m_1,l_3m_3} = \sum_{m=0}^{\infty} \mathbf{T}^{l_2m_2,m}_{l_1m_1,l_3m_3} = \frac{\sqrt{\pi}}{4a\mu\Gamma(l_1+3/2)\Gamma(l_2+3/2)}\left(\frac{a}{R_{jk}}\right)^{l_1+l_2+1}$$
$$\times\mathbf{K}^{l_2m_2}_{l_1m_1,l_3m_3}\sum_{m=0}\frac{(l_1+l_2+2m+1/2)\Gamma(l_1+l_2+m+1/2)}{m!}\frac{\Gamma(\zeta)}{\Gamma(Z+1)}$$
$$\times F_4\left[-m, l_1+l_2+m+1/2;\ l_1+3/2, l_2+3/2;\ \left(\frac{a}{R_{jk}}\right)^2, \left(\frac{a}{R_{jk}}\right)^2\right],$$

for the cases $|l_1+l_2+2m-l_3| = 0$.

In turn, from the properties of the tensors $\mathbf{K}^{l_2m_2}_{l_1m_1,l_3m_3}$ [9] one can deduce that only the following sets of indices are admissible for the case of Re $\Rightarrow 0$:

$$(4.11)\qquad l_1+l_2+l_3 = 2n, \qquad l_1+l_2-l_3 \geq -2.$$

Thus the Stokes hydrodynamic interactions are described by means of the tensors

$$(4.12)\qquad \begin{array}{llll} \text{(i)} & \mathbf{T}^{l_2m_2,0}_{l_1m_1,l_3m_3} & \text{for the cases} & l_3 = l_1+l_2, \\ \text{(ii)} & \mathbf{T}^{l_2m_2,1}_{l_1m_1,l_3m_3} & \text{for the cases} & l_3 = l_1+l_2+2. \end{array}$$

Taking into account the properties of the function F_4, we see that the Stokes tensors behave as follows:

(4.13) they are of the leading order $(a/R_{jk})^{l_1+l_2+1}$,

(4.14) the tensors with $m = 1$ contain terms $(a/R_{jk})^{l_1+l_2+3}$.

Hence we have recovered the dependence on the inverse powers of the interparticle distances, characteristic for the Stokes conditions.

Similarly to the self-interaction case, we are interested in these mutual-interaction tensors, which contain contributions linear in Re.

As follows from Eq. (4.9), we have in that group the mutual-interaction tensors, being of the leading order in Re. Their indices fulfil the relation

$$(4.15)\qquad |l_1+l_2-l_3+2m| = 1.$$

Hence, the respective contributions to the tensors $\mathbf{T}^{l_2m_2}_{l_1m_1,l_3m_3}$ are given by

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$$(4.16)\qquad \mathbf{T}^{l_2m_2}_{l_1m_1,l_3m_3} = \sum_{m=0}^{\infty} \mathbf{T}^{l_2m_2,m}_{l_1m_1,l_3m_3} = \frac{\sqrt{\pi}}{8a\mu\Gamma(l_1+3/2)\Gamma(l_2+3/2)}\left(\frac{a}{R_{jk}}\right)^{l_1+l_2+1}$$
$$\times i^{l_1-l_2-l_3+3} R_{jk}\alpha \int d\hat{\mathbf{k}}(\mathbf{1}-\hat{\mathbf{k}}\hat{\mathbf{k}}) Y_{l_1}^{-m_1} Y_{l_2}^{m_2} Y_{l_3}^{-m_3} \cos(\hat{\mathbf{U}},\hat{\mathbf{k}})$$
$$\times \sum_{m=0}^{\infty} \frac{(l_1+l_2+2m+1/2)\Gamma(l_1+l_2+m+1/2)}{m!}\,\frac{\Gamma(\zeta)}{\Gamma(Z+1)}$$
$$\times F_4\left[-m, l_1+l_2+m+1/2;\ l_1+3/2, l_2+3/2;\ \left(\frac{a}{R_{jk}}\right)^2, \left(\frac{a}{R_{jk}}\right)^2\right],$$

for the sets of indices $|l_1+l_2+2m-l_3| = 1$.

Taking into account the properties of the tensors **K**, one can deduce that the following relations should be fulfilled:

$$(4.17)\qquad l_1+l_2+l_3 \pm 1 = 2n, \qquad l_1+l_2-l_3 \mp 1 \geq -2.$$

Thus the hydrodynamic interactions linear in Re are described in terms of the following tensors:

(4.18)

(i) $\mathbf{T}^{l_2m_2}_{l_1m_1,l_3m_3} = \mathbf{T}^{l_2m_2,0}_{l_1m_1,l_3m_3}$ for $l_1+l_2-l_3 = 1$,

(ii) $\mathbf{T}^{l_2m_2}_{l_1m_1,l_3m_3} = \sum_{m=0,1} \mathbf{T}^{l_2m_2,m}_{l_1m_1,l_3m_3}$ for $l_1+l_2-l_3 = -1$,

(iii) $\mathbf{T}^{l_2m_2}_{l_1m_1,l_3m_3} = \sum_{m=1,2} \mathbf{T}^{l_2m_2,m}_{l_1m_1,l_3m_3}$ for $l_1+l_2-l_3 = -3$,

(iv) for the sets of the indices $l_1+l_2-l_3 \neq 1, -1, -3,$ the tensors $\mathbf{T}^{l_2m_2}_{l_1m_1,l_3m_3}$ do not contain terms linear in Re.

On the basis of Eq. (4.8) we see that tensors linear in Re behave as follows:

(4.19)

(i) they are of the leading order of $(a/R_{jk})^{l_1+l_2+1}\cdot \mathrm{Re}_m$;

(ii) in addition to the leading order terms, the tensors with $m=1$ contain terms of the order of $(a/R_{jk})^{l_1+l_2+3}\cdot \mathrm{Re}_m$, and the tensors with $m=2$ contain, respectively, terms of the order of $(a/R_{jk})^{l_1+l_2+5}\cdot \mathrm{Re}_m$.

We note that, similarly to the case of the Stokes hydrodynamic interactions, the hydrodynamic interactions linear in Re are described by means of a finite number of the tensors $\mathbf{T}^{l_2m_2,m}_{l_1m_1,l_3m_3}(m=0,1,2)$.

Similarly to the self-interaction tensors, we have also the second source of the linearity in Re. Namely, expanding the functions $F_{l_1l_2,l_1+l_2}$, and $F_{l_1l_2,l_1+l_2+2}$

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into a power series in Re, we obtain the linear contributions for the case of the function $F_{00,0}$. The resulting expression for the tensor $\mathbf{T}^{00}_{00,00}$ is of the form

$$(4.20)\qquad \mathbf{T}^{00}_{00,00} Y^0_0 = \frac{1}{6\pi\mu R_{jk}} \left[\mathbf{1} - \frac{3}{8\pi} \mathrm{Re}_m \int d\hat{\mathbf{k}} (\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}) |\cos(\widehat{\mathbf{U}}, \hat{\mathbf{k}})| \right] + \ldots .$$

We see that the contribution resembles that for the self-interaction tensors; however, the characteristic length is now the distance between the centres of two spheres, instead of the radius of the sphere.

5. Concluding remarks

Once the described properties of the hydrodynamic interaction tensors are known, the basic set of algebraic equations (3.2) can be analysed within the assumed approximation with respect to Re and σ. Here, as an example, we consider the relation of the forces $\mathbf{F}_j$, exerted on the spheres by the fluid, to the velocity of the fluid at infinity $\mathbf{U}$, retaining contributions up to $0(\mathrm{Re})$, and $0(\sigma^0)$. To that end, the quantities $\mathbf{F}_j$ and $\mathbf{U}$ are expressed in terms of the expansion coefficients $\mathbf{f}_{j,lm}$ and $\mathbf{V}_{j,lm}$, respectively:

$$(5.1)\qquad \begin{aligned} \mathbf{F}_j &= -\int \nabla \cdot \mathbf{P}(\mathbf{r}_j)\, d\mathbf{r}_j = -\mathbf{f}_{j,00}, \\ \mathbf{V}_{j,lm} &= -\mathbf{v}^0_{j,lm} = -\mathbf{U}, \qquad l = 0, \\ &= \mathbf{0}, \qquad l > 1. \end{aligned}$$

Thus the set of algebraic equations leads to the following form of the relations between the forces $\mathbf{F}_j$ and the fluid velocity $\mathbf{U}$:

$$(5.2)\qquad \mathbf{F}_j = \widetilde{\mathbf{T}}^{00}_{00}(\mathbf{0}_j) \cdot \mathbf{U} - \widetilde{\mathbf{T}}^{00}_{00}(\mathbf{0}_j) \cdot \sum_{k \neq j} \mathbf{T}^{00}_{00}(\mathbf{R}_{jk}) \cdot \widetilde{\mathbf{T}}^{00}_{00}(\mathbf{0}_k) \cdot \mathbf{U} + \ldots ,$$

where $\widetilde{\mathbf{T}}^{00}_{00}(\mathbf{0}_j)$ is the inverse self-interaction tensor [11].

It follows from Eqs. (3.4), (4.10) and (4.16) that the mutual-interaction tensor $\mathbf{T}^{00}_{00}(\mathbf{R}_{jk}, \mathrm{Re})$ can be written in the following form:

$$(5.3)\qquad \mathbf{T}^{00}_{00}(\mathbf{R}_{jk}, \mathrm{Re}) = \sum_{l_3 m_3} \mathbf{T}^{00}_{00,l_3 m_3}(|R_{jk}|, \mathrm{Re}) Y^{m_3}_{l_3}(\Theta_{jk}, \Phi_{jk}),$$

where, in the range of $\mathrm{Re}_m < 1$, the terms of the zero order with respect to Re, coming from $\mathbf{T}^{00}_{00,00}$, and $\mathbf{T}^{00}_{00,2m}$, describe the Stokes hydrodynamic interactions, whereas the terms of the order of Re, coming from $\mathbf{T}^{00}_{00,00}$, and $\mathbf{T}^{00}_{00,lm}$, $l \neq 0, 2$

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describe the hydrodynamic interactions linear in Re. The Stokes contribution to the tensors $\mathbf{T}^{00}_{00}(\mathbf{R}_{jk}, \mathrm{Re})$, calculated using Eq. (4.10), reads:

$$\begin{aligned} \mathbf{T}^{00}_{00}|_{\text{Stokes}} &= \frac{1}{8\pi\mu R_{jk}}\left[\mathbf{1} + \mathbf{e}_{jk}\mathbf{e}_{jk} + \frac{2a^2}{R^2_{jk}}\left(\frac{1}{3} - \mathbf{e}_{jk}\mathbf{e}_{jk}\right)\right], \\ \mathbf{e}_{jk} &= \mathbf{R}_{jk}/|\mathbf{R}_{jk}|. \end{aligned} \tag{5.4}$$

The contributions to $\mathbf{T}^{00}_{00,00}$ linear in Re are given by Eq. (4.20). It follows from Eq. (4.8) that the contributions to the tensor $\mathbf{T}^{00}_{00}(\mathbf{R}_{jk})$, being of the leading order in Re, are equal to

$$\begin{aligned} \mathbf{T}^{00}_{00}|_{\text{leading order Re}} = -\frac{\mathrm{Re}\widehat{U}}{4a\mu\pi}\sum_{\substack{l_3=1\\ l_3\neq 2}}^{\infty} Y^{m_3}_{l_3}(\Omega_{jk})(i)^{-l_3}\int d\hat{\mathbf{k}}(\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}})Y^{-m_3}_{l_3}Y^0_1 \\ \times\frac{1}{\sqrt{3}}\cdot\sum_{m=0}^{\infty}\frac{\left(2m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)\Gamma(\zeta)}{m!\,\Gamma(Z+1)}i^{|2m-l_3|} \\ \times F_4\left[-m, m+\frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \left(\frac{a}{R_{jk}}\right)^2, \left(\frac{a}{R_{jk}}\right)^2\right], \end{aligned} \tag{5.5}$$

where $\zeta = \min(2m + 1/2, l_3 + 1/2)$,

$$Z = \max(2m + 1/2, l_3 + 1/2), \quad \mathbf{U}(0, 0, U).$$

The integral

$$\int d\hat{\mathbf{k}}(\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}})Y^{-m_3}_{l_3}Y^0_1 = i^{1+l_3}2\sqrt{\pi}\mathbf{K}^{10}_{00,l_3m_3}$$

is different from zero only for $l_3 = 1, 3$ [9].

Hence it follows from Eq. (4.18) that the considered contributions of the order of Re to the tensors $\mathbf{T}^{00}_{00}$ are given by:

(i) for the case of $l_3 = 1$:

$$\mathbf{T}^{00}_{00,1m_3} = \mathbf{T}^{00,0}_{00,1m_3} + \mathbf{T}^{00,1}_{00,1m_3}, \tag{5.6}$$

(ii) for the case of $l_3 = 3$:

$$\mathbf{T}^{00}_{00,3m_3} = \mathbf{T}^{00,1}_{00,3m_3} + \mathbf{T}^{00,2}_{00,3m_3}.$$

Using the formula (5.5), we obtain:

$$\begin{aligned} \mathbf{T}^{00}_{00,1m_3} &= \frac{\widehat{U}}{2\sqrt{3}a\mu}\mathrm{Re}\,\mathbf{K}^{10}_{00,1m_3} + \text{h.o.t.}, \\ \mathbf{T}^{00}_{00,3m_3} &= \frac{\widehat{U}}{8\sqrt{3}a\mu}\mathrm{Re}\,\mathbf{K}^{10}_{00,3m_3} + \text{h.o.t.}. \end{aligned} \tag{5.7}$$

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As an example of the hydrodynamic interactions linear in Re, we consider the linear contributions to the lift forces $\mathbf{F}_j^L$, $\mathbf{F}_j^L \cdot \mathbf{U} = 0$, $j = 1, 2$ exerted by the fluid on two rigid spheres, being at rest in an unbounded fluid, having at infinity the streaming velocity $\mathbf{U}(0, 0, U)$. The line joining the centres of the spheres is perpendicular to the streaming velocity $\mathbf{U}$.

Taking into acount the properties of the tensors $\mathbf{T}_{00}^{00}(\mathbf{0}_j)$, and $\mathbf{T}_{00}^{00}(\mathbf{R}_{jk})$, we conclude that the linear contributions to the lift forces, considered up to σ^0, depend only on the Stokes contributions to the tensor $\widetilde{\mathbf{T}}_{00}^{00}(\mathbf{0}_j)$

$$\mathbf{F}_j^L = -(6\pi\mu a)^2 \sum_{k\neq j} \mathbf{T}_{00}^{00}(\mathbf{R}_{jk}, \mathrm{Re}) \cdot \mathbf{U}, \qquad k = 1, 2, \qquad j = 1, 2. \tag{5.8}$$

To calculate the relevant components of the tensor $\mathbf{T}_{00}^{00}(\mathbf{R}_{12}, \mathrm{Re})$, and $\mathbf{T}_{00}^{00}(\mathbf{R}_{21}, \mathrm{Re})$, we specify the relative distances between the spheres in terms of the vectors $\mathbf{R}_{12}(R_{12}, \Theta_{12} = 90°, \Phi_{12} = 0°)$ and, respectively, $\mathbf{R}_{21}(R_{21}, \Theta_{21} = 90°, \Phi_{21} = 180°)$.

Using Eq. (5.7), we arrive at

$$\begin{aligned} \mathbf{F}_1^L &= -6\pi\mu a U \left[\frac{3}{16}\mathrm{Re} + \mathrm{h.o.t.}\right], \\ \mathbf{F}_2^L &= -\mathbf{F}_1^L . \end{aligned} \tag{5.9}$$

The result (5.9) is obtained under the three assumptions:

$$a/R_{12} < \frac{1}{2}, \qquad \frac{aU}{\nu} < 1, \qquad \frac{R_{12}U}{\nu} < 1.$$

Thus it concerns the case where the distance between the spheres is finite, and where the inertia effects are weak.

Hence, for the considered case of two spheres, the Smoluchowski's method of description of the hydrodynamic interactions, used by OSEEN [3], the method of matched asymptotic expansions, developed by KANEDA and ISHII [4], and the present method lead to the same results for the lift forces, with accuracy up to the terms of the order of σ^0 and Re.

We see that the relations of the forces $\mathbf{F}_j$ to the fluid velocity $\mathbf{U}$ in the approximation considered are not influenced by the lack of the pairwise additivity of the hydrodynamic interactions [6]. To discuss this influence, the basic set of algebraic equations will be regarded in the forthcoming paper to $\mathbf{0}(\mathrm{Re})$, and $\mathbf{0}(\sigma^2)$.

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Transfer matrix for random system of elastic layers

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TWO TYPES of elementary cells are placed at random in a long chain of cells. Harmonic scalar wave (shear wave, Mode 3) propagates along the chain. The transfer matrix equals the product of the transfer matrices for the elementary cells. Assuming that the probability of finding the cell of definite type at the n-th place is given, the probability of the particular value of the transfer matrix for the whole chain is calculated. In computer simulation the transfer matrix for the chain consisting of 170 cells was calculated for several different probabilities.

1. Homogeneous layers

CONSIDER THE SYSTEM of N, in general different, homogeneous elastic layers, Fig. 1. The layer situated between x_k and x_{k+1} is identified by the natural

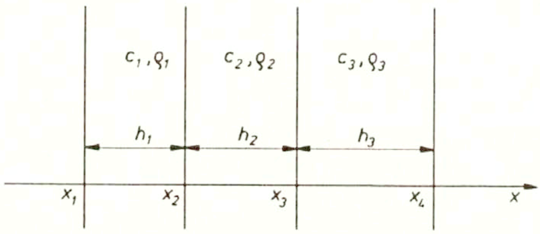

FIG. 1.

number k, $k = 1, 2, 3, \dots N$. The density, thickness and elastic wave propagation speed of the k-th layer are denoted by ρ_k, h_k and c_k, respectively. In this layer two sinusoidal waves of frequency ω propagate, one of amplitude A_k in the x direction, and an other one of amplitude B_k in the $-x$ direction. The problem of such waves was considered e.g. in [1–3]. The displacement in the layer k is

$$(1.1) \qquad u_k = A_k \exp i\omega\left[t - (x - x_k)/c_k\right] + B_k \exp i\omega\left[t + (x - x_k)/c_k\right],$$

where t is time, $x_k \le x \le x_{k+1}$, and c_k is the wave speed in the k-th layer. The displacement u_k satisfies the equation of motion

$$(1.2) \qquad c_k^2 u_{k,xx} = u_{k,tt}\,.$$

At both sides of the boundary between layers k and $k+1$ both the displacement and the stress vector have the same values. This continuity leads to the following

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relation between the wave amplitudes in the layer k and wave amplitudes in the layer $k+1$

$$\begin{bmatrix} A_{k+1} \\ B_{k+1} \end{bmatrix} = L_k \begin{bmatrix} A_k \\ B_k \end{bmatrix}, \tag{1.3}$$

where

$$L_k = \begin{bmatrix} P_k & Q_k \\ R_k & S_k \end{bmatrix}, \tag{1.4}$$

$$\begin{aligned} P_k &= \frac{1}{2}(1+\kappa_k)\exp(-i\alpha_k), \\ Q_k &= \frac{1}{2}(1-\kappa_k)\exp(i\alpha_k), \end{aligned} \tag{1.5}$$

$$R_k = \overline{Q_k}, \qquad S_k = \overline{P_k}, \tag{1.6}$$

$$\alpha_k = \omega h_k / c_k, \qquad \kappa_k = \rho_k c_k / \rho_{k+1} c_{k+1}, \tag{1.7}$$

$$h_k = x_{k+1} - x_k . \tag{1.8}$$

The symmetry (1.6) will be called w-symmetry. A product of w-symmetric matrices is w-symmetric.

The set of elastic layers defined above will be called a chain. Assume that there exists an access to the ends of the chain. Some external time-dependent forces act at the ends of the chain. In this case the displacement at one end of the chain may be an arbitrary function of time. The displacement of the other end may then be calculated, taking into account the total number of layers, their dimensions and elastic properties. In particular, the displacement at one end of the system may be given in advance as a harmonic function of arbitrary frequency ω.

The transfer matrix L_k for the layer k is a function of the propagation speed and thickness of the layer k, and additionally of the propagation speed in the layer $k+1$, cf. Eqs. (1.7). This fact makes the numerical calculations rather awkward. In order to remove the dependence on $k+1$, add after each layer a virtual layer of zero thickness and of a fixed propagation speed and density. Now between each two neighbouring layers a virtual layer of zero thickness is situated. It is known that such a layer of zero thickness does not change the dynamics of the chain.

The virtual layer and the k-th layer constitute the elementary cell. The cells will be identified by the numbers of the layer (real, not virtual) $k = 1, 2, 3, 4, \ldots$. Transfer matrix M_k for one cell is a product of the transfer matrices for the layers constituting the cell, and therefore it is w-symmetric.

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2. Random distribution of layers

Assume that there are two kinds of cells denoted by α, β. Consider the chain consisting of N cells, made of N_α cells of kind α, and N_β cells of kind β, $N_\alpha + N_\beta = N$. The distribution of cells is not deterministic, but random. We face two different cells distributed randomly, e.g. $[\alpha, \alpha, \beta, \alpha, \beta, \beta, \ldots, \beta, \beta, \alpha]$. The particular cells are repeated several times at different places. Further analysis is based on the excellent paper by H. SMITH, who considered a chain of randomly distributed interacting masses, [4]. Closely connected with the present problems are the papers [5–8].

The w-symmetric transfer matrices for the cells α and β are

$$(2.1) \qquad M_\alpha = \begin{bmatrix} P_\alpha & Q_\alpha \\ R_\alpha & S_\alpha \end{bmatrix}, \qquad M_\beta = \begin{bmatrix} P_\beta & Q_\beta \\ R_\beta & S_\beta \end{bmatrix}.$$

They are the products of the transfer matrices of the layers constituting the cell. We assume that the virtual layer was taken into account, therefore M_α depends on the cell α only, and does not depend on the neighbouring cells. M_β is characterized by an analogous property.

The two above kinds of cells are distributed over N places. The probability that the place k is occupied by the cell of type α will be denoted by $p_{k\alpha}$, and the probability that the place k is occupied by the cell of type β will be denoted by $p_{k\beta}$. Obviously $p_{k\alpha} + p_{k\beta} = 1$. In the special case of homogeneous distribution, the probability is independent of k and the index k may be omitted, $p_{k\alpha} = p_\alpha$, $p_{k\beta} = p_\beta$.

Consider a large number of chains, each consisting of N cells with random distribution of the two above kinds of cells. Calculate for each particular chain the matrices K defined by the relation

$$(2.2) \qquad K_k = \begin{bmatrix} P_{k-1} & Q_{k-1} \\ R_{k-1} & S_{k-1} \end{bmatrix} \cdots \begin{bmatrix} P_3 & Q_3 \\ R_3 & S_3 \end{bmatrix} \begin{bmatrix} P_2 & Q_2 \\ R_2 & S_2 \end{bmatrix} \begin{bmatrix} P_1 & Q_1 \\ R_1 & S_1 \end{bmatrix},$$

$k = 1, 2, 3, \ldots, N$. In general, for this chain we face N different complex-valued 2×2 matrices. Each transfer matrix at the right-hand side of Eq. (2.2) equals either M_α or M_β. For each chain the matrix K_k is in general another function of frequency ω.

Note that the matrix K_k is w-symmetric. Therefore it is completely characterized by its two components $(K_k)_{11}$ and $(K_k)_{12}$. These two complex numbers are equivalent to four real numbers. The complex-valued matrix K_k may therefore be represented by a point in the 4-dimensional real space R_4.

Concentrate attention on the fixed place k (fixed cell k) in a set of chains consisting of N cells. In the situation considered in this chapter the components P_k, Q_k, R_k, S_k of the transfer matrix M_k are equal either to P_α, Q_α, R_α, S_α if the k-th cell is a cell of type α, or to P_β, Q_β, R_β, S_β, if the k-th cell is the

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cell of type β. In general, for each chain the matrix K_k has another value, since K_k is a product of different matrices for different chains. Consider an arbitrary fixed region dv_k in the 4-dimensional space R_4 mentioned above. The measure (volume) of the region dv_k will be denoted by dV_k. For some chains the point K_k is situated inside dv_k, for other chains – outside dv_k. Define the probability density distribution $w_k(K_k)$ of the points K_k in the space R_4

$$w_k(K_k)dV_k = \frac{\text{number of chains for which } K_k \in dv_k}{\text{total number of chains}}. \tag{2.3}$$

In order to derive the equations for $w_k(K_k)$ make a temporary assumption, that in all chains the k-th place is occupied by a cell of the type α, $M_k = M_\alpha$. Then, in accord with Eq. (2.2), the matrix K_{k+1} (point in R_4) is a definite function of K_k (point in R_4)

$$K_{k+1} = \begin{bmatrix} P_\alpha & Q_\alpha \\ R_\alpha & S_\alpha \end{bmatrix} K_k, \tag{2.4}$$

$$K_k = \begin{bmatrix} S_\alpha & -Q_\alpha \\ -R_\alpha & P_\alpha \end{bmatrix} K_{k+1}. \tag{2.5}$$

The second relation is the inverse of the first one. Denote by dv_{k+1} the region into which the above function transforms the region dv_k. In further calculations we need the expression for the ratio of their measures (4-dimensional volumes) dV_{k+1}/dV_k. Derivation in complex variables is difficult, therefore we prefer to replace each complex variable by two real variables. Write the relation (2.4) in the form

$$\begin{bmatrix} X+iY & Z+iT \\ Z-iT & X-iY \end{bmatrix} = \begin{bmatrix} A+iB & C+iD \\ C-iD & A-iB \end{bmatrix} \begin{bmatrix} x_1+ix_2 & x_3+ix_4 \\ x_3-ix_4 & x_1-ix_2 \end{bmatrix}.$$

Obvious notation introduced here serves only for the derivation of the ratio of dV_k and dV_{k+1} (Eq. (2.7)). The above relation, equivalent to Eq. (2.4), is equivalent to 4 real relations

$$\begin{aligned} X &= Ax - By + Cz + Dt, \\ Y &= Ay + Bx - Ct + Dz, \\ Z &= Az - Bt + Cx + Dy, \\ T &= At + Bz - Cy + Dx, \end{aligned} \tag{2.6}$$

defining a transformation of R_4 into R_4. The corresponding Jacobian of the transformation is

$$\Delta = \begin{vmatrix} A & -B & C & D \\ B & A & D & -C \\ C & D & A & -B \\ D & -C & B & A \end{vmatrix}.$$

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Calculation of the determinant leads to the expression

$$\Delta = A^4 + B^4 + C^4 + D^4 + 2A^2B^2 - 2A^2C^2 - 2A^2D^2 - 2B^2C^2 - 2B^2D^2 + 2C^2D^2.$$

The above expression may be transformed into the expression

$$\Delta = (P_\alpha S_\alpha + Q_\alpha R_\alpha)^2. \tag{2.7}$$

Take dv_k to be a 4-dimensional box with infinitesimal edges dx_1, dx_2, dx_3, dx_4 parallel to the axes $\mathrm{Re}(K_{11})$, $\mathrm{Im}(K_{11})$, $\mathrm{Re}(K_{12})$, $\mathrm{Im}(K_{12})$. Volume dV_k of the box dv_k equals $dx_1\,dx_2\,dx_3\,dx_4$. The function (2.4) transforms dv_k into a parallelepiped dv_{k+1} of volume dV_{k+1}. The ratio of the volumes equals the Jacobian of the transformation (2.6). This ratio dV_{k+1}/dV_k is therefore given by the formula

$$\frac{dV_{k+1}}{dV_k} = (P_\alpha S_\alpha + Q_\alpha R_\alpha)^2. \tag{2.8}$$

In the special case, when in each chain the k-th cell is of type α, the transformation is not stochastic but deterministic. If the point K_k is inside (outside) the region dv_k, then the point K_{k+1} is situated inside (outside) dv_{k+1}. Therefore the probability of finding the point K_{k+1} inside the region dv_{k+1} equals the probability of finding the point K_k inside the region dv_k

$$w_{k+1}(K_{k+1})\,dV_{k+1} = w_k(K_k)\,dV_k.$$

Note that K_k and K_{k+1} are related by Eq. (2.5), therefore

$$w_{k+1}(K_{k+1})\,dV_{k+1} = w_k\left(\begin{bmatrix} S_\alpha & -Q_\alpha \\ -R_\alpha & P_\alpha \end{bmatrix} K_{k+1}\right) dV_k.$$

Since K_{k+1} is in fact an independent variable, in what follows we omit the subscript $k+1$ and write H instead of K_{k+1}. Take into account the expression (2.8) for the ratio dV_{k+1}/dV_k. There follows the relation between the probability densities w_k and w_{k+1}

$$w_{k+1}(H) = (P_\alpha S_\alpha + Q_\alpha R_\alpha)^2\, w_k\left(\begin{bmatrix} S_\alpha & -Q_\alpha \\ -R_\alpha & P_\alpha \end{bmatrix} H\right). \tag{2.9}$$

The above calculations were performed assuming that at the k-th place a cell of type α is situated. If at the k-th place the cell of type β was situated, an analogous formula would be obtained, namely

$$w_{k+1}(H) = (P_\beta S_\beta + Q_\beta R_\beta)^2\, w_k\left(\begin{bmatrix} S_\beta & -Q_\beta \\ -R_\beta & P_\beta \end{bmatrix} H\right). \tag{2.9'}$$

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Therefore the general formula for the situation, when in all realizations at the k-th place is located a cell of the same type is

$$w_{k+1}(H) = (P_k S_k + Q_k R_k)^2 w_k \left(\begin{bmatrix} S_k & -Q_k \\ -R_k & P_k \end{bmatrix} H \right). \tag{2.10}$$

For the parameters with subscript k the data should be taken for the cell located at the k-th place in all realizations, either α or β.

Actually the k-th place may be occupied either by the cell of type α with probability $p_{k\alpha}$ (and then the transfer matrix is M_α), or by the cell of type β with probability $p_{k\beta}$ (and then the transfer matrix is M_β), $p_{k\alpha} + p_{k\beta} = 1$. The actual relation between $w_{k+1}(H)$ and $w_k(H)$ is therefore

$$\begin{aligned} w_{k+1}(H) = p_{k\alpha} (P_\alpha S_\alpha + Q_\alpha R_\alpha)^2 w_k \left(\begin{bmatrix} S_\alpha & Q_\alpha \\ -R_\alpha & P_\alpha \end{bmatrix} H \right) \\ + p_{k\beta} (P_\beta S_\beta + Q_\beta R_\beta)^2 w_k \left(\begin{bmatrix} S_\beta & Q_\beta \\ -R_\beta & P_\beta \end{bmatrix} H \right). \end{aligned} \tag{2.11}$$

Since in each realization $K_1 = M_\alpha$ or $K_1 = M_\beta$, cf. Eq. (2.1), therefore w_1 is not equal zero only if $K_1 = M_\alpha$ or $K_1 = M_\beta$. Taking into account the fact that the integral over the whole 4-dimensional R_4 must be equal 1, and that the probabilities $p_{1\alpha}$ and $p_{1\beta}$ are known, we have in the shorthand notation

$$w_1(H) = p_{1\alpha}\delta_4(H - M_\alpha) + p_{1\beta}\delta_4(H - M_\beta), \tag{2.12}$$

where δ^4 is the 4-dimensional Dirac delta. In accord with the above definition of a 4-dimensional volume dV_k, we have

$$\delta_4(H) = \delta(\mathrm{Re}H_{11})\,\delta(\mathrm{Im}H_{11})\,\delta(\mathrm{Re}H_{12})\,\delta(\mathrm{Im}H_{12}), \tag{2.13}$$

where δ is the one-dimensional Dirac delta. Formulae (2.11) and (2.12) completely determine the probability distribution $w_k(K)$ for each k. Note that $w_k(K_k)$ for each k is not a function, but a generalized function (distribution). Almost everywhere there is $w_k(K_k) = 0$, in at most 2^k points the function differs from zero. The calculations will be confined to finding the support of $w_k(K_k)$.

Perform the numerical calculations for the layers α and β characterized by the following wave speeds, densities and thicknesses:

$$c_\alpha = 1, \quad c_\beta = 2, \quad c_v = 1, \quad \rho_\alpha = \rho_\beta = \rho_v = 1, \quad h_\alpha = h_\beta = 1. \tag{2.14}$$

The propagation speed and density of the virtual layer are denoted by c_v, ρ_v, respectively. The following values of the components of the corresponding transfer matrices M_α, M_β for the two kinds of cells have been calculated

$$\begin{aligned} \mathrm{Re}(M_\alpha)_{11} &= 0.995, & \mathrm{Im}(M_\alpha)_{11} &= -0.100, \\ \mathrm{Re}(M_\alpha)_{12} &= 0, & \mathrm{Im}(M_\alpha)_{12} &= 0, \\ \mathrm{Re}(M_\beta)_{11} &= 0.999, & \mathrm{Im}(M_\beta)_{11} &= 0.062, \\ \mathrm{Re}(M_\beta)_{12} &= 0, & \mathrm{Im}(M_\beta)_{12} &= -0.037. \end{aligned} \tag{2.15}$$

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The remaining components are determined by the w-symmetry.

Fix the number of cells in each chain $N = 170$. Consider the harmonic wave of frequency $\omega = 0.1$ propagating across the random chain consisting of 170 cells. The probability p_α, p_β of finding at the n-th place the cell of type α or β is assumed to be independent of n, $p_\beta = 1 - p_\alpha$. Calculate the transfer matrices K_k for particular realizations of the chain.

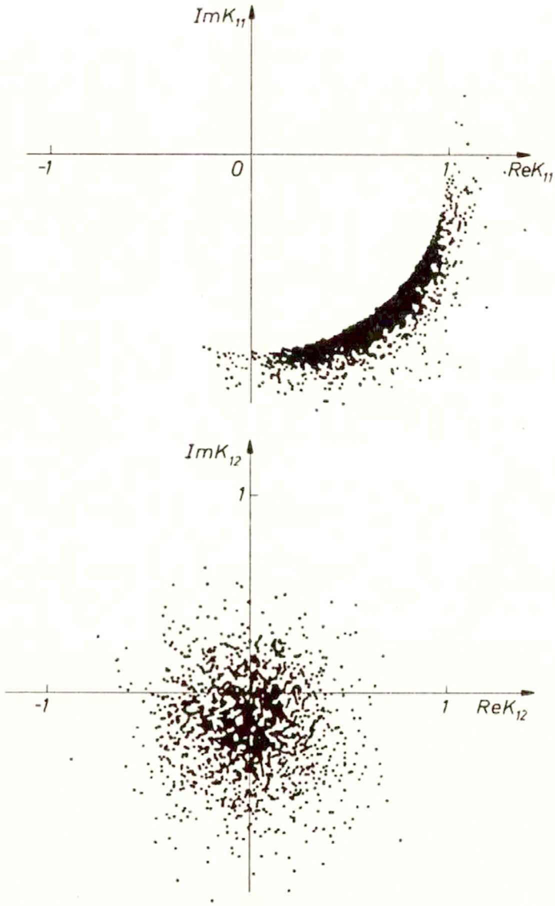

FIG. 2.

Consider first the case, when the probability of finding at the n-th place the cell of type α equals the probability of finding the cell of type β, $p_\alpha = p_\beta = 0.5$. Complex component G_{11} of the global transfer matrix $G = K_{170}$ for 2000 realizations is given in Fig. 2a. The corresponding complex component G_{12} of the global transfer matrix $G = K_{170}$ is shown in Fig. 2b. The average values are

$$\langle G_{11} \rangle = 0.637 - 0.792i, \qquad \langle G_1 2 \rangle = 0.031 - 0.183i.$$

The remaining components are determined by the w-symmetry. It is seen that

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the distribution of G is non-uniform. In some regions the density of points is large, in other regions it is small. If all cells are of the same type, only a single point (G_{11}, G_{12}) is obtained. There is

$$G_{11} = -0.602 - 0.998i, \quad G_{12} = -0.5989i \quad \text{if} \quad p_\alpha = 0, \quad p_\beta = 1,$$
$$G_{11} = -0.275 - 0.961i, \quad G_{12} = 0 \quad \text{if} \quad p_\alpha = 1, \quad p_\beta = 0.$$

Take in turn $p_\alpha = 0.75$, $p_\beta = 0.25$. The corresponding values of G are shown in Fig. 3. The average values are

$$< G_{11} >= -0.926 - 0.380i, \qquad < G_{12} >= 0.001 - 0.039i.$$

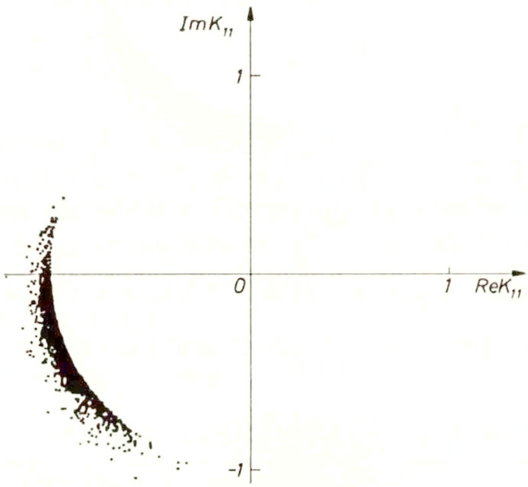

FIG. 3.

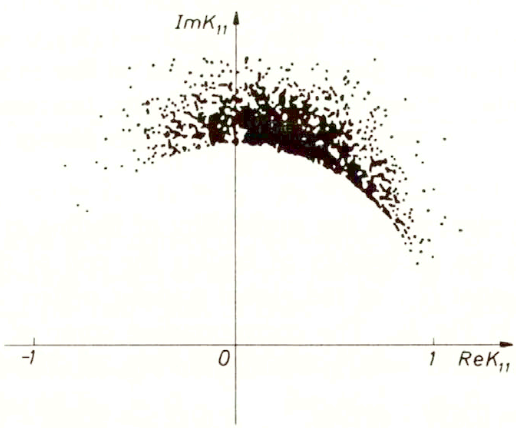

FIG. 4.

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For $p_\alpha = 0.25$, $p_\beta = 0.75$ the realizations are shown in Fig. 4. The average values are

$$<G_{11}>= 0.257 + 1.05i, \qquad <G_{12}>= 0.004 + 0.419i.$$

The data for $p_\alpha = 0.05$, $p_\beta = 0.95$ are presented in Fig. 5. It is seen that the distribution of points is similar to that given in Fig. 3. The same average values may be obtained for different densities of the cells α.

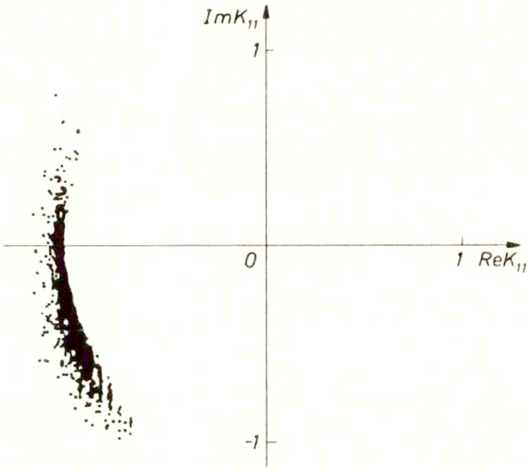

FIG. 5.

The transparency τ is the ratio of the transmitted energy flux to the incident energy flux. The average transparency has been plotted in Fig. 6 as a function of p_α. It is seen that the same average value of τ may be obtained for different p_α.

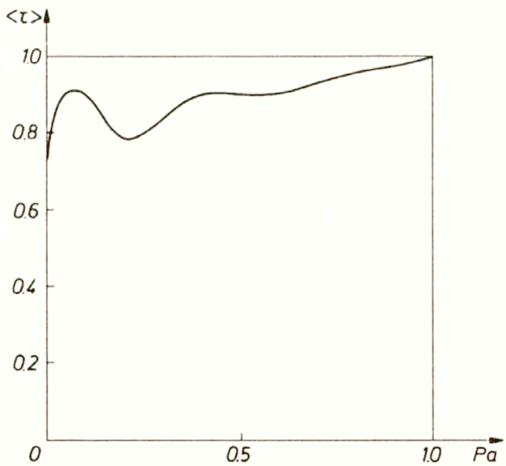

FIG. 6.

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Quasi-isobaric solutions of the Hiemenz equation

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SOLUTIONS $U(x)$, obtained from the ordinary differential Hiemenz Equation of the third order, determine particular solutions of the Navier–Stokes Equations for stationary, plane symmetric flows of incompressible viscous liquid, between parallel plane surfaces, on which the pressure in quasi-isobaric case remains constant. Analytically continuated power series, with estimation of the truncation errors, have been applied to compute $U(x)$ with boundary conditions $C_0 = U(0)$, $C_1 = U'(0)$, $C_2 = U''(0)$. Solutions $U(x)$ are invariant with respect to translation and affine transformation, by means of which they may be divided into classes of similar functions with parameters distributed over two-dimensional surfaces in the three-dimensional space C_0, C_1, C_2. The surface $\mathcal{R} = 0$, where $\mathcal{R} = C_1^3 + C_2^2 - C_0C_1C_2$, represents unstable Riabouchinsky's solutions $U(x; C_0, C_1, C_2, 0) = -\lambda(1 \mp Be^{-\lambda x})$ and separates the parameters defining other classes of similarity. Monotonous solutions $U(x; C_0, C_1, C_2, 0)$ with $\mathcal{R} < 0$ seem to be not interesting for hydrodynamics, and thus the main attention is paid to solutions with $\mathcal{R} > 0$. They may be applied to solve some hydrodynamical problems mentioned above, but they may be also helpful to study solutions $U(x; C_0, C_1, C_2, \sigma)$ with $\sigma \neq 0$, describing more general flows between parallel surfaces with pressure depending on two variables x, y.

1. Hiemenz Equation

CONSIDERING STAGNATION POINTS in laminar, stationary, plane symmetric flows of incompressible, viscous liquids, HIEMENZ [1] found in 1911 a particular exact solution of the Navier–Stokes equations. By assuming nondimensional parameters: the velocity components $u(x,y)$, $v(x,y)$, the pressure $p(x,y)$ and the stream function $\Psi(x,y)$ in the form:

$$\begin{aligned} u &= U(x), \qquad \nu = -yU'(x), \qquad \Psi = yU(x), \\ p &= U'(x) - U^2(x)/2 - \sigma \cdot y^2/2 + \text{const}, \end{aligned} \tag{1.1}$$

we may satisfy the N-S equations if $U(x)$ fulfills the ordinary differential Hiemenz equation

$$U''' - U \cdot U'' + U' \cdot U' = \sigma, \qquad \text{where} \qquad \sigma = \text{const}. \tag{1.2}$$

For the considered case of flow in a half-space bounded by a plane, rigid wall, Hiemenz introduced the boundary conditions $U(0) = U'(0) = 0$ on the impermeable, rigid surface and the asymptotic condition $U'(\infty) = \text{const}$ at infinity.

The four-parameter solutions $U(x; C_0, C_1, C_2, \sigma)$ of the Hiemenz equation (1.2) may also have other hydrodynamic applications, fulfilling other boundary conditions. RIABOUCHINSKY [2] in 1924 and CRANE [3] in 1970 considered the flow with a stagnation point at an extensible, isobaric surface for $\sigma = 0$ with

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boundary conditions $U(0) = 0$, $U'(0) \neq 0$. From 1953 other solutions of Eq. (1.2) were introduced to describe flows between parallel, plane surfaces: rigid and permeable [4–8], extensible and impermeable [9], or the mixed conditions with tangential slip at the permeable boundary [10, 11].

There may be many different ways of selecting the arbitrary constants C_0, C_1, C_2, which determine the same function $U(x)$. Here, in further considerations, they will be determined by the initial conditions

$$C_0 = U(0), \qquad C_1 = U'(0), \qquad C_2 = U''(0). \tag{1.3}$$

Not all solutions $U(x)$ of Eq. (1.2) may describe realizable flows. To this aim they should fulfill hydrodynamic boundary conditions on the wall, $U(*) = 0$ for impermeability, $U'(*) = 0$ for inextensibility, $U''(*) = 0$ for no tangential stress (free surface), and possibly the asymptotic condition: $U''(\infty) = 0$. So, for hydrodynamic application the solutions $U(x)$ with some zero values are most interesting, but other solutions $U(x; C_0, C_1, C_2, 0)$ with any arbitrary values C_0, C_1, C_2 will be also considered here.

To find $U(x)$, finite difference numerical methods were mainly applied [1, 4, 6–10], and in [5] the power series expansion was also used. However, finite differences are not always sufficiently accurate and the accuracy of power series expansion is limited by its radius of convergence. The evaluation of errors related to the rules of convergence allowed here to apply the truncated power series not only within their radii of convergence, but also in a larger region of their analytic continuations. Taking into account the truncation errors, an algorithm with a numerical program allowing to compute $U(x)$ for a large range of x has been prepared here.

Mainly the particular cases of solutions $U(x; C_0, C_1, C_2, 0)$ fulfilling Eq. (1.2) with $\sigma = 0$ will be considered here. They describe quasi-isobaric flows, where the pressure p (1.1) depends on one variable x only, and the planes $x = \text{const}$ are isobaric.

2. Invariance rules, similar solutions

Each solution $U(x)$ of Eq. (1.2) is invariant with respect to the translation:

$$U(\tilde{x}; \tilde{C}_0, \tilde{C}_1, \tilde{C}_2, \sigma) = U(x; C_0, C_1, C_2, \sigma), \qquad x = \tilde{x} + x_0, \tag{2.1}$$

where

$$x_0 = \text{const}, \qquad \tilde{C}_0 = U(x_0; C_0, C_1, C_2, \sigma), \qquad \tilde{C}_1 = U'(x_0; C_0, C_1, C_2, \sigma),$$
$$\tilde{C}_2 = U''(x_0; C_0, C_1, C_2, \sigma),$$

and to the transformation of affinity:

$$U(\tilde{x}; \tilde{C}_0, \tilde{C}_1, \tilde{C}_2, \tilde{\sigma}) = \lambda U(x; C_0, C_1, C_2, \sigma), \qquad x = \lambda \tilde{x}, \tag{2.2}$$

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where

$$\widetilde{C}_0 = \lambda C_0, \qquad \widetilde{C}_1 = \lambda^2 C_1, \qquad \widetilde{C}_2 = \lambda^3 C_2, \qquad \widetilde{\sigma} = \lambda^4 \sigma, \qquad \lambda = \text{const} \neq 0.$$

Choosing here the particular case $\lambda = -1$, we obtain the invariance of $U(x)$ with respect to the transformation of symmetry about the center $x = U = 0$,

$$U(-x;\, -C_0, C_1, -C_2, \sigma) = -U(x;\, C_0, C_1, C_2, \sigma). \tag{2.3}$$

These transformations define the classes of similarity within which the solutions may be expressed by each other. The transformation (2.1) defines the identity of shifted functions as their similarity rule, and allows to choose for them any arbitrary center x_0 of the coordinate axis x. On the basis of the transformation (2.2), any solution $U(x;\, C_0, C_1, C_2, 0)$ determines its similarity class of affinity by

$$U(x;\, \lambda C_0, \lambda^2 C_1, \lambda^3 C_2, 0) = \lambda U(\lambda x;\, C_0, C_1, C_2, 0). \tag{2.4}$$

By superposing both these transformations we obtain

$$U(x;\, \widetilde{C}_0, \widetilde{C}_1, \widetilde{C}_2, 0) = \lambda U(x_C + \lambda x;\, C_0, C_1, C_2, 0), \tag{2.5}$$

where

$$\widetilde{C}_0 = \lambda U(x_C;\, C_0, C_1, C_2, 0), \qquad \widetilde{C}_1 = \lambda^2 U'(x_C;\, C_0, C_1, C_2, 0),$$

$$\widetilde{C}_2 = \lambda^3 U''(x_C;\, C_0, C_1, C_2, 0).$$

Each $U(x;\, C_0, C_1, C_2, 0)$ determines here by x_C, λ, a two-parameter manifold of similar functions $U(x;\, \widetilde{C}_0, \widetilde{C}_1, \widetilde{C}_2, 0)$.

In the three-dimensional space of parameters C_0, C_1, C_2, the similarity classes are generally defined by parameters $\widetilde{C}_0$, $\widetilde{C}_1$, $\widetilde{C}_2$, distributed over two-dimensional surfaces. We may arbitrarily choose some solutions as basic, by which the other ones, belonging to the same class of similarity, are determined. The analysis of main properties of any similarity class $U(x;\, \widetilde{C}_0, \widetilde{C}_1, \widetilde{C}_2, 0)$ may be reduced also to the analysis of its basic solution $U(x;\, C_0, C_1, C_2, 0)$.

For the cases when $U(x_0;\, \widetilde{C}_0, \widetilde{C}_1, \widetilde{C}_2, 0) = 0$, it is possible and it seems to be convenient to choose x_0 as the coordinate center of the basic solution, which may be presented in the form

$$U(x;\, 0, C_1, C_2, 0), \tag{2.6}$$

where

$$C_1 = C_1(\mathcal{C}), \qquad C_2 = C_2(\mathcal{C}),$$

are arbitrary, but suitably chosen functions of one parameter $\mathcal{C}$ only. The curves $C_0 = 0$, $C_1 = C_1(\mathcal{C})$, $C_2 = C_2(\mathcal{C})$, may be obtained here from the intersection

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of the two-dimensional manifold of the corresponding similarity class with the surface $C_0 = 0$. Thus, any three-parameter function $U(x; \widetilde{C}_0, \widetilde{C}_1, \widetilde{C}_2, 0)$ fulfilling the condition

$$(2.7) \qquad U(x_0; \widetilde{C}_0, \widetilde{C}_1, \widetilde{C}_2, 0) = 0$$

may be expressed by a one-parameter basic solution (2.6) according to the following rules.

At first we find from Eq. (2.1): $U(x - x_0; 0, \widehat{C}_1, \widehat{C}_2, 0) = U(x; \widetilde{C}_0, \widetilde{C}_1, \widetilde{C}_2, 0)$, with x_0 determined by Eq. (2.7) and

$$\widehat{C}_1 = U'(x_0; \widetilde{C}_0, \widetilde{C}_1, \widetilde{C}_2, 0), \qquad \widehat{C}_2 = U''(x_0; \widetilde{C}_0, \widetilde{C}_1, \widetilde{C}_2, 0).$$

By means of Eq. (2.4), we may transform it to the form:

$$(2.8) \qquad U(x; \widetilde{C}_0, \widetilde{C}_1, \widetilde{C}_2, 0) = U(x - x_0; 0, \lambda^2 C_1, \lambda^3 C_2, 0) = \lambda U(\lambda(x - x_0); 0, C_1, C_2, 0),$$

where x_0 is found from Eq. (2.7), and the formulae

$$(2.9) \qquad \lambda^2 C_1(\mathcal{C}) = U'(x_0; \widetilde{C}_0, \widetilde{C}_1, \widetilde{C}_2, 0), \qquad \lambda^3 C_2(\mathcal{C}) = U''(x_0; \widetilde{C}_0, \widetilde{C}_1, \widetilde{C}_2, 0),$$

allow to choose suitably λ and two functions $C_1(\mathcal{C})$, $C_2(\mathcal{C})$.

As a simple example of such a choice let us take $C_1(\mathcal{C}) = \mathcal{C}$, $C_2(\mathcal{C}) = 1$, allowing to define the basic solutions and their affine similarity class by

$$(2.10) \qquad U(x; 0, \mathcal{C}, 1, 0) \quad \text{and} \quad U(x; 0, C_1, C_2, 0),$$

where

$$C_1 = \lambda^2 \mathcal{C}, \qquad C_2 = \lambda^3.$$

It is seen that on the plane $C_0 = 0$ similar solutions have the constants C_1, C_2, distributed on the curve

$$(2.11) \qquad C_2^2 = (C_1/\mathcal{C})^3 .$$

The representation of a similarity class with one set of constants x_0, $\mathcal{C}$ is not unique, because the solution $U(x)$ may have many zeros x_0 and $C_1(\mathcal{C})$, $C_2(\mathcal{C})$ are chosen arbitrarily.

3. Power series expansion

Let us assume $U(x; C_0, C_1, C_2, 0)$ in the form of the power series

$$(3.1) \qquad U(x - x_i; a_{i,0}, a_{i,1}, 2a_{i,2}, 0) = \sum_{n=0}^{\infty} a_{i,n}(x - x_i)^n$$

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developed at an arbitrarily chosen origin x_i. Three initial coefficients: $a_{i,0}$, $a_{i,1}$, $a_{i,2}$, should be given here. By introducing $U(x-x_i)$ (3.1) into Eq. (1.2) with $\sigma = 0$ and by comparing identical powers of $(x - x_i)$, we obtain for the consecutive coefficients $a_{i,n}$ the recurrent formula

$$a_{i,n} = \sum_{k=0}^{n-1} c_{n,k} a_{i,k} a_{i,n-k-1}, \qquad n = 3,4,5,\ldots, \tag{3.2}$$

with

$$c_{n,k} = \frac{k(2k-n)}{(n-2)(n-1)n}, \qquad k = 0,1,\ldots n-1, \qquad n = 3,4,5,\ldots . \tag{3.3}$$

When we combine in Eq. (3.2) the terms with identical factors

$$a_{i,n-k-1} a_{i,k} = a_{i,k} a_{i,n-k-1},$$

then, instead of $c_{n,k}$, we find from (3.3)

$$\tilde{c}_{n,k} = \left\{ \begin{array}{lll} c_{n,k} + c_{n,n-k-1} = \dfrac{1}{n}\left[1 - \dfrac{4k(n-k-1)}{(n-1)(n-2)}\right] & \text{if} & 0 \le 2k < n-1 \\ c_{n,k} = -\dfrac{1}{2n(n-2)} & \text{if} & 2k = n-1 \\ 0 & \text{if} & n-1 < 2k < 2n \end{array} \right\}, \tag{3.3$'$}$$

$$n > 2.$$

Introducing either $c_{n,k}$ Eq. (3.3) or $\tilde{c}_{n,k}$ Eq. (3.3′) into Eq. (3.2), we obtain for both cases the same values $a_{i,n}$, but $\tilde{c}_{n,k}$ is more convenient for faster computation, giving the final formula:

$$a_{i,n} = \sum_{0 \le 2k < n-1} \frac{1}{n}\left[1 - \frac{4k(n-k-1)}{(n-1)(n-2)}\right] a_{i,k} a_{i,n-k-1} - \left\{ \begin{array}{ll} \dfrac{1}{2n(n-2)} a^2_{i,\frac{n-1}{2}}, & n = 3,5,\ldots, \\ 0, & n = 4,6,\ldots, \end{array} \right. \tag{3.4}$$

which allows us to compute consecutively $a_{i,3}$, $a_{i,4}$, $a_{i,5}$, ..., if the first three coefficients $a_{i,0}$, $a_{i,1}$, $a_{i,2}$ are known.

There may exist also particular solutions, developed about the center $x_0 = 0$, with many vanishing coefficients $a_n = a_{0,n}$. Omitting the vanishing terms, their power series expansion may be presented in the form obtained from Eqs. (3.1), (3.2), (3.3) by putting there $2m-1$, $2k-1$ or $3m-1$, $3k-1$ $(m, k = 1, 2\ldots)$ instead of $n, k = 0, 1\ldots$. The symmetric solutions contain odd terms only

$$U(x; 0, a_1, 0, 0) = \sum_{m=1}^{\infty} a_{2m-1} x^{2m-1}, \tag{3.5}$$

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with not-vanishing coefficients a_{2m-1} determined by

$$(3.6) \qquad a_{2m-1} = \sum_{k=1}^{m-1} c_{2m-1,2k-1} a_{2k-1} a_{2m-2k-1}$$
$$= \sum_{0 \le 2k < m} \frac{1}{2m-1} \left[1 - 4 \frac{(2k-1)(2m-2k-1)}{(2m-2)(2m-3)} \right] a_{2k-1} a_{2m-2k-1}$$
$$- \begin{cases} \dfrac{a_{m-1}^2}{2(2m-1)(2m-3)}, & m = 2, 4, ..., \\ 0, & m = 3, 5... . \end{cases}$$

Yet a larger number of coefficients a_n vanish in solutions with the following power series:

$$(3.7) \qquad U(x;\, 0, 0, 2a_2, 0) = \sum_{m=1}^{\infty} a_{3m-1} x^{3m-1},$$

where

$$(3.8) \qquad a_{3m-1} = \sum_{k=1}^{m-1} c_{3m-1,3k-1} a_{3k-1} a_{3m-3k-1}$$
$$= \sum_{0 \le 2k < m} \frac{1}{3m-1} \left[1 - 4 \frac{(3k-1)(3m-3k-1)}{(3m-2)(3m-3)} \right] a_{3k-1} a_{3m-3k-1}$$
$$- \begin{cases} \dfrac{a_{3m/2-1}^2}{2(3m-1)(3m-3)}, & m = 2, 4... , \\ 0, & m = 3, 5... . \end{cases}$$

For solutions (3.5), (3.7), the formulae (3.2), (3.3) are also valid and they should give, at the expense of more computing work, the same results.

4. Truncation errors

In computing $U(x)$, all power series must be truncated after a finite number $1 + N$ of terms. Such truncation gives only an approximation of $U(x)$ and thus, the evaluation of the resulting errors is needed. Let us consider the truncated power series expansion (cf.Eq. (3.1)):

$$(4.1) \qquad U(x - x_i;\, a_{i,0}, a_{i,1}, 2a_{i,2}, 0) = \sum_{n=0}^{N} a_{i,n}(x - x_i)^n + R_{i,N}(x),$$

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with the remainder

(4.2) $$R_{i,N}(x) = \sum_{n=N+1}^{\infty} a_{i,n}(x - x_i)^n.$$

By computing such series, reduced to a polynomial of N-th degree, we will bound the sum (4.2) and its $L = 2$ derivatives by means of the allowable error ε,

(4.3) $$|R^{(l)}_{i,N}(x)| < \varepsilon \qquad \text{for} \quad l = 0, 1, ..., L, \quad L < N.$$

It is proved in the Appendix, that the coefficients $a_{i,n}$, Eq. (3.2), satisfy the estimate (A.1) $|a_{i,n}| < \gamma_i / r_i^{n+1}$ with determinable $\gamma_i, r_i > 0$. In consequence, the remainder $|R_{i,N}(x)|$ may be bounded for $|x - x_i| < r_i$ by a geometric series

$$|R_{i,N}(x)| < \sum_{n=N+1}^{\infty} |a_{i,n}| \cdot |x - x_i|^n < \sum_{n=N+1}^{\infty} \frac{\gamma_i}{r_i} \left| \frac{x - x_i}{r_i} \right|^n = \frac{\gamma_i}{r_i - |x - x_i|} \left| \frac{x - x_i}{r_i} \right|^{N+1}.$$

Introducing here

(4.4) $$\xi_i = |x - x_i| / r_i < 1,$$

we may estimate the remainder $|R_{i,N}(x)|$ and its L derivatives by the inequality

(4.5) $$\left| R^{(l)}_{i,N}(x) \right| < \frac{\gamma_i}{r_i^{l+1}} \left(\frac{1}{1 - \xi_i} \xi_i^{N+1} \right)^{(l)}., \qquad l = 0, 1, \ldots, L.$$

For $l < N$ we find

$$\left(\frac{1}{1 - \xi_i} \xi_i^{N+1} \right)^{(l)} = \frac{l!}{(1 - \xi_i)^{l+1}} \xi_i^{N+1} + \frac{l}{1} \cdot \frac{(l-1)!}{(1 - \xi_i)^l} \cdot (N+1) \xi_i^N + \ldots + \frac{1}{1 - \xi_i} \cdot (N+1) \cdot N \cdot \ldots (N + 2 - l) \xi_i^{N+1-l}$$
$$= \frac{l!}{1 - \xi_i} \xi_i^{N+1} \cdot \left[\left(\frac{1}{1 - \xi_i} \right)^l + \frac{l}{1} \cdot \frac{N+1}{l \cdot \xi_i} \cdot \left(\frac{1}{1 - \xi_i} \right)^{l-1} + \ldots + \frac{(N+1) \cdot \ldots (N + 2 - l)}{l \cdot (l-1) \cdot \ldots 1 \cdot \xi_i^l} \right].$$

Taking account of

$$l(l-1)(l-2) \ldots (l-i) > 1, \quad (N+1)N(N-1) \ldots (N+1-i) < (N+1)^{i+1}, \quad i < l < N$$

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and

$$\left[\left(\frac{1}{1-\xi_i}\right)^l + \frac{l}{1}\cdot\frac{N+1}{\xi_i}\left(\frac{1}{1-\xi_i}\right)^{l-1} + \ldots + \left(\frac{N+1}{\xi_i}\right)^l\right] = \left(\frac{1}{1-\xi_i} + \frac{N+1}{\xi_i}\right)^l,$$

we may obtain

$$\left(\frac{1}{1-\xi_i}\cdot\xi_i^{N+1}\right)^{(l)} < \mathcal{F}(\xi_i; l, N),$$

where the estimating function is:

$$\mathcal{F}(\xi_i; l, N) = \left(\frac{1}{1-\xi_i}\cdot\xi_i^{N+1}\right) l! \left(\frac{1}{1-\xi_i} + \frac{N+1}{\xi_i}\right)^l, \qquad l = 0, 1, \ldots, L, \qquad L < N. \tag{4.6}$$

The two conditions (4.3) and (4.5) may be now replaced by one inequality

$$\mathcal{F}(\xi_i; l, N) \le E_{i,l}, \qquad \text{where} \quad E_{i,l} = \frac{r_i^{l+1}}{\gamma_i}\varepsilon. \tag{4.7}$$

In the Fig. 1a some diagrams of the function $\mathcal{F}(\xi_i, l, N)$ are given. The rules for choosing γ_i, r_i are presented in the Appendix.

The inequality (4.7) allows us to determine the minimal value $N_{i\,\min,l}(\xi_i)$ or the maximal value $\xi_{i\,\max,l}(N)$ (plotted in the Fig. 1b) from the condition

$$\mathcal{F}(\xi_i; l, N_{i\,\min,l}) = \mathcal{F}(\xi_{i\,\max,l}; l, N) = E_{i,l}, \tag{4.8}$$

satisfying the inequality (4.3) $|R_{i,N}^{(l)}(x)| < \varepsilon$ for the values $l = 0, 1, \ldots, L$. But by forming analytic continuations in Sec. 5 we should satisfy the inequality (4.3) for all values of $l \le L$. Since $\mathcal{F}(\xi_i; l, N) \le \mathcal{F}(\xi_i; L, N)$ for $\xi_i < 1$, $l < L$, we should determine the region of validity of x by

$$|x - x_i|/r_i = \xi_i \le \xi_{i\,\max}, \tag{4.9}$$

where the new $\xi_{i\,\max}(N)$ or $N_{i\,\min}(\xi_i)$ are defined by

$$\mathcal{F}(\xi_i; L, N_{i,\min}) = \mathcal{F}(\xi_{i\,\max}; L, N) = \begin{cases} E_{i,0}, & r_i > 1, \\ E_{i,L}, & r_i < 1. \end{cases} \tag{4.10}$$

For known values of L, N, $E_{i,l}$, we find from Eq. (4.10) the value $\xi_{i\,\max}$, determining by (4.9) the region of x, where the truncated power series (4.1) approximates the solution $U(x)$ and its L derivatives with an error not exceeding ε. For the third-order differential equation (1.1), the value $L = 2$ should be introduced. Since the influence of r_i upon the range of validity $|x - x_i| < r_i\xi_{i\,\max}$ is important, much attention should be paid to the proper choice of the possibly largest r_i, formulae (A.5), (A.8), (A.9).

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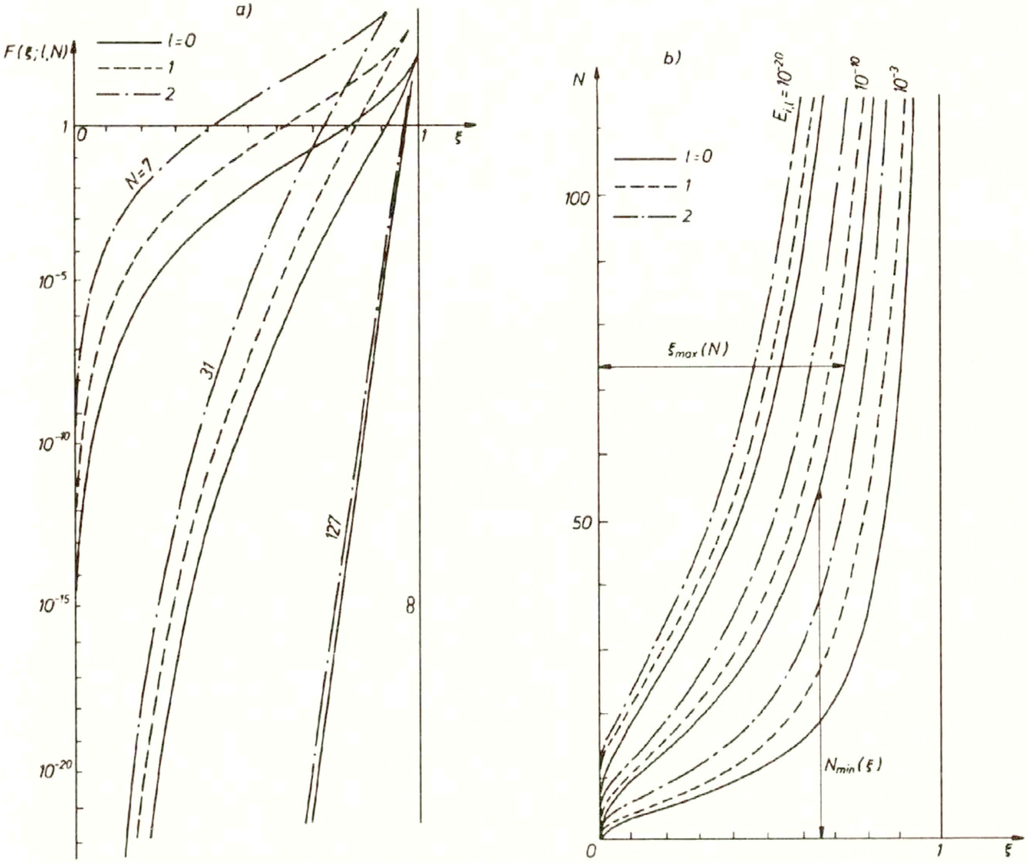

FIG. 1.

5. Analytic continuations of the power series

The truncated power series, expanded at an arbitrarily chosen center x_i, gives a solution of the Eq. (1.2) in a local region (4.9) $|x - x_i| \le r_i \xi_{i\,\max}$ only. But we seek the global solution $U(x; C_0, C_1, C_2, 0)$ in an extended, larger region of x, always with truncation errors of $U(x)$, $U'(x)$, $U''(x)$ smaller than ε. To this end we should determine a set of $1 + I_+ - I_-$ local solutions (4.1) $U(x - x_i; a_{i,0}, a_{i,1}, 2a_{i,2}, 0)$, which are analytic continuations of $U(x; C_0, C_1, C_2, 0)$ and which should succesively match each other for growing $i = 0, 1, \ldots I_+$ on the positive side, and for decreasing negative $i = 0, -1, -2, \ldots I_-$. These local solutions, with their first $L = 2$ derivatives, are defined by the truncated power series (4.1) and they should approximate one global continuous function with its derivatives. This function is defined by

$$U(x; C_0, C_1, C_2, 0) = U(x - x_i; a_{i,0}, a_{i,1}, 2a_{i,2}, 0) \quad \text{for} \quad i = I_-, \ldots, -1, 0, 1, \ldots, I_+, \tag{5.1}$$

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in the global region

$$x_{\min} \leq x \leq x_{\max}, \tag{5.2}$$

where

$$x_{\min} = x_{I_-} - r_{I_-} \cdot \xi_{I_- \max}, \qquad x_{\max} = x_{I_+} + r_{I_+} \cdot \xi_{I_+ \max}.$$

Beginning the computation of $U(x; C_0, C_1, C_2, 0)$ from $i = 0$, where $x_0 = 0$ and

$$U(0; C_0, C_1, C_2, 0) = C_0, \qquad U'(0, C_0, C_1, C_2, 0) = C_1,$$

$$U''(0; C_0, C_1, C_2, 0) = C_2,$$

we must at first find $1 + L = 3$ initial coefficients

$$\begin{aligned} a_{i,0} &= U(x_i; C_0, C_1, C_2, 0), \\ a_{i,1} &= U'(x_i; C_0, C_1, C_2, 0), \\ a_{i,2} &= U''(x_i; C_0, C_1, C_2, 0)/2, \end{aligned} \tag{5.3}$$

for each i-th analytic continuation. The succeeding $N - L$ coefficients $a_{i,3}$, $a_{i,4}, \ldots a_{i,N}$ should be found from the recurrent formula (3.4). Then we choose γ_i, r_i according to the rules formulated in the Appendix and we find from Eq. (4.10) the value $\xi_{i\max}$, determining by (4.9) the region of validity of the local solution

$$U(x - x_i; a_{i,0}, a_{i,1}, 2a_{i,2}, 0) = U(x; C_0, C_1, C_2) \quad \text{for} \quad |x - x_i| \leq r_i \xi_{i\max}. \tag{5.4}$$

In this local region the succeeding center x_j of the j-th analytic continuation should be chosen, fulfilling the condition

$$|x_j - x_i| = d_i \leq r_i \xi_{i\max}, \qquad \text{where} \qquad j = \begin{cases} i+1 & \text{for} \quad 0 \leq i < I_+, \\ i-1 & \text{for} \quad 0 \geq i > I_-. \end{cases} \tag{5.5}$$

For this succeeding step $j \to i$, the values

$$\begin{aligned} a_{j,0} = U(x_j; C_0, C_1, C_2, 0) &= U(x_j - x_i; a_{i,0}, a_{i,1}, 2a_{i,2}, 0) \\ &= \sum_{n=0}^{N} a_{i,n}(x_j - x_i)^n + R_{i,N}(x_j), \end{aligned}$$

$$a_{j,1} = U'(x_j; C_0, C_1, C_2, 0) = \sum_{n=0}^{N} n a_{i,n}(x_j - x_i)^{n-1} + R'_{i,N}(x_j), \tag{5.6}$$

$$a_{j,2} = U''(x_j; C_0, C_1, C_2, 0) = \sum_{n=0}^{N} n(n-1) a_{i,n}(x_j - x_i)^{n-2} + R''_{i,N}(x_j),$$

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should be introduced into formulae (5.3) as the approximate initial coefficients with the truncation errors (4.3) $|R_{i,N}^{(l)}(x_j)| \le \varepsilon$ for $l \le L = 2$. The value x_j in Eq. (5.5) may be either defined by the formula

$$(5.7A) \qquad x_j = x_i + (j-i) r_i \xi_{i\,\max},$$

or as one of the roots of the equations

$$(5.7B) \qquad U(x_j) = 0 \quad \text{or} \quad U'(x_j) = 0 \quad \text{or} \quad U''(x_j) = 0 \quad \text{with} \quad |x_j - x_i| < r_i \xi_{i\,\max}.$$

This procedure is repeated successively $(1 + I_+ - I_-)$ times, for $i = 0, 1, \dots, I_+$ and $i = -1, \dots, I_-$:

a) we find from Eqs. (5.3) and (3.2) the coefficients $a_{i,n}$, $n = 0, 1, 2, 3, \dots, N$;

b) we choose from (A.9) the values γ_i, r_i;

c) we determine by Eq. (4.10) and the condition (4.9) the local region $|x - x_i| \le r_i \xi_{i\,\max}$, where errors of truncation are smaller than ε;

d) in this region we choose the successive center (5.7) x_j, where we prepare $U(x_j)$, $U'(x_j)$, $U''(x_j)$ for (a) in the following step $j \to i$.

We start at $i = 0$ and go I_+ steps in the positive direction, then we must start for the second time from $i = 0$, and go $-I_-$ steps in the negative direction, taking everywhere account of local regions of validity (5.5), or using the roots (5.7B).

In this manner, the solutions $U(x; C_0, C_1, C_2, 0)$ of Eq. (1.2) may be obtained with the accuracy defined by the truncation error ε. The initial values $x_0 = 0$, $C_0 = U(0)$, $C_1 = U'(0)$, $C_2 = U''(0)$ are given and sets of $(1 + I_+ - I_-)$ analytically continued, truncated power series (5.1) are found. It should be emphasized that then the previous large number N of coefficients $a_{i,n}$ will not be needed. The smaller numbers $N_i = N_{i\,\min,l}(\xi_i)$, found from Eq. (4.8), may be sufficient to compute the truncated series (4.1), with the same accuracy, for many values of

$$(5.8) \qquad |x - x_i| < b_i \qquad \text{or} \qquad |x - x_j| < b_j,$$

where

$$b_i + b_j \le d_i = |x_j - x_i|.$$

Consequently, the main result for $U(x; C_0, C_1, C_2, 0)$ may now be reduced to finding $1 + I_+ - I_-$ local solutions (5.1) with the following basic local data:

$1 + I_+ - I_-$	centers x_i $(i = I_-, \dots, -1, 0, 1, \dots, I_+)$ of analytic continuations;
$3(1 + I_+ - I_-)$	initial coefficients $a_{i,l}$ $(l = 0, 1, 2)$ only;
$2(1 + I_+ - I_-)$	evaluation constants γ_i, r_i.

These data allow us to obtain from Eqs. (4.8), (3.4) all N_i, $a_{i,n}$, necessary to compute $U(x)$ in the regions $|x - x_i| < b_i$ or $|x - x_j| < b_j$, (5.8).

On the basis of the method presented above, the program HIM0 for obtaining solutions of the Hiemenz Equation, in two options, has been prepared. This

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program allows to compute, with an accuracy of about 18 decimal digits, the functions $U(x; C_0, C_1, C_2, 0)$ for the given input data $C_0 = U(0)$, $C_1 = U'(0)$, $C_2 = U''(0)$ and allowable maximal values of N, I_-, I_+.

In the first (A) option, the centers x_j ($j = I_-, \ldots, -1, \ldots, I_+$) are obtained from (5.7A) without taking into account the existence of zeros of $U(x)$. As these zeros are not known in advance, so, taking them into account from (5.7B) in the (B) option, we should also find their distribution by a more sophisticated program, where smaller N_i may be allowed. Unfortunately, such program with larger number of separation centers x_j may generate larger numerical errors and its application to some particular cases $U(x)$ with densely distributed zeros may cause additional difficulties. However, the knowledge of the zero's distribution may be often very important and thus the preference of the option (B) is rather suggested.

By means of the program HIM0 we may also exchange the output data with the memory, or we may transform different functions into each other, belonging to the same class of similarity. The data memory contains three files: NAGL.DAT, WSP.DAT, ZERA.DAT, with sets of chosen basic solutions. Main input data of these solutions are collected in NAGL.DAT, which contains also additional information concerning the distribution of zeros in ZERA.DAST and of local values: x_i, $a_{i,0}$, $a_{i,1}$, $a_{i,2}$, γ_i, r_i in WSP.DAT.

6. Solutions of Eq. (1.2) with $\sigma = 0$

6.1. Trivial solutions U = const

The considered Eq. (1.2) is fulfilled by the simplest solutions

$$U(x; C_0, 0, 0, 0) = C_0 = \text{const}. \tag{6.1}$$

By applying the transformations (2.1), (2.2) to the "immobility" case $U(x; 0, 0, 0, 0) = 0$ with $C_0 = 0$, its similarity class is reduced to one element with $C_0 = C_1 = C_2 = 0$ only. The second trivial solutions (6.1), with $C_0 \neq 0$ describing uniform flows, are distributed in the space C_0, C_1, C_2, on the C_0-axis.

6.2. Singular solution $U_\infty = -6/x$

Equation (1.2) with $\sigma = 0$ is fulfilled by the singular solution

$$U_\infty(x) = U(x; \pm\infty, \infty, \pm\infty, 0) = -6/x, \tag{6.2}$$

which does not vanish for any x. According to Eq. (1.1), $u = -6/x$, $v = -6y/x^2$, $\psi = -6y/x$, it would describe a non-realistic strong plane "negative source" in the center $x = y = 0$, with straight stream-lines (Fig. 2).

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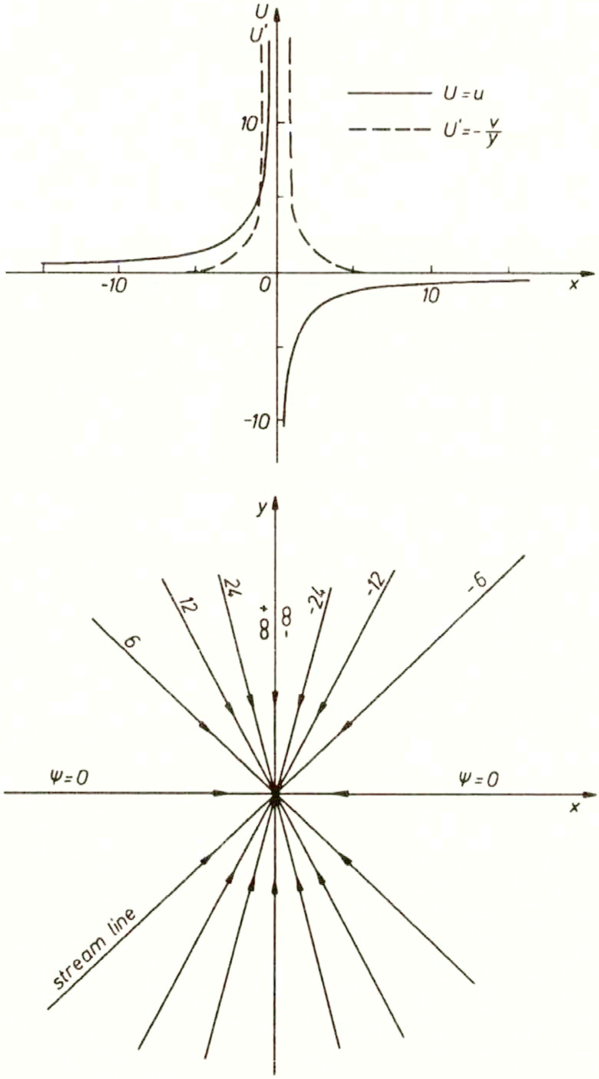

FIG. 2.

The solution (6.2) is not affected by the affine transformation (2.5), $\lambda U_\infty(\lambda x) = -6/x = U_\infty(x)$. But from Eq. (2.1) we obtain the translation similarity class

(6.3) $$U(x; C_0, C_1, C_2, 0) = -\frac{6}{x + x_0},$$

where

$$C_0 = -\frac{6}{x_0}, \qquad C_1 = \frac{6}{x_0^2}, \qquad C_2 = -\frac{12}{x_0^3}.$$

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This class is determined by constants distributed in the space C_0, C_1, C_2, on the three-dimensional curve defined by the equations

$$C_1 = C_0^2/6, \qquad C_2 = C_0^3/18. \tag{6.4}$$

6.3. Riabouchinsky's solutions

RIABOUCHINSKY [2] found in 1924 an exact solution of the Navier–Stokes equations, fulfilling also the Hiemenz equation (1.2) for $\sigma = 0$, which may be presented in the basic form (Fig. 3)

$$U_R(x) = U(x; 0, -1, 1, 0) = -(1 - e^{-x}). \tag{6.5}$$

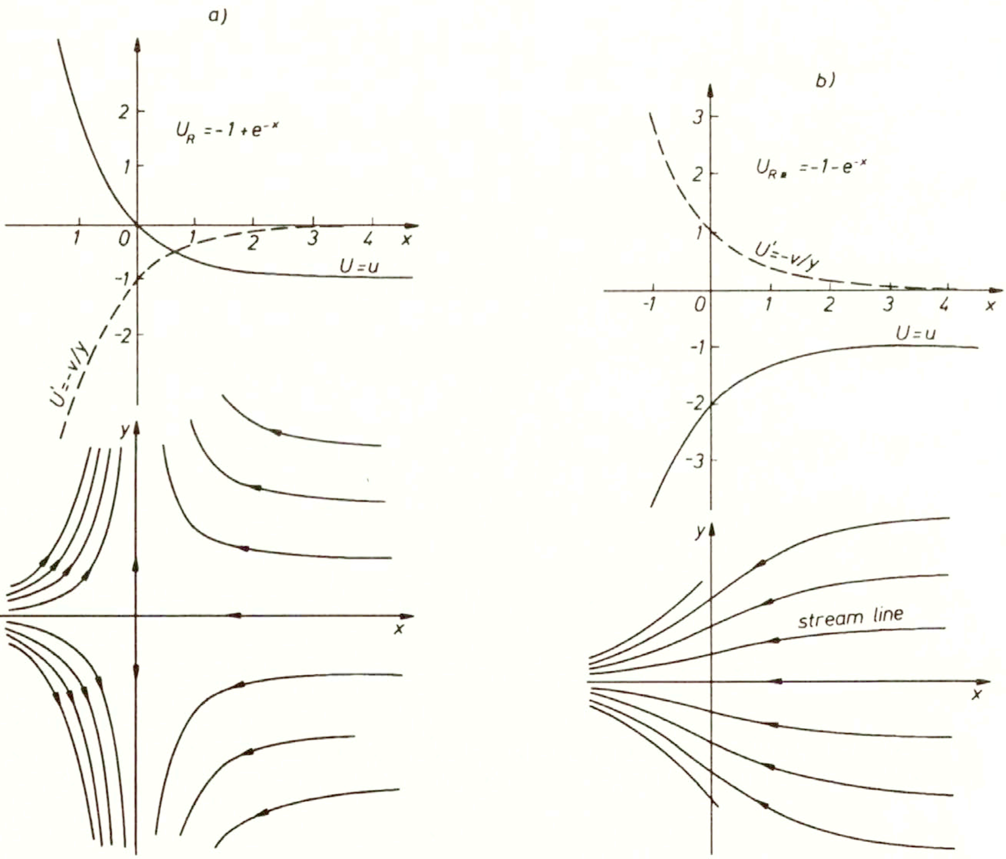

FIG. 3.

An analogous solution may be obtained by changing the sign,

$$U_{R^*}(x) = U(x; -2, 1, -1, 0) = -(1 + e^{-x}). \tag{6.6}$$

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These solutions define by means of Eqs. (2.1) and (2.2) two classes of similarity:

$$\lambda U_R(\lambda(x-x_0)) = U(x;-\lambda(1-B),-\lambda^2 B,\lambda^3 B,0) = -\lambda\left(1-Be^{-\lambda x}\right), \tag{6.7}$$

$$\lambda U_{R^*}(\lambda(x-x_0)) = U(x;-\lambda(1+B),\lambda^2 B,-\lambda^3 B,0) = -\lambda\left(1+Be^{-\lambda x}\right), \tag{6.8}$$

where $B = e^{\lambda x_0} > 0$.

They approach the asymptotic value $-\lambda$ for $\lambda x \to \infty$, the functions $\lambda U_{R^*}(\lambda(x-x_0))$ do not reach zero anywhere, while $U_R(x)$ defines a class of similarity (6.7) with a single zero for $x = x_0$.

By means of Eq. (1.1), the flow field (Fig. 3) may be obtained with parameters

$$\begin{aligned} u &= -\lambda(1-Be^{-\lambda x}), & \nu &= \lambda^2 Bye^{-\lambda x},\\ p &= -\frac{\lambda^2}{2}B^2e^{-2\lambda x} + \text{const}, & \Psi &= -\lambda y(1-Be^{-\lambda x}),\\ u &= -\lambda(1+Be^{-\lambda x}), & \nu &= -\lambda^2 Bye^{-\lambda x},\\ p &= -\frac{\lambda^2}{2}B^2e^{-2\lambda x} + \text{const}, & \Psi &= -\lambda y(1+Be^{-\lambda x}). \end{aligned}$$

The first solution was used by RIABOUCHINSKY [2] and CRANE [3] for the flow past a stretching plate.

In Eqs. (6.7) and (6.8) $U(x;C_0,C_1,C_2,0) = -\lambda(1\mp Be^{-\lambda x})$ we have the constants $C_0 = -\lambda(1\mp B)$, $C_1 = \mp\lambda^2 B$, $C_2 = \pm\lambda^3 B$, defining both similarity classes and distributed on a two-dimensional surface in three-dimensional space C_0, C_1, C_2. Eliminating here $B = \mp C_1^3/C_2^2$ and $\lambda = -C_2/C_1$ from C_0, we obtain the equation in the form

$$\mathcal{R} \equiv C_1^3 + C_2^2 - C_0C_1C_2 = 0, \tag{6.9}$$

or

$$C_2 = C_1\cdot\left(C_0 \pm \sqrt{C_0^2-4C_1}\right)/2. \tag{6.10}$$

It will be shown later, that Riabouchinsky's functions (6.7) and (6.8), with the surface $\mathcal{R} = 0$ Eq. (6.9), play an important role in the analysis of other solutions $U(x;C_0,C_1,C_2,0)$. The majority of these solutions may be asymptotically approached by $-\alpha_\pm U_R(-\alpha_\pm(x-x_\pm))$

$$U(x;C_0,C_1,C_2,0) \to \alpha_\pm\left(1-e^{\alpha_\pm(x-x_\pm)}\right) \qquad \text{for} \quad x\to\pm\infty, \tag{6.11}$$

where

$$\alpha_\pm = \lim\left(\frac{U''(x)}{U'(x)}\right), \qquad x_\pm = \lim\left\{x - \frac{U'(x)}{U''(x)}\ln\left(-\frac{[U'(x)]^3}{[U''(x)]^2}\right)\right\} \tag{6.12}$$
$$\text{for} \qquad x\to\pm\infty.$$

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In consequence, the local data of analytic continuations obtained in Sec. 5, with the asymptotic formulae (6.11), (6.12), may allow to compute $U(x; C_0, C_1, C_2, 0)$ for any x in any arbitrary range $-\infty < x < \infty$.

6.4. Symmetric cases $U(-x) = -U(x)$

There exist two classes of similarity of the symmetric solutions, which may be defined by two basic solutions:

$$U_-(x) = U(x; 0, -1, 0, 0), \tag{6.13}$$

or

$$U_+(x) = U(x; 0, +1, 0, 0). \tag{6.14}$$

In the central region $-\xi_{\max} r < x < \xi_{\max} r$ their values may be computed by means of the power series $U(x; 0, a_1, 0, 0)$ Eq. (3.5) with odd coefficients only. Their first non-vanishing values a_{2m-1} for $m = 1, 2, \ldots, 15$ are:

$n = 2m-1$	1	3	7	9	11	13	15	17	19
$n! \cdot a_{2m-1}$	∓ 1	-1	-2	± 8	-32	∓ 64	3312	∓ 18048	-964992

21	23	25	27	29
± 27197440	146549760	∓ 30560231424	668929236984	± 25325353803600

Two diagrams, presenting parts of symmetric curves $U_-(x)$, $U'_-(x)$, $U''_-(x)$ or $U_+(x)$, $U'_+(x)$, $U''_+(x)$, are shown on both sides of the Fig. 4. From Eq. (3.5), taking into account 30 terms only ($m_{\max} = 15$, $n_{\max} = 29$), we may compute $U_\mp(x)$, $U'_\mp(x)$, $U''_\mp(x)$, for $|x| < 2.5$, with errors ε Eq. (4.3) smaller than 10^{-5}, 10^{-4}, 10^{-3}, respectively.

The solution $U_-(x)$ obtained from the option (A) with $N = 127$ yields

i	r_i	γ_i	x_i	$U_-(x_i)$	$U'_-(x_i)$	$U''_-(x_i)$
0	4.24	54	0	0	-1	0
± 1	3.03	59	± 2.754609910	∓ 6.567801477	-5.557663257	∓ 4.215903455
± 2	2.41	143	± 4.716089780	∓ 31.682463060	-24.609177657	∓ 18.668067949

It reaches zero $U_-(0) = U''_-(0) = 0$ at the center $x_0 = 0$ only, and for $x \to \pm\infty$ it may be asymptotically approached by Riabouchinsky's solution (6.11) with

$$\alpha_\pm = \pm 0.758581542572728, \qquad x_\pm = \mp 0.2348999287870.$$

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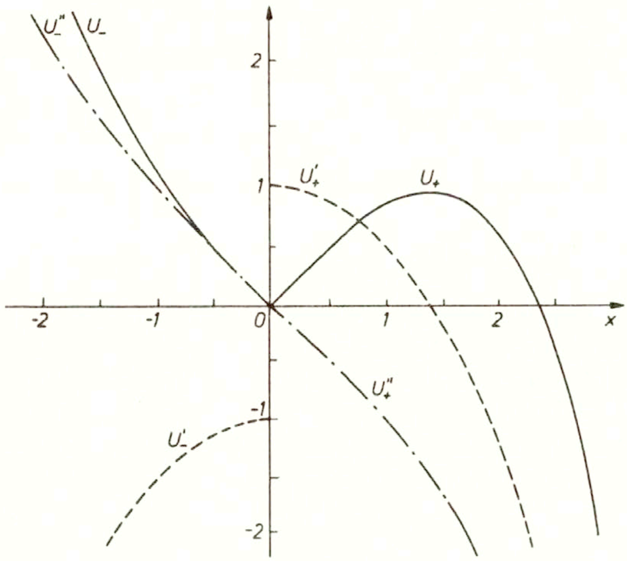

FIG. 4.

The solution $U_+(x)$ seems to be more interesting for hydrodynamics and it was computed here by means of both options of HIM0. The option (A) with $N = 127$ gave

i	r_i	γ_i	x_i	$U_+(x_i)$	$U'_+(x_i)$	$U''_+(x_i)$
0	4.24	54	0	0	1	0
± 1	2.09	38	± 2.754609910	∓ 1.448820288	-4.814100016	∓ 7.779260281

with the asymptotic values

$$\alpha_\pm = \pm 1.59124076522024, \qquad x_\pm = \pm 2.34842678820127.$$

In the option (B) larger $I_+ = -I_- = 3$ with smaller N_i have been obtained

i	N_i	r_i	γ_1	x_i	$U_+(x_i)$	$U'_+(x_i)$	$U''_+(x_i)$
0	51	4.184	51	0	0	1	0
± 1	51	3.012	24	± 1.39728669028	± 0.93800239840	0	∓ 1.5042499245
± 2	127	2.388	39	± 2.33915869848	0	-2.40299296054	∓ 4.2498130492
± 3	127	1.475	75	± 3.88739750540	∓ 16.82783786109	-29.30918867538	∓ 46.6379777796

The basic function $U_\mp(x)$ define two similarity classes $\lambda U_\mp(x_C + \lambda x)$, depending on λ, x_C. The constants C_0, C_1, C_2, determining the first similar solutions

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$U(x;C_0,C_1,C_2,0) = \lambda U_-(x_C + \lambda x)$, are distributed on a two-dimensional surface, intersecting the coordinate planes $C_0 = 0$, $C_2 = 0$, at the negative semi-axis $C_1 < 0$, $C_0 = C_2 = 0$.

The parameters

$$C_0 = \lambda U_+(x_C), \qquad C_1 = \lambda^2 U'_+(x_C), \qquad C_2 = \lambda^3 U''_+(x_C),$$

determining the second similar solutions

$$U(x;C_0,C_1,C_2,0) = \lambda U_+(x_C + \lambda x),$$

are distributed on the surface intersecting the same planes at the positive semi-axis $C_1 > 0$, $C_0 = C_2 = 0$, but it has also other lines of intersection. To show them, let us choose the translations passing through the zeros x_i for $i = 0, \pm 1, \pm 2$:

$$U(x;C_{i,0},C_{i,1},C_{i,2},0) = \lambda U_+(x_i + \lambda x),$$

$$C_{i,0} = \lambda U_+(x_i), \qquad C_{i,1} = \lambda^2 U'_+(x_i), \qquad C_{i,2} = \lambda^3 U''_+(x_i).$$

The positive C_1-axis is here obtained for $i = 0 : C_0 = C_2 = 0$, $C_1 = \lambda^2$. For $i = \pm 1, \pm 2$, we find the respective equations of other intersection lines:

$$C_1 = 0, \qquad C_2 = U''_+(x_1)/U^3_+(x_1) \cdot C_0^3$$

and cf. Eq. (2.11)

$$\text{(6.15)} \qquad C_0 = 0, \qquad C_1 = C_+ \sqrt[3]{C_2^2},$$

with

$$C_+ = U'_+(x_2)/\sqrt[3]{U''_+(x_2)^2} = -0.915881488790480047 8.$$

The solutions $\lambda U_+(x_i + \lambda x)$ determine flows in channels between plane, parallel walls with symmetric boundary conditions. As an example, let us consider a quasi-isobaric flow between rigid, porous walls with uniform suction, and let us find conditions necessary for its realization.

To solve this problem, we should satisfy the no-slip boundary condition $U'(\pm 1) = 0$ for the symmetric solution

$$U(x;0,C_1,0,0) = \lambda U_+(\lambda x).$$

Since

$$U'_+(\pm x_1) = 0,$$

it is sufficient here to take $\lambda = x_1$ and we obtain

$$U(x) = x_1 U_+(x_1 x).$$

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From Eq. (1.1) we find the velocity $u = U(x)$, $v = -yU'(x)$, and the stream function $\Psi = yU(x)$. The obtained field of quasi-isobaric flow $\Psi(x, y) = \text{const}$ is shown in Fig. 5 and to realize it, the suction velocity u_w on the wall should be equal to its quasi-isobaric value, $u_q = U(1) = x_1 U_+(x_1) = 1.310658266$. In our dimensionless reference system, the value u_q is equal to a Reynolds number referred to the dimensional suction velocity. If $u_w \neq u_q$, then the obtained solution with $\sigma = 0$ is not valid and the pressure gradient $\partial p/\partial y$ with $\sigma \neq 0$ should be taken into account.

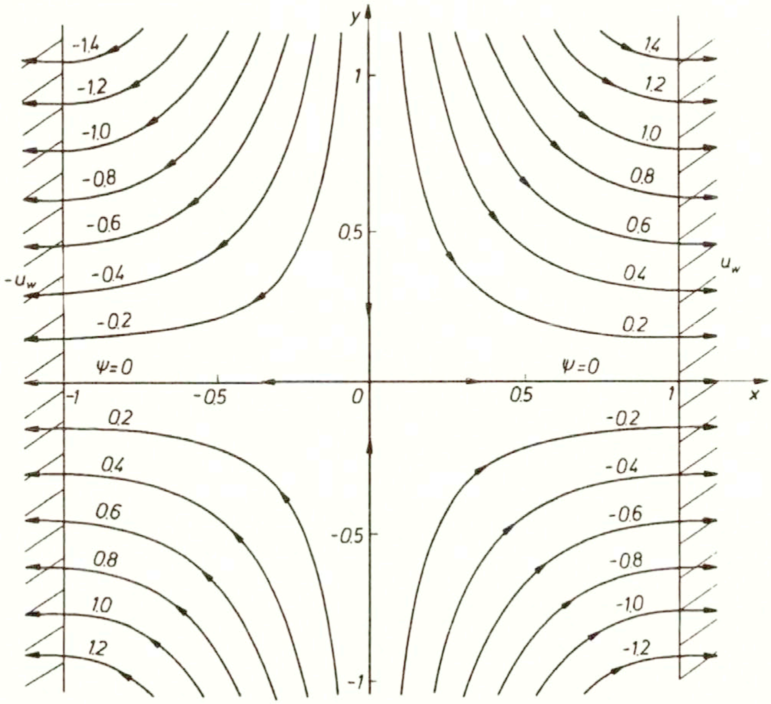

FIG. 5.

6.5. Case $U(0) = U'(0) = 0$

Let us introduce for our case the basic function

$$U_A(x) = U(x; 0, 0, 1, 0). \tag{6.16}$$

The power series $U(x; 0, 0, 0, 2a_2, 0)$ Eq. (3.5), expanded here at the center $x_0 = 0$ for $a_2 = 1/2$, has also several non-vanishing coefficients a_{3m-1}, for the first values of $m = 1, 2, 3, 4, 5, 6, 7, 8$ they are given in the table below

$n = 3m - 1$	2	5	8	11	14	17	20	23
$n! \cdot a_{3m-1}$	1	−1	−1	−27	−951	−51465	−3355857	−151875891

We will apply here both options of HIM0.

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From the option (A) with $N = 127$ we obtain the values at the centers x_i of analytic continuations:

i	r_i	γ_i	x_i	$U_A(x_i)$	$U'_A(x_i)$	$U''_A(x_i)$
−2	2.27	125	−4.599564890	27.652066714	−24.282529941	20.686287834
−1	2.90	53	−2.772533721	5.159131567	−5.120743824	4.362763158
0	4.39	60	0	0	0	1
1	1.84	27	2.772533721	2.318085943	−0.263962035	−4.537369331
2	0.98	52	3.933984044	−8.976345513	−35.882443310	−107.55628806

and the asymptotic values

$$\alpha_- = -0.8519000237691, \qquad x_- = -0.4789764813954 \qquad \text{for} \qquad x \to -\infty$$

and

$$\alpha_+ = 2.996961192059, \qquad x_+ = 3.471823699089 \qquad \text{for} \qquad x \to +\infty.$$

No zeros exist here for $i < 0$ and thus it is not necessary to use the identical values of $N_i = 127$, r_i, γ_i, x_i, $U_A(x_i)$, $U'_A(x_i)$, $U''_A(x_i)$, obtained from both options. For $i \geq 0$ option (B) yields

i	N_i	r_i	γ_i	x_i	$U_A(x_i)$	$U'_A(x_i)$	$U''_A(x_i)$
0	70	4.374	58	0	0	0	1
1	70	2.698	21	1.78014746588	1.43256891674	1.34780768981	0
2	70	1.871	25	2.71001917889	2.32613658590	0	−3.92268342579
3	127	1.293	37	3.47040219504	0	−8.84825811071	−27.35379056568

The diagrams of $U_A(x)$, $U'_A(x)$, $U''_A(x)$, with stream-lines $\Psi(x, y) = yU_A(x) =$ const are shown in Fig. 6. Such flow could be realized in a liquid layer on a rigid impermeable plane wall, when its upper free surface is stretched by linearly variable surface tension (for instance, due to the Marangoni effect).

The function $U_A(x)$ defines similar solutions

$$U(x; C_0, C_1, C_2, 0) = \lambda U_A(x_C + \lambda x)$$

with parameters distributed on

$$C_0 = \lambda U_A(x_C), \qquad C_1 = \lambda^2 U'_A(x_C), \qquad C_2 = \lambda^3 U''_A(x_C).$$

The obtained surface intersects for $x_C = x_i$ $(i = 0, 1, 2, 3)$ the coordinate planes: at the axis $C_2 \neq 0$, $C_0 = C_1 = 0$, at the line $C_1 = U'_A(x_1)/U^2_A(x_1) \cdot C_0^2$ on the

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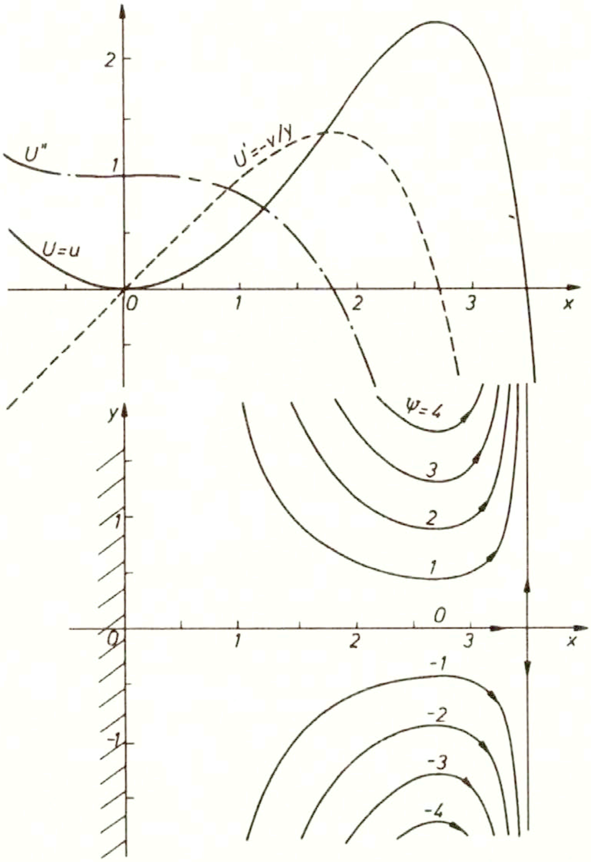

FIG. 6.

plane $C_2 = 0$, at the line $C_2 = U''_A(x_2)/U^3_A(x_2) \cdot C_0^3$ on the plane $C_1 = 0$ and at the line

$$C_0 = 0, \qquad C_1 = \mathcal{C}_A \sqrt[3]{C_2^2}, \tag{6.17}$$

with

$$\mathcal{C}_A = U'_A(x_3)/\sqrt[3]{U''_A(x_3)^2} = -0.974644204225262984.$$

6.6. General solutions $U(x; C_0, C_1, C_2, 0)$

The solutions $U(x; C_0, C_1, C_2, 0)$ of the Hiemenz equation (1.2) depend on three parameters C_0, C_1, C_2, and in such a general form they are more difficult to analyze. However, due to the invariance rules (Sec. 2), by translations (2.1) and affine transformations (2.2) they may be expressed by each other from the same class of similarity. So, instead of analyzing all similar functions, we may choose any of them as their basic representative for studying their main properties. Three

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constants C_0, C_1, C_2 of these basic solutions may be determined by one parameter $C_0(\mathcal{C})$, $C_1(\mathcal{C})$, $C_2(\mathcal{C})$, only (cf. Sec. 2). Main attention will be paid to such basic functions, their similarity class being defined.

In Sec. 6.3 the surface $\mathcal{R} = 0$, Eq. (6.9), determining constants C_0, C_1, C_2 from Riabouchinsky's solutions was obtained. In the Fig. 7 the contour lines of this compound surface for positive $C_0 = 0, 1, 2, 3$ are plotted, and for negative C_0 they are situated symmetrically with respect to the C_1-axis, as it is shown for $C_0 = -2$, $C_1 > 0$. Outside this surface the expression $\mathcal{R}(C_0, C_1, C_2)$ of Eq. (6.9) is different from zero, but its sign does not change for similar solutions. In the space C_0, C_1, C_2, the surface $\mathcal{R} = 0$ separates regions with different similarity classes of solutions with different numbers of zeros.

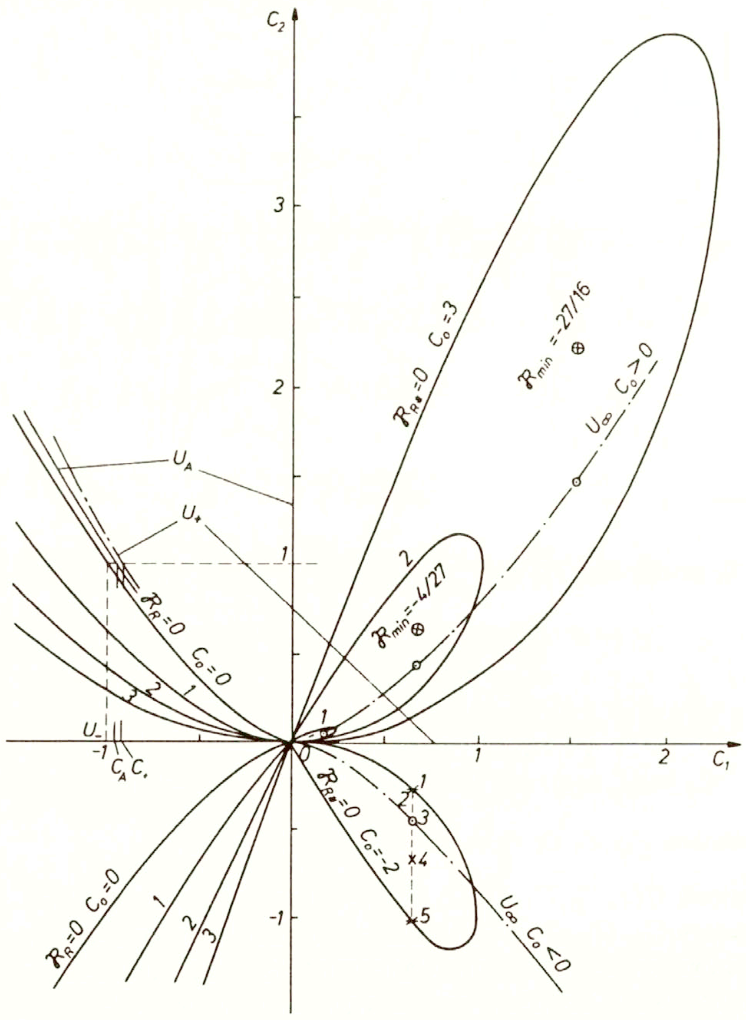

FIG. 7.

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To show the sign invariance of $\mathcal{R}(C_0, C_1, C_2)$ for similar solutions, let us introduce the function

$$\mathcal{S}(x; C_0, C_1, C_2) = (U')^3 + (U'')^2 - UU'U'' , \tag{6.18}$$

where

$$U(x) = U(x; C_0, C_1, C_2, 0).$$

Putting $U''' = UU'' - U'U'$ Eq. (1.2) into

$$\mathcal{S}' = 3U'U'U'' + 2U''U''' - U'U'U'' - UU''U'' - UU'U''',$$

we obtain the differential equation $\mathcal{S}' = U\mathcal{S}$ with the integral

$$\mathcal{S}(x_c; C_0, C_1, C_2) = \mathcal{R}(C_0, C_1, C_2) \cdot \exp\left(\int_0^{x_c} (x; C_0, C_1, C_2, 0)\, dx\right) . \tag{6.19}$$

Since the sign of $\mathcal{R}(U(x_c), U'(x_c), U''(x_c)) = \mathcal{S}(x_c; C_0, C_1, C_2)$ remains unchanged for any x_c and the transformation (2.4), $\mathcal{R}(\lambda C_0, \lambda^2 C_1, \lambda^3 C_2) = \lambda^6 \mathcal{R}(C_0, C_1, C_2)$ does not change it as well, this sign remains invariant for all similar solutions and it becomes one of their main attributes.

By the condition $\mathcal{R} < 0$ two "negative" regions of C_0, C_1, C_2 may be defined: for $C_1 > 0$ without zeros of $U(x; C_0, C_1, C_2, 0) \neq 0$, and for $C_1 < 0$ with only one x_0 fulfilling $U(x_0; C_0, C_1, C_2, 0) = 0$, where $-\infty < x,\, x_0 < \infty$. The functions with parameters C_0, C_1, C_2, from the first inside region of two surfaces of conical shape (Fig. 7) $\mathcal{R} \leq 0$, $C_1 > 0$, do not reach zero anywhere. The function $\mathcal{R}(C_0, C_1, C_2)$ decreases here for each $C_0 = \text{const}$ from $\mathcal{R}_{R^*} = 0$ for some $\lambda U_{R^*}(\lambda(x - x_0))$ (cf. Sec. 6.3), through $\mathcal{R}_\infty = -C_0^6/648$ for $U(x; C_0, C_0^2/6, C_0^3/18, 0)$ (cf. Sec. 6.2), until its lowest value $\mathcal{R}_{\min} = -C^6/432$ for $U(x; C_0, C_0^2/6, C_0^3/12, 0)$. The diagrams of the functions $U(x; C_0, C_0^2/6, C_2, 0)$ for $C_0 = -2$ and for C_2 given below

Mark	1	2	3	4	5
C_2	$-(1 - 1/\sqrt{3}) \cdot 2/3$ $-0.28176648...$	-0.28177	$-4/9$ $-0.44...$	$-6/9$ $-0.66...$	$-(1 + 1/\sqrt{3}) \cdot 2/3$ $-1.05156684...$
$\mathcal{R}$	$\mathcal{R}_{R^*} = 0$	$-2.7 \cdot 10^{-6}$	$R_\infty = -8/81$	$\mathcal{R}_{\min} = -4/27$	$\mathcal{R}_{R^*} = 0$

are plotted in the Fig. 8.

All solutions $U(x; C_0, C_1, C_2, 0)$ with C_0, C_1, C_2, belonging to the second region $\mathcal{R} \leq 0$, $C_1 < 0$ may be expressed by the basic functions $U(x; 0, -1, \mathcal{C}, 0)$ with $0 \leq \mathcal{C} \leq 1$. They also do not seem suitable for hydrodynamics. Their diagrams (Fig. 9) show that they intersect the x-axis at one point only and their first derivatives are negative, $U'(x) < 0$, for all $-\infty < x < \infty$. For $x \to \pm\infty$ they approach asymptotically $\alpha_\pm(1 - e^{\alpha_\pm(x - x_\pm)})$, Eq. (6.11).

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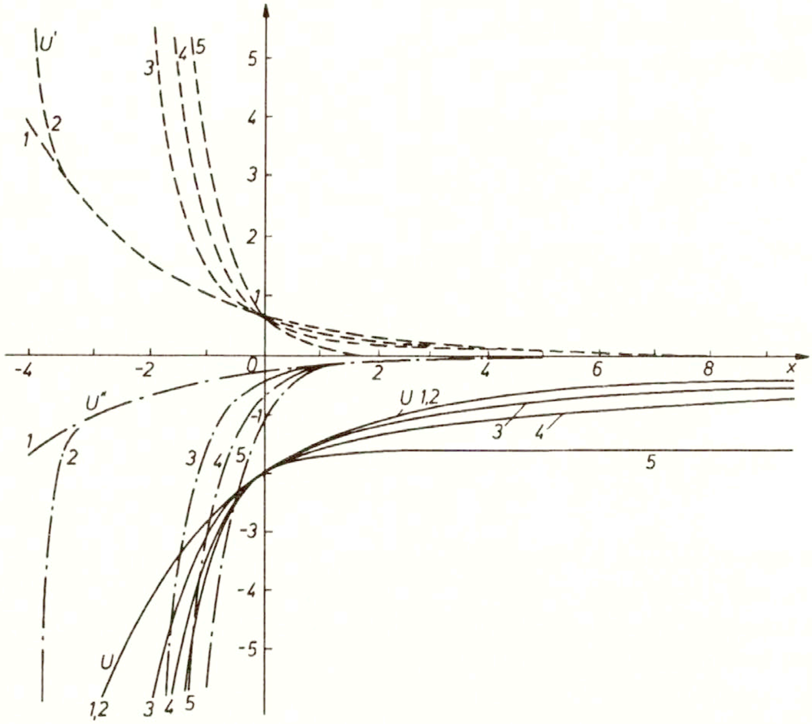

FIG. 8.

In contrast to the above "negative" cases, the solutions $U(x; C_0, C_1, C_2, 0)$ with C_0, C_1, C_2, fulfilling $\mathcal{R} > 0$, may satisfy boundary conditions for hydrodynamics. Using the transformation

(6.20) $$U(x; C_0, C_1, C_2, 0) = \lambda U(x_C + \lambda x; 0, \mathcal{C}, 1, 0),$$

where

(6.21) $$\begin{aligned} C_0 &= \lambda U(x_C; 0, \mathcal{C}, 1, 0), \\ C_1 &= \lambda^2 U'(x_C; 0, \mathcal{C}, 1, 0), \\ C_2 &= \lambda^3 U''(x_C; 0, \mathcal{C}, 1, 0), \end{aligned}$$

we may express all these solutions by means of the basic functions

(6.22) $$U(x; 0, \mathcal{C}, 1, 0) \quad \text{with} \quad -1 < \mathcal{C} \le \mathcal{C}_+ = -0.915881\ldots .$$

Also this "positive" region C_0, C_1, C_2, defined by Eqs. (6.21) and (6.22), shall be divided here into two parts, with

(6.23) $$\mathcal{C}_A = -0.974644\ldots \le \mathcal{C} \le \mathcal{C}_+ = -0.915881\ldots ,$$

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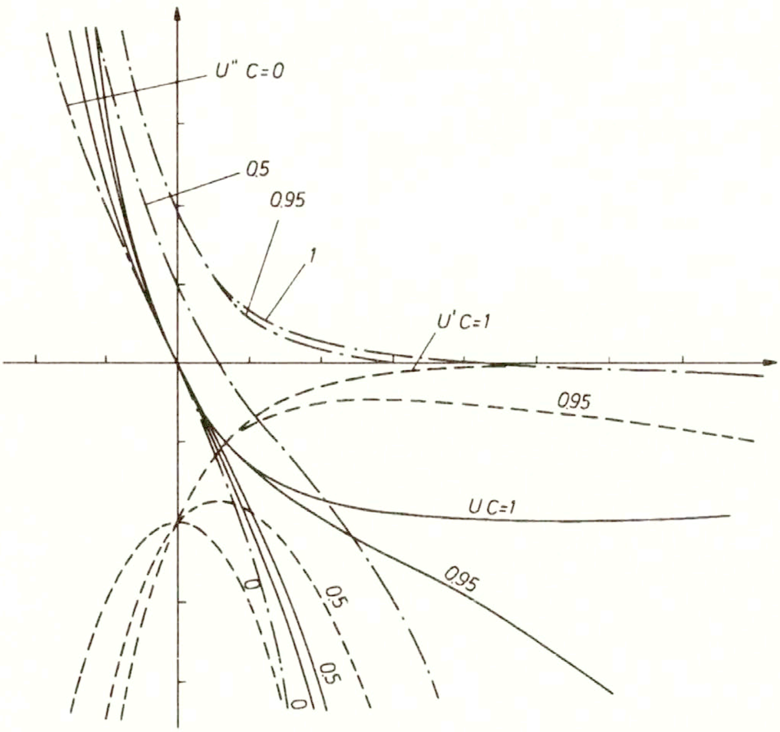

FIG. 9.

and

(6.24) $$-1 < \mathcal{C} < \mathcal{C}_A = -0.974644\ldots.$$

The main hydrodynamically interesting solutions are obtained from the first region (6.20), (6.23), with the functions $U(x)$ and their derivatives $U'(x)$ intersecting the x-axis at three and two points, respectively. Their examples are shown in the Fig. 10 with the basic functions:

1) $U(x; 0, \mathcal{C}_A, 1, 0)$;
2) $U(x; 0, -0.95, 1, 0)$;
3) $U(x; 0, \mathcal{C}_+, 1, 0)$.

By introducing $C_0 = 0$, $C_2 = 1$, into Eqs. (6.20) and (6.21) we may find similar solutions of the same form, but with $C_1 > \mathcal{C}_+$,

(6.25) $$U(x; 0, C_1, 1, 0) = \lambda U(x_k + \lambda x; 0, \mathcal{C}, 1, 0),$$

where

(6.26) $$U(x_k; 0, \mathcal{C}, 1, 0) = 0,$$
$$\lambda = 1/\sqrt[3]{U''(x_k; 0, \mathcal{C}, 1, 0)},$$
$$C_1 = \lambda^2 U'(x_k; 0, \mathcal{C}, 1, 0).$$

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In Fig. 10 these similar solutions are denoted by the same numbers as their basic functions, but they are marked by single or double primes. They are determined by the constants obtained from Eqs. (6.25) and (6.26)

1′) $x_1 = 10.4564833...,\quad \lambda = -3.01304653...,\quad C_1 = 0;$

2′) $x_2 = 9.21531218...,\quad \lambda = -1.52881783...,\quad C_1 = -0.839523460630094446...;$

2″) $x_3 = 5.27078097...,\quad \lambda = -2.61153498...,\quad C_1 = 1.56193441358539807....$

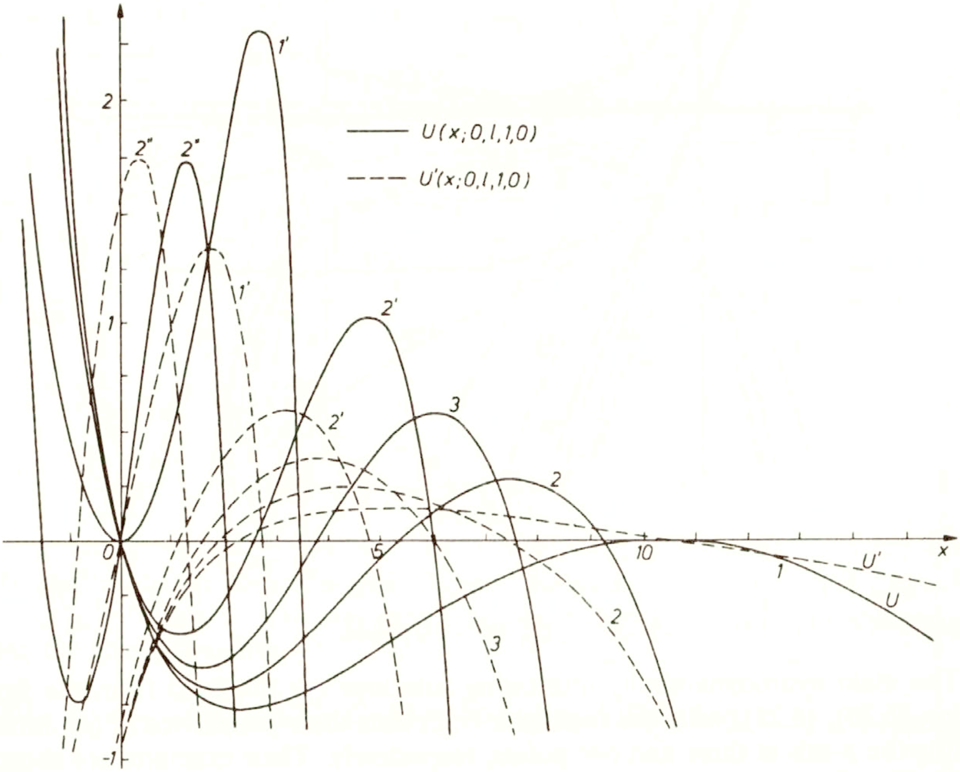

FIG. 10.

The functions $U(x;0,C,1,0)$ with $C_+ \le C \le 0$ instead of $C_+ \ge C \ge C_A$ could be chosen also as basic functions to define the same similarity classes. An exemplary field of flow between two stretched, parallel surfaces, obtained from 2′, is shown in the Fig. 11.

The solutions $U(x;C_0,C_1,C_2,0)$ with constants C_0, C_1, C_2, from the "positive" layer $\mathcal{R} > 0$, adjacent to the surface $\mathcal{R} = 0$, are not monotonous, and their first derivatives intersect the x-axis twice. Examples of basic solutions $U(x;0,C,1,0)$ with $C = -0.99$, -0.985, supplemented by $C = -1$, C_A, are plotted in Fig. 12. In the Fig. 13 are shown the resulting stream-lines of a quasi-isobaric flow between two parallel surfaces, first of them being stretched and second one — porous, for $C = -0.985$.

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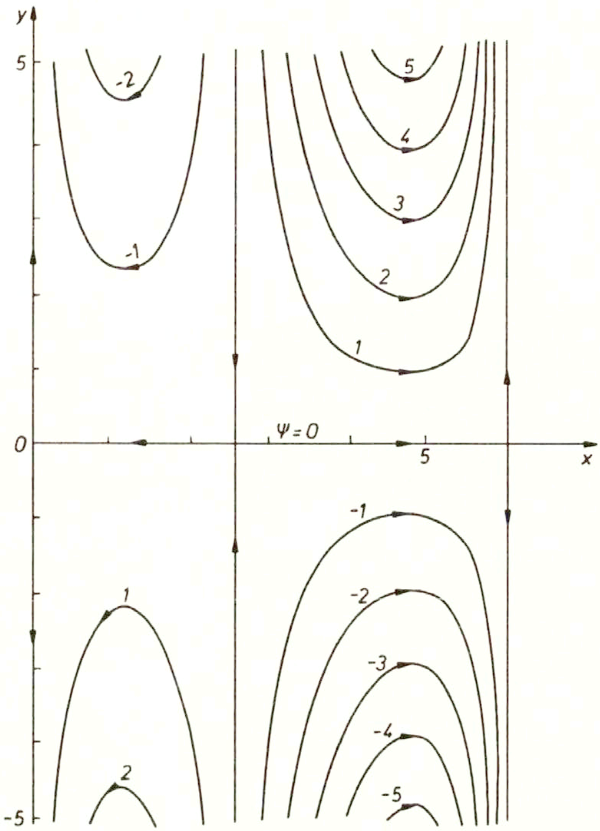

FIG. 11.

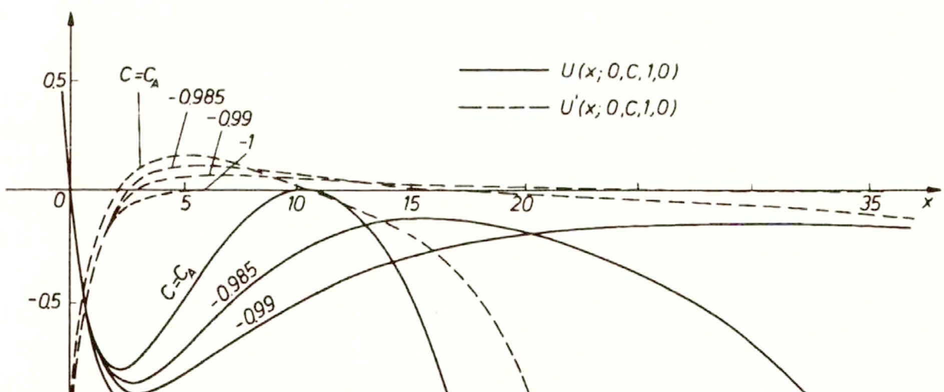

FIG. 12.

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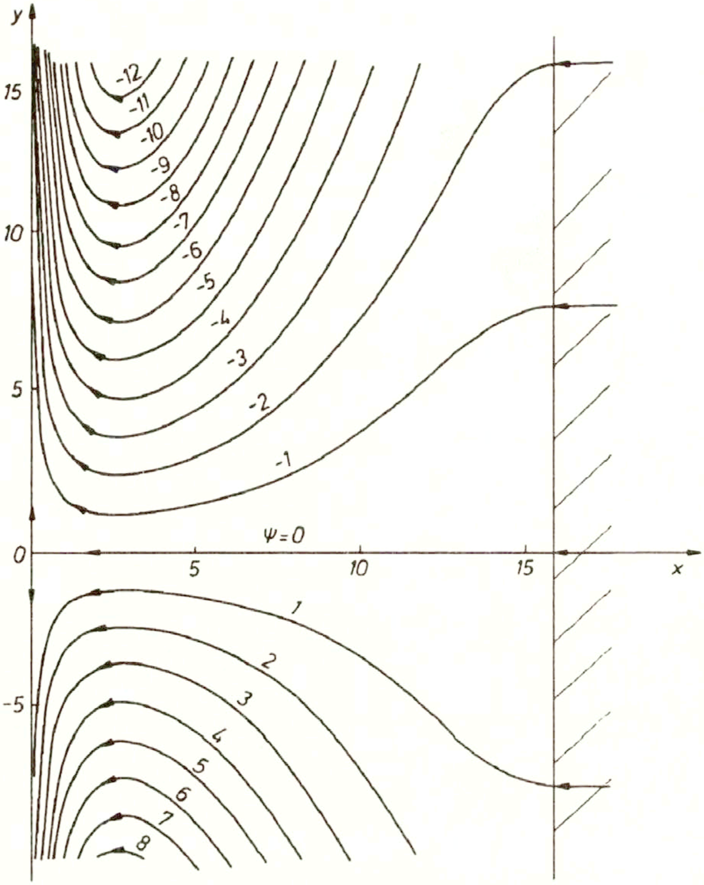

FIG. 13.

Solutions $U(x) = U(x;0,C,1,0)$ are very sensitive to small changes of C around $C = -1$. Although for $C = -1$ we obtain $U_R(x) = -(1-e^{-x})$ with a finite asymptotic value $U_R(\infty) = -1$, an infinitesimal perturbation $|\delta| \ll 1$ in $C = -1+\delta$ is sufficient to produce an infinite decreasing to $-\infty$ of $U(x;C_0,C_1,C_2,0)$, giving $U(x;0,C,1) \to \alpha_+ U_R(-\alpha_+(x-x_+)) = \alpha_+(1-e^{\alpha_+(x-x_+)})$ with $\alpha_+ > 0$ for $x \to \infty$.

The constants C_0, C_1, C_2, Eqs. (6.21), defined in similar solutions (6.20) $U(x) = U(x;C_0,C_1,C_2,0)$ by $U(x;0,C,1)$ with $-1 < C < C_A$ (6.24), are distributed on both sides of the coordinate surface $C_1 = 0$. In the close vicinity of Riabouchinsky's surface for $C_1 < 0$, the solutions $U(x)$ with $C_0 \approx -\lambda(1-B)$, $C_1 \approx -\lambda^2 B$, $C_2 \approx \lambda^3 B$, (cf. Eq. (6.7)) tend to infinity instead of the bounded asymptotical value $-\lambda$ for $\lambda x \to \infty$. But the influence of infinitesimal perturbations of solutions $\lambda U_{R^*}(\lambda(x-x_0)) = -\lambda(1+Be^{-\lambda x})$ Eq. (6.8) with $C_0 \approx -\lambda(1+B)$, $C_1 \approx \lambda^2 B$, $C_2 \approx -\lambda^3 B$, on their asymptotical behaviour is much stronger. For

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$C_1 > 0$ and $0 < \mathcal{R} \ll 1$ such perturbations provoke not only tending of $U(x)$ to infinity instead of the asymptotical value $-\lambda$ for $\lambda x \to \infty$ on one side, but additionally also on the other side the perturbed solutions tend to infinity $U \to \pm\infty$ with the sign opposite to the unperturbed solutions $\lambda U_{R^*} \to \pm\infty$ for $\lambda x \to -\infty$. By perturbations with $\mathcal{R} > 0$ of both solutions (6.19) and (6.8), we may obtain all solutions belonging to the classes of similarity defined by basic functions $U(x; 0, \mathcal{C}, 1, 0)$ with $-1 < \mathcal{C} < \mathcal{C}_A$, or even with $-1 < \mathcal{C} < \mathcal{C}_+$. An analogous rule for the case $\mathcal{R} < 0$ is not valid: solutions with $C_1 < 0$ are not similar to solutions with $C_1 > 0$.

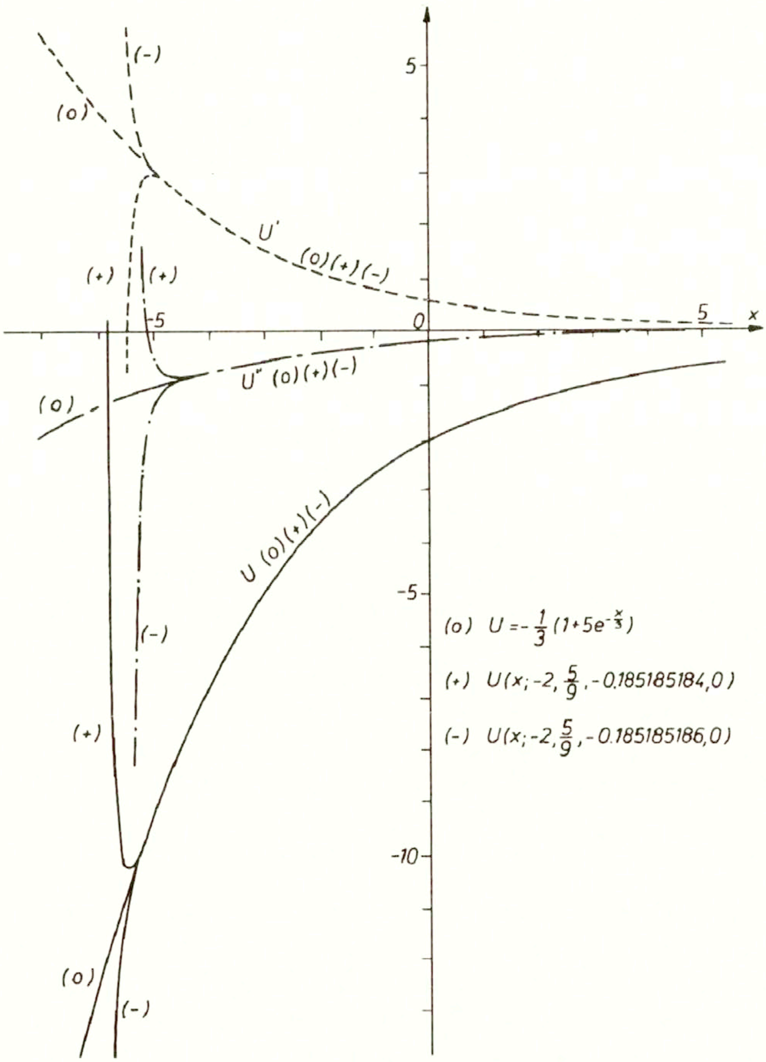

FIG. 14.

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High sensitivity of $U(x;-2,5/9,-5/27,0) = -\dfrac{1}{3}(1+5e^{-x/3})$, $\mathcal{R}=0$ to small perturbations of $C_2 = -0.185185185...$ for $x < 0$ is shown in Fig. 14. The main exponential formula was used to plot $U(x)$, $U'(x)$, $U''(x)$ denoted by (0). Signs (+) and (−) denote the perturbed solutions with

$$(+)\ C_2 = -0.185185184, \qquad \mathcal{R} = 1.3\cdot 10^{-14} > 0$$

and

$$(-)\ C_2 = -0.185185186, \qquad \mathcal{R} = -8.9\cdot 10^{-15} < 0,$$

respectively. It may be observed that all solutions almost coincide for moderate values of x, but for higher $|x|$, exceeding some characteristic values, they differ from each other not only quantitatively, but also qualitatively. The solution (+) $U(x;-2,0.555...,-0.185185184,0)$, being similar to $U(x;0,-0.9967...,1,0)$, may be also obtained by means of Eqs. (6.20) and (6.21) with:

$$\mathcal{R} = 0.00979..., \qquad \mathcal{C} = -0.996724521256897738...,$$

$$x_C = 64.10609306..., \qquad \lambda = 11.00385215... .$$

7. Final remarks

The solutions $U(x;C_0,C_1,C_2,0)$ of the Hiemenz equation (1.2) may be divided into different classes of similarity (Sec. 2) with parameters $C_0 = U(0)$, $C_1 = U'(0)$, $C_2 = U''(0)$, (1.3) distributed on two-dimensional surfaces in the three-dimensional space C_0, C_1, C_2. By means of the transformation (2.5), every solution may be obtained from a similar one, which eventually may be chosen as their basic representative function. These basic functions may be also used to analyze main properties of their similar solutions and they may be chosen in such a way that they depend on one parameter only (Sec. 2).

Two cardinal classes of similarity are represented on the surface $\mathcal{R} = C_1^3 + C_2^2 - C_0C_1C_2 = 0$, Eq. (6.9), by the Riabouchinsky solutions (6.7) and (6.8), which may be obtained from the basic functions (Sec. 6.3, Fig. 3):

$$U_R(x) = -(1-e^{-x}) \quad \text{for} \quad C_1 < 0 \quad \text{and} \quad U_{R^*}(x) = -(1+e^{-x}) \quad \text{for} \quad C_1 > 0.$$

Being defined by C_0, C_1, C_2, with $\mathcal{R}(C_0,C_1,C_2) = 0$, they are unstable and very sensitive to small perturbations of $\mathcal{R}$. Solutions $U(x;C_0,C_1,C_2,0)$ perturbed with $0 < |\mathcal{R}(C_0,C_1,C_2| \ll 1$ differ much not only quantitatively but also qualitatively from Riabouchinsky's solutions (6.7) and (6.8) with $\mathcal{R} = 0$. This is shown in the Fig. 14 for the example $U(x;-2,5/9,-5/27,0)$ of Eq. (6.8) with $\mathcal{R} = 0$, $C_1 > 0$, where perturbation of $C_2 \approx -5/27$ with $|\mathcal{R}| \approx 10^{-14}$ gave rise to infinite differences of $U(x)$ for $|x| \to \infty$ and to discrepancies in $U(-6)$ and in $U''(-6)$

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of the order of 100 and 10^5, respectively. But such perturbed solution fulfills also the Hiemenz equation (1.2) and it belongs to a similarity class defined by the basic functions (6.22) $U(x; 0, -0.9967..., 1, 0)$ with $\mathcal{R} = 0.00987...$ and $C_1 < 0$. The Riabouchinsky solutions take intermediate position between the main classes of similarity represented by $\mathcal{R}(C_0, C_1, C_2) < 0$ and $\mathcal{R}(C_0, C_1, C_2) > 0$. Thus, the sign of $\mathcal{R}$ is here considered as the main attribute of Hiemenz solutions $U(x; C_0, C_1, C_2, 0)$.

Monotonous solutions $U(x; C_0, C_1, C_2, 0)$ from the "negative" side $\mathcal{R} < 0$, with $C_1 > 0$ (Sec. 6.2, Fig. 8) or with $C_1 < 0$ (Eq. (6.13), Figs. 4, 9), do not seem interesting for hydrodynamics and here they will not be considered.

All solutions $U(x; C_0, C_1, C_2, 0)$ with $\mathcal{R} > 0$ are not monotonous and their derivatives $U'(x)$ have two zeros (Figs. 4, 6, 10, 12). Each of them decreases from infinity, where $U(x) \to \alpha_-(1 - e^{\alpha_-(x-x_-)})$ for $x \to -\infty$ with $\alpha_- < 0$, then it goes through local minimum and maximum and, finally, it decreases to minus infinity, as $U(x) \to \alpha_+(1 - e^{\alpha_+(x-x_+)})$ for $x \to \infty$ with $\alpha_+ > 0$, Eq. (6.11). So, they all tend asymptotically to Riabouchinsky's function $\alpha_\pm(1 - e^{\alpha_\pm(x-x_\pm)})$ for $\alpha_\pm x \to \infty$ and they may be defined by the basic functions $U(x; 0, \mathcal{C}, 1, 0)$, Eq. (6.22), with $\mathcal{C}$ from the close vicinity $-1 < \mathcal{C} \le \mathcal{C}_+$ of the Riabouchinsky's value $\mathcal{C} = -1$. Examples of flow fields $\Psi(x, y) = \text{const}$, obtained from the considered cases of solutions $U(x)$, are shown in Figs. 5, 6, 11, 13.

Since for computing $U(x; C_0, C_1, C_2, 0)$ a high accuracy was needed, much attention had to be paid to the evaluation of errors. The analytically continued power series, with their truncation as the source of errors, was chosen to prepare the numerical program HIM0, which was tested by comparing similar solutions computed in different ways with both options. Exact solutions from Secs. 6.2 and 6.3 were compared also with their approximations computed by means of HIM0. Although the results obtained were in general satisfactory, discrepancies were also observed, but they could be explained by instabilities, as in the case of $U(x; -2, 5/9, -5/27, 0)$ (Fig. 14).

Hydrodynamic parameters u, v, p, Eq. (1.1) expressed by $U(x)$ of Eq. (1.2) may describe some stationary, symmetric flows between two parallel plane surfaces $x_\pm = \text{const}$ of incompressible, viscous liquid, if they satisfy at these surfaces the suitable boundary conditions, mainly four kinematic conditions concerning normal $u(x_\pm, y) = U(x_\pm) = \mathcal{U}_\pm$ and tangent $\partial v(x_\pm, y)/\partial y = -U'(x_\pm) = -\mathcal{V}_\pm$ components of velocity with $\mathcal{U}_\pm, \mathcal{V}_\pm = \text{const}$. Such problem may enable us to find four unknown values C_0, C_1, C_2, σ from four equations $U(x_\pm; C_0, C_1, C_2, \sigma) = \mathcal{U}_\pm$, $U'(x_\pm; C_0, C_1, C_2, \sigma) = \mathcal{V}_\pm$, but in order to find here for $\sigma = 0$ three unknown values C_0, C_1, C_2, we must satisfy an additional relation between $\mathcal{U}_-, \mathcal{U}_+, \mathcal{V}_-, \mathcal{V}_+$. It means that the quasi-isobaric flow, with pressure depending on one variables x, may be realized for particular boundary conditions only. Generally, the pressure (1.1) found from $U(x; C_0, C_1, C_2, \sigma)$ with $\sigma \neq 0$ should depend on two variables x, y but analyzing the flows between parallel surfaces, an introductory analysis of properties of $U(x; C_0, C_1, C_2, \sigma)$ with $\sigma = 0$ seems to be also useful.

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Appendix

The aim of this Appendix is to prove the existence of two positive numbers $\gamma, r > 0$, allowing to estimate the coefficients defined by Eq. (3.2)

$$|a_n| \leq \gamma / r^{n+1} \qquad n = 0, 1, 2, 3, \dots, \tag{A.1}$$

and then to formulate the rules of the proper choice of γ, r. As the inequality (A.1) allows to bound and to estimate the remainder Eq. (4.2) by means of a geometric series with the radius of convergence r (cf. Sec. 4), we will try to find a possibly large value of r.

In the first part we will prove the following:

THEOREM. *For a_k determined by* Eq. (3.2), *there exist two positive numbers $\gamma, r > 0$, such that if a_k fulfills* (A.1) *for $k = 0, 1, 2, \dots, n-1$, where $n > 2$, then the condition* (A.1) *is valid for $k = n$. In consequence, the inequality* (A.1) *becomes valid for any positive integer $k = 0, 1, 2, 3$, with the same γ, r.*

In the proof we will use Γ_n, $n = 3, 4, \dots$, defined by means of $c_{n,k}$ (3.3′),

$$\frac{1}{\Gamma_n} = \sum_{k=0}^{n-1} |c_{n,k}|, \tag{A.2}$$

where

$$c_{n,k} = \frac{k(2k-n)}{(n-2)(n-1)n}, \qquad k = 0, 1, \dots n-1, \quad n = 3, 4, \dots,$$

and we will need there the conclusion of the auxiliary

LEMMA. The progression $\Gamma_n > 0$ is not decreasing, i.e

$$\Gamma_n \leq \Gamma_{n+1} \qquad \text{for} \quad n = 3, 4, \dots . \tag{A.3}$$

To prove this Lemma, let us introduce the difference

$$\frac{\Gamma_{n+1} - \Gamma_n}{\Gamma_n \cdot \Gamma_{n+1}} = \frac{1}{\Gamma_n} - \frac{1}{\Gamma_{n+1}} = \sum_{k=0}^{n-1} |c_{n,k}| - \sum_{k=0}^{n} |c_{n+1,k}|.$$

Taking into account (A.2) $c_{n,k} < 0$ for $2k < n$, $c_{n,k} = 0$ for $2k = n$, $c_{n,k} > 0$ for $2k > n$ and $c_{n,0} = 0$, let us apply here the following transformations:

$$\sum_{k=0}^{n-1} |c_{n,k}| - \sum_{k=0}^{n} |c_{n+1,k}| = \sum_{k=1}^{\nu} (c_{n,n-k} - c_{n,k} - c_{n+1,n+1-k} + c_{n+1,k})$$
$$= \sum_{k=1}^{\nu} \frac{(n+2) - 4k}{(n-2)(n-1)n} = \frac{\nu \cdot (n+2) - \nu \cdot (2\nu + 2)}{(n-2) \cdot (n-1)n},$$

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where

$$\nu = \begin{cases} \dfrac{n-1}{2} & \text{for} \quad n = 3, 5, \dots, \\ \dfrac{n}{2} & \text{for} \quad n = 4, 6, \dots . \end{cases}$$

In consequence, all differences

$$\Gamma_{n+1} - \Gamma_n = \Gamma_n \cdot \Gamma_{n+1} \cdot \begin{cases} \dfrac{1}{2n \cdot (n-2)} & \text{for} \quad n = 3, 5, 7, \dots, \\ 0 & \text{for} \quad n = 4, 6, 8, \dots . \end{cases}$$

are not negative and the relation (A.3) is satisfied, what had to be proved.

Now we return to the main Theorem. Taking account of (A.1) in Eq. (3.2), we may obtain the rough estimates

$$\text{(A.4)} \qquad |a_n| \le \sum_{k=0}^{n-1} |c_{n,k}| \cdot |a_k| \cdot |a_{n-1-k}| \le \frac{\gamma^2}{r^{n+1}} \sum_{k=0}^{n-1} |c_{n,k}| = \left(\frac{\gamma}{\Gamma_n}\right) \cdot \frac{\gamma}{r^{n+1}} .$$

From Eq. (A.2) we find $\Gamma_3 = 2$ and from the Lemma we obtain $\Gamma_3 < \Gamma_n$ for $n > 3$, hence in order to prove the existence of $\gamma, r > 0$ in the Theorem, it is sufficient to choose for (A.1) two numbers:

$$\text{(A.5)} \qquad \gamma = \Gamma_3 = 2 \qquad \text{and} \qquad r = \text{Min}\left(\gamma/|a_0|, \sqrt{\gamma/|a_1|}, \sqrt[3]{\gamma/|a_2|}\right) .$$

The inequality (A.1) is now fulfilled for $n = 0, 1, 2$. Satisfaction of (A.1) for any larger $n = 3, 4, \dots$ results from $\gamma/\Gamma_n = \Gamma_3/\Gamma_n \le 1$ in Eq. (A.4). Thus the existence of two numbers $\gamma, r > 0$ has been proved, but their values, determined by (A.5), seem to be too small. The following part of this Appendix will be devoted to the proper choice of γ, r.

It may be observed that we could also use $\widetilde{\Gamma}_n = \left(\sum_{k=0}^{n-1} |\widetilde{c}_{n,k}|\right)^{-1}$ instead of the slightly smaller $\Gamma_n \le \widetilde{\Gamma}_n$. Some values of nondecreasing progressions Γ_n, $\widetilde{\Gamma}_n$, obtained from Eqs. (3.3), (3.3′), are given in the table below

n	3	4	8	16	32	64	99	128	∞
Γ_n	2	3	3.5	3.75	3.875	3.937	3.959	3.968	4
$\widetilde{\Gamma}_n$	2	3	4.9	5.60	5.863	5.952	5.975	5.983	6

In the same way as in (A.5), we may find higher γ, r, satisfying the inequalities

$$\text{(A.6)} \qquad \gamma \le \widetilde{\Gamma}_n, \qquad r \le \left|\frac{\gamma}{a_k}\right|^{1/(k+1)}, \qquad k = 0, 1, \dots, N,$$
$$n = N+1, N+2, \dots, \qquad N > 2, \qquad a_k \ne 0,$$

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but their increment ($\widetilde{\Gamma}_n - \Gamma_3 < 4$) is not large. The value γ is determined by N and by the matrix $\widetilde{c}_{n,k}$ (3.3′) only, and r additionally depends, through a_k, on the particular solution $U(x; a_0, a_1, 2a_2, 0)$. Such γ, r are obtained from the rough estimations (A.4), which take into account neither the posible vanishing of some $a_{i,n}$, nor the signs of $\widetilde{c}_{n,k}a_{i,k}a_{i,n-k-1}$. Application of better evaluations would be difficult and so we will try to obtain higher values of γ, r by a heuristic method.

At first let us transform (A.1),

$$\text{(A.7)} \qquad r \leq \gamma^{1/(n+1)} \cdot |a_n|^{-1/(n+1)} \qquad \text{and} \qquad \gamma \geq |a_n| \cdot r^{n+1} \qquad \text{for} \quad n = 0, 1, 2, 3, \ldots .$$

As $\lim_{n\to\infty} \gamma^{1/(n+1)} = 1$, we may introduce the asymptotic values

$$\text{(A.8)} \qquad r_\infty = \liminf_{n\to\infty}(|a_n|^{-1/(n+1)}), \qquad \gamma_\infty = \limsup_{n\to\infty}(|a_n| \cdot r_\infty^{n+1}),$$

which should majorize all coefficients:

$$|a_n| \leq \gamma/r^{n+1} \leq \gamma_\infty/r_\infty^{n+1} \qquad \text{for} \qquad n = 0, 1, 2, 3, \ldots .$$

Our aim is here to find the largest possible r and γ, approaching their asymptotic values r_∞ and γ_∞.

From the error evaluation (4.7), (4.8) and from Fig. 1 it is seen, that a considerably large number $1 + N$ of terms is advised to be taken in the truncated power series (4.1). To find the convenient, higher values γ, fulfilling (A.1) with a sufficiently large r, the following procedure is proposed.

We begin by choosing a large truncation number N and the initial input values: not too large $p_{-1} = \gamma_{\text{ini}}$ (for instance $\gamma_{\text{ini}} = \widetilde{\Gamma}_{N+1}$) and large $q_{N+1} \gg p_{-1}$. We determine the output values r, γ by means of the recursive procedure:

$$\text{(A.9)} \qquad \begin{aligned} q_{N-n} &= \begin{cases} \operatorname{Min}\left(q_{N-n+1}, (p_{n-1}/|a_{N-n}|)^{1/(N-n+1)}\right) & \text{for} \quad a_n \neq 0, \\ q_{N-n+1} & \text{for} \quad a_n = 0, \end{cases} \\ p_n &= \operatorname{Max}\left(p_{n-1}, |a_n| q_{N-n}^{n+1}\right), \end{aligned}$$

$$r = q_{N-M}, \qquad \gamma = p_M, \qquad \text{where} \quad n = 0, 1, 2, \ldots, M \quad \text{and} \quad N - M \leq M < N.$$

Since

$$p_0 \leq p_1 \leq \ldots \leq p_M \qquad \text{and} \qquad q_N \geq q_{N-1} \geq \ldots q_{N-M},$$

we obtain also from (A.9) the following inequalities:

$$\begin{aligned} |a_n| &\leq \frac{p_n}{q_{N-n}^{n+1}} \leq \frac{p_M}{q_{N-M}^{n+1}} = \frac{\gamma}{r^{n+1}}, \\ |a_{N-n}| &\leq \frac{p_{n-1}}{q_{N-n}^{N-n+1}} \leq \frac{p_M}{q_{N-M}^{N-M+1}} = \frac{\gamma}{r^{n+1}} \end{aligned}$$

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for $n = 0, 1, \ldots, M$.

In consequence, the main inequality (A.1) $|a_n| \leq \gamma/r^{n+1}$ is here satisfied for all $1 + N$ coefficients a_n $(n = 0, 1, \ldots, N)$. The number M of steps in the recurrence formulae (A.9) should be not less than $N/2$, but not necessarily much greater than that. The recurrence (A.9) may also be repeated many times with larger N and with the input value γ_{ini} taken from the previous output result: $\gamma_{\text{ini}} = p_M$. With growing $N \to \infty$, the values r, γ approach their asymptotic values $r \to r_\infty$ and $\gamma \to \gamma_\infty$.

The suitable values γ_i, r_i depend not only on solutions $U(x; C_0, C_1, C_2, 0)$, but also on their analytic continuations $U(x - x_i; a_{i,0}, a_{i,1}, 2a_{i,2}, 0)$, the errors of which are determined in Sec. 4. Thus, the proper choice of γ_i, r_i is here very important for the accuracy of computing analytical continuations. The heuristic procedure presented was "empirically verified" by analyzing several examples.

As an example of choosing γ, r, let us consider $U_A(x) = U(x; 0, 0, 1, 0)$ (Sec. 6.5), expanded at the center $x_0 = 0$ into the power series (3.7). Their first non-vanishing coefficients a_n $(n = 3m - 1$ for $m = 0, 1, 2, \ldots, 11)$ with $|a_n|^{-1/(n+1)} = |a_{3m-1}|^{-1/3m}$ and $\widetilde{\Gamma}_{3m-1}$, obtained from (A.2), (3.3′) with $\widetilde{c}_{n,k} = \widetilde{c}_{3m-1,3k-1}$, are presented in the following table:

m	$3m-1$	a_{3m-1}	$\lvert a_{3m-1}^{-1/3m}\rvert$	$\widetilde{\Gamma}_{3m-1}$
1	2	$1/2!$	1.260	–
2	5	$-1/5!$	2.221	30
3	8	$-1/8!$	3.249	168
4	11	$-27/11!$	3.267	31.93
5	14	$-951/14!$	3.394	30.33
6	17	$-51465/17!$	3.520	26.15
7	20	$-1.3793638168 \cdot 10^{-12}$	3.671	24.60
8	23	$-5.874817912 \cdot 10^{-15}$	3.917	23.60
9	26	$7.435121112 \cdot 10^{-17}$	3.956	22.67
10	29	$1.730926415 \cdot 10^{-18}$	3.909	22.33
11	32	$1.311332744 \cdot 10^{-20}$	4.004	21.66

....

It is seen that in this case, with many vanishing coefficients a_n, the progression $\widetilde{\Gamma}_{3m-1}$ non-monotonously depends on m. With m growing to infinity it decreases to its asymptotic value $\widetilde{\Gamma}_\infty = 18$, obtained from Eq. (3.8),

$$\lim_{m\to\infty} \frac{1}{\widetilde{\Gamma}_{3m-1}} = \lim_{m\to\infty} \frac{1}{3} \sum_{0\leq 2k<m} \left|1 - 4\frac{3k-1}{3m-3}\left(1 - \frac{3k-1}{3m-2}\right)\right| \frac{3}{3m-1}$$

$$= \frac{1}{3}\int_0^{1/2} |1 - 4\kappa(1-\kappa)|\, d\kappa = \frac{1}{18}\,.$$

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The values $\gamma = 2$ and $r = \sqrt[3]{2/0.5} = \sqrt[3]{4} = 1.5874$ may be found here from Eq. (A.5), basing on the rough estimates Eq. (A.4) with (A.3) for $N = 2$.

Introducing the same truncation number $N = 2$, but applying (A.6) and taking into account $\widetilde{\Gamma}_\infty < \widetilde{\Gamma}_{3m-1}$ for $m = 1, 2, \ldots$, we find higher values of γ and r,

$$\gamma = 18 \qquad \text{and} \qquad r = \sqrt[3]{18/0.5} = \sqrt[3]{36} = 3.3019.$$

Let us pay more attention to the heuristic method based on the recursive formulae (A.9). At first let us assume $N = 19$, $M = 11$ and $\gamma_{\text{ini}} = 18$. At these assumptions we obtain from (A.9)

$$\begin{aligned}
q_{18} &= q_{19} \gg 18,\\
p_1 &= p_0 = 18,\\
q_8 &= q_9 = q_{10} = q_{11} = q_{12} = q_{13} = q_{14} = q_{15} = q_{16} = q_{17} = 4.134,\\
p_4 &= p_3 = p_2 = 35.32,\\
p_{11} &= p_{10} = p_9 = p_8 = p_7 = p_6 = p_5 = 41.595,\\
r &= 4.134,\\
\gamma &= 41.595.
\end{aligned}$$

The same procedure may be repeated for $N = 34$, $M = 17$ and $\gamma_{\text{ini}} = 41.595$, giving (some q_{N-n} and p_n have not been written here):

$$q_{32} = 4.48, \quad q_{20} = q_{23} = q_{26} = q_{29} = 4.4379, \quad q_{17} = 4.4346, \quad r = 4.4346,$$

$$p_0 = 41.595, \quad p_2 = 45.045, \quad p_{17} = p_{14} = p_{11} = p_8 = p_5 = 63.66, \quad \gamma = 63.66.$$

It should be emphasized that the value γ_{ini} may not be assumed to be arbitrarily large. Choosing in the considered case arbitrarily $\gamma_{\text{ini}} = 100$, by means of (A.9) for $N = 34$, $M = 17$, we obtain $r = q_{17} = 4.5473$ and $\gamma = p_{17} = p_0 = 100$. It may be easily shown that such results are wrong and do not fulfill the conditions (A.1) $|a_{3m-1}| < \gamma/r^{3m}$ for many $n > N$.

The examples of comparison of a_{3m-1} with γ/r^{3m} $(3m-1 > N)$, for different γ and r obtained above, are presented in the table. It is seen that $\gamma = 2$, $r = \sqrt[3]{2/0.5} = 1.587$ Eq. (A.5) and $\gamma = 18$, $r = 3.302$ from (A.6) give correct but rough estimates of $|a_{3n-1}|$. More convenient for computing are $\gamma = 63.66$, $r = 4.4346$, obtained by the heuristic method (A.9). But the same method with erroneous initial value $\gamma = \gamma_{\text{ini}} = 100$ yields too large $r = 4.5473$ giving, in general, wrong results with $|a_{3m-1}| > \gamma/r^{3m}$.

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$3m-1$	a_{3m-1}	$\lvert a_{3m-1}^{-1/3m} \rvert$	γ/r^3m			
			$\gamma = 2$ $r = 1.587$	18 3.302	63.66 4.4346	100 4.5473
35	$-6.527662 \cdot 10^{-23}$	4.133	$1.2 \cdot 10^{-7}$	$3.8 \cdot 10^{-18}$	$3.29 \cdot 10^{-22}$	$2.1 \cdot 10^{-22}$
47	$2.864759 \cdot 10^{-30}$	4.125	$4.7 \cdot 10^{-10}$	$2.3 \cdot 10^{-24}$	$5.69 \cdot 10^{-30}$	$2.7 \cdot 10^{-30}$
65	$-5.333759 \cdot 10^{-42}$	4.220	$1.1 \cdot 10^{-13}$	$1.0 \cdot 10^{-33}$	$1.29 \cdot 10^{-41}$	$3.9 \cdot 10^{-42}$
95	$-1.593358 \cdot 10^{-61}$	4.298	$1.1 \cdot 10^{-19}$	$2.8 \cdot 10^{-49}$	$5.08 \cdot 10^{-61}$	$7.2 \cdot 10^{-62}$
125	$-4.541070 \cdot 10^{-81}$	4.341	$1.0 \cdot 10^{-25}$	$7.8 \cdot 10^{-65}$	$2.00 \cdot 10^{-80}$	$1.3 \cdot 10^{-81}$
152	$-2.829481 \cdot 10^{-98}$	4.340	$3.9 \cdot 10^{-31}$	$7.6 \cdot 10^{-79}$	$6.84 \cdot 10^{-98}$	$2.3 \cdot 10^{-99}$
188	$1.566638 \cdot 10^{-121}$	4.358	$2.3 \cdot 10^{-38}$	$1.6 \cdot 10^{-97}$	$2.3 \cdot 10^{-121}$	$5 \cdot 10^{-123}$

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On interaction between internal defects and external surface in the ductile fracture mechanics

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SLIP-LINE TECHNIQUE is used for the analysis of the ideally ductile interaction between internal defects in metals close to the external surface of the body and that surface. During plastic deformation of the body local reduction of the working area to a point may occur. Such a separation of the material is referred to as the ideally ductile fracture.

1. Introduction

THE MECHANICS of ductile fracture in metals is complex and not yet fully examined. Depending on the temperature, which in some cases (rocket engines, gas turbines, chemical installations) may be very high, on the rate of deformation and on the internal structure of the metal itself, various mechanisms may play a crucial role in the fracture process understood as a process leading to a total or local separation of the material. Coalescence of voids, various mechanisms of interaction between internal defects, interaction between such defects and external surface of the body may contribute to the progressing process of ductile fracture. Ductile fracture processes cannot be analysed in terms of the brittle fracture mechanics based on the assumption of the elastic model of the material.

In numerous theoretical studies the model of the elastic-plastic material has been used for numerical calculation of the propagation of plastic zones at the front of the crack. Usually such calculations are performed with the use of the FEM technique.

In most commercial metals the fracture mechanisms is accompanied by both the ductile and brittle phenomena. The interaction between the two factors is complex and still not fully examined. Depending on the circumstances, either brittle or ductile phenomena are dominating during the course of the fracture processes. In some cases both of them play almost the same role. In the extreme cases plastic phenomena may prevail and the fracture consists in reducing the area of the critical cross-section of the body to a point as the result of the process of local plastic flow. Such a local separation of the material will be referred to as the ideally ductile fracture. It may be interpreted as an extreme contrast to the ideally brittle fracture which is not accompanied by any plastic deformation.

Numerous problems of ideally ductile fracture can be analysed by using the standard slip-line technique based on the assumption of an idealized rigid-plastic model of the material. It has been shown, for example in a previous paper [1], that in the presence of certain systems of defects the ductile fracture mechanisms may

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lead to a total separation of the material, even when its total plastic deformation is rather small. Interaction between variously oriented systems of cracks or defects was analysed in [2] also with the use of the slip-line technique.

In the present study we shall analyse, using the slip-line technique, certain configurations of cracks and voids interacting with the external surface of the body. It will be demonstrated that the plastic deformation process leads to the local separation of the material even in such cases when a linear crack is oriented parallelly to the surface of the body under uniaxial tension. A crack of such orientation does not play any role in the brittle fracture theory.

2. Interaction between a linear crack and the surface of the body

Let us begin with the particular case important for the mechanics of ideally ductile fracture when a linear crack of the length $2a \geq m$ is parallel to the stress-free surface of a half-space subject to uniaxial tension (Fig. 1a). The process of plastic deformation will be analysed under plane strain conditions. Thus it will be assumed that the thickness in the direction perpendicular to the plane of the figure is large as compared with the characteristic dimensions m and a. Since the plastic deformation is strongly localized, only a certain domain $A - B - C - D$ indicated in Fig. 1a will be analysed in the following figures.

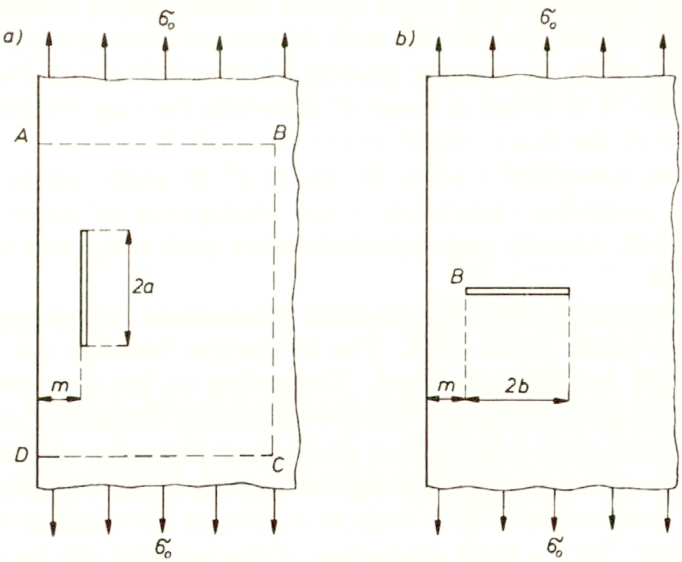

FIG. 1.

In Fig. 2 the complete slip-line solution to the limiting case when $m = 2a$ is presented. The mechanism of plastic deformation consists is rigid blocks motion. Initial configuration of slip-lines constituting simultaneously the lines of velocity discontinuity is shown by dashed lines in Fig. 2a. In Fig. 2b is presented an

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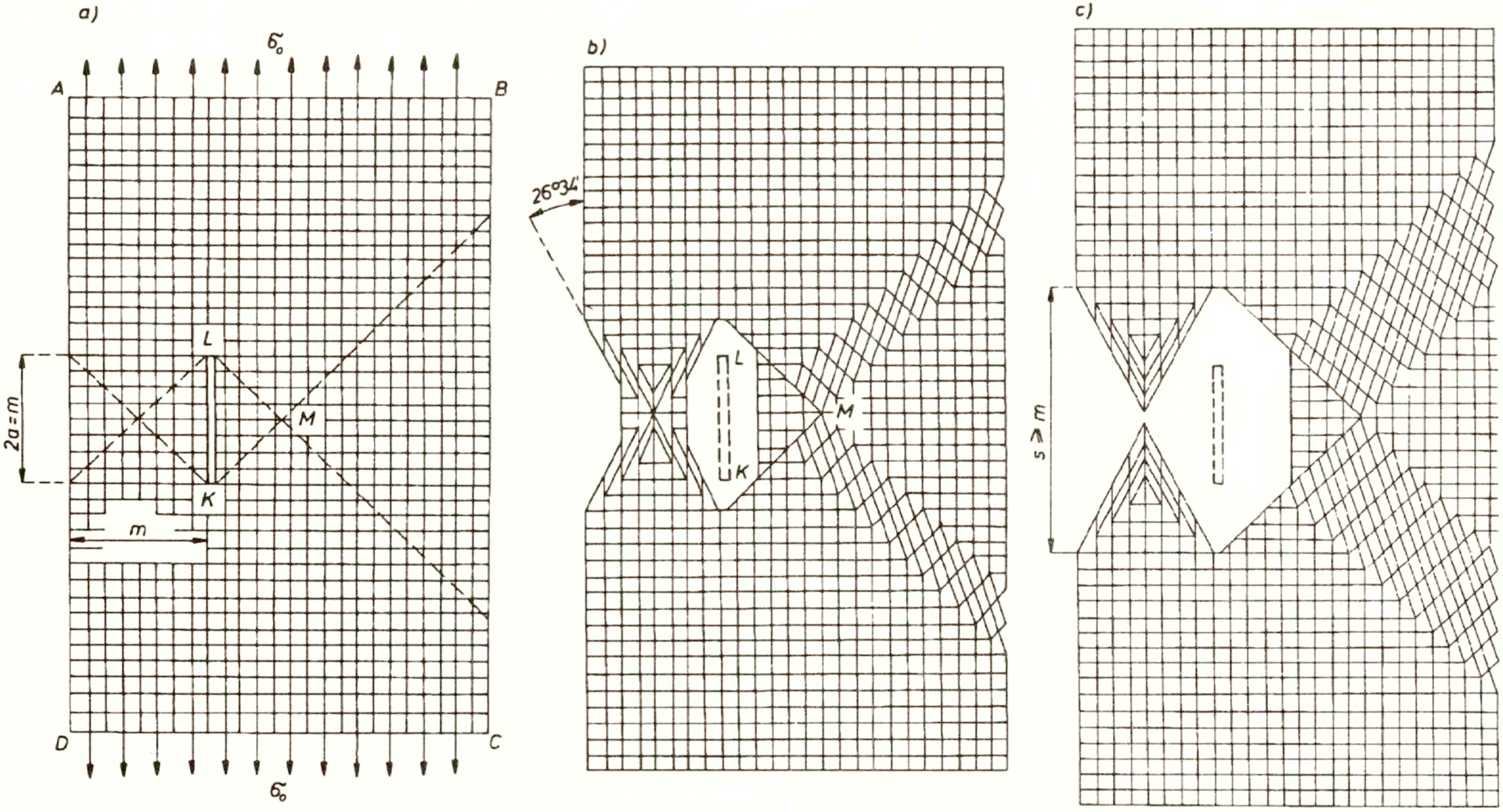

FIG. 2.

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advanced stage of plastic deformation. In the ligament between the crack and the external surface the well-known ideally plastic symmetrical necking takes place. We shall not discuss the details of the theoretical solution to this necking process since they are given in other papers (cf. e.g. [2]).

For that part of the material which is located to the right of the crack, an infinite number of complete solutions satisfying all the static and kinematic conditions can be constructed. However, that of them which is shown in Fig. 2b requires the smallest energy expenditure to carry out the deformation process. In this particular solution the plastic deformation consists in localized shearing along the two slip-lines $L - M$ and $K - M$, while the triangular region KLM moves to the right as a rigid block. Such a shearing mechanism can occur in the ideally plastic material without fracture (cf. e.g. [3]). Solution of this type looks rather artificial, even within the ideal plasticity theory. However, it can be viewed as a limit case of more realistic solutions such as that shown in Fig. 3, where a shearing band of a finite width is formed.

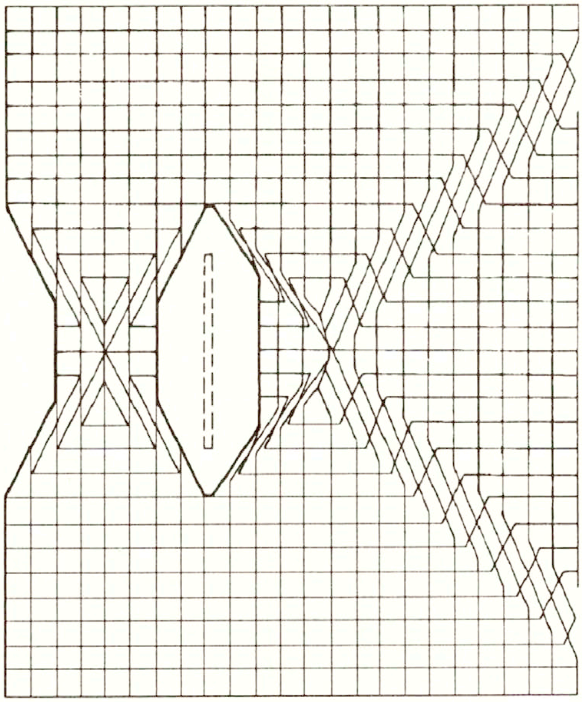

FIG. 3.

The necking process in the ligament leads finally to the local separation of the material as shown in Fig. 2c.

For short cracks when $2a < m$ the solution must be modified. The initial configuration of slip-lines for such a case is shown in Fig. 4a. Contrary to the

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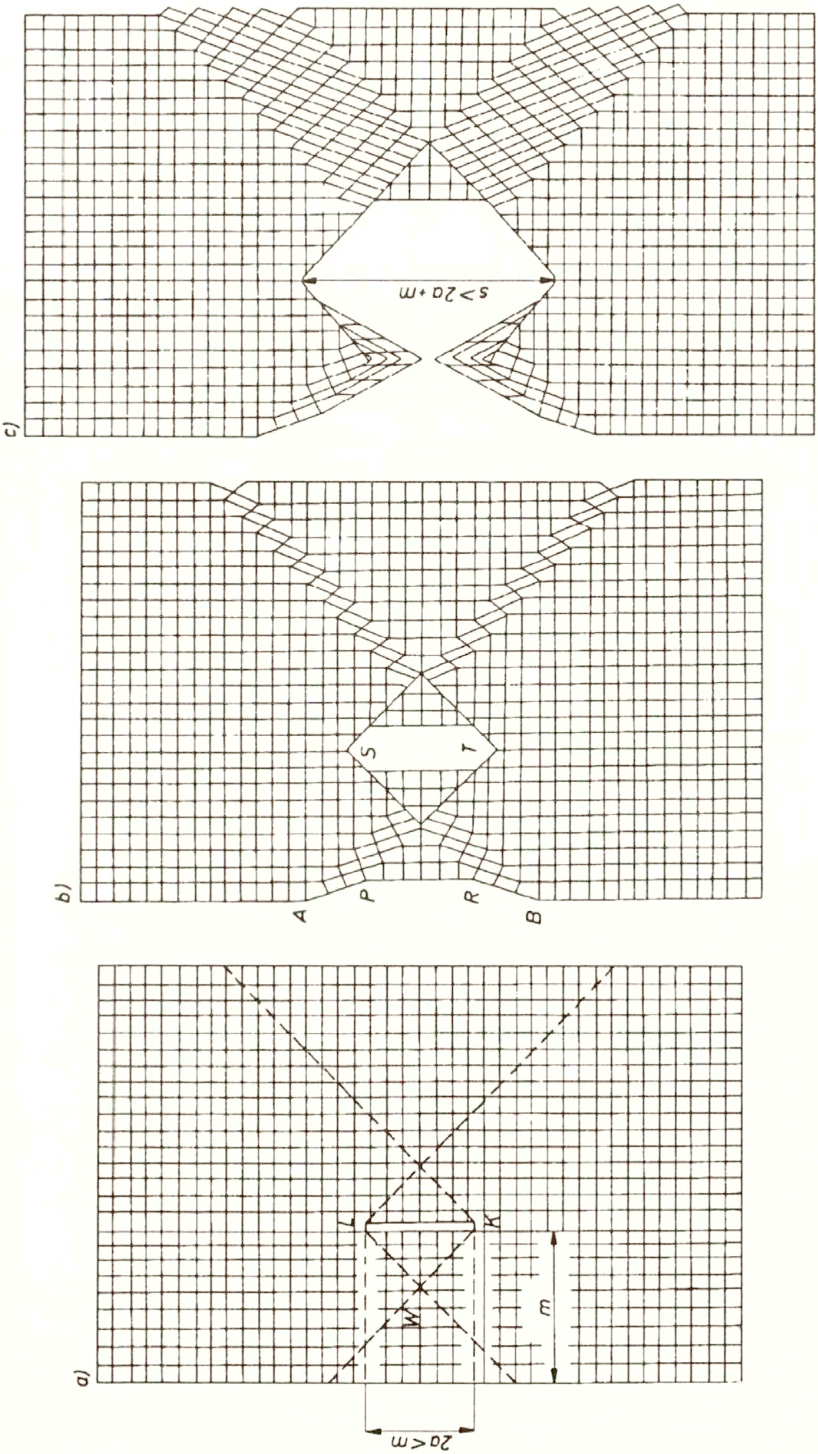

FIG. 4.

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previous case, now this configuration is not symmetric. An advanced stage of the deformation process shown in Fig. 4b was reached by assuming the mechanisms of rigid blocks motion. On both sides of the crack the mechanism consists in localized shearing along particular slip-lines. As before, such a deformation mode should be treated as a limit case of other more realistic possible solutions when shearing bands of finite width are formed. Note, however, that the particular solution from Fig. 4b requires the smallest energy to carry out the process of plastic deformation.

From the moment when the sectors $P-R$ and $S-T$ reach the same length, as shown in Fig. 4b, the symmetrical mode of necking may be assumed. This second stage of the process leads finally to the local reduction of the working area to a point, as shown in Fig. 4c. Geometrical parameters of the whole solutions are shown in Fig. 5.

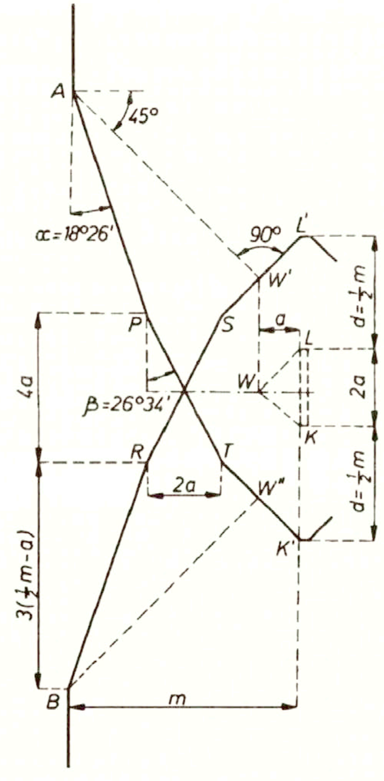

FIG. 5.

Total separation of the ligament shown in the figure takes place when the plastic elongation of the whole body reaches the value $2d = m$. The initial end points K and L of the crack are shifted then to the new positions K' and L'. The initial triangle $K - L - W$ shown in Fig. 4a has moved during the initial stage of the deformation process to the left, sliding along the shearing lines $L' - S$

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and $K' - T$ to assume the configuration shown in Fig. 4b, when the two segments $P - R$ and $S - T$ are of the same length. The segments $A - P$ and $B - R$ of the external surface are, after the initial stage of deformation, inclined at the angles

$$\alpha = \operatorname{arc\,tan}\frac{1}{3} = 18°26'$$

to the vertical direction. The final stage of the necking process is symmetrical, being identical with that shown in Fig. 2.

This type of slip-line solution may be used also in other cases of the ideally ductile fracture mechanics. An example is shown in Fig. 6 when a plastic strip weakened by a short ($2a < m$) crack parallel to the longitudinal direction is loaded by uniaxial tension. In the theory of brittle fracture mechanics such a crack has no influence on the behaviour of the strip loaded, as shown in the figure. However, when the problem is analysed in terms of the ideally ductile fracture mechanics, such configuration of the crack leads to the local strain concentration and, finally, to the total separation of the strip.

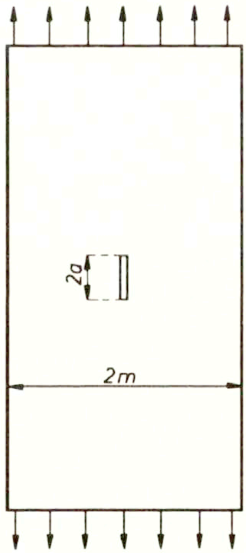

FIG. 6.

Complete solution to this problem composed of two slip-line solutions of the type shown in Fig. 5 is presented in Fig. 7a. A portion of the solution corresponding to the stage of deformation when symmetrical necking begins, is shown in Fig. 7b.

Analogous solutions may be used for the analysis of the ideally ductile fracture for such cases when the internal crack is perpendicular to the surface of the body (Fig. 1b) or is inclined to it.

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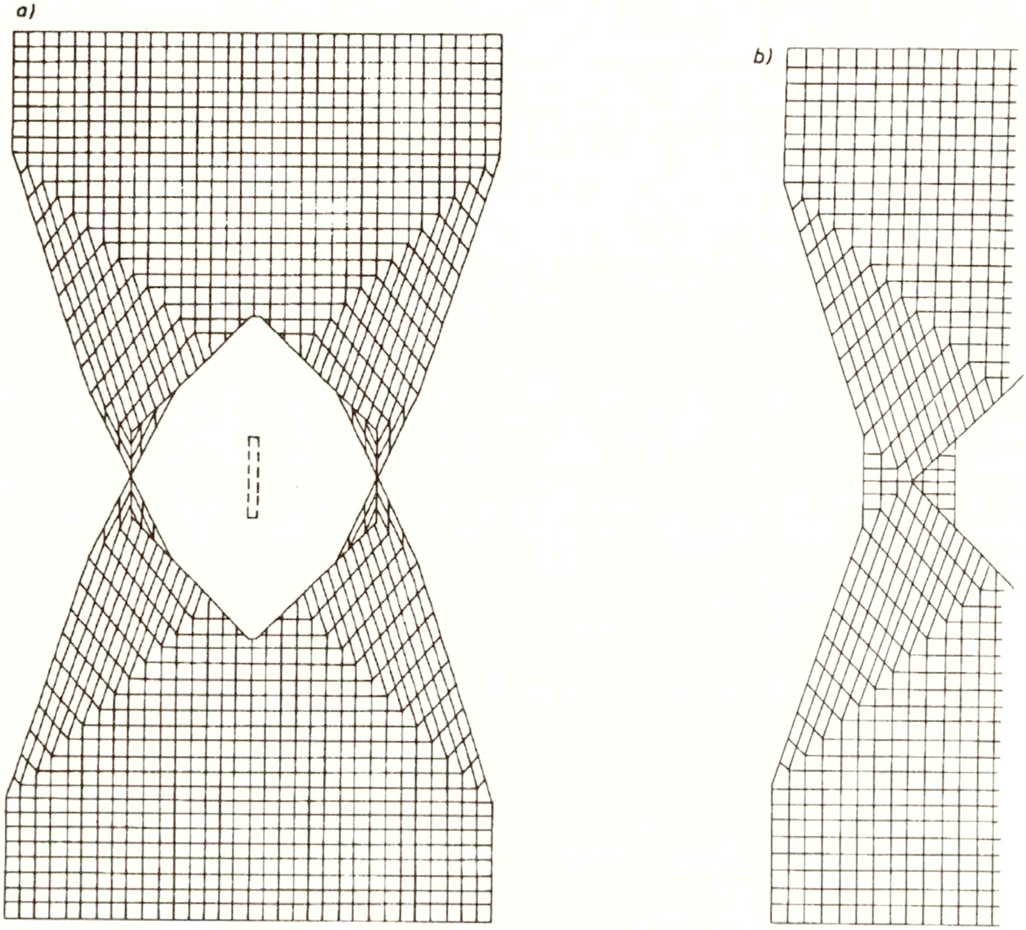

FIG. 7.

3. Interaction of a cavity with the surface of the body

Consider now an important case of interaction between the external surface of the body and a cavity located at a short distance from that surface (Fig. 8). The stress distribution in the elastic state around a circular hole lying close to the boundary of a semi-infinite plate loaded in tension was analysed by G.B. Jeffery and later by R.D. MINDLIN [4], cf. also [5]. The maximum stress occurs on the hole boundary at point E nearest to the straight edge. From that point plastic zones will begin to propagate if the value of the maximum stress reaches the yield point.

Analysing the problem in terms of the ideally ductile fracture mechanics, we shall assume as before the rigid-plastic model of the material. We shall analyse a certain domain $A-B-C-D$ indicated in Fig. 8, because the plastic deformation is strongly localized.

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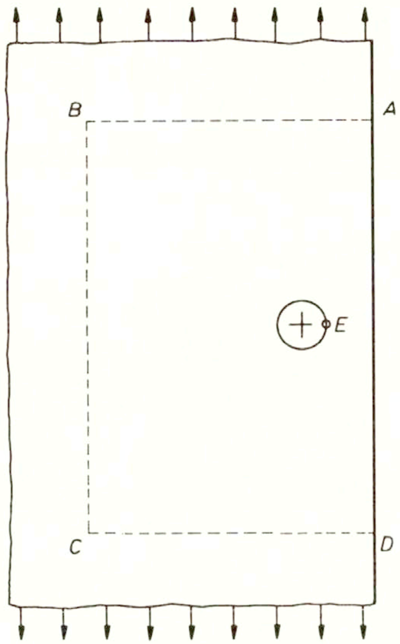

FIG. 8.

In Fig. 9 the complete slip-line solution to this problem is presented. The mechanism of plastic deformation consists in rigid blocks motion. Initial configuration of slip-lines is shown in Fig. 9a. In Fig. 9b is presented the advanced stage of plastic deformation, when the width of the ligament between the hole and the external surface of the body was reduced to zero. The deformation in the ligament itself has been shown in the figure only schematically. However, it has been analysed according to the solution of the type shown in Fig. 5 by assuming initially a small value of the dimension $2a$.

Deformation mode of an experimental model corresponding to the above theoretical solution is presented in Fig. 10. On the left is shown the specimen made of a rolled bar of an Al-2% Mg aluminium alloy in the state before the test. The thickness of the specimen was five times larger than the width of the ligament between the hole and the edge. Thus the deformation mode was close to the plane strain conditions. On the right-hand side of the figure the specimen after tensile loading up to the stage corresponding to that in the slip-line solution (Fig. 9b) is shown. It can be seen that the predicting ability of the slip-line theory is satisfactory when applied to the problems of the mechanics of ductile fracture of metals. Enlarged part of the fractured zone (Fig. 11) shows the similarity of the real fracture mode to that predicted by the theoretical solution.

A similar test was performed with the use of a specimen made of the same material but with two holes drilled symmetrically close to the edges. The specimen after tension test is shown in Fig. 12. Note that plastic zones shown in

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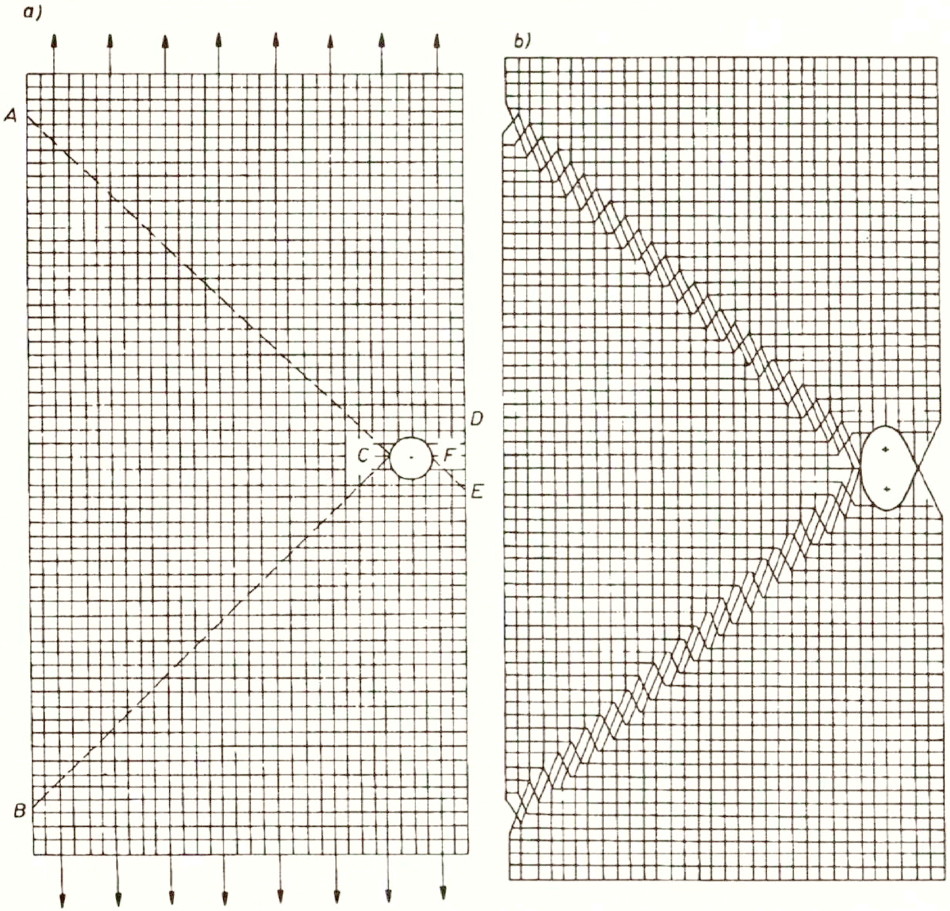

FIG. 9.

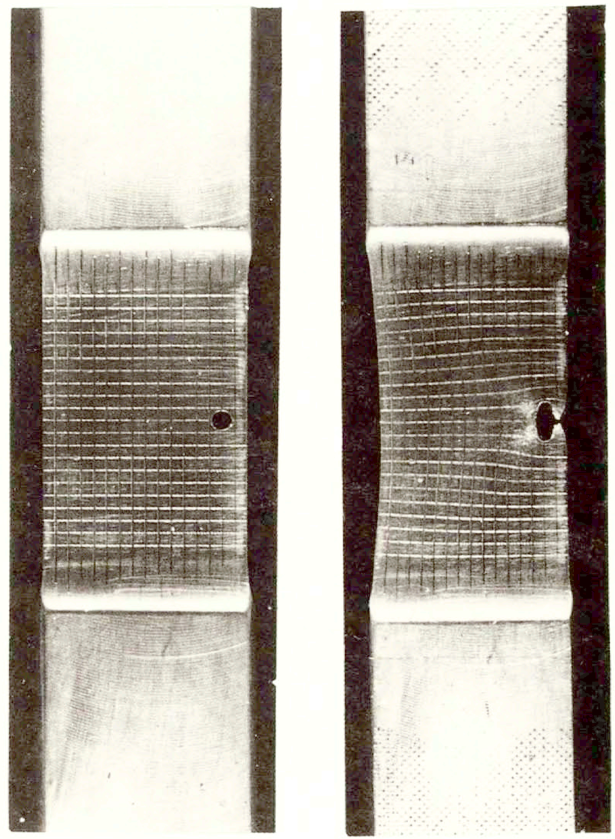

FIG. 10.

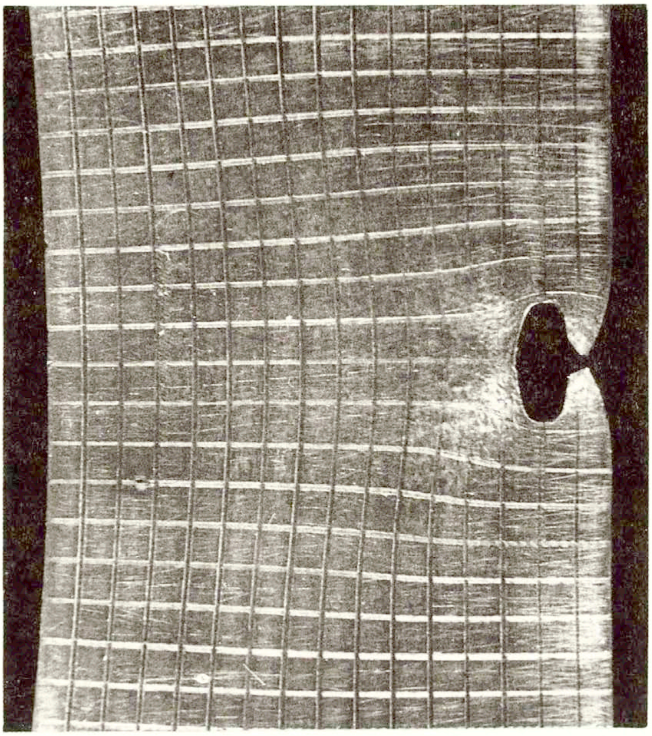

FIG. 11.

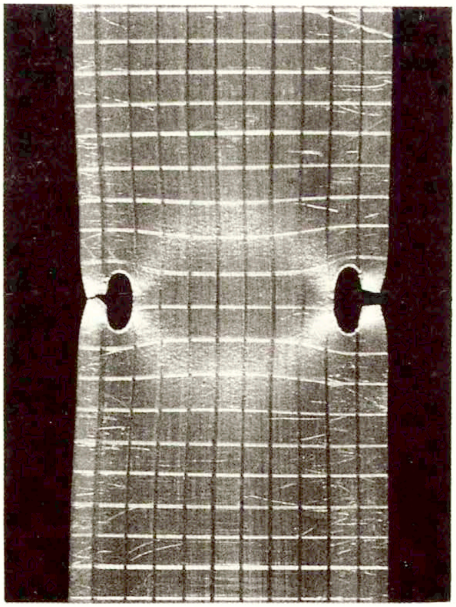

FIG. 12.

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the photograph as white bands caused by reflection of the light from the slightly deformed surface, coincide with the corresponding shear bands inclined at the angles of 45° which have been obtained in theoretical solutions. In spite of certain differences caused by the strain hardening effect neglected in the theoretical slip-line solution, the general layout of deformations is similar.

4. Final remarks

The theoretical slip-line analysis of the mechanisms of ductile fracture in the presence of internal cracks or voids close to the external surface of the body shows that during plastic deformation strong local strain concentrations appear, leading to the local reduction of the working area to a point. Such local separations of the material is referred to as the ideally ductile fracture. Simple experiments show that such fracture mechanisms are of real practical significance.

Acknowledgements

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Reflection of oblique shock waves in Murnaghan material

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THE SEMI-INVERSE METHOD of solution is used to examine a 2-D reflection problem for a finite oblique shock wave reflected from a plane boundary of a half-space filled with Murnaghan material. The incident wave is assumed to be a longitudinal plane shock wave. There are, in general, two reflected waves, each of them can be a simple wave or a shock wave. Using a perturbation procedure, the equations for the wave fronts of the reflected waves from a clamped boundary are derived. A numerical analysis of the reflection solution shows that each pair of the reflected waves consists of one simple wave and one shock wave. For small values of the angle of incidence the first reflected wave is a shock wave and the second one is a simple wave; the order is reversed for the incident angle passing through a certain transition value.

1. Introduction

WRIGHT IN HIS PAPER [11] on reflection of an oblique finite shock wave at a plane boundary of a nonlinear elastic solid presented a general procedure for combining the reflected simple (or shock) waves to obtain the reflection solution. In this procedure (often referred to as a *semi-inverse method*) it is assumed that, if the angle of incidence does not exceed a certain *critical* value (which, in general, would depend on the incident shock strength, the initial state of the solid, the material properties and the boundary conditions), then the reflection solution consists of a family of plane simple waves [12], centred on a moving line of contact between the incident wave and the boundary. Each wave in this family connects a fixed state ahead of the wave with a one-parameter family of states behind the wave. In anisotropic solids there are three possible reflected waves, so that a sequence of such waves connects the state behind the incident shock with a three-parameter family of states adjacent to the boundary. In general, there are three independent boundary conditions from which the parameters specifying the reflected waves can be determined. The assumed solution pattern reduces the reflection problem to an initial-boundary value problem for a system of ordinary differential equations governing the variation of the deformation gradient and the velocity fields in the region of simple waves. Its solution determines the wedge-shaped regions of simple waves and the distribution and strength of the wavelets within each wave. If the assumed reflection pattern fails the admissibility test, it is modified to include shocks as well; for shocks, the reflection problem is then reduced to solving a system of algebraic equations for the direction of propagation and strength of the reflected shocks.

We will apply the semi-inverse method to study the reflection problem for a plane shock wave incident on a plane boundary of an elastic half-space. The

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medium initially is unstrained and at rest. Bearing in mind the complexity of the analysis of finite amplitude waves in nonlinear solids, we confine our attention to a special kind of an isotropic compressible elastic material of second order [9]. The motion is assumed to be plane strain and hence the reflection solution pattern should include only two reflected waves, both centred on the line of incidence.

Section 2 contains a summary of the necessary theory and derivation of the propagation condition for shocks and simple waves. In Sec. 3 a description of the incident shock, the constant state of the region behind the shock and the assumed solution pattern is given. An approximate reflection solution, derived by using a perturbation method, is presented in Sec. 4, and a numerical analysis of the solution is conducted in Sec. 5.

2. Basic equations

The motion of the continuum is given by a set of functions

(2.1) $$x_i = x_i(X_\alpha, t),$$

where (x_1, x_2, x_3) and (X_1, X_2, X_3) are the coordinates of a material particle in the present configuration B at time t and in an unstressed reference state B_R, at $t = 0$, respectively, both with respect to a global Cartesian coordinate system. The deformation gradient and the particle velocity are denoted by

(2.2) $$F_{i\alpha} = x_{i,\alpha} = \frac{\partial x_i}{\partial X_\alpha}, \qquad \dot{x}_i = u_i = \frac{\partial x_i}{\partial t}.$$

It is assumed that $\det(\mathbf{F}) \neq 0$. It is also assumed that the material is homogeneous and hyperelastic. The Piola–Kirchhoff stress tensor for such materials is

(2.3) $$T_{i\alpha} = \rho_R \frac{\partial \sigma}{\partial F_{i\alpha}},$$

where $\sigma = \sigma(\mathbf{F})$(1) denotes internal energy per unit mass in B_R and ρ_R is the mass density in B_R.

If the stress and velocity fields are differentiable, then the equations expressing balance of momentum and moment of momentum are

(2.4) $$T_{i\alpha,\alpha} = \rho_R \dot{u}_i,$$

(2.5) $$F_{i\alpha} T_{j\alpha} = F_{j\alpha} T_{i\alpha}.$$

(1) Index notation as well as direct notation will be used when convenient.

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Simple waves

Simple waves are defined (VARLEY [10]) to be regions of space-time in which all field quantities are continuous functions of a single parameter. This means that in the region of a simple wave all field quantities can be expressed as functions of one of them. Hence, if one of the field quantities is constant in this region, the remaining quantities are also constant throughout this region.

It is known (cf. [10, 11]) that simple waves are one-parameter families of planes

$$g(\gamma) = N_\alpha(\gamma)X_\alpha - U(\gamma)t \tag{2.6}$$

called wavelets, propagating at speed $U(\gamma)$ in the direction of the unit vector $\mathbf{N}(\gamma)$. Equation (2.6) defines implicitly the wave parameter γ as a function of X_α and $t : \gamma = \psi(X_\alpha, t)$. We have then

$$N_\alpha(\gamma) = \frac{\psi_{,\alpha}}{|\Delta\psi|}, \qquad U(\gamma) = -\frac{\dot{\psi}}{|\Delta\psi|} \tag{2.7}$$

and the equation of motion (2.4) and a compatibility condition in the region of simple wave are now

$$\frac{\partial T_{i\alpha}}{\partial F_{j\beta}} F'_{j\beta}\psi_{,\alpha} = \rho u'_i \dot{\psi}, \tag{2.8}$$

$$F'_{j\beta}\dot{\psi} = u'_j \psi_{,\beta}, \tag{2.9}$$

where the prime indicates differentiation with respect to γ. If $\dot{\psi} \neq 0$, Eqs. (2.8) and (2.9) can be rewritten in the form

$$(Q_{ij} - \rho_R U^2 \delta_{ij})u'_j = 0, \tag{2.10}$$

$$U F'_{i\alpha} + u'_j N_\alpha = 0, \tag{2.11}$$

where

$$Q_{ij} = \sigma_{i\alpha j\beta} N_\alpha N_\beta \tag{2.12}$$

is the acoustic tensor and

$$\sigma_{i\alpha j\beta} = \frac{\partial^2\sigma}{\partial F_{i\alpha}\partial F_{j\beta}} \tag{2.13}$$

are the material elasticities. For simple waves to propagate it is necessary that the eigenvalues of Q_{ij}, that is the roots of the equation

$$\det(Q_{ij} - \rho_R U^2 \delta_{ij}) = 0, \tag{2.14}$$

are real monotone functions of the wave parameter (cf. [11]).

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Shock waves

If the functions (2.1) are continuous everywhere but have discontinuous first derivatives on some propagating surface $\Sigma(\mathbf{X}, t) = 0$, Eqs. (2.4) must be replaced by the jump conditions on this surface [9]

$$[T_{i\alpha}]N_\alpha = -\rho V[u_i], \tag{2.15}$$

$$[F_{i\alpha}] = a_i N_\alpha , \qquad [u_i] = -a_i V. \tag{2.16}$$

Such a surface is called a shock wave. Vector **N** is a material unit normal to the wave, V is the speed of propagation along **N** and **a** is the amplitude vector of the jump. The square brackets indicate the jump in the quantity enclosed across the surface; thus

$$[\cdot] = (\cdot)^B - (\cdot)^F ,$$

where the letters F and B refer to the limit values taken in front and rear sides of Σ, respectively. Eliminating the velocity jump $[u_i]$ from Eqs. (2.15) and (2.16), we obtain

$$[T_{i\alpha}]N_\alpha N_\beta = \rho_R V^2 [F_{i\beta}]. \tag{2.17}$$

Since **T** is a given function of the deformation gradient, Eq. (2.17) represents a system of nine equations for $F^B_{i\alpha}$, in terms of $F^F_{i\alpha}$, V and **N**.

The propagating shock wave is assumed to be stable. With no thermal effects included in the constitutive equation (2.3), criteria other than the thermodynamic one should be considered to determine the shock stability. According to LAX [4], for a shock wave to be stable it is necessary that it must travel faster than the corresponding type of acceleration wave ahead of the shock and slower than the corresponding type of acceleration wave behind the shock.

Plane longitudinal shock wave

For plane longitudinal shocks equations (2.17) can be considerably simplified by a suitable choice of the coordinate system. Without loss of generality we can choose the coordinate axes such that the wave normal **N** is (ref. Fig. 1)

$$\mathbf{N} = (\sin\theta, -\cos\theta, 0). \tag{2.18}$$

Since for a longitudinal shock to be stable it is necessary that the parallel vectors **N** and **a** have opposite directions (cf. [6]), that is

$$\mathbf{N} = -\mathbf{a}/a, \tag{2.19}$$

where $a = |\mathbf{a}|$ is the shock strength, we conclude from Eqs. (2.15) and (2.16) that

$$[F_{i\alpha}] = -aN_i N_\alpha , \qquad [u_i] = aVN_i , \tag{2.20}$$

$$V^2 = -\frac{1}{\rho_R a}[T_{i\alpha}]N_i N_\alpha . \tag{2.21}$$

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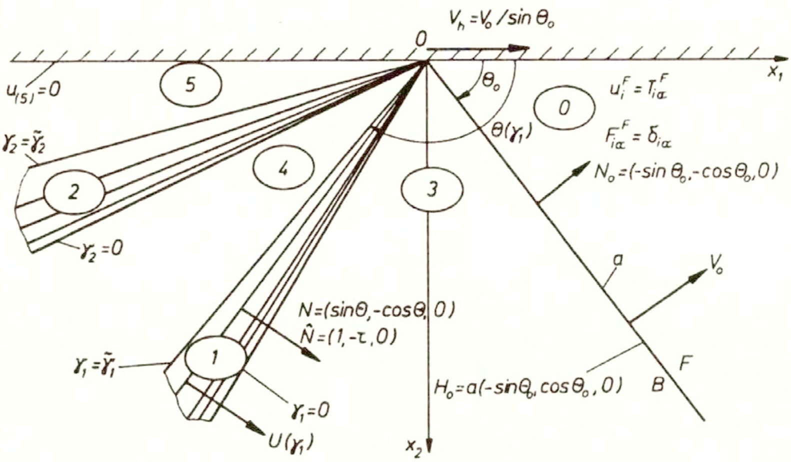

FIG. 1. Incident shock and assumed reflection pattern.

Thus, the region behind the propagating shock is in a state of plane strain deformation specified by

(2.22) $$F_{i\alpha}^B = F_{i\alpha}^F - aN_iN_\alpha ,$$

(2.23) $$u_i^B = u_i^F + aVN_i$$

and, for a fixed state of the region in front of the shock and a fixed direction of propagation **N**, completely determined by a single parameter a, the shock strength.

Simple wave in plane strain

Suppose a simple wave (2.10) propagates through a region of constant state given by Eq. (2.22) in the direction of its unit normal

(2.24) $$\mathbf{N}(\lambda) = (\sin\theta(\lambda), -\cos\theta(\lambda), 0).$$

Since in a plane strain deformation the components $Q_{3i} = Q_{i3}$, $i = 1,2$, of the acoustic tensor (2.12) are zero and since $Q_{33} - \rho_R U^2$ is arbitrary (because of $u_3' = 0$), the propagation condition (2.14) is reduced to a quadratic equation in U^2:

(2.25) $$\det(Q_{ij} - \rho_R U^2 \delta_{ij}) = 0, \qquad i,j = 1,2.$$

3. Incident shock

In general, a propagating wave incident on a boundary of an elastic medium does not meet the boundary conditions. If it is the only wave, the medium is not

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in a state of dynamic equilibrium; this is the reason for some additional waves, called reflected waves, being formed in association with the incident wave.

Suppose the incident wave is a plane longitudinal shock wave, as described in Sec. 2, and it is propagating through an elastic half-space $X_2 \geq 0$. The angle of incidence $\theta_0 \in (0, \theta_c)$ on the boundary $X_2 = 0$ and the shock strength a are known, and the material region ahead of the shock is unstrained and at rest:

$$F^F_{i\alpha} = \delta_{i\alpha}, \qquad u^F_i = \dot{u}^F_i = \sigma^F = T_{i\alpha} = 0. \tag{3.1}$$

Then, the constant state of region 3 just behind the shock (ref. Eqs. (2.22) and (2.23)) is completely determined by

$$\mathbf{F}^B_{(3)} = \begin{pmatrix} 1 - a\sin^2\theta_0 & a\sin\theta_0\cos\theta_0 & 0 \\ a\sin\theta_0\cos\theta_0 & 1 - a\cos^2\theta_0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.2}$$

$$\mathbf{u}^B_{(3)} = (\sin\theta_0 - \cos\theta_0, 0)aV_0 \tag{3.3}$$

with speed V_0 given by formula (2.21); the *critical* angle θ_c ($< \pi/2$) will be specified later. The problem now is to find out what combination of additional waves is required in order that the appropriate conditions at the boundary behind the incident shock should be satisfied.

Reflection pattern

To solve this problem we will follow a procedure proposed by WRIGHT [11].

The reflection solution is assumed in a form of a sequence of simple waves centred on a moving line of incidence (point O in Fig. 1). This sequence of waves connects the state just fixed by the travelling incident wave with the constant state adjacent to the boundary, compatible with the boundary conditions. Since the motion is restricted to plane strain, only two such waves are possible. Each wave is specified by a single parameter, so there are two parameters, say γ_1 and γ_2, to consider and there are two boundary conditions to satisfy.

Suppose a simple wave (2.10) is centred at point O on the X_1 axis (Fig. 1) and travels through the region of constant state specified by Eqs. (3.2) and (3.3). The wavelet normal (2.24) varies with the wave parameter γ, and the corresponding family of wavelets can be depicted as a wedge-shaped region in the (X_1, X_2)-plane, connected with the boundary at O. For such case $g(\gamma) = 0$ in Eq. (2.6). Since point O moves along the boundary with speed

$$V_h = V_0/\sin\theta_0, \tag{3.4}$$

the normal speed of the wavelet at angle $\theta(\gamma)$ is

$$U(\gamma) = V_h \sin\theta(\gamma) \tag{3.5}$$

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and Eq. (2.6) which determines the wavelet planes may be written

$$X_1 \sin\theta(\gamma) - X_2 \cos\theta(\gamma) = V_h t \sin\theta(\gamma). \tag{3.6}$$

It is expected (causality requirement, (cf. [1], p. 165)) that the reflected waves will propagate away from the boundary, hence

$$\frac{\pi}{2} < \theta(\gamma) < \pi\,. \tag{3.6a}$$

Let us denote

$$\tau = \cot\theta(\gamma). \tag{3.7}$$

We have then from Eqs. (2.24), (2.10), (2.11) and (3.5)

$$\widehat{\mathbf{N}} = \mathbf{N}/\sin\theta(\gamma) = (1, -\tau, 0), \tag{3.8}$$

$$(\widehat{Q}_{ij} - \rho_R V_h^2 \delta_{ij}) u_j' = 0, \tag{3.9}$$

$$V_h F_{i\alpha}' + u_i' \widehat{N}_\alpha = 0, \tag{3.10}$$

where $i, j = 1, 2$, $\widehat{Q}_{ij} = \sigma_{i\alpha j\beta} \widehat{N}_\alpha \widehat{N}_\beta$, and V_h is independent of $\theta(\gamma)$ (ref. Eqs. (3.4)). At every point in the wave region the following conditions must be satisfied:

$$\pi(\tau) = \det(\widehat{Q}_{ij} - \rho_R V_h^2 \delta_{ij}) = 0, \tag{3.11}$$

$$u_i' = k r_i\,, \tag{3.12}$$

where $\pi(\tau)$ is a fourth degree polynomial in τ, $\mathbf{r}$ is a right eigenvector of the acoustic tensor $\mathbf{Q} = \sin^2\theta(\gamma)\widehat{\mathbf{Q}}$ associated with a particular root τ, and k is a scalar function of the deformation gradient; it is convenient to assume that $\mathbf{r}$ is a unit vector. The corresponding eigenvalue of $\mathbf{Q}$, the characteristic speed of the simple wave, is $\rho_R V_h^2/(1 + \tau^2)$. Thus, if τ and $\mathbf{r}$ correspond to the reflected simple wave under consideration, then by Eqs. (3.12) and (3.10) we have

$$\mathbf{u}' = k\mathbf{r}, \qquad \mathbf{F}' = -V_h^{-1} k \mathbf{r} \otimes \widehat{\mathbf{N}}. \tag{3.13}$$

Each simple wave is completely described by a one-parameter set of functions, the variation of which is governed by the above system of ordinary differential equations. Since the velocity and the deformation gradient are continuous throughout the regions behind the incident shock, the initial values for Eqs. (3.13) are the constant values of the region in front of the wave. The constant state of the region just behind the wave is fixed by the values at the trailing edge of the wave.

A detailed discussion and geometric interpretation of the roots of $\pi(\tau)$ can be found in [11]. Here we shall only state that for a simple wave to propagate, τ must be a real decreasing function of $\gamma \in [0, \widetilde{\gamma}]$; this means that its wavelets (rays) diverge with increasing γ (ref. Eq. (3.7)). If $\tau(\gamma)$ increases, then the assumed reflection pattern should be modified to include shocks as well.

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4. Shock reflection in Murnaghan material

Further analysis of the problem described in Sec. 3 is restricted to a special kind of *second order* elastic material, called Murnaghan material. According to MURNAGHAN [7], the internal energy function σ for isotropic compressible elastic materials under moderate strain can be approximated by

$$\rho_R\sigma = \frac{1+2m}{24}(I_1-3)^3 + \frac{\lambda+2\mu+4m}{8}(I_1-3)^2 + \frac{8\mu+n}{8}(I_1-3) - \frac{m}{4}(I_1-3)(I_2-3) - \frac{4\mu+n}{8}(I_2-3) + \frac{n}{8}(I_3-1), \tag{4.1}$$

where $I_1 = B_{ii}$, $I_2 = 2^{-1}(B_{ii}B_{jj} - B_{ij}B_{ij})$, $I_3 = \det(B_{ij})$ are the invariants of the left Cauchy-Green strain tensor $\mathbf{B}$, λ and μ are Lamé coefficients, and l, m, n are the elastic constants of second order. In view of Eq. (3.2), for a fixed shock parameter θ_0, σ can be seen as a polynomial function of the other shock parameter, the shock strength a.

Numerical analysis (cf. [6]) shows that only shocks of a relatively small (of order up to 10^{-3}) strength can propagate in materials defined by Eq. (4.1). It is reasonable then to consider perturbation methods as a means of finding an approximate solution of the reflection problem.

We assume now that the half-space $X_2 \geq 0$ is filled with Murnaghan material (4.1). Expanding $\widehat{\mathbf{Q}}$, V_h^2 and $\mathbf{r}$ in powers of a and retaining the linear parts only, we have in region 3

$$\widehat{Q}_{ij} = \sigma_{i\alpha j\beta}\widehat{N}_\alpha\widehat{N}_\beta = \overset{0}{Q}_{ij} + a\overset{1}{Q}_{ij}, \tag{4.2}$$

$$V_h^2 = \overset{0}{V}{}_h^2 + a\overset{1}{V}{}_h^2, \tag{4.3}$$

$$r_i = \overset{0}{r}_i + a\overset{1}{r}_i, \qquad |\mathbf{r}| = |\overset{0}{\mathbf{r}}| = 1, \tag{4.4}$$

where

$$\overset{0}{\mathbf{Q}} = \begin{pmatrix} \lambda+2\mu+\mu\tau^2 & -\tau(\lambda+\mu) & 0 \\ -\tau(\lambda+\mu) & \lambda+2\mu+\mu\tau^2 & 0 \\ 0 & 0 & \mu(1+\tau^2) \end{pmatrix}, \tag{4.5}$$

and the non-zero elements of matrix $\overset{1}{\mathbf{Q}}$ are

$$\begin{aligned} \overset{1}{Q}_{11} &= -(\phi_1 + 2\phi_2\sin^2\theta_0 + 2\tau\phi_2\sin\theta_0 + \tau^2\phi_3), \\ \overset{1}{Q}_{12} &= \overset{1}{Q}_{21} = (1+\tau^2)\phi_2\sin\theta_0\cos\theta_0 + (\phi_1+\mu+m)\tau, \\ \overset{1}{Q}_{22} &= -(\phi_3 + 2\tau\phi_2\sin\theta_0\cos\theta_0 + (\phi_1 + 2\phi_2\cos^2\theta_0)\tau^2), \\ \overset{1}{Q}_{33} &= -\phi_3 + \left(2\mu+\frac{n}{2}\right)\cos^2\theta_0 + \left(-\phi_3 + \left(2\mu+\frac{n}{2}\right)\sin^2\theta_0\right)\tau^2, \end{aligned} \tag{4.6}$$

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and

$$\phi_1 = \lambda + 2l, \qquad \phi_2 = \lambda + 3\mu + 2m, \qquad \phi_3 = \lambda + 2\mu + m. \tag{4.7}$$

We note here that the $\widehat{Q}_{33}$ component does not contribute to the setting of the problem and it will be ignored in the further steps.

Substituting Eqs. (4.2), (4.3) and (4.4) into Eq. (3.9) and using Eq. (3.12), we obtain

$$\left(\left(\overset{0}{\mathbf{Q}} + a\overset{1}{\mathbf{Q}}\right) - \rho_R\left(\overset{0}{V}{}_h^2 + a\overset{1}{V}{}_h^2\right)\mathbf{I}\right)\left(\overset{0}{\mathbf{r}} + a\overset{1}{\mathbf{r}}\right) = 0 \tag{4.8}$$

and subsequently

$$(\overset{0}{\mathbf{Q}} - \rho_R \overset{0}{V}{}_h^2\mathbf{I})\overset{0}{\mathbf{r}} = 0, \tag{4.9}$$

$$(\overset{1}{\mathbf{Q}} - \rho_R \overset{1}{V}{}_h^2\mathbf{I})\overset{0}{\mathbf{r}} + (\overset{0}{\mathbf{Q}} - \rho_R \overset{0}{V}{}_h^2\mathbf{I})\overset{1}{\mathbf{r}} = 0. \tag{4.10}$$

Equation (4.9) can be easily solved to obtain the eigenvalues and eigenvectors in the zero-order approximation

$$\begin{aligned} \overset{0}{V}{}_{h1}^2 &= c_L^2(1+\tau^2), \qquad \overset{0}{\mathbf{r}}_1 = \left(\frac{1}{\sqrt{1+\tau^2}}, -\frac{\tau}{\sqrt{1+\tau^2}}, 0\right), \\ \overset{0}{V}{}_{h2}^2 &= c_T^2(1+\tau^2), \qquad \overset{0}{\mathbf{r}}_2 = \left(\frac{\tau}{\sqrt{1+\tau^2}}, \frac{1}{\sqrt{1+\tau^2}}, 0\right), \end{aligned} \tag{4.11}$$

where $c_L^2 = (\lambda + 2\mu)/\rho_R$ and $c_T^2 = \mu/\rho_R$ are the squared speeds of infinitesimal waves in linear theory. The directions θ_i of propagation of the reflected waves in *zero* approximation can be now found from Eq. (3.4) after substituting Eqs. (4.11). Two pairs of solutions $\pm\overset{0}{\tau}_i$ are obtained, but only the negative solutions correspond to the reflected waves (cf. [11])

$$\begin{aligned} \overset{0}{\tau}_1 &= \cot\theta_1 = -\left[\left(\frac{V_0}{c_L \sin\theta_0}\right)^2 - 1\right]^{1/2}, \\ \overset{0}{\tau}_2 &= \cot\theta_2 = -\left[\left(\frac{V_0}{c_T \sin\theta_0}\right)^2 - 1\right]^{1/2}. \end{aligned} \tag{4.12}$$

We note here that since $V_0 \geq c_L > c_T$, both solutions are real for arbitrary values of the angle of incidence, and that $\theta_1 < \theta_2$.

To find the coefficient $\overset{1}{V}{}_h^2$ in Eq. (4.3) we pre-multiply Eq. (4.10) by $\overset{0}{\mathbf{r}}_i$; matrix $\overset{0}{\mathbf{Q}}$ is symmetric, so the second term can be written as $\overset{1}{\mathbf{r}}_i\,(\overset{0}{\mathbf{Q}} - \rho_R \overset{0}{V}{}_{hi}^2\mathbf{I})\,\overset{0}{\mathbf{r}}_i$,

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(no sum) and by Eq. (4.9), it is equal to zero. The remaining term gives $\overset{0}{\mathbf{r}}_i \left(\overset{1}{\mathbf{Q}} - \rho_R \overset{1}{V}{}^2_{hi} \mathbf{I} \right) \overset{0}{\mathbf{r}}_i = 0$, and hence

$$\rho_R \overset{1}{V}{}^2_{hi} = \overset{0}{\mathbf{r}}_i \cdot \left(\overset{1}{\mathbf{Q}} \overset{0}{\mathbf{r}}_i \right), \qquad i = 1, 2 \qquad \text{(no sum)}. \tag{4.13}$$

The first term in Eq. (4.10) is now known. Denoting it by $\mathbf{z}_i$:

$$\mathbf{z}_i = (\overset{1}{\mathbf{Q}} - \rho_R \overset{1}{V}{}^2_{hi} \mathbf{I}) \overset{0}{\mathbf{r}}_i \tag{4.14}$$

we obtain a nonhomogeneous system of linear equations for the unknown vectors $\overset{1}{\mathbf{r}}_i$

$$(\overset{0}{\mathbf{Q}} - \rho_R \overset{0}{V}{}^2_{hi} \mathbf{I}) \overset{1}{\mathbf{r}}_i = -\mathbf{z}_i \,. \tag{4.15}$$

To solve Eq. (4.15), first we replace $\overset{1}{\mathbf{r}}_i$ and $\mathbf{z}_i$ by their linear combinations of $\overset{0}{\mathbf{r}}_i$:

$$\overset{1}{\mathbf{r}}_i = \alpha_{ij} \overset{0}{\mathbf{r}}_j \,, \qquad \mathbf{z}_i = \beta_{ij} \overset{0}{\mathbf{r}}_j \,, \tag{4.16}$$

$$(\overset{0}{Q} - \rho_R \overset{0}{V}{}^2_{hi} \mathbf{I}) \alpha_{ij} \overset{0}{\mathbf{r}}_j = -\beta_{ij} \overset{0}{\mathbf{r}}_j \,. \tag{4.17}$$

Then, the first equation in Eq. (4.17) is pre-multiplied by $\overset{0}{\mathbf{r}}_2$, and the second equation by $\overset{0}{\mathbf{r}}_1$. Noting that

$$\overset{0}{\mathbf{r}}_i \cdot \left(\overset{0}{\mathbf{Q}} \overset{0}{\mathbf{r}}_i \right) = \rho_R \overset{0}{V}{}^2_{hi} \qquad \text{(no sum)} \tag{4.18}$$

and using Eq. (4.9), we can write the resulting equations for $i \neq j$ in the form

$$\rho_R \left(\overset{0}{V}{}^2_{hj} - \overset{0}{V}{}^2_{hi} \right) = -\beta_{ij} \qquad \text{(no sum)} \tag{4.19}$$

and α_{11}, α_{22} are arbitrary; it is convenient to assume $\alpha_{11} = \alpha_{22} = 0$. The coefficients β_{ij} are determined by Eq. (4.14). After inserting $\mathbf{z}_i = \beta_{ik} \overset{0}{\mathbf{r}}_k$ in Eq. (4.14) and pre-multiplying the resulting equation by $\overset{0}{\mathbf{r}}_j$ we obtain

$$\beta_{ij} = \overset{0}{\mathbf{r}}_j \cdot (\overset{1}{\mathbf{Q}} - \rho_R \overset{1}{V}{}^2_{hi} \mathbf{I}) \overset{0}{\mathbf{r}}_i \,. \tag{4.20}$$

The required in Eq. (4.19) coefficients β_{12}, β_{21} are

$$\beta_{12} = \beta_{21} = \overset{0}{\mathbf{r}}_1 \cdot \left(\overset{1}{\mathbf{Q}} \overset{0}{\mathbf{r}}_2 \right) \tag{4.21}$$

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and hence

$$\text{(4.22)} \qquad \alpha_{12} = -\alpha_{21} = \frac{\overset{0}{\mathbf{r}}_1 \cdot \left(\overset{1}{\mathbf{Q}}\overset{0}{\mathbf{r}}_2\right)}{\rho_R \left(\overset{0}{V}{}^2_{h1} - \overset{0}{V}{}^2_{h2}\right)} .$$

Finally, after substituting Eq. (4.22) into Eq. (4.16) and then into Eq. (4.4), we obtain the linear approximation of the eigenvectors $\mathbf{r}_i$ expressed in terms of $\overset{0}{\mathbf{r}}_i$

$$\text{(4.23)} \qquad \mathbf{r}_1 = \overset{0}{\mathbf{r}}_1 + a\alpha_{12}\overset{0}{\mathbf{r}}_2, \qquad \mathbf{r}_2 = \overset{0}{\mathbf{r}}_2 - a\alpha_{12}\overset{0}{\mathbf{r}}_1 .$$

We note that $\mathbf{r}_1 \cdot \mathbf{r}_2 = 0$ and $|\mathbf{r}_i|^2 = 1 + a^2\alpha_{12} \approx 1$.

So far the quantities (4.2)–(4.6) have been expressed as functions of the yet unspecified values of the parameter $\tau = \cot\theta(\gamma)$ defining the propagation direction of the reflected wave. As in the case of *zero* approximation, we will use the *centred wave* condition (3.4) to establish the equations determining τ. We have now two equations to consider

$$\text{(4.24)} \qquad \overset{0}{V}{}^2_{hi} + a\,\overset{1}{V}{}^2_{hi} = \frac{V_0^2}{\sin^2\theta_0}, \qquad i = 1,2$$

which, after using Eqs. (4.11), (4.13) and (4.5), (4.6), take up a form of two polynomial equations in τ_i

$$\text{(4.25)} \qquad \begin{aligned} p(\tau_1) &= \overset{0}{p}(\tau_1) + a\,\overset{1}{p}(\tau_1) = \sum_{k=0}^{4}\left(\overset{0}{p}_k + a\,\overset{1}{p}_k\right)\tau_1^k = 0, \\ q(\tau_2) &= \overset{0}{q}(\tau_2) + a\,\overset{1}{q}(\tau_2) = \sum_{k=0}^{4}\left(\overset{0}{q}_k + a\,\overset{1}{q}_k\right)\tau_2^k = 0, \end{aligned}$$

where τ_i^k is the k-th power of τ_i and

$$\text{(4.26)} \qquad \begin{aligned} \overset{0}{p}_4 &= \lambda + 2\mu, & \overset{1}{p}_4 &= -(\phi_1 + 2\phi_2\cos^2\theta_0), \\ \overset{0}{p}_3 &= \overset{0}{p}_1 = 0, & \overset{1}{p}_3 &= \overset{1}{p}_1 = -4\phi_2\sin\theta_0\cos\theta_0, \\ \overset{0}{p}_2 &= 2\lambda + 4\mu - \rho_R V_0^2/\sin^2\theta_0, & \overset{1}{p}_2 &= 2(2l - \mu), \\ \overset{0}{p}_0 &= \lambda + 2\mu - \rho_R V_0^2/\sin^2\theta_0, & \overset{1}{p}_0 &= -(\phi_1 + 2\phi_2\sin^2\theta_0), \\ \overset{0}{q}_4 &= \mu, & \overset{1}{q}_4 &= -\phi_3, \\ \overset{0}{q}_3 &= \overset{0}{q}_1 = 0, & \overset{1}{q}_3 &= \overset{1}{q}_1 = 0, \\ \overset{0}{q}_2 &= 2\mu - \rho V_0^2/\sin^2\theta_0, & \overset{1}{q}_2 &= -\phi_2 + 2\mu + 2m - \phi_2\cos^2\theta_0, \\ \overset{0}{q}_0 &= \mu - \rho_R V_0^2/\sin^2\theta_0, & \overset{1}{q}_0 &= -\phi_3. \end{aligned}$$

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The solutions $\tau_i(a)$ of Eqs. (4.25) are assumed in the form

$$\tau_i(a) = \overset{0}{\tau}_i + a\,\overset{1}{\tau}_i\,, \qquad i = 1, 2, \tag{4.27}$$

where $\overset{0}{\tau}_i = \tau_i(0)$ is given by Eq. (4.12). Substituting Eq. (4.27) into $(4.25)_1$ and using Eq. (4.26) we obtain

$$p\left(\overset{0}{\tau}_1 + a\,\overset{1}{\tau}_1\right) = p(\overset{0}{\tau}_1) + p'(\overset{0}{\tau}_1)a\,\overset{1}{\tau}_1 = 0, \tag{4.28}$$

$$\overset{0}{p}(\overset{0}{\tau}_1) + \left(\overset{1}{p}(\overset{0}{\tau}_1) + \overset{1}{\tau}_1\overset{0}{p}'(\overset{0}{\tau}_1)\right) a = 0, \tag{4.29}$$

where the prime denotes differentiation. The first term in Eq. (4.29) is zero (ref. Eqs. (4.11)) and the second term yields

$$\overset{1}{\tau}_1 = -\frac{\overset{1}{p}(\overset{0}{\tau}_1)}{\overset{0}{p}'(\overset{0}{\tau}_1)}\,. \tag{4.30}$$

In a similar way we find the remaining unknown coefficient in Eq. (4.27)

$$\overset{1}{\tau}_2 = -\frac{\overset{1}{q}(\overset{0}{\tau}_2)}{\overset{0}{q}'(\overset{0}{\tau}_2)}\,. \tag{4.31}$$

We now examine the solution of the differential equation (3.13).

Let the i-th reflected simple wave be specified by the parameter $\gamma_i \in [0, \tilde{\gamma}_i]$ where $\gamma_i = 0$ indicates the leading wavelet and $\gamma_i = \tilde{\gamma}_i$ indicates the trailing wavelet of the wave. It will be shown later that $\gamma_i = O(a)$ for small (but finite) a. By definition, all field quantities in the region of simple wave are continuous functions of the wave parameter. Following the perturbation procedure used here, each such quantity is represented by its linear approximation [8]. Denoting by $\mathcal{H}(\gamma_i)$ a typical field quantity, we have then: $\mathcal{H}(\gamma_i) = \mathcal{H}(0) + \gamma_i\,\overset{2}{\mathcal{H}}$ where $\mathcal{H}(0)$ is the value of $\mathcal{H}$ in the preceding region and $\overset{2}{\mathcal{H}}$ is to be determined.

First reflected wave

We have then in region 1:

$$\mathbf{u}_{(1)}(\gamma_1) = \mathbf{u}_{(3)} + \gamma_1\,\overset{2}{\mathbf{u}}_{(1)}\,, \qquad \mathbf{F}_{(1)}(\gamma_1) = \mathbf{F}_{(3)} + \gamma_1\,\overset{2}{\mathbf{F}}_{(1)}\,, \qquad 0 \le \gamma_1 \le \tilde{\gamma}_1\,, \tag{4.32}$$

$$\tau_1(\gamma_1) = \tau_1 + \gamma_1\,\overset{2}{\tau}_1\,, \qquad 0 \le \gamma_1 \le \tilde{\gamma}_1\,. \tag{4.33}$$

The leading wavelet is determined by Eqs. (4.27) and (4.30). Substitution of

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Eq. (4.32) into Eq. (2.10) leads to the following system of equations:

$$(4.34)\qquad (\widehat{\mathbf{Q}}(0) - \rho_R V_{h1}^2(0)\mathbf{I})\,\overset{2}{\mathbf{u}}_{(1)} = 0,$$

$$(4.35)\qquad \overset{2}{\mathbf{F}}_{(1)} = -V_{h1}^{-1}\,\overset{2}{\mathbf{u}}_{(1)} \otimes \widehat{\mathbf{N}}_{(1)},$$

where $\widehat{\mathbf{Q}}(0) = \overset{0}{\mathbf{Q}} + a\,\overset{1}{\mathbf{Q}}$, $V_h^2(0) = \overset{0}{V}{}_h^2 + a\,\overset{1}{V}{}_h^2$ and $\widehat{\mathbf{N}}_{(1)} = \left(1, -\left(\overset{0}{\tau}_1 + \overset{1}{\tau}_1\right), 0\right)$. The eigensolutions of Eq. (4.34) are now independent of the wave parameter and Eqs. (3.13) can be easily integrated.

A particular choice of the coefficient k in Eqs. (3.13) affects only the parametrization of the wave. It is convenient to retain k as an arbitrary scalar, but with opposite sign; we will use it later as a *scaling* factor for the wave parameter γ. We have then for the i-th wave:

$$(4.36)\qquad \mathbf{u}'_{(i)} = -k\mathbf{r}_i\,, \qquad \mathbf{F}'_{(i)} = kV_{hi}^{-1}\mathbf{r}_i \otimes \widehat{\mathbf{N}}_{(i)}\,.$$

The initial conditions for the differential equations (4.36) describing the first reflected wave are provided by the constant values of the region behind the incident shock (region 3). Thus, integrating the first equation (4.36) ($i = 1$) with respect to γ_1 and using Eq. (4.32) we obtain

$$(4.37)\qquad \mathbf{u}_{(1)}(\gamma_1) = -\gamma_1 k\mathbf{r}_1 + \mathbf{u}_{(3)}\,, \qquad \mathbf{F}_{(1)}(\gamma_1) = \gamma_1 kV_{h1}^{-1}\mathbf{r}_1 \otimes \widehat{\mathbf{N}}_{(1)} + \mathbf{F}_{(3)}\,,$$
$$0 \le \gamma_1 \le \widetilde{\gamma}_1\,.$$

The acoustic tensor, the eigenvalues and eigenvectors in region 1 are assumed in the form

$$(4.38)\qquad \widehat{\mathbf{Q}} = \overset{0}{\mathbf{Q}} + a\,\overset{1}{\mathbf{Q}} + \gamma_1\,\overset{2}{\mathbf{Q}},$$

$$(4.39)\qquad V_{hi}^2 = \overset{0}{V}{}_{hi}^2 + a\,\overset{1}{V}{}_{hi}^2 + \gamma_1\,\overset{2}{V}{}_{hi}^2,$$

$$(4.40)\qquad \mathbf{r}_i = \overset{0}{\mathbf{r}}_i + a\,\overset{1}{\mathbf{r}}_i + \gamma_1\,\overset{2}{\mathbf{r}}_i\,.$$

The non-zero components of $\overset{2}{\mathbf{Q}}$, calculated from Eqs. (4.1) and (4.2) after substituting Eq. (4.38), are

$$\overset{2}{Q}_{11} = (\phi_1 + 2\phi_2)F'_{(1)11} + \phi_1 F'_{(1)22} - 2((\mu+m)F'_{(1)21} + (\phi_2 - \mu - m)F'_{(1)12})\tau$$
$$+ \phi_3(F'_{(1)11} + F'_{(1)22})\tau^2,$$

$$(4.41)\qquad \overset{2}{Q}_{22} = \phi_3(F'_{(1)11} + F'_{(1)22}) - 2((\mu + m)F'_{(1)12} + (\phi_2 - \mu - m)F'_{(1)21})\tau$$
$$+ (\phi_1 F'_{(1)11} + (\phi_1 + 2\phi_2)F'_{(1)22})\tau^2,$$

$$\overset{2}{Q}_{12} = \overset{2}{Q}_{21} = (\phi_2 - \mu - m)F'_{(1)21} + (\mu + m)F'_{(1)12} - (\phi_1 + \mu + m)(F'_{(1)11} + F'_{(1)22})\tau$$
$$+ ((\phi_2 - \mu - m)F'_{(1)12} + (\mu + m)F'_{(1)21})\tau^2,$$

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where the components of $\mathbf{F}'_{(1)}$ are given by Eq. (4.36). To find $\overset{2}{V}{}^2_{hi}$ and $\overset{2}{\mathbf{r}}_i$ we consider equations (3.9) and (3.12), with $\widehat{\mathbf{Q}}$, V_h^2 and $\mathbf{r}$ replaced by expressions (4.38), (4.39) and (4.40). We have then

$$\left(\left(\overset{0}{\mathbf{Q}} + a\,\overset{1}{\mathbf{Q}}\right) - \rho_R\left(\overset{0}{V}{}^2_{hi} + a\,\overset{1}{V}{}^2_{hi}\right)\mathbf{I}\right)\left(\overset{0}{\mathbf{r}}_i + a\,\overset{1}{\mathbf{r}}_i\right) = 0, \tag{4.42}$$

$$\left(\overset{2}{\mathbf{Q}} - \rho_R\,\overset{2}{V}{}^2_{hi}\mathbf{I}\right)\left(\overset{0}{\mathbf{r}}_i + a\,\overset{1}{\mathbf{r}}_i\right) + \left(\left(\overset{0}{\mathbf{Q}} + a\,\overset{1}{\mathbf{Q}}\right) - \rho_R\left(\overset{0}{V}{}^2_{hi} + a\,\overset{1}{V}{}^2_{hi}\right)\mathbf{I}\right)\overset{2}{\mathbf{r}}_i = 0. \tag{4.43}$$

Equation (4.42) is satisfied identically (ref. Eq. (4.8)). Then, following the steps as in Eqs. (4.13) – (4.23), and retaining only the terms linear in a, $\widetilde{\gamma}_1$, we obtain from Eq. (4.43)

$$\rho_R\,\overset{2}{V}{}^2_{hi} = \overset{0}{\mathbf{r}}_i \cdot \left(\overset{2}{\mathbf{Q}}\overset{0}{\mathbf{r}}_i\right) + 2a\,\overset{0}{\mathbf{r}}_i \cdot \left(\overset{2}{\mathbf{Q}}\overset{1}{\mathbf{r}}_i\right) \qquad \text{(no sum)}, \tag{4.44}$$

$$\overset{2}{\mathbf{r}}_1 = \overline{\alpha}_{12}\,\overset{0}{\mathbf{r}}_2, \qquad \overset{2}{\mathbf{r}}_2 = -\overline{\alpha}_{12}\,\overset{0}{\mathbf{r}}_1, \tag{4.45}$$

where

$$\overline{\alpha}_{12} = \frac{\overset{0}{\mathbf{r}}_1 \cdot \left(\overset{2}{\mathbf{Q}}\overset{0}{\mathbf{r}}_2\right)}{\rho_R\left(\overset{0}{V}{}^2_{h1} + a\,\overset{1}{V}{}^2_{h1} - \overset{0}{V}{}^2_{h2} - a\,\overset{1}{V}{}^2_{h2}\right)}. \tag{4.46}$$

The second terms in Eq. (4.44) become, after substitution in Eq. (4.39), small of order $O(a^2)$, so they can be ignored. The expressions (4.39) and (4.40) are now

$$V^2_{hi} = \overset{0}{\mathbf{r}}_i \cdot \left(\overset{0}{\mathbf{Q}} + a\,\overset{1}{\mathbf{Q}} + \gamma_1\,\overset{2}{\mathbf{Q}}\right)\overset{0}{\mathbf{r}}_i \qquad \text{(no sum)}, \tag{4.47}$$

$$\mathbf{r}_1 = \overset{0}{\mathbf{r}}_1 + (a\alpha_{12} + \gamma_1\overline{\alpha}_{12})\,\overset{0}{\mathbf{r}}_2, \qquad \mathbf{r}_2 = \overset{0}{\mathbf{r}}_2 - (a\alpha_{12} + \gamma_1\overline{\alpha}_{12})\,\overset{0}{\mathbf{r}}_1. \tag{4.48}$$

The propagation directions of the wavelets in the reflected wave are given by

$$\tau_1(a, \gamma_1) = \overset{0}{\tau}_1 + a\,\overset{1}{\tau}_1 + \gamma_1\,\overset{2}{\tau}_1, \qquad 0 \le \gamma_1 \le \widetilde{\gamma}_1. \tag{4.49}$$

To find $\overset{2}{\tau}_1$ we will use again the *centred wave* condition.

Substitution of (4.47) into (3.4) leads to two equations

$$\overset{0}{\mathbf{r}}_i \cdot \left(\overset{0}{\mathbf{Q}} + a\,\overset{1}{\mathbf{Q}} + \gamma_1\,\overset{2}{\mathbf{Q}}\right)\overset{0}{\mathbf{r}}_i = \frac{\rho_R V_0^2}{\sin^2\theta_0} \qquad \text{(no sum)} \tag{4.50}$$

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which, after using Eqs. (4.5), (4.6) and (4.41), take up a form of two polynomial equations in τ_i:

(4.51)
$$p(\tau_1) = \overset{0}{p}(\tau_1) + a\overset{1}{p}(\tau_1) + \gamma_1\overset{2}{p}(\tau_1) = \sum_{k=0}^{4}\left(\overset{0}{p}_k + a\overset{1}{p}_k + \gamma_1\overset{2}{p}_k\right)\tau_1^k = 0,$$
$$q(\tau_2) = \overset{0}{q}(\tau_2) + a\overset{1}{q}(\tau_2) + \gamma_1\overset{2}{q}(\tau_2) = \sum_{k=0}^{4}\left(\overset{0}{q}_k + a\overset{1}{q}_k + \gamma_1\overset{2}{q}_k\right)\tau_2^k = 0,$$

where $\overset{0}{p}_k$, $\overset{1}{p}_k$, $\overset{0}{q}_k$, $\overset{1}{q}_k$ are given by Eq. (4.26) and

(4.52)
$$\begin{aligned}
\overset{2}{p}_4 &= \phi_1 F'_{(1)11} + (\phi_1 + 2\phi_2)F'_{(1)22},\\
\overset{2}{q}_4 &= \phi_3(F'_{(1)11} + F'_{(1)22}),\\
\overset{2}{p}_3 &= \overset{2}{p}_1 = -2\phi_2(F'_{(1)12} + F'_{(1)21}),\\
\overset{2}{q}_3 &= \overset{2}{q}_1 = 0,\\
\overset{2}{p}_2 &= 2(F'_{(1)11} + F'_{(1)22})(\phi_1 + \phi_3 + m + \mu),\\
\overset{2}{q}_2 &= 2(\phi_2 - m - \mu)(F'_{(1)11} + F'_{(1)22}),\\
\overset{2}{p}_0 &= (\phi_1 + 2\phi_2)F'_{(1)11} + \phi_1 F'_{(1)22},\\
\overset{2}{q}_0 &= \phi_3(F'_{(1)11} + F'_{(1)22}).
\end{aligned}$$

Substituting Eqs. (4.49) into $(4.51)_1$ and solving the resulting equation for $\overset{2}{\tau}_1$ we obtain

$(4.53)_1$
$$\overset{2}{\tau}_1 = -\frac{\overset{2}{p}(\tau_1(a,0))}{\overset{0}{p}{}'(\tau_1(a,0)) + a\overset{1}{p}{}'(\tau_1(a,0))}.$$

Equation $(4.51)_2$ is concerned with region 2; its solution

$(4.53)_2$
$$\tau_2(a,\gamma_1) = \overset{0}{\tau}_2 + a\overset{1}{\tau}_2 + \gamma_1\overset{2}{\tau}_2, \qquad \overset{2}{\tau}_2 = -\frac{\overset{2}{q}(\tau_2(a,0))}{\overset{0}{q}{}'(\tau_2(a,0)) + a\overset{1}{q}{}'(\tau_2(a,0))}$$

will be used in determining the direction of propagation of the second reflected simple wave.

The final values of region 1, for $\gamma_1 = \tilde{\gamma}_1$, define the constant state of region 4:

(4.54)
$$\mathbf{u}_{(4)} = \mathbf{u}_{(1)}(\tilde{\gamma}_1), \qquad \mathbf{F}_{(4)} = \mathbf{F}_{(1)}(\tilde{\gamma}_1).$$

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Second reflected wave

Region 2 connects the constant state of region 4 defined by Eqs. (4.54) and the constant state of region 5, adjacent to the boundary $X_2 = 0$. Integrating the second equation (4.36) ($i = 2$) with respect to γ_2 and using Eqs. (4.54) as the initial conditions we obtain

$$\mathbf{u}_{(2)} = -\gamma_2 k \mathbf{r}_2 + \mathbf{u}_{(4)}, \quad \mathbf{F}_{(2)} = \gamma_2 V_{h2}^{-1} k \mathbf{r}_2 \otimes \widehat{\mathbf{N}}_{(2)} + \mathbf{F}_{(4)}, \quad 0 \le \gamma_2 \le \widetilde{\gamma}_2 \tag{4.55}$$

with $\mathbf{r}_2$ and V_{h2} subject to a correction due to a change of field values across region 1. The wave normal is $\widehat{\mathbf{N}}_{(2)} = (1, -\tau_2(a, \widetilde{\gamma}_1), 0)$, and the *corrected* expressions for $\mathbf{r}_2$, V_{h2} are given by Eqs. (4.47) and (4.48) for $\gamma_1 = \widetilde{\gamma}_1$.

We have in region 2

$$\widehat{\mathbf{Q}} = \overset{0}{\mathbf{Q}} + a \overset{1}{\mathbf{Q}} + \widetilde{\gamma}_1 \overset{2}{\mathbf{Q}} + \gamma_2 \overset{3}{\mathbf{Q}}, \tag{4.56}$$

$$V_{hi}^2 = \overset{0}{V}{}_{hi}^2 + a \overset{1}{V}{}_{hi}^2 + \widetilde{\gamma}_1 \overset{2}{V}{}_{hi}^2 + \gamma_2 \overset{3}{V}{}_{hi}^2 . \tag{4.57}$$

The non-zero components of $\overset{3}{\mathbf{Q}}$, calculated from Eqs. (4.1) and (4.2) after substituting Eqs. (4.55), are

$$\begin{aligned}
\overset{3}{Q}_{11} &= (\phi_1 + 2\phi_2) F'_{(2)11} + \phi_1 F'_{(2)22} - 2((\mu + m) F'_{(2)21} + (\phi_2 - \mu - m) F'_{(2)12})\tau \\
&\qquad + \phi_3 (F'_{(2)11} + F'_{(2)22}) \tau^2, \\
\overset{3}{Q}_{22} &= \phi_3 (F'_{(2)11} + F'_{(2)22}) - 2(\mu + m) F'_{(2)12} \tau + (\phi_1 F'_{(2)11} + (\phi_1 + 2\phi_2) F'_{(2)22}) \tau^2, \\
\overset{3}{Q}_{12} = \overset{3}{Q}_{21} &= (\phi_2 - \mu - m) F'_{(2)21} + (\mu + m) F'_{(2)12} - (\phi_1 + \mu + m)(F'_{(2)11} + F'_{(2)22}) \tau \\
&\qquad + ((\phi_2 - \mu - m) F'_{(2)12} + (\mu + m) F'_{(2)21}) \tau^2 .
\end{aligned} \tag{4.58}$$

The correct value of the eigenvector $\mathbf{r}_2$ associated with this region is given by Eqs. (4.48). The *correcting* term $\overset{3}{V}{}_{hi}^2$ in Eq. (4.57) is found from Eq. (3.9), in a similar way as in the case of region 1:

$$\rho_R \overset{3}{V}{}_{hi}^2 = \overset{0}{\mathbf{r}}_2 \cdot \left(\overset{3}{\mathbf{Q}} \overset{0}{\mathbf{r}}_2 \right). \tag{4.59}$$

The propagation directions of the wavelets of the second reflected wave are given by

$$\tau_2(a, \widetilde{\gamma}_1, \gamma_2) = \overset{0}{\tau}_2 + a \overset{1}{\tau}_2 + \widetilde{\gamma}_1 \overset{2}{\tau}_2 + \gamma_2 \overset{3}{\tau}_2, \qquad 0 \le \gamma_2 \le \widetilde{\gamma}_2 \tag{4.60}$$

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with the last term to be found from the *centred wave* condition. Substitution of Eq. (4.57) into Eq. (3.4) leads to the equation

$$(4.61)\qquad \overset{0}{\mathbf{r}}_2 \cdot \left(\overset{0}{\mathbf{Q}} + a\,\overset{1}{\mathbf{Q}} + \widetilde{\gamma}_1\,\overset{2}{\mathbf{Q}} + \gamma_2\,\overset{3}{\mathbf{Q}}\right) \overset{0}{\mathbf{r}}_2 = \frac{V_0^2}{\sin^2\theta_0}$$

which, after using Eqs. (4.5), (4.6), (4.41) and (4.58), takes up a form of a polynomial equation in τ_2

$$(4.62)\qquad q(\tau_2) = \overset{0}{q}(\tau_2) + a\,\overset{1}{q}(\tau_2) + \widetilde{\gamma}_1\,\overset{2}{q}(\tau_2) + \gamma_2\,\overset{3}{q}(\tau_2) = \sum_{k=0}^{4}\left(\overset{0}{q}_k + a\,\overset{1}{q}_k + \widetilde{\gamma}_1\,\overset{2}{q}_k + \gamma_2\,\overset{3}{q}_k\right)\tau_2^k,$$

where $\overset{0}{q}_k$, $\overset{1}{q}_k$, $\overset{2}{q}_k$ are given by Eqs. (4.46) and (4.55), and

$$(4.63)\qquad \overset{3}{q}_4 = \phi_3(F'_{(2)11} + F'_{(2)22}), \qquad \overset{3}{q}_3 = \overset{3}{q}_1 = 0,$$

$$(4.64)\qquad \overset{3}{q}_2 = 2(\phi_2 - \mu - m)(F'_{(2)11} + F'_{(2)22}), \qquad \overset{3}{q}_0 = \phi_3(F'_{(2)11} + F'_{(2)22}).$$

Substituting Eq. (4.60) into Eq. (4.62) and solving the resulting equation for $\overset{3}{\tau}_2$ we obtain

$$(4.64)\qquad \overset{3}{\tau}_2 = -\frac{\overset{3}{q}(\tau_2(a,\widetilde{\gamma}_1,0))}{\overset{0}{q}'(\tau_2(a,\widetilde{\gamma}_1,0)) + a\,\overset{1}{q}'(\tau_2(a,\widetilde{\gamma}_1,0)) + \widetilde{\gamma}_1\,\overset{2}{q}'(\tau_2(a,\widetilde{\gamma}_1,0))}.$$

The constant state of region 5 is defined by the final values of region 2:

$$(4.65)\qquad \mathbf{u}_{(5)} = \mathbf{u}_{(2)}(\widetilde{\gamma}_2), \qquad \mathbf{F}_{(5)} = \mathbf{F}_{(2)}(\widetilde{\gamma}_2).$$

The requirement that this state must meet the appropriate boundary conditions will provide a set of equations for the two parameters $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$.

Boundary conditions

Let us assume that the incident shock is reflected from a rigidly constrained boundary. This means that

$$(4.66)\qquad \mathbf{u}_{(5)} = 0 \qquad \text{on} \qquad X_2 = 0.$$

Substitution of Eqs. (4.65), (4.54) and (4.37) into Eq. (4.66) leads to the equation

$$(4.67)\qquad \widetilde{\gamma}_1\mathbf{r}_1 + \widetilde{\gamma}_2\mathbf{r}_2 = k^{-1}\mathbf{u}_{(3)},$$

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which can be easily solved for $\tilde{\gamma}_i$ (ref. Eq. (3.3)):

$$\tilde{\gamma}_i = \frac{\mathbf{r}_i \cdot \mathbf{u}_{(3)}}{|\mathbf{r}_i|^2} k^{-1} = \mathbf{r}_i \cdot \mathbf{N}_0 k^{-1} a V_0 , \tag{4.68}$$

where $\mathbf{r}_1$ is given by Eqs. (4.23) and $\mathbf{r}_2$ by Eqs. (4.48).

As we can see from Eq. (4.68), a suitable choice of k can make the values of $\tilde{\gamma}_i$ and a to be of the same order.

Propagation condition for simple waves

Once the wavefronts in the assumed solution pattern are determined, it is necessary to test the solution against the propagation condition. This condition, in the case discussed here, is

$$\tau_1(a, \tilde{\gamma}_1) < \tau_1(a, 0), \qquad \tau_2(a, \tilde{\gamma}_1, \tilde{\gamma}_2) < \tau_2(a, \tilde{\gamma}_1, 0) \tag{4.69}$$

for the first and the second reflected wave, respectively. The problem is to establish for what values of the incident shock parameters θ_0 and a the above inequalities are satisfied (a) simultaneously, (b) first or second, (c) neither is satisfied. In event (b) the assumed solution pattern should be modified to include a shock as well; in event (c) the assumed solution should consist of two shock waves.

Modified reflection pattern

If τ increases with γ, the travelling pencil of wavelets converges to the leading wavelet, thus forming a shock wave (Figure 2). The corresponding differential equations (4.36) should be replaced by the jump conditions (2.16)

$$[u_i] = -\overline{V}\overline{a}_i , \qquad [F_{i\alpha}] = -\overline{a}_i \overline{N}_\alpha , \tag{4.70}$$

where the unit normal $\overline{\mathbf{N}} = (\sin\overline{\theta}, -\cos\overline{\theta}, 0)$, the amplitude vector $\overline{a}$ and the normal speed $\overline{V}$ of the reflected shock are the quantities to be determined.

Suppose the first reflected wave becomes a shock. According to Eqs. (4.37) and (4.55), the jumps of the velocity and the deformation gradient across the wave are

$$[u_{(1)i}] = -\tilde{\gamma}_1 k r_{1i} , \qquad [F_{(1)i\alpha}] = \tilde{\gamma}_1 V_h^{-1} k r_{1i} \widehat{N}_{(1)\alpha} . \tag{4.71}$$

Also, since the reflection angle $\overline{\theta}$ of the shock is equal to the angle $\theta(0)$ of the corresponding simple wave leading wavelet, we have

$$\overline{\mathbf{N}} = \sin\theta(0) \widehat{\mathbf{N}}_{(1)} . \tag{4.72}$$

This means that the reflected shock is longitudinal. Further, from Eqs. $(4.70)_2$ and $(4.71)_2$ we find that

$$\frac{[u_{(1)1}]}{[u_{(1)2}]} = \frac{\overline{a}_1}{\overline{a}_2} = \frac{r_{11}}{r_{12}} \tag{4.73}$$

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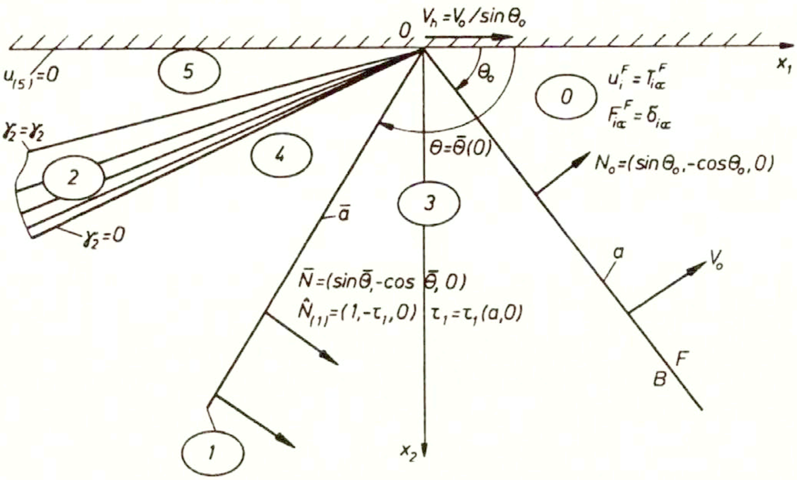

FIG. 2. Modified shock reflection pattern: the reflection solution consists of a shock wave and a simple wave.

that is, the vectors $\bar{\mathbf{a}}$ and $\mathbf{r}_1$ are parallel. We can write $\bar{\mathbf{a}} = \pm\bar{a}\mathbf{r}_1$ where $\bar{a} = |\bar{\mathbf{a}}|$ is the shock strength. For reason of stability we choose (ref. Eq. (2.19))

$$\bar{\mathbf{a}} = -\bar{a}\mathbf{r}_1 . \tag{4.74}$$

Then, using Eq. (4.72) and (4.74) we obtain from Eqs. $(4.70)_1$ and $(4.71)_1$

$$\bar{a} = \tilde{\gamma}_1 V_h^{-1} k\sqrt{1 + \tau_1^2} , \tag{4.75}$$

where $\tau_1 = \tau_1(\mathbf{a}, 0)$ is given by Eq. (4.49) (ref. also Eq. (4.27)), and k is selected positive.

In a similar way we can determine the parameters of the second reflected shock wave which appears to be a transverse wave.

5. Numerical analysis

The reflection solutions discussed in Sec. 4, are examined numerically for a certain kind of steel and for two values of the incident shock wave strength $a = 0.0085$ and $a = 0.0045$. The elasticity constants of the first and second order were taken from [6].

The calculations were realized under the previous assumption that the incident shock is reflected from a rigid boundary and relations (4.65), (4.68) and subsequent holds. The expression for the components of the acoustic tensor (4.2) in the region behind the incident shock wave meets the standard perturbation analysis condition. The Lamé constants and the second order constants have the same order of quantity, the parameter a – shock strength has the order $o(10^{-3})$. For

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this reason the matrix aQ_{ij} can be treated as a small perturbation of the values of components of the matrix Q_{ij}. The form of the expressions (4.32), (4.54) suggests that the components of the deformation gradient $\mathbf{F}_{(3)}$ and $\mathbf{F}_{(1)}$ have the same order of magnitude and the parameter $\gamma_1 \ll 0$. In fact, according to the cases just calculated and Fig. 6, the parameter γ_1 fluctuates in the limits ± 50. The values of the elements of the matrix $\mathbf{F}_{(3)}$ (which represent the deformation gradient in the region 3) outside the main diagonal have the order of magnitude 10^2 greater than the values of the elements of second matrix in the sum (4.32) $\mathbf{F}_{(1)}$.

Formally the treatment of the problem does not change. The elements of both matrices differ in the order of quantity so much, that the problem can be treated as a typical perturbation problem. In order to retain its perturbation character, we can use a scaling factor for the wave parameter γ_1. Introducing the scaling factor equal $\gamma_1^* = 10^{-2}$, the expression (4.54) takes the form

$$\mathbf{F}_{(4)} = \mathbf{F}_{(1)}(\gamma_1) = \mathbf{F}_{(3)} + \gamma_1\gamma_1^* \left(1/\gamma_1^*\mathbf{F}_{(1)}\right).$$

The elements of both above matrices $\mathbf{F}_{(3)}$ and $\mathbf{F}_{(1)}/\gamma_1^*$ have the same order of magnitude and the parameter which is in the form of the product $\gamma_1\gamma_1^* \ll 1$.

Figure 3 shows the relation between the incident angle and the final values of the parameters γ_1, γ_2 for two values of strength of the incident wave ($a = m_0$).

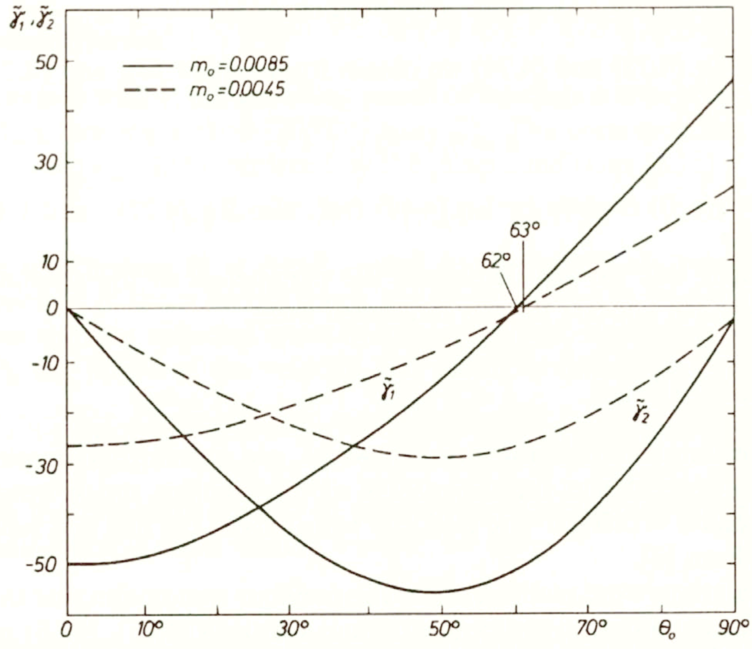

FIG. 3. Final values of the simple waves parameters as function of the incident angle.

In the plane problem the wave surface consists of two branches. The front of the first reflected wave intersects the first wave surface (external) at an inci-

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dence angle $\Theta_0 = 85°$ for $a = 0.0085$ and at angle $\Theta_0 = 80°$ for $a = 0.0045$, respectively, (Fig. 4a) and the assumed reflection pattern no longer holds true. The parameter γ_1 is equal to zero for the incident angle $\Theta_0 = 62°$ or $\Theta_0 = 63°$ depending on the incident shock wave strength. According to Eq. (4.75), the reflected shock wave strength in this case is equal to zero and one of the reflected waves disappears. According to [5], the front of the incident wave and the front of the single reflected wave are perpendicular. Compare Fig. 8, for $a = 0.0085$ the incidence angle $\Theta_0 = 62°$, and the reflection angle for the single reflected wave $\Theta_2 = 152°$. The second equivalence theorem [9] admits the propagation of the reflected infinitesimal progressive wave for the values $\gamma_1 = a = 0$. It results from the numerical calculations that, for small incidence angles, the first reflected wave is a shock wave (Fig. 4b) and the second reflected wave is a simple wave. If the incidence angle increases, there is a transition in such reflection pattern for the value of the parameter $\gamma_1 = 0$, the first reflected shock wave changes into a simple wave, and the second reflected simple wave changes into a shock wave, respectively. Figure 4b presents a graph of the reflected shock strengths $\overline{a}_1$, $\overline{a}_2$ as functions (ref. Eq. (4.75)) of the incident angle. The dotted curves in the graph are representative of simple waves. In the case treated here there is no possibility to obtain the reflection pattern in the form of two reflected simple waves for any value of the incidence angle. Some of the results presented here for Murnaghan's material agree with the results obtained by Y. LI and T.C.T. TING [5] for a different class of hyperelastic materials. The value of the incidence angle at which the transition in the reflection pattern from shock to simple wave takes place is the same as in [5], for the incident shock range considered here. Figure 4a presents the difference in values of the cotangent functions, for angles fixing the position of the extreme wavelets in the second (A), and first (B) reflected simple wave, respectively.

Both graphs are in the same figure. The ordinates of graph A have the multiplier 10^{-4} and the ordinates of the graph B – the multiplier 10^{-1}. Figure 5 presents graphs for the components of the right proper vectors $\mathbf{r}_1$, $\mathbf{r}_2$ of the acoustic tensor and polarisation vectors $\hat{\mathbf{d}}(a_1)$, $\hat{\mathbf{d}}(a_2)$ for the first and second reflected shock wave.

The solid lines mark the relationship between the proper vector and the polarisation vector, for such intervals of the incidence angle, which are connected with the existence of the reflected shock waves. The other parts of the graphs, which are marked with the dotted line, are connected with the simple waves in the reflection pattern. For comparison, in the Fig. 5 are shown the graphs of functions $\sin \Theta_0$ and $\cos \Theta_0$, when the components are reduced to the proper vectors in the "zero" approximation. The graphs of these trigonometric functions are shown only for the value $a = 0.0085$, because of very small differences for the lower value of incident shock strength. It is interesting to notice the symmetry of the graphs for components of the qualitatively different vectors.

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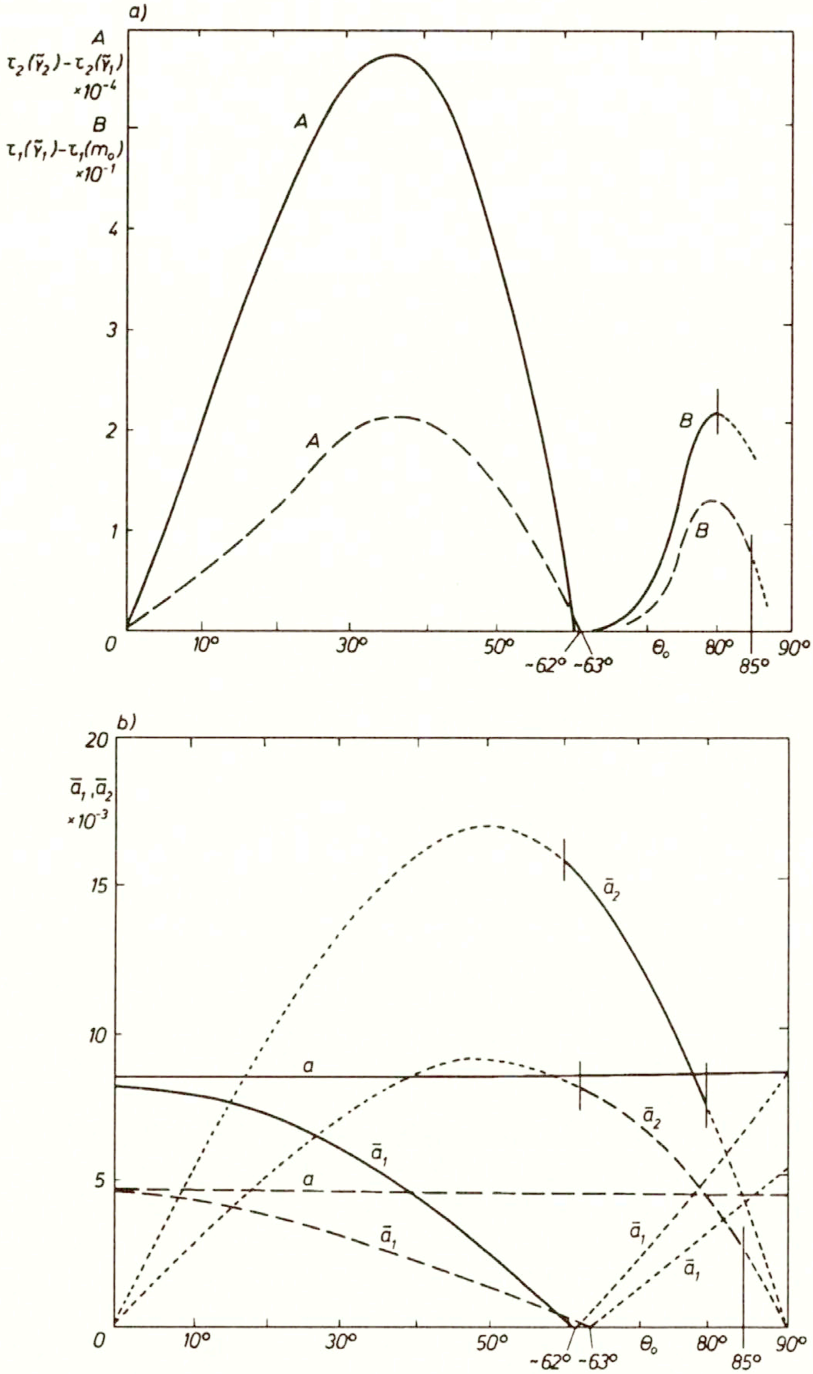

FIG. 4. a) Angular opening of the "fan" for reflected simple waves; b) reflected shock strength as function of the incidence angle.

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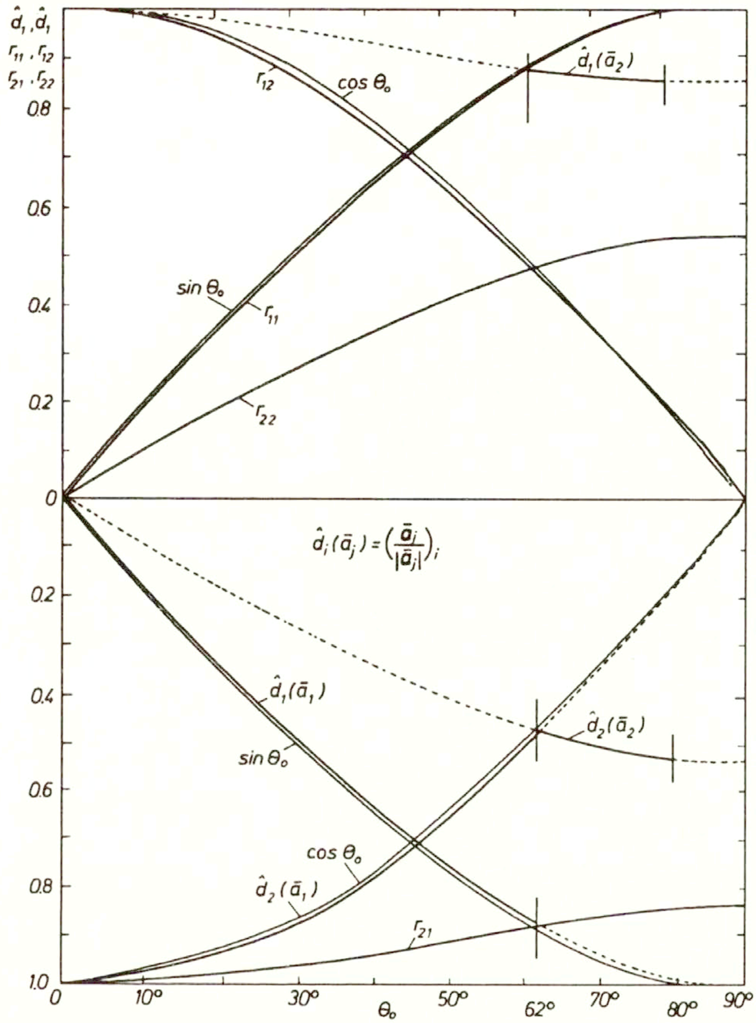

FIG. 5. Components of the right proper vectors and polarisations vectors for both reflected shock waves.

Figures 6 and 7 show the relationship between the components of the deformation gradient and the incidence angle. We also note the fact that at the transition point, when one reflection pattern changes to another, the relationship remains continuous at this approximation.

Figure 8 presents the dependence of the angles Θ_1, Θ_2, determining the fronts of the reflected waves (for the approximation $\tau_1(a)$, $\tau_2(a)$) on the angle of incidence. The graphs for the next approximations do not practically differ in shape from that presented here, in the assumed scale of figure.

Characteristic in the Fig. 9 is the width of region between the two reflected waves. The graphs are realised by two cotangent functions for angles, characteristic for wavelets which envelop this region.

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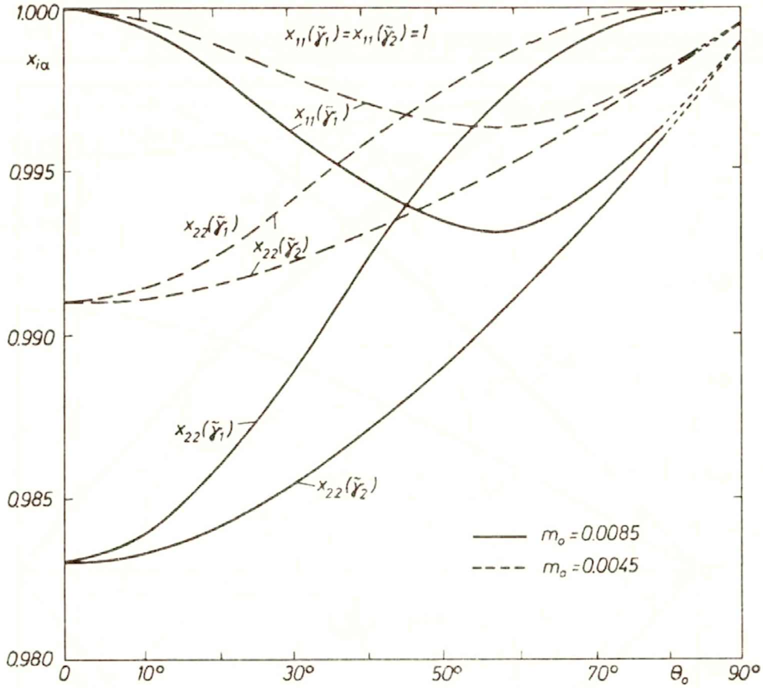

FIG. 6. Components of the deformation gradient for final values of the simple wave parameters as functions of the incidence angle.

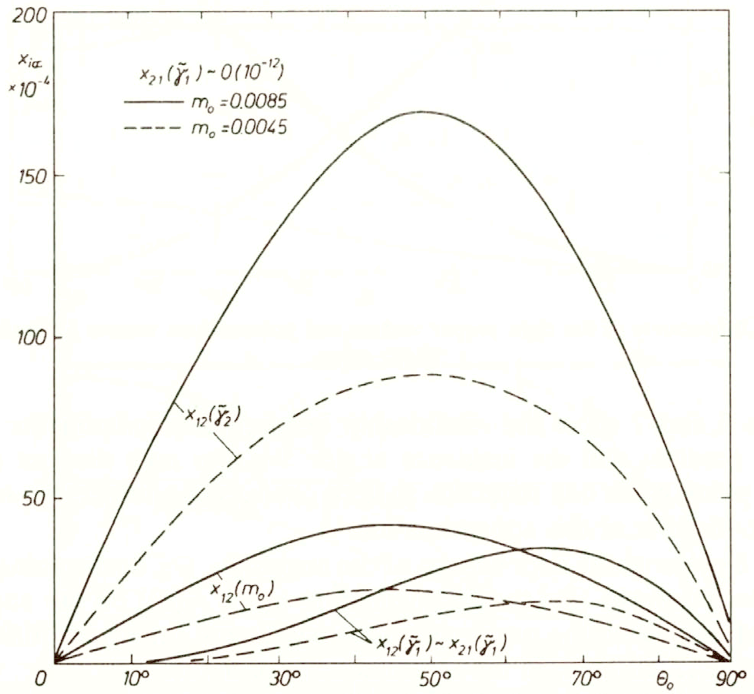

FIG. 7. Components of the deformation gradient for final values of the simple wave parameters as functions of the incidence angle.

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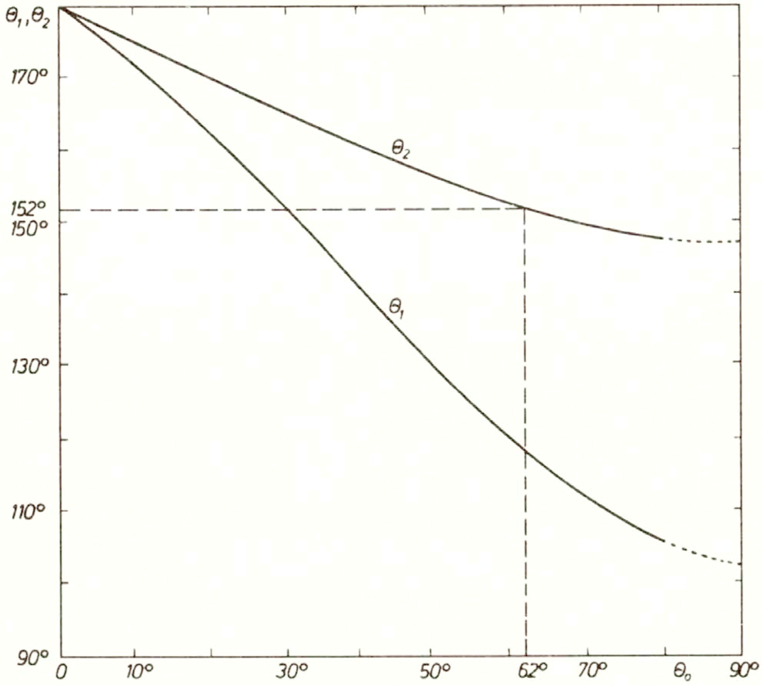

FIG. 8. Relation between the reflected shock wave fronts and the angle of incidence.

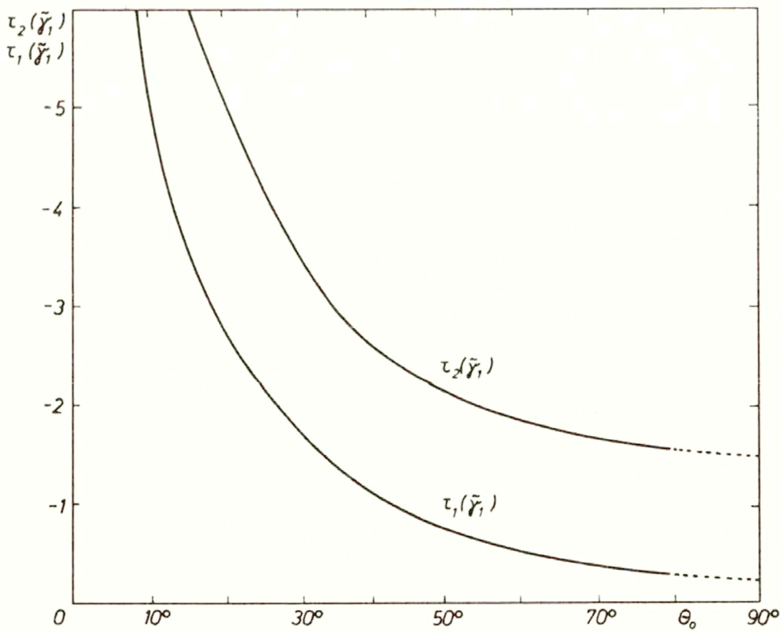

FIG. 9. The width of the region between the reflected waves as a function of the angle of incidence.

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Figures 10 and 11 present the speed of propagation as a function of the incidence angle. In the two intervals: $< 0 - 62° >$ and $< 0 - 63° >$ for the

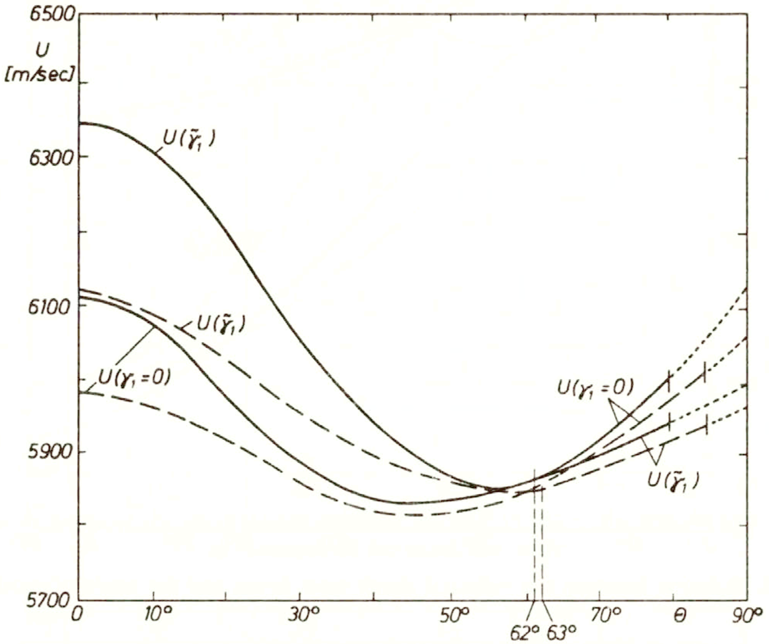

Fig. 10. Speed of propagation as a function of the incidence angle.

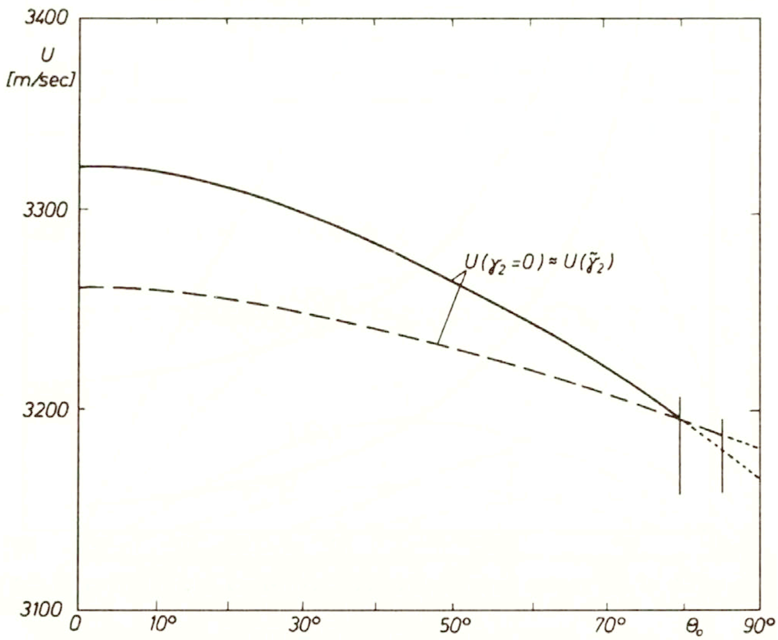

Fig. 11. Speed of propagation as a function of the incidence angle.

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incident angle in the Fig. 10 (for the considered here incident shock strength $U(\gamma_1 = 0) = V_1$), if the incidence angle increases beyond the transition angle, the inequality $U(\gamma_1 = 0) > U(\gamma_1)$ holds and the "fan" of wavelets in the first reflected wave can expand. The differences in the propagation speed in the second reflected wave are so small, that practically $U(\gamma_2) \approx U(\gamma_2 = 0)$, Fig. 10.

6. Concluding remarks

The initial deformations, and the material region behind the incident wave front should remain elastic. Hence the discontinuity jump can not be arbitrary and the appropriate estimate for a – shock strength should be established [6]. In the elastic materials as described by Murnaghan's potential, when the incident shock wave strength is of the order of 10^{-3} (treating the problem as elastic), the expansion into power series of all the quantities describing the wave process in the body is justified.

The differences in values of the parameters between the unstrained state and the state after the propagation of two reflected waves are in the limits 2 – 4% depending of the angle of incidence. The components of the proper vectors are practically the same for all approximations, Fig. 5. The values of the final parameters γ_1, γ_2 are much greater than 1, but the proposed perturbation procedure is valid, because all elements of the matrices which are multiplied by γ_1 and γ_2 are by some orders of magnitude greater than the values of the basic matrix.

For the value $\gamma_1 = 0$ a transition occurs, and the first reflected shock wave changes into a simple wave (the incidence angle increases), and for the same angle the second reflected wave changes into the shock wave, and the strength of it has immediately a finite value. When the fronts of the incident wave and the second reflected wave are perpendicular, the first reflected wave vanishes and this angle of incidence simultaneously becomes the angle of transition from one reflection pattern to another for both reflected waves. When the incident shock wave strength tends to zero, the solutions are exactly the same as those known from the linear theory of elasticity.

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BRIEF NOTES

Asymptotic behaviour of derivatives for systems of second order Ordinary Differential Equations

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IN THIS WORK we analyze systems of the form $u'' = G(\xi, u, u')$, where $u \in R^n$ and $\xi \in (0, \infty)$. We prove that, if certain conditions are fulfilled, then every solution v, such that $G(\infty, v(\infty), \mathbf{0}) = \mathbf{0}$, must have its first (and second) derivatives tending to 0 for $\xi \to \infty$.

1. Introduction

IN MANY APPLICATIONS, when we are dealing with ordinary differential equations (e.g. in asymptotic theory of nonlinear reaction-diffusion systems), arises a question of asymptotic behaviour of derivatives of their solutions. To be more precise, let us consider a second-order system of ODEs:

$$u'' = G(\xi, u, u'), \tag{1.1}$$

where

$$u := (u_1, \ldots, u_n)^T, \qquad u' := (u'_1, \ldots, u'_n)^T, \qquad G := (G_1, \ldots, G_n)^T, \qquad n \in N_+.$$

Let us assume that

$$G(\infty, \mathbf{0}, \mathbf{0}) = \mathbf{0} \tag{1.2}$$

and suppose that we have found a solution $u(\xi)$ such that $||u(\xi)|| \underset{\xi \to \infty}{\longrightarrow} 0$.

Here and below $||\ ||$ *denotes the norm on* $\mathbb{R}^n$ *defined by*

$$||u(\xi)|| := \max\{|u_1(\xi)|, \ldots, |u_n(\xi)|\}.$$

A natural question arises:

Does the above condition imply that $u'(\xi) \underset{\xi \to \infty}{\longrightarrow} \mathbf{0}$?

As one may suspect, this problem is essentially connected with the possibility of *a priori* estimating $||u'||$ (in terms of $||u||$) on compact subsets of the half-line $(0, \infty)$. Thus if

$$||G(\xi, u, z)|| \le C(u, z)||z||^2,$$

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and $C(u, z)$ is sufficiently small for $\|z\| \to \infty$, then for any natural k and $B = (k, k+1)$ we have an estimate

$$\|u'\|_{C(B)} \leq Q\left(\|u\|_{C(B)}\right), \tag{1.3}$$

where

$$\|u\|_{C(B)} = \sup_{\xi \in B} \|u(\xi)\|$$

with the function Q independent of k (cf. e.g. [1], Sec. VIII 1–4). This time a simple argument allows to prove that $u'(\xi)$ tends to $\mathbf{0}$ if only $u(\xi)$ tends to $\mathbf{0}$ (for $\xi \to \infty$) (point a of the proof of the Theorem 1). However, condition (1.3) becomes invalid when C = const (see the counter-example in [5]). In this paper we are going to prove that even in this case the question has a positive answer.

2. Main result

We have the following theorem:

THEOREM 1. *Let us assume that:*
H1: $G(\infty, \mathbf{0}, \mathbf{0}) = \mathbf{0}$.
H2: *G is of C^0 class.*
H3: *there exist real nonnegative numbers M, p, ϑ and C such that for*

$$\xi > M, \qquad \|u\| \leq p, \qquad \text{and} \qquad \|z\| \geq \frac{1}{2}\vartheta$$

we have

$$\|G(\xi, u, z)\| \leq C\|z\|^2. \tag{2.1}$$

Suppose that $u(\xi)$ is a solution of Eq. (1.1), *such that $\|u(\xi)\| \xrightarrow[\xi\to\infty]{} 0$. Then $\|u'(\xi)\| \xrightarrow[\xi\to\infty]{} 0$ and also $\|u''(\xi)\| \xrightarrow[\xi\to\infty]{} 0$.* ■

Proof. As we are interested in the asymptotic behaviour of u' then, without losing generality, we may confine ourselves in the following proof to $\xi > M$.

a. First we will prove that, if the components of $u'(\xi)$ remain bounded for $\xi \to \infty$, then the theorem is valid. Thus, from Eq. (1.1) we conclude that $u''_k(\xi)$, $k = 1, \ldots, n$ is also bounded. Now, suppose to the contrary that there exists an index k such that $u'_k(\xi)$ does not tend to 0 for $\xi \to \infty$. Then, there would exist a sequence $(\xi)_j$ diverging to infinity, such that $u'_k(\xi_j) = \vartheta_j$ and $|\vartheta_j| > \sigma > 0$. By choosing an appropriate subsequence we may assume that, for all j, ϑ_j is of one sign, say, positive. As $u_k(\xi) \to 0$, then for every $\varepsilon > 0$ there exists $J = J(\varepsilon)$ such

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that $|u_k(\xi)| < \varepsilon/2$ for all $\xi > \xi_J$. Due to that, the value of u'_k must decrease from $u'_k(\xi_J)$ to $\sigma/2$ in an interval shorter than $2\varepsilon/\sigma$. However, for $\xi \in (\xi_j, \xi_j + 2\varepsilon/\sigma)$

$$u'_k(\xi) \geq u'_k(\xi_J) - \frac{2\varepsilon}{\sigma}S \geq \sigma - \frac{2\varepsilon}{\sigma}S,$$

where S is a bound for $\|u''\|$. For ε sufficiently small we arrive at a contradiction. Thus $\|u'(\xi)\| \underset{\xi\to\infty}{\longrightarrow} 0$ and $\|u''(\xi)\| \underset{\xi\to\infty}{\longrightarrow} 0$.

b. Now, let us suppose that the sequence $\|z(\xi_j)\| = \|u'(\xi_j)\| := \vartheta_j$, $j = 1, 2, \ldots, \infty$, diverges to infinity. Without losing generality we may assume that $\vartheta_j > \vartheta$ and $\|u(\xi)\| < p$ for all $\xi \geq \xi_1$. It is easy to note that $\|z'(\xi)\| > |\|z(\xi)\|'|$ if the derivatives are eventually understood as the right or left ones (comp. [2] Sec. III. 3 Lemma 3.2). So, from Eq. (2.1) we have $\|z(\xi)\|' \leq C\|z(\xi)\|^2$ and

$$(2.2)_1 \qquad \|z(\xi)\| \leq \left(\|z(\xi_j)\|^{-1} - C(\xi - \xi_j)\right)^{-1}$$

for all $\xi > \xi_j$ sufficiently close to ξ_j (i.e. such that $\|z(\xi)\| \geq \frac{1}{2}\vartheta$). From (2.1) we obtain consequently

$$(2.2)_2 \qquad \|z'(\xi)\| \leq C\|z(\xi)\|^2 \leq C\left(\|z(\xi_j)\|^{-1} - C(\xi - \xi_j)\right)^{-2}.$$

Now, for all j there exist $k \in \{1, \ldots, n\}$, such that $|z_k(\xi_j)| = \vartheta_j$. By choosing appropriate subsequence, and renumbering the components if necessary, we may take $k = 1$ and assume that $z_k(\xi_j) > 0$. (If $z_k(\xi_j) < 0$, the proof would be the same). For all sufficiently small $\varepsilon > 0$ we have (according to the definition of $J(\varepsilon)$ as in point a)

$$(2.3) \qquad |u_1(\xi) - u_1(y)| < \varepsilon$$

for all $\xi, y > \xi_J$, $J = J(\varepsilon)$. As for $\xi > \xi_J$

$$(2.4) \qquad z_1(\xi) \geq z_1(\xi_J) - \int_{\xi_J}^{\xi} \|z'(y)\|\,dy,$$

then the value of z_1 according to (2.3) must decrease from $z_0 = z_1(\xi_J)$ to $z_0/2$ in an interval shorter than $d = 2\varepsilon/z_0$. If $\varepsilon < C/4$), then $\frac{1}{z_0} - \frac{2C\varepsilon}{z_0} > \frac{1}{2z_0}$ for all $z_0 > 0$. Hence, from Eq. $(2.2)_2$ for all $\xi < \xi_J + d$, $J = J(\varepsilon)$ we have

$$(2.5) \qquad \int_{\xi_J}^{\xi} \|z'(y)\|\,dy < \int_{\xi_J}^{\xi} C(2z_0)^2\,dy < 4Cz_0^2 d = 4Cz_0\varepsilon.$$

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Let $\eta_J > \xi_J$ denote the first point, where z_1 is equal to $z_0/2$. As $\eta_J - \xi_J < d$, then from Eqs. (2.4) and (2.5) we obtain

$$\frac{1}{2} > 1 - 4C\varepsilon. \tag{2.6}$$

For ε suficiently small Eq. (2.6) cannot be fulfilled, so we arrive at a contradiction. Thus $\|u'(\xi)\| \underset{\xi\to\infty}{\longrightarrow} 0$ and $\|u''(\xi)\| \underset{\xi\to\infty}{\longrightarrow} 0$. The theorem is proved. ■

3. Counter-example

The assumption $H3$ of the Theorem is in a sense optimal, i.e. it cannot be weakened. To show it let us consider the following system of two equations:

$$\begin{aligned} u_1'' &= \lambda r b(r)(\alpha u_2 + u_2') + \alpha^2 u_1, \\ u_2'' &= \lambda r b(r)(\alpha u_1 + u_1') + \alpha^2 u_2, \end{aligned} \tag{3.1}$$

where $\lambda, \alpha \in \mathbb{R}^1$, $\alpha > 0$, $r^2 := (\alpha u_1 + u_1')^2 + (\alpha u_2 + u_2')^2$. We assume that $b \in C^1[0,\infty)$, $b > 0$, $b(r) \underset{r\to\infty}{\longrightarrow} \infty$, $b(r)r^{-1}$ is monotonically decreasing and $b(r)r^{-1} \underset{r\to\infty}{\longrightarrow} 0$. Despite its nonlinearity, the system can be integrated after being written in polar coordinates. So, if $z_i := \alpha u_i + u_i'$, $i = 1, 2$, then $r^2 = z_1^2 + z_2^2$ and Eqs. (3.1) are equivalent to:

$$\begin{aligned} z_1' &= \lambda r b(r) z_2 + \alpha z_1, \\ z_2' &= -\lambda r b(r) z_1 + \alpha z_2. \end{aligned}$$

Then, if $z_1 = r\cos\phi$, $z_2 = r\sin\phi$, we obtain the system

$$\begin{aligned} r' &= \alpha r, \\ \phi' &= -\lambda r b(r). \end{aligned}$$

The last system has the following solution:

$$r = S\exp(\alpha\xi), \qquad \phi = -\lambda \int_0^\xi r(y) b(r(y))\, dy + \phi_0,$$

where ϕ_0 and S are constants. Let us take $S = 1$ and $\phi_0 = 0$. Then we obtain

$$u_1(\xi) = (r(\xi))^{-1} \int_0^\xi r^2(y) \cos\left(-\lambda \int_0^y r(\zeta) b(r(\zeta))\, d\zeta\right) dy + C_1 \exp(-\alpha\xi).$$

In the expression for $u_2(\xi)$ we must put sine instead of cosine.

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Let us make still another change of variables:

$$s(\xi) := \int_0^{\xi} r(y)b(r(y))\,dy.$$

Then s may be treated as a function of r:

$$s(r) := s(\xi(r)) = \alpha^{-1}\int_1^{r(\xi)} b(\zeta)\,d\zeta.$$

As $b(r) > 0$, then for $\xi, r \in (0,\infty)$ we have:

$$\frac{ds}{dr} > 0, \qquad \frac{ds}{d\xi} > 0, \qquad \frac{dr}{ds} > 0, \qquad \frac{d\xi}{ds} = [r(s)b(r(s))]^{-1} > 0.$$

Thus, the expression for $u_1(\xi)$ may be written in the following form:

$$\begin{aligned} u_1(\xi(s)) &= (r(s))^{-1}\int_0^{s} r(s^*)(b(r(s^*)))^{-1}\cos(-\lambda s^*)\,ds^* + C_1\exp(-\alpha\xi(s)) \\ &= b(r(s))^{-1}\int_0^{s}\frac{r(s^*)b(r(s))}{r(s)b(r(s^*))}\cos(-\lambda s^*)\,ds^* + C_1\exp(-\alpha\xi(s)) \\ &:= b(r(s))^{-1}I(s) + C_1\exp(-\alpha\xi(s)). \end{aligned}$$

In the integral $I(s)$ let $s^* := s - \eta$. Then

$$\begin{aligned} I(s) &= \int_0^{s} h(s,\eta)\cos(-\lambda(s-\eta))\,d\eta \\ &= \sin(\lambda s)\int_0^{s} h(s,\eta)\sin(\lambda\eta)\,d\eta + \cos(\lambda s)\int_0^{s} h(s,\eta)\cos(\lambda\eta)\,d\eta, \end{aligned}$$

where

$$h(s,\eta) := \frac{r(s-\eta)b(r(s))}{r(s)b(r(s-\eta))}.$$

Now, for $J = \Pi\lambda^{-1}$, $k = 1, 2, \ldots$, let

$$a_k := \int_{kJ}^{(k+1)J} h((k+1)J,\eta)\sin(\lambda\eta)\,d\eta.$$

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We see that

$$h((k+1)J,(k+1)J) = r(0)b(r((k+1)J))\,[r((k+1)J)b(r(0))]^{-1}$$

and

$$h((k+1)J,kJ) = r(J)b(r((k+1)J))\,[r((k+1)J)b(J)]^{-1}\,.$$

Due to the assumptions satisfied by b we infer that $b(r(s))r(s)^{-1}$ tends monotonically to 0 as $s \to \infty$, so $|a_{k+1}| < |a_k|$. Thus, due to the Leibnitz theorem (concerning alternating sequences) and the fact that $\cos(\lambda\eta) = \sin(\lambda\eta + (\pi/2))$, we conclude that $I(\infty)$ is finite. In this way we have $u_1(\xi) \to 0$ as $\xi \to \infty$. Similarly we can prove that $u_2(\xi) \to 0$ as $\xi \to \infty$. However, $r^2(\xi) = e^{2\alpha\xi}$, so due to the above definitions we have $(u_1'(\xi))^2 + (u_2'|\xi))^2 \to \infty$ as $\xi \to \infty$.

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On the phenomenon of wave breaking

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THE DU BOIS-REYMOND equations are used to investigate the wave breaking phenomenon. Various relationships between the components of wave slopes are determined and it is shown that abrupt discontinuities in the bottom profile initiates breaking.

1. Introduction

THE BREAKING phenomena in wave profiles lead to the development of singularities in them. Many scientists have done extensive work in this field such as BANNER and PHILIPS [1], BENNEY [2] and LONGUET-HIGGINS and COKELET [3]. WHITHAM [4] has given a good account of this phenomenon in detail in his book. The discontinuity in the wave profile begins with steepening of the front and it persists until the front becomes vertical and then curls over. This can be explained in terms of changes in wave slopes on both sides of the wave crest. To model this phenomenon, an extended form of the Du Bois-Reymond equations of variational methods [5] are assumed to be suitable. These equations are applicable only when the oscillations are normal to the shoreline. In shallow water, it is not necessary that the bottom contours should always be parallel to the coastlines. Thus in most of the cases the wave crests approach the shoreline obliquely. We extend the Du Bois-Reymond equations to incorporate this effect.

2. The Du Bois – Reymond equations

The Du Bois-Reymond equations give the conditions for the curve $C : y = g(x)$ to have discontinuous slopes at a given point or points on a line. Here we intend to extend this idea to plane surfaces defined by the coordinate system r, θ and $\eta(r, \theta)$. For this purpose, we introduce the dynamical system with Lagrange's functional I and Lagrange's density $L(r, \theta, \eta, p, q)$ which are connected by the relationship

$$I = \int\limits_{-\pi/2}^{\pi/2} \int\limits_{0}^{\infty} L(r, \theta, \eta, p, q)\, dr\, d\theta, \tag{2.1}$$

where $p = \eta_r$, $q = \eta_\theta$.

For sufficiently small α, we write

$$\eta(r, \theta) = \eta_0(r, \theta) + \alpha\tilde{\eta}(r, \theta), \tag{2.2}$$

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so that we have,

$$
(2.3)\qquad \begin{aligned} p &= \eta_{0r} + \alpha\tilde{\eta}_r(r,\theta), \\ q &= \eta_{0\theta} + \alpha\tilde{\eta}_\theta(r,\theta). \end{aligned}
$$

We specify the conditions

$$
(2.4)\qquad \tilde{\eta}(r, \pm\pi/2) = \tilde{\eta}(0,\theta) = 0,
$$

and require $\tilde{\eta}_r$ and $\tilde{\eta}_\theta$ to vanish along the shoreline.

Using Eqs. (2.2) and (2.3), we have,

$$
(2.5)\qquad \frac{dI(\alpha)}{d\alpha} = \int\limits_{-\pi/2}^{\pi/2}\int\limits_{0}^{\infty}\left(\tilde{\eta}\frac{\partial L}{\partial \eta} + \tilde{\eta}_r\frac{\partial L}{\partial p} + \tilde{\eta}_\theta\frac{\partial L}{\partial \theta}\right) dr\, d\theta.
$$

We now introduce $\eta(r,\theta)$, $\tilde{\eta}(r,\theta)$, in the region $0 \le r < \infty$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ for functions which are continuous on the specified plane but their derivatives are discontinuous at breaking point. Thus p and q are discontinuous across a breaker zone. We therefore introduce

$$
(2.6)\qquad F(r,\theta) = \int\limits_{\theta}^{\pi/2}\int\limits_{r}^{\infty}\frac{\partial L}{\partial \eta} dr\, d\theta,
$$

where

$$
(2.7)\qquad \frac{\partial^2 F}{\partial r \partial \theta} = \frac{\partial L}{\partial \eta},
$$

$$
(2.8)\qquad F(r,\pi/2) = F(\infty,\theta) = 0.
$$

Now,

$$
(2.9)\qquad \frac{\partial^2(\tilde{\eta}F)}{\partial r \partial \theta} = \tilde{\eta}F_{r\theta} + \tilde{\eta}_r F_\theta + \tilde{\eta}_\theta F_r + F\tilde{\eta}_{r\theta}.
$$

From Eqs. (2.7) and (2.9), we have

$$
(2.10)\qquad \tilde{\eta}L_\eta = \frac{\partial^2(\tilde{\eta}F)}{\partial r \partial \theta} - \tilde{\eta}_r F_\theta - \tilde{\eta}_\theta F_r - F\tilde{\eta}_{r\theta}.
$$

Substituting Eq. (2.10) in Eq. (2.5), we obtain

$$
(2.11)\qquad \frac{dI}{d\alpha} = \int\limits_{-\pi/2}^{\pi/2}\int\limits_{0}^{\infty}\left[\tilde{\eta}_r L_p + \tilde{\eta}_\theta L_q + (\tilde{\eta}F)_{r\theta} - \tilde{\eta}_r F_\theta - \tilde{\eta}_\theta F_r - F\tilde{\eta}_{r\theta}\right] dr\, d\theta.
$$

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From Eqs. (2.8) and (2.4), we have

(2.12)
$$\int_{-\pi/2}^{\pi/2}\int_{0}^{\infty}(\tilde{\eta}F)_{r\theta}\,dr\,d\theta = 0,$$

and

(2.13)
$$\int_{-\pi/2}^{\pi/2}\int_{0}^{\infty}F\tilde{\eta}_{r\theta}\,dr\,d\theta = \int_{0}^{\infty}F\tilde{\eta}_{r}\Big|_{-\pi/2}^{\pi/2}dr - \int_{-\pi/2}^{\pi/2}\int_{0}^{\infty}\tilde{\eta}_{r}F_{\theta}\,dr\,d\theta$$
$$= -\int_{-\pi/2}^{\pi/2}\int_{0}^{\infty}\tilde{\eta}_{r}F_{\theta}\,dr\,d\theta.$$

Therefore, using Eqs. (2.12) and (2.13), Eq. (2.11) becomes

(2.14)
$$\frac{dI}{d\alpha} = \int_{-\pi/2}^{\pi/2}\int_{0}^{\infty}\tilde{\eta}_{r}L_{p}\,dr\,d\theta + \int_{-\pi/2}^{\pi/2}\int_{0}^{\infty}\tilde{\eta}_{\theta}(L_{q} - F_{r})\,dr\,d\theta.$$

Since $\tilde{\eta}(r,\theta)$ is the region $(0,\infty;-\pi/2,\pi/2)$ and $\tilde{\eta}(r,\pm\pi/2) = \tilde{\eta}(0,\theta) = 0$, Du Bois - Reymond's theorem gives us the results concerning the extreme values of a function of two variables from Eq. (2.14), that is,

(2.15)
$$L_{p} = C_{1}(\theta),$$

(2.16)
$$L_{q} - F_{r} = C_{2}(r).$$

Since F_r is continuous across any discontinuity in the plane under consideration,

(2.17)
$$L_{q} = C_{2}(r) + F_{r} = C_{3}(r).$$

Here, C_1, C_2 and C_3 are arbitrary functions. Now, we define r and θ by

(2.18)
$$r = \rho + \alpha_{1}\lambda(\rho),$$

(2.19)
$$\theta = \phi + \alpha_{2}\mu(\phi),$$

where $\lambda(0) = \mu\left(\pm\frac{\pi}{2}\right) = 0$ and $\lambda(\rho)$, $\mu(\phi)$ are in $\left(0,\infty;-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Corresponding to the extrema at $\alpha_1 = \alpha_2 = 0$, we have

(2.20)
$$\frac{\partial I}{\partial\alpha_{1}} = \int_{-\pi/2}^{\pi/2}\int_{0}^{\infty}(L\lambda_{r} + \lambda L_{r} + \lambda_{r}pL_{p})\,d\rho\,d\phi = 0,$$

(2.21)
$$\frac{\partial I}{\partial\alpha_{2}} = \int_{-\pi/2}^{\pi/2}\int_{0}^{\infty}(L\mu_{\theta} + \mu L_{\theta} + \mu_{\theta}qL_{q})\,d\rho\,d\phi = 0.$$

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Now, we introduce the two functions $G(r,\theta)$ and $H(r,\theta)$ given by

$$G(r,\theta) = \int_r^\infty L_r\, dr, \qquad G(\infty,\theta) = 0, \tag{2.22}$$

$$H(r,\theta) = \int_\theta^{\pi/2} L_\theta\, d\theta, \qquad H(r,\pi/2) = 0. \tag{2.23}$$

Using $G_r = L_r$,

$$\int_{-\pi/2}^{\pi/2}\int_0^\infty \lambda L_r\, dr\, d\theta = -\int_{-\pi/2}^{\pi/2}\int_0^\infty \lambda_r G\, dr\, d\theta. \tag{2.24}$$

In the same manner, using $H_\theta = L_\theta$, we have,

$$\int_0^\infty\int_{-\pi/2}^{\pi/2} \mu L_\theta\, d\theta\, dr = -\int_{-\pi/2}^{\pi/2}\int_0^\infty \mu_\theta H\, dr\, d\theta. \tag{2.25}$$

Therefore, from Eqs. (2.20) and (2.24), we get

$$\frac{\partial I}{\partial \alpha_1} = \int_{-\pi/2}^{\pi/2}\int_0^\infty \lambda_r (L + pL_p - G)\, dr\, d\theta = 0, \tag{2.26}$$

and from Eqs. (2.21) and (2.25), we get

$$\frac{\partial I}{\partial \alpha_2} = \int_{-\pi/2}^{\pi/2}\int_0^\infty \mu_\theta (L + qL_q - H)\, dr\, d\theta = 0. \tag{2.27}$$

Equations (2.26) and (2.27) establish the conditions concerning the extrema of the functional (2.1) at $\alpha_1 = \alpha_2 = 0$. Now, application of Du Bois-Reymond's results leads us to

$$L + pL_p - G = C_4(\theta), \tag{2.28}$$

$$L + qL_q - H = C_5(r). \tag{2.29}$$

Since G and H are continuous across any specified plane, we conclude that

$$L + pL_p = C_6(\theta), \tag{2.30}$$

$$L + qL_q = C_7(r). \tag{2.31}$$

Here, C_4, C_5, C_6 and C_7 are arbitrary functions.

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3. Long waves in shallow water

We consider the equation governing the propagation of linear long waves in shallow water, which is

$$\frac{\partial}{\partial r}\left(rh\frac{\partial \eta}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{h}{r}\frac{\partial \eta}{\partial \theta}\right) + \frac{r\Omega^2\eta}{g} = 0, \tag{3.1}$$

where $\eta = \eta(r,\theta)$ is the wave profile, $h = h(r,\theta)$ is the bottom topography of the sea-bed, Ω is the wave frequency and g is the acceleration due to gravity. We now multiply this elliptic equation (3.1) by $(\overline{\eta} - \eta)$ and integrate with respect to r from 0 to ∞ and with respect to θ from $-\pi/2$ to $\pi/2$, using the conditions $\eta(r,\theta) \to 0$ as $r \to \infty$, $h(r,\theta) \to 0$, $h(r,\theta)\eta(r,\theta) \to 0$ as $r \to 0$ and $\theta \to \pm\pi/2$. Then

$$I(\overline{\eta}) > I(\eta), \tag{3.2}$$

where

$$I(\eta) = \int\limits_0^\infty \int\limits_{-\pi/2}^{\pi/2} L(r,\theta,\eta,p,q)\,d\theta\,dr, \tag{3.3}$$

$$L(r,\theta,\eta,p,q) = rhp^2 + \frac{h}{r}q^2 + \frac{2r\Omega^2\eta^2}{g}. \tag{3.4}$$

Here, η is a solution of Eq. (3.1) while $\overline{\eta}$ is any admissible function.

Now, let (r_1, θ_1) be the point on the wave crest where the front has a tendency to be steeper progressively. Also, let q_2, p_2 and q_3, p_3 be respective wave slopes in front and behind this point. Then, we have $q_3 < q_2$, $p_3 < p_2$, and h_1 is the water depth measured from the undisturbed level at $h = h_1$. Thus

$$h_1 r_1 p_2 = h_1 r_1 p_3, \tag{3.5}$$

$$h_1 r_1 q_2 = h_1 r_1 q_3, \tag{3.6}$$

$$r_1(p_2^2 - p_2^3) = q_2^2 - q_2^3. \tag{3.7}$$

On the shoreline, where $r = 0$ and $h = 0$, Eqs. (3.5), (3.6) are satisfied and Eq. (3.7) is satisfied if $q_2 = q_3$ at the beginning. This is a known fact concerning wave breaking along the shoreline, which is a kind of singularity that does not propagate away from the source.

Now, along the curve $\Gamma : r = R(\theta)$ across which $h(r,\theta)$ is discontinuous in shallow water,

$$\left[h\frac{\partial \eta}{\partial n}\right]_1 = \left[h\frac{\partial \eta}{\partial n}\right]_2, \tag{3.8}$$

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where $\hat{n}$ is the unit vector normal to Γ and subscripts 1 and 2 denote two sections separated by Γ. This result (3.8) is the well-known law of conservation of mass and gives an indication of the fact that the discontinuity in wave profile can be brought about by an abrupt change in the bottom profile.

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A note on dynamic modelling of periodic composites(*)

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In this note it is shown under what conditions the refined macrodynamics for linear-elastic micro-periodic composites [1], can be obtained from the linear elastodynamics as an asymptotic approximation.

1. Preliminaries

Let Ω be a regular region in the referential 3-space occupied by a linear-elastic micro-periodic composite body. By $V = (0, \delta_1) \times (0, \delta_2) \times (0, \delta_3)$ we denote the representative volume element of this body. Setting $\delta \equiv \max\{\delta_1, \delta_2, \delta_3\}$ we shall consider δ as *the microstructure parameter*, sufficiently small as compared to the smallest characteristic length of Ω. The second small parameter is *the modelling accuracy parameter* λ which represents the admissible computation accuracy of functions describing (in the dimensionless form) the properties or the behaviour of the composite. For the known values of δ and λ, function $F(\cdot)$ defined and continuous on $\overline{\Omega}$ will be called *the macro-function* if for every $\mathbf{x}, \mathbf{z} \in \Omega$ condition $|\mathbf{x} - \mathbf{z}| < \delta$ implies $|F(\mathbf{x}) - F(\mathbf{z})| < \lambda$. Similarly, function $F(\cdot)$ defined and continuous on $\overline{\Omega}$ and having in Ω continuous derivatives up to k-th order is said to be the macro-function of the k-th order (or regular macro-function) if $F(\cdot)$, together with all its derivatives, are macro-functions.

Let $f(\cdot)$ be an integrable V-periodic function (a function with periods δ_1, δ_2, δ_3) defined on $\mathbb{R}^3$. If $< f >$ stands for the value of $f(\cdot)$ averaged over V (constant) and $F(\cdot)$ is an arbitrary macro-function, then we obtain

$$\text{(1.1)} \qquad \int_\Omega f F \, dv = <f> \int_\Omega F \, dv + 0(\lambda), \qquad dv \equiv dx_1 \, dx_2 \, dx_3,$$

where $0(\lambda) \to 0$ together with $\lambda \to 0$. Moreover, if $f(\cdot) \equiv f_0(\cdot) + f_\delta(\cdot)$, where $f_\delta(\mathbf{x}) \in 0(\delta)$ and $f_0(\cdot)$ is independent of δ, then we also obtain

$$\text{(1.2)} \qquad \int_\Omega f F \, dv = < f_0 > \int_\Omega F \, dv + 0(\delta).$$

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Formulas (1.1) and (1.2) will be applied in the subsequent section. Following [1], by *the micro-shape functions* $h_a(\cdot)$, $a = 1, \ldots, n$, we shall mean the linearly independent system of continuous V-periodic functions having picewise continuous derivatives $\partial h_a / \partial x_i$ such that $< \partial h_a / \partial x_i >= 0$, $h_a(\mathbf{x}) \in 0(\delta)$ and $\partial h_a / \partial x_i(\mathbf{x})$ are independent of δ. Functions $h_a(\cdot)$ describe, roughly speaking, the shape of expected micro-vibrations related to the micro-heterogeneous periodic material structure of a composite body.

In the sequel subscripts i, j, k, l run over 1, 2, 3 and are referred to the orthogonal Cartesian coordinate system in the referential space. Points of this space are represented by triples $\mathbf{x} = (x_1, x_2, x_3)$ and τ is the time coordinate. Indices a, b run over $1, \ldots, n$, where n is the number of micro-shape functions. Summation convention holds both for i, j, k, l and a, b.

2. Analysis

Let $a_{ijkl}(\cdot)$ and $\rho(\cdot)$ be the V-periodic elasticity tensor field and the V-periodic mass density scalar field, respectively, in the composite body. Let t_i be the boundary tractions on $\partial\Omega$, b_i be the constant body force vector and define the V-periodic vector field $f_i(\cdot) \equiv \rho(\cdot) b_i$. Moreover, let $u_i(\cdot, \tau)$ be the vector field of displacement from the natural configuration of a body. As it is known, the governing equations of the linear elastodynamics can be derived from the principle of stationary action for the action functional

$$\mathcal{A} = \int_{\tau_0}^{\tau_f} (\mathcal{K} - \mathcal{P})\, d\tau, \tag{2.1}$$

where $\mathcal{K}$ and $\mathcal{P}$ are kinetic and potential energy functionals, respectively, given by

$$\begin{aligned} \mathcal{K} &= \frac{1}{2} \int_{\Omega} \rho(\mathbf{x})\, \dot{u}_i\, (\mathbf{x}, \tau)\, \dot{u}_i\, (\mathbf{x}, \tau)\, dv, \\ \mathcal{P} &= \frac{1}{2} \int_{\Omega} a_{ijkl}(\mathbf{x}) u_{(i,j)}(\mathbf{x}, \tau) u_{(k,l)}(\mathbf{x}, \tau)\, dv - \int_{\Omega} f_i(\mathbf{x}) u_i(\mathbf{x}, \tau)\, dv \\ &\quad - \oint_{\partial\Omega} t_i(\mathbf{x}, \tau) u_i(\mathbf{x}, \tau)\, da. \end{aligned} \tag{2.2}$$

Following [1], we introduce the *Micro-Macro Kinematic Hypothesis* by imposing on displacements $u_i(\cdot, \tau)$ in Eqs. (2.2) constraints of the form

$$u_i(\mathbf{x}, \tau) = U_i(\mathbf{x}, \tau) + h_a(\mathbf{x}) Q_i^a(\mathbf{x}, \tau), \qquad \mathbf{x} \in \Omega, \quad \tau \in (\tau_0, \tau_f), \tag{2.3}$$

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where $U_i(\cdot,\tau)$, $Q_i^a(\cdot,\tau)$ are arbitrary, sufficiently regular macro-functions (together with their time derivatives) and $h_a(\cdot)$ are the postulated *a priori* micro-shape functions such that $< \rho h_a >= 0$. Substituting the right-hand sides of Eqs. (2.3) into Eq. (2.2), applying formula (1.1) to functional $\mathcal{K}$ and formula (1.2) to functional $\mathcal{P}$, we obtain

$$\mathcal{K} = \int\limits_{\Omega} \Big[\frac{1}{2} <\rho> \dot{U}_i\,(\mathbf{x},\tau)\,\dot{U}_i\,(\mathbf{x},\tau) + <\rho h_a> \dot{U}_i\,(\mathbf{x},\tau)\,\dot{Q}_i^a\,(\mathbf{x},\tau) + \frac{1}{2} <\rho h_a h_b> \dot{Q}_i^a\,(\mathbf{x},\tau)\,\dot{Q}_i^b\,(\mathbf{x},\tau) \Big]\, dv + 0(\lambda),$$

$$\tag{2.4} \mathcal{P} = \int\limits_{\Omega} \Big[\frac{1}{2} < a_{ijkl} > U_{(i,j)}(\mathbf{x},\tau) U_{(k,l)}(\mathbf{x},\tau) + < a_{ijkl} h_{a,k} > U_{(i,j)}(\mathbf{x},\tau) Q_l^a(\mathbf{x},\tau) + \frac{1}{2} < a_{ijkl} h_{a,j} h_{b,l} > Q_i^a(\mathbf{x},\tau) Q_k^b(\mathbf{x},\tau) \Big]\, dv - \int\limits_{\Omega} < f_i > U_i(\mathbf{x},\tau)\, dv - \oint\limits_{\partial\Omega} t_i(\mathbf{x},\tau) U_i(\mathbf{x},\tau)\, da + 0(\delta).$$

The crucial point of the procedure proposed in this note is the *Macro-Modelling Condition* which states that in Eqs. (2.4) terms $0(\lambda)$ in the kinetic energy functional $\mathcal{K}$ and terms $0(\delta)$ in the potential energy functional $\mathcal{P}$ can be neglected. Using this condition as the modelling hypothesis, we obtain from Eqs. (2.1) and (2.4) the new action functional $\mathcal{A}_0$ given by

$$\tag{2.5} \mathcal{A}_0 = \int\limits_{\tau_0}^{\tau_f} (\mathcal{K}_0 - \mathcal{P}_0)\, d\tau,$$

where we have denoted

$$\mathcal{K}_0 \equiv \int\limits_{\Omega} \Big(\frac{1}{2} <\rho> \dot{U}_i \dot{U}_i + <\rho h_a> \dot{U}_i \dot{Q}_i^a + \frac{1}{2} <\rho h_a h_b> \dot{Q}_i^a \dot{Q}_i^b \Big)\, dv,$$

$$\tag{2.6} \mathcal{P}_0 \equiv \int\limits_{\Omega} \Big(\frac{1}{2} < a_{ijkl} > U_{(i,j)} U_{(k,l)} + < a_{ijkl} h_{a,k} > U_{(i,j)} Q_l^a + \frac{1}{2} < a_{ijkl} h_{a,j} h_{b,l} > Q_i^a Q_k^b - < f_i > U_i \Big)\, dv - \oint\limits_{\partial\Omega} t_i U_i\, da.$$

The material and inertial properties of the composite are described in Eqs. (2.6) by constant (averaged) elastic and inertial moduli $<\rho>, \ldots, < a_{ijkl} h_{a,j} h_{b,l} >$. The new dynamic variables are represented by macro-functions $U_i(\cdot,\tau)$ and $Q_i^a(\cdot,\tau)$ which are called macro-displacements and correctors, respectively, [1].

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3. Conclusions

Applying the principle of stationary action $\delta \mathcal{A}_0 = 0$ to the action functional (2.5), we obtain the system of field equations

$$
(3.1)\qquad
\begin{aligned}
&< a_{ijkl} > U_{k,jl}(\mathbf{x},\tau) - <\rho> \ddot{U}_i(\mathbf{x},\tau) + < a_{ijkl} h_{a,k} > Q^a_{l,j}(\mathbf{x},\tau) \\
&\qquad\qquad - < f_i > = 0, \\
&< \rho h_a h_b > \ddot{Q}^b_i(\mathbf{x},\tau) + < a_{ijkl} h_{a,j} h_{b,l} > Q^b_k(\mathbf{x},\tau) \\
&\qquad\qquad + < a_{ijkl} h_{a,j} > U_{k,l}(\mathbf{x},\tau) = 0
\end{aligned}
$$

which have to hold in Ω, and the natural boundary condition

$$
(3.2)\qquad [< a_{ijkl} > U_{k,l}(\mathbf{x},\tau) + < a_{ijkl} h_{a,k} > Q^a_l(\mathbf{x},\tau)]\, n_j(\mathbf{x}) = t_i(\mathbf{x},\tau)
$$

for almost every $\mathbf{x} \in \partial\Omega$, where $n_j(\mathbf{x})$ is an outward unit normal to $\partial\Omega$ at $\mathbf{x}$. Equations (3.1), (3.2) constitute governing equations of the refined macrodynamics which were obtained independently in [1]. For a discussion and applications of these equations the reader is referred to [1] and the related papers. Here we shall restrict ourselves to the general conclusion that the refined macrodynamics of linear-elastic micro-periodic composites can be obtained from the linear elastodynamics as an asymptotic theory, provided that the constraints (2.3) hold and the Macro-Modelling Condition is taken into account. It has to be notified that, under this condition, terms of order $0(\delta)$ are neglected in the asymptotic approximation $\delta \to 0$ of the potential energy but they are retained in the kinetic energy where we deal with asymptotic approximation $\lambda \to 0$. Retention of the terms $0(\delta)$ in the kinetic energy has a clear physical sense; they are terms involving inertia moduli $< \rho h_a h_b >$ which in Eqs. (3.1) are responsible for the description of the scale and dispersion effects that are typical for a dynamic behaviour of inhomogeneous media [1–3]. On the other hand, neglect of terms $0(\delta)$ in potential energy is motivated by the expected local (i.e. independent of the microstructure parameter δ) character of internal interactions (stresses), as well as interactions with external fields (body forces and boundary tractions). Retention of terms $0(\delta)$ also in the potential energy leads to additional $3n$ boundary conditions for correctors Q^a_i which do not have any physical sense and cannot be properly formulated in the boundary value problems encountered in engineering applications of the theory. That is why in the proposed approach we have introduced two small parameters λ, δ and two different asymptotic approximations for the kinetic and potential energies.

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