Contents of issue 6 vol. XLVII

1015 A. Berezowski and F. Kaplanski, *Vorticity distributions for thick and thin viscous vortex pairs and rings*

1027 B. Nsom, *Stability of Couette-Poiseuille flow of fiber suspensions*

1043 I.Yu. Popov, *Operator approach to three problems of fluid mechanics*

1057 T.C.N. Monteiro and G.J.M. Copeland, *A hydrodynamic model of unsteady tidal flow in coastal waters based on an inverse method*

1073 M. Puzyrewski and K. Namieśnik, *Comparison between strictly conical and paraboloidal flow fields in the presence of angular momentum*

1089 K. Piechór, *A four-velocity model for van der Waals fluids*

1113 P.G. Kalitzova-Kurteva, S.G. Slavtchev and I. Kurtev, *Unsteady heat transfer in a liquid film on a rotating disk*

1125 C. Huet, *Bounds for the overall properties of viscoelastic heterogeneous and composite materials*

1157 Suripeddi Srinivas and N.Ch.P. Ramacharyulu, *Peristaltic transport of a viscoelastic fluid (a long wavelength approximation)*

1169 V.V. Mityushev, *Generalized method of Schwarz and addition theorems in mechanics of materials containing cavities*

Brief Notes

1183 N. Kuznetsov, *Trapping of waves by horizontal cylinders in a channel containing two-layer fluid*

1189 S. Zahorski, *Comments on instability of disturbed elongational flows of viscoelastic fluids*

Polish Academy of Sciences

Institute of Fundamental Technological Research

Archives of Mechanics

Archiwum Mechaniki Stosowanej

volume 47 issue 6

Polish Scientific Publishers PWN

Warszawa 1995

ARCHIVES OF MECHANICS IS DEVOTED TO
Theory of elasticity and plasticity • Theory of nonclassical continua • Physics of continuous media • Mechanics of discrete media • Nonlinear mechanics • Rheology • Fluid gas-mechanics • Rarefied gas • Thermodynamics

---

Printed in Poland, Editorial Office: Świętokrzyska 21,
00-049 Warszawa (Poland)

Arkuszy wydawniczych 12,25. Arkuszy drukarskich 11,25
Papier offset. kl. III 70 g. B1. Oddano do składania w październiku 1995 r.
Druk ukończono w grudniu 1995 r.
Druk i oprawa: Drukarnia Braci Grodzickich, Żabieniec ul. Przelotowa 7

Arch. Mech., 47, 6, pp. 1015–1026, Warszawa 1995

Vorticity distributions for thick and thin viscous vortex pairs and rings

A. BEREZOVSKI and F. KAPLANSKI (TALLINN)

UNSTEADY VORTICITY distributions inside the viscous core of vortex pairs and rings are presented. All of them are solutions of the Stokes slow-flow equations, which are valid for the viscous core in the first approximation even in the case of slender vortices. It is explicitly shown that the condition of conservation for the impulse of vorticity is fulfilled for such distributions. The obtained solutions for thick viscous vortices are the direct generalizations of the classical self-similar distribution. Another form of vorticity distribution for slender vortices is identical both in the plane and in the axi-symmetric case.

1. Introduction

VORTICITY IS A NECESSARY element in almost every real flow. All the diversity of incompressible fluid flows is provided by the presence of different vorticity distributions. The description of vorticity evolution has fascinated researchers for over one hundred years. Nevertheless, only a few exact forms of vorticity distributions inside viscous vortex cores are known at present, encompassing both unsteady flows as well as steady solutions.

A physically realistic distribution of vorticity in an unbounded viscous fluid must satisfy the following requirements:

- It should be a solution of the Navier–Stokes equations.
- It should decay at infinity.
- The impulse of vorticity should be conserved in time (BATCHELOR [1]); that is

$$I = \frac{\varrho}{2} \int r \Omega \, dr \, dz = \text{const} \tag{1.1}$$

in the plane case, and

$$I = \varrho \pi \int r^2 \Omega \, dr \, dz = \text{const} \tag{1.2}$$

in the axisymmetric case, respectively. Here, ϱ is the fluid density, r is the radial and z is the axial coordinate in the cylindrical system of coordinates located at the center of vortex ring (longitudinal and transversal coordinates in the case of vortex pair), and Ω is the azimuthal (or out-of-plane) component of vorticity (see Fig. 1).

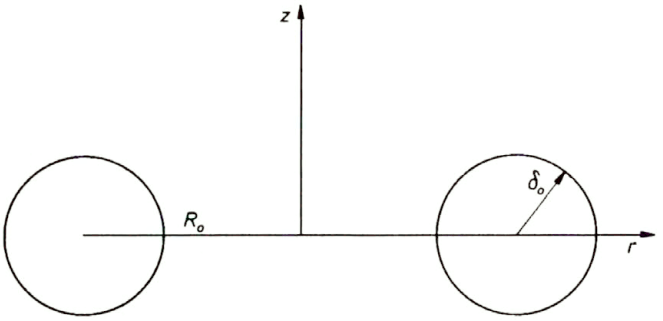


FIG. 1. Schematic drawing of the cross-section of a vortex pair or ring.

In the case of a vortex ring, dimensional analysis and the condition of conservation of the impulse of vorticity determine the self-similar vorticity distribution in the long-time limit,

$$\Omega(r,z,t) = \frac{M}{4\sqrt{2}\varrho\pi^{3/2}(2\nu t)^2}\left[\frac{r}{(2\nu t)^{1/2}}\exp\left(-\frac{r^2+z^2}{4\nu t}\right)\right], \tag{1.3}$$

where M is the initial value of the impulse of vorticity.

The analogue of this formula for a vortex pair in the plane can be expressed in the form

$$\Omega(r,z,t) = \frac{M}{\varrho\pi(2\nu t)^{3/2}}\left[\frac{r}{(2\nu t)^{1/2}}\exp\left(-\frac{r^2+z^2}{4\nu t}\right)\right]. \tag{1.4}$$

It is easy to see that dimensionless distributions (in square brackets) are the same in both cases, the difference in the geometries being contained in the different powers of time and constant factors in the denominators of the leading terms.

The distribution (1.3) appears for the first time in the paper by PHILLIPS [8] and remains the unique example of the solution of the Stokes slow-flow equations satisfying the conservation of the impulse of vorticity (see, for instance, TING and BAUER [11], ROTT and CANTWELL [9]), excluding the present work.

Attempts to generalize the self-similar vorticity distribution to a wider range of times by KAMBE and OSHIMA [6] for vortex rings and by CANTWELL and ROTT [5] for vortex pairs both take into account (at second order in a perturbation expansion) the effects of nonlinear vorticity convection, but lead to the non-uniform validity of solution.

However, it must be noted that the vorticity distribution (1.3) is not the sole solution of the Stokes equations conserving the impulse of vorticity. This can be easily shown for the plane case. It is well known that the superposition of two equal but counter-rotating Lamb–Oseen vortices satisfies the Stokes equations. The corresponding vorticity distribution can be represented in the following

form LAMB [7]:

$$\Omega(r,z,t) = \frac{M}{4\pi\varrho\nu t R_0}\left[\exp\left(-\frac{z^2+(r-R_0)^2}{4\nu t}\right) - \exp\left(-\frac{z^2+(r+R_0)^2}{4\nu t}\right)\right] = \frac{M}{2\pi\varrho\nu t R_0}\exp\left(-\frac{z^2+r^2+R_0^2}{4\nu t}\right)\sinh\left(\frac{rR_0}{2\nu t}\right), \tag{1.5}$$

where $2R_0$ is the initial distance between the vortices (Fig. 1).

A direct calculation shows that the conservation of the impulse of vorticity takes place in this case. The relation for the impulse of vorticity (1.1) can be rewritten in the following way, using (1.5):

$$I = \frac{\varrho}{2}\int_{-\infty}^{\infty} r\Omega\, dr\, dz = \frac{M}{2\pi\varrho R_0(2\nu t)^{1/2}}\int_{-\infty}^{\infty}\exp\left(-\frac{r^2+R_0^2}{4\nu t}\right)\sinh\left(\frac{rR_0}{2\nu t}\right) r\, dr \int_{-\infty}^{\infty}\exp\left(-\frac{\zeta^2}{2}\right) d\zeta, \tag{1.6}$$

where $\zeta = z/\sqrt{2\nu t}$. Afer integration with respect to ζ we obtain the expression

$$I = \frac{M}{2(2\pi)^{1/2}\eta}\int_{-\infty}^{\infty}\left[\exp\left(-\frac{(\sigma-\eta)^2}{2}\right) - \exp\left(-\frac{(\sigma+\eta)^2}{2}\right)\right]\sigma\, d\sigma, \tag{1.7}$$

where $\sigma = r/\sqrt{2\nu t}$ and $\eta = R_0/\sqrt{2\nu t}$. The substitutions $\sigma - \eta = \tau_1$, $\sigma + \eta = \tau_2$ transform the last integral to the difference between very similar integrals

$$I = \frac{M}{2(2\pi)^{1/2}\eta}\left[\int_{-\infty}^{\infty}(\tau_1+\eta)\exp\left(-\frac{\tau_1^2}{2}\right)d\tau_1 - \int_{-\infty}^{\infty}(\tau_2-\eta)\exp\left(-\frac{\tau_2^2}{2}\right)d\tau_2\right], \tag{1.8}$$

which can be combined to obtain the well known Gaussian integral, which is independent of time:

$$I = \frac{M}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\exp\left(-\frac{\tau^2}{2}\right)d\tau = M = \text{const}. \tag{1.9}$$

Thus, the superposition of two counter-rotating Lamb – Oseen vortices satisfies both the Stokes equations and the condition of conservation for the impulse of vorticity. The similarity solution (1.4) is the limiting case of solution (1.5) for large times. In fact, since for large times $\sinh(rR_0/2\nu t) \sim rR_0/2\nu t$, we have

$$\frac{M}{2\pi\varrho\nu t R_0}\exp\left(-\frac{z^2+r^2+R_0^2}{4\nu t}\right)\sinh\left(\frac{rR_0}{2\nu t}\right) \sim \frac{M}{\pi\varrho(2\nu t)^{3/2}}\left[\frac{r}{(2\nu t)^{1/2}}\exp\left(-\frac{r^2+z^2}{4\nu t}\right)\right]. \tag{1.10}$$

The generalization of the solution (1.5) to the axisymmetric case is

$$\Omega(r,z,t) = \frac{M}{\varrho(2\pi\nu t)^{3/2}R_0}\exp\left(-\frac{r^2+z^2+R_0^2}{4\nu t}\right) I_1\left(\frac{rR_0}{2\nu t}\right), \tag{1.11}$$

where I_1 is the modified Bessel function of the first order. The derivation of this solution can be found in the paper by BEREZOVSKI and KAPLANSKI [2].

The presence of additional solutions of the Stokes equations which conserve the impulse of vorticity allows one to supplement the self-similar vorticity distributions classification given by CANTWELL [4]. The aim of this paper is to present such new vorticity distributions which are valid after the initial formation of vortices has finished and before the vortex pairs or rings have finally decayed. This corresponds to entrainment phase of the development of laminar vortices in terminology by SHARIFF and LEONARD [10].

In the Sec. 2 of the paper, thick viscous vortex pairs are considered in detail. The equation of motion is reduced to a linear diffusion-like equation for the vorticity in the case of small Reynolds numbers. The obtained distribution of vorticity and circulation are direct generalizations of self-similar expressions. The analogous results for thick vortex rings are briefly presented.

In the Sec. 3 we describe slender vortices. The flow field is separated into two distinct regions: the viscous vortex core and the inviscid outer flow, and the problem is treated via the method of matched asymptotic expansions. A linear diffusion-like equation for the vorticity is obtained for the vortex core in the first approximation. The vorticity distributions for slender vortices are shown to be the same both for vortex pairs and rings. Conservation of the impulse of vorticity leads here to a time-independent circulation and distance between vortices (or ring diameter).

In the Sec. 4 we present our conclusions.

2. Viscous vortices

2.1. Viscous vortex pairs

For simplicity, we deal first with vortex pairs. The two-dimensional Navier–Stokes equations in vorticity-stream function form are

$$\begin{aligned} &\frac{\partial\Omega}{\partial t} + \frac{\partial}{\partial r}\left(-\Omega\frac{\partial\Psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\Omega\frac{\partial\Psi}{\partial r}\right) = \nu\left(\frac{\partial^2\Omega}{\partial r^2} + \frac{\partial^2\Omega}{\partial z^2}\right), \\ &\frac{\partial^2\Psi}{\partial r^2} + \frac{\partial^2\Psi}{\partial z^2} = -\Omega, \end{aligned} \tag{2.1}$$

where Ω is the out-of-plane component of vorticity vector and Ψ is the stream function, related to Cartesian velocity components u and v in the r and z direc-

tions, respectively, by

$$u = -\frac{\partial \Psi}{\partial z}, \qquad v = \frac{\partial \Psi}{\partial r}. \tag{2.2}$$

The vorticity field is antisymmetric about the z-axis in this case

$$\Omega(-r, z, t) = -\Omega(r, z, t). \tag{2.3}$$

The boundary conditions include no flow through symmetry plane

$$\Psi(0, z, t) = \Omega(0, z, t) = 0. \tag{2.4}$$

The far-field conditions must satisfy the assumption of the decay at infinity

$$\Omega \to 0, \qquad \Psi \to 0, \quad \text{as} \quad r^2 + z^2 \to \infty. \tag{2.5}$$

For initial conditions, we specify an initial radius of each vortex core δ_0 and an initial distance between the centers of the vortices $2R_0$ at time t_0 (Fig. 1):

$$R(t_0) = R_0, \qquad \delta(t_0) = \delta_0. \tag{2.6}$$

The initial vorticity distribution inside the core remains indetermined, because we seek a self-similar solution of the problem.

A non-trivial solution of the problem is provided by the conservation of a non-zero value of the impulse of vorticity I in time. A prescribed initial value M of the impulse of vorticity determines the self-similar solution completely.

It must be noted that there exist two different length scales in the problem. One of them is the viscous diffusion scale $\delta(t) = (2\nu)^{1/2}$, and another is the distance between vortices $2R(t)$. For viscous vortices at sufficiently large times, the condition $\delta/R(t) \gg 1$ is fulfilled. The initial distance between vortices becomes, therefore, an insignificant parameter, and its influence may be neglected. This is the common way to obtain the self-similar solution (CANTWELL and ROTT [5]). Nevertheless, we take into account the influence of the initial distance as well.

Passing to a laboratory frame of reference, we introduce dimensionless variables:

$$\sigma = \frac{r}{\delta}, \qquad \zeta = \frac{z - Z(t)}{\delta}, \qquad \eta = \frac{R_0}{\delta}, \tag{2.7}$$

where $Z(t)$ is a current axial position of the vortex pair, $dZ/dt = V(t)$, and $V(t)$ is the propagation velocity of the vortex pair. It must be noted, that the relations connecting the stream function and velocity components change in the following way

$$u = -\frac{\partial \Psi}{\partial z}, \qquad v = \frac{\partial \Psi}{\partial r} + V(t). \tag{2.8}$$

Introducing dimensionless vorticity and stream function to be determined:

$$\omega(\sigma,\zeta,\eta)=\frac{\Omega(r,z,t)}{\Omega_0(t)},\qquad \psi(\sigma,\zeta,\eta)=\frac{\Psi(r,z,t)}{\Omega_0(t)\delta^2}, \tag{2.9}$$

we can rewrite the equation of motion $(2.1)_1$ in the dimensionless form

$$-2\alpha\omega-\left(\sigma\frac{\partial\omega}{\partial\sigma}+\zeta\frac{\partial\omega}{\partial\zeta}+\eta\frac{\partial\omega}{\partial\eta}\right)-\mathrm{Re}\left[\frac{\partial\psi}{\partial\zeta}\frac{\partial\omega}{\partial\sigma}-\frac{\partial\psi}{\partial\sigma}\frac{\partial\omega}{\partial\zeta}\right]=\frac{\partial^2\omega}{\partial\sigma^2}+\frac{\partial^2\omega}{\partial\zeta^2}, \tag{2.10}$$

where $\mathrm{Re}=\Omega_0\delta^2/\nu$, and it is assumed that the scaling function for the vorticity has a power-law decay with time, i.e. $\Omega_0(t)\sim t^{-\alpha}$.

For small values of Reynolds number, which are characteristic of viscosity-dominated flows, we seek a solution in the form of an asymptotic expansion in powers of the Reynolds number,

$$\omega(\sigma,\zeta,\eta;\mathrm{Re})=\omega_1(\sigma,\zeta,\eta)+\mathrm{Re}\,\omega_2(\sigma,\zeta,\eta)+\dots. \tag{2.11}$$

At lowest order, we obtain a linear equation for the vorticity

$$-2\alpha\omega_1-\left(\sigma\frac{\partial\omega_1}{\partial\sigma}+\zeta\frac{\partial\omega_1}{\partial\zeta}+\eta\frac{\partial\omega_1}{\partial\eta}\right)=\left[\frac{\partial^2\omega_1}{\partial\sigma^2}+\frac{\partial^2\omega_1}{\partial\zeta^2}\right]. \tag{2.12}$$

The solution of Eq. (2.12) which conserves the impulse of vorticity and decays at infinity has the form

$$\omega_1(\sigma,\zeta,\eta)=\exp\left(-\frac{\zeta^2+\sigma^2+\eta^2}{2}\right)\sinh(\sigma\eta), \tag{2.13}$$

and corresponds to a time-decay exponent $\alpha=1$. In dimensional form, this solution is just the same as that for the superposition of two counter-rotating Lamb - Oseen vortices (1.5)

$$\Omega_1(r,z,t)=\frac{M}{2\varrho\pi\nu tR_0}\exp\left(-\frac{z^2+r^2+R_0^2}{4\nu t}\right)\sinh\left(\frac{rR_0}{2\nu t}\right). \tag{2.14}$$

One of the main features concerns the circulation of the vortices. Since the vorticity field in the plane case is antisymmetrical, vortices have identical values of circulation, which are of opposite sign. Therefore, it is sufficient to determine circulation only for one half-plane. Direct integration, using the expression (2.14), gives

$$\Gamma_1(t)=\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{M}{2\varrho\pi\nu tR_0}\exp\left(-\frac{z^2+r^2+R_0^2}{4\nu t}\right)\sinh\left(\frac{rR_0}{2\nu t}\right)dz\,dr$$
$$=\frac{M}{\varrho R_0}\mathrm{erf}\left(\frac{R_0}{\sqrt{4\nu t}}\right). \tag{2.15}$$

This means that circulation decreases with time, but, unlike the vorticity, its time-dependence is not of a power-law type.

Plots of dimensionless form for the obtained vorticity distribution

$$\omega_1(\sigma,\eta,\zeta)=\frac{1}{2\eta}\left[\exp\left(-\frac{\zeta^2+(\sigma-\eta)^2}{2}\right)-\exp\left(-\frac{\zeta^2+(\sigma+\eta)^2}{2}\right)\right], \tag{2.16}$$

are shown in the Fig. 2 for different time values. In addition, the dimensionless self-similar distribution

$$\omega(\sigma,\zeta)=\sigma\exp\left(-\frac{\zeta^2+\sigma^2}{2}\right) \tag{2.17}$$

is shown. The smaller value of η, the longer time corresponds to it. As expected, the obtained solution has approached the self-similar distribution with time.

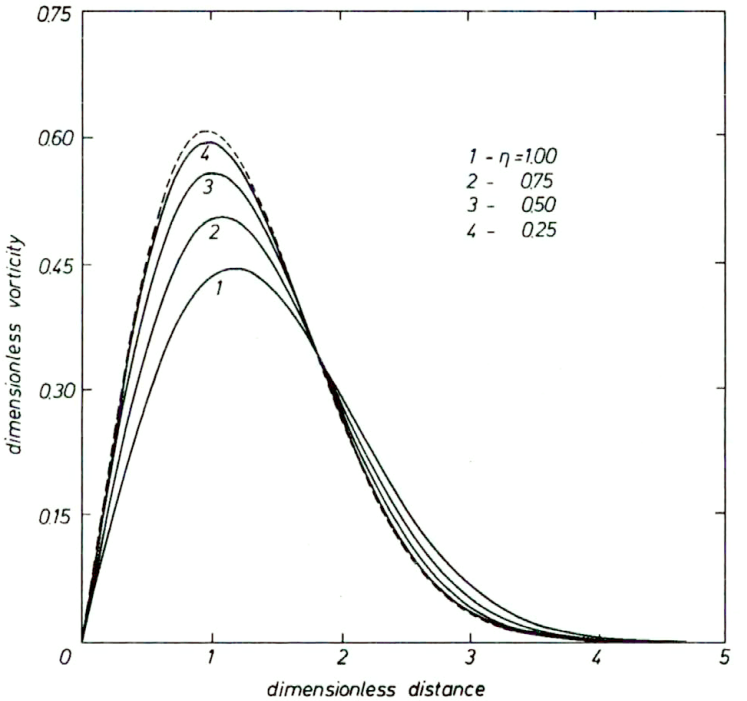


FIG. 2. Variation of vorticity distribution with time for a viscous vortex pair. Dashed curve corresponds to the self-similar solution.

2.2. Viscous vortex rings

The linearized equation of vorticity diffusion at lowest order in Re in this case is

$$-2\alpha\omega-\left(\sigma\frac{\partial\omega}{\partial\sigma}+\zeta\frac{\partial\omega}{\partial\zeta}+\eta\frac{\partial\omega}{\partial\eta}\right)=\frac{\partial^2\omega}{\partial\sigma^2}+\frac{\partial^2\omega}{\partial\zeta^2}+\frac{1}{\sigma}\frac{\partial\omega}{\partial\sigma}-\frac{\omega}{\sigma^2}. \tag{2.18}$$

The vorticity distribution for the vortex ring with conservation of the impulse of vorticity has the form (BEREZOVSKI and KAPLANSKI [2])

$$\Omega_1(r,z,t) = \frac{M}{\varrho(2\pi\nu t)^{3/2}R_0}\exp\left(-\frac{\sigma^2+\zeta^2+\eta^2}{4\nu t}\right)I_1(\sigma\eta), \tag{2.19}$$

where I_1 is the modified Bessel function of the first order. Such a solution corresponds to a time-decay exponent $\alpha = 3/2$. The circulation in the case of the vortex ring is again found by direct integration, using the vorticity distribution (2.19)

$$\Gamma_1(t) = \int_{\infty}^{\infty}\int_{0}^{\infty}\frac{M}{\varrho(2\pi\nu t)^{3/2}R_0}\exp\left(-\frac{\sigma^2+\zeta^2+\eta^2}{4\nu t}\right)I_1(\sigma\eta)\,dr\,dz \tag{2.20}$$

$$= \frac{M}{\varrho\pi R_0^2}\left(1-e^{-\eta^2/2}\right).$$

In the long-time limit ($\eta \to 0$) the vorticity distribution (2.19) approaches asymptotically the self-similar form (1.3), the circulation (2.20) tends to the self-similar form as well.

3. Slender vortices

In the case of slender vortices, the flow-field is divided into two regions: the outer potential flow and the viscous vortex core. Each of these regions is characterized by its own length scale. The viscous diffusion scale $\delta = \sqrt{2\nu(t-t_0)}$ remains valid for the vortex core, whereas the outer flow has the length scale compared with the distance between vortices $R(t)$. Slenderness means that the length scale ratio is a small parameter.

$$\varepsilon = \frac{\delta}{R} \ll 1. \tag{3.1}$$

In the context of a matched asymptotic expansion method, we introduce two kinds of dimensionless variables:

- outer variables

$$\sigma = \frac{r}{R}, \qquad \zeta = \frac{z-Z}{R}, \qquad \eta = \frac{R_0}{R}, \tag{3.2}$$

- inner variables

$$\bar{\sigma} = \frac{r-R}{\delta}, \qquad \bar{\zeta} = \frac{z-Z}{\delta}, \qquad \bar{\eta} = \frac{\delta_0}{\delta}, \tag{3.3}$$

where R, Z are the coordinates of the (dimensional) location of the center of a vortex core.

We seek a solution of the Navier-Stokes equations (2.1) in the form of the asymptotic expansions, each of which depends on its own kind of variables:

- outer expansion

$$\begin{aligned} \frac{\Omega(r,z,t)}{\Omega_0(t)} &= \omega_1(\sigma,\zeta,\eta) + \tau^{1/2}\omega_2(\sigma,\zeta,\eta) + O(\tau^1), \\ \frac{\Psi(r,z,t)}{\Omega_0(t)R^2(t)} &= \psi_1(\sigma,\zeta,\eta) + \tau^{1/2}\psi_2(\sigma,\zeta,\eta) + O(\tau^1), \end{aligned} \tag{3.4}$$

where Ω_0 is a scaling function for the vorticity and $\tau = (t - t_0)$. This expansion is valid as $\tau \to 0$ for fixed σ, ζ, η.

- inner expansion

$$\begin{aligned} \Omega(r,z,t) &= \frac{1}{\tau}\overline{\omega}_1\left(\overline{\sigma},\overline{\zeta},\overline{\eta}\right) + \frac{1}{\tau^{1/2}}\overline{\omega}_2\left(\overline{\sigma},\overline{\zeta},\overline{\eta}\right) + O(\tau^0), \\ \frac{\Psi(r,z,t)}{\nu} &= \overline{\psi}_1\left(\overline{\sigma},\overline{\zeta},\overline{\eta}\right) + \tau^{1/2}\overline{\psi}_2\left(\overline{\sigma},\overline{\zeta},\overline{\eta}\right) + O(\tau^1), \end{aligned} \tag{3.5}$$

which is valid as $\tau \to 0$ for fixed $\overline{\sigma}$, $\overline{\zeta}$, $\overline{\eta}$.

Substituting these expansions into the equation of motion $(2.1)_1$, we obtain in the lowest order for the outer problem

$$\begin{aligned} \frac{\omega_1}{\Omega_0^2}\frac{d\Omega_0}{dt} - \frac{\dot{R}}{R\Omega_0}\left(\sigma\frac{\partial\omega_1}{\partial\sigma} + \zeta\frac{\partial\omega_1}{\partial\zeta} + \eta\frac{\partial\omega_1}{\partial\eta}\right) - \frac{\dot{Z}}{R\Omega_0}\frac{\partial\omega_1}{\partial\zeta} \\ - \frac{\partial\psi_1}{\partial\zeta}\frac{\partial\omega_1}{\partial\sigma} + \frac{\partial\psi_1}{\partial\sigma}\frac{\partial\omega_1}{\partial\zeta} = \frac{1}{Re}\left[\frac{\partial^2\omega_1}{\partial\sigma^2} + \frac{\partial^2\omega_1}{\partial\zeta^2}\right]. \end{aligned} \tag{3.6}$$

For the outer flow, viscous terms are negligible because the Reynolds number $Re = \Omega_0 R^2/\nu$ cannot be assumed small as it was for thick vortices. Only trivial solution satisfies zero boundary conditions at infinity in this case, i.e.

$$\omega_1(\sigma,\zeta,\eta) \equiv 0. \tag{3.7}$$

For the inner problem, we obtain in the lowest order the following equation

$$\begin{aligned} -2\overline{\omega}_1 - \left(\overline{\sigma}\frac{\partial\overline{\omega}_1}{\partial\overline{\sigma}} + \overline{\zeta}\frac{\partial\overline{\omega}_1}{\partial\overline{\zeta}} + \overline{\eta}\frac{\partial\overline{\omega}_1}{\partial\overline{\eta}}\right) - \frac{\partial\overline{\psi}_1}{\partial\overline{\zeta}}\frac{\partial\overline{\omega}_1}{\partial\overline{\sigma}} + \frac{\partial\overline{\psi}_1}{\partial\overline{\sigma}}\frac{\partial\overline{\omega}_1}{\partial\overline{\zeta}} \\ = \frac{\partial^2\overline{\omega}_1}{\partial\overline{\sigma}^2} + \frac{\partial^2\overline{\omega}_1}{\partial\overline{\zeta}^2}. \end{aligned} \tag{3.8}$$

We make the common assumption that the vorticity distribution in the viscous core is circular in the lowest order. Introducing the polar system of coordinates ξ, θ:

$$\overline{\sigma} = \xi\cos\theta, \qquad \overline{\zeta} = \xi\sin\theta, \tag{3.9}$$

and representing the vorticity in the inner problem in the form

$$\overline{\omega}_1\left(\overline{\sigma}, \overline{\zeta}, \overline{\eta}\right) = \overline{\omega}_1(\xi, \overline{\eta}), \tag{3.10}$$

we obtain a linear equation for the vorticity in this approximation

$$-2\alpha\overline{\omega}_1 - \xi\frac{\partial\overline{\omega}_1}{\partial\xi} - \overline{\eta}\frac{\partial\overline{\omega}_1}{\partial\overline{\eta}} = \frac{\partial^2\overline{\omega}_1}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\overline{\omega}_1}{\partial\xi}. \tag{3.11}$$

Thus, at the lowest order, the governing equation for the vorticity in the viscous vortex core of a slender vortex pair is a linear diffusion-like equation similar to that which we obtained for the viscosity-dominated case.

It is easy to see that the solution of Eq. (3.11) decaying at infinity, is the following

$$\overline{\omega}_1(\xi, \overline{\eta}) = \exp\left(-\frac{\xi^2 + \overline{\eta}^2}{2}\right) I_0(\xi\overline{\eta}), \tag{3.12}$$

where I_0 is the modified Bessel function of zeroth order. Such a solution corresponds to the exponent $\alpha = 1$. This distribution tends to that of the Lamb–Oseen vortex

$$\overline{\omega}(\zeta) = \exp\left(-\frac{\xi^2}{2}\right) \tag{3.13}$$

only in the long-time limit ($\overline{\eta} \to 0$).

Returning to dimensional variables, we have

$$\Omega_1 = \frac{1}{t - t_0}\exp\left(-\frac{(r - R)^2 + (z - Z)^2 + \delta_0^2}{4\nu(t - t_0)}\right) \times I_0\left(\frac{\delta_0\sqrt{(r - R)^2 + (z - Z)^2}}{2\nu(t - t_0)}\right). \tag{3.14}$$

The circulation is

$$\Gamma_1(t) = \int_{-\infty}^{\infty}\int_{0}^{\infty} \Omega_1\, dz\, dr = \frac{\delta^2}{t - t_0}\int_0^{2\pi}\int_0^{\infty}\exp\left(-\frac{\xi^2 + \overline{\eta}^2}{2}\right) I_0(\xi\overline{\eta})\xi\, d\xi\, d\theta = 4\pi\nu. \tag{3.15}$$

The value of the impulse of vorticity is expressed through the circulation of the vortices Γ and the distance between them by the relation

$$M = \varrho R \Gamma. \tag{3.16}$$

We see that conservation of the impulse of vorticity requires the invariance in time for the circulation and the distance between the vortices in this case:

$$\Gamma_1(t) = \frac{M}{\varrho R_0} = \text{const}, \qquad R(t) = R_0 = \text{const}. \tag{3.17}$$

Plots of the dimensionless vorticity distribution inside the vortex core are shown in the Fig. 3 for different time values. For the comparison, the Lamb–Oseen distribution is also presented.

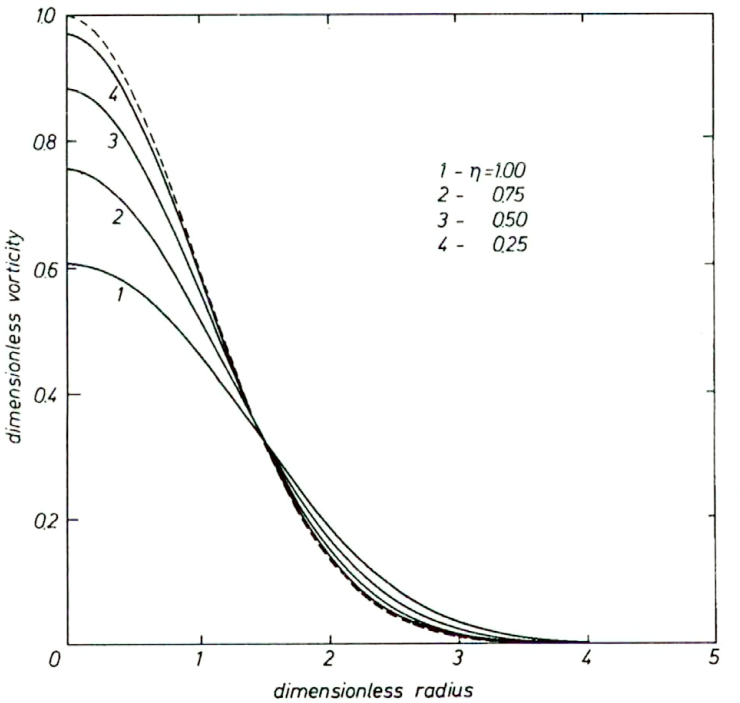


FIG. 3. Vorticity distribution inside the core of a slender vortex. Dashed curve corresponds to the Lamb–Oseen vortex.

A similar procedure for a vortex ring (BEREZOVSKI and KAPLANSKI [3]) gives, in the first approximation, just the same form of vorticity distribution (3.12). The outer radius of the ring, as well as its circulation, are forced to be invariant:

$$R(t) = R_0 = \text{const}, \qquad \Gamma(t) = \frac{M}{\pi \varrho R_0^2} = \text{const}. \tag{3.18}$$

4. Conclusion

We have demonstrated that there exist additional vorticity distributions for viscosity-dominated vortex rings and pairs which satisfy conservation of the impulse of vorticity, besides the classical self-similar one. All of distributions are the lowest-order approximations in a perturbation expansion of the solutions of the full Navier-Stokes equations, but their validity is not restricted to the long-time limit only.

Another kind of vorticity distribution is valid in the core of slender vortices. Its form is the same both in the plane and axisymmetric (ring) case. Conservation of the impulse of vorticity makes both the circulation and the distance between vortices (radius of the ring) constant in time.

References

1. G.K. BATCHELOR, *An introduction to fluid dynamics*, Cambridge University Press, Cambridge 1967.
2. A.A. BEREZOVSKI and F.B. KAPLANSKI, *Diffusion of ring vortex*, Izv. Akad. Nauk SSSR Mekh. Zhidk. Gaza, No 6, 10–15, 1987 (English translation: Fluid Dynamics (USA), **22**, 832–837, 1987.)
3. A.A. BEREZOVSKI and F.B. KAPLANSKI, *Dynamics of thin vortex rings in a low-viscosity fluid*, Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No 5, 52–59, 1992 (English translation: Fluid Dynamics (USA), **27**, 643–649, 1992.)
4. B.J. CANTWELL, *Viscous starting jets*, J. Fluid Mech., **173**, 159–189, 1986.
5. B. CANTWELL and N. ROTT, *The decay of a viscous vortex pair*, Phys. Fluids, **31**, 3213–3224, 1988.
6. T. KAMBE and Y. OSHIMA, *Generation and decay of viscous vortex rings*, J. Phys. Soc. Japan, **38**, 271–280, 1975.
7. H. LAMB, *Hydrodynamics*, Cambridge University Press, Cambridge 1932.
8. O.M. PHILLIPS, *The final period of decay of non-homogeneous turbulence*, Proc. Cambridge Phil. Soc., **52**, 135–151, 1956.
9. N. ROTT and B. CANTWELL, *Vortex drift. I. Dynamic interpretation*, Phys. Fluids A, **5**, 1443–1450, 1993.
10. K. SHARIFF and A. LEONARD, *Vortex rings*, Annual Rev. Fluid Mech., **24**, 235–279, 1992.
11. L. TING and F. BAUER, *Viscous vortices in two- and three-dimensional space*, Computers Fluids, **22**, 565–588, 1993.

DEPARTMENT OF MECHANICS AND APPLIED MATHEMATICS
INSTITUTE OF CYBERNETICS
and
DEPARTMENT OF AEROMECHANICS
INSTITUTE OF ENERGY RESEARCH, TALLINN, ESTONIA.

Received October 3, 1994.

Arch. Mech., 47, 6, pp. 1027–1041, Warszawa 1995

Stability of Couette-Poiseuille flow of fiber suspensions

B. NSOM (GRENOBLE)

WE STUDY the linear stability of the flow of fiber suspensions between concentric rotating cylinders with an axial pressure gradient. In the dilute concentration range, the critical wave number, Taylor number and wave velocity depend on the cylinders radii ratio, the axial Reynolds number and the fibers aspect ratio. In the semi-concentrated range, the critical parameters depend on the cylinders radii ratio, the axial Reynolds number and the fiber concentration and density.

Notations

Latin letters

- a reduced wavenumber based on the gap-width, Eq. (3.10),
- A, B constants of integration, Eq. (3.8),
- c total voluminal concentration of the fibers,
- b gap-width $(R_2 - R_1)$,
- $\mathbf{d}$ unit-vector along the fiber length,
- E constant, Eq. (1.3),
- $F(x)$ dimensionless axial velocity distribution in the annulus, Eq. (3.12),
- $G(x)$ dimensionless angular velocity distribution in the annulus, Eq. (3.12),
- k wavenumber, Eq. $(3.1)_{1-3}$,
- L fiber length,
- n number of fibers per unit volume,
- p perturbation pressure field, Eq. $(3.1)_4$,
- P pressure field in the basic Couette–Poiseuille flow, Eq. $(2.7)_3$,
- P_0 integration constant, Eq. $(2.7)_3$,
- r radial coordinate in the usual cylindrical coordinate system,
- R_0 mid-gap radius,
- R_1, R_2 inner and outer cylinders radii,
- Re axial Reynolds number, Eq. $(3.8)_1$,
- $\mathbf{v}$ perturbation velocity field in the usual cylindrical coordinate system with components (u, v, w), Eq. $(3.1)_{1-3}$,
- $\mathbf{V}$ velocity field of the basic Couette–Poiseuille flow in the usual cylindrical coordinate system with components $(0, V, W)$, Eqs. (2.1), $(2.7)_{1,2}$,
- T Taylor number based on the gap-width, Eq. (3.15),
- $W(x)$ axial velocity distribution in the annulus,
- W_{av} average axial velocity, Eq. $(3.8)_2$,
- x dimensionless radial coordinate,
- z axial coordinate.

Greek letters

β disturbance frequency, Eq. (3.15),
δ dimensionless gap-width, Eq. (3.10),
Δ fiber density (nL^3),
$\dot{\varepsilon}$ rate of strain tensor in the basic Couette-Poiseuille flow,
$(\dot{\varepsilon})'$ perturbation rate of strain tensor,
ζ fiber aspect ratio (L/D),
η_0 dynamic viscosity,
η_2 suspension rheological parameter, Eq. (1.1),
θ tangential coordinate,
κ cylinders radii ratio (R_1/R_2),
ν kinematic viscosity,
ϱ fluid density,
$\dot{\sigma}$ stress tensor in the basic Couette-Poiseuille flow,
σ' perturbation stress tensor,
Σ perturbation frequency,
ϕ fiber diameter,
ω wave velocity relative to W_{av}, Eq. (4.8),
$\Omega(x)$ angular velocity distribution in the annulus,
Ω_1 inner cylinder angular velocity.

Subscripts

c critical value in the suspension,
DP from the literature,
N from this paper,
0 critical value in water.

1. Background

THE COUETTE-TAYLOR flow remains a classical teaching tool of the Fluid Dynamics Stability topics. The arrangement obtained by superimposing to it a pressure gradient in the axial direction (Couette-Poiseuille flow) has many industrial applications such as oil and gas exploration. Since the first experimental study (CORNISH, [4]) and theoretical study (GOLDSTEIN, [10]), a large number of papers have been devoted to that flow configuration. CHANDRASEKHAR [3] and DIPRIMA [5] included an axial flow in the annulus to extend Taylor's analysis of the stability of circular Couette flow. They considered the case of axisymmetric disturbances in a narrow gap. The arbitrary gap situation was studied later by HASOON and MARTIN [11] and by DIPRIMA and PRIDOR [6]. The resulting flow consists of toroidal Taylor vortices translating with the axial flow. In fact, when the instability is relaxed to allow asymmetric disturbances and for arbitrary (axial) Reynolds numbers, LUEPTOW *et al.* [15] pointed out flow regimes known to exist for circular Couette flow with no axial flow which were identified by FENSTERMACHER [9]. Furthermore, they identified flow regimes which were not previously described.

So, they could furnish a map of the flow regimes in the Taylor number – Reynolds number plane, completing thus the results provided earlier by KATAOKA *et al.* [13] and BÜHLER and POLIFKE [2]. From LUEPTOW *et al.* [15], at low angular speeds, a laminar circular Couette flow superimposed on a laminar axial Poiseuille flow exists in the annulus. From the critical value of that angular speed, the laminar Taylor vortices move with the axial flow, being unaffected by the axial flow for low axial Reynolds numbers. At higher Reynolds numbers, the vortices are no longer individual toroids, but form a pair of helical vortices which translate with the axial flow.

In a recent paper (NSOM, [6]), we considered the stability of the Couette flow of suspensions of macroscopic fibers in a Newtonian liquid. Many industrial processes deal with these fluids (paper making, water treatment ...). In the present work, we consider the superposition of that Couette flow of fiber suspensions with a constant axial pressure gradient when the Reynolds number is low. So, from a linear stability analysis, we characterize the occurence of laminar Taylor vortices in the basic Couette - Poiseuille flow of a fiber suspension. The fibers considered are rigid rods with a large aspect ratio ζ ($\zeta = L/\phi$, L being the fiber length and ϕ their diameter). Let c denote the fibers volumetric concentration and Δ their number density ($\Delta = nL^3$), n being the number of fibers per unit volume. From a rheological point of view, the suspension is said to be dilute if each solid particle is unaffected by the presence of the others. This situation occurs if c is less than $(\phi/L)^2$ and Δ is much less than unity. It is semi-concentrated when c lies between $(\phi/L)^2$ and (ϕ/L), (DOI and EDWARDS, [7] and [8]). In the dilute range, the suspension obeys the Ericksen anisotropic fluid equation of state AUSIAS *et al.*, [1]

$$\sigma = -p\delta + 2\eta_0 \dot{\varepsilon} + \eta_2 \dot{\varepsilon}\,\mathbf{d}, \tag{1.1}$$

where σ is the stress tensor, p is the isotropic pressure field, $\dot{\varepsilon}$ is the rate of strain tensor, $\mathbf{d}$ is a unit vector along the direction of a fiber length; η_0 is the dynamic viscosity of the suspending liquid and η_2 is a rheological coefficient given by

$$\eta_2 = \frac{(\zeta)^2 \eta_0}{\ln(\zeta)}. \tag{1.2}$$

In the semi-concentrated range, the medium is the seat of the "fiber-fiber" and "fiber-boundaries" interactions. SHAQFEH and FREDRICKSON [18] summed these interactions. They established the two following results:

- The "fiber-boundaries" interactions are negligible with respect to the "fiber-fiber" interactions;
- Introducing the "fiber-fiber" interactions in the dynamics of the medium, its constitutive equation preserves the form (1.1), but the rheological coefficient η_2 is given by the following expression

$$\eta_2 = \frac{\pi \Delta \eta_0}{3\,[\ln(1/c) + \ln(\ln(1/c)) + E]}. \tag{1.3}$$

In this relation E is a constant with value -0.66 for suspension in which all orientations are equiprobable. When all the particles are aligned in a common direction, E takes the value 0.16. The latter situation which is a usual closure hypothesis in the study of fiber suspension flow will be assumed (PAPANASTASIOU and ALEXANDRIOU, [17]; STOVER *et al.*, [19]). Thus the dilute and semi-concentrated suspensions can be treated in a similar way since they are described by rheological equations of the same form. The details of the two cases will appear in the numerical results, in the respective ranges of the concentration.

In the second section of this paper, we calculate the basic Couette - Poiseuille flow.

The third section states the equations governing the occurence of laminar Taylor vortices in this flow. Then we solve them in the fourth section. Throughout the whole paper, the axial Reynolds number is low as we said previously.

2. Basic flow

Consider a system of two concentric cylinders, the outer one with radius R_2 being fixed and the inner one with radius R_1 and angular speed Ω_1. κ denotes the ratio R_1/R_2 and b the gap-width i.e. $R_2 - R_1$. When Ω_1 is low, a circular Couette flow superimposed over a laminar Poiseuille flow exists. In a cylindrical system of coordinates with basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$, z lying along the common axis of the cylinders, it is described by a velocity-field of the form

$$\mathbf{V} = V(r)\mathbf{e}_\theta + W(r)\mathbf{e}_z\,, \tag{2.1}$$

and the orientation-state is given by

$$\mathbf{d} = \frac{V}{\sqrt{(V^2 + W^2)}}\mathbf{e}_\theta + \frac{W}{\sqrt{(V^2 + W^2)}}\mathbf{e}_z\,, \tag{2.2}$$

while the components of the rate of strain tensor (which is symmetric) are

$$\dot{\varepsilon}_{r\theta} = \frac{1}{2}r\frac{d}{dr}\left(\frac{V}{r}\right), \qquad \dot{\varepsilon}_{rz} = \frac{1}{2}\frac{dW}{dr}\,, \tag{2.3}$$

and zero otherwise. Then using the constitutive equation (1.1), the components of the stress tensor (also symmetric) are given by

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = -P(r), \tag{2.4}$$

$$\sigma_{r\theta} = \eta_0 r\frac{d}{dr}\left(\frac{V}{r}\right), \qquad \sigma_{rz} = \eta_0 r\frac{dW}{dr}\,, \qquad \sigma_{rz} = 0. \tag{2.5}$$

The continuity equation is identically satisfied, while the equations of motion reduce to

$$\begin{aligned} \frac{\partial}{\partial r}P(r) &= \varrho\frac{V^2}{r}, \\ \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rV)\right] &= 0, \\ \frac{1}{r}\frac{d}{dr}\left(r\frac{dW}{dr}\right) &= \frac{1}{\eta_0}\left(\frac{\partial P}{\partial z}\right)_0, \end{aligned} \tag{2.6}$$

where $\left(\frac{\partial P}{\partial z}\right)_0$ is the axial pressure gradient.

Substituting Eqs. (2.5) for the stress tensor components, a straightforward calculation leads to

$$\begin{aligned} V(r) &= Ar + B/r, \\ W(r) &= -\frac{1}{4\varrho\nu}\frac{\partial P}{\partial z}\left[R_1^2 - r^2 + (R_2^2 - R_1^2)\frac{\ln(r/R_1)}{\ln(R_2/R_1)}\right], \\ P(r) &= \varrho\left[\frac{A^2r^2}{2} - 2\frac{2B^2}{r^2} + AB\ln(r)\right] + \left(\frac{\partial P}{\partial z}\right)_0 z + P_0, \end{aligned} \tag{2.7}$$

where A and B, the integration constants are determined by the no-slip condition of the fluid on the cylinder walls. Their expressions are the following

$$A = -\frac{\Omega_1\kappa^2}{1-\kappa^2}, \qquad B = \frac{\Omega_1 R_1^2}{1-\kappa^2}, \tag{2.8}$$

while P_0 is an arbitrary constant. This hydrodynamic field which is a superposition of a circular Couette flow over a Poiseuille flow in the annulus, has the same expression as in a Newtonian fluid. This is not surprising, as it is a general result of basic theoretical rheology that Eqs. (2.7) describe the basic flow in hand, for any constitutive law of the fluid. Let us consider now the linear stability of this fiber suspension flow which, to our knowledge, has never been studied.

3. Equations governing the occurence of toroidal Taylor vortices

If the angular speed of the internal cylinder is progressively increased, the Taylor vortex flow, which is known to be developed by the flow in the absence of the pressure gradient, occurs. The axisymmetric toroidal vortices move downstream with the axial flow remaining unaffected by it.

3.1. Equations

The flow is described by the superposition of the basic Couette-Poiseuille flow on a small perturbation of the form

$$
\begin{aligned}
\mathbf{v}_r(r,z) &= u(r)\exp[i(\Sigma t - ikz)], \qquad i^2 = -1,\\
\mathbf{v}_\theta(r,z) &= v(r)\exp[i(\Sigma t - ikz)],\\
\mathbf{v}_z(r,z) &= w(r)\exp[i(\Sigma t - ikz)],\\
p(r,z) &= p(r)\exp[i(\Sigma t - ikz)].
\end{aligned} \tag{3.1}
$$

The continuity equation in this disturbance leads to

$$
w = -\frac{i}{k} D_{1*} u\,, \tag{3.2}
$$

where

$$
D_{1*} \equiv \frac{d}{dr} + \frac{1}{r}\,. \tag{3.3}
$$

Assuming again the alignment of the fibers along the streamlines, i.e. collinearity of the orientation vector **d** with the resulting velocity field (**V**+**v**), the components of the perturbation rate of strain $(\dot{\varepsilon})'$ have the following expressions

$$
\begin{aligned}
(\dot{\varepsilon})'_{rr} &= D_1 u \exp[i(\Sigma t - kz)],\\
(\dot{\varepsilon})'_{\theta\theta} &= \frac{u}{r}\exp[i(\Sigma t - kz)],\\
(\dot{\varepsilon})'_{zz} &= -ikw\exp[i(\Sigma t - kz)],\\
(\dot{\varepsilon})'_{r\theta} &= \frac{1}{2} r D_1\left(\frac{v}{r}\right)\exp[i(\Sigma t - kz)],\\
(\dot{\varepsilon})'_{\theta z} &= -\frac{1}{2} ikv\exp[i(\Sigma t - kz)],\\
(\dot{\varepsilon})'_{rz} &= \frac{1}{2}(-iku + D_1 w)\exp[i(\Sigma t - kz)].
\end{aligned} \tag{3.4}
$$

In these relations, D_1 denotes the ordinary derivative with respect to r.

If σ' denotes the perturbation stress tensor, the equations of motion of the superposition of the resulting flow (the basic flow with the disturbance) have the form

$$
\begin{aligned}
\frac{\partial \sigma'_{rr}}{\partial r} + \frac{\partial \sigma'_{rz}}{\partial z} + \frac{\sigma'_{rr} - \sigma'_{\theta\theta}}{r} &= \varrho\left(\frac{\partial u}{\partial t} + W\frac{\partial u}{\partial z} - \frac{2vV}{r}\right),\\
\frac{\partial \sigma'_{r\theta}}{\partial r} + \frac{\partial \sigma'_{\theta z}}{\partial z} + 2\frac{\sigma'_{r\theta}}{r} &= \varrho\left(\frac{\partial v}{\partial t} + u\frac{\partial V}{\partial r} + W\frac{\partial v}{\partial z} + \frac{uV}{r}\right),\\
\frac{\partial \sigma'_{rz}}{\partial r} + \frac{\partial \sigma'_{zz}}{\partial z} + \frac{\sigma'_{rz}}{r} &= \varrho\left(\frac{\partial w}{\partial t} + u\frac{\partial W}{\partial r} + W\frac{\partial w}{\partial z}\right),
\end{aligned} \tag{3.5}
$$

where t denotes the time.

Since the secondary motion takes place in the same medium as the basic flow, it is governed by the same behaviour law, say Eq. (1.1). Inserting its components in Eqs. (3.5), the following system of equations is obtained after linearization

$$
\begin{aligned}
&-D_1 p + \left[i\varrho(Wk - \Sigma) + \eta_0(2D_1 D_{1*} + k^2) - \eta_2\right] u - i\eta_0 k D_1 w = -\frac{2\varrho v V}{r}, \\
&(D_{1*}V)u + \left[k^2\nu - \nu D_1 D_{1*} + i(kW + \Sigma\right] v = 0, \\
&p = \left(\eta_0 D_{1*} - i\varrho\frac{D_1 W}{k}\right) u + \left[\frac{\sigma\varrho}{k} + W\varrho - 2ik\eta_0 + \frac{i\eta_0}{k} D_{1*} D_1\right] w,
\end{aligned}
\tag{3.6}
$$

where ν is the kinematic viscosity and D_1 denotes the differential operator d/dr.

These equations are associated with the no-slip condition of the fluid on the cylinder walls, which are written in the form

$$
u = v = D_{1*}u = 0 \qquad \text{at} \qquad r = R_1 \qquad \text{and} \qquad r = R_2 . \tag{3.7}
$$

Now we define R_e, the axial Reynolds number, and W_{av}, the average axial velocity, defined as

$$
\begin{aligned}
\mathrm{R}_e &= \frac{W_{\mathrm{av}} b}{\nu}, \\
W_{\mathrm{av}} &= -\frac{1}{8\eta_0}\frac{\partial p}{\partial z}\left[(R_2)^2 + (R_1)^2 - \frac{(R_2)^2 + (R_1)^2}{\ln(\kappa)}\right].
\end{aligned}
\tag{3.8}
$$

Let now R_0 and δ denote the mid-gap radius and the dimensionless gap-width, defined respectively by

$$
R_0 = \frac{R_1 + R_2}{2} \qquad \text{and} \qquad \delta = \frac{b}{R_0} . \tag{3.9}
$$

We also define dimensionless radial coordinate x and wave number a based on the gap-width by

$$
x = \frac{r - R_0}{b} \qquad \text{and} \qquad a = kb . \tag{3.10}
$$

Define now two following functions

$$
\gamma(x) = -\frac{1}{1 + \delta x} \qquad \text{and} \qquad \psi(x) = \frac{1 + \delta x}{1 - (\delta/2)} . \tag{3.11}
$$

Then, $G(x)$ and $F(x)$ denoting the dimensionless angular speed distribution and axial velocity distribution in the annulus, i.e.

$$
\Omega(x) = \Omega_1 G(x) \qquad \text{and} \qquad W(x) = W_{\mathrm{av}} F(x), \tag{3.12}
$$

these two functions having the following expressions:

$$
\begin{aligned}
G(x) &= -\frac{\kappa^2}{1-\kappa^2} + \frac{1}{1-\kappa^2}\frac{1}{\psi(x)^2}, \\
F(x) &= -\frac{2(1-\kappa^2)\ln(\psi(x))}{(1-\kappa^2)+(1+\kappa^2)\ln(\kappa)} + \kappa^2\left[\psi(x)^2 - 1\right]\ln(\kappa).
\end{aligned} \tag{3.13}
$$

Now let D and D_* denote the following differential operators:

$$
D \equiv \frac{d}{dx} \qquad \text{and} \qquad D_* \equiv \left[\frac{d}{dx} + \delta\gamma(x)\right], \tag{3.14}
$$

and T and β the Taylor number and the disturbance frequency, respectively, defined by

$$
T = -\frac{4A\Omega_1 b^4}{\nu^2}, \qquad \beta = \frac{\Sigma b^2}{\nu}. \tag{3.15}
$$

After linearization, we obtain the following equations of motion in a dimensionless form

$$
\begin{aligned}
&DD_* - a^2 - i\left[\beta - a\mathrm{Re}\,F(x)\right](DD_* - a^2)u \\
&\qquad ia\mathrm{Re}\left[D^2F(x) - \delta\gamma(x)DF(x)\right]u + \left[\frac{\eta_2}{\eta_0}\frac{a^2\delta^2}{\gamma(x)^2}\right]u = -T, \\
&\left[DD_* - a^2 - i\left[\beta - a\mathrm{Re}\,F(x)\right]\right]v = u.
\end{aligned} \tag{3.16}
$$

Equations (3.16) form the system of equations governing the occurence of the Taylor vortices in the annulus. The non-dimensional form of the no-slip condition (Eq. (3.7)) which is associated with them is the following

$$
u = v = D_*u = 0 \qquad \text{at} \qquad x = -\frac{1}{2} \qquad \text{and} \qquad x = +\frac{1}{2}. \tag{3.17}
$$

Putting $\eta_2 = 0$ in that system (no fibers in the medium), we obtain again the equations governing the transition in a Newtonian liquid (DIPRIMA and PRIDOR, [6], TAKEUCHI and JANKOWSKI, [20]).

So, $\dfrac{\eta_2}{\eta_0}\dfrac{a^2\delta^2}{\gamma(x)^2}$ can be seen as a perturbation term, characterizing the presence of the fibers in the medium. This perturbation term depends on the fiber aspect ratio, centration and density through η_2.

4. Solution

4.1. Solution method

The eigenvalue problem defined by Eqs. (3.16) and (3.17) is solved by a shooting method (HARRIS and REID, [11]; KRUEGER *et al.*, [14]). For numerical purposes, it is helpful to convert this two-point boundary-value problem into an initial-value problem. To do this, we define the following six unknowns

$$
\begin{aligned}
&Y_1 = u, \qquad Y_2 = v, \qquad Y_3 = D_* u\,, \\
&Y_4 = D_* v, \qquad Y_5 = (DD_* - a^2)u, \qquad Y_6 = D_*(DD_* - a^2)u.
\end{aligned} \tag{4.1}
$$

Using them, the system of equations of motion (3.16) is converted into the following set of six first order differential equations:

$$
D_* Y_1 = Y_3, \qquad D_* Y_2 = Y_4, \qquad D_* Y_5 = Y_6\,; \tag{4.2}
$$

$$
\begin{aligned}
&DY_3 - a^2 Y_1 = Y_5, \qquad DY_4 - M(x)Y_2 = Y_1\,, \\
&DY_6 - M(x)Y_5 - N(x)Y_1 = -Ta^2 G Y_2\,,
\end{aligned} \tag{4.3}
$$

where $M(x)$ and $N(x)$ denote the following functions

$$
\begin{aligned}
M(x) &= a^2 + i\,[\beta - a\mathrm{Re}\,F(x)]\,, \\
N(x) &= ia\mathrm{Re}\left[D^2 F(x) - \delta\gamma(x) DF(x)\right].
\end{aligned}
$$

The associated boundary conditions are obtained from Eqs. (3.17) in the form

$$
Y_i = 0 \qquad \text{at} \qquad x = -\frac{1}{2} \qquad \text{and} \qquad x = +\frac{1}{2} \qquad (i = 1,2,3). \tag{4.4}
$$

A set of three linearly independent solutions $(Y_i)_j$, $(i = 1,2,3,4,5,6)$ and $(j = a,b,c)$ of the system of differential equations (4.3) which satisfy the boundary conditions at $x = -1/2$ can be built by imposing arbitrarily the following three initial conditions

$$
\begin{aligned}
((Y_1)_a, (Y_2)_a, (Y_3)_a, (Y_4)_a, (Y_5)_a, (Y_6)_a) &= (0,0,0,1,0,0), \\
((Y_1)_b, (Y_2)_b, (Y_3)_b, (Y_4)_b, (Y_5)_b, (Y_6)_b) &= (0,0,0,0,1,0), \\
((Y_1)_c, (Y_2)_c, (Y_3)_c, (Y_4)_c, (Y_5)_c, (Y_6)_c) &= (0,0,0,0,0,1).
\end{aligned} \tag{4.5}
$$

A solution of the system (4.3) which satisfies the boundary conditions at $x = -1/2$ can therefore be written as a linear combination of these three solutions

$$
Y = \sum_{l=1}^{l=3} C_i Y_i\,. \tag{4.6}
$$

The necessary condition of this linear combination to satisfy the boundary conditions at $x = -1/2$ is vanishing of the determinant

$$H(T, a, \beta, \kappa, \mathrm{Re}, \eta_2) \equiv \det[(Y_i)_j] = 0 \quad \text{at} \quad x = \frac{1}{2} \quad (i = 1,2,3, \quad j = a,b,c). \tag{4.7}$$

This is the required characteristic equation. This determinant is complex-valued and requiring that both its real and imaginary parts vanish simultaneously. The basic solutions Y_i $(i = 1, 2, \ldots, 6)$ are obtained by integration of the system (4.3) by using a fourth order Ringe-Kutta method.

4.2. Numerical results

First of all, we solved the stability problem in hand for the Newtonian fluid case ($\eta_2 = 0$). Typically, the case where the instability is characterized by the occurence of laminar Taylor vortices occurs if the Reynolds number is less than 10, (LUEPTOW *et al.*, [15]). This value will define the maximum value of our study.

Table 1 shows the good agreement of our numerical results with those of DIPRIMA and PRIDOR [6]. It furnishes the critical Taylor number T_0 and Wave number a_0, and wave velocity of the Taylor vortices moving in the direction of the axial flow ω_0 relative to W_{av}, defined as

$$\omega = \frac{\beta}{a\mathrm{Re}}. \tag{4.8}$$

Table 1. Critical parameters in a Newtonian fluid from our calculations $(a_0, T_0, c_0)_N$ compared to those of Diprima and Pridor (1979), $(a_0', T_0', c_0)_{DP}$.

Re	κ	$(a_0)_N$	$(a_0)_{DP}$	$(T_0)_N$	$(T_0)_{DP}$	$(\omega_0)_N$	$(\omega_0)_{DP}$
0.01	1	3.13	3.13	1694.3	1695	1.170	1.170
0.01	0.95	3.13	3.13	1754	1754.9	1.170	1.170
0.01	0.9	3.13	3.13	1824	1823.4	1.170	1.170
10	1	3.14	3.14	1827.5	1826.8	1.169	1.169
10	0.95	3.14	3.14	1893.5	1891.3	1.169	1.169
10	0.9	3.14	3.14	1964	1965.1	1.169	1.169

Then we considered the case of the suspension flow. It was found that the presence of the fibers in the medium does not change the critical wavenumber a and wave velocity ω in the investigated ranges of Reynolds number, fiber-aspect ratio, concentration and density. But it increases the critical Taylor number with respect to the Reynolds number.

Furthermore, in the dilute case, the critical Taylor number increases with both the gap-width and the fiber-aspect ratio, for a given Reynolds number (Fig. 1). From Fig. 2 (drawn for Re = 1.0), the critical Taylor number also increases with the Reynolds number for a given gap-width and fibers aspect ratio (Fig. 2).

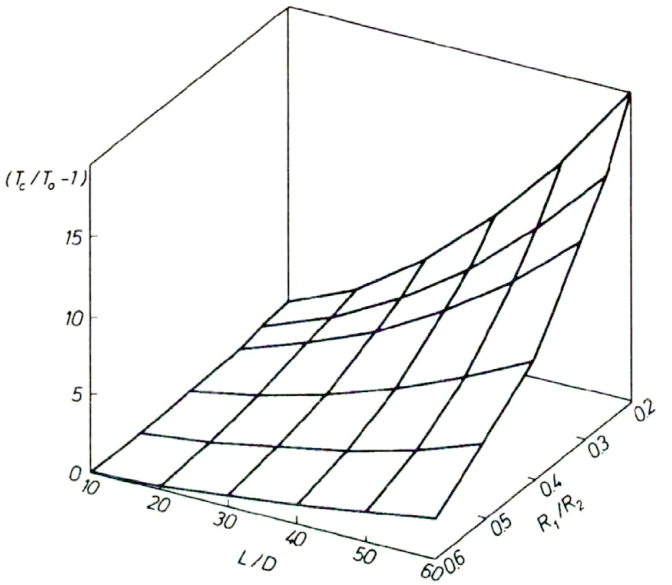


FIG. 1. Variation of T_c with fiber aspect ratio and R_1/R_2.

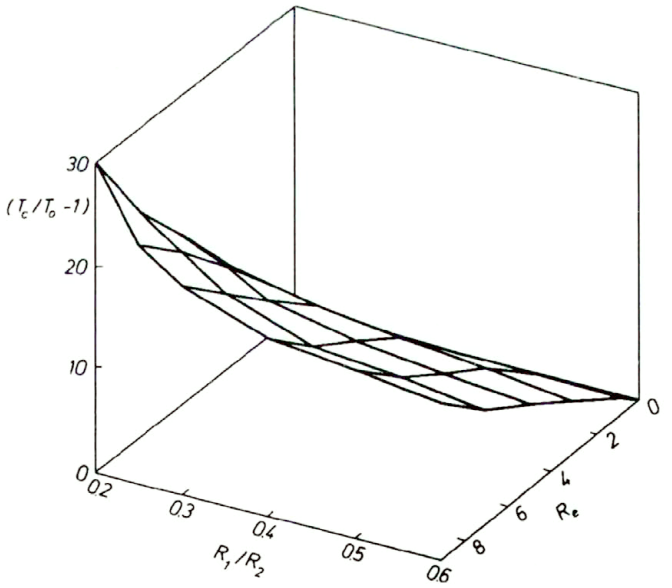


FIG. 2. Variation of T_c with Reynolds number and R_1/R_2.

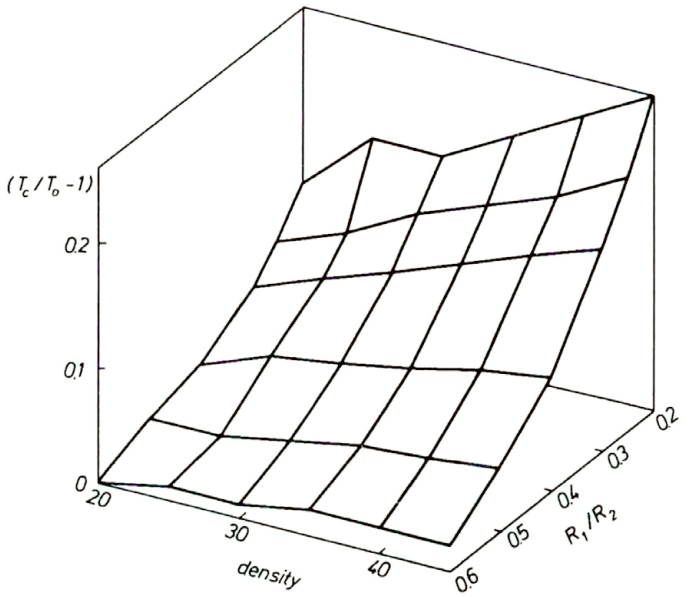


FIG. 3. Variation of T_c with R_1/R_2 and fibers density.

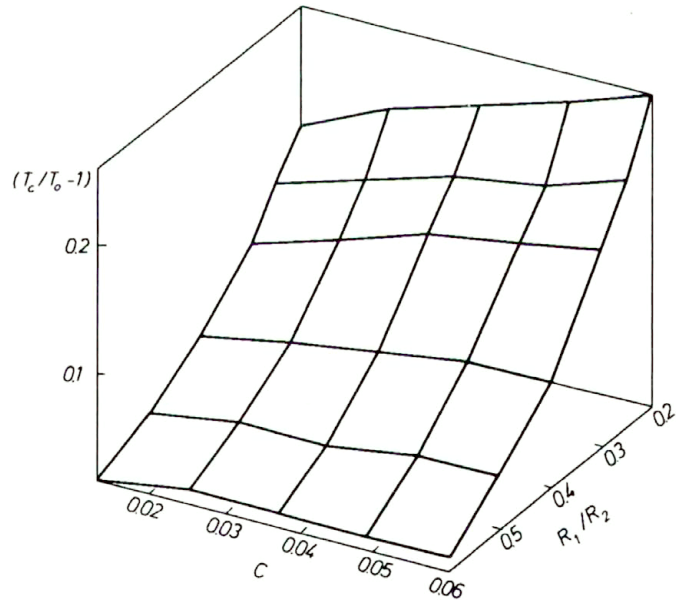


FIG. 4. Variation of T_c with R_1/R_2 and fibers concentration.

In the semi-concentrated case, from Fig. 3 (drawn for Re = 1.0 and $c = 0.06$), and from Fig. 4 (drawn for Re = 1.0 and $\Delta = 45$), the critical Taylor number increases with the gap-width and the concentration and density of the fibers, for

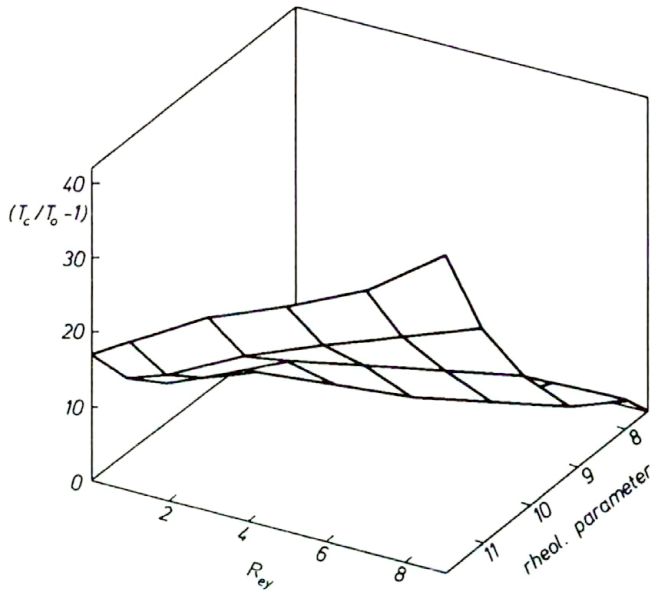


FIG. 5. Variation of T_c with Rey and rheological parameter.

a given Reynolds number. Finally, from Fig. 5, the critical Taylor number also increases with the Reynolds number for given fiber concentration and density (rheological parameter η_2 fixed).

5. Conclusion

A theoretical study of the stability of suspensions of stiff macroscopic fibers of large aspect ratio in a wide-gap Couette – Poiseuille configuration is presented for the dilute and semi-concentrated ranges.

In the dilute range of concentration, we have used the Ericksen anisotropic fluid equation of state, with rheological parameters related to the fiber aspect ratio.

In the semi-concentrated case, the "fiber-fiber" interactions being taken into account, the Ericksen law remains valid but the rheological parameters depend on the concentration of the solid particles and on their density.

When the axial Reynolds number is low, the instability appears in the form of laminar Taylor vortices, unaffected by the superimposed pressure gradient. They just move downstream with the axial flow. This is the case that we considered in a linear analysis.

In all the situations which have been considered, we found that

• the presence of the fibers in the medium does not change the critical wavenumber and wave velocity, but it increases the critical Taylor number.

• In the dilute case, increasing of the gap-width or the fiber aspect ratio stabilizes the Couette - Poiseuille flow.

In the semi-concentrated case, increasing of the gap-width or the fiber concentration or density also stabilizes the Couette - Poiseuille flow.

Superposition of the Reynolds number stabilizes the suspension Couette - Taylor flow in both the dilute and the semi-concentrated cases.

References

1. G. AUSIAS, J.F. AGASSANT and M. VINCENT, *Rheology of short glass fiber reinforced polypropylene*, J. Rheol, **36**, 4, 525–542, 1992.
2. K. BÜHLER and N. POLIFKE, *Dynamical behavior of Taylor vortices with superimposed axial flow*, [in:] Nonlinear Evolution of Spatio-Temporal Structures in Dissipative Continuous Systems, F.H. BUSSE and K.L. KRAMER [Eds.], Plenum, New York 1961.
3. S. CHANDRASEKHAR, *Hydrodynamic and hydromagnetic stability*, Dover Publications Inc. Eds., New York 1961.
4. R.J. CORNISH, *Flow of water through fine clearances with relative motion of the boundaries*, Proc. Roy. Soc. Lond. A, **140**, 227–240, 1933.
5. R.C. DIPRIMA, *The stability of a viscous fluid between rotating cylinders with an axial flow*, J. Fluid Mech., **9**, 621–631, 1960.
6. R.C. DIPRIMA and A. PRIDOR, *The stability of viscous flow between rotaring concentric cylinders with an axial flow*, Proc. Roy. Soc. Lond. A, **366**, 555–573, 1979.
7. M. DOI and S.F. EDWARDS, *Dynamics of rod-like macromolecules in concentrated solution*, J. Chem. Soc. Faraday Trans. II, **74**, 560–570, 1978.
8. M. DOI and S.F. EDWARDS, *Dynamics of rod-like macromolecules in concentrated solution. Part 2*, J. Chem. Soc. Faraday Trans. II, **74**, 918–932, 1978.
9. P.R. FENSTERMACHER, H.L. SWINNEY and J.P. GOLLUB, *Dynamical instabilities and the transition to chaotic Taylor vortex flow*, J. Fluid Mech., **94**, 103–128, 1979.
10. S. GOLDSTEIN, *The stability of viscous fluid flow between rotating cylinders*, Proc. Camb. Phil. Soc., **33**, 41–61, 1937.
11. D.L. HARRIS and W.H. REID, *On the stability of viscous flow between rotating cylinders*, J. Fluid Mech., **20**, 95–101, 1964.
12. M.A. HASOON and B.W. MARTIN, *The stability of viscous axial flow in an annulus with a rotating inner cylinder*, Proc. Roy. Soc. Lond. A, **352**, 351–380, 1977.
13. K. KATAOKA, H. DOI and T. KOMAI, *Heat/mass transfer in Taylor vortex flow with constant axial flow rates*, Int. Heat Mass Transfer, **20**, 57–63, 1977.
14. E.R. KRUEGER, A. GROSS and R.C. DIPRIMA, *On the relative importance of Taylor-vortex and non-axisymmetric modes in flow between rotating cylinders*, J. Fluid Mech., **24**, 521–538, 1966.
15. R.M. LUEPTOW, A. DOCTER and K. MIN, *Stability of axial flow in an annulus with a rotating inner cylinder*, Phys. Fluids A, **4**, 11, 2446–2455, 1992.
16. B. NSOM, *Transition from circular Couette flow to Taylor vortex flow in dilute and semi-concentrated suspensions of stiff fibers*, J. Phys. II (France), **4**, 9-22, 1994.
17. T.C. PAPANASTASIOU and A.N. ALEXANDRIOU, *Isothermal extrusion of non-dilute fiber suspensions*, J. Non. Newt. Fluid Mech., **25**, 313–328, 1987.

18. E.S.G. SHAQFEH and G.H. FREDRICKSON, *The hydrodynamic stress in a suspension of rods*, Phys. Fluids A, **2**, 1, 7–24, 1990.

19. C.A. STOVER, D.L. KOCH and C. COHEN, *Observations of fibre orientation in simple shear flow of semi-dilute suspensions*, J. Fluid Mech., **238**, 277–296, 1992.

20. D.I. TAKEUCHI and D.F. JANKOWSKI, *A numerical and experimental investigation of the stability of spiral Poiseuille flow*, J. Fluid Mech., **102**, 101–126, 1981.

LABORATOIRE DES ECOULEMENTS GÉOPHYSIQUES ET INDUSTRIELS
INSTITUT DE MÉCANIQUE, GRENOBLE, FRANCE.

Received October 18, 1994.

Arch. Mech., 47, 6, pp. 1043–1056, Warszawa 1995

Operator approach to three problems of fluid mechanics

I. Yu. POPOV (ST. PETERSBURG)

OPERATOR VERSION of the stokeslet approach is suggested. The creeping flow in coupled cavities is described. Operator extension theory model for the description of stratified flows is described. The Orr – Sommerfeld problem is analyzed as a problem of perturbation of the Schrödinger operator.

1. Introduction

THE PAPER DEALS with three different fluid mechanics problems connected by the operator theory approach. The first problem is the description of Stokes flows in complex domains. It is known that creeping flow past a small obstacle or small rotating body may be described by means of a stokeslet, i.e. singular solution of the Stokes equations [1]. The operator version of the stokeslet approach based on the theory of self-adjoint extensions of symmetric operators is suggested in the paper. It is analogous to the well-known zero-range potential approach in quantum mechanics [2, 3] and zero-width slit model in diffraction theory [4, 5]. It allows one to reveal general mathematical features of the stokeslet and, as a result, to expand the range of applications of the conventional stokeslet. Namely, it allows one to simulate creeping flows in domains coupled through small apertures. An example of such a flow in a ring structure is considered.

The second problem is the description of two-dimensional stratified flow of inviscid incompressible dielectric medium in gravitational and electric fields. We derive the modified Dubreil – Jacotin equation and reduce it to the equation analogous to the Schrödinger one. The operator extension theory model for the description of such flows past a small obstacle or an obstacle with small opening is suggested.

The third problem is that of hydrodynamic stability. It is shown that the Orr – Sommerfeld problem in different situations (the conventional one, the problem for the case when a transversal component of the flow velocity is taken into account, and the problem for the flow between two cylinders) can be considered as a problem of perturbation of the Schrödinger operator. It allows one to use powerful methods of the theory of operator perturbations. Particularly, the completeness of the set of eigenfunctions and associated functions is proved.

2. Operator version of the stokeslet approach

2.1. Stokeslet

We consider a two-dimensional flow. In this case it is convenient to use a

stream function ψ instead of the velocity $\mathbf{v} = (v_1, v_2)$ ($\mathbf{x} = (x_1, x_2)$, x_1, x_2 are the Cartesian coordinates):

$$v_1 = \frac{\partial \psi}{\partial x_2}, \qquad v_2 = -\frac{\partial \psi}{\partial x_1}. \tag{2.1}$$

Hence, the Stokes equations reduce to the biharmonic one: $\Delta^2 \psi = 0$.

To give correct mathematical description of a point-like interaction for such equation, we use a method analogous to that applied in zero-range potential method [3, 4], i.e. we use the operator extension theory [6, 7]. Let us consider the operator Δ_0^2 in the space $L_2(\mathbb{R}^2)$ and defined on the set of smooth finite functions from $L_2(\mathbb{R}^2)$ vanishing near some fixed point r_0. The closure of this operator is a symmetric operator with the domain

$$D(\Delta_0^2) = \Big\{ u\colon\ u \in L_2(\mathbb{R}^2), \quad \Delta^2 u \in L_2(\mathbb{R}^2), \\ u(r_0) = u'_{x_i}(r_0) = u''_{x_j x_i}(r_0) = 0, \quad i,j = 1,2 \Big\}. \tag{2.2}$$

The operator Δ_0^2 has deficiency indices $(6,6)$. An element u from the domain of the operator Δ_0^{2*} takes the form:

$$u(x) = \sum_{i,j=1}^{2} c_{i,j}^u g_{x_i,x_j}(x) + \sum_{i=1}^{2} c_i^u g_{x_i}(x) + c_0 g(x) \\ + \xi(x) \left(a_0 - \sum_{i=1}^{2} a_i x_i + \sum_{i,j=1}^{2} a_{ij} g_{ij} x_i x_j \right) + u_0(x). \tag{2.3}$$

Here $u_0 \in D(\Delta_0)$, $g_{ij} = 1$, $i \neq j$, $g_{ii} = 2^{-1}$, $i,j = 1, 2$, $\xi(x)$ is a smooth cutting function: $\xi(x) = 1$, $|x| \leq 1$, $\xi(x) = 0$, $|x| \geq 2$, g is a fundamental solution. The function $(\Delta_0^{2*} u)(x)$ at the point x ($x \neq 0$) is computed as $(\Delta_0^{2*} u)(x) = \Delta^2 u(x)$. The bilinear form $I(u,v) = (\Delta_0^{2*} u, v) - (u, \Delta_0^{2*} v)$ on the elements from $D(\Delta_0^{2*})$ can be computed by taking into account the representation (2.3):

$$I(u,v) = a_0^u \overline{c_0^v} - c_0^u \overline{a_0^v} + \sum_{i=1,2} \left(a_i^u \overline{c_i^v} - c_i^u \overline{a_i^v} \right) + \sum_{i,j=1,2} \left(a_{ij}^u \overline{c_{ij}^v} - c_{ij}^u \overline{a_{ij}^v} \right).$$

To construct a domain of self-adjoint extension of the operator Δ_0^2. It is necessary to select such linear subset of $D(\Delta_0^{2*})$, that the form I vanishes on the elements of this subset. It is an ordinary problem of linear algebra in a space $\mathbb{C}^6$. As a result, we obtain

Theorem 1. *The operator Δ_s^2 (extension) is self-adjoint if and only if $\Delta_0^2 \subset \Delta_s^2 \subset \Delta_0^{2*}$. Here $D(\Delta_s^2)$ is such a linear subset of $D(\Delta_0^{2*})$ having no extensions that*

one of the following conditions is valid for the boundary vectors of any functions from the set $D(\Delta_s^2)$:

1. $U = (U_0, U_1)$, $U_0 = AU_1$, $U_1 = (c_0^u, c_1^u, c_2^u, c_{11}^u, c_{12}^u, c_{22}^u)$,
   $U_0 = (a_0^u, a_1^u, a_2^u, a_{11}^u, a_{12}^u, a_{22}^u)$, $A\colon \mathbb{C}^6 \to \mathbb{C}^6$, $A = A^*$,
2. $U_1 = AU_0$, $A\colon \mathbb{C}^6 \to \mathbb{C}^6$, $A = A^*$,
3. $U_0 = \alpha + \gamma$, $U_1 = \beta + A\gamma$; *here* α, β *are vectors from arbitrary orthogonal fixed subspaces* N^+ *and* N^-, $N^+, N^- \in \mathbb{C}^6$, $\gamma \in N$, $N = \mathbb{C}^6 \ominus N^+ \ominus N^-$, $A\colon N \to N$, *the operator* A *is self-adjoint and reversible.*

It is this theorem that gives us the correct mathematical description of the stokeslet.

2.2. Distokeslet

To describe the stokeslets of higher orders, i.e. singular solutions having strong singularity, it is necessary to modify the scheme. Namely, it is necessary to extend the initial space L_2 by adding functions with strong singularities which do not belong to the space L_2. As a result, we have to work in the space with indefinite metrics, but it does not destroy the construction. Let us describe how to take into account such a singular solution. The general case may be considered without essential problems. Let A_1 be the following set of functions:

$$A_1 = \left\{ f(x)\colon\ f \in L_2(\mathbb{R}^2),\quad \Delta^2 f \in L_2(\mathbb{R}^2),\ \int\limits_{\mathbb{R}^2} f(x)|x - x_0|^{-4}dx \ -\ \text{converges} \right\},$$

$$h_{-1}(x) = g^{(j_1,j_2)}(x), \qquad h_1 = (-\Delta^2 - \lambda_0)^{-1} h_{-1},$$

where x_0 is a fixed point, λ_0 is some positive value (a regular point of the operator $-\Delta^2$), $g^{(j_1,j_2)}(x)$ ($j_1 + j_2 = 3$, say) is the corresponding derivative of the fundamental solution g. Let $\mathfrak{U}_1$ be a set of elements which can be represented in a form

$$f = f_1 + c_1 h_1 + c_{-1} h_{-1},$$

where $f_1 \in A_1$. We define an inner product in $\mathfrak{U}_1$ by the following expression:

$$(f, \varphi)_{\mathfrak{U}_1} = (f_1, \varphi_1)_{A_1} + c_1^f \int\limits_{\mathbb{R}^2} h_1 \overline{\varphi_1}\, dx + \overline{c_1^\varphi} \int\limits_{\mathbb{R}^2} f_1 \overline{h_1}\, dx + \left(c_1^f \overline{c_{-1}^\varphi} + c_{-1}^f \overline{c_1^\varphi} \right) \int\limits_{\mathbb{R}^2} h_1 \overline{h_{-1}}\, dx.$$

The topology in the set $\mathfrak{U}_1$ is defined in a standard manner [8]. The form $(f, \varphi)_{\mathfrak{U}_1}$ should be transformed to a diagonal form. The obtained expression contains one "negative squares". One obtains a positive form by replacing the corresponding sign "minus" by "plus". This form gives an expression for a positive

definite scalar product, which is closely linked with a topology. The closure of the set $\mathfrak{U}_1$ in a space with such a topology is a Pontryagin's space Π_1.

Let us consider the operator $\tilde{\Delta}^2$ with the domain

$$D(\tilde{\Delta}^2) = \left\{ f \in \tilde{A}_1, \quad f_1 \in W_2^{4,\mathrm{loc}}, \quad f_1 = f_2 + ch_1, \quad h_{-1} \notin D(\tilde{\Delta}^2) \right\},$$

where f_2 is such function from A_1, that $(-\tilde{\Delta}^2 - \lambda_0)f_2 \in \mathfrak{U}_1$. The operator $-\tilde{\Delta}^2$ acts as a square of the Laplace operator on the set A_1, and the image of the element h_1 is such that

$$(-\tilde{\Delta}^2 - \lambda_0)h_1 = h_{-1}.$$

The operator $-\tilde{\Delta}^2$ is a symmetric one. Moreover, it is a self-adjoint operator because the following relation takes place:

$$(-\tilde{\Delta}^2 - \lambda_0)D(\tilde{\Delta}^2) = \mathfrak{U}_1.$$

Let $\Delta_{1,0}^2$ be a restriction of the operator $-\tilde{\Delta}^2$ onto the set

$$D(\Delta_{1,0}^2) = \left\{ f\colon f \in D(\Delta_0^2), \quad ((-\tilde{\Delta}^2 - \lambda_0)f, h_{-1}) = 0 \right\}.$$

The obtained operator $\Delta_{1,0}^2$ is a symmetric one and has deficiency indices $(1,1)$. Indeed, the condition means that the element h_{-1} is orthogonal to the set of images of the operator $-\tilde{\Delta}^2 - \lambda_0$. There are no other deficiency elements because the set $\mathfrak{U}_1$ is a dense subset of the space Π_1 with respect to the described topology. The operator has self-adjoint extensions.

Theorem 2. *The domain of self-adjoint extension of the operator $\Delta_{1,0}^2$ consists of all elements f from the set $D(\Delta_{1,0}^{2*})$, satisfying the condition $c_1^f = ac_{-1}^f$, where a is a real number,*

$$c_1^f = \left((-\tilde{\Delta}^2 - \lambda_0)f_1, h_{-1} \right).$$

2.3. The model for coupled cavities

Various hydrodynamic applications of stokeslets are described in the review of Hasimoto and Sano [1]. The operator version allows one to extend the field of applications of the stokeslet, for example to simulate the creeping flow in the cavities with small apertures. To construct this model one must change slightly the described scheme. Let us consider these alterations for concrete problem. Let $\Omega = \{(\varrho, \phi) : |\varrho| \leq R\}$. Here ϱ, ϕ are the polar coordinates. We shall consider the square of the Laplace operator with the following boundary condition:

$$\psi|_{\varrho=R} = 0, \qquad \psi'_\varrho|_{\varrho=R} = 0.$$

Let $(\Delta_0^{\mathrm{in}})^2$ be the restriction of the initial operator to the set of smooth functions, which satisfy the conditions (2.2) near some point $(R, \phi_0) = r_0$ of the boundary.

The deficiency elements may be obtained by the following procedure. Let r_0' be an internal point of Ω. One can find the solution of the problem:

$$\Delta^2 g(r) + k^2 g(r) = \delta(r - r_0'), \qquad g(r)|_{\varrho=R} = g_\varrho'|_{\varrho=R} = 0.$$

Here $k^2 > 0$. The asymptotics of the solution near the point r_0' is the same as that of the fundamental solution. It is necessary to look for the solution g and its derivatives when $r_0' \to r_0$. It occurs that the function $g''_{\varrho_0\varrho_0}$ (second normal derivative) only gives a non-zero limit (the derivatives of higher order don't belong to the space L_2). This function is the deficiency element of the operator $(\Delta_0^{\text{in}})^2$. That is, the deficiency indices of the operator $(\Delta_0^{\text{in}})^2$ are $(1,1)$. One can show that the main term of the asymptotics of the deficiency element near the point r_0 is $\cos^2\theta$, where θ is the angle between the vector $r - r_0$ and the normal at the point r_0.

The analogous construction (the operator $(\Delta_0^{\text{ex}})^2$) is found for the exterior domain $\mathbb{R}^2 \setminus \Omega$. Let the operator Δ_0^2 be the orthogonal sum of the internal and external operators. It is the symmetric one with the deficiency indices $(2,2)$. The domain of the adjoint operator consists of the following elements:

$$u = (u^{\text{in}}, u^{\text{ex}}), \qquad u^{\text{in,ex}} = \zeta_+^{\text{in,ex}} w_+^{\text{in,ex}} + \zeta_-^{\text{in,ex}} w_-^{\text{in,ex}} + u_0^{\text{in,ex}},$$

where

$$w_\pm^{\text{in,ex}} = 2^{-1}(h_\lambda^{\text{in,ex}} \pm h_{\overline{\lambda}}^{\text{in,ex}}),$$

$\lambda = k^2$ is a regular point for the operators $\Delta_0^{\text{in,ex}}$, $h_\lambda^{\text{in,ex}}$ is the corresponding deficiency element, $\zeta_+^{\text{in,ex}}$, $\zeta_-^{\text{in,ex}}$ are some numbers, $u_0^{\text{in,ex}} \in D(\Delta_0^{\text{in,ex}})$. One can obtain the boundary form I for the adjoint operator:

$$I(u,v) = \zeta_{+u}^{\text{in}}\overline{\zeta_{-v}^{\text{in}}} - \zeta_{-u}^{\text{in}}\overline{\zeta_{+v}^{\text{in}}} + \zeta_{+u}^{\text{ex}}\overline{\zeta_{-v}^{\text{ex}}} - \zeta_{-u}^{\text{ex}}\overline{\zeta_{+v}^{\text{ex}}}.$$

The domain of self-adjoint extension is a linear subset of the domain of the adjoint operator, on the elements of which the boundary form vanishes. It is easy to show that this domain consists of all functions from the domain of the adjoint operator, which satisfy the condition:

$$\begin{pmatrix} \zeta_-^{\text{in}} \\ \zeta_-^{\text{ex}} \end{pmatrix} = A \begin{pmatrix} \zeta_+^{\text{in}}(s) \\ \zeta_+^{\text{ex}}(s) \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \zeta_+^{\text{in}}(s) \\ -\zeta_-^{\text{ex}}(s) \end{pmatrix} = B \begin{pmatrix} \zeta_-^{\text{in}}(s) \\ \zeta_+^{\text{ex}}(s) \end{pmatrix}.$$

Here A and B are Hermitian matrices.

One can realize the analogous construction for the case when $\Omega = \{x\colon R_1 < \varrho < R_2\}$, $|r_0| = R_2$. It is possible to describe a situation when there are several point-like apertures at the points r_i, $i = 1, 2, \ldots p$. Here the procedure starts from the restriction $(\Delta_0^{\text{in}})^2$ of the Laplace operator onto the set of smooth functions vanishing near the points r_i. The operator has the deficiency indices (p,p).

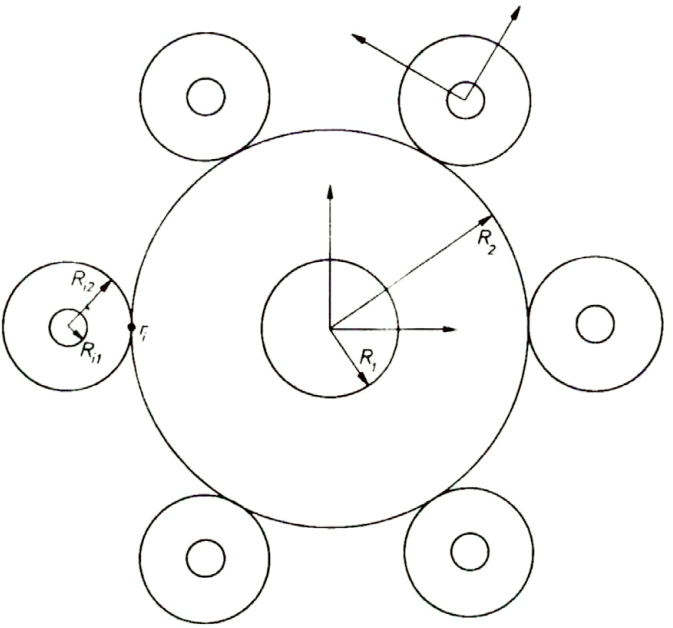


Fig. 1. The geometrical configuration of hydrotron.

Let us consider a more complicated structure – a ring with a system of tangent (at points r_i) rings (Fig. 1). Let $(\Delta_{i0}^{\mathrm{ex}})^2$ be the restriction of the Laplace operator in the circle Ω_i onto the set of smooth functions vanishing near the point $r_i \in \overline{\Omega_i} \cap \overline{\Omega^{\mathrm{in}}}$, and near an internal point r_{i0}. This operator is analogous to the one which has been described above. Let

$$\Delta_0^h = \overline{\sum_{i=1}^{p} \oplus (\Delta_{i0}^{\mathrm{ex}})^2 \oplus (\Delta_0^{\mathrm{in}})^2}\,.$$

Remark 1. The reason of using the superscript h in this notation is the following. The geometrical configuration of our system is analogous to that used for the well-known electronic device – a magnetron. That is why the name "hydrotron" is natural for this system. The investigated flow is caused by the rotating axes (the internal circles in the domains Ω_i). If the boundary of each domain Ω_i consists of two concentric circles, we can find the stream-function for Ω_i by using the procedure which has been described above (for the ring). To obtain the solution corresponding to the case when the axis rotates, one must assume the inhomogeneous boundary condition $\psi'_\varrho|_{\varrho=R_{i1}} = \text{const}$. If the circles are not concentric, the procedure fails. But we can use a stokeslet to simulate the rotating axis. This situation is described below. We add the restriction at the points r_{i0} to simulate the small rotating circle.

The operator Δ_0^h is a symmetric one with the deficiency indices $(3p, 3p)$. It has self-adjoint extensions. We shall not analyze here the dependence of the picture

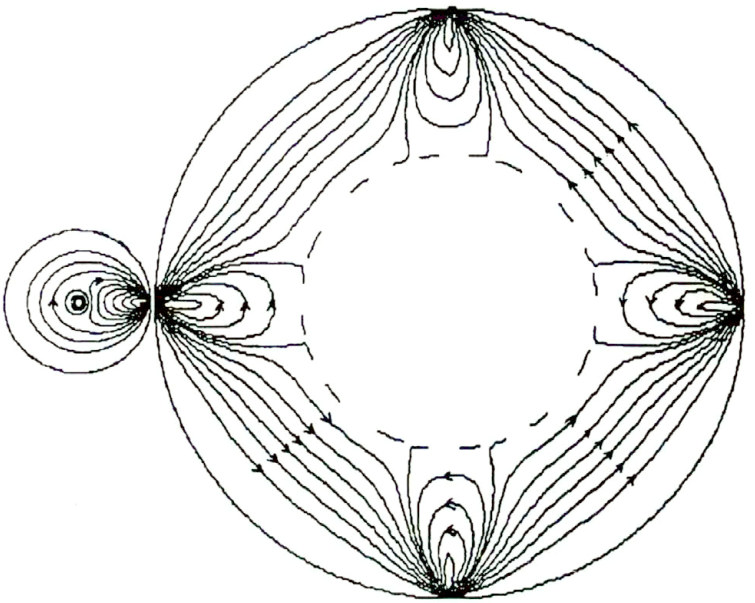


FIG. 2. Cellular structure of the flow in hydrotron, $R_1/R_2 = 0.5$, $p = 4$.

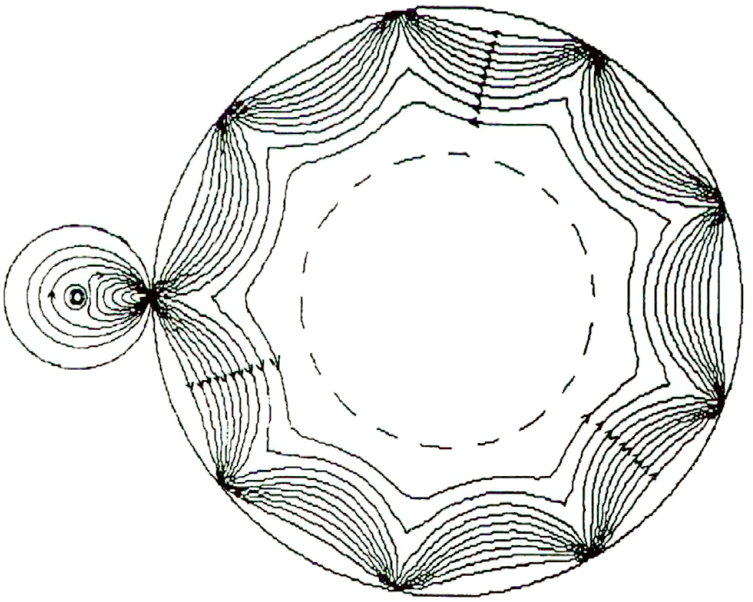


FIG. 3. The flow in hydrotron for $R_1/R_2 = 0.5$, $p = 9$.

of streamlines on the choice of the extension (one can make this analysis without any problems). Let us fix the extension of B-type with the following matrix B: $B = \{B_{ij}\}$, $B_{11} = B_{22} = 0$, $B_{12} = B_{21} = -1$. The character of the flow depends

on the number of circles. If the number is small, the flow in the ring has a cellular structure (Fig. 2), but for greater numbers the separating streamlines are eliminated (Fig. 3).

3. Stratified flows

3.1. Modified Dubreil–Jacotin equation

Let us consider the two-dimensional steady flow of inviscid incompressible stratified medium in gravitational field. It is known that such flow can be described by the Dubreil–Jacotin equation [9]. We suppose that the medium is a dielectric one (with the molecules of dipole type). We shall study a case when there exists not only gravitational but also an electric field. Let ψ be the stream function (see (2.1)), p is pressure, ϱ is density of mass distribution, g is the free fall acceleration, $F = (F_1, F_2) = \alpha \nabla U$, $U = E^2$, E is the strength of the electric field, the coefficient α is related to the parameters of the dipoles and the density of their distribution. Then one obtains the modified Dubreil–Jacotin equation:

$$\Delta\psi + \varrho' \varrho^{-1}(2^{-1}|\nabla\psi|^2 + g x_2 + \alpha U) = \Phi(\psi).$$

In particular problems the function $\Phi(\psi)$ can be determined by taking into account the condition at infinity.

Let us consider the flow described by the equation. We assume that the function U vanishes at infinity (for example, it has a bounded support $\operatorname{supp} U$) and L is the size of a domain where it differs from zero considerably enough in comparison with the other corresponding parameters of the equation (for example, $L = \operatorname{diam} \operatorname{supp} U$), and that the flow is unperturbed far from the origin: $(\psi - V_0 x_2)|_{r\to\infty} \to 0$, where V_0 is the velocity of the unperturbed flow at infinity. The equation takes the form:

$$\Delta\psi + \varrho^{-1}\varrho'(2^{-1}(|\nabla\psi|^2 - V_0^2) + g V_0^{-1}(V_0 x_2 - \psi) + \alpha U) = 0.$$

Let u, $\psi = L V_0 u$, be the dimensionless stream function. Then

$$\Delta u + (2\varrho_1)^{-1}\varrho_1'(|\nabla u|^2 - 1 + \alpha V_0^{-2} U) = \varepsilon^2 \varrho_1^{-1}\varrho_1'(u - x_2),$$

$$(u - x_2)|_{r\to\infty} \to 0, \qquad r^2 = x_1^2 + x_2^2,$$

where $\varepsilon^2 = g L V_0^{-2} = (\mathrm{Fr})^{-1}$, Fr is the Froude number, $\varrho_1(u) = \varrho_0(Lu)$. The approximation $\varepsilon^2 \ll 1$ should be discussed. Then the equation simplifies to

$$\Delta u + (2\varrho_1)^{-1}\varrho_1'(|\nabla u|^2 - 1 + \alpha V_0^{-2} U) = 0.$$

For an exponential dependence of the density on x_2 (i.e. $\varrho_0(x_2) = A \exp(-2\beta x_2)$, A, β are some constants):

$$\Delta u + \mu^2(|\nabla u|^2 - 1 + \alpha V_0^{-2} U) = 0.$$

Here $\mu^2 = \beta L$. One can replace the function u by the function v in accordance with the relation: $u = -\mu^{-2} \ln v$. Hence, we obtain the following equation for the function v:

$$\begin{aligned} &\Delta v - \mu^4(1 - \alpha V_0^{-2} U)v = 0, \\ &(v - \exp(-\mu^2 x_2))|_{r\to\infty} \to 0. \end{aligned} \tag{3.1}$$

The equation is similar to the stationary Schrödinger one with an attractive potential ($\alpha V_0^{-2} U$ is positive). Hence, the eigenstates can exist (the situation depends on the values of the parameters α, μ, V_0 and the form of the potential, i.e. on the electric field distribution; as for estimates for the number of eigenstates, see [10] and references therein). In accordance with the relation described above it means that there exist local eddies in the corresponding flow caused by the electric field.

3.2. Operator extension theory model for stratified flows

Operator extension theory model can be constructed for the description of the flow near obstacles. Particularly, if we have a small obstacle, one should make simple modification of the conventional zero-range potential method [3, 4]. Namely, let us restrict the operator $(-\Delta + \mu^4(1 - \alpha V_0^{-2} U))$ in $L_2(\mathbb{R}^2)$ to the set of smooth functions vanishing near a fixed point r_0. One can show that the closure $-\Delta_0$ of this operator is a symmetric one with the deficiency indices (1,1). The Green function $G_U(r, r_0, k_1)$ of the initial operator with the source at the point r_0 is a deficiency element corresponding to the regular point k_1^2. One uses the "real basis" $w_+(r)$, $w_-(r)$:

$$\begin{aligned} w_+(r) &= \operatorname{Re} G_U(r, r_0, k_1), \\ w_-(r) &= \operatorname{Im} G_U(r, r_0, k_1)(\operatorname{Im} G_U(r_0, r_0, k_1))^{-1} \end{aligned}$$

in the deficiency subspace for the simplification of the construction (here we take into account that the function $\operatorname{Im} G_U(r, r_0, k_1)$ has no singularity). The domain of the adjoint operator consists of elements of the form:

$$u = \zeta_+ w_+ + \zeta_- w_- + u_0, \tag{3.2}$$

where $u_0^{\text{in,ex}} \in D(\Delta_0^{\text{in,ex}})$, ζ_+, ζ_- are some constants. After brief calculation one obtains the boundary form:

$$I(u, v) = (-\Delta_0^* u, v) - (u, -\Delta_0^* v) = \zeta_{+u}\overline{\zeta_{-v}} - \zeta_{-u}\overline{\zeta_{+v}}.$$

The domain of a self-adjoint extension is a linear subset of the domain of the adjoint operator for elements of which the boundary form vanishes. In our case the problem of the subset description reduces to the simple problem of linear algebra. The consideration resulted in the theorem

Theorem 3. *The set of self-adjoint extensions of the operator Δ_0 is a one-parameter family of operators Δ_α, $\Delta_0 \subset \Delta_\alpha \subset \Delta_0^*$, the domains of which consist of all functions of the form* (3.2) *satisfying the condition: $\zeta_+ = \alpha\zeta_-$, where α is real parameter.*

Let us consider the case of absence of an electric field. In this situation we have the Helmholtz equation instead of the Schrödinger one [11]. The flow near an obstacle with small opening may be simulated using zero-width slits model [5, 6]. Stratified flow near the surface with small aperture under the assumptions made above is described by the Dubreil-Jacotin equation. We reduce this problem to the Dirichlet problem for the Helmholtz equation (instead of (3.1)):

$$\begin{gathered} \Delta v - \mu^4 v = 0, \\ (v - \exp(-\mu^2 x_2))|_{r\to\infty} \to 0, \qquad v|_\Gamma = \exp(-\mu^2 u_0). \end{gathered} \tag{3.3}$$

Here Γ is the boundary, u_0 is the parameter of the boundary condition for the dimensionless stream function u: $u|_\Gamma = u_0$. To simulate the aperture at the point x_0, $x_0 \in \Gamma$, we use zero-width slits model, i.e. we replace the ordinary Laplace operator in (3.3) by the extension constructed earlier [5]. The corresponding solution is:

$$v = \begin{cases} \alpha^{\text{in}} \dfrac{\partial}{\partial n_y}(G_D^{\text{in}}(x, y, k))\Big|_{y=x_0}, & x \in \Omega^{\text{in}}, \\ \alpha^{\text{ex}} \dfrac{\partial}{\partial n_y}(G_D^{\text{ex}}(x, y, k))\Big|_{y=x_0} + V, & x \in \Omega^{\text{ex}}. \end{cases}$$

Here V is the solution of the corresponding unperturbed (without aperture) problem

$$\alpha^{\text{in}} = \alpha^{\text{ex}} = -V(x_0)(D^{\text{in}}(k) + D^{\text{ex}}(k))^{-1},$$

where

$$D^{\text{in}}(k) = (\lambda - \lambda_0) \sum_m \frac{1}{(\lambda_m - \lambda)(\lambda_m - \lambda_0)} \left| \frac{\partial \phi_m^{\text{in}}(x_0)}{\partial n} \right|^2,$$

$$D^{\text{ex}}(k) = \frac{1}{4\pi} \int_0^\infty \frac{s^{1/2}(\lambda - \lambda_0) ds}{(s - \lambda)(s - \lambda_0)} \int_0^{2\pi} \left| \frac{\partial \phi^{\text{ex}}(x_0, \nu, s^{1/2})}{\partial n} \right|^2 d\nu,$$

$\phi_m^{\text{in}}(x)$ is the eigenfunction of the operator $-\Delta^{\text{in}}$ (unperturbed operator), i.e. the operator for the internal problem (without opening) corresponding to the eigenvalue λ_m, $\phi^{\text{ex}}(x, \nu, s^{1/2})$ is the eigenfunction corresponding to the point s of the continuous spectrum of the operator $-\Delta^{\text{ex}}$. Hence, one obtains the model stream function u in an explicit form. The model can be constructed also for the case of an electric field ($\alpha \neq 0$). The only modification is that one must use the Green function for the Schrödinger operator instead of the Green function for the Laplace operator.

While studying the scattering problem for the Helmholtz equation in the exterior of bounded domain, P. LAX and R. PHILLIPS [12] have shown that there exist a series of resonances (scattering frequencies) on the imaginary axis of the spectral parameter k complex plane, i.e. such values k_m that there exists a non-trivial solution v_m of the equation

$$\Delta v_m + k_m^2 v_m = 0,$$

satisfying the Dirichlet boundary condition. It is not the eigenvalue of the operator because v_m does not belong to the space $L_2(\Omega^{\text{ex}})$ – it has an exponential growth at infinity. In accordance with the obtained relation, these solutions correspond to the solutions of the Dubreil-Jacotin equation, i.e. the stream functions with a linear growth at infinity ($r \to \infty$).

4. Hydrodynamic stability and perturbation of Schrödinger operator

The hydrodynamic stability problem is analyzed from the point of view of the operator perturbation theory. This problem is difficult and is usually studied numerically. Our approach gives a new view on the investigation of spectral properties of the Orr-Sommerfeld equation. The work [13] in which the Rayleigh equation (the limiting case of the Orr-Sommerfeld one for an inviscid fluid) is studied from the point of view of the operator theory should be mentioned. We deal with a two-dimensional parallel flow of viscid incompressible fluid. In this case one seeks the perturbed stream function for boundary layer in the form of a Tollmien-Schlichting wave: $\Psi(x,z,t) = \Phi(x)\exp(i\alpha(z-ct))$. Hence, the Orr-Sommerfeld equation (in dimensionless form) is obtained for the function Φ:

$$\Phi'''' - 2\alpha^2\Phi'' + \alpha^4\Phi = i\alpha\mathrm{Re}((u-c)(\Phi''-\alpha^2\phi) - u''\Phi), \tag{4.1}$$

where u, $u = u(x)$ ($u(x) = u_\infty = \text{const}$ for $x > 1$), is the velocity of the unperturbed flow, Re is the Reynolds number [14, 15].

4.1. Conventional Orr–Sommerfeld problem

We consider the Orr-Sommerfeld problem in three situations. The first one is the conventional Orr-Sommerfeld equation for a boundary layer (4.1). This equation is studied usually on the half-axis $[0,\infty)$ or on the segment $\Delta = [0,1]$ with zero boundary conditions. We shall deal with both the cases. Let us denote by ψ the following function $\psi = \Phi'' - \alpha^2\Phi$. Then $\Phi = G\psi$ ($\Phi = G_\Delta\psi$), where $G(G_\Delta)$ is the corresponding operator. Then the equation can be rewritten in the form

$$(-d^2/dx^2 + \alpha^2 + i\alpha\mathrm{Re}u)\psi - i\alpha\mathrm{Re}u''G\psi = i\alpha\mathrm{Re}c\psi.$$

Let L_Δ be the following operator $L_\Delta = H_\Delta + V_\Delta$, where H_Δ is the Schrödinger operator determined by the differential expression $-d^2/dx^2 + \alpha^2 + i\alpha\mathrm{Re}u$ and zero

boundary conditions at the points 0 and 1, $V_\Delta = -i\alpha \mathrm{Re} u''G$. The corresponding operators for the half-axis will be denoted as L, H, V, correspondingly. In such a way the problem is reduced to the eigenvalue problem for the operator $L(L_\Delta)$.

Remark 2. Using conventional approach to the problem of hydrodynamic stability, one obtains the Schrödinger operator with complex potential ($\mathrm{Im}\,\alpha = 0$).

Since the support of the function u'' belongs to the segment $[0, 1]$, the operators G and G_Δ are trace operators (of course, it is not a complete information about the operators, but it is sufficient for the purpose of the present paper). Hence, using results of [16, 17] one obtains the following theorem:

Theorem 4. a. *The operators L and L_Δ are closed operators.* b. *The essential spectra of the operators L and H (L_Δ and H_Δ) coincide.* c. *The eigenfunctions and associated functions of the operator L_Δ are complete.*

Remark 3. It is interesting to note that the important problem of influence of the main flow velocity profile $u(x)$ on the stability of boundary layer is now related to the influence of the potential profile ($\alpha^2 + i\alpha \mathrm{Re} u$) on the spectrum displacement.

Remark 4. The described approach allows one to use theorems concerning operator perturbations to estimate the shift of the eigenvalues and, consequently, to estimate the critical value of the Reynolds number (it is the main goal of the investigation of hydrodynamic stability).

As for the problem of completeness for the oprator L, we can use the wave operator approach for non-self-adjoint perturbation of self-adjoint operator [18]. Then we obtain

Theorem 5. a. *The absolutely continuous spectra of the operators L and H coincide.* b. *There exist complete wave operators $W_\pm(H, L_0)$ ($W_\pm(L_0, H)$), $L_0 = -d^2/dx^2$.*

4.2. Modified Orr–Sommerfeld equation

The second interesting situation occurs in the case when we take into account the transversal component of the velocity (u_1, u_2), under the supposition that these components depend on the transversal coordinate only $u_1 = u_1(x)$, $u_2 = u_2(x) \not\equiv 0$. Then one obtains [19] the modified Orr–Sommerfeld equation:

$$\Phi'''' - 2\alpha^2\Phi'' + \alpha^4\Phi = i\alpha \mathrm{Re}((u_1 - c)(\Phi'' - \alpha^2\phi) - u_i''\Phi) + \mathrm{Re}(u_2(\Phi''' - \alpha^2\Phi') - u_2''\Phi').$$

Using the same notation as above we reduce the equation to the form:

$$(-d^2/dx^2 + \alpha^2 + i\alpha \mathrm{Re} u_1)\psi - i\alpha \mathrm{Re} u_1'' G\psi + \mathrm{Re} u_2\psi' - \mathrm{Re} u_2''(G\psi)' = i\alpha \mathrm{Re} c\psi.$$

The unperturbed Schrödinger operator H^m in this case is the same as in the first one (with $u = u_1$) and the perturbation is

$$V^m\psi = i\alpha \mathrm{Re} u_1'' G\psi + \mathrm{Re} u_2\psi' - \mathrm{Re} u_2''(G\psi)'.$$

Let $M = H^m + V^m$ (and for the segment: $M_\Delta = H_\Delta^M + V_\Delta^m$). The perturbation in this case is not compact, but using the resuls of [20, 16, 17], one can prove

THEOREM 6 a. *The operator $V^m(V_\Delta^m)$ is $H^m(H_\Delta^m)$-compact.* b. *The essential spectra of the operators M and H^m (M_Δ and H_Δ^m) coincide.*

THEOREM 7. *The operator H_Δ^m has a resolvent of Hilbert–Schmidt type and its system of eigenfunctions and associated functions is complete.*

4.3. The flow between two concentric cylinders

The third case is the Orr–Sommerfeld problem for the flow between two concentric cylinders $r = a$ and $r = b$ ((r, φ, z) are the cylindrical coordinates of a point). In this situation the Tollmien–Schlichting wave has the form $\Psi(z, r, t) = \exp(i\alpha(z - ct))\Phi(r)$. Then we get [21] the following equation:

$$2i\mathrm{Re}_c\alpha\left(-(u - c)P\Phi - r\frac{d}{dr}\left(r^{-1}\frac{du}{dr}\right)\Phi\right) = P^2\Phi,$$

where the operator P in $L_2(a, b)$ is determined by the expression

$$P = -d^2/dr^2 + r^{-1} - d/dr + \alpha^2$$

and zero boundary condition, Re_c is the Reynolds number for the problem, $u = u(r)$. Using the ordinary notation $P\Phi = \psi$, $\Phi = G_c\psi$, where G_c is the Green operator P, one obtains the spectral problem for the operator L_c:

$$L_c\psi = 2i\alpha\mathrm{Re}_c c\psi, \qquad L_c = H_c + V_c,$$

$$H_c\psi = P\psi + 2i\alpha\mathrm{Re}_c u\psi, \qquad V_c\psi = -r\frac{d}{dr}\left(r^{-1}\frac{du}{dr}\right)G_c\psi.$$

The situation here is analogous to the first one and we have

THEOREM 8. *The eigenfunctions and associated functions of the operator L_c are complete.*

Acknowledgements

The author is very grateful to the organizers of the 2nd European Fluid Mechanics Conference for the possibility of making this report and to all participants who are inerested in this mathematical approaches to fluid mechanics problems. The work is partly supported by the grant No 94-2.7-1067 of the Ministry of Higher Education of Russia, Soros Foundation and ANS RF.

References

1. H. HASIMOTO and O. SANO, *Stokeslets and eddies in creeping flow*, Ann. Rev. Fluid Mech., **12**, 335–363, 1980.
2. S. ALBEVERIO, F. GESZTESY, R. HØEGH-KRON and H. HOLDEN, *Solvable models in quantum mechanics*, Springer-Verlag, New York 1988.
3. B.S. PAVLOV, *The theory of extensions and explicitly-solvable models*, Uspechi Mat. Nauk., **42**, 6, 99–131, 1987.
4. I.YU. POPOV, *The extension theory and diffraction problems*, Lect. Notes in Physics, **324**, 218–229, 1989.
5. I.YU. POPOV, *The resonator with narrow slit and the model based on the operator extensions theory*, J. Math. Phys., **33**, 11, 3794–3801, 1992.
6. YU.E. KARPESHINA and B.S. PAVLOV, *Zero-range interaction for a biharmonic and poliharmonic equations* [in Russian], Mat. Zametki., **40**, 1, 49–59, 1986.
7. I.YU. POPOV, *Operator extensions theory and eddies in creping flow*, Physica Scripta, **47**, 682–686, 1993.
8. YU.G. SHONDIN, *Quantum models in $\mathbb{R}_n$ related with the extensions of energy operator in the Pontryagin space* [in Russian], Teor. Mat. Fiz., **74**, 3, 331–344, 1988.
9. M.L. DUBREIL-JACOTIN, *Sur les theorems d'exisstence relatifs sur ondes permanentes periodiques a deux dimensions dans les liquides heterogenes*, J. Math. Pures Appl., **13**, 217–291, 1937.
10. M.SH. BIRMAN and M.Z. SOLOMYAK, *Estimates for the number of negative eigenvalues of the Schrödinger operator and its generations*, [in:] Advances in Soviet Mathematics, V. 7, M.SH. BIRMAN [Ed.], pp. 1–57, Amer. Math. Soc., 1991.
11. S.A. GABOV and M.B. TVERSKOI, *On the problem of stratified flow over obstacle*, Zh. Vich. Mat. i Mat. Fiz., **28**, 4, 608–613, 1988.
12. P. LAX and R.S. PHILLIPS, *Scattering theory*, Academic Press, New York 1967.
13. S.A. STEPIN, *On the eigenfunction expansion for continuous spectrum of the Rayleigh equation*, Funk. Anal. i Pril., **27**, 3, 87–89, 1993.
14. C.C. LIN, *The theory of hydrodynamic stability*, Cambridge University Press, Cambridge 1955.
15. H. SCHLICHTING, *Boundary layer theory*, McGraw Hill, New York 1960.
16. T. KATO, *Perturbation theory for linear operators*, Springer Verlag, Berlin 1966.
17. V.B. LIDSKY, *The completeness condition for the system of subspaces of eigenvectors and associated vectors for non-self-adjoint operators with discrete spectra*, Trudy Mosk. Mat. Ob., **8**, 82–120, 1959.
18. S.N. NABOKO, *Functional model of perturbation theory and is applications to the scattering theory*, Trudy Steklov's Mat. Inst. AN SSSR, **147**, 86–114, 1980.
19. T.H. HUGHES and W.H. REID, *On the stability of the asymptotic suction boundary layer profile*, J. Fluid Mech., **23**, 4, 715–735, 1965.
20. E. BALSLEV, *Perturbation of ordinary differential operators*, Math. Scand., **11**, 2, 131–148, 1962.
21. D.D. JOSEPH, *Stability of fluid motion*, Springer Verlag, Berlin 1971.

DEPARTMENT OF HIGHER MATHEMATICS
ST. PETERSBURG INSTITUTE OF FINE MECHANICS AND OPTICS, ST. PETERSBURG, RUSSIA.

Received October 20, 1994.

Arch. Mech., 47, 6, pp. 1057–1071, Warszawa 1995

A hydrodynamic model of unsteady tidal flow in coastal waters based on an inverse method

T.C.N. MONTEIRO (RIO DE JANEIRO)
and G.J.M. COPELAND (GLASGOW)

THIS PAPER PRESENTS a numerical method for solving the linearised shallow water equations of motion based on a variational principle. The solution found is that which lies closest to the flow field defined by measurements or other data. The process of minimising the difference between the solution and the measurements leads to the use of the Euler–Lagrange equations. A solution is presented for an unsteady, one-dimensional (x-space and time) example obtained using a finite difference formulation with the Successive Over-Relaxation (SOR) method.

Nomenclature

h [m]	vertical displacement of water from mean surface elevation,
h_0 [m]	measured or estimated vertical displacement of water from mean surface elevation used as an initial trial solution,
q [m^2s^{-1}]	rate of flow per unit width,
q_0 [m^2s^{-1}]	measured or estimated rate of flow per unit width used as an initial trial solution,
g [$9.81\,ms^{-2}$]	gravitational constant,
H [m]	total depth,
S [ms^{-1}]	celerity,
t [s]	time,
x [m]	horizontal distance, in direction of wave propagation,
w	weights, represent the relative accuracy of measured data,
λ_1 [m^3s^{-1}]	Lagrange multiplier,
λ_2 [m^3s^{-1}]	Lagrange multiplier,
C	the Chezy friction coefficient.

Note: other notations are introduced locally as convenient.

1. Introduction

THE USE OF MATHEMATICAL MODELS to predict tidal flows and effluent dispersion in coastal waters is well established, for example LIU and LEEDERTSE [1], FALCONER [2], COPELAND [3]. These models either use finite differences, FALCONER [2], or finite element methods, JANIN *et al.* [4], and are able to incorporate data only as boundary conditions or as calibration parameters. Good quality current data on tidal flows are now available from remote sensing equipment such as the Ocean Surface Current Radar System (OSCR), PRANDLE [5], MARCONI [6]. Such measurements cover extensive areas of sea providing data at points regularly distributed over a significant part of an area to be modelled. The need has arisen,

therefore, to develop models capable of incorporating large amounts of real data in such a way that the calculated flows follow very closely the measurements and so are more realistic.

LAM [7] presented the theoretical basis of a model, developed by SASAKI [8] and SHERMAN [9] from the variational principle, which had the ability to include into the flow simulation large amounts of real data. However, this model used only the steady state continuity equation to define the flow, and so took a kinematic approach to the problem which has also been described by COPELAND [3]. A complete flow representation should include both the unsteady continuity and momentum equations.

This paper describes an inverse method for a one-dimensional unsteady hydrodynamic model, with both continuity and momentum equations, to produce a solution for an unsteady flow field, using the linearised equations of motion.

Inverse methods have been widely used in the field of meteorology and oceanography. Early work performed by SASAKI [8] based on the variational analysis, defines the formalism from which the approach described here was developed. The main principle is to minimise a functional composed of the variance of the difference between the observed and analysed values and extra terms. The approach described here shows exactly how to build these extra terms.

Another method minimises the variance by including a filter in the minimisation process. HEEMINK and KLOOSTERHUIS [10] showed how to use a Kalman filter for incorporating data into a numerical shallow water flow model by using a stochastic process (Kalman filter) embedded into a dynamic model (shallow water equations). Thus a stochastic-dynamic model is combined with measurements and the agreement between the solution and the measurements is optimised in a least-squares sense.

A second method which has received several contributions in the literature has the functional composed of the variance and constraint conditions. The functional is then minimised according to a least-squares criterion using a conjugate-gradient method, NAVON and LEGLER [11]. A typical application uses the inverse methodology to simulate open ocean circulation using as constraint conditions the linearized shallow water equations, BENNETT and MCINTOSH [12].

Due to the difficulty in using inverse modelling with the full 3-D equations for tidal circulation, an important application is to smooth the values of observed parameters such as friction coefficients, topography, or eddy viscosity and then to solve the full 3-D equations for tidal circulation using the conventional methodology as finite differences, finite elements or method of characteristics, PAMCHANG and RICHARDSON [13].

The term "inverse method", see BENNETT [14], is used in this context to refer to a calculation procedure which combines measured flow data from the model area with the governing equations which describe the movement of the tidal wave in coastal waters.

2. The governing equations

The well-known equations used to predict tidal flows in coastal waters can be developed from the three-dimensional Navier–Stokes equations and the three-dimensional mass conservation equation. Because coastal waters are usually vertically well mixed, it is a common practice to integrate these equations through the vertical to produce a model formulation which is two-dimensional, FALCONER [2].

In order to demonstrate the development of the inverse method, an unsteady model in one spatial dimension has been used for this study in which the equations are also integrated over one horizontal direction to produce a cross-sectionally averaged formulation. However, in the example presented here further simplifications have been made so that the main features of the method could be investigated without undue complication. To this end the following nonlinear terms have been omitted from the governing equations: advective accelerations, turbulent diffusion of momentum and wind shear stress terms. These simplifications lead to the following continuity and momentum equations:

$$\frac{\partial \eta}{\partial t} + \frac{\partial q}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial q}{\partial t} + g(H + \eta)\frac{\partial \eta}{\partial x} + \frac{g|q|q}{(H + \eta)^2 C^2} = 0, \tag{2.2}$$

where all the notation used is defined in the list of nomenclature. In the following analysis the friction term, $[g|q|]/[(H + \eta)^2 C^2]$ is replaced by a term kf.

Note there are two nonlinear terms remaining. These nonlinear terms are simplified by the use of η^* and q^*, resulting $(H + \eta^*)[(\partial\eta)/(\partial x)]$ and $[g|q|q]/[(H + \eta)^2 C^2]$. Both q^* and η^* are obtained from adjusted values as soon as they became available in the iterations of the numerical solution.

These simplifications permit a linear mathematical treatment of the equations. However, the nonlinear behaviour caused by the terms is still present because the magnitudes of both these terms are close to those of q and η.

The Coriolis acceleration, which depends on the angular speed of rotation of the Earth and the latitude of the fluid under study, is a linear term and can be easily included in a two-dimensional model.

3. The inverse method

Unsteady tidal flow in coastal waters can be described by both direct and inverse methods. The conventional modelling method, here called the direct method, solves the tidal governing equations subject to boundary conditions which define the flow. On the other hand, the inverse method uses data, measured or estimated, at each grid point in the study domain, together with the tidal governing equations, in order to predict a tidal flow field which is in close agreement

with measurements. In this case, the solution found is that which is closest to the measurements (that is, the difference between solution and measurements is minimised), being subject to the constraints of the governing equations.

The problem consists in finding a conditional minimum and so can be solved using the variational principle. In general, this optimises the solution for a function f subject to constraint conditions g_j; SASAKI [8] describes this problem in which the constraints are satisfied exactly as strong constraints on the problem. Here, f is the variance of the deviation between the required solution and the measured or estimated data subject to the continuity and momentum equations given above which form the constraint conditions g_j.

The variational formalism used is the Lagrange Multiplier Method. This seeks a solution to the generalised function:

$$f(v_1, v_2, \ldots, v_m),$$

where variables v_l, $l = 1, m$ are $q = q(x, t)$ and $\eta = \eta(x, t)$, in this particular context, subject to constraints:

$$\begin{aligned} g_1(v_1, v_2, \ldots, v_m) &= 0, \\ g_2(v_1, v_2, \ldots, v_m) &= 0, \\ &\ldots\ldots\ldots\ldots\ldots, \\ g_k(v_1, v_2, \ldots, v_m) &= 0. \end{aligned} \tag{3.1}$$

Note that x and t are the independent variables represented, in general, by x_i, $i = 1, n$ defined in n-dimensional vector space R^n (in this context $n = 2$).

Suppose that f and g are smooth functions of m dependent variables. If $(v_{0_1}, v_{0_2}, \ldots, v_{0_m})$ provide a relative minimum for the function f under the g_j, $j = 1, k$ constraint conditions (that is, $(v_{0_1}, v_{0_2}, \ldots, v_{0_m})$ is the required solution), and if the gradient of g at the point $(v_{0_1}, v_{0_2}, \ldots, v_{0_m})$ is not the zero vector, then there are k real numbers λ_j, called Lagrange Multipliers, which form part of the definition of the functional F:

$$F = f^2 + \sum_{j=1}^{k} \lambda_j g_j \,. \tag{3.2}$$

The relative minimum for the function f under the constraints conditions g_j, is achieved when the following functional is minimised:

$$F^* = \int F \, dx_i \,. \tag{3.3}$$

That is, the functional F is integrated over the whole solution domain.

The minimisation of the function f must satisfy the Euler-Lagrange differential equation, which can be developed according to the variational principle, see BRECHTKEN-MANDERSCHEID [15].

$$\sum_{ii=1}^{m} \left(\frac{\partial F}{\partial \nu_{ii}} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial F}{\partial \dot{\nu}_{x_i}} \right) = 0, \tag{3.4}$$

where

$$\dot{\nu}_{x_i} = \frac{\partial \nu_{ii}}{\partial x_i}.$$

Then there are m Euler-Lagrange equations, plus k constraint equations, g_j, $j = 1, k$, for the determination of the Lagrange multipliers λ_j, $j = 1, \ldots, k$ and hence of the solution v_{0_l}, $l = 1, \ldots, m$.

In applying the above theoretical basis to solve an inverse problem of unsteady tidal flows in a one-dimensional model, the function f is the variance of deviation between the required solution (the adjusted variables) and the measured or estimated variables. It will be shown that the required solution is found by making adjustments to the measured or estimated variables which are determined from the Lagrange multipliers.

It is the overall variance, also known as a residual, taken over the whole model domain (all x and t in this context) which is required to be minimised. If the required solution is $q(x,t)$ and $\eta(x,t)$ and the measured or estimated flow field is represented by $q_0(x,t)$ and $\eta_0(x,t)$, then the following residual $R(q,\eta)$ is to be minimised:

$$R(q,\eta) = \int_x \int_t \left[\omega_1^2 (q - q_0)^2 + S^2 \omega_2^2 (\eta - \eta_0)^2 \right] dx\, dt. \tag{3.5}$$

Now by comparison with Eqs. (3.2) and (3.3) we see that:

$$f^2 = \left[\omega_1^2 (q - q_0)^2 + s^2 \omega_2^2 (\eta - \eta_0)^2 \right], \tag{3.6}$$

where

$$q = q(x,t), \qquad \eta = \eta(x,t), \qquad S = (g.H)^{0.5},$$

w_1 and w_2 are weights that represent the relative accuracy of measured or estimated variables in the flow field, larger values (5, say) indicate a more accurate measurement than a low value of weight (1, say).

Now by considering the constraints g_j to be the momentum and continuity equations, Eqs. (2.1) and (2.2), the functional F is rewritten according Eqs. (3.2), (3.3) and (3.6) as:

$$F^* = \int_x \int_t \left[\omega_1^2 (q - q_0)^2 + S^2 \omega_2^2 (\eta - \eta_0)^2 + \lambda_1 \left(\frac{\partial \eta}{\partial t} + \frac{\partial q}{\partial x} \right) \right. \tag{3.7}$$

$$\left. + \lambda_2 \left(\frac{\partial q}{\partial t} + g(H + \eta_0) \frac{\partial \eta}{\partial x} + kf \cdot q \right) \right] dx\, dt.$$

The Euler-Lagrange equations, Eq. (3.4), when applied to the appropriate variables v_l (which are q and η) give the following equations:

$$\frac{\partial F}{\partial q} - \frac{\partial}{\partial x}\frac{\partial F}{\partial \dot{q}_x} - \frac{\partial}{\partial t}\frac{\partial F}{\partial \dot{q}_t} = 0 \tag{3.8}$$

and

$$\frac{\partial F}{\partial \eta} - \frac{\partial}{\partial x}\frac{\partial F}{\partial \dot{\eta}_x} - \frac{\partial}{\partial t}\frac{\partial F}{\partial \dot{\eta}_t} = 0 \tag{3.9}$$

in which, from Eqs. (3.2) and (3.3), F is defined by the kernel of Eq. (3.7). Then by substituting f into the Euler-Lagrange equations (3.8) and (3.9), the following results for q and η are found:

$$\begin{aligned} q &= q_0 - \frac{kf}{2\omega_1^2}\lambda_2 + \frac{1}{2\omega_1^2}\frac{\partial \lambda_1}{\partial x} + \frac{1}{2\omega_1^2}\frac{\partial \lambda_2}{\partial t}, \\ \eta &= \eta_0 - \frac{g}{2S^2\omega_2^2}\frac{\partial}{\partial x}(\lambda_2(H+\eta_0)) + \frac{1}{2S^2\omega_2^2}\frac{\partial \lambda_1}{\partial t}. \end{aligned} \tag{3.10}$$

The above equations produce adjusted values to both q and η using measured or estimated values q_0 and η_0 also using gradients of the Lagrange Multipliers. The adjusted values are then substituted into the constraint equations (2.1) and (2.2), resulting in two elliptical equations. The first, Eq. (3.11), is the continuity condition:

$$\begin{aligned} &\frac{1}{2\omega_2^2 S^2}\frac{\partial^2 \lambda_1}{\partial t^2} + \frac{\partial}{\partial x}\left(\frac{1}{2\omega_1^2}\frac{\partial \lambda_1}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{kf\cdot\lambda_2}{2\omega_1^2}\right) + \frac{\partial}{\partial x}\left(\frac{1}{2\omega_1^2}\frac{\partial \lambda_2}{\partial t}\right) \\ &\qquad + \frac{g}{2\omega_2^2 S^2}\frac{\partial}{\partial t}\frac{\partial}{\partial x}(\lambda_2(H+\eta_0)) + \left[\frac{\partial \eta_0}{\partial t} + \frac{\partial q_0}{\partial x}\right] = 0, \end{aligned} \tag{3.11}$$

and the second, Eq. (3.12), from the momentum condition:

$$\begin{aligned} &\frac{1}{2\omega_1^2}\frac{\partial^2 \lambda_2}{\partial t^2} + g^2(H+\eta_0)\frac{\partial}{\partial x}\left[\frac{1}{2\omega_2^2 s^2}\frac{\partial}{\partial x}(\lambda_2(H+\eta_0))\right] + \frac{kf}{2\omega_1^2}\frac{\partial \lambda_2}{\partial t} \\ &\quad - \frac{1}{2\omega_1^2}\frac{\partial kf\cdot\lambda_2}{\partial t} - \frac{kf^2\lambda_2}{2\omega_1^2} + \frac{1}{2\omega_1^2}\frac{\partial^2 \lambda_1}{\partial t \partial x} + g(H+\eta_0)\frac{\partial}{\partial x}\left[\frac{1}{2\omega_2^2 S^2}\frac{\partial \lambda_1}{\partial t}\right] \\ &\qquad + \left[\frac{\partial q_0}{\partial t} + g(H+\eta_0)\frac{\partial \eta_0}{\partial x} + kf\cdot q_0\right] = 0. \end{aligned} \tag{3.12}$$

A dimensional analysis of Eqs. (3.11) and (3.12) shows that the dimensions of λ_1 and λ_2 are L^3T^{-1} and L^2, respectively.

The problem now reduces to that of finding solutions for of λ_1 and λ_2 at each point in the model domain and then of using them to find values for the "adjustments" to q_0 and η_0 defined in Eqs. (3.10) such that the required solutions for q and η can be determined. A numerical solution for both of λ is given in the next section.

4. Numerical scheme

The elliptical equations (3.11) and (3.12) define a boundary value problem, for which stable solutions are relatively easy to achieve without reference to stability parameters such as the well-known Courant condition.

The equations are solved numerically for both of λ_1 and λ_2, subject to either Dirichlet or Neumann boundary conditions. When λ is zero on a boundary, which is the Dirichlet condition, the variables q and η are adjusted by a term which is a function of the normal derivative of λ, see Eqs. (3.10), and so the flow variables are free to "adjust". However, when there is a Neumann condition, that is the normal derivative of λ is zero, no adjustment is made to the measured variables. Thus, the Neumann condition is appropriate to closed boundaries such as the shoreline where the normal flow must remain zero, or to particular situations when flow data are specified and is required to be used without adjustment such as a river discharge or tidal elevation.

The derivatives in the unsteady elliptical equations (3.11) and (3.12), are approximated by finite differences according to the staggering procedure defined by Fig. 1. The space step in the x-direction is Δx and the time step is Δt; the indices i and j denote grid positions in the x and t coordinates, respectively. A long channel represents the one-dimensional model investigated in this study. This allows a complete time evaluation of the tidal flow over a full period at each x-position.

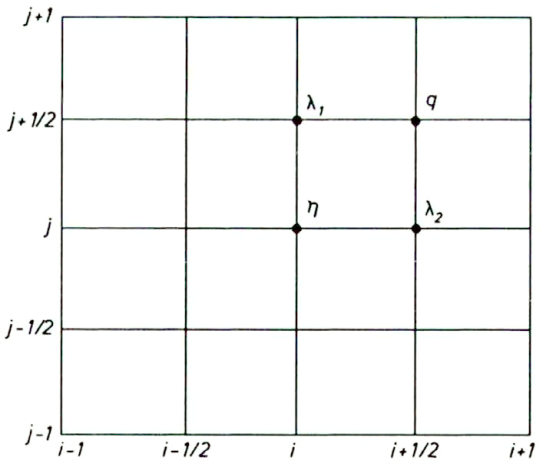


FIG. 1. The staggering procedure.

The mean water depth H, water surface elevation η, weight coefficients w_1 and w_2, and celerity S are located at the centre of each cell with index i, j. The flux q in the x-direction is located at $i+1/2$, $j+1/2$ and the Lagrange multipliers λ_1 and λ_2 are located at i, $j+1/2$ and $i+1/2$, j, respectively. The derivatives from the partial differential equations are replaced by difference quotients as in

the following example of a second-derivative term:

$$\frac{\partial^2 \lambda}{\partial x^2} = \frac{(\lambda(i+1,j) - 2\lambda(i,j) + \lambda(i-1,j))}{\Delta x^2}. \tag{4.1}$$

The result from this discretization procedure of Eqs. (3.11) and (3.12) is a system of $2 \times nx \times nt$ linear algebraic equations for $2 \times nx \times nt$ unknowns, where nx and nt are respectively the number of grid cells in the x-direction and number of time steps in the $x - t$ domain. Their solution is not a trivial procedure and is achieved by numerical computation. Writing the differential form of Eqs. (3.11) and (3.12) in their matrix form, we obtain

$$A\lambda - f = 0, \tag{4.2}$$

where λ is a column vector of the Lagrange Multipliers with components $\lambda 1_{1,1}$, $\lambda 1_{2,1}$,..., $\lambda 1_{nx,nt}$, $\lambda 2_{1,1}$, $\lambda 2_{2,1}$, ..., $\lambda 2_{nx,nt}$. The column vector f with each component being the residual of continuity and momentum equations the first $nx \times nt$ components ($f1_k$) are residuals of continuity equation, Eq. (4.3), followed by $nx \times nt$ components ($f2_k$) which are residuals of the momentum equation, Eq. (4.4). The sparse matrix A is composed of coefficients which are functions of known parameters for each grid, these do not change as the solution progresses, such as: weights, celerity, depth and gravitational constant.

The definitions of components f_k of the column vector f are:

$$f1_k = \left[\frac{\partial \eta_0}{\partial t} + \frac{\partial q_0}{\partial x}\right], \qquad k = 1 \text{ to } nx \cdot nt, \tag{4.3}$$

$$f2_k = \left[\frac{\partial q_0}{\partial t} + g(H + \eta_0)\frac{\partial \eta_0}{\partial x} + kf \cdot q_0\right], \qquad k = nx \cdot nt + 1 \text{ to } 2 \cdot nx \cdot nt. \tag{4.4}$$

The solution of the matrix equation (4.2) was obtained using a Successive Over-Relaxation (SOR) methods, PRESS *et al.* [16]. As an iterative process, the first approximation of column vector λ, Eq. (4.2), was used to calculate the second approximation, and so on. The process continued until the desired convergence was reached. The set of residuals $[A\lambda - f]_k$ each approach zero to within a specified tolerance.

The process in this way produces after each iteration a new value to both q and η, which tend to values that are correct to within a specified accuracy. Based on these successive iterations the approximations to the nonlinear terms of Eq. (2.2) using q^* and η^* are rewritten progressively. The frequency of updating q^* and η^* was reduced as the iterations progressed to ensure monotonic reduction in the residual. Then when good convergence is reached, both q^* and η^* are close enough to q and η to allow the nonlinear effect to be well represented.

The final step in the calculation used difference forms of Eqs. (3.10) to adjust the measured flow field to give the required solution. An example of the results from such a calculation for a tidal wave in a long one-dimensional channel is given in the next section.

5. Application

In order to demonstrate the validity of the inverse method, it was applied to calculate unsteady tidal flow in a one-dimensional channel as described by Eqs. (2.1) and (2.2). This enabled us to show the development of a numerical solution for λ_1 and λ_2, and hence the development of the quantitative improvement to the initial estimated solution by finding the adjusted values q and η.

Real data were not used as the initial trial solution. Instead, a simplified analytical solution to both the continuity equation and the linearised momentum equation was used. That is, the solution was used for cases when $\eta \ll H$ and the friction effects were neglected reducing Eq. (2.2) to

$$\frac{\partial q}{\partial t} + gH\frac{\partial \eta}{\partial x} = 0. \tag{5.1}$$

If the trial solution for the discharge per unit width is given by $q_0 = v_{\text{amp}} H \cos(kx - \sigma t)$, then Eq. (5.1) has a solution $\eta_0 = q_0/S$, where $k = 2\pi/L$, $\sigma = 2\pi/T$, v_{amp} is the amplitude of the tidal current and $S = (gH)^{0.5}$ the wave speed. Clearly, when η is not negligible compared with H and the friction effect can not be dismissed, an initial solution based on the linearised equations will not be a good solution to the nonlinear equations (2.1) and (2.2) which are incorporated into the model. In these circumstances, the inverse method will find the solution to the nonlinear equations which is close to the linear solution q_0 and η_0.

Demonstrating the method in this way has the advantage of offering a clear appreciation of the errors in the initial solution and of the improvements made.

The model represented a one-dimensional channel of length 500 km and parabolic bottom profile, subject a progressive tidal wave of period 12.5 hours (45000 s). The domain $(x - t)$ was represented in the model using a mesh of 50×25 cells, with $\Delta x = 10000$ m and $\Delta t = 1800$ s.

A simulation was performed with a parabolic bed profile for which the depth varied from 108.40 m to 15.70 m. The amplitude of the flow was 162.54 m^2/s. The wave number ($k = 2\pi/L$) used in the trial solution (estimated data) was computed at each position in order to consider the shoaling effect, and so in that simulation the wavelength was significantly reduced in the shallow end of the channel, however the trial solution did not include amplitude shoaling.

In order to make a good estimate of the artifical "measured data" which was required to be included in the input data set, a pre-simulation was performed. At two locations over the channel, four points assumed to represent "measured data" were taken from the pre-simulation output. The full input data set composed data from these two locations embedded in the linear solution. All points with data which were estimated using the linear solution were given a weight equal to one, and the remaining points composed of "measured data" from the pre-simulation received weights equal to five.

Figure 2 and Fig. 3 shows respectively residuals of the continuity equation and of the momentum equation over the channel length. Three residuals were computed over the whole domain $x-t$. The sum of the absolute values of residual over the time period at each spatial location was plotted against distance along the longitudinal channel.

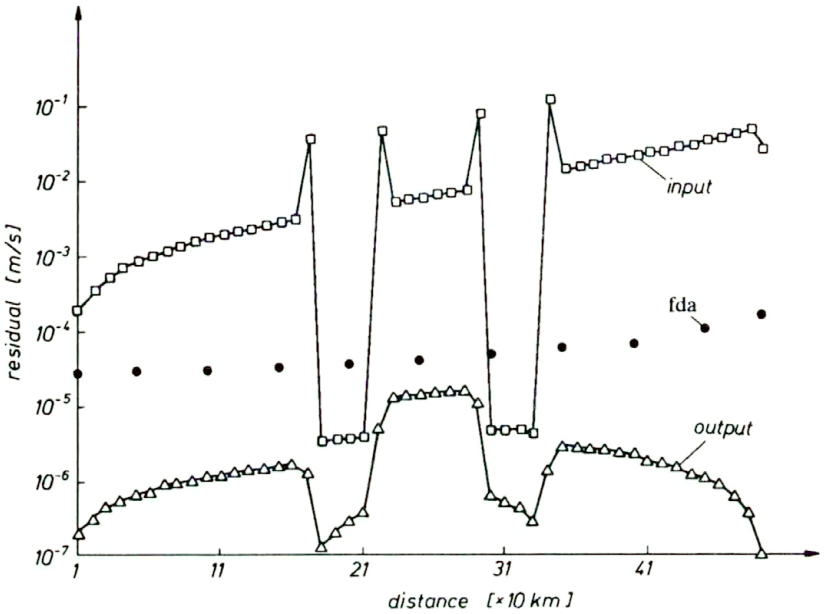


FIG. 2. Residuals from the continuity equation along the length of the channel. Each residual is the sum of 24 absolute residuals over a period of 12.5 hours.

The first residual in each labelled "input" refers to the residual of Eq. (2.1) or Eq. (2.2) using data from the initial input data set.

The second residual named "fda", was obtained from the analysis of the *finite difference approximation* (fda) by applying the Taylor series to the linear equations (2.1) and (5.1), for a channel of linearly varying depth, as developed by IPPEN [17]. According to the staggering procedure, Fig. 1, the numerical approximation has a second-order truncation error. These errors for Eq. (2.1) and Eq. (5.1), respectively, (continuity equation and momentum equation) are as follows:

$$\frac{\Delta t^2}{24}\frac{\partial^3 \eta_0}{\partial t^3} + \frac{\Delta x^2}{24}\frac{\partial^3 q_0}{\partial x^3} = \text{fda} \quad \text{(continuity)}, \tag{5.2}$$

$$\frac{\Delta t^2}{24}\frac{\partial^3 q_0}{\partial t^3} + g \cdot H \frac{\Delta x^2}{24}\frac{\partial^3 \eta_0}{\partial x^3} = \text{fda} \quad \text{(momentum)}. \tag{5.3}$$

The fda's were evaluated using values of η_0 and q_0 obtained using computer algebra for a solution over the parabolic depth profile, allowing for shoaling effects in amplitude and wavelength.

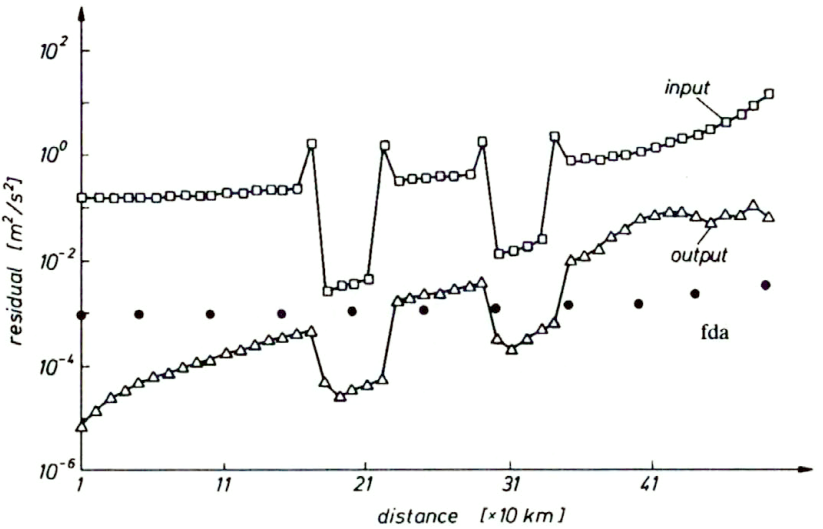


FIG. 3. Residuals from the momentum equation along the length of the channel. Each residual is the sum of 24 absolute residuals over a period of 12.5 hours.

The final residuals plotted, labelled "output" in both Figs. 2 and 3, were the actual residual of Eqs. (2.1) and (2.2) after application of the inverse method.

In the present work the fda's were used as reference values for the residual analysis. The fda residuals, Fig. 2 and Fig. 3, slightly increase when the depth is reduced, as a consequence of the reduction of the wavelength, i.e. as the ratio $\Delta x/L$ increases, but the ratio $\Delta t/L$ is greater than $\Delta x/L$ so the fda in the time derivative is greater than spatial derivative. Therefore the major contribution to the total fda is from the spatial derivatives.

The residual from the trial solution is relatively large because the trial solution did not allow for any amplitude shoaling nor any nonlinear effects. The theoretical value of the fda represents the magnitude of the residual below which improvements achieved by the inverse method are not actually justified. However, in the inverse methodology only one convergence criterion was used over the whole domain $(x-t)$. Consequently, in some areas where the nonlinearity is less important, the final residuals produced by the optimisation process were smaller than the fda residuals (especially in the continuity equation) in order to satisfy the accuracy requirements at the critical areas (shallow end with the momentum equation). This, in effect, is modifying the solution to comply with the numerical accuracy requirements.

The final residuals obtained from adjusted data following the inverse methodology were reduced over the whole channel even in the shallow areas where the nonlinear effects are more important. The final residuals in the shallow area were reduced by two orders of magnitude, showing that the simplification in the nonlinear terms of Eq. (2.2) was successful.

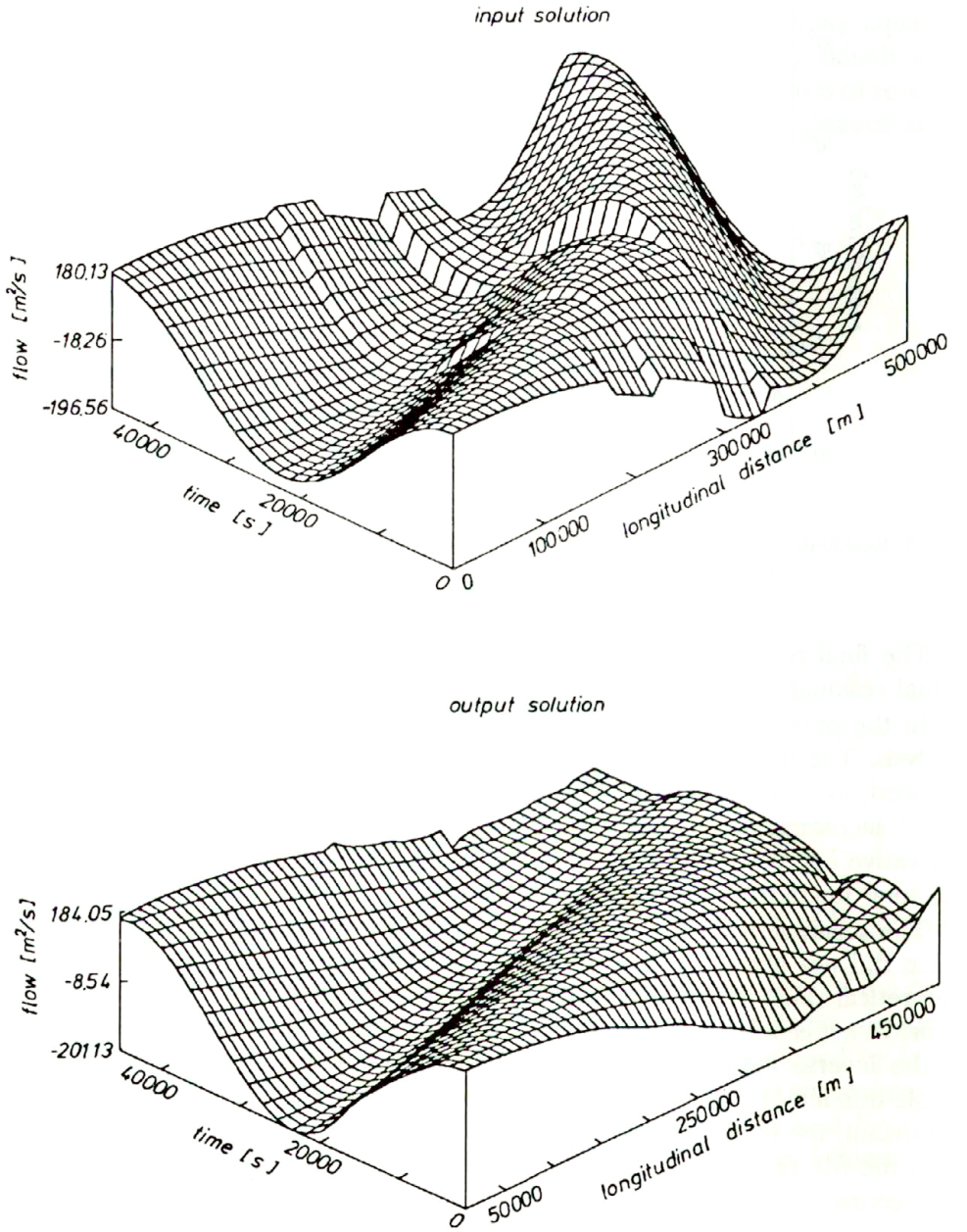


FIG. 4. Flow over domain $(x - t)$ showing the input and output solution.

Figure 4 and Fig. 5 show the input data (full initial data set) and output solution to the inverse method in terms of discharges and elevations over the domain $(x - t)$, respectively. As the final solution was composed of both "measurements"

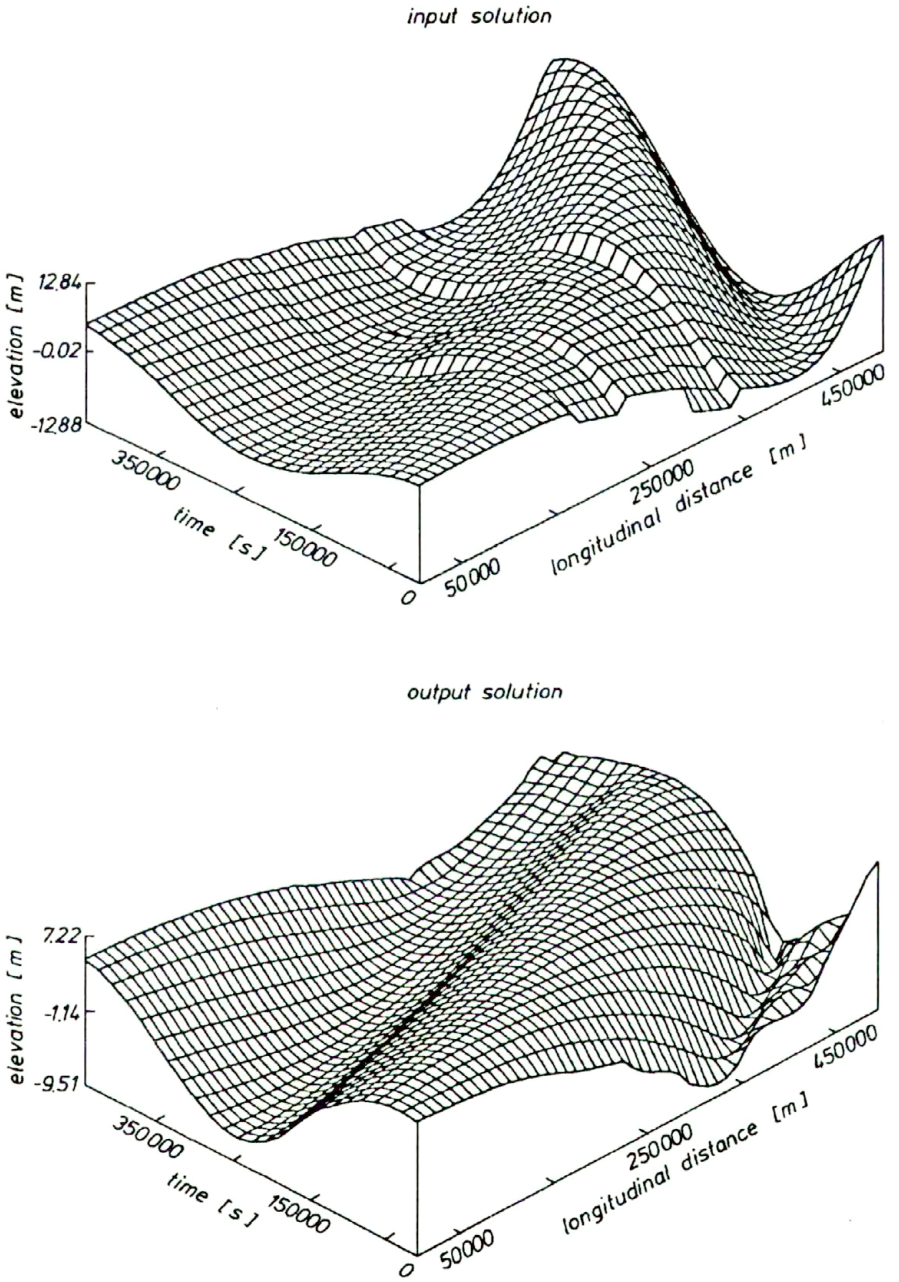


FIG. 5. Elevation over domain $(x - t)$ showing the input and output solution.

and estimated data, the graphs of both the flow and elevation inputs show discontinuities. However, the model was able to produce a smooth final solution except for some minor effects on the boundaries.

6. Conclusions

It is suggested that a new modelling method is required to predict tidal flows in coastal seas which makes maximum use of the measured data obtained from within the model area. The term "inverse modelling" is used to describe this approach in which the flow calculations are controlled by data points within the model domain rather than by data along the boundaries only as in conventional models. The increase in availability of flow data from remote sensing equipment with a wide spatial coverage is seen as driving the need for inverse models. This paper describes a method of this kind, based on the variational principle, in which a solution to the governing continuity and momentum equations is found which lies closest to the initial estimated solution determined by measurements or by other data.

The problem is analysed for a one-dimensional long wave motion entering a shallow area according to a parabolic bed profile. The simplified nonlinear terms in the momentum equation, Eq. (2.2), had the ability to simulate properly the strong linearity effect at the shallow areas.

A solution method is presented based on a finite difference scheme in which a S.O.R method resulted in a stable and asymptotic rate of convergence even when the nonlinearity becomes stronger in the case of a large amplitude motion in shallow water.

In order to demonstrate the effectiveness of the method, the case of a tidal wave in a 500 km channel was studied. The trial solution was composed of artificial "measurements" and data estimated from the analytical solution to the completely linear problem neglecting the damping in shallow areas. As the flow was strongly affected by the nonlinear terms in the shallow areas, this trial solution became increasingly less satisfactory in shallow water and so the inverse method was required to compute larger corrections.

The results from the demonstration of the model were assessed in terms of the residuals obtained when both the trial solution and the corrected solution were substituted into the momentum equation. The residual error was greatly reduced to the levels of the finite difference error well within the tolerances required for engineering studies of tidal flows.

The results presented show that the method is valid and so it supports further development to models with two spatial dimensions and with the extra nonlinear term of convective accelerations.

References

1. S.K. LIU and J.J. LEENDERTSE, *Multidimensional modelling of estuaries and coastal seas*, Adv. Hydroscience, Ed. Chon., **11**, 95–164, 1978.

2. R.A. FALCONER and Y. CHEN, *Modelling flow and pollutant transport from sea outfalls using higher order finite difference schemes*, University of Bradford – Dep. Civil Engineering, September 1992.

3. GRAHAM J.M. COPELAND, *An inverse method of kinetic flow modelling based on measured currents*, Proc. Inst. Civ. Engrs Wat., Marit. and Energy, 106, Sept., 249–258, 1994.
4. J.M. JANIN, LEPEINTRE and Ph. PECHON, *Three-dimensional modelling of a free-surface stratified flow and of a thermal discharge*, Proc. of the 5th International Symp. "Refined Flow Modelling and Turbulence Measurements", pp. 679–686, Paris, 7-10 September 1993.
5. D. PRANDLE, *A new view of near-shore dynamics based on observations from HF*, Radar. Prog. Oceanog., **25**, pp. 403–439, 1991.
6. *Marconi Radar Systems Ltd., Ocean Surface Current Radar (OSCRII)*, Rittle Rd. Chelnsford, CM1 3BN, UK.
7. D.C.L LAM, C.R. MURTHY and R.B. SIMPSON, *Lecture Notes on Coastal and Estuarine Studies*, Vol. 5, Springer-Verlag, 1984.
8. Y. SASAKI, *Some basic formalism in numerical variational analysis*, Monthly Weather Review, **98**, 12, pp. 875–883, 1970.
9. C.A. SHERMAN, *A mass-consistent model for wind fields over complex terrain*, L.L. Laboratory, University of California – UCRL-76171, 1975.
10. A.W. HEEMINK and H. KLOOSTERHUIS, *Data assimilation for nonlinear tidal models*, Inter. J. for Numerical Methods in Fluids, **11**, 1097–1112, 1990.
11. I.M. NAVON and D.M. LEGLER, *Conjugate-gradient for large-scale minimization in meteorology*, Monthly Weather Review, **115**, 1479–1502, 1987.
12. A.F. BENNETT and P.C. MCINTOSH, *Open ocean modelling as an inverse problem. Tidal theory*, J. Physical Oceanography, **12**, 1004–1018, 1982.
13. V.G. PANCHANG and J.E. RICHARDSON, J. Hydraulic Engng., **119**, 4, 505–524, April 1993.
14. ANDREW F. BENNETT, *Inverse methods in physical oceanography*, Cambridge Monographs on Mechanics and Applied Mathematics, 1992.
15. U. BRECHTKEN-MANDERSCHEID, *Introduction to the calculus of variations*, Chapman and Hall Mathematics.
16. W.H. PRESS *et al.*, *Numerical recipes in FORTRAN: the art of scientific computing*, 2nd ed., Cambridge University Press, 1992.
17. A.T. IPPEN, *Estuary and coastline hydrodynamics*, McGraw-Hill Book Company, Inc., 1966.

DEPARTMENT OF ENVIRONMENTAL HEALTH
PUBLIC HEALTH NATIONAL SCHOOL, RIO DE JANEIRO, BRAZIL
and
DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF STRATHCLYDE, GLASGOW, SCOTLAND, U.K.

Received October 31, 1994.

Arch. Mech., 47, 6, pp. 1073–1088, Warszawa 1995

Comparison between strictly conical and paraboloidal flow fields in the presence of angular momentum (*)

R. PUZYREWSKI and K. NAMIEŚNIK (GDAŃSK)

NONVISCID, compressible, stationary model of the flow along conical and paraboloidal stream surfaces is considered. The aim of the work is to create such a pressure field which is able to hold the assumed shape of stream surfaces in the presence of angular momentum. The flow is described in the nonorthogonal curvilinear system of coordinates based on the assumed stream surfaces. Due to this specific coordinate system, the flow can be treated as quasi-twodimensional. Only the velocity components tangential to the conical or paraboloidal surfaces are the non-zero components. This system of conservation equations can be reduced to the hyperbolic type. There exist two families of characteristics. The first family is a family of constant values of the mass flow function and total energy function. Along the second family of characteristics, the pressure gradient is in equilibrium with accelerations of three-dimensionally curved trajectories of fluid elements. The existence of characteristics is the basis for an algorithm of solving of the whole flow field in a given axisymmetrical space domain. The examples of computation of the flow field are shown in the paper. The regions of nonexisting solutions have been discovered in some cases. The obtained solution may be applied to the design of the turbomachinery vanes.

1. Introduction

THE STARTING POINT for the problem discussed in the paper is the layout of the flow field in turbomachinery stages. The approach presented here can be classified among the inverse methods when no blading exists, but desirable stream lines in a given meridional cross-section of the stage are expected. Then the blading should be determined by stream lines given *a priori*. The main physical assumptions of the presented inverse model are widely recognized in the literature [1, 2]. They are commonly accepted in the field of turbomachinery. The present paper is a continuation of the proposed general method of solution [3] and follows the works presented already in [4, 5, 6].

2. Description of the model

2.1. Geometry of the flow

Let us consider the geometry as it is shown in Fig. 1 a and 1 b.

Figure 1 a shows the family of cones which are focussed at the circle r_w in the plane $z = 0$. Figure 1 b presents the family of paraboloidal surfaces converging also in the circle r_w in the plane $z = 0$.

(*) Presented at the 2-nd European Fluid Mechanics Conference, Warsaw, 20–24 Sept., 1994.

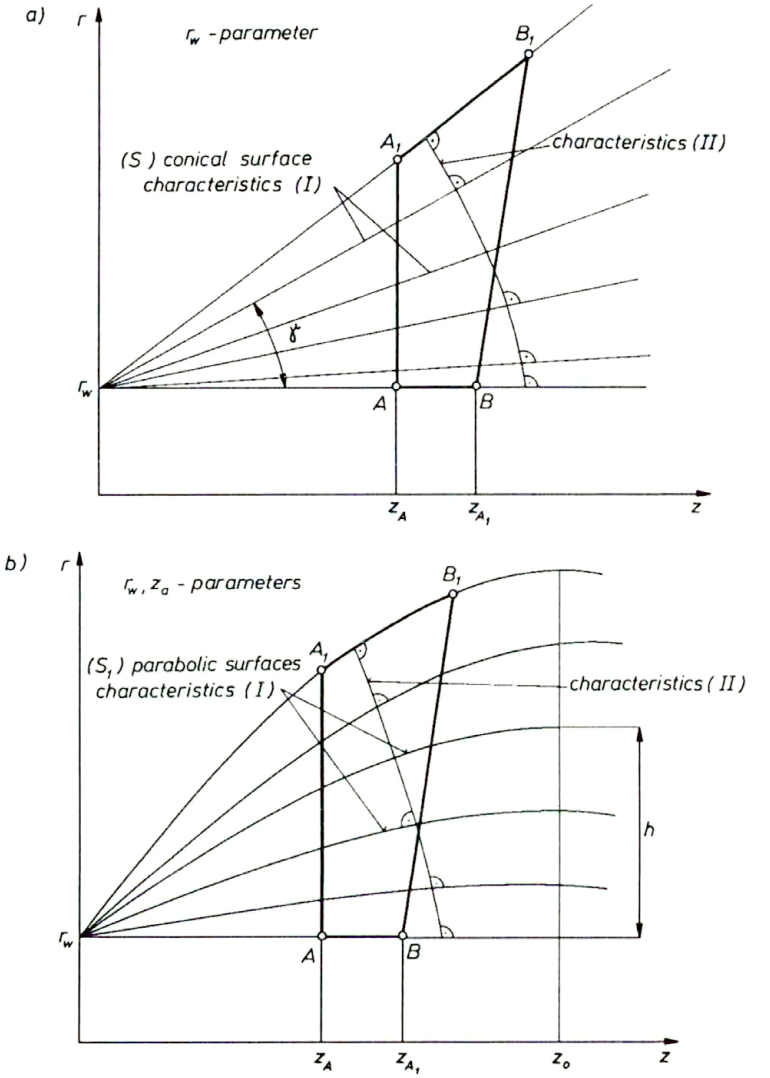


FIG. 1. Meridional cross-section of conical a) and paraboloidal b) stream surfaces.

In the case of Fig. 1 a the coordinates $q = \operatorname{tg}\gamma$ and z describe the cones according to the formula

$$r = r_w + qz\,. \tag{2.1}$$

For the case 1 b we have coordinates h and z which determine the paraboloidal surface in the form

$$r = r_w + \frac{hz}{z_0}\left(2 - \frac{z}{z_0}\right). \tag{2.2}$$

Here, besides r_w, we have the second parameter z_0.

In the sequel, the conical (2.1) and paraboloidal surfaces (2.2) will be treated as stream surfaces of type S1. The aim of the considerations is to determine the flow parameters on these surfaces.

The domain of the considerations will be restricted to the region between points ABB_1A_1. This region is supposed to be occupied by a stationary blading (nozzles) of a designed stage.

The following notations will be used for the non-orthogonal system of coordinates based on surfaces (2.1) or (2.2).

Coordinates:

$$x^{(1)} = q \quad \text{or} \quad h, \text{ as in Fig. 1 a and 1 b,}$$
$$x^{(2)} = \varphi, \quad \text{along the circumference,}$$
$$x^{(3)} = z, \quad \text{as in Fig. 1a and 1 b.}$$

Nonzero velocity components in the system of coordinates are

$$U_{x^{(2)}}, U_{x^{(3)}},$$

whereas $U_{x^{(1)}} \equiv 0$ follows from the definition of stream surfaces of S1-type for $x^{(1)} = \text{const}$.

2.2. Physical assumptions

The flow along surfaces $x^{(1)} = \text{const}$ will be considered under the following assumptions:

stationary flow

$$\frac{\partial}{\partial t} = 0,$$

axisymmetrical flow

$$\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial x^{(2)}} = 0,$$

compressible

$$\varrho = \text{var},$$

no viscosity, hence the flow is nondissipative.

The influence of the unknown blading on the flow field will be simulated in the mass conservation equation and in the momentum conservation equation. The assumption of lack of dissipation implies the energy conservation equation such as for ideal flow without energy subtraction, because we consider the domain of the nozzle.

In the mass conservation equation the blading will be taken into account by means of blockage coefficient, which subtracts from the full circumference a certain fraction $2\pi(1 - \tau(x^{(1)}, x^{(3)}))$.

In the momentum conservation equation, the influence of blading will be expressed by the force components as

$$F_{x^{(1)}} \equiv 0, \qquad F_{x^{(2)}} \neq 0, \qquad F_{x^{(3)}} \neq 0.$$

This means that the blades influence the flow by means of body forces only in the stream surfaces $x^{(1)} = \text{const}$, but not in the directions perpendicular to them.

2.3. Conservation equations

The conservation equations will be presented in the non-orthogonal system of coordinates $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, parallel for conical and paraboloidal flows.

Mass conservation equation

For a conical flow

$$\varrho(r_w + x^{(1)}x^{(3)})x^{(3)}U_{x^{(3)}}(1 - \tau(x^{(1)}, x^{(3)})) = m(x^{(1)}), \tag{2.3}$$

for a paraboloidal flow

$$\varrho \frac{\left(r_w + \frac{x^{(1)}x^{(3)}}{z_0}\left(2 - \frac{x^{(3)}}{z_0}\right)\right)\frac{x^{(3)}}{z_0}\left(2 - \frac{x^{(3)}}{z_0}\right)}{\sqrt{1 + \left(\frac{2x^{(1)}}{z_0}\left(1 - \frac{x^{(3)}}{z_0}\right)\right)^2}} U_{x^{(3)}}\left(1 - \tau(x^{(1)}, x^{(3)})\right) = m(x^{(1)}). \tag{2.4}$$

It is worth noting that the equations have one term on the left-hand side and mass flow function on the right-hand side which depend only on $x^{(1)}$ coordinate.

Momentum conservation equation

For conical flow

$$-\frac{\varrho U^2_{x^{(2)}}}{r_w + x^{(1)}x^{(3)}} = -\frac{1 + x^{(1)2}}{x^{(3)}}\frac{\partial p}{\partial x^{(1)}} + x^{(1)}\frac{\partial p}{\partial x^{(3)}}, \tag{2.5}$$

$$\frac{U_{x^{(3)}}}{\sqrt{1 + x^{(1)2}}}\left(\frac{x^{(1)}}{r_w + x^{(1)}x^{(3)}}U_{x^{(2)}} + \frac{\partial U_{x^{(2)}}}{\partial x^{(3)}}\right) = F_{x^{(2)}}, \tag{2.6}$$

$$\varrho U_{x^{(3)}}\frac{\partial U_{x^{(3)}}}{\partial x^{(3)}} = \varrho F_{x^{(3)}}\sqrt{1 + x^{(1)2}} - (1 + x^{(1)2})\frac{\partial p}{\partial x^{(3)}} + \frac{x^{(1)}(1 + x^{(1)2})}{x^{(3)}}\frac{\partial p}{\partial x^{(1)}}, \tag{2.7}$$

for paraboloidal flow

$$(2.8)\qquad -\frac{\varrho U_{x^{(2)}}^2}{r_w+\dfrac{x^{(1)}x^{(3)}}{z_0}\left(2-\dfrac{x^{(3)}}{z_0}\right)}-\frac{\varrho U_{x^{(3)}}^2\dfrac{2x^{(1)}}{z_0^2}}{1+\left(2\dfrac{x^{(1)}}{z_0}\left(1-\dfrac{x^{(3)}}{z_0}\right)\right)^2}$$
$$=-\frac{1+\left(\dfrac{2x^{(1)}}{z_0}\left(1-\dfrac{x^{(3)}}{z_0}\right)\right)^2}{\dfrac{x^{(3)}}{z_0}\left(2-\dfrac{x^{(3)}}{z_0}\right)}\frac{\partial p}{\partial x^{(1)}}+\frac{2x^{(1)}}{z_0}\left(1-\frac{x^{(3)}}{z_0}\right)\frac{\partial p}{\partial x^{(3)}},$$

$$(2.9)\qquad \frac{\varrho U_{x^{(3)}}}{\sqrt{1+\left(\dfrac{2x^{(1)}}{z_0}\left(1-\dfrac{x^{(3)}}{z_0}\right)\right)^2}}$$
$$\times\left(\frac{\partial U_{x^{(2)}}}{\partial x^{(3)}}+\frac{\dfrac{2x^{(1)}}{z_0}\left(1-\dfrac{x^{(3)}}{z_0}\right)U_{x^{(2)}}}{r_w+\dfrac{x^{(1)}x^{(3)}}{z_0}\left(2-\dfrac{x^{(3)}}{z_0}\right)}\right)=\varrho F_{x^{(2)}},$$

$$(2.10)\qquad \frac{\varrho U_{x^{(3)}}}{\sqrt{1+\left(\dfrac{2x^{(1)}}{z_0}\left(1-\dfrac{x^{(3)}}{z_0}\right)\right)^2}}\left(\frac{\partial U_{x^{(3)}}}{\partial x^{(3)}}+\frac{\dfrac{4x^{(1)2}}{z_0^3}\left(1-\dfrac{x^{(3)}}{z_0}\right)U_{x^{(3)}}}{1+\left(\dfrac{2x^{(1)}}{z_0}\left(1-\dfrac{x^{(3)}}{z_0}\right)\right)}\right)$$
$$+\varrho F_{x^{(3)}}+\sqrt{1+\left(\frac{2x^{(1)}}{z_0}\left(1-\frac{x^{(3)}}{z_0}\right)\right)^2}\left(\frac{\dfrac{2x^{(1)}}{z_0}\left(1-\dfrac{x^{(3)}}{z_0}\right)}{\dfrac{x^{(3)}}{z_0}\left(2-\dfrac{x^{(3)}}{z_0}\right)}\frac{\partial p}{\partial x^{(1)}}-\frac{\partial p}{\partial x^{(3)}}\right).$$

Energy conservation equation

$$(2.11)\qquad \frac{1}{2}\left(U_{x^{(2)}}^2+U_{x^{(3)}}^2\right)+\frac{k}{k-1}\frac{p}{\varrho}=h(x^{(1)}).$$

Isentropic condition

$$(2.12)\qquad \frac{p}{\varrho^k}=c(x^{(1)}).$$

The above system of equations is closed with respect to

$$\varrho,\ U_{x^{(2)}},\ U_{x^{(3)}},\ F_{x^{(2)}},\ F_{x^{(3)}},\ p.$$

3. Method of solution

It is not difficult to note that the system of above equations can be reduced to a hyperbolic one [5].

Two families of real characteristics can be found for the system.

The first family is

$$x^{(1)} = \text{const},$$

or simply stream surfaces of (1) or (2)-type (S1). Along them $m(x^{(1)})$, the mass flow rate functions (2.3) and (2.4) are constant. Also $h(x^{(1)})$, the energy function (2.11), is constant.

The second family for a conical flow is

$$x^{(3)}\sqrt{1 + x^{(1)_2}} = \text{const}, \tag{3.1}$$

and for a paraboloidal flow is

$$x^{(1)} - \frac{z_0^3}{x^{(3)}(2z_0 - x^{(3)})} \times \sqrt{\frac{1}{2}\left(\left(\frac{z'}{z_0}\right)^2 - \left(\frac{x^{(3)}}{z_0}\right)^2\right) - \left(\left(\frac{z'}{z_0}\right) - \left(\frac{x^{(3)}}{z_0}\right)\right) - \ln\frac{z_0 - z'}{z_0 - x^{(3)}}} = \text{const}. \tag{3.2}$$

where $z' = x^{(3)}$ at $x^{(1)} = 0$.

Along the characteristics (3.1), the ordinary differential equation of the form

$$\frac{dp}{dx^{(1)}} = \frac{\varrho U_{x^{(2)}}^2 x^{(3)}}{(r_w + x^{(1)}x^{(3)})(1 + x^{(1)_2})} \tag{3.3}$$

has to be satisfied.

For the characteristics (3.2), the equation reads:

$$\frac{dp}{dx^{(1)}} = \frac{\varrho U_{x^{(2)}}^2}{r_w + \frac{x^{(1)}x^{(3)}}{z_0}\left(2 - \frac{x^{(3)}}{z_0}\right)} \frac{\frac{x^{(3)}}{z_0}\left(2 - \frac{x^{(3)}}{z_0}\right)}{1 + \left(\frac{2x^{(1)}}{z_0}\left(1 - \frac{x^{(3)}}{z_0}\right)\right)^2} + \frac{\varrho U_{x^{(3)}}^2 \frac{2x^{(1)}}{z_0^2}\frac{x^{(3)}}{z_0}\left(2 - \frac{x^{(3)}}{z_0}\right)}{\left(1 + \left(\frac{2x^{(1)}}{z_0}\left(1 - \frac{x^{(3)}}{z_0}\right)\right)^2\right)^2}. \tag{3.4}$$

The characteristics (3.1) and 3.2) are perpendicular to $x^1 = \text{const}$. Along them the presssure gradient is balanced by the centrifugal force which is expressed by the Eqs. (3.3) and (3.4).

The method of solution relies on solving, by means of the Runge-Kutta method, the differential equation (3.3) or (3.4) along the characteristics (3.1) or (3.2), starting from the lines A_1B_1 and AA_1 shown in Fig. 2.

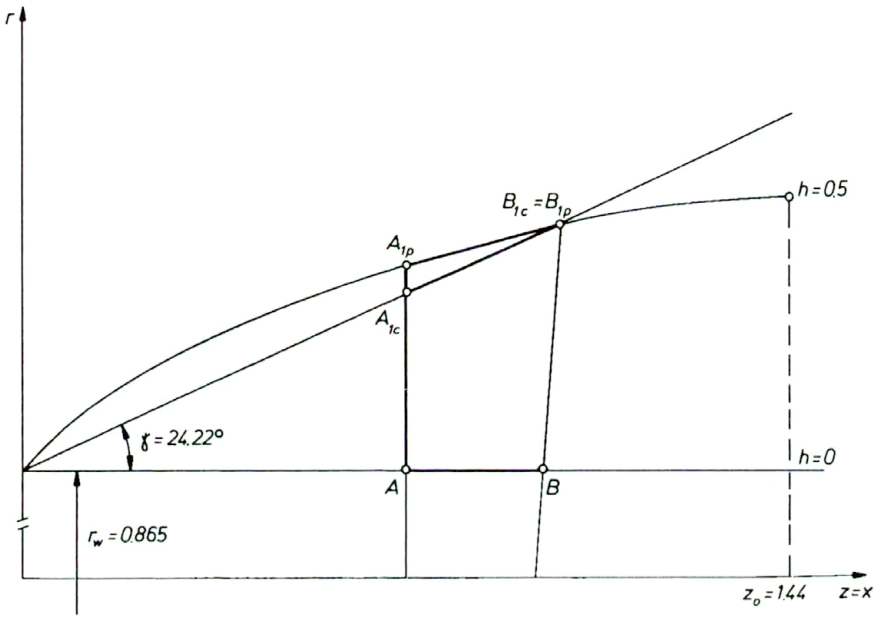


FIG. 2. Geometry of compared domains for conically and paraboloidally shaped stream lines.

This means that the boundary conditions for pressure p should be given along these lines.

4. Examples of numerical calculation

In order to show the difference in the solutions for different geometry from Fig. 1 a and Fig. 1 b, we will choose the following boundary conditions as they are illustrated in Fig. 2.

For paraboloidal and conically shaped stream surfaces, the distance BB_1 is the same. At this oulet station the flow is prepared to work in the rotor which is placed behind the nozzles in the real stage.

The boundary conditions should be given along the lines: $AA_{1c}B_{1c}$ for the conical stream surfaces, and $AA_{1p}B_{1p}$ for the paraboloidal stream surfaces.

At the inlets to the domain AA_1B_1B let us assume the same pressure and temperature, e.g.

$$p_0 = 1500\,\text{Pa}, \qquad T_0 = 327^\circ\,\text{K},$$

and the gas constants

$$R = 460\frac{\text{J}}{\text{kg}\,^\circ\text{C}}, \qquad k = 1.3.$$

For both cases the same mass flow rate was chosen,

$$m = 24.404\,\text{kg/s},$$

which means that normal velocity at the inlet of conical flow is

$$U_{nc} = 115.257\,\text{m/s},$$

and for paraboloidal flow is

$$U_{np} = 97.623\,\text{m/s}.$$

No circumferential components of the inlet velocity were assumed.

To create angular momentum in the region of vanes, one has to introduce the circumferential velocity change along bounds $A_{1p}B_{1p}$ and $A_{1c}B_{1c}$. It was assumed to be linear (as a function of z) increase of $U_x^{(2)}$ components along these arcs from $U_x^{(2)} = 0$ at points A_{1c}, A_{1p} up to $U_x^{(2)} = 400\,\text{m/s}$ in the points B_{1c} and B_{1p}. The known circumferential component along these arcs enabled us to determine also the pressure distribution and to start the computation of flow parameters in the whole domain.

The first comparison has been done for zero blockage coefficient $\tau(x^{(1)},x^{(3)}) \equiv 0$. It corresponds to an infinitesimal thickness of blading.

The graphical presentation of trajectories at different stream surfaces is shown in Fig. 3 a and Fig. 3 b. The stream surfaces of the second family (S2), created by the trajectories starting from the lines AA_{1c} and AA_{1p}, are shown for conical flow in Fig. 3 a and for paraboloidal flow – in Fig. 3 b.

Figure 4 shows the distribution of circumferential velocity at the outlet, line BB_1. It is worth noting that in the case of a paraboloidal flow field, the distribution of circumferential velocity component is more uniform and the velocity is higher at the inner diameter.

If we define the moment of circumferenial momentum over the outlet area S_{BB1} in Fig. 2 like

$$L = \int_{S_{BB_1}} \varrho U_{x^{(2)}} U_n r \, dS, \tag{4.1}$$

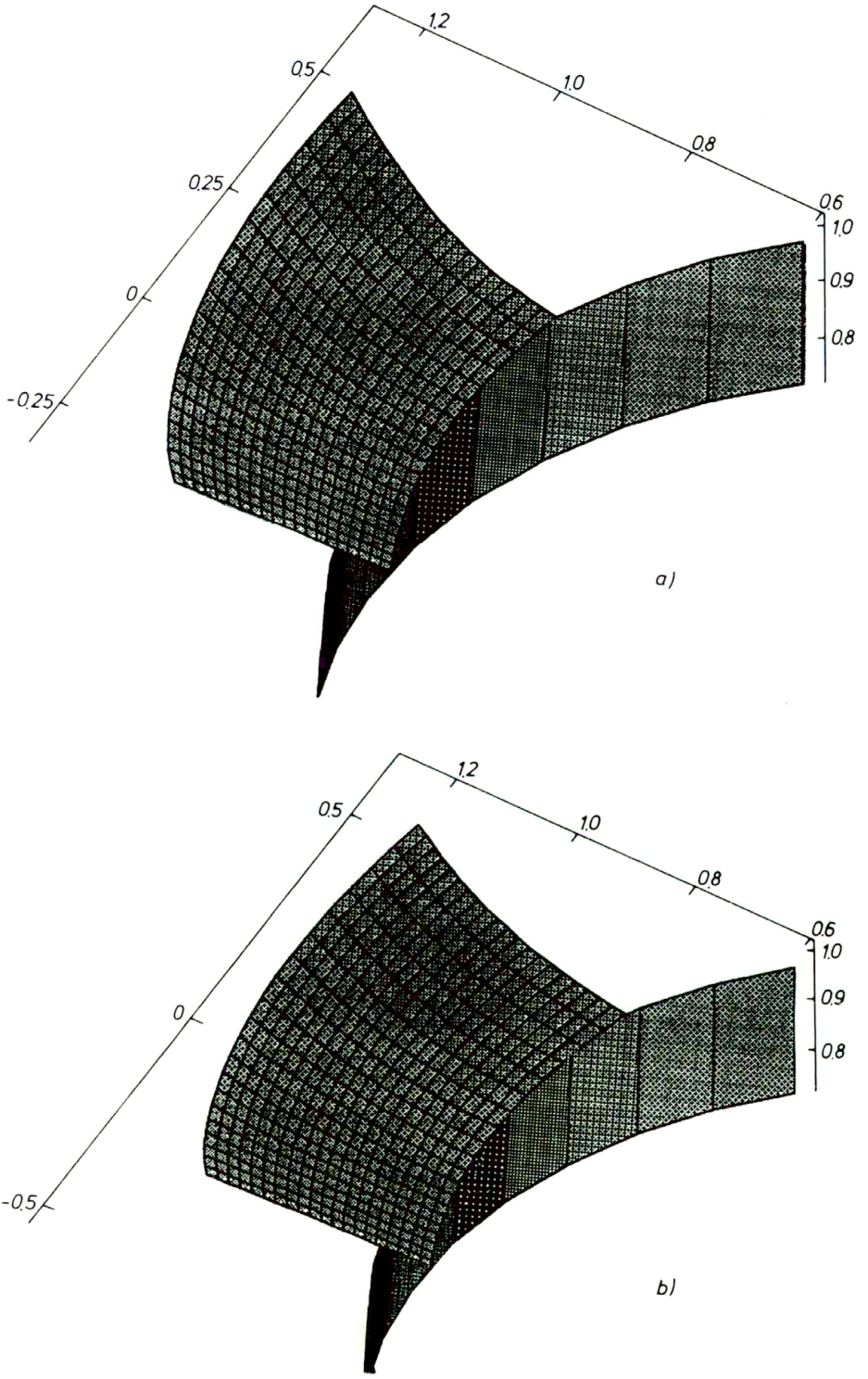


FIG. 3. Solution of S2 surfaces for both cases at zero blockage coefficient τ.

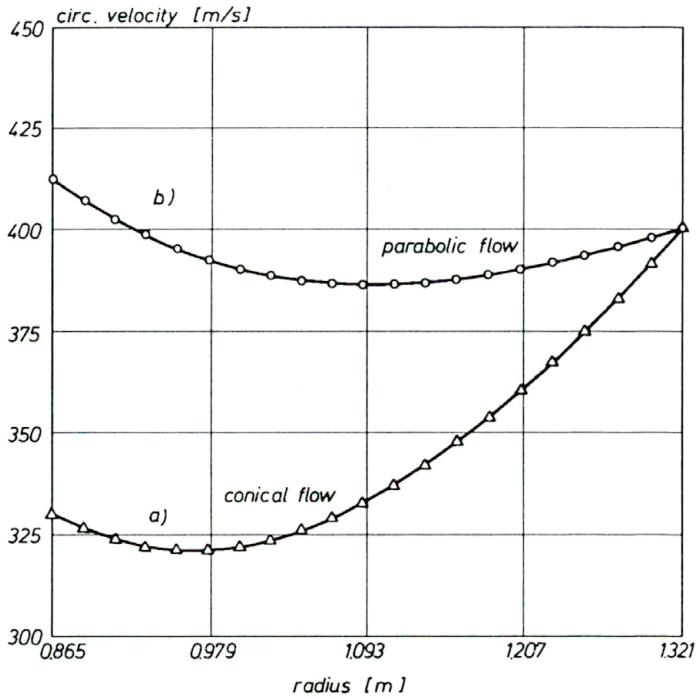


FIG. 4. Comparison of circumferential velocity at the outlet of conical and paraboloidal stream surfaces.

then for a paraboloidal flow one gets

$$L_p = 10626.654\,\mathrm{Nm},$$

whereas for a conical flow the moment is

$$L_c = 9393.558\,\mathrm{Nm}.$$

In the case of paraboloidal flow, the same mass flow rate generates a moment higher by about 13% than the one for the conical flow.

The second point which has been checked in the present example is the sensitivity of the flow to the blockage coefficient $\tau(x^{(1)}, x^{(3)})$.

The following function was chosen for τ:

$$\tau(x^{(1)}, \overline{x}^{(3)}) = \left(\tau_1 + \tau_2 \frac{t_{\max} - x^{(1)}}{t_{\max} - t_{\min}}\right) \left(\frac{1 - \overline{x}^{(3)}}{1 - \overline{x}^{(3)}_{\max}}\right)^{n_1} \left(\frac{\overline{x}^{(3)}}{\overline{x}^{(3)}_{\max}}\right)^{n_2}, \tag{4.2}$$

$$\overline{x}^{(3)} = \frac{x^{(3)} - z_A}{z_B - z_A}, \qquad \overline{x}^{(3)}_{\max} = 0.5.$$

The function (4.2) with the above parameters localizes the maximum thickness of blading at $\overline{x}^{(3)}_{\max} = 0.5$. Altogether five parameters can be chosen to distribute τ

in the domain of the flow, namely n_1, n_2, τ_1, τ_2, $\bar{x}^{(3)}_{\max}$. The set of parameters enable us to approximate a large class of blade thicknesses. The aim of the present analysis is to show the existence or nonexistence of the solution, depending on the level of blockage coefficient $\tau(x^{(1)}, \bar{x}^{(3)})$. Out of many possible combinations of parameters of the function (4.2), let us assume the following ones to be constant:

$$z_A = 0.73, \qquad z_B = 0.98 + 0.0755(r - 0.865),$$

$$\begin{aligned} t_{\max} &= h_{\max} = 0.5 && \text{for paraboloidal flow,} \\ t_{\min} &= h_{\min} = 0.0 && \text{for paraboloidal flow,} \\ t_{\max} &= q_{\max} = 0.4498 && \text{for conical flow,} \\ t_{\min} &= q_{\min} = 0.0 && \text{for conical flow,} \\ n_1 &= 2, \quad n_2 = 2, \end{aligned}$$

and vary only τ_1, τ_2. These two parameters control the change of maximum blade thickness of the blade in the middle of the width at $\bar{x}^{(3)} = 0.5$. In Fig. 5, curve C for the conical flow ($\tau_1 = \tau_2 = 0.086$) and curve P, for the paraboloidal flow ($\tau_1 = \tau_2 = 0.11219$), show the limiting blockage coefficient τ at the root of blade $x^{(1)} = 0$. They split the plane ino two sub-domains. The inner sub-domain provides the existence of solution for the whole range of $x^{(1)}$. The regions of no solution – "white spots" – appear in the outer subdomains. This is valid for the set of parameters chosen for the discussion. But qualitatively, it characterizes the general feature of the problem, namely – due to a too high blockage the solution may fail.

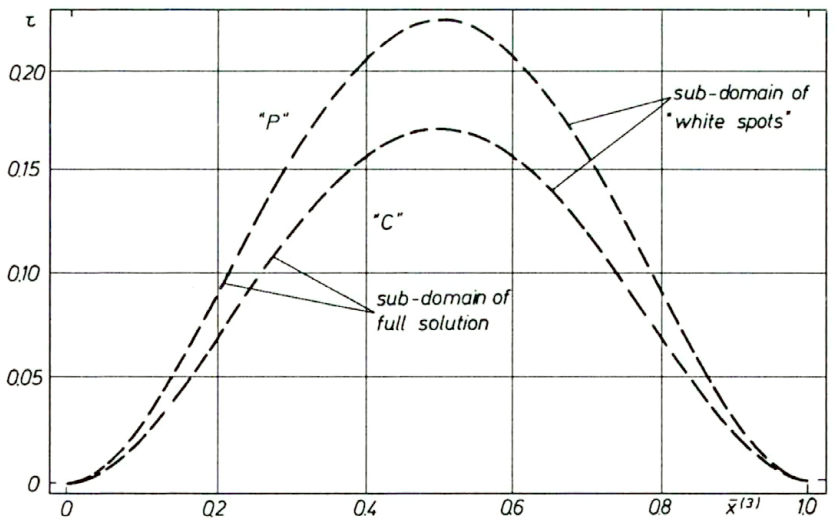


FIG. 5. Limiting curves for the existence of "white spots" in conical and paraboloidal flows.

To illustrate the influence of blockage, the following parameters were chosen for the numerical examples:

$$\text{curve } A \quad - \quad \tau_1 = \tau_2 = 0.08,$$
$$\text{curve } B \quad - \quad \tau_1 = \tau_2 = 0.10,$$
$$\text{curve } C \quad - \quad \tau_1 = \tau_2 = 0.12.$$

They are shown among the limiting curves in Fig. 6.

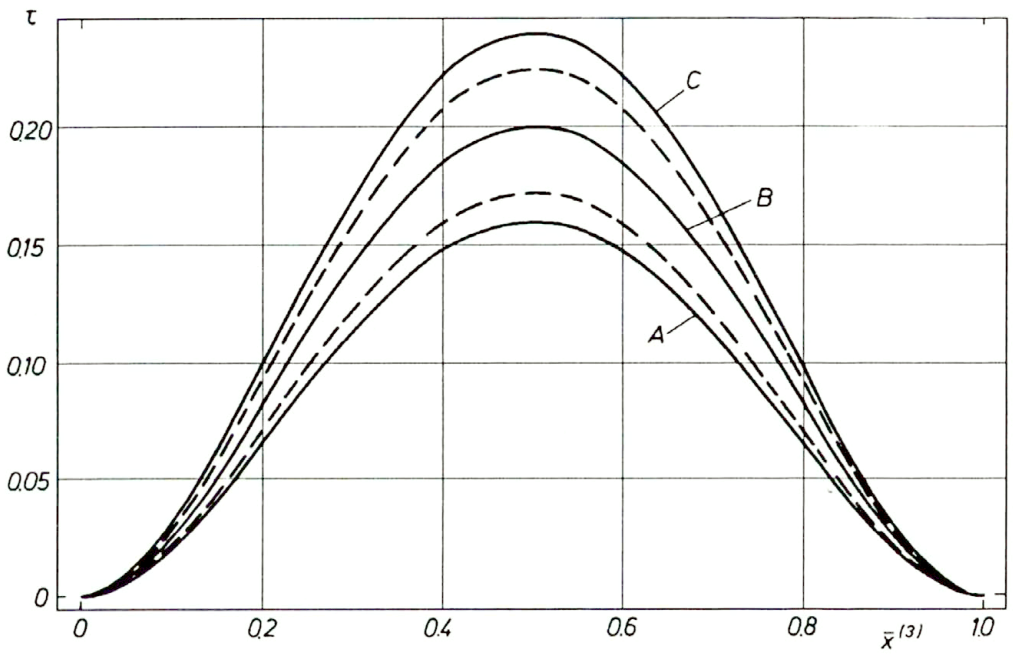


FIG. 6. Low, medium and high blockage coefficients in comparison to limiting curves.

The stream surfaces S2, for the parameters $\tau_1 = \tau_2 = 0.08$, for a conical flow (Fig. 7 a) and for a paraboloidal flow (Fig. 7 b) are shown. The solution for S2 exists for the whole range of $x^{(1)}$.

For the parameters $\tau_1 = \tau_2 = 0.10$ there is no full solution at the root for S2 in the conical case, but for the paraboloidal case the solution exists in the whole range of $x^{(1)}$. It is shown in Fig. 8 a and in Fig. 8 b.

For the case of $\tau_1 = \tau_2 = 0.12$ there is no full solution at the root in both the cases, conical and paraboloidal. The size of "white spot" is larger in the case of conical flow – Fig. 9 a than that in the case of paraboloidal flow – Fig. 9 b.

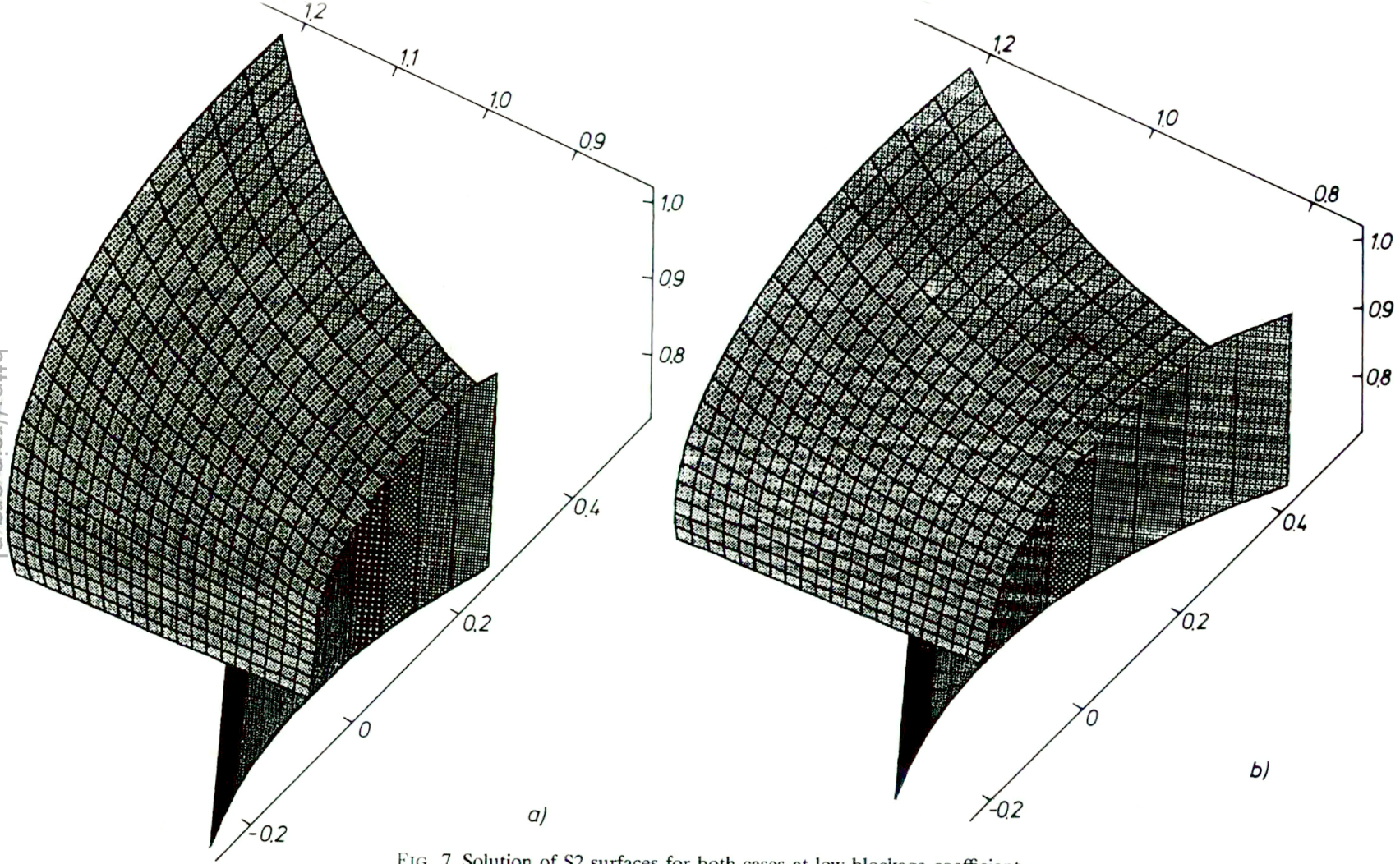


FIG. 7. Solution of S2 surfaces for both cases at low blockage coefficient τ.

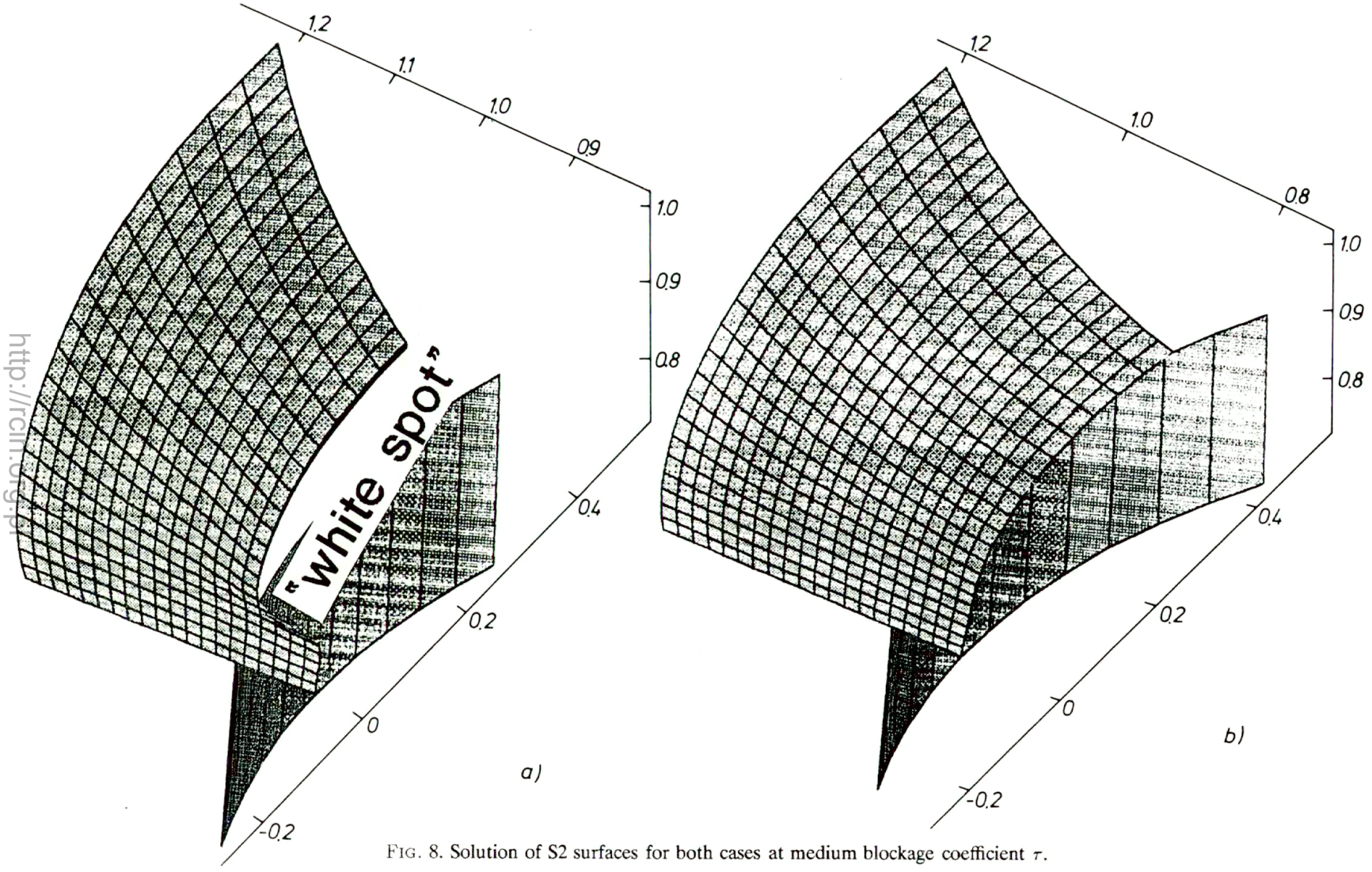


FIG. 8. Solution of S2 surfaces for both cases at medium blockage coefficient τ.

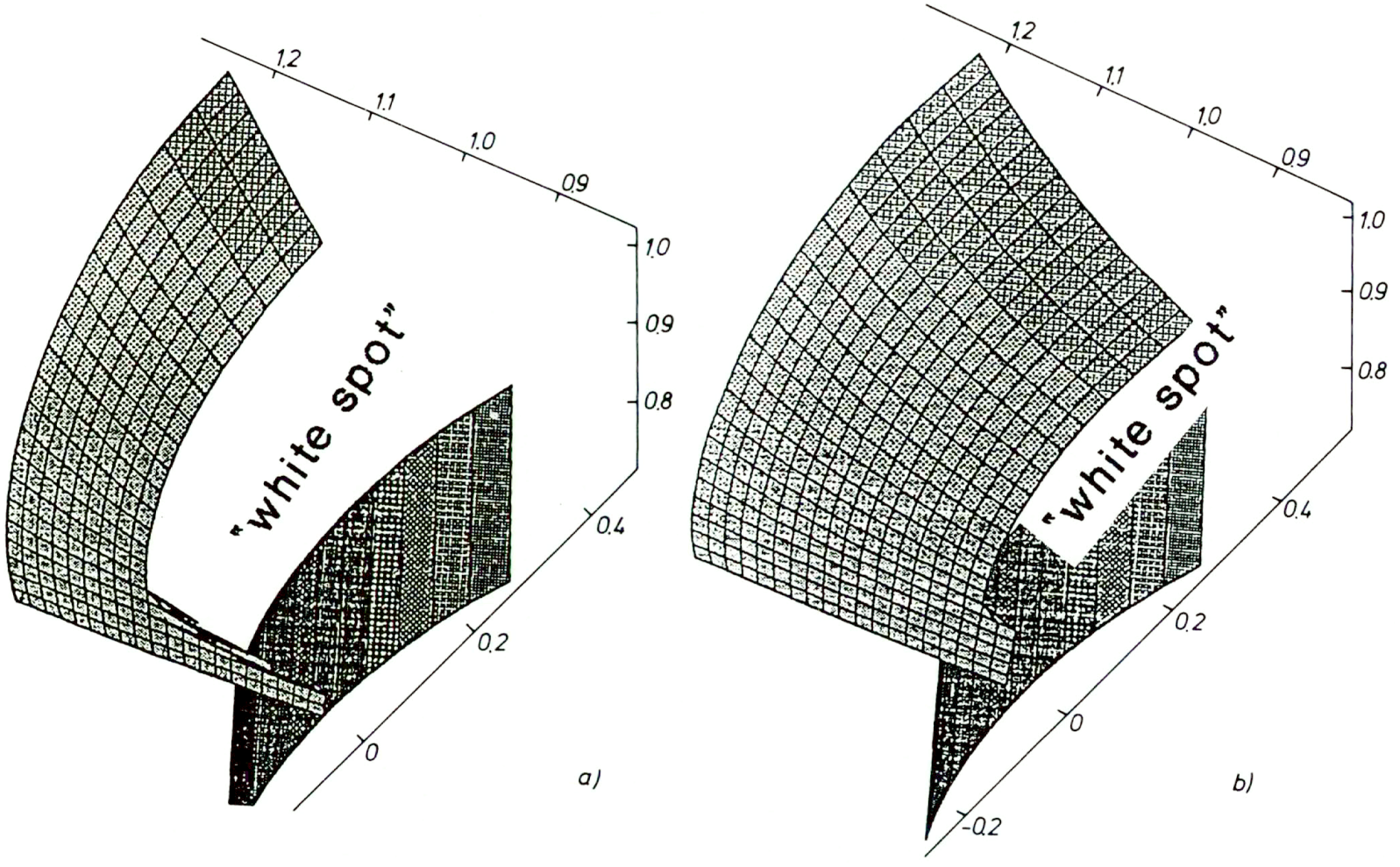


FIG. 9. Solution of S2 surfaces for both cases at high blockage coefficient.

5. Conclusions

For the presented examples of paraboloidal and conical flows, if one assumes the conditions of comparison of these two flows, the following conclusions can be drawn:

1. The parabolic flow produces higher moment of circumferential momentum than the conical flow.

2. The high blockage of the flow due to a too thick blading makes the solution impossible in the region close to the inner diameter – "white spot".

3. The area of "white spot" is larger for a conical flow than for a paraboloidal flow.

From the points of view presented above one can state that the paraboloidal flow is more favorable than the conical flow. In practical applications the advantages of paraboloidal flows have to be confronted with the broader aspects of a technical character.

References

1. L.H. SMITH, J.R., *The radial equation of turbomachinery*, J. Engng. for Power, January, 1966.
2. A.J. WENNERSTROM, *On the treatment of body forces in the radial equilibrium equation of turbomachinery*, Traupel-Fesschrift, Juris-Verlag, Zurich 1974.
3. R. PUZYREWSKI, *Hyperbolic model of a turbomachine stage (inverse problem)* [in Polish], Proc. 10th Polish Conf. of Fluid Mech., Gdańsk University of Technology, Gdańsk-Sarnówek, IX 1992.
4. R. PUZYREWSKI and J. POZORSKI, *Stream surfaces of inviscid conical flow model for turbine stage*, Proc. Intern. Symposium on Experimental and Computational Aerothermodynamics of Internal Flows, Prague, July 12-15, 1993.
5. R. PUZYREWSKI and K. NAMIEŚNIK, *Paraboloidal-planar coordinate system applied to the inverse problem of a turbomachine stage* [in Polish], Rep. of the Fluid Machinery Inst. No. 95, 1993.
6. R. PUZYREWSKI and J. POZORSKI, *Conical flow model for turbine stage*, Arch. Bud. Maszyn, **40**, 3-4, 1993.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FLUID FLOW MACHINERY.

Received October 7, 1994.

Arch. Mech., 47, 6, pp. 1089–1111, Warszawa 1995

A four-velocity model for van der Waals fluids

K. PIECHÓR (WARSZAWA)

A SIMPLE, four-velocity model for van der Waals fluids is presented. The true kinetic equation we imitate is that of Enskog–Vlasov. A closed system of equations for the number density n and the mean velocity $\mathbf{u}$ is obtained in exactly the same way as in the case of discrete kinetic theory of the Boltzmann equation, i.e. under the only assumption that the Knudsen number based on the mean-free-path of hard-core collisions is a small parameter, and without expanding the model Enskog operator. Consequently, the obtained equations of hydrodynamics are integro-differential. If, however, we expand the terms representing the hard-core collisions and those representing the attractive tail in a power series in suitably defined small parameters, then the equations of hydrodynamics are shown to take the usual differential form with capillarity terms included. Also, one-dimensional forms of both the kinetic model equations and the resulting hydrodynamic equations are given. As an application we show that the Maxwell equal area rule for liquid-vapour transitions results directly from the approximate kinetic equations as well as from the hydrodynamics ones, and that it is of the same, classical form.

1. Introduction

SINCE THE USE OF THE BOLTZMANN equation [1–4] is limited to sufficiently rarefied gases, Enskog [1–6] proposed a kinetic equation for the one-particle distribution function by modifying Boltzmann's assumptions in two ways:

i. He assumed that the molecules are hard spheres and took into account that the centres of two colliding spheres are a distance σ apart, where σ is the sphere diameter;

ii. He increased the collision frequency by a factor H, which is a function of the gas density at the contact point of the two colliding spheres, and represents the pair correlation function corresponding to the system in uniform equilibrium.

The original Enskog equation, now referred to as a standard one, was used to evaluate transport coefficients in moderately dense gases [1–3], and quite recently to determine the shock wave structure [7].

Although the results concerning the transport coefficients were more accurate than the ones provided by the Boltzmann equation, the standard Enskog equation exhibited some disagreement with irreversible thermodynamics. This was removed by van BEIJEREN and ERNST [8], who replaced the function H used originally by Enskog by the exact local equilibrium pair distribution function which takes fully into account the spatial nonuniformities in the local equilibrium state. This form of the Enskog equation is called revised.

The Enskog equation or its mutants have been the object of intensive mathematical studies for the last years, and the results with long lists of earlier references are presented in [5] and [6].

In order to have an equation suitable to liquid dynamics and phase changes, GRMELA [9, 10] was presumably the first who proposed the so-called Enskog–Vlasov equation. The intermolecular potential is split into a hard-core and an attractive tail. The hard-core is treated as in the standard Enskog equation, whereas the tail enters the kinetic equation only linearly in a mean-field term. Later, this equation was derived from the Liouville equation by applying the principle of maximization of entropy [11, 12]. The hard-core term was obtained as in the revised Enskog equation. This equation was used in [11] to evaluate the transport coefficients. A comparison with experimental data exhibited a pretty good agreement even in the case of quite complex molecules and in a wide range of pressures. Recently this equation was used by the present author [13] to deduce so-called thermocapillarity equations which earlier have been derived on phenomenological grounds.

The Enskog–Vlasov equation reads:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_r f = -\mathbf{F}\cdot\nabla_{\mathbf{v}} f + E(f), \tag{1.1}$$

where $f = f(t, \mathbf{r}, \mathbf{v})$ is the distribution function, t is the time, $\mathbf{r} \in \mathbb{R}^3$ is the position, $\mathbf{v} \in \mathbb{R}^3$ is the molecular velocity. The self-consistent force of attraction $\mathbf{F}$ is assumed to be of the mean-field form

$$\mathbf{F} = \int \frac{\mathbf{y} - \mathbf{r}}{|\mathbf{y} - \mathbf{r}|} \Phi'(|\mathbf{y} - \mathbf{r}|) n(t, \mathbf{y})\, d\mathbf{y}, \tag{1.2}$$

where $n(t, \mathbf{r})$ is the number density,

$$n(t, \mathbf{r}) = \int f(t, \mathbf{r}, \mathbf{v})\, d\mathbf{v}, \tag{1.3}$$

$\Phi(r)$ is the attractive potential, and $\Phi'(r)$ is its derivative with respect to r.

As the collision operator $E(f)$ we take the revised Enskog operator [4–6, 8, 11, 12]

$$E(f) = \sigma^2 \int \theta[\mathbf{n}\cdot(\mathbf{v}_* - \mathbf{v})] \Big[H(\mathbf{r}, \mathbf{r} + \sigma]\mathbf{n}) f'(\mathbf{r}) f'_*(\mathbf{r} + \sigma\mathbf{n}) - H(\mathbf{r}, \mathbf{r} - \sigma\mathbf{n}) f(\mathbf{r}) f_*(\mathbf{r} - \sigma\mathbf{n}) \Big] d\mathbf{n}\, d\mathbf{v}_*, \tag{1.4}$$

where $f(\mathbf{r}) = f(t, \mathbf{r}, \mathbf{v})$, $f_*(\mathbf{r}) = f(t, \mathbf{r}, \mathbf{v}_*)$, $f'(\mathbf{r}) = f(t, \mathbf{r}, \mathbf{v}')$, $f'(\mathbf{r}) = f(t, \mathbf{r}, \mathbf{v}'_*)$ are values of the distribution function at the indicated velocities. Here, as usual, $\mathbf{v}'$, $\mathbf{v}'_*$ are the post-collision velocities related to the pre-collision ones $\mathbf{v}$, $\mathbf{v}_*$ by

$$\begin{aligned} \mathbf{v}' &= \mathbf{v} + \mathbf{n}[\mathbf{n}\cdot(\mathbf{v}_* - \mathbf{v})], \\ \mathbf{v}'_* &= \mathbf{v}_* - \mathbf{n}[\mathbf{n}\cdot(\mathbf{v}_* - \mathbf{v})], \end{aligned} \tag{1.5}$$

with $\mathbf{n}$ being a unit vector bisecting the angle between $\mathbf{v}'_* - \mathbf{v}'$ and $\mathbf{v}_* - \mathbf{v}$, and directed from the molecule with the velocity $\mathbf{v}_*$ to the molecule with the velocity $\mathbf{v}$ when in contact. Next, $\theta(x)$ in (1.4) is a function of one real variable defined by

$$\theta(x) = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x \leq 0. \end{cases} \tag{1.6}$$

Finally, $H(\mathbf{r}, \mathbf{y})$ is a functional of the number density as well as of the space variables, usually denoted as follows

$$H(\mathbf{r}, \mathbf{y}) = H(\mathbf{r}, \mathbf{y} \mid n(t)), \tag{1.7}$$

to mark that it depends on n. We use a shorthand, but (1.7) has to be remembered. Usually it is assumed that H is positive, and that it is symmetric under the interchange of variables $\mathbf{r}$ and $\mathbf{y}$,

$$H(\mathbf{r}, \mathbf{y}) = H(\mathbf{y}, \mathbf{r}). \tag{1.8}$$

It is obvious that the Enskog equation, and the more the Enskog–Vlasov one, is much more complicated than the Boltzmann equation, so that a radical but reasonable simplification is necessary. We will show in a series of papers that this can be achieved by extending the basic ideas of discrete kinetic theory of the Boltzmann equation ([14–17]) to the present case.

Discrete velocity kinetic theory considers such mathematical models of kinetic equations in which the velocity space is not all $\mathbb{R}^d$ (where $d = 1, 2, 3$) but a finite, fixed in advance set of d-dimensional vectors.

These ideas were extended in [18] to construct a quite general class of discrete models of the Enskog–Vlasov equation (1.1). Contrary to this paper, we consider here a specific, four-velocity model. Hence, we supply the general considerations of [18] with an example, lacking in this paper. We pursue the study much further and obtain the explicit form of the Navier–Stokes type of approximation to the considered model. It turns out that to receive this approximation, it is not necessary to assume that the sphere diameter σ is much smaller than the macroscopic length scale, what is usually done [2, 3, 6]. The equations of the model hydrodynamics obtained in this way are integro-differential, what reflects the basic postulate of the Enskog theory that the molecular interactions are non-local. If we take the additional assumption that σ is small, we obtain pure differential equations of the hydrodynamic character.

As an introductory applications we show that the Maxwell equal area rule can be deduced both directly from the kinetic model as well as from its hydrodynamic approximation. The detailed and complete analysis of our model and its various approximations will be the object of future papers.

2. The model

We consider a model "gas" whose molecules move in planes parallel to a certain fixed plane in such a way that their motion is independent of the distance between the actual plane of motion and that of reference. Hence such a motion can be totally described by two two-dimensional vectors: $\mathbf{r} = (x_1, x_2)$ of the position, and $\mathbf{c} = (c_1, c_2)$ of the molecular velocity.

The key assumption is that the molecules can move with only one of the following four velocities:

$$\mathbf{c}_0 = (c_1, c_2), \qquad c_1 = (-c_2, c_1), \qquad \mathbf{c}_2 = (-c_1, -c_2), \qquad \mathbf{c}_3 = (c_2, -c_1), \tag{2.1}$$

where c_1, c_2 are constants.

We denote by $N_i(t, \mathbf{r})$, $i = 0, 1, 2, 3$, the probability density of finding a molecule moving with the velocity $\mathbf{c}_i$, i.e. the i-th molecule, in a volume $d\mathbf{r}$ around $\mathbf{r}$, at the time t.

If we take the Boltzmann's concept of gas ([1–4]) according to which: i) the molecules are mass-points, ii) in the absence of external forces they move freely with the exception of collisions with other molecules, iii) the collisions are instantaneous, then $N_i(t, \mathbf{r})$ satisfy a system of equations of the form

$$\frac{\partial}{\partial t} N_i + \mathbf{c}_i \cdot \nabla N_i = Q_i(N), \qquad i = 0, 1, 2, 3, \tag{2.2}$$

where the left-hand side describes changes of N_i due to the free motion, and $Q_i(N)$ on the right-hand side describes changes of N_i resulting from the collisions.

In (2.2), as well as in all forthcoming formulae, the dot between two symbols of vectors denotes their scalar product in the sense of $\mathbb{R}^2$.

In order to write down an expression for $Q_i(N)$ we take the Boltzmann's ideas of gas. It is obvious that due to the collisions, some molecules will leave the considered portion of the gas, but some other, not present before the collision, can enter it. Consequently, $Q_i(N)$ is the difference between the "gain" and "loss" terms. Each of these terms is proportional to the product of probability densities of molecules participating in the collision since, by assumption, the molecules move independently in the stochastic sense. In the Boltzmann theory of rarefied gases only two molecules can collide and the outcome of the collision are two identical molecules moving with new, post-collision velocities. These velocities are such that the total momentum and energy before and after the collision remain unchanged. In our case of (2.1) the momentum conservation principle is satisfied in the following four cases.

$$\mathbf{c}_i + \mathbf{c}_{i+2} = \mathbf{c}_{i+1} + \mathbf{c}_{i+3}, \qquad i = 0, 1, 2, 3. \tag{2.3}$$

The energy conservation principle

$$\mathbf{c}_i^2 + \mathbf{c}_{i+2}^2 = \mathbf{c}_{i+1}^2 + \mathbf{c}_{i+3}^2, \qquad i = 0, 1, 2, 3, \tag{2.4}$$

is satisfied identically owing to (2.1).

Here and in the future we take the following convention. Let $\{Z_i\}$, $i = 0,1,2,\ldots$, be a finite or infinite sequence of some quantities. We assume that if $i \geq 4$, then

$$Z_i = Z_j\,,$$

where $i = j(\mathrm{mod}\,4)$, i.e. j is the remainder of the division of i by 4.

Hence, the only admissible binary collisions are such that the pre-collision velocities are $\mathbf{c}_i$ and $\mathbf{c}_{i+2}$, and the post-collision ones are $\mathbf{c}_{i+1}$ and $\mathbf{c}_{i+3}$.

Therefore we write (c.f. [14–17])

$$Q_i^B(N) = \sigma c(N_{i+1}N_{i+3} - N_iN_{i+2}), \qquad i = 0,1,2,3, \tag{2.5}$$

where the superscript B is used to mark that this form of Q was obtained on the basis of the Boltzmann's ideas. In (2.5) σ is a positive constant coefficient called the collision cross-section, and c is defined by

$$c = \sqrt{c_1^2 + c_2^2}\,. \tag{2.6}$$

In the case of dense gases the problem is much more subtle. We keep the hypothesis that the state of the gas can be described by the one-particle distributions N_i $(i = 0,1,2,3)$, as well as we keep the assumptions leading to Eqs. (2.2). The question is, what is the form of Q_i. To solve it we follow the Enskog's ideas, and therefore we supply Q_i with the superscript E, i.e. we write Q_i^E instead of Q_i. The basic difference between the Boltzmann's and Enskog's approaches is that in the latter case the size of the molecules is not ignored. Hence, the centres of two colliding molecules are at different points, and consequently the distributions, say N_i and N_j, are evaluated at different positions but at the same time.

If the location of the molecule moving at the velocity $\mathbf{c}_i$ is at $\mathbf{r}$, then the location of another molecule, say $\mathbf{c}_j$, when in contact with $\mathbf{c}_i$ is at

$$\mathbf{r} - \sigma\mathbf{n},$$

where $\mathbf{n}$ is a unit vector directed from the centre of $\mathbf{c}_j$ to that of $\mathbf{c}_i$. Their velocities after the collision, say $\mathbf{c}_i'$ and $\mathbf{c}_j'$ respectively, are evaluated by using (1.5) with $\mathbf{v} = \mathbf{c}_i$, and $\mathbf{v}_* = \mathbf{c}_j$. This guarantees that the principles of conservation of momentum and energy are satisfied. But $\mathbf{c}_i'$ and $\mathbf{c}_j'$ have to belong to the set (2.1) of admissible velocities, so $\mathbf{n}$ cannot be arbitrary. Hence, we are forced to discretize the range of vectors $\mathbf{n}$ as well.

Let us notice that we did tacitly the same when modelling the Boltzmann collision operator either. But then we did not need explicitly the vector $\mathbf{n}$ since the probability densities of both particles participating in a collision were evaluated at the same point.

In this paper we take into account only such collisions that the angle between the line joining the centres of two molecules in contact and the line parallel to $\mathbf{c}_i$ is equal to $\pm 45°$.

We have four such collisions:

i. Before the collision the molecule $\mathbf{c}_i$ is at $\mathbf{r}$, the molecule $\mathbf{c}_{i+2}$ is at

$$\mathbf{r} - \sigma \frac{(\mathbf{c}_{i+2} - \mathbf{c}_{i+3}) - (\mathbf{c}_i - \mathbf{c}_{i+1})}{|(\mathbf{c}_{i+2} - \mathbf{c}_{i+3}) - (\mathbf{c}_i - \mathbf{c}_{i+1})|}, \tag{2.7}$$

after the collision $\mathbf{c}_i$ changes to $\mathbf{c}_{i+1}$, and $\mathbf{c}_{i+2}$ changes to $\mathbf{c}_{i+3}$ (Fig. 1).

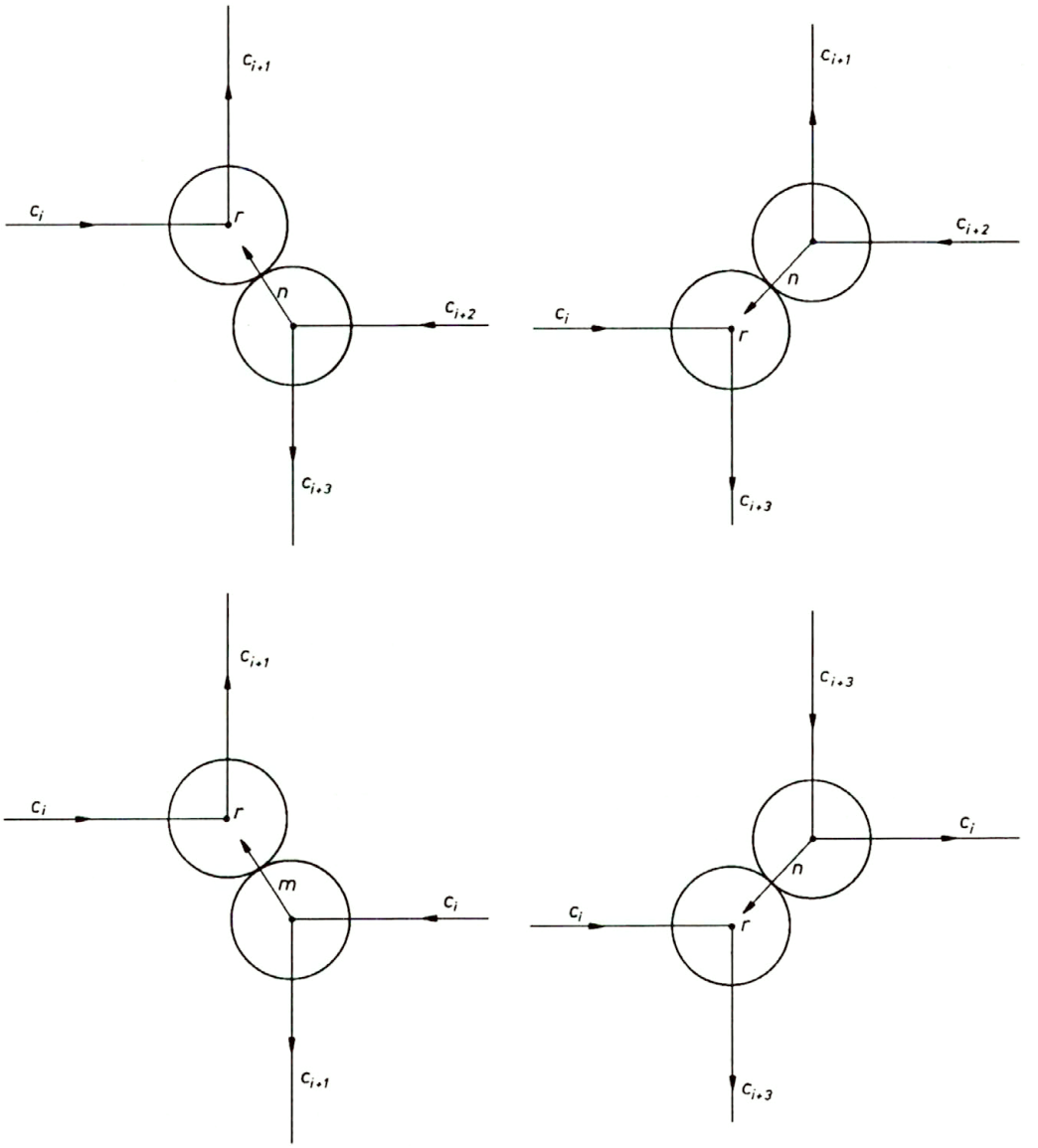


FIG. 1.

ii. Before the collision the molecule $\mathbf{c}_i$ is at $\mathbf{r}$, the molecule $\mathbf{c}_{i+2}$ is at

$$\mathbf{r} - \sigma \frac{(\mathbf{c}_{i+2} - \mathbf{c}_{i+1}) - (\mathbf{c}_i - \mathbf{c}_{i+3})}{|(\mathbf{c}_{i+2} - \mathbf{c}_{i+1}) - (\mathbf{c}_i - \mathbf{c}_{i+3})|}, \tag{2.7'}$$

after the collision $\mathbf{c}_i$ changes to $\mathbf{c}_{i+3}$, and $\mathbf{c}_{i+2}$ changes to $\mathbf{c}_{i+1}$.

NOTE. In the previous, Boltzmann case it was sufficient to declare that the pair $(\mathbf{c}_i, \mathbf{c}_j)$ changes to the pair $(\mathbf{c}_i', \mathbf{c}_j')$. Now, due to the distinction of the positions of the centres of the colliding molecules, we have to distinguish between collisions of the type: $\mathbf{c}_i$ changes to $\mathbf{c}_i'$, $\mathbf{c}_j$ changes to $\mathbf{c}_j'$, and $\mathbf{c}_i$ changes to $\mathbf{c}_j'$, $\mathbf{c}_j$ changes to $\mathbf{c}_i'$.

iii. Before the collision the molecule $\mathbf{c}_i$ is at $\mathbf{r}$, the molecule $\mathbf{c}_{i+1}$ is at

$$\mathbf{r} - \sigma \frac{(\mathbf{c}_{i+1} - \mathbf{c}_i) - (\mathbf{c}_i - \mathbf{c}_{i+1})}{|(\mathbf{c}_{i+1} - \mathbf{c}_i) - (\mathbf{c}_i - \mathbf{c}_{i+1})|}, \tag{2.7''}$$

after the collision $\mathbf{c}_i$ changes to $\mathbf{c}_{i+1}$, and $\mathbf{c}_{i+1}$ changes to $\mathbf{c}_i$.

iv. Before the collision the molecule $\mathbf{c}_i$ is at $\mathbf{r}$, the molecule $\mathbf{c}_{i+3}$ is at

$$\mathbf{r} - \sigma \frac{(\mathbf{c}_{i+3} - \mathbf{c}_i) - (\mathbf{c}_i - \mathbf{c}_{i+3})}{|(\mathbf{c}_{i+3} - \mathbf{c}_i) - (\mathbf{c}_i - \mathbf{c}_{i+3})|}, \tag{2.7'''}$$

after the collision $\mathbf{c}_i$ changes to $\mathbf{c}_{i+3}$, and $\mathbf{c}_{i+3}$ changes to $\mathbf{c}_i$.

NOTE. Collisions of the type iii. and iv. are called exchange collisions, since the molecules participating in such an event simply exchange their velocities. In the Boltzmann case such collisions are trivial because they do not contribute to Q_i^B. It is no longer true in the case of dense gases due to the shift of the positions of the two molecules in contact.

The next question concerns the factor H in (1.4). We solve this problem as follows. We define the number density n by

$$n(t, \mathbf{r}) = \frac{1}{4}(N_0 + N_1 + N_2 + N_3). \tag{2.8}$$

Assuming that H depends functionally on the number density n only, we use simply the same H as in the continuous case of (1.4).

As the four-velocity model of the Enskog operator we take

$$\begin{aligned} Q_i^E(N) = \frac{\sigma c}{2} \Bigg[& H\left(\mathbf{r}, \mathbf{r} + \sigma \frac{\mathbf{c}_{i+1} - \mathbf{c}_i}{|\mathbf{c}_{i+1} - \mathbf{c}_i|}\right) N_{i+1}(\mathbf{r}) N_{i+3}\left(\mathbf{r} + \sigma \frac{\mathbf{c}_{i+1} - \mathbf{c}_i}{|\mathbf{c}_{i+1} - \mathbf{c}_i|}\right) \\ & - H\left(\mathbf{r}, \mathbf{r} + \sigma \frac{\mathbf{c}_{i+1} - \mathbf{c}_i}{|\mathbf{c}_{i+1} - \mathbf{c}_i|}\right) N_i(\mathbf{r}) N_{i+2}\left(\mathbf{r} + \sigma \frac{\mathbf{c}_{i+1} - \mathbf{c}_i}{|\mathbf{c}_{i+1} - \mathbf{c}_i|}\right) \Bigg] \\ & + \frac{\sigma c}{2} \Bigg[H\left(\mathbf{r}, \mathbf{r} - \sigma \frac{\mathbf{c}_{i+1} + \mathbf{c}_i}{|\mathbf{c}_{i+1} + \mathbf{c}_i|}\right) N_{i+3}(\mathbf{r}) N_{i+1}\left(\mathbf{r} - \sigma \frac{\mathbf{c}_{i+1} + \mathbf{c}_i}{|\mathbf{c}_{i+1} + \mathbf{c}_i|}\right) \end{aligned} \tag{2.9}$$

(2.9) [cont.)

$$
\begin{aligned}
&-H\left(\mathbf{r}, \mathbf{r}+\sigma \frac{\mathbf{c}_{i+1}+\mathbf{c}_i}{|\mathbf{c}_{i+1}+\mathbf{c}_i|}\right) N_i(\mathbf{r}) N_{i+2}\left(\mathbf{r}+\sigma \frac{\mathbf{c}_{i+1}+\mathbf{c}_i}{|\mathbf{c}_{i+1}+\mathbf{c}_i|}\right)\Bigg] \\
&+\frac{\sigma c}{2}\Bigg[H\left(\mathbf{r}, \mathbf{r}+\sigma \frac{\mathbf{c}_{i+1}-\mathbf{c}_i}{|\mathbf{c}_{i+1}-\mathbf{c}_i|}\right) N_{i+1}(\mathbf{r}) N_i \cdot\left(\mathbf{r}+\sigma \frac{\mathbf{c}_{i+1}-\mathbf{c}_i}{|\mathbf{c}_{i+1}-\mathbf{c}_i|}\right) \\
&\quad -H\left(\mathbf{r}, \mathbf{r}-\sigma \frac{\mathbf{c}_{i+1}-\mathbf{c}_i}{|\mathbf{c}_{i+1}-\mathbf{c}_i|}\right) N_i(\mathbf{r}) N_{i+1}\left(\mathbf{r}-\sigma \frac{\mathbf{c}_{i+1}-\mathbf{c}_i}{|\mathbf{c}_{i+1}-\mathbf{c}_i|}\right)\Bigg] \\
&+\frac{\sigma c}{2}\Bigg[H\left(\mathbf{r}, \mathbf{r}-\sigma \frac{\mathbf{c}_{i+1}+\mathbf{c}_i}{|\mathbf{c}_{i+1}+\mathbf{c}_i|}\right) N_{i+3}(\mathbf{r}) N_i \left(\mathbf{r}-\sigma \frac{\mathbf{c}_{i+1}+\mathbf{c}_i}{|\mathbf{c}_{i+1}+\mathbf{c}_i|}\right) \\
&\quad -H\left(\mathbf{r}, \mathbf{r}+\sigma \frac{\mathbf{c}_{i+1}+\mathbf{c}_i}{|\mathbf{c}_{i+1}+\mathbf{c}_i|}\right) N_i(\mathbf{r}) N_{i+3}\left(\mathbf{r}+\sigma \frac{\mathbf{c}_{i+1}+\mathbf{c}_i}{|\mathbf{c}_{i+1}+\mathbf{c}_i|}\right)\Bigg],
\end{aligned}
$$

$i = 0, 1, 2, 3$, where the momentum conservation principle (2.3) was used.

NOTE. The multipliers of each of the terms in the square brackets [ ] in (2.9) are called the transition rates. Here all of them are taken to be equal $\sigma c/2$. In general, they can be arbitrary nonnegative numbers. However, as shown in [18], some restrictions have to be imposed on them to obtain the correct form of the hydrodynamic approximation. "Correct" means that the equations of hydrodynamics deduced from a discrete kinetic theory should resemble the hydrodynamic equations deduced from the true kinetic theory. The choice of the transition rates used in (2.9) is correct in this sense, as we will see it later.

In order to have a discrete theory suitable for phase transitions, we have to add a term representing long-range forces of intermolecular attraction, as suggested by Eq. (1.1). The core of the problem is to model the gradient $\partial f/\partial \mathbf{v}$. It is exactly the same difficulty which is faced when modelling the Boltzmann equation containing external forces. In our opinion, this problem has no satisfactory solution. Here, we follow the ideas of our previous paper [18]; other approaches (inapplicable to our needs) can be found in [19, 20].

Our idea is as follows. Let f_M be the Maxwell distribution of local equilibrium. We approximate $\partial f/\partial \mathbf{v}$ in Eq. (1.1) by $\partial f_M/\partial \mathbf{v}$. But $\partial f_M/\partial \mathbf{v} = -\partial f_M/\partial \mathbf{u}$ where $\mathbf{u}$ is the mean velocity of the ordered motion.

In the four-velocity kinetic theory, the mean velocity $\mathbf{u}$ is defined by

$$
\mathbf{u} = \frac{1}{4n} \sum_{i=0}^{3} N_i \mathbf{c}_i \,, \tag{2.10}
$$

where n is the number density (2.8), whereas the Maxwell distribution M_i, $i = 0, 1, 2, 3$, satisfies

$$
M_0 M_3 = M_1 M_2 \,. \tag{2.11}
$$

Usually, the components of the Maxwellian are expressed through the number density n and the mean velocity $\mathbf{u}$. To this end we set $N_i = M_i$ in (2.8) and (2.10),

and using (2.11) we obtain

$$M_i = n\left[\left(1+\frac{\mathbf{c}_i\cdot\mathbf{u}}{c^2}\right)^2 - \left(\frac{\mathbf{c}_{i+1}\cdot\mathbf{u}}{c^2}\right)^2\right], \qquad i=0,1,2,3. \tag{2.12}$$

As the four-velocity model of the Enskog–Vlasov equation (1.1) we take Eqs. (2.2) with

$$Q_i(N) = Q_i^{E-V}(N) = \mathcal{F}_i + Q_i^E, \; i=0,1,2,3, \tag{2.13}$$

where Q_i^E is given by (2.9), and

$$\mathcal{F}_i = \frac{\partial M_i}{\partial \mathbf{u}}\cdot\mathbf{F}, \tag{2.14}$$

where $\mathbf{F}$ has the same form as in (1.2), but with $\mathbf{y}$ and $\mathbf{r}$ being two-dimensional vectors, and the domain of integration being now $\mathbb{R}^2$ instead of $\mathbb{R}^3$.

Explicitly, $\mathcal{F}_i$ are given by

$$\mathcal{F}_i = 2n\left[\left(1+\frac{\mathbf{c}_i\cdot\mathbf{u}}{c^2}\right)\left(\frac{\mathbf{c}_i\cdot\mathbf{F}}{c^2}\right) - \left(\frac{\mathbf{c}_{i+1}\cdot\mathbf{u}}{c^2}\right)\left(\frac{\mathbf{c}_{i+1}\cdot\mathbf{F}}{c^2}\right)\right], \tag{2.15}$$

$i=0,1,2,3.$

3. Moment equations

It is convenient to express the kinetic equations (2.2), (2.15) in an equivalent, moment form.

The moments of the distribution (N_0, N_1, N_2, N_3) are n, $\mathbf{u}$ and R, with

$$R = \frac{1}{4n}(N_0 - N_1 + N_2 - N_3). \tag{3.1}$$

R is not an observable quantity; it is the only nonhydrodynamic moment of N in our model.

From (2.8), (2.12), (2.13) and (3.1) we get

$$N_i = n\left(1 + 2\frac{\mathbf{c}_i\cdot\mathbf{u}}{c^2} + (-1)^i R\right), \qquad i=0,1,2,3. \tag{3.2}$$

The system of equations for n, $\mathbf{u}$, and R can be obtained from (2.2), (2.13) in the following three steps.

i. Mass conservation equation

We take the scalar product of (2.2), (2.13) with the vector $(1,1,1,1)$, make use of (3.2), of the two identities

$$\sum_{i=0}^{3} \mathcal{F}_i = 0, \qquad \sum_{i=0}^{3} Q_i^E = 0, \tag{3.3}$$

and obtain the standard equation

$$\frac{\partial}{\partial t} n + \nabla \cdot (n\mathbf{u}) = 0. \tag{3.4}$$

ii. Momentum conservation equation

Now, we take the scalar product of Eqs. (2.2), (2.13) with $(\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$.

The following identities along with Definitions (2.11), (2.12) are crucial for the derivation of these equations.

From (2.12) and (3.2) it follows that

$$\frac{1}{2} \sum_{i=0}^{3} N_i \mathbf{c}_i \mathbf{c}_i = n(\mathbf{c}_0\mathbf{c}_0 + \mathbf{c}_1\mathbf{c}_1) + nR(\mathbf{c}_0\mathbf{c}_0 - \mathbf{c}_1\mathbf{c}_1).$$

It is convenient to introduce the following unit vectors

$$\mathbf{k} = \frac{\mathbf{c}_0 + \mathbf{c}_1}{|\mathbf{c}_0 + \mathbf{c}_1|}, \qquad \mathbf{l} = \frac{\mathbf{c}_0 - \mathbf{c}_1}{|\mathbf{c}_0 - \mathbf{c}_1|}. \tag{3.5}$$

Using these we can write

$$\frac{1}{4} \sum_{i=0}^{3} N_i \mathbf{c}_i \mathbf{c}_i = \frac{c^2}{2} \mathbb{I} n + (\mathbf{k}\mathbf{l} + \mathbf{l}\mathbf{k}) nR, \tag{3.6}$$

where $\mathbb{I}$ is the unit 2×2 matrix.

Next we have

$$\frac{1}{4} \sum_{i=0}^{3} \mathcal{F}_i \mathbf{c}_i = n\mathbf{F}, \tag{3.7}$$

for any force term $\mathbf{F}$.

If, however, $\mathbf{F}$ is given by (1.2), the $n\mathbf{F}$ can be represented as the divergence of a 2×2 tensor.

Indeed, from (1.2) it follows that

$$n\mathbf{F} = \frac{1}{2} \int \frac{\mathbf{y}}{|\mathbf{y}|} \Phi'(|\mathbf{y}|)[n(\mathbf{r} + \mathbf{y})n(\mathbf{r}) - n(\mathbf{r})n(\mathbf{r} - \mathbf{y})] d\mathbf{y}.$$

We use the following identity, which holds true for any function $G(\mathbf{r}, \mathbf{s})$

$$G(\mathbf{r}+\mathbf{y}, \mathbf{r}) - G(\mathbf{r}, \mathbf{r}-\mathbf{y}) = \mathbf{y}\cdot\nabla_r \int_0^1 G(\mathbf{r}+\lambda\mathbf{y}, \mathbf{r}+(\lambda-1)\mathbf{y})d\lambda. \tag{3.8}$$

Hence, we have

$$n\mathbf{F} = c^2\nabla\cdot\mathbb{A}, \tag{3.9}$$

where

$$\mathbb{A} = \frac{1}{2c^2}\int_0^1 d\lambda \int \frac{\mathbf{y}\,\mathbf{y}}{|\mathbf{y}|}\Phi'(|\mathbf{y}|)n(\mathbf{r}+\lambda\mathbf{y})n(\mathbf{r}+(\lambda-1)\mathbf{y})\,d\mathbf{y}. \tag{3.10}$$

The most cumbersome it is to derive the following formula

$$\frac{1}{4}\sum_{i=0}^{3} Q_i^E \mathbf{c}_i = -c^2\nabla\cdot[\mathbf{k}\,\mathbf{k}\mathcal{K} + \mathbf{l}\,\mathbf{l}\mathcal{L}], \tag{3.11}$$

where

$$\mathcal{K} = \frac{\sigma^2}{\sqrt{2}}\int_0^1 H(\mathbf{r}+\sigma\lambda\mathbf{k}, \mathbf{r}+\sigma(\lambda-1)\mathbf{k})n(\mathbf{r}+\sigma\lambda\mathbf{k})n(\mathbf{r}+\sigma(\lambda-1)\mathbf{k}) \times\left[1-\frac{\sqrt{2}}{c}\mathbf{k}\cdot\mathbf{u}(\mathbf{r}+\sigma\lambda\mathbf{k})\right]\left[1+\frac{\sqrt{2}}{c}\mathbf{k}\cdot\mathbf{u}(\mathbf{r}+\sigma(\lambda-1)\mathbf{k})\right]d\lambda, \tag{3.12}$$

and

$$\mathcal{L} = \frac{\sqrt{2}}{c}\int_0^1 H(\mathbf{r}+\sigma\lambda\mathbf{l}, \mathbf{r}+\sigma(\lambda-1)\mathbf{l})n(\mathbf{r}+\sigma\lambda\mathbf{l})n(\mathbf{r}+\sigma(\lambda-1)\mathbf{l}) \times\left[1-\frac{\sqrt{2}}{c}\mathbf{l}\cdot\mathbf{u}(\mathbf{r}+\sigma\lambda\mathbf{l})\right]\left[1+\frac{\sqrt{2}}{c}\mathbf{l}\cdot\mathbf{u}(\mathbf{r}+\sigma(\lambda-1)\mathbf{l})\right]d\lambda. \tag{3.13}$$

Use of (3.8) was made when deducing (3.12) and (3.13).

Hence, the momentum conservation equation is of the form

$$\frac{\partial}{\partial t}(n\mathbf{u}) + \nabla\cdot c^2\left[\frac{1}{2}n\mathbb{I} + \frac{1}{2}nR(\mathbf{k}\,\mathbf{l}+\mathbf{l}\,\mathbf{k}) + \mathbf{k}\,\mathbf{k}\mathcal{K} + \mathbf{l}\,\mathbf{l}\mathcal{L}\right] = n\mathbf{F}. \tag{3.14}$$

If $\mathbf{F}$ is given by (1.2), this equation takes a conservative form (c.f. (3.9))

$$\frac{\partial}{\partial t}(n\mathbf{u}) + \nabla\cdot c^2\left[\frac{1}{2}n\mathbb{I} + \frac{1}{2}nR(\mathbf{k}\,\mathbf{l}+\mathbf{l}\,\mathbf{k}) + \mathbf{k}\,\mathbf{k}\mathcal{K} + \mathbf{l}\,\mathbf{l}\mathcal{L} - \mathbb{A}\right] = 0. \tag{3.15}$$

Let us notice that $\mathcal{K}$, $\mathcal{L}$, and $\mathbb{A}$ are nonlinear operators acting on hydrodynamic moments n, $\mathbf{u}$ only.

iii. Closing equation

The fourth moment equation is obtained by taking the scalar product of Eqs. (2.2), (2.13) with $(1, -1, 1, -1)$. We have the following auxiliary formulae

$$\frac{1}{4}\sum_{i=1}^{3}(-1)^i N_i \mathbf{c}_i = n\mathbf{u}\cdot(\mathbf{k}\mathbf{l} + \mathbf{l}\mathbf{k}), \tag{3.16}$$

$$\frac{1}{4}\sum_{i=1}^{3}(-1)^i \mathcal{F}_i = \frac{2n}{c^2}\mathbf{F}\cdot(\mathbf{k}\mathbf{l} + \mathbf{l}\mathbf{k})\cdot\mathbf{u}, \tag{3.17}$$

and

$$\frac{1}{4}\sum_{i=1}^{3}(-1)^i Q_i^E(N) = -4\sigma c n^2 \mathfrak{h}\left[R\left(\frac{\sqrt{2}\mathbf{k}\cdot\mathbf{u}}{c}\right)\left(\frac{\sqrt{2}\mathbf{l}\cdot\mathbf{u}}{c}\right)\right] - 2\mathbf{u}\cdot(\mathbf{l}\mathbf{k}\cdot\nabla\mathcal{K} + \mathbf{k}\mathbf{l}\cdot\nabla\mathcal{L}), \tag{3.18}$$

where

$$\begin{aligned}\mathfrak{h} = \frac{1}{4n(\mathbf{r})}\Big\{ & H(\mathbf{r}, \mathbf{r} + \sigma\mathbf{k})n(\mathbf{r} + \sigma\mathbf{k})\left[1 - \frac{\sqrt{2}}{c}\mathbf{k}\cdot\mathbf{u}(\mathbf{r} + \sigma\mathbf{k})\right] \\ & + H(\mathbf{r}, \mathbf{r} - \sigma\mathbf{k})n(\mathbf{r} - \sigma\mathbf{k})\left[1 + \frac{\sqrt{2}}{c}\mathbf{k}\cdot\mathbf{u}(\mathbf{r} - \sigma\mathbf{k})\right] \\ & + H(\mathbf{r}, \mathbf{r} + \sigma\mathbf{l})n(\mathbf{r} + \sigma\mathbf{l})\left[1 - \frac{\sqrt{2}}{c}\mathbf{l}\cdot\mathbf{u}(\mathbf{r} + \sigma\mathbf{l})\right] \\ & + H(\mathbf{r}, \mathbf{r} - \sigma\mathbf{l})n(\mathbf{r} - \sigma\mathbf{l})\left[1 + \frac{\sqrt{2}}{c}\mathbf{l}\cdot\mathbf{u}(\mathbf{r} - \sigma\mathbf{l})\right]\Big\}.\end{aligned} \tag{3.19}$$

Hence the fourth equation reads

$$\frac{\partial}{\partial t}(nR) + \nabla\cdot[n(\mathbf{k}\mathbf{l} + \mathbf{l}\mathbf{k})\cdot\mathbf{u}] = -4\sigma c n^2 \mathfrak{h}\left[R - \left(\frac{\sqrt{2}\mathbf{k}\cdot\mathbf{u}}{c}\right)\left(\frac{\sqrt{2}\mathbf{l}\cdot\mathbf{u}}{c}\right)\right] + 2(\nabla\cdot\mathbb{A})\cdot(\mathbf{k}\mathbf{l} + \mathbf{l}\mathbf{k})\cdot\mathbf{u} - 2\mathbf{u}\cdot(\mathbf{l}\mathbf{k}\cdot\nabla\mathcal{K} + \mathbf{k}\mathbf{l}\cdot\nabla\mathcal{L}). \tag{3.20}$$

Equations (3.4), (3.15), (3.20) form the sought system of moment equations. Let us notice that R enters this system linearly, and that it does not appear in any of the complicated nonlinear operators $\mathcal{K}$, $\mathcal{L}$, $\mathfrak{h}$, and $\mathbb{A}$. This fact will be very helpful in deriving the hydrodynamic equations.

4. Equations of hydrodynamics

As usual, to deduce the hydrodynamic approximation to Eqs. (3.4), (3.15), (3.20), we pass to dimensionless variables. Since this is a routine procedure, we omit the details and give only the quantities of reference. Let L be the typical macroscopic length-scale; as the typical time we use L/c; the mean velocity $\mathbf{u}$ is referred to c, and the number density n is referred to a typical value of it denoted by n_0. To bring the self-consistent force $\mathbf{F}$ to a dimensionless form we use the characteristic value Φ_0 of the attractive potential and its range r_0. The dimensionless quantities are denoted by the same symbols as their dimensional counterparts. It should not lead to a confusion since from now on all quantities are treated as nondimensional. If however the changes are serious (for example, if a quantity contains dimensionless parameters), then their dimensionless form is given explicitly, but it is denoted by the same character.

Prior to applying an asymptotic procedure, it is convenient to eliminate $\mathbb{A}$ from Eq. (3.20). It is done by means of Eqs. (3.4) and (3.15).

Hence, the moment equations are transformed to

$$\frac{\partial}{\partial t} n + \nabla\cdot(n\mathbf{u}) = 0, \tag{4.1}$$

$$\frac{\partial}{\partial t}(n\mathbf{u}) + \nabla\left[\frac{1}{2} n\mathbb{I} + n(\mathbf{kl} + \mathbf{lk})(\mathbf{k}\cdot\mathbf{u})(\mathbf{l}\cdot\mathbf{u}) + \frac{1}{2}(\mathbf{kl} + \mathbf{lk})\mathcal{C} + \mathbf{kk}\mathcal{K} + \mathbf{ll}\mathcal{L} - \mathbb{A}\right] = 0, \tag{4.2}$$

and

$$\frac{\partial}{\partial t}\mathcal{C} - \mathbf{u}\cdot\nabla\mathcal{C} + n\mathbb{B} : \nabla\mathbf{u} = -\frac{4n}{\varepsilon}\flat\mathcal{C}, \tag{4.3}$$

where $\mathcal{C}$ is defined by

$$\mathcal{C} = n[R - 2(\mathbf{k}\cdot\mathbf{u})(\mathbf{l}\cdot\mathbf{u})]. \tag{4.4}$$

$\mathcal{C}$ can be interpreted as the measure of deviation of the distribution from local equilibrium, because the distribution is local Maxwellian if and only if $\mathcal{C} = 0$.

Next, $\mathcal{K}$, $\mathcal{L}$, $\flat$ and $\mathbb{A}$ be dimensionless counterparts of the quantities denoted by the same symbols and introduced in the preceding section. Since their dimensionless forms contain some dimensionless parameters, we list them below to avoid a confusion.

$$\mathcal{K} = \frac{\beta}{\sqrt{2}} \int_0^1 H(\mathbf{r} + \beta\varepsilon\lambda\mathbf{k}, \mathbf{r} + \beta\varepsilon(\lambda - 1)\mathbf{k}) n(\mathbf{r} + \beta\varepsilon\mathbf{k}) n(\mathbf{r} + \beta\varepsilon(\lambda - 1)\mathbf{k}) \times \left[1 - \sqrt{2}\mathbf{k}\cdot\mathbf{u}(\mathbf{r} + \beta\varepsilon\lambda\mathbf{k})\right]\left[1 + \sqrt{2}\mathbf{k}\cdot\mathbf{u}(\mathbf{r} + \beta\varepsilon(\lambda - 1)\mathbf{k})\right] d\lambda, \tag{4.5}$$

$$\mathcal{L} = \frac{\beta}{\sqrt{2}} \int_0^1 H(\mathbf{r} + \beta\varepsilon\lambda\mathbf{l}, \mathbf{r} + \beta\varepsilon(\lambda - 1)\mathbf{l}) n(\mathbf{r} + \beta\varepsilon\lambda\mathbf{l}) n(\mathbf{r} + \beta\varepsilon(\lambda - 1)\mathbf{l}) \times \left[1 - \sqrt{2}\mathbf{l}\cdot\mathbf{u}(\mathbf{r} + \beta\varepsilon\lambda\mathbf{l})\right] \left[1 + \sqrt{2}\mathbf{l}\cdot\mathbf{u}(\mathbf{r} + \beta\varepsilon(\lambda - 1)\mathbf{l})\right] d\lambda, \tag{4.6}$$

$$\begin{aligned} \mathfrak{h} = \frac{1}{4n(\mathbf{r})} \Big\{ & H(\mathbf{r}, \mathbf{r} + \beta\varepsilon\mathbf{k}) n(\mathbf{r} + \beta\varepsilon\mathbf{k}) [1 - \sqrt{2}\mathbf{k}\cdot\mathbf{u}(\mathbf{r} + \beta\varepsilon\mathbf{k})] \\ & + H(\mathbf{r}, \mathbf{r} - \beta\varepsilon\mathbf{k}) n(\mathbf{r} - \beta\varepsilon\mathbf{k}) [1 + \sqrt{2}\mathbf{k}\cdot\mathbf{u}(\mathbf{r} - \beta\varepsilon\mathbf{k})] \\ & + H(\mathbf{r}, \mathbf{r} + \beta\varepsilon\mathbf{l}) n(\mathbf{r} + \beta\varepsilon\mathbf{l}) [1 - \sqrt{2}\mathbf{l}\cdot\mathbf{u}(\mathbf{r} + \beta\varepsilon\mathbf{l})] \\ & + H(\mathbf{r}, \mathbf{r} - \beta\varepsilon\mathbf{l}) n(\mathbf{r} - \beta\varepsilon\mathbf{l}) [1 + \sqrt{2}\mathbf{l}\cdot\mathbf{u}(\mathbf{r} - \beta\varepsilon\mathbf{l})] \Big\}, \end{aligned} \tag{4.7}$$

$$\mathbb{A} = \frac{\Phi_0 n_0 r_0^2}{2c^2} \int_0^1 d\lambda \int \frac{\mathbf{y}\,\mathbf{y}}{|\mathbf{y}|} \Phi'(|\mathbf{y}|) n(\mathbf{r} + \alpha\varepsilon\lambda\mathbf{y}) n(\mathbf{r} + \alpha\varepsilon(\lambda - 1)\mathbf{y})\, d\mathbf{y}. \tag{4.8}$$

Here α, β, ε are dimensionless parameters which are defined by

$$\alpha = n_0 r_0 \sigma, \tag{4.9}$$

$$\beta = n_0 \sigma^2, \tag{4.10}$$

$$\varepsilon = \frac{1}{L n_0 \sigma}. \tag{4.11}$$

Finally, the 2×2 matrix $\mathbb{B}$, which appears on the left-hand side of Eq. (4.3) is defined by

$$\mathbb{B} = \left[1 - 2(\mathbf{k}\cdot\mathbf{u})^2\right] \mathbf{k}\,\mathbf{l} + \left[1 - 2(\mathbf{l}\cdot\mathbf{u})^2\right] \mathbf{l}\,\mathbf{k}. \tag{4.12}$$

For comparison, the moment form of the four-velocity model of the Boltzmann equation (2.2), (2.5) consists of Eq. (4.1) without any change; Eq. (4.2) but with $\mathcal{K} \equiv \mathcal{L} \equiv 0$ and $\mathbb{A} \equiv 0$; and Eq. (4.3) with $\mathfrak{h} \equiv 1$. We see that Eq. (4.3) differs only a little from its ideal gas counterpart. Therefore we can proceed in exactly the same way as in the case of the four-velocity model of the Boltzmann equation to obtain a hydrodynamic approximation, i.e a system of equations for n and $\mathbf{u}$ only, to our model. To this end we assume that ε is a small parameter

$$\varepsilon \ll 1, \tag{4.13}$$

and in the first order approximation we get from Eq. (4.3)

$$\mathcal{C} = 0, \tag{4.14}$$

i.e. the distribution is locally Maxwellian. Setting $\mathcal{C} = 0$ in Eq. (4.2) we obtain the Euler-type approximation to our model. However, the obtained equations differ

significantly from what is usually called the Euler equations since the operators $\mathcal{K}$ and $\mathcal{L}$ contain some dissipation, what will be shown later.

In the second order approximation we obtain from Eq. (4.3)

$$\mathcal{C} = \frac{\varepsilon}{4\mathfrak{h}}\mathbb{B}:\nabla\mathbf{u} + O(\varepsilon^2). \tag{4.15}$$

Inserting (4.15) into Eq. (4.2) we obtain the Navier–Stokes-type approximation.

Let us notice that we have obtained the hydrodynamic approximation to the considered kinetic equations without expanding the Enskog collision operator in a power series in $\beta\varepsilon$, as it is usually done (see [1–3, 5, 6], and as it was done by the present author in [13, 18]).

Assuming, however, additionally that

$$\beta\varepsilon \ll 1, \tag{4.16}$$

we expand $\mathcal{K}$, $\mathcal{L}$ and $\mathfrak{h}$ in a Taylor series in $\beta\varepsilon$. Truncating the expansion at the first two nonvanishing terms we obtain

$$\mathcal{K} = \frac{\beta}{\sqrt{2}}\left[n^2h(1 - 2(\mathbf{k}\cdot\mathbf{u})^2) - \sqrt{2}\beta\varepsilon h(\mathbf{k}\cdot\nabla)(\mathbf{k}\cdot\mathbf{u})\right] + O((\beta\varepsilon)^2), \tag{4.17}$$

$$\mathcal{L} = \frac{\beta}{\sqrt{2}}\left[n^2h(1 - 2(\mathbf{l}\cdot\mathbf{u})^2) - \sqrt{2}\beta\varepsilon h(\mathbf{l}\cdot\nabla)(\mathbf{l}\cdot\mathbf{u})\right] + O((\beta\varepsilon)^2), \tag{4.18}$$

and

$$\mathfrak{h} = h - \frac{\beta\varepsilon}{\sqrt{2}n}\nabla\cdot(nh\mathbf{u}) + O((\beta\varepsilon)^2), \tag{4.19}$$

where

$$h(r) = h(n(\mathbf{r}, t)) = H(\mathbf{r}, \mathbf{r}). \tag{4.20}$$

We take also the next assumption, namely

$$\alpha\varepsilon \ll 1. \tag{4.21}$$

Then

$$\mathbb{A} = an^2\mathbb{I} + A(\alpha\varepsilon)^2\left[(2n\Delta n + (\nabla n)^2\mathbb{I} - 2\nabla n\nabla n\right] + O((\alpha\varepsilon)^4), \tag{4.22}$$

where

$$a = \frac{\Phi_0 n_0 r_0^2}{2c^2}\pi\int_0^\infty y^2\Phi'(y)\,dy, \tag{4.23}$$

$$A = \frac{\Phi_0 n_0 r_0^2}{2c^2}\pi\int_0^\infty y^4\Phi'(y)\,dy, \tag{4.24}$$

are positive constants, and Δ is the two-dimensional Laplace operator.

Using (4.15), (4.17)–(4.19) and (4.22) we obtain the purely differential form of the hydrodynamic equations. They read

$$\frac{\partial}{\partial t} n + \nabla\cdot(n\mathbf{u}) = 0, \tag{4.25}$$

$$\frac{\partial}{\partial t}(n\mathbf{u}) + \nabla\cdot\left[\frac{1}{2}n(1+\sqrt{2}\beta nh)[2\mathbf{k}\,\mathbf{k}(\mathbf{k}\cdot\mathbf{u})^2 - 2\mathbf{l}\,\mathbf{l}(\mathbf{l}\cdot\mathbf{u})^2] - an^2\mathbb{I} + n\mathbf{u}\,\mathbf{u}\right] \tag{4.26}$$

$$= (\alpha\varepsilon)^2\nabla\cdot\mathbb{C} + \varepsilon\nabla\cdot\left[\frac{1}{8h}(\mathbf{k}\,\mathbf{l} + \mathbf{l}\,\mathbf{k})[(1-2(\mathbf{k}\cdot\mathbf{u}))^2\mathbf{k}\,\mathbf{l} + (1-2(\mathbf{l}\cdot\mathbf{u}))^2]:\nabla\mathbf{u}\right.$$

$$\left. + \beta^2 n^2 h(\mathbf{k}\,\mathbf{k}\,\mathbf{k}\,\mathbf{k} + \mathbf{l}\,\mathbf{l}\,\mathbf{l}\,\mathbf{l}):\nabla\mathbf{u}\right],$$

where

$$\mathbb{C} = A\left[(2n\Delta n + (\nabla n)^2)\mathbb{I} - 2\nabla n\nabla n\right], \tag{4.27}$$

is called the capillarity tensor.

These equations are written in the traditional form. To this end we kept the first terms on the right-hand side of (4.17), (4.18) on the left-hand side of Eq. (4.26) and counted them to the pressure tensor, whereas the second terms in the expansions (4.17), (4.18) were transferred to right-hand side of (4.26) and treated as a contribution to the dissipation tensor. Similarly, the first term in the expansion (4.22) was treated as a part of the pressure tensor, whereas the second part of it was transferred to the right-hand side of (4.27) and treated as representing the capillarity term. Due to this dualism of treating the expansions of $\mathcal{K}$, $\mathcal{L}$, and $\mathbb{A}$, we wrote that the Euler type approximation (4.14) does not produce equations which are traditionally called the Euler equations.

If we truncate the expansion of $\mathbb{A}$ at a higher number of terms, we obtain model equations of so-called fluids of grade N (c.f. [21]).

Finally, let us notice that we can use the expansions (4.17), (4.18), (4.19), and (4.22) directly in the model kinetic equations (4.1)–(4.3) prior to making the asymptotic procedure related to (4.13). This is the traditional form of treating the Enskog-type equations ([1–3]).

5. One-dimensional equations

In the future we will be interested in a one-dimensional version of our equations. This is such a case when the molecular velocities $\mathbf{c}_i$ (in dimensionless form) are

$$\mathbf{c}_0 = (1,0), \qquad \mathbf{c}_1 = (0,1), \qquad \mathbf{c}_2 = (-1,0), \qquad \mathbf{c}_3 = (0,-1). \tag{5.1}$$

Then (cf. (3.5))

$$\mathbf{k} = \frac{1}{\sqrt{2}}(1,1), \qquad \mathbf{l} = \frac{1}{\sqrt{2}}(1,-1). \tag{5.2}$$

Additionally we assume that $v \equiv 0$, and that all other quantities, i.e. n, u, and $\mathcal{C}$ depend on x and t only, and do not depend on y. In this case Eqs. (4.1)–(4.9) reduce to

$$\frac{\partial}{\partial t} n + \frac{\partial}{\partial x}(nu) = 0, \tag{5.3}$$

$$\frac{\partial}{\partial t}(nu) + \frac{\partial}{\partial x}\left[\frac{n(1-u^2)}{2} + \mathcal{K} + \frac{1}{2}\mathcal{C} - \mathcal{A} + nu^2\right] = 0, \tag{5.4}$$

and

$$\frac{\partial}{\partial t}\mathcal{C} - u\frac{\partial}{\partial x}\mathcal{C} + n(1-u^2)\frac{\partial}{\partial x}u = -\frac{4n\mathfrak{h}}{\varepsilon}\mathcal{C}, \tag{5.5}$$

where

$$\mathcal{K} = \frac{\beta}{\sqrt{2}}\int_0^1 H\left(x + \frac{\beta\varepsilon\lambda}{\sqrt{2}}, x + \frac{\beta\varepsilon(\lambda-1)}{\sqrt{2}}\right) n\left(x + \frac{\beta\varepsilon\lambda}{\sqrt{2}}\right) n\left(x + \frac{\beta\varepsilon(\lambda-1)}{\sqrt{2}}\right) \times \left[1 - u\left(x\frac{\beta\varepsilon\lambda}{\sqrt{2}}\right)\right]\left[1 + u\left(x + \frac{\beta\varepsilon(\lambda-1)}{\sqrt{2}}\right)\right] d\lambda, \tag{5.6}$$

$$\mathcal{A} = \frac{\Phi_0 n_0^2 r_0}{2c^2}\int_0^1 d\lambda \int_{-\infty}^{\infty} |y| V'(|y|) n(x + \alpha\varepsilon\lambda y) n(x + \alpha\varepsilon(\lambda-1)y) dy \tag{5.7}$$

with

$$V(|y|) = \int_{-\infty}^{\infty} \Phi\left(\sqrt{y^2 + x^2}\right) dz. \tag{5.8}$$

Finally,

$$\mathfrak{h} = \frac{1}{2n(x)}\left\{H\left(x, x + \frac{\beta\varepsilon}{\sqrt{2}}\right) n\left(x + \frac{\beta\varepsilon}{\sqrt{2}}\right)\left[1 - u\left(x + \frac{\beta\varepsilon}{\sqrt{2}}\right)\right] + H\left(x, x - \frac{\beta\varepsilon}{\sqrt{2}}\right) n\left(x - \frac{\beta\varepsilon}{\sqrt{2}}\right)\left[1 + u\left(x - \frac{\beta\varepsilon}{\sqrt{2}}\right)\right]\right\}. \tag{5.9}$$

The equations of hydrodynamics (4.25), (4.26) now become

$$\frac{\partial}{\partial t} n + \frac{\partial}{\partial x}(nu) = 0, \tag{5.10}$$

$$\frac{\partial}{\partial t}(nu) + \frac{\partial}{\partial x}\left[\frac{n(1-u^2)}{2} + \mathcal{K} + \mathcal{A} + nu^2\right] = \varepsilon\frac{\partial}{\partial x}\left(\frac{1-u^2}{8\mathfrak{h}}\frac{\partial}{\partial x}u\right). \tag{5.11}$$

The expansions of (5.6), (5.9) for $\beta\varepsilon \to 0$, and that of (5.7) for $\alpha\varepsilon \to 0$, are simplified and take the form

$$\mathcal{K} = \frac{\beta}{\sqrt{2}} n^2 h(1-u^2) - \frac{\beta^2 \varepsilon n^2 h}{2} \frac{\partial}{\partial x} u + O((\beta\varepsilon)^2), \tag{5.12}$$

$$\mathfrak{h} = h + O(\beta\varepsilon^2), \tag{5.13}$$

where $h(x) = H(x, x)$, and

$$\mathcal{A} = an^2 + A(\alpha\varepsilon)^2 \left[2n \frac{\partial^2}{\partial x^2} n - \left(\frac{\partial}{\partial x} n \right)^2 \right]. \tag{5.14}$$

It is convenient to rewrite Eqs. (5.3)–(5.5) and (5.10)–(5.11), along with the approximations (5.12)–(5.14), in a Lagrangian coordinate system. To this end we follow the ideas of COURANT and FRIEDRICHS [22]. These ideas are based on the law of mass conservation. We define the Lagrange coordinate X as the mass of the fluid moving along the x-axis in a tube of unit cross-section between any definite section and the current one travelling together with the fluid. Hence

$$X = \int_{\chi(0,t)}^{\chi(X,t)} n(x,t)\, dx, \tag{5.15}$$

where $x = \chi(X, t)$ denotes the position of the particle.

The kinetic equations (5.3)–(5.5) along with (5.12)–(5.14) can be written as

$$\frac{\partial}{\partial t} w - \frac{\partial}{\partial X} u = 0, \tag{5.16}$$

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial X} \left[\frac{1-u^2}{2w} \left(1 + \frac{b}{w} \varrho \right) - \frac{a}{w^2} + \frac{1}{2} \mathcal{C} \right] = \varepsilon \frac{\partial}{\partial X} \left(\frac{b^2}{4w^3} \frac{\partial}{\partial X} u \right) + A(\alpha\varepsilon)^2 \frac{\partial}{\partial X} \left[\frac{5}{w^6} \left(\frac{\partial}{\partial X} w \right)^2 - \frac{2}{w^5} \frac{\partial^2}{\partial X^2} w \right], \tag{5.17}$$

and

$$w \frac{\partial}{\partial t} \mathcal{C} - 2u \frac{\partial}{\partial X} \mathcal{C} + \frac{1-u^2}{w} \frac{\partial}{\partial X} u = -\frac{4\varrho}{\varepsilon} \mathcal{C}, \tag{5.18}$$

where

$$b = \sqrt{2}\beta, \tag{5.19}$$

$$w = \frac{1}{n}, \tag{5.20}$$

is the specific volume, and

$$\varrho(w) = h\left(\frac{1}{w}\right). \tag{5.21}$$

The hydrodynamic equations (5.10), (5.11) along with (5.12)–(5.14) take the form

$$\frac{\partial}{\partial t}w - \frac{\partial}{\partial X}u = 0, \tag{5.22}$$

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial X}\left[\frac{1-u^2}{2w}\left(1+\frac{b}{w}\varrho\right) - \frac{a}{w^2}\right] = \varepsilon\frac{\partial}{\partial X}\left(\mu\frac{\partial}{\partial X}u\right) + A(\alpha\varepsilon)^2\frac{\partial}{\partial X}\left[\frac{5}{w^6}\left(\frac{\partial}{\partial X}w\right)^2 - \frac{2}{w^5}\frac{\partial^2}{\partial X^2}w\right], \tag{5.23}$$

where

$$\mu = \frac{1-u^2+2b^2\varrho^2w^2}{8w^2\varrho}. \tag{5.24}$$

The form of Eq. (5.24) suggests the following interpretation: μ is the viscosity coefficient, and p defined by

$$p = \frac{1-u^2}{2w}\left(1+\frac{b}{w}\varrho\right) - \frac{a}{w^2}, \tag{5.25}$$

is the pressure in our model. If we take

$$\varrho = \frac{w}{w-b}, \tag{5.26}$$

and denote

$$T = \frac{1-u^2}{2}, \tag{5.27}$$

then (5.25) takes the form of the van der Waals formula

$$p = \frac{T}{w-b} - \frac{a}{w^2}, \tag{5.28}$$

of course, if T is interpreted as the temperature (see [14÷17], as well as [23]).

6. Maxwell equal area rule

As an example of applications of our model we consider the problem of stagnant phase boundary. To this end we look for stationary solutions of (5.16)–(5.18), i.e. we assume $n, u, \mathcal{C}$ to be independent of t. Under this assumption Eqs. (5.16)–(5.18) reduce to

$$\frac{d}{dX} w = 0, \tag{6.1}$$

$$\frac{d}{dX}\left\{A(\alpha\varepsilon)^2\left[\frac{2}{w^5}\frac{d^2}{dX^2}w - \frac{5}{w^6}\left(\frac{d}{dX}w\right)^2\right] - \varepsilon\frac{b^2\varrho}{4w^3}\frac{d}{dX}u + \frac{1-u^2}{2w}\left(1+\frac{b}{w}\varrho\right) - \frac{a}{w^3} + \frac{1}{2}\mathcal{C}\right\} = 0, \tag{6.2}$$

and

$$2u\frac{d}{dX}\mathcal{C} - \frac{4\varrho}{\varepsilon}\mathcal{C} = \frac{1-u^2}{w}\frac{d}{dX}u. \tag{6.3}$$

We look for solutions of (6.1)–(6.3) satisfying the following limit conditions

$$\lim_{X\to-\infty}(w, u, \mathcal{C}) = (w_l, u_l, 0), \tag{6.4}$$

and

$$\lim_{X\to\infty}(w, u, \mathcal{C}) = (w_r, u_r, 0), \tag{6.5}$$

where (w_l, u_l) and (w_r, u_r) are some constants.

As we can easily see from (6.1) and (6.4), (6.5), the existence of solutions to our problem demands

$$u_l = u_r\,, \tag{6.6}$$

what we assume, of course.

Then we have

$$u(X) = u_l = u_r\,. \tag{6.7}$$

Integrating Eq. (6.2) with the use of (6.4), (6.7) leads to

$$A(\alpha\varepsilon)^2\left[\frac{2}{w^5}\frac{d^2}{dX^2}w - \frac{5}{w^6}\left(\frac{d}{dX}w\right)\right] + \frac{1}{2}\mathcal{C} = -p(w) + p(w_l), \tag{6.8}$$

where

$$p(w) = \frac{1-u_l^2}{2w}\left(1+\frac{b}{w}\varrho\right) - \frac{a}{w^2}\,. \tag{6.9}$$

Owing to (6.7), Eq. (6.3) reduces to

$$u_l \frac{d}{dX}\mathcal{C} - \frac{2\varrho}{\varepsilon}\mathcal{C} = 0. \tag{6.10}$$

If $u_l = 0$, then $\mathcal{C} = 0$, since $\varrho \neq 0$. Let us assume then that $u_l \neq 0$. In this case

$$\mathcal{C} = \mathcal{C}_0 \exp\left\{\frac{2}{\varepsilon u_1}\int_{X_0}^{X} \varrho\left[n(\zeta)\right] d\zeta\right\}, \tag{6.11}$$

where $\mathcal{C}_0$ and X_0 are some constants.

The general solution (6.11) satisfies $(6.4)_3$ and $(6.5)_3$ if and only if $\mathcal{C}_0 = 0$. Hence, in any case we obtain

$$\mathcal{C}(X) = 0. \tag{6.12}$$

Consequently, Eq. (6.8) reduces to

$$A(\alpha\varepsilon)^2\left[\frac{2}{w^5}\frac{d^2}{dX^2}w - \frac{5}{w^6}\left(\frac{d}{dX}w\right)^2\right] = -p(w) + p(w_l). \tag{6.13}$$

As it is seen, the necessary condition for (6.13) to have a solution satisfying $(6.4)_1$, $(6.5)_1$ is

$$p(w_r) = p(w_l). \tag{6.14}$$

We assume that this equation for w_r has a solution different from w_l

$$w_r \neq w_l, \tag{6.15}$$

otherwise the problem becomes trivial.

Multiply Eq. (6.13) by dw/dX, and integrate the result over X from $X = -\infty$ to the current value of X. It gives

$$A(\alpha\varepsilon)\left(\frac{d}{dX}w\right)^2 = -w^5\int_{w_l}^{w}\left[p(\zeta) - p(w_l)\right] d\zeta. \tag{6.16}$$

Since we need $dw/dX \to 0$ as $X \to \infty$, it must be that

$$\int_{w_l}^{w_r}\left[p(w, u_l) - p(w_l, u_l)\right] dw = 0. \tag{6.17}$$

This is the Maxwell equal area rule [24].

Let us notice that this principle, playing the fundamental role in the equilibrium phase transitions, was obtained directly from our kinetic model. It can be deduced from the hydrodynamic approximation as well.

The hydrodynamic equations (5.22), (5.23) in that case of a steady motion simplify to

$$\frac{d}{dX}u = 0, \tag{6.18}$$

and

$$\frac{d}{dX}\left\{A(\alpha\varepsilon)^2\left[\frac{2}{w^5}\frac{d^2}{dX^2}w - \frac{5}{w^6}\left(\frac{d}{dX}w\right)^2\right] - \varepsilon\mu\frac{d}{dX}u + \frac{1-u^2}{2w}\left(1+\frac{b}{w}\varrho\right) - \frac{a}{w^2}\right\} = 0. \tag{6.19}$$

We look for solutions of Eqs. (6.18), (6.19) such that

$$\lim_{X\to-\infty}(w,u) = (w_l, u_l), \tag{6.20}$$

and

$$\lim_{X\to\infty}(w,u) = (w_r, u_r), \tag{6.21}$$

where (w_l, u_l) and (w_r, u_r) are the same as previously.

An analysis similar to the above leads to (6.7), and therefore Eq. (6.19) reduces to Eq. (6.13), from which we obtain the Maxwell rule (6.17).

References

1. S. CHAPMAN and T.G. COWLING, *The mathematical theory of non-uniform gases*, Cambridge University Press, Cambridge 1952.
2. J.H. FERZIGER and H.G. KAPER, *Mathematical theory of transport processes in gases*, North-Holland, Amsterdam 1972.
3. P. RESIBOIS and M. de LEENER, *Classical kinetic theory of fluids*, Wiley, London 1977.
4. N. BELLOMO, A. PALCZEWSKI and G. TOSCANI, *Mathematical topics in nonlinear kinetic theory*, World Scientific, Singapore 1988.
5. N. BELLOMO, M. LACHOWICZ, J. POLEWCZAK and G. TOSCANI, *The Enskog equation. Mathematical topics in nonlinear kinetic theory II.*, World Scientific, Singapore 1991.
6. M. LACHOWICZ, *Asymptotic analysis of nonlinear kinetic equations: the hydrodynamic limits*, Uniwersytet Warszawski, Wydział Matematyki, Informatyki i Mechaniki, Preprint 1/94, Warsaw, February 1994.
7. A. FREZOTTI and C. SGARRA, *Numerical analysis of a shock-wave solution of the Enskog equation obtained via a Monte Carlo method*, J. Statistical Physics, **75**, 1/2, 193–207, 1993.
8. H. van BEIJEREN and M.H. ERNST, *The modified Enskog equation*, Physica (Utrecht), **68**, 437–456, 1973.
9. M. GRMELA, *Kinetic approach to phase transitions*, J. Statistical Physics, **3**, 3, 347–364, 1971.
10. M. GRMELA, *On the approach to equilibrium in kinetic theory*, J. Math. Physics, **15**, 1, 35–40, 1974.

11. J. KARKHECK and G. STELL, *Kinetic mean-field theories*, J. Chemical Physics, **75**, 3, 1475–1486, 1981.
12. J. KARKHECK and G. STELL, *Maximization of entropy, kinetic equations, and irreversible thermodynamics*, Physical Review A, **25**, 6, 3302–3326, 1982.
13. K. PIECHÓR, *Kinetic theory and thermocapillarity equations*, Arch. Mech., **46**, 6, 937–951, 1994.
14. R. GATIGNOL, *Theorie cinetique des gaz a repartition discrete de vitesses*, Lecture Notes in Physics, No. 36, Springer-Verlag, Berlin 1975.
15. U.M. SULTANGAZIN, *Discrete nonlinear models of the Boltzmann equation* [in Russian], Nauka, Alma-Ata 1985.
16. T. PŁATKOWSKI and R. ILLNER, *Discrete velocity models of the Boltzmann equation: A survey on the mathematical aspects of the theory*, SIAM Review, **30**, 213–255, 1988.
17. R. MONACO and L. PREZIOSI, *Fluid dynamic applications of the discrete Boltzmann equation*, World Scientific, Singapore 1991.
18. K. PIECHÓR, *Discrete velocity models of the Enskog–Vlasov equation*, Transport Theory and Statistical Physics, **23**, 1-3, 39–74, 1994.
19. A.S. ALVES, *Discrete kinetic models with external forces*, European J. Mech., B/Fluids, **9**, 5, 457–467, 1990.
20. Y. CHOQUET-BRUHAT and G. PICHON, *Plasmas with discrete velocities*, [in:] Discrete Models of Fluid Dynamics, A.S. ALVES [Ed.], World Scientific, Singapore, 85–96, 1991.
21. P. CASAL and H. GOUIN, *Invariance properties of inviscid fluids of grade N*, [in:] PDEs and Continuum Models of Phase Transitions, M. RASCLE, D. SERRE, M. SLEMROD [Eds.], Lecture Notes in Physics, No. 344, Springer-Verlag, Berlin, 85–98, 1989.
22. R. COURANT and K.O. FRIEDRICHS, *Supersonic flow and shock waves*, Wiley, Interscience, New York 1948.
23. C. CERCIGNANI, *On the thermodynamics of discrete velocity gas*, Transport Theory and Statistical Physics, **23**, 1-3, 1–8, 1994.
24. R. BALESCU, *Equilibrium and nonequilibrium statistical mechanics*, vol. I, Wiley, New York 1975.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received November 25, 1994.

Arch. Mech., 47, 6, pp. 1113–1124, Warszawa 1995

Unsteady heat transfer in a liquid film on a rotating disk

P. G. KALITZOVA-KURTEVA, S. G. SLAVTCHEV
and I. KURTEV (SOFIA)

HEAT TRANSFER in an unsteady film flow spreading on horizontal rotating disk is studied. The disk is rotating with a prescribed non-steady angular velocity and is subjected to a temperature expressed by a given function of time and the radial coordinate. The free surface of the film is assumed to remain flat and thermally insulated during the motion. By means of a similarity transformation the equations of continuity, momentum and energy are reduced to a system of six ordinary differential equations, four of them for the fluid velocity and the pressure, and the last two for the temperature. Basing on the solution of the dynamic problem obtained in a previous paper, the temperature equations are solved by the Runge–Kutta method and the temperature distribution in the film is obtained. The results show that when the disk temperature decreases with increasing time, the heat flows from the liquid towards the disk for large values of the Prandtl number. For small and moderate Prandtl numbers, the heat flux changes its sign at some distance R, and outside of a circle of radius R it is directed from the rigid surface towards the film. The circle radius increases with time.

Nomenclature

a	thermal diffusivity,
T_0	initial constant temperature,
T_w	disk temperature,
T	local liquid temperature,
u, v, w	velocity components,
r	radial coordinate,
z	normal coordinate,
p	pressure,
p_0	gas pressure above the film,
h	film thickness,
h_0	initial film thickness,
F, G, H, P	functions defined in (2.13),
s	non-dimensional parameter ($= \alpha/\omega$),
q_w	heat flux through the disk surface,
R	radius defined by (3.9),
Re	Reynolds number,
Pr	Prandtl number.

Greek symbols

α	acceleration parameter,
$\gamma_1, \gamma_2, \gamma_3, \gamma_4$	constants to be determined,
ε	constant equal to 1 or -1,
ζ	similarity variable defined by (2.13),
ζ_h	coordinate of the free surface,
θ_1 and θ_2	functions satisfying Eqs. (2.24) and (2.25),

μ dynamic viscosity,
ν kinematic viscosity,
ϱ density,
τ dimensionless time,
φ azimuthal coordinate,
ω initial angular velocity of the disk,
Ω angular velocity.

1. Introduction

HEAT TRANSFER in film flows driven by rotating disks is of major importance in spin coating processes when the disk surface is covered by a thin liquid film which solidifies after spreading. Fluid dynamics and heat transfer problems arise, for example, in producing integrated circuits, computer storage devices, etc.

Usually, the film flows on a rotating disk are three-dimensional and unsteady. Moreover, in many cases the liquid free surface does not remain flat during the motion. These features of the flows produce some difficulties in solving the physical problems of predicting the velocity and temperature distributions in the film, and limit the possibility of obtaining the solutions by means of analytical methods.

Unsteady film flows of a viscous fluid on a disk subject to steady rotation was studied by HIGGINS [1]. He performed an asymptotic method of solution for the cases of low and high Reynolds numbers, as well as for the case of small times but of arbitrary Reynolds numbers. A class of similarity flows in a film with a uniform non-steady thickness at a given unsteady angular velocity of the disk was obtained in [2]. A similarity solution to the equations of motion has been also presented in [3] for the case of a rotating disk with angular velocity proportional to $(1 + \alpha t)^{-1}$ (α is a constant, t the time). Applying a transformation of the variables similar to that used in Kármán's problem [4, 5] for steady rotating disk in an infinite fluid medium, the problem is reduced to a system of four ordinary differential equations which are solved numerically.

The cooling problem for a film on a disk starting stationary rotation from the state of rest is considered in [6] because of its technological importance. The analysis is based on the approach applied in [1] to the corresponding hydrodynamic problem.

In the present paper, a similarity solution to the unsteady energy equation is obtained for the film flow on a disk rotating with the angular velocity considered in [3]. The disk surface is assumed to be at temperature expressed by a given function of time and the radial position. For the case of viscous incompressible fluid, the corresponding hydrodynamical problem separates from the thermal one and the solution of the former one obtained in [3] is used to solve the latter one. By use of a similarity transformation of the variables, the energy equation is reduced to a system of ordinary differential equations for two uknown

functions. The temperature distribution in the film and the rate of heat flow through the disk surface are found from the numerical solution of the system with the corresponding boundary conditions. It is shown that at a fixed time there is a circle on the disk inside which the heat flows from the rigid surface to the liquid, while the reverse heat flux takes place outside of it. The radius of the circle increases with time.

2. Formulation of the problem

Consider a film of viscous heat-conducting liquid on a horizontal disk of the radius which is much larger than the initial uniform thickness h_0 of the film. The system is initially subject to a constant temperature T_0. Suppose that at the time $t = 0$ the disk starts to rotate with an angular velocity

$$\Omega(t) = \frac{\omega}{1 + \alpha t} \tag{2.1}$$

about an axis perpendicular to the disk surface. Here ω is the initial angular velocity, and α is a positive or negative constant. After the initial rotation with velocity ω, the disk accelerates at $\alpha < 0$ and decelerates at $\alpha > 0$.

Assume that the temperature on the disk surface varies as the function

$$T_w = T_0 \left[\frac{1}{1 + \alpha t} + \varepsilon \frac{r^2}{(1 + \alpha t)^2} \frac{\omega}{\nu} \right], \tag{2.2}$$

where r is the distance from the axis of rotation, ν is the kinematic viscosity and ε is a constant equal to 1 or -1. For $\varepsilon = 1$ the surface temperature increases with r at a fixed time, while for $\varepsilon = -1$ it decreases and becomes negative at a certain distance. For positive α the wall temperature at a given point decreases in time, i.e. the disk is cooled, while for negative (and small) values of α it increases, i.e. the disk surface is heated.

Due to the centrifugal forces, the liquid spreads outwards from the axis of rotation and the film thickness decreases with increasing time. We restrict ourselves to the film flows with a free surface maintaining uniform, but unsteady thickness, i.e. the thickness $h(t) = h_0 l(t)$ is a function of time only ($l(0) = 1$).

The physical properties of the liquid are taken to be constant and independent of the temperature. The viscous dissipation is neglected. In a non-rotating, cylindrical polar coordinate system (r, φ, z) with the axis z coinciding with the axis of rotation, the equations of continuity, momentum and energy are written as follows:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{2.3}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} = -\frac{1}{\varrho}\frac{\partial p}{\partial r} + \nu \left[\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2} \right], \tag{2.4}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{uv}{r} + w\frac{\partial v}{\partial z} = v\left[\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2}\right], \tag{2.5}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\varrho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right], \tag{2.6}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z} = a\left[\frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} + \frac{\partial^2 T}{\partial z^2}\right], \tag{2.7}$$

where the velocity components u, v, w are in the direction of increasing r, φ, z, respectively, p is the pressure, ϱ the density, a the thermal diffusivity, T the local temperature.

The solution of Eqs. (2.3) – (2.6) satisfies the following boundary conditions

$$u(r,0,t) = 0, \qquad v(r,0,t) = r\Omega(t), \qquad w(r,0,t) = 0 \qquad \text{at} \quad z = 0, \tag{2.8}$$

$$(p - p_0) - 2\mu\frac{\partial w}{\partial z} = 0, \qquad \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} = 0, \qquad \frac{\partial v}{\partial z} = 0 \qquad \text{at} \quad z = h(t), \tag{2.9}$$

where p_0 is the gas pressure above the free surface and μ is the dynamic viscosity. Motion of the gas phase is neglected.

The condition of continuity of the moving fluid surface is

$$\frac{\partial h}{\partial t} = w(r, h(t), t), \tag{2.10}$$

and the initial conditions are as follows:

$$u = w = 0, \qquad v = \omega r, \qquad h = h_0 \qquad \text{at} \quad t = 0. \tag{2.11}$$

The boundary and initial conditions for the temperature equation (2.7) are:

$$\begin{aligned} &T = T_w \qquad \text{at} \qquad z = 0, \qquad \frac{\partial T}{\partial z} = 0 \qquad \text{at} \quad z = h(t), \\ &T = T_0 \qquad \text{at} \qquad t = 0. \end{aligned} \tag{2.12}$$

The liquid free surface is considered to be thermally insulated.

We seek a class of similarity solutions to the momentum and energy equations in the film on a rotating disk. The solution of the dynamic problem (2.3) – (2.6) and (2.8) – (2.11) has been found in [3] in the following form:

$$\begin{aligned} &u = r\Omega(t)F(\zeta), \qquad v = r\Omega(t)G(\zeta), \qquad w = -\sqrt{\nu\Omega(t)}\,H(\zeta), \\ &p - p_0 = \mu\Omega(t)P(\zeta), \qquad \zeta = z\sqrt{\frac{\Omega(t)}{\nu}}\,. \end{aligned} \tag{2.13}$$

After substitution of expressions (2.13) into (2.3)–(2.6) one obtains a system of ordinary differential equations for the unknown functions

$$H' - 2F = 0, \tag{2.14}$$

$$F'' + F'H + G^2 - F^2 + s\left(\frac{\zeta}{2}F' + F\right) = 0, \tag{2.15}$$

$$G'' + G'H - 2FG + s\left(\frac{\zeta}{2}G' + G\right) = 0, \tag{2.16}$$

$$P' + HH' + H'' + \frac{s}{2}(\zeta H' + H) = 0, \tag{2.17}$$

where $s = \alpha/\omega$ is a non-dimensional parameter and primes denote differentiation with respect to ζ. For $s = 0$ the system formally coincides with the von Kármán system [5].

Boundary conditions (2.8) and (2.9) are written as

$$F = 0, \quad G = 1, \quad H = 0 \quad \text{at} \quad \zeta = 0, \tag{2.18}$$

$$F' = 0, \quad G' = 0, \quad P = -2H' \quad \text{at} \quad \zeta = \zeta_h, \tag{2.19}$$

where the dimensionless variable

$$\zeta_h = \zeta_h(t) = h(t)\sqrt{\frac{\Omega(t)}{\nu}} \tag{2.20}$$

depends, in general, on the time. The equation for $\zeta_h(\tau)$ follows from the kinematic condition (2.10), namely

$$\frac{d\zeta_h}{d\tau} + \frac{1}{1 + s\tau}\left[\frac{s}{2}\zeta_h + H(\zeta_h)\right] = 0, \tag{2.21}$$

where $\tau = \omega t$ is the dimensionless time. The solution of (2.21) satisfies at $\tau = 0$ the condition

$$\zeta_h = \zeta_h(0) = \zeta_h^0 = h_0\sqrt{\frac{\omega}{\nu}}. \tag{2.22}$$

In the case under consideration the dynamic problem has been formulated without using initial conditions (2.11), because the possible solutions of the unsteady Navier–Stokes equations for axisymmetric flows due to a rotating disk (with angular velocity (2.1)) belong to the class of solutions at "large times" after the start of the motion (see the discussion on this matter in [7]). Such solutions describe the boundary-layer type flows. It means that the flow is approximately uniform everywhere except for a boundary layer near the disk. Moreover, in the case of acceleration or deceleration of the disk, the flow becomes quasi-steady in the sense that it behaves like a steady flow with unsteady film thickness.

Owing to the temperature distribution (2.2) on the disk surface, the solution of the energy equation is assumed in the relevant form

$$T = T_1 + T_2\frac{\omega}{\nu}r^2 = T_0\left[\frac{\theta_1(\zeta)}{1+\alpha t} + \frac{\theta_2(\zeta)}{(1+\alpha t)^2}\frac{\omega}{\nu}r^2\right]. \tag{2.23}$$

After substituting (2.23) into equation (2.7), the thermal problem is reduced to a system of two ordinary differential equations for the unknown functions $\theta_1(\zeta)$ and $\theta_2(\zeta)$:

$$\frac{1}{\mathrm{Pr}}\theta_2'' + \left[H + s\frac{\zeta}{2}\right]\theta_2' + 2[s - F]\theta_2 = 0, \tag{2.24}$$

$$\frac{1}{\mathrm{Pr}}\theta_1'' + \left[H + s\frac{\zeta}{2}\right]\theta_1' + s\theta_1 = -\frac{4}{\mathrm{Pr}}\theta_2, \tag{2.25}$$

where $\mathrm{Pr} = \nu/a$ is the Prandtl number.

The boundary conditions at the rigid and free surfaces of the film are as follows

$$\theta_1(0) = 1, \qquad \theta_2(0) = \varepsilon \qquad \text{at} \quad \zeta = 0, \tag{2.26}$$

$$\theta_1'(\zeta_h) = \theta_2'(\zeta_h) = 0 \qquad \text{at} \quad \zeta = \zeta_h. \tag{2.27}$$

The system (2.24)–(2.27) is solved by the approach used in [3] to obtain the solution of the dynamic problem (2.14)–(2.19).

3. Method of solution and results

The main difficulty in solving the mathematical problems (2.14)–(2.19) and (2.24)–(2.27) is that the right-hand boundary of the interval $[0, \zeta_h]$ is unknown in advance. It depends implicitly on the time through the general solution of equation (2.21)

$$\tau = \frac{1}{s}\left[-1 + \exp\left(\int_{\zeta_h}^{\zeta_h^0} \frac{s\,d\zeta}{H(\zeta) + \frac{s}{2}\zeta}\right)\right], \tag{3.1}$$

where the function $H(\zeta)$ under the integral is taken on the film surface (at $\zeta = \zeta_h$) and should be calculated from the solution of the system (2.14)–(2.16).

In [3], in order to overcome this difficulty, the two-point boundary problem (2.14)–(2.19) has been replaced by a proper initial value problem whose solutions were expected to satisfy the original problem for some period of time, i.e. in the interval $[0, \zeta_h]$ with $\zeta_h(\tau)$ calculated from (2.19). The system (2.14)–(2.16) with (2.18) and additional boundary conditions

$$F' = \gamma_1, \qquad G' = \gamma_2 \qquad \text{at} \quad \zeta = 0, \tag{3.2}$$

has been solved by the fifth-order Runge–Kutta method. Using a shooting technique, the values of γ_1 and γ_2 have been predicted from the condition of vanishing of the functions F' and G' in a certain interval of the independent variable. Equation (2.17) for the pressure can be integrated analytically [3].

The numerical calculations show that for a given s there exists only one pair of values of γ_1 and γ_2 determining the solution of the formulated initial problem with vanishing first derivatives of F and G in some interval $[\zeta_1, \zeta_0]$, where $\zeta_1 < \zeta_0$. In this interval the functions F and G are zero and H is a non-zero constant. Moreover, for positive s, the right-hand boundary of the interval tends to infinity, i.e. $F = G = 0$ and $H = b_1 = \text{const}$ for every ζ larger than a certain value ζ_1.

For two values of s, the functions F, G and H are presented in Figs. 1 and 2. For $s = 0.2$ the angular velocity of the disk decreases from the initial value, while for $s = -0.2$ the disk accelerates after the start.

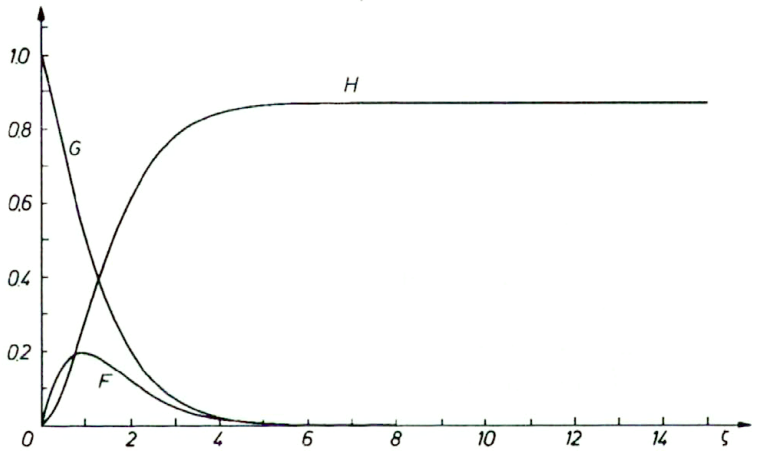


FIG. 1. Functions F, G, H against ζ for $s = 0.2$.

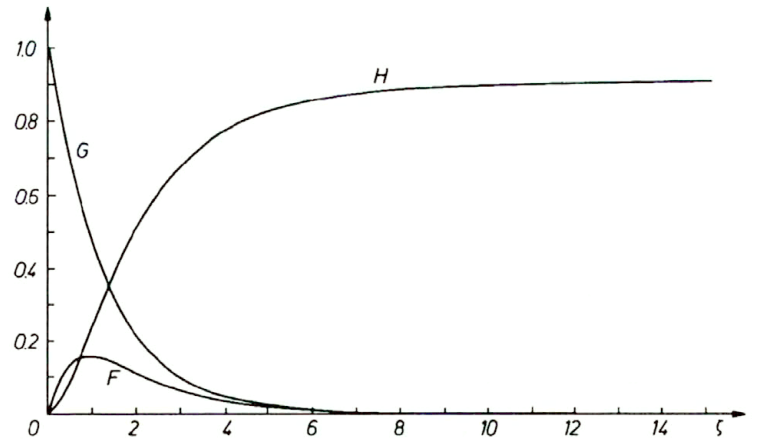


FIG. 2. Functions F, G, H against ζ for $s = -0.2$.

Consider an interval $[\zeta_1, \zeta_0]$ where the solution of the initial value problem satisfies $F = G = 0$ and $H = b_1 = sb$ (b is a new constant). Substituting the value of H into (3.1) and replacing ζ_h^0 by ζ_0 and ζ_h by ζ_1, one can calculate the time period (say τ_1) during which the film thickness varies as

$$\frac{h}{h_0} = l(\tau) = 1 + A - A(1 + s\tau)^{1/2}, \tag{3.3}$$

where $A = 2b/\zeta_0$. The thickness decreases generally as $(1 + s\tau)^{1/2}$, but at very small times ($s\tau < 1$) it is a linear function of time. Note that the similarity unsteady flow is characterized by the fluid surface velocity presented by one non-zero component directed to the disk.

The thermal equations (2.24) – (2.25) are solved with the boundary conditions (2.26) and the additional conditions

$$\theta_1' = \gamma_3, \qquad \theta_2' = \gamma_4 \qquad \text{at} \quad \zeta = 0 \tag{3.4}$$

using also the Runge – Kutta method. The constants γ_3 and γ_4 are calculated in such a way which allows us to satisfy the condition of vanishing derivatives θ_1' and θ_2' at least in the same interval $[\zeta_1, \zeta_0]$. The equation (2.24) for θ_2 is solved first, and then the solution of (2.25) is obtained.

Representative solutions for Pr = 1 and $\varepsilon = 1$ are given in Figs. 3 and 4 for three values of s. For the same Prandtl number and $\varepsilon = -1$, the function θ_1 is shown in Fig. 5, while θ_2 coincides with the function presented in Fig. 4, but it is taken with the opposite sign. For different values of Pr ($s = 0.2$ and $\varepsilon = 1$) these functions are plotted in Figs. 6 and 7, and some values of γ_3 and γ_4 are reported in Table 1.

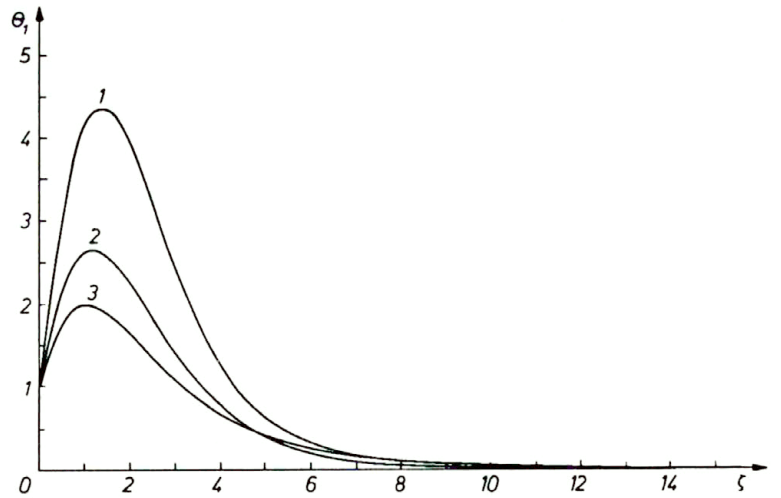


FIG. 3. Function θ_1 against ζ for $\varepsilon = 1$, Pr = 1 and different values of s: *1* — $s = 0.2$; *2* — $s = 0$; *3* — $s = -0.2$.

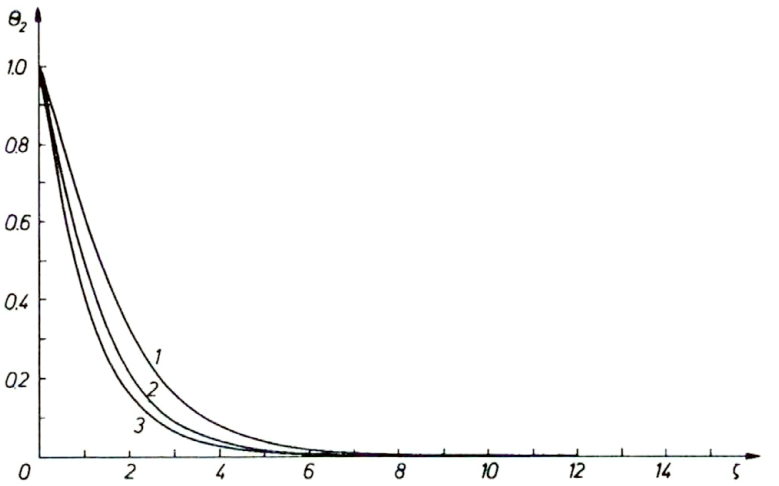


FIG. 4. Function θ_2 against ζ for $\varepsilon = 1$, Pr = 1 and different values of s: *1* — $s = 0.2$; *2* — $s = 0$; *3* — $s = -0.2$.

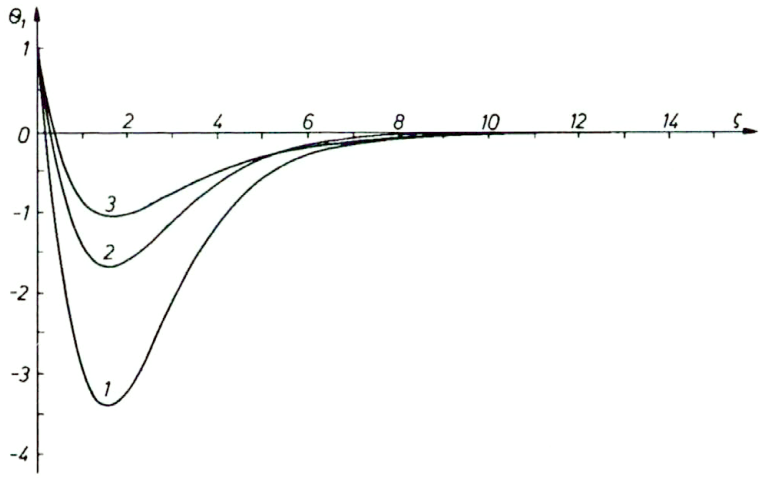


FIG. 5. Function θ_1 against ζ for $\varepsilon = -1$, Pr = 1 and different values of s: *1* — $s = 0.2$; *2* — $s = 0$; *3* — $s = -0.2$.

It is not surprising that there exist values of γ_3 and γ_4 for which the solutions of Eqs. (2.24) – (2.25) vanish together with their first derivatives in some interval coinciding with (or even being larger than) the interval $[\zeta_1, \zeta_0]$, where the functions F' and G' are zero. Such a behaviour of the solutions can be predicted from the simplified form of these equations obtained by substituting $F = 0$ and $H = sb$. For example, introducing a new independent variable

$$\eta = s^{1/2}\left(b + \frac{\zeta}{2}\right), \tag{3.5}$$

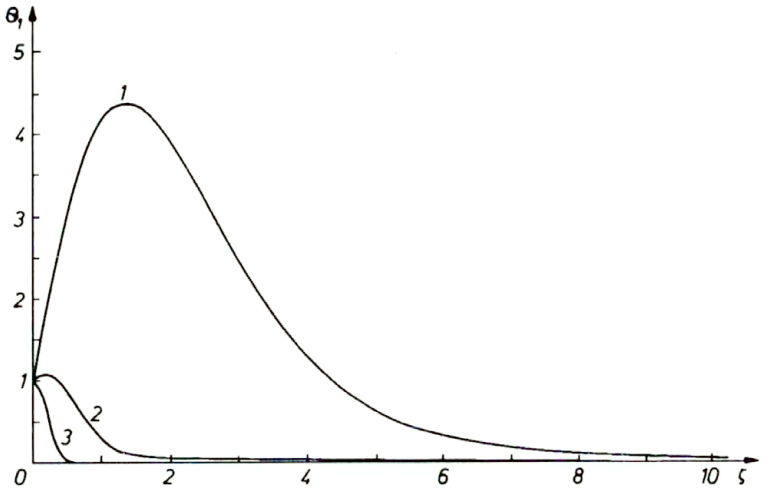


FIG. 6. Function θ_1 against ζ for $\varepsilon = 1$, $s = 0.2$ and different values of Pr: *1* — Pr = 1; *2* — Pr = 10; *3* — Pr = 100.

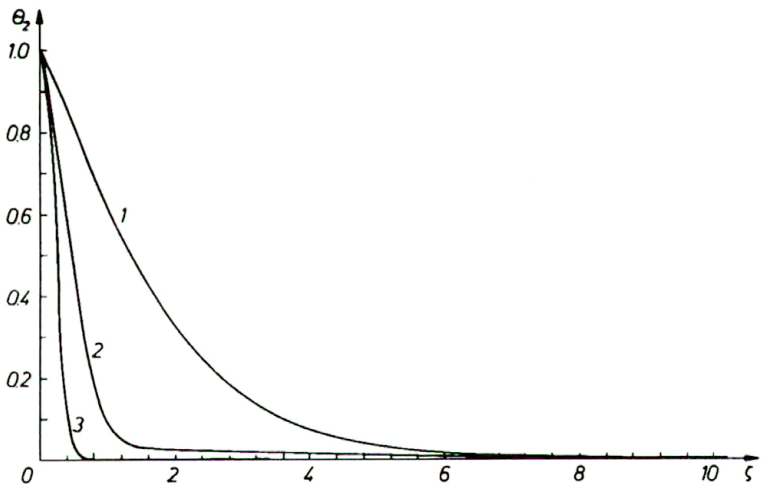


FIG. 7. Function θ_2 against ζ for $\varepsilon = 1$, $s = 0.2$ and different values of Pr: *1* — Pr = 1; *2* — Pr = 10; *3* — Pr = 100.

one obtains the reduced equation (2.24) in the form (Pr = 1)

$$\theta_2'' + 2\eta\theta_2' + 8\theta_2 = 0. \tag{3.6}$$

A partial solution of this equation is the third derivative of the function $\exp(-x^2)$. So, the solution and its derivatives vanish very quickly when the independent variable increases. For this reason the fluid temperature varies, in general, in a sublayer lying on the disk surface. The sublayer thickness becomes small with increasing Prandtl number (see Figs. 6 and 7).

Table 1.

Pr	γ_3	γ_4
1	5.321	−0.329
10	0.996	−0.679
20	0.496	−0.629
40	0.196	−0.279
100	0.1125	0.950

The heat flux across the disk surface is calculated by the expression

$$q_w = -\lambda \frac{\partial T}{\partial z}\bigg|_{z=0} = -\lambda T_0 \left(\frac{\omega}{\nu}\right)^{1/2} \left[\gamma_3 + \frac{\gamma_4}{(1+s\tau)} \frac{\omega}{\nu} r^2\right] \frac{1}{(1+s\tau)^{3/2}}. \tag{3.7}$$

Defining a local Nusselt number as $\mathrm{Nu} = q_w r/\lambda T_0$, the formula (3.7) takes the form

$$\mathrm{Nu} = -\mathrm{Re}_r \left[\gamma_3 + \mathrm{Re}_r^2 \frac{\gamma_4}{1+s\tau}\right] \frac{1}{(1+s\tau)^{3/2}}. \tag{3.8}$$

Here $\mathrm{Re}_r = \sqrt{\omega/\nu}\, r$ is the Reynolds number.

The formula (3.7) shows that at a fixed time, the heat flux depends on the radial coordinate and does not change its direction if the constants γ_3 and γ_4 have the same sign. When these parameters have different signs, the flux changes its sign at r equal to

$$R = \left[-\frac{\gamma_3}{\gamma_4} \frac{\nu}{\omega}(1+s\tau)\right]^{1/2}. \tag{3.9}$$

For example, at $\varepsilon = -1$, when $\gamma_3 < 0$ (see Fig. 5) and $\gamma_4 > 0$, the flux is directed from the wall towards the liquid inside the circle of radius R and in opposite direction outside of it. Note that the radius of the circle increases with time.

In the case of $\varepsilon = 1$ and positive α, the surface temperature (2.2) at a fixed position decreases with increasing time. As it is seen from Table 1, for small and moderate values of the Prandtl number, the heat flows from the liquid towards the disk for $r < R$, and in opposite direction for $r > R$. In the case of $\mathrm{Pr} = 100$, when the thickness of the thermal boundary layer developing on the rigid surface is quite small, the heat flux is directed towards the disk.

For small values of the Prandtl number, Eqs. (2.24) and (2.25) can be simplified and the reduced equations have the analytical solution

$$\theta_2 \equiv \varepsilon, \qquad \theta_1 = 2\varepsilon\zeta(2\zeta_h - \zeta) + 1. \tag{3.10}$$

Then, the Nusselt number is presented by the expression

$$\text{Nu} = -\varepsilon \text{Re}_r \frac{4\zeta_h}{(1 + s\tau)^{3/2}} \tag{3.11}$$

showing that the heat flux on the disk has only one direction depending on ε.

Acknowledgements

This work was partially supported by the National Science Foundation of Bulgaria under Contract No. MM-12/1991.

References

1. B.G. HIGGINS, *Film flow on a rotating disk*, Phys. Fluids, **29**, 11, 3522–3529, 1986.
2. O.M. LAVRENTIEVA, *Steady and similarity viscous flows in a film on a rotating disk* [in Russian], [in:] Theor. Appl. Mech., Proc. Sixth National Congr. Theor. Appl. Mech., Sept. 1989, Varna. Publ. House Bulg. Acad. Sci., Sofia, **3**, 161–164, 1990.
3. S.G. SLAVTCHEV and P.G. KALITZOVA-KURTEVA, *An exact solution to the Navier-Stokes equations for unsteady flow in a liquid film on a rotating disk*, [in:] Some Applied Problems in Fluid Mechanics, Ind. Statistical Inst., Calcutta, 45–57, 1993.
4. Th. VON KÁRMÁN, *Über laminare und turbulente Reibung*, Z. Angew. Math. Mech., **1**, 233, 1921.
5. H. SCHLICHTING, *Boundary layer theory*, McGraw-Hill, New York 1968.
6. B.S. DANDAPAT and P.C. RAY, *Film cooling on a rotating disk*, Int. J. Non-Linear Mech., **25**, 5, 569–582, 1990.
7. L. ROSENHEAD [Ed.], *Laminar boundary layers*, Oxford University Press, London 1963.

INSTITUTE OF MECHANICS
BULGARIAN ACADEMY OF SCIENCE
and
SOFIA UNIVERSITY OF TECHNOLOGY, SOFIA, BULGARIA.

Received December 19, 1994.

Arch. Mech., 47, 6, pp. 1125–1155, Warszawa 1995

Bounds for the overall properties of viscoelastic heterogeneous and composite materials

C. HUET (LAUSANNE)

THE PROBLEM of finding bounds for the creep and relaxation functions tensors for heterogeneous, and possibly anisotropic, viscoelastic materials is considered. It is shown that extensive and unrestrictive solutions can be given to this problem. The derivation makes use of an associated elasticity problems approach and involves several ingredients of thermodynamic character that are explained in the paper. In particular, it involves the viscoelastic counterpart, derived by Brun (1965), of the classical local Clapeyron equation for the elastic strain energy together with the pseudo-convolutive formalism previously introduced by Huet (1988), (1992). Through pseudoconvolutive extensions of the classical Hill condition widely used for elastic heterogeneous media, the derivation makes use of the classical minimum theorems of elasticity applied to special kinds of boundary conditions. Strict unilateral bounds for the rates of the relaxation functions tensor and of the creep functions tensor are thus obtained together with strict bilateral bounds for the relaxation functions tensor and for the creep functions tensor themselves. These bounds involve for all times the computation of appropriate elastic effective properties only, allowing thus the numerous results of the elasticity theory to be used.

1. Introduction

THE PROBLEM of studying and predicting the influence of the composition of materials on their overall or macroscopic properties and/or the distribution of internal stresses through micromechanical approaches going beyond purely empirical ones, is one of the most important goals of material science in various fields of engineering, see for instance HUET and ZAOUI [1], HUET [2]. This is specially true for viscoelastic properties, the assessment of which always requires long and difficult experiments, even in the isotropic case.

In fact, most of the developments made about the theoretical evaluation of the overall mechanical properties of random heterogeneous materials following from the knowledge of the properties of their constituents relate to linear elastic materials, for which many results have been obtained over the past thirty years. For random materials, they relate to effective properties, defined as the properties of a so-called representative volume, for which the overall properties are considered to be independent of the boundary conditions, and thus can be used as material properties in structural calculations, the heterogeneous material being replaced by an equivalent homogeneous one. Among the numerous approaches that have been used, the derivation of bounds through some variational approach involving appropriate minimum theorems has emerged as a very useful and powerful tool, see HILL [3, 4], HASHIN and SHTRIKMAN [5], WALPOLE [6, 7], etc. They have culminated in the systematic theory derived by KRÖNER [8–11] showing that, depending of the amount of statistical information introduced in

the formulation of the problem, these bounds can be organized and generalized in form of a full series of bounds of increasing orders, approaching the real solution when the amount of statistical information introduced as data into the problem is increased.

Since almost the very beginning, several attempts were made in order to extend the results obtained for the elastic case to the effective properties of heterogeneous viscoelastic materials. This has been done mainly through quasi-elastic approximations, for special values of time, in the Laplace-transform domain and/or for complex moduli or compliances, see namely HASHIN [12–14], SCHAPERY [15–17], MINSTER [18], ROSCOE [19, 20], CHÉTOUI [21], CHÉTOUI *et al.* [22], HUET [23], GIBIANSKY and MILTON [24].

Very few results were obtained concerning the bounds of the creep and relaxation properties in the time domain, and only under very restrictive conditions, CHRISTENSEN [25, 26] or as pseudo-elastic approximations, SCHAPERY [17]. In particular, the idea used by Christensen was to look for minimum theorems in linear viscoelasticity problems that might be the viscoelastic counterparts of the minimum theorems of linear elasticity on which most of the various bounds obtained for elastic media are based. But no sufficiently general theorems have been obtained that might provide us with bounds of the same degree of generality as the elastic ones. This is discussed for instance in CHRISTENSEN [26]. The variational theorems of GURTIN [27] do not provide the information needed since they are stationary theorems only, and not the required minimum theorems. Many particular minimum theorems which have been derived since the Gurtin stationary ones are not able to provide the expected results. This is so because, as discussed extensively in HUET [28], most of them relate to bodies involving one scalar viscoelastic parameter only, or give results for the instantaneous elasticity only. Even the more general viscoelastic minimum theorems that we recently derived in [28, 29] through a so-called pseudo-convolutive approach, are submitted to restricting symmetry conditions that have to be checked for each particular problem. This strongly restricts the domain of validity of the bounding results that we published in [30].

This led us more recently to another approach, still using, as in [28, 29], the general thermodynamic properties of linear viscoelastic materials, first derived by STAVERMAN and SCHWARZL [31], but avoiding the use of viscoelastic minimum theorems. The purpose of this paper is to present this approach, and to show that it allows us to derive various kinds of bounds that apply to the creep and relaxation functions of heterogeneous materials of any kind. In Sec. 2, we recall elementary but basic concepts about the local properties of viscoelastic materials and introduce the mechanical definitions of their effective properties. In Sec. 3 we recall, following mainly HUET [28], the basic thermodynamic concepts for viscoelastic materials that are used in the sequel. In particular, we recall the Brun-Clapeyron equation that leads to our pseudo-convolutive formalism, the latter extending to the thermodynamics of viscoelastic materials the classi-

cal Stieltjès convolution currently applied to viscoelasticity at least since GURTIN [27], and in particular by LAWS and MCLAUGHLIN [32], LAWS [33]. This allows us to provide pseudo-convolutive extensions of the classical Hill conditions that are key points in the derivation of bounds. The results of these two sections are explicitly used in the subsequent derivations. In Sec. 4, we sketch the principles of the approach leading to the results that are to be presented and we recall the results given by the classical variational theorems when applied to appropriate boundary conditions. In Secs. 5 and 6, we successively derive unilateral bounds for the relaxation functions tensor, its rate, the rate of the creep functions tensor and the creep functions tensor itself. In Sec. 7, we derive bilateral bounds for the creep and relaxation functions tensors. Some discussion and possible extensions of the results are provided in a concluding section.

2. Effective properties for viscoelastic materials

2.1. Local relaxation and creep functions tensors

We consider a heterogeneous viscoelastic body with linear viscoelastic constituents without aging. Some of them may be possibly purely linear elastic or purely viscous-Newtonian – as limiting cases of linear viscoelasticity. We restrict ourselves to small strains. As well known, the behaviour of each such linear viscoelastic constituent is completely defined, in the observable variables approach, by expressing the stress tensor as a linear functional of the history of the strain tensor ε. Thus, for each point x inside the body, one may write two dual constitutive equations which are of the simple Volterra – Stieltjès integral form, with convolutive kernels, that can be interpreted as expressing the BOLTZMANN [34] superposition principle:

$$\sigma_{ij}(t) = \int_{o^-}^{t} r_{ijkl}(t-u) : d\varepsilon_{kl}(u),$$

$$\varepsilon_{ij}(t) = \int_{o^-}^{t} f_{ijkl}(t-u) : d\sigma_{kl}(u). \tag{2.1}$$

In Eqs. (2.1) shown above, r is the local relaxation function tensor, a tensor of the fourth rank with components being the functions $r_{ijkl}(t)$ of the time t; f is the local creep function tensor, also of the fourth rank with components $f_{ijkl}(t)$. Throughout the paper, the summation convention upon repeated subscripts is used and it is understood that r, f and the field variables are not only functions of the time t, but also functions of the position x of the considered point inside the body, including its boundary.

We will also write Eqs. (2.1) in the convolutive form:

$$\sigma_{ij} = r_{ijkl} \circ \varepsilon_{kl}, \qquad \varepsilon_{ij} = f_{ijkl} \circ \sigma_{kl} \tag{2.2}$$

or, in a symbolic tensor form:

$$\sigma = r \circ \varepsilon, \qquad \varepsilon = f \circ \sigma, \tag{2.3}$$

where $a \circ b$ denotes the temporal Stieltjès convolution between scalars or tensors:

$$a \circ b = \int_{o^-}^{t} a(t-u) : db(u). \tag{2.4}$$

As it is usually done, we assume that the tensors r and f are zero for negative times and have at every instant the same symmetries as in linear elasticity:

$$r_{ijkl}(t) = r_{jikl}(t) = r_{ijlk}(t) = r_{klij}(t) \quad \forall t. \tag{2.5}$$

No restriction is made about the material symmetries of the constituents of the body, that may involve any kind of anisotropy. For times going to infinity, the constituents may behave indifferently as solids or fluids unless otherwise specified. All the evolutions are supposed to be quasistatic in small deformations.

The two dual constitutive equations, Eqs. (2.3), are not independent. As well known, the associativity property of the Stieltjès convolution yields that r and f are convolutive reciprocals, meaning that they satisfy the following Stieltjès–Volterra's equations:

$$r \circ f = f \circ r = I^4 Y, \tag{2.6}$$

where $Y(t)$ is the Heaviside step function of time, and where I^4 is the unit tensor of the fourth rank, with components expressed in terms of the Kronecker δ by:

$$I^4_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{2.7}$$

In fact, as it is also well known, these Stieltjès–Volterra equations are Volterra equations of the second kind, for which a unique solution always exists.

The Stieltjès convolution has thus all the properties of a commutative algebraic field.

In addition to the properties recalled above, it has been shown by BIOT [35] that every viscoelastic behaviour defined by a set of linear differential equations in terms of the observable variables and of a set of internal variables with the classical Onsager symmetry admits a spectral representation even in the anisotropic case. This spectral representation has also been studied by MANDEL [36] and

more recently by COUSSY [37]. For most real viscoelastic materials, the relaxation and retardation spectra are continuous. In particular, this is the case for the behaviours based on simple or multiple power-law creep elements. They are specially important since they are very often observed in real materials, see for instance HUET [38, 39], RUDDOCK *et al.* [40].

For materials with a continuous spectrum, the relaxation and creep function tensors are given by

$$r(t) = \int_0^\infty \varrho(\tau) e^{-1/\tau}\, d\tau\,, \tag{2.8}$$

$$f(t) = \int_0^\infty \psi(\tau) \left[1 - e^{-1/\tau}\right] d\tau\,, \tag{2.9}$$

where $\psi(\tau)$ and $\varrho(\tau)$ are positive definite tensor functions of the fourth rank, and where the curves defined by the functions $\varrho_{ijkl}(\tau)$ and $\psi_{ijkl}(\tau)$ in terms of the scalar variable τ define the relaxation spectra and the retardation spectra, respectively.

The case of a discrete spectrum corresponds to taking $\psi(\tau)$ and $\varrho(\tau)$ in the following forms, making use of the Dirac δ (see for instance [41]):

$$\psi(\tau) = \sum_k \psi_k \delta(\tau - \tau_k), \tag{2.10}$$

$$\varrho(\tau) = \sum_h \varrho_h \delta(\tau - \tau_h). \tag{2.11}$$

An important consequence of the existence of a spectral representation is that the time derivatives of the relaxation function tensor and of the creep function tensor are monotonic functions of time: the second derivative of the relaxation tensor is positive definite while that of the creep tensor is negative definite. In fact, as can be seen from Eqs. (2.8) and (2.9), the successive time derivatives of $r(t)$ are of alternating signs (in the tensorial definite positiveness sense) and the same holds for $f(t)$.

As a consequence, one has from Eqs. (2.6), the following inequalities for the algebraic product of the present values $r(t)$ and $f(t)$ of r and f at the same single instant t:

$$r : f < I^4\,, \tag{2.12}$$

$$f : r < I^4\,. \tag{2.13}$$

This is because, from

$$0 < r(t) < r(t-u) \quad \text{and} \quad \dot{f}(u) > 0 \quad \forall\, u \in]0, t[\,, \tag{2.14}$$

$$0 < f(t-u) < f(t) \quad \text{and} \quad \dot{r}(u) > 0 \quad \forall\, u \in]0, t[\,, \tag{2.15}$$

one finds immediately

$$r(t) : f(t) = r(t) : \int_{o^-}^{t} df(u) = \int_{o^-}^{t} r(t) : df(u) < \int_{o^-}^{t} r(t-u) : df(u) = I^4, \tag{2.16}$$

and

$$f(t) : r(t) = f(t) : \int_{o^-}^{t} dr(u) = f(t) : r(0^+) + \int_{o^+}^{t} f(t) : \dot{r}(u)\, d(u)$$
$$< f(t) : r(0^+) + \int_{o^+}^{t} f(t-u) : \dot{r}(u)\, du = I^4 \tag{2.17}$$

for all viscoelastic materials exhibiting instantaneous elasticity, meaning that $r(0^+)$ is finite. This latter assumption will be made all over the paper.

All these classical properties of the relaxation and creep function tensors and their rates will be used in the sequel.

2.2. Effective relaxation and creep functions tensors: mechanical approach

In a fashion similar to what is currently done in the literature for the elastic case, we assume that we may define the effective properties of a heterogeneous linear viscoelastic material through linear convolutive relations between volume averages of the stress and strain tensors taken over a representative volume:

$$\langle r \circ \varepsilon \rangle = \langle \sigma \rangle = r^{\text{eff}} \circ \langle \varepsilon \rangle \tag{2.18}$$

and

$$\langle f \circ \sigma \rangle = \langle \varepsilon \rangle = f^{\text{eff}} \circ \langle \sigma \rangle, \tag{2.19}$$

where r^{eff} and f^{eff} stand here for the effective relaxation and creep function tensors, respectively. They are also tensors of the fourth rank, with corresponding current components r^{eff}_{ijkl} and f^{eff}_{ijkl} being also functions of time. This usual assumption is made possible by the linear character of the viscoelasticity problem involved.

Equivalent indicial formulations of the above equations read thus:

$$\begin{aligned} \langle \sigma_{ij} \rangle &= r^{\text{eff}}_{ijkl} \circ \langle \varepsilon_{kl} \rangle, \\ \langle \varepsilon_{ij} \rangle &= f^{\text{eff}}_{ijkl} \circ \langle \sigma_{kl} \rangle. \end{aligned} \tag{2.20}$$

As shown by LAWS [33] in 1980, r^{eff} and f^{eff} have at least the same general symmetries – but generally, of course not the particular material ones – as the local r and f tensors, even for the most general anisotropic case. In particular, they also remain invariant under exchange of the pairs of subscripts (ij) and (kl).

The problem which is addressed in this paper is to find relationships between these effective viscoelasticity tensors of the heterogeneous material and the local viscoelasticity tensors of its constituents.

Of course, no dependence upon the space variable x is involved in the effective Equations (2.18) to (2.20). From Eqs. $(2.20)_1$ one gets, because of the commutativity property of the volume average and the temporal convolution operators:

$$\langle \sigma \rangle = \langle r \circ \varepsilon \rangle = \langle r \rangle \circ \langle \varepsilon \rangle + \langle r' \circ \varepsilon' \rangle \tag{2.21}$$

and

$$\langle \varepsilon \rangle = \langle f \circ \sigma \rangle = \langle f \rangle \circ \langle \sigma \rangle + \langle f' \circ \sigma' \rangle, \tag{2.22}$$

where a' defines the local fluctuation $a - \langle a \rangle$ of the variable a around its average $\langle a \rangle$. This shows that – like for the elastic case – the viscoelastic effective properties differ in general from the volume averages of the corresponding local properties.

From Eqs. (2.18) and (2.19) it can be seen that, for bodies with the representative volume, and by definition, the effective properties are supposed to be well defined and independent of the spatial distribution of boundary conditions. Then, the property of r and f being convolutive reciprocals, extends to the effective properties r^{eff} and f^{eff}:

$$r^{\text{eff}} \circ f^{\text{eff}} = f^{\text{eff}} \circ r^{\text{eff}} = I^4 Y \,. \tag{2.23}$$

Of course, for the same reasons as for the local properties above, this yields the effective analogs of inequalities (2.12) and (2.13):

$$\begin{aligned} r^{\text{eff}} &: f^{\text{eff}} < I^4 \,, \\ f^{\text{eff}} &: r^{\text{eff}} < I^4 \,. \end{aligned} \tag{2.24}$$

These properties will also be used later on in this paper. Of course, the response of the viscoelastic representative volume depends on the history of the imposed variables, and not only on their current values. This makes a difference as compared with the elastic case.

2.3. Effective relaxation rate and effective creep rate tensors: mechanical definitions

Let us consider again the definitions of the effective relaxation functions tensors, Eqs. (2.18) and (2.19). Taking their time derivatives and denoting the time derivative of r^{eff} by:

$$\dot{r}^{\text{eff}} = \frac{d}{dt} r^{\text{eff}} \tag{2.25}$$

we get, from the convolutive derivation rule,

$$\langle r(0^+) : \dot{\varepsilon} \rangle + \langle \dot{r} \circ \varepsilon \rangle = \langle \dot{\sigma} \rangle = r^{\text{eff}}(0^+) : \langle \dot{\varepsilon} \rangle + \dot{r}^{\text{eff}} \circ \langle \varepsilon \rangle. \tag{2.26}$$

Since one has, for the instantaneous elastic part of the response:

$$\langle r(0^+) : \dot{\varepsilon} \rangle = r^{\text{eff}}(0^+) : \langle \dot{\varepsilon} \rangle, \tag{2.27}$$

it follows that

$$\langle \dot{r} \circ \varepsilon \rangle = \dot{r}^{\text{eff}} \circ \langle \varepsilon \rangle = \langle \dot{\sigma} \rangle - r^{\text{eff}}(0^+) \circ \langle \dot{\varepsilon} \rangle. \tag{2.28}$$

This shows that the effective tensor associated to $\dot{r}$ is simply the time derivative of r^{eff}.

Similarly, from Eq. (2.19), denoting the time derivative of the effective creep functions tensor f^{eff} by

$$\dot{f}^{\text{eff}} = \frac{d}{dt} f^{\text{eff}}, \tag{2.29}$$

we get

$$\langle f(0^+) : \dot{\sigma} \rangle + \langle \dot{f} \circ \sigma \rangle = \langle \dot{\varepsilon} \rangle = f^{\text{eff}}(0^+) : \langle \dot{\sigma} \rangle + \dot{f}^{\text{eff}} \circ \langle \sigma \rangle. \tag{2.30}$$

Together with

$$\langle f(0^+) : \dot{\sigma} \rangle = f^{\text{eff}}(0^+) : \langle \dot{\sigma} \rangle \tag{2.31}$$

this yields:

$$\langle \dot{f} \circ \sigma \rangle = \dot{f}^{\text{eff}} \circ \langle \sigma \rangle = \langle \dot{\varepsilon} \rangle - f^{eff}(0) \circ \langle \dot{\sigma} \rangle. \tag{2.32}$$

This shows that the effective tensor associated to $\dot{f}$ is simply the time derivative of f^{eff}. This will also be used later on in this paper.

3. Thermodynamics of viscoelastic behaviour for heterogeneous media

3.1. Integral expressions for the local free energy, free enthalpy and dissipation densities in isothermal linear viscoelasticity

In order to make further progress, we need to go beyond the mechanical definitions given above and consider the energetics of our viscoelastic bodies.

For linear viscoelasticity without aging, the thermodynamic framework has been developed mainly in the fifties and sixties, through the works by STAVERMAN and SCHWARZL [31], BIOT [35], MANDEL [36], BRUN [42, 43], and in another context by COLEMAN [44] and some others.

Staverman and Schwarzl showed – in the one-dimensional case and through an internal variables approach – that all the viscoelastic models that are mechanically equivalent are also energetically equivalent. Moreover, they showed – in the one-dimensional case also – that the free energy density φ per unit volume

and the dissipated power density D per unit volume (briefly named *dissipation* in the sequel) at time t are quadratic functionals of the macroscopic strain history up to time t, involving as their kernels the relaxation function tensor r up to time $2t$ and its rate $\dot{r}$ up to $2t$, respectively.

These results were then extended by several authors to the three-dimensional case in the following form in which the time origin is set at the first loading:

$$\begin{aligned} \varphi(t) &= \frac{1}{2}\int_{o^-}^{t}\int_{o^-}^{t} r(2t-u_1-u_2):d\varepsilon(u_2):d\varepsilon(u_1), \\ D(t) &= -\int_{o^-}^{t}\int_{o^-}^{t} \dot{r}\,(2t-u_1-u_2):d\varepsilon(u_2):d\varepsilon(u_1). \end{aligned} \tag{3.1}$$

In these equations, the brackets needed to separate the two contracted products in each integrand have been omitted to simplify the formulas. This is done all along this paper except when specially stated. It should be remembered that the first product has to be read first.

Similarly, three-dimensional Staverman and Schwarzl formulas can be written for the free enthalpy density γ and the dissipation D as quadratic functionals of the stress history up to time t involving as their kernels the creep function tensor f up to time $2t$ and its rate $\dot{f}$ up to $2t$, respectively,

$$\begin{aligned} \gamma(t) &= -\frac{1}{2}\int_{o^-}^{t}\int_{o^-}^{t} f(2t-u_1-u_2):d\sigma(u_2):d\sigma(u_1), \\ D(t) &= \int_{o^-}^{t}\int_{o^-}^{t} \dot{f}\,(2t-u_1-u_2):d\sigma(u_2):d\sigma(u_1), \end{aligned} \tag{3.2}$$

where γ is defined, as usually, as the local Legendre transform of φ relative to σ:

$$\gamma = \varphi - \sigma:\varepsilon\,. \tag{3.3}$$

In case of one-step histories with amplitudes ε^0 and σ^0, respectively, Eqs. (3.1) and (3.2) become, in terms of ε^0:

$$\begin{aligned} \varphi(t) &= \frac{1}{2}r(2t):\varepsilon^0:\varepsilon^0, \\ D(t) &= -\dot{r}\,(2t):\varepsilon^0:\varepsilon^0, \\ \gamma(t) &= -\frac{1}{2}[2r(t)-r(2t)]:\varepsilon^0:\varepsilon^0 \end{aligned} \tag{3.4}$$

and, in terms of σ^0:

$$\gamma(t) = -\frac{1}{2} f(2t) : \sigma^0 : \sigma^0,$$

$$D(t) = \dot{f}(2t) : \sigma^0 : \sigma^0, \tag{3.5}$$

$$\varphi(t) = \frac{1}{2}[2f(t) - f(2t)] : \sigma^0 : \sigma^0.$$

From the second principle of thermodynamics, expressed here as the Clausius–Duhem inequality in local form:

$$D = -\left(\dot{\varphi} - \sigma : \dot{\varepsilon}\right) \geq 0, \tag{3.6}$$

the dissipation should be positive for every evolution and zero for equilibrium. Thus, for isothermal relaxation, $D(t)$ – and thus $(-\dot{\varphi})$ in Eq. $(3.4)_2$ – is positive for every ε^0, meaning that $\dot{r}$ is a definite negative tensor for all finite times, conforming thus to the common experimental observations. On the other hand, for time going to infinity, φ tends to its final equilibrium value which is the non-negative final elastic energy of the body. From this and the above result it can be concluded that, for viscoelastic materials with the Staverman and Schwarzl form of the free energy, the relaxation function tensor is positive-definite and decreasing for all finite times.

Further, taking the time derivative of φ in Eq. $(3.1)_1$ yields:

$$\dot{\varphi}(t) = \sigma(t) : \dot{\varepsilon}(t) + \int_{o^-}^{t} \int_{o^-}^{t} \dot{r}(2t - u_1 - u_2) : d\varepsilon(u_2) : d\varepsilon(u_1) \tag{3.7}$$

which is compatible with $(3.1)_2$ through (3.6).

Similarly, the positivity of the dissipation during the evolution involves, from its dual expression in terms of the stress history, that $\dot{f}$ is a positive definite tensor kernel. Thus, f is an increasing monotonic tensor function of time.

All this is compatible with the properties obtained in Sec. 2 from the spectral representation, which itself leads to the Staverman and Schwarzl formulas, see for instance [28].

3.2. Brun–Clapeyron equation and associated pseudo-convolutive calculus

In HUET [30, 28], we introduced a pseudo-convolutive formalism based on the above results and on further results obtained by BRUN in [42, 43]. In particular, Brun gave a generalization of the local Clapeyron equation for the non-aging viscoelastic case. It can be written in the form:

$$\varphi(t) = \frac{1}{2}\left(\int_{o^-}^{t} - \int_{t}^{2t^+}\right) \sigma(2t - u) : d\varepsilon(u). \tag{3.8}$$

This is a bilinear form in σ and ε, which restitutes the Staverman and Schwarzl formula when replacing σ by its value given by the constitutive equation, and for which we have introduced the following pseudo-convolutive notation:

$$\varphi(t) = \frac{1}{2}\sigma \,\square\, \varepsilon \tag{3.9}$$

showing clearly its similarity to the elastic case and also to the convolutive notation classically used in viscoelasticity.

Generally speaking, the pseudo-convolution of the tensor a by the the tensor b is the non-commutative bilinear operator defined by:

$$a \,\square\, b = \left(\int_{o^-}^{t} - \int_{t}^{2t^+} \right) a(2t-u) : db(u). \tag{3.10}$$

The properties of this pseudo-convolution involve a set of identities that define the rules of the pseudo-convolutive calculus. Some of them were given by BRUN [43] without use of the pseudo-convolutive notation, and also by HUET [28] in the pseudo-convolutive form. Others were derived by ourselves and are here published for the first time. The most important ones, that will be used in the sequel, are recapitulated below and can be easily verified:

$$(g \circ b) \,\square\, b = \left(\int_{o^-}^{t} - \int_{t}^{2t^+} \right) \int_{o^-}^{2t-u} g(2t-u-\nu) : db(\nu) : db(u) \tag{3.11}$$

$$= \int_{o^-}^{t} \int_{o^-}^{t} g(2t-u-\nu) : db(\nu) : db(u);$$

$$\frac{1}{2}(a \,\square\, b + b \,\square\, a) = a : b,$$

$$a \,\square\, a = a : a,$$

$$\dot{a} \,\square\, b + \dot{b} \,\square\, a = \dot{a} \,\square\, a = 0, \tag{3.12}$$

$$(k_0 : a) \,\square\, b = k_0 : (a \,\square\, b) = k_0 : a \,\square\, b,$$

$$(k_0 : \dot{a}) \,\square\, b + (k_0 : \dot{b}) \,\square\, a = (k_0 : \dot{a}) \,\square\, a = 0,$$

for every fourth rank tensor which remains constant in time.

Moreover, taking the time derivative of Eq. (3.10) gives:

$$\frac{d}{dt}\left(\frac{1}{2} a \,\square\, b\right) = a : \dot{b} + \dot{a} \,\square\, b, \tag{3.13}$$

which appears as the pseudo-convolutive extension of the Leibnitz rule (with the 1/2 factor not to be forgotten). It is also a pseudo-convolutive analogue of the time derivative of the Stieltjès convolution:

$$\frac{d}{dt}(g \circ b) = g_0 : \dot{b} + \dot{a} \circ b . \tag{3.14}$$

In the two equations above, $a : b$ is written for its value $a(t) : b(t)$ at time t while g_0 stands for the value $g(0^+)$ of $g(t)$ at time zero. The same notation is used throughout the paper.

One must pay attention to the fact that the bilinear pseudo-biconvolution $(g \circ e) \square b$ differs in general from the corresponding bilinear double integral associated to the Staverman and Schwarzl formula:

$$(g \circ e) \square b = \left(\int_{o^-}^{t} - \int_{t}^{2t^+} \right) \int_{o^-}^{2t-u} g(2t - u - \nu) : de(\nu) : db(u) \tag{3.15}$$

$$\neq \int_{o^-}^{t} \int_{o^-}^{t} g(2t - u - \nu) : de(\nu) : db(u)$$

and that one has in general:

$$(g \circ e) \square b \neq (g \circ b) \square e . \tag{3.16}$$

This means that, in contrast with the bilinear form associated to the Staverman and Schwarzl quadratic one, the pseudo-biconvolution is not symmetric even for kernels g having the universal symmetries of linear elasticity. This is the basic reason for which the minimum theorems derived in [28] are submitted to restrictions.

Nevertheless, one still may mention that – for such symmetric kernels – the equalities are recovered for those special histories that keep both e and b constant in the time interval $(t, 2t)$. Moreover one has, for all cases,

$$(g \circ e) \square b + (g \circ b) \square e = 2 \int_{o^-}^{t} \int_{o^-}^{t} g(2t - u - \nu) : de(\nu) : db(u) \tag{3.17]}$$

$$= 2 \int_{o^-}^{t} \int_{o^-}^{t} g(2t - u - \nu) : db(\nu) : de(u)$$

which is twice the symmetric-bilinear form associated to the Staverman and Schwarzl quadratic form.

On the other hand, using Eqs. (3.11) and $(3.12)_4$ shows that algebraic quadratic forms can be expressed as pseudo-convolutive ones, since:

$$(k : a) \,\square\, a = k : (a \,\square\, a) = k : a : a \,. \tag{3.18}$$

This remark will simplify the derivations in the final sections of this paper. It will be also useful to extend the algebraic definition of the order relationship between two viscoelastic kernels to a pseudo-convolutive one as follows:

$$h < g \;\Leftrightarrow\; (g - h) \circ a \,\square\, a > 0 \quad \text{for all histories } a \text{ over } [0, 2t]. \tag{3.19}$$

For this relationship being valid, it is enough that the same one will be valid in the algebraic sense for all times. From this the following general inequality can be derived, valid for any kernel tensor $g(t)$ which is a monotonic decreasing function of time in the algebraic tensor sense:

$$g(2t) : b(t) : b(t) \;<\; (g \circ b) \,\square\, b \,. \tag{3.20}$$

The proof is straightforward owing to

$$\begin{aligned} &g(2t) < g(2t - u - \nu) \quad \forall t > 0, \quad \forall u \in]0, 2t], \forall \nu \in]0, 2t - u] \quad \Rightarrow \\ &g(2t) : b(t) : b(t) = g(2t) : \int_{0^-}^{t} \int_{0^-}^{t} db(\nu) : db(u) \\ &\qquad = \int_{0^-}^{t} \int_{0^-}^{t} g(2t) : db(\nu) : db(u) \\ &\qquad < \int_{0^-}^{t} \int_{0^-}^{t} g(2t - u - \nu) : db(\nu) : db(u) \\ &\qquad = \left(\int_{0^-}^{t} - \int_{t}^{2t^+} \right) \int_{0^-}^{2t-u} g(2t - u - \nu) : db(\nu) : db(u) \\ &\qquad = (g \circ b) \,\square\, b \,. \end{aligned} \tag{3.21}$$

We will make an extensive use of Ineq. (3.20) in Secs. 5 to 7.

3.3. Thermodynamics of viscoelastic behaviour in pseudo-convolutive form

From the rules given above it can be easily verified that, in the pseudo-convolutive formalism, all the local thermodynamic quantities studied in Sec. 3.1 may,

at time t, be written in the following symbolic forms:

$$\varphi = \frac{1}{2}\sigma \,\square\, \varepsilon = \frac{1}{2}(r \circ \varepsilon) \,\square\, \varepsilon = \sigma : \varepsilon - \frac{1}{2}\varepsilon \,\square\, \sigma = \sigma : \varepsilon + \gamma ,$$

$$\gamma = -\frac{1}{2}\varepsilon \,\square\, \sigma = -\frac{1}{2}(f \circ \sigma) \,\square\, \sigma = \frac{1}{2}\sigma \,\square\, \varepsilon - \sigma : \varepsilon = \varphi - \sigma : \varepsilon ,$$

$$D = -\dot{\sigma} \,\square\, \varepsilon = -(\dot{r} \circ \varepsilon) \,\square\, \varepsilon = \dot{\varepsilon} \,\square\, \sigma = (\dot{f} \circ \sigma) \,\square\, \sigma , \tag{3.22}$$

$$\dot{\varphi} = \frac{d}{dt}\left(\frac{1}{2}\sigma \,\square\, \varepsilon\right) = \sigma : \dot{\varepsilon} + \dot{\sigma} \,\square\, \varepsilon = \sigma : \dot{\varepsilon} + (\dot{r} \circ \varepsilon) \,\square\, \varepsilon ,$$

$$-\dot{\gamma} = \frac{d}{dt}\left(\frac{1}{2}\varepsilon \,\square\, \sigma\right) = \varepsilon : \dot{\sigma} + \dot{\varepsilon} \,\square\, \sigma = \varepsilon : \dot{\sigma} + (\dot{f} \circ \sigma) \,\square\, \sigma ,$$

where the time dependence is implicitly understood.

In Eqs. $(3.22)_3$, the second Identity $(3.12)_5$ has been used in the form:

$$(r_0 : \dot{\varepsilon}) \,\square\, \varepsilon = (f_0 : \dot{\sigma}) \,\square\, \sigma \equiv 0. \tag{3.23}$$

Combination of Eqs. $(3.22)_3$, $(3.22)_4$ and $(3.22)_5$ restitutes the two classical forms of the isothermal Clausius-Duhem inequality when written in terms of φ and γ, respectively:

$$D = -(\dot{\varphi} - \sigma : \dot{\varepsilon}) = -(\dot{\gamma} + \varepsilon : \dot{\sigma}) > 0, \tag{3.24}$$

showing thus the consistency of this formalism.

3.4. Macroscopic expressions for the viscoelastic thermodynamic quantities of heterogeneous materials and pseudo-convolutive Hill conditions

In order to be able to perform viscoelasticity calculations for heterogeneous bodies through the use of an equivalent homogeneous medium, it is useful – just as for the elastic case currently studied in the literature, see for instance [45] – to know the macroscopic expressions for the thermodynamic quantities in terms of the volume averages of the stress tensor, of the strain tensor and of their rates only. In addition to that, one requests the corresponding viscoelastic effective properties being the same as the ones introduced in Sec. 2.2 through mechanical definitions. For instance, the macroscopic free energy will be required to be the volume average of the local one and, at the same time, to be expressed in terms of the strain average and effective relaxation functions tensor by:

$$\langle \varphi \rangle = \frac{1}{2}(r^{\mathrm{eff}} \circ \langle \varepsilon \rangle) \,\square\, \langle \varepsilon \rangle = \frac{1}{2}\langle \sigma \rangle \,\square\, \langle \varepsilon \rangle . \tag{3.25}$$

When comparing this macroscopic thermodynamic expression with the volume averages of the corresponding local equations obtained in Sec. 3.3 above:

$$\langle \varphi \rangle = \frac{1}{2}\langle (r \circ \varepsilon) \,\square\, \varepsilon \rangle = \frac{1}{2}\langle \sigma \,\square\, \varepsilon \rangle , \tag{3.26}$$

one sees that it implies the following pseudo-convolutive extension of the classical Hill condition, see HILL [4], KRÖNER [8], widely used for the elastic case:

$$\langle \sigma \,\square\, \varepsilon \rangle = \langle \sigma \rangle \,\square\, \langle \varepsilon \rangle \;\Leftrightarrow\; \langle \sigma' \,\square\, \varepsilon' \rangle = 0, \tag{3.27}$$

where here again a' denotes the fluctuation of the variable a around its volume average.

Doing the same for the other thermodynamic quantities considered in Sec. 3.3 yields also:

$$\begin{aligned} \langle \varepsilon \,\square\, \sigma \rangle &= \langle \varepsilon \rangle \,\square\, \langle \sigma \rangle, \\ -\langle \dot{\sigma} \,\square\, \varepsilon \rangle &= -\langle \dot{\sigma} \rangle \,\square\, \langle \varepsilon \rangle = \langle \dot{\varepsilon} \,\square\, \sigma \rangle = \langle \dot{\varepsilon} \rangle \,\square\, \langle \sigma \rangle. \end{aligned} \tag{3.28}$$

As can be easily seen from the general pseudo-convolutive identity (3.13) given in Sec. 3.2 above, this is compatible with the macroscopic forms of the Clausius–Duhem inequality:

$$\begin{aligned} \langle D \rangle &= -\left(\langle \dot{\varphi} \rangle - \langle \sigma : \dot{\varepsilon} \rangle \right) = -\left(\langle \dot{\varphi} \rangle - \langle \sigma \rangle : \langle \dot{\varepsilon} \rangle \right) \\ &= -\left(\langle \dot{\gamma} \rangle + \langle \varepsilon : \dot{\sigma} \rangle \right) = -\left(\langle \dot{\gamma} \rangle + \langle \varepsilon \rangle : \langle \dot{\sigma} \rangle \right) > 0. \end{aligned} \tag{3.29}$$

Moreover, combining Eq. (3.13), Eq. (3.27) and the first Eq. $(3.28)_2$ shows that, when the pseudo-convolutive Hill conditions above are verified, the algebraic Hill condition for the strain power density:

$$\langle \sigma : \dot{\varepsilon} \rangle = \langle \sigma \rangle : \langle \dot{\varepsilon} \rangle \tag{3.30}$$

is also fulfilled, as requested for the definition of an equivalent homogeneous medium for heterogeneous materials with behaviours of any kind, see HUET [46, 47, 48].

We shall see now that the pseudo-convolutive Hill conditions are indeed verified for the viscoelastic counterparts of two special kinds of boundary conditions currently considered in the elastic case. For this we need first the pseudo-convolutive counterparts of the classical general relationships existing between the volume averages of the mechanical and thermodynamic variables and the boundary conditions.

3.5. Relationships between volume averages and boundary conditions

We denote by x the material point coordinates vector in a given fixed frame of reference. By applying the Gauss theorem, in the form of the gradient theorem, to the strain tensor field $\varepsilon(x)$ inside a domain D with volume V, external boundary ∂D, and perfect interface, one has for the volume average of the strain at any time, the classical universal relationship:

$$\langle \varepsilon \rangle = \frac{1}{V} \int_{\partial D} \operatorname{sym} (\xi \times n)\, d\Sigma, \tag{3.31}$$

where ξ is the displacement vector, "grad" the gradient operator, "sym" the operator giving the symmetrical part of a tensor (one half of the sum of a tensor and its transpose), $\times$ the tensor product (dyadic), $d\Sigma$ the area element of ∂D. Thus, when the displacement ξ is and remains continuous on the interface, with no slips or cracks, the strain average at a given time is completely determined by the knowledge of ξ at all points of the external boundary at the same time.

By employing the same theorem, in the form of the divergence theorem applied to the product $(x \times \sigma)$, and by taking account of the conservation equation for the momentum in the assumed absence of body forces, one also has for impervious interfaces, the other classical relationship, valid only under these two simplifying conditions:

$$\langle \sigma \rangle = \frac{1}{V} \int_{\partial D} \operatorname{sym} (P \times x)\, d\Sigma , \tag{3.32}$$

where P is the (prescribed or not) boundary stress vector defined at any time by

$$P = \sigma \cdot n , \tag{3.33}$$

with n being the outer unit normal to the external boundary.

For the strain power in small deformations one has in a similar fashion, when the two simplifying conditions above are fulfilled, the relation

$$\langle \sigma : \dot{\varepsilon} \rangle = \frac{1}{V} \int_{\partial D} (\sigma \cdot n) \cdot \dot{\xi}\, d\Sigma = \frac{1}{V} \int_{\partial D} P \cdot \dot{\xi}\, d\Sigma , \tag{3.34}$$

which is nothing else, for this particular case, than the principle of virtual power applied to the real solution fields at a given time.

But the same derivation may be applied to the stress and the strain taken at two different times t_1 and t_2:

$$\langle \sigma(t_1) : \dot{\varepsilon}(t_2) \rangle = \frac{1}{V} \int_{\partial D} [\sigma(t_1) \cdot n] \cdot \dot{\xi}(t_2)\, d\Sigma == \frac{1}{V} \int_{\partial D} P(t_1) \cdot \dot{\xi}(t_2)\, d\Sigma . \tag{3.35}$$

Through time integration and permutation of the time and domain integrals, this gives the pseudo-convolutive versions of this formula:

$$\begin{aligned} \langle \sigma \,\square\, \varepsilon \rangle &= \frac{1}{V} \int_{\partial D} (\sigma \cdot n) \,\square\, \xi\, d\Sigma = \frac{1}{V} \int_{\partial D} P \,\square\, \xi\, d\Sigma , \\ \langle \varepsilon \,\square\, \sigma \rangle &= \frac{1}{V} \int_{\partial D} \xi \,\square\, (\sigma \cdot n)\, d\Sigma = \frac{1}{V} \int_{\partial D} \xi \,\square\, P\, d\Sigma . \end{aligned} \tag{3.36}$$

Similar formulas apply when the real field variables are replaced by admissible virtual ones or by their rates in the equations of this section, and conversely. In the

presence of body forces and imperfect interfaces, more general formulas have to be used. They may be found for instance in [45] for the usual algebraic case, and in [28] for the pseudo-convolutive case. In the latter case, they can be considered as pseudo-convolutive extensions of the virtual work and virtual power theorems.

3.6. The case of kinematic uniform boundary conditions (ε_0–KUBC)

Let us first impose to the heterogeneous viscoelastic body the kinematic uniform boundary conditions. In the sequel they are denoted ε_0–KUBC for brevity and defined by prescribed displacements ξ^d applied to the whole boundary ∂D of the body D in the form:

$$\xi^d(x,u) = \varepsilon_0(u)\cdot x \qquad \forall x \text{ in } \partial D, \quad \forall u \text{ in } [0^-, 2t^+], \tag{3.37}$$

where ε_0 is uniform on the whole external boundary, but can be a piecewise continuous function of time in the interval $[0^-, 2t^+]$.

Taking account of the formulas given in the preceding section this yields, as usual, the universal formula:

$$\begin{aligned}\langle \varepsilon(u)\rangle = \frac{1}{V}\int_D \varepsilon(u)\,dV &= \frac{1}{V}\int_{\partial D} \xi \times n\,d\Sigma \\ &= \frac{1}{V}\int_{\partial D} \varepsilon_0(u)\cdot x \times n\,d\Sigma = \frac{1}{V}\varepsilon_0(u)\cdot \int_{\partial D} x \times n\,d\Sigma \\ &= \varepsilon_0(u)\cdot \frac{1}{V}\int_D \operatorname{grad} x\,dV = \varepsilon_0(u),\end{aligned} \tag{3.38}$$

which is the well known formula equating the average strain to the uniform one imposed on the boundary, and where the equality of $\operatorname{grad} x$ to the unit tensor δ has been used.

With the free energy density given in the pseudo-convolutive form, Eq. (3.22), one derives similarly:

$$\begin{aligned}2\langle \varphi(t)\rangle = \langle (r \circ \varepsilon) \square \varepsilon\rangle &= \langle \sigma \square \varepsilon\rangle \\ &= \frac{1}{V}\int_{\partial D} P \square (\varepsilon_0\cdot x)\,d\Sigma = \frac{1}{V}\int_{\partial D} (P \times x)\,d\Sigma \square \varepsilon_0 \\ &= \langle \sigma\rangle \square \varepsilon_0 = (r^{\text{eff}} \circ \varepsilon_0) \square \varepsilon_0 .\end{aligned} \tag{3.39}$$

Replacing, in the above sequence of equations, σ by $\dot\sigma$ and P by $\dot P$ yields also:

$$\begin{aligned}\langle D\rangle = -\langle (\dot r \circ \varepsilon) \square \varepsilon\rangle &= -\langle \dot\sigma \square \varepsilon\rangle \\ &= -\langle \dot\sigma\rangle \square \varepsilon_0 = -\langle \dot\sigma\rangle \square \langle \varepsilon\rangle = -(\dot r^{\text{eff}} \square \varepsilon_0) \square \varepsilon_0 .\end{aligned} \tag{3.40}$$

Thus, as announced above, the required pseudo-convolutive Hill conditions are indeed satisfied in ε_0–KUBC for both the macroscopic free energy and the dissipation densities.

3.7. Static uniform boundary conditions (σ_0–SUBC)

Similarly, let us consider the same body in *static uniform boundary conditions of the kind* σ_0, briefly named σ_0–SUBC in the sequel, and defined by a prescribed stress vector surface density P^d, applied to the whole boundary ∂D of the body D in the form:

$$P^d(u) = \sigma_0(u)\cdot n \qquad \forall x \text{ in } \partial D \quad \forall u \text{ in } [0^-, 2t^+], \tag{3.41}$$

where σ_0 is a given symmetric tensor with dimension of stress and with values taken in the order of magnitude compatible with the linearly viscoelastic domain of the material. In this case, the volume forces and accelerations are still supposed to be zero or negligible, as appropriate for the assessment of constitutive equations.

As can be easily verified by replacing P with its given value in (3.26), and through application of the Gauss theorem in the reverse way, one obtains the also well known universal result

$$\langle \sigma(u) \rangle = \frac{1}{V} \int_{\partial D} \operatorname{sym} [(\sigma_0 \cdot n) \times x] \, d\Sigma = \sigma_0(u). \tag{3.42}$$

Moreover, for σ_0 still uniform on the whole external boundary, but a piecewise continuous function of time u in the interval $[0^-, 2t^+]$, one gets:

$$\begin{aligned} -2\langle \gamma(t) \rangle = \langle (f \circ \sigma) \square \sigma \rangle &= \langle \varepsilon \square \sigma \rangle \\ &= \frac{1}{V} \int_{\partial D} \xi \square (\sigma_0 \cdot n) \, d\Sigma = \frac{1}{V} \int_{\partial D} (\xi \times n) \, d\Sigma \square \sigma_0 \\ &= \langle \varepsilon \rangle \square \sigma_0 = (f^{\text{eff}} \circ \sigma_0) \square \sigma_0 . \end{aligned} \tag{3.43}$$

Replacing in this sequence of equations ε by $\dot{\varepsilon}$ and ξ by $\dot{\xi}$ yields also:

$$\langle D \rangle = \langle (\dot{f} \circ \sigma) \square \sigma \rangle = \langle \dot{\varepsilon} \square \sigma \rangle = \langle \dot{\varepsilon} \rangle \square \sigma_0 = (\dot{f}^{\text{eff}} \circ \sigma_0) \square \sigma_0 . \tag{3.44}$$

Thus, in σ_0–SUBC, the required pseudo-convolutive Hill conditions are fulfilled for the macroscopic free enthalpy and dissipation densities. All the above preliminary results will be used in the final derivations to be presented now. Nevertheless, in order to simplify the notation we will avoid, in the sequel, the use of the brackets in the pseudo-biconvolutions such as $(g \circ e) \square b$: the formula will be written simply $g \circ e \square b$ thereafter. One should remind that the left-hand product has to be read first. The same holds for the algebraic bi-products of the form $(g :. e) : b = g :: (e \times b)$ written simply $g : e : b$.

4. The associated elasticity problems approach

4.1. Principles of the method

The basic idea of this approach is to make use of auxiliary elasticity problems for which, at some given time t taken as a fixed parameter during the calculations, the elastic properties are taken as some appropriate functions of the viscoelastic ones, and for which the boundary conditions are taken the same as for the vicoelasticity problems at the same time t.

Then, it can be observed that the displacement, strain and stress fields which are solutions of the viscoelastic evolution problem at the considered instant t in the considered viscoelastic body are admissible fields for the associated elasticity problem. Thus the various already known minimum theorems of elasticity can be used. Here we will make use of the classical minimum theorems for the potential energy and for the complementary energy. In fact, it will be enough to write the theorems for the kinematic and static uniform boundary conditions, respectively.

4.2. Potential energy minimum theorems for elastic bodies in uniform kinematic boundary conditions

As well known, the elasticity modulus tensor C and the corresponding compliance tensor S have symmetry and positivity properties associated with the existence of elastic potentials like the free energy and the free enthalpy, and with the stability of elastic equilibrium. Together with the classical virtual work theorem, this yields two minimum theorems for elasticity problems. The first one relates to the potential energy $F_{\tilde{\varepsilon}}$ of the elastic system.

Let us denote by $\tilde{\xi}$ and $\tilde{\varepsilon}$ any admissible virtual displacement and strain fields satisfying the conditions

$$\text{(4.1)} \qquad \begin{aligned} &\tilde{\xi} \text{ continuous and } && \tilde{\varepsilon} = \operatorname{sym}(\operatorname{grad}\tilde{\xi}) && \forall\, x \text{ in } D\,, \\ &&& \tilde{\xi} = \xi^d && \forall\, x \text{ on } \partial D_\xi\,. \end{aligned}$$

The potential energy minimum theorem states that the elastic solution displacement and strain fields ξ and ε minimize $F_{\tilde{\varepsilon}}$ among all the corresponding admissible fields. In ε_0–KUBC the displacement is prescribed over all ∂D, and we have:

$$\text{(4.2)} \qquad \partial D_\sigma = \emptyset\,, \qquad \partial D_\xi = \partial D\,.$$

Thus the classical expression for the virtual potential energy $F_{\tilde{\varepsilon}}$ reduces here to

$$\text{(4.3)} \qquad F_{\tilde{\varepsilon}} = \frac{1}{2}\int_D C : \tilde{\varepsilon} : \tilde{\varepsilon}\, dV = \frac{1}{2} V \langle C : \tilde{\varepsilon} : \tilde{\varepsilon} \rangle,$$

and the same without tilde for the potential energy of the real elastic solution. Thus, the minimum theorem gives here

$$\langle C : \varepsilon : \varepsilon \rangle \leq \langle C : \tilde{\varepsilon} : \tilde{\varepsilon} \rangle, \tag{4.4}$$

the volume averages being taken again on the domain D.

Through the classical elastic Hill condition and the effective Hooke law, this still reads:

$$\langle \sigma : \varepsilon \rangle = \langle \sigma \rangle : \langle \varepsilon \rangle = C^{\text{eff}} : \varepsilon_0 : \varepsilon_0 \leq \langle C : \tilde{\varepsilon} : \tilde{\varepsilon} \rangle. \tag{4.5}$$

4.3. Complementary energy minimum theorems for elastic bodies in static uniform boundary conditions

The second classical minimum theorem of elasticity relates to the complementary energy $F_{\tilde{\sigma}}$ of the elastic body. Let us denote by $\tilde{\sigma}$ any admissible virtual stress field, satisfying

$$\begin{aligned} \operatorname{div} \tilde{\sigma} &= -F^d \qquad \forall\, x \text{ in } D, \\ \tilde{\sigma} \cdot n &= P^d \qquad \forall\, x \text{ on } \partial D_\sigma . \end{aligned} \tag{4.6}$$

The complementary energy minimum theorem states that the elastic solution stress field σ minimizes $F_{\tilde{\sigma}}$ among all the corresponding admissible stress fields. In σ_0–SUBC, the surface traction $\sigma_0 \cdot n$ is prescribed over all the boundary ∂D of D, and we have

$$\partial D_\sigma = \partial D, \qquad \partial D_\xi = \emptyset . \tag{4.7}$$

Thus the virtual complementary energy $F_{\tilde{\sigma}}$ reduces here to

$$F_{\tilde{\sigma}} = \frac{1}{2} \int_D S : \tilde{\sigma} : \tilde{\sigma}\, dV = \frac{1}{2} V \langle S : \tilde{\sigma} : \tilde{\sigma} \rangle, \tag{4.8}$$

and the same without tilde for the real solution. Thus, the minimum theorem gives here

$$\langle S : \sigma : \sigma \rangle \leq \langle S : \tilde{\sigma} : \tilde{\sigma} \rangle. \tag{4.9}$$

Using the elastic Hill condition and Hooke law again, we get:

$$\langle \sigma : \varepsilon \rangle = \langle \sigma \rangle : \langle \varepsilon \rangle = S^{\text{eff}} : \sigma_0 : \sigma_0 \leq \langle S : \tilde{\sigma} : \tilde{\sigma} \rangle. \tag{4.10}$$

We are now ready to determine the sought bounds for the effective viscoelastic properties.

5. Bounds for the relaxation properties

Let us consider first the relaxation functions tensor and its rate.

5.1. Lower bounds for the effective relaxation functions tensor

In this section, for the sake of clarity, we present the derivation step by step for the first bound. Since similar arguments will be used in the subsequent sections, we will simplify the derivations for the other ones in order to shorten the paper.

Let us consider the viscoelastic body D subjected to ε_0–KUBC with ε_0 being a known function $\varepsilon_0(t)$ of time. The real displacement and strain fields $\xi^{\text{vel}}(t)$ and $\varepsilon^{\text{vel}}(t)$ at time t for this viscoelasticity problem can be taken as kinematic admissible fields for any elastic problem of the corresponding body with same geometry, with any modulus tensor C and subjected to the same ε_0–KUBC, for ε_0 taking the value $\varepsilon_0(t)$. This is so because, at a fixed value of time t, $\xi^{\text{vel}}(t)$ is continuous over D and satisfies the boundary conditions at the same time value t. Moreover, $\varepsilon^{\text{vel}}(t)$ is the symmetric part of the gradient of $\xi^{\text{vel}}(t)$.

Thus, from Eq. (4.4) one has, for every time t:

$$C^{\text{eff}} : \varepsilon_0 : \varepsilon_0 \leq \langle C : \varepsilon^{\text{vel}} : \varepsilon^{\text{vel}} \rangle, \tag{5.1}$$

where C^{eff} is the effective elasticity modulus tensor corresponding to C for the associated elastic body subjected to the $\varepsilon_0(t)$–KUBC defined above. Let us take for C the value taken by the relaxation functions tensor $r(2t)$ at time $2t$. Thus we get:

$$r^{\text{eff el}}(2t) : \varepsilon_0 : \varepsilon_0 \leq \langle r(2t) : \varepsilon^{\text{vel}}(t) : \varepsilon^{\text{vel}}(t) \rangle, \tag{5.2}$$

where $r^{\text{eff el}}$ is the corresponding effective elasticity modulus tensor. But, from the monotonic decreasing property of r derived in Sec. 3.1 and Ineq. (3.20), one has:

$$\langle r(2t) : \varepsilon^{\text{vel}}(t) : \varepsilon^{\text{vel}}(t) \rangle < \langle r \circ \varepsilon^{\text{vel}} \square \varepsilon^{\text{vel}} \rangle = r^{\text{eff}} \circ \varepsilon_0 \square \varepsilon_0 , \tag{5.3}$$

where Eq. (3.39) has also been used. This yields finally

$$r^{\text{eff el}}(2t) : \varepsilon_0 : \varepsilon_0 < r^{\text{eff}} \circ \varepsilon_0 \square \varepsilon_0 . \tag{5.4}$$

Taking the history $\varepsilon_0(t)$ as a one-step function $\varepsilon_0^0 Y(t)$, see Sec. 3.1 Eq. $(3.4)_1$, the RHS becomes equal to:

$$2\langle \varphi(t) \rangle = r^{\text{eff}}(2t) : \varepsilon_0^0 : \varepsilon_0^0 . \tag{5.5}$$

Thus, for this history of ε_0, Ineq. (5.4) becomes

$$r^{\text{eff el}}(2t) : \varepsilon_0^0 : \varepsilon_0^0 < r^{\text{eff}}(2t) : \varepsilon_0^0 : \varepsilon_0^0 . \tag{5.6}$$

This, being true for all non vanishing ε_0^0, means that the following order relationship, in the tensor algebraic sense, has been established between the viscoelastic effective relaxation function tensor $r^{\text{eff}}(2t)$ and the associated effective elastic moduli tensor $r^{\text{eff el}}(2t)$, the latter being, for each value of the time, strictly smaller than the former one taken at the same finite time value:

$$r^{\text{eff el}}(2t) < r^{\text{eff}}(2t). \tag{5.7}$$

We have thus obtained the first bound, in fact a lower one, for the effective relaxation function tensor r^{eff}.

Moreover, every lower bound for the associated effective elastic modulus at time t is also a lower bound for the effective relaxation function tensor at time t. By this result, all the results of the previous efforts aimed at finding lower bounds for the effective elasticity moduli, and in particular the works by Hill, Hashin, Walpole, Kröner mentioned in the Introduction, can be applied to the relaxation function tensor of the viscoelasticity problems through the result obtained here.

In the light of the remarks made at the end of Sec. 3.2, the order relationship, Ineq. (5.7), may also be understood in the pseudo-convolutive sense.

5.2. Upper bounds for the effective relaxation rates tensor

In Sec. 2.3 we have defined the effective tensor associated with the relaxation rate tensor and have shown that it equals the time derivative of the effective relaxation function tensor. From its definition, it has the same general symmetries as r itself. Thus, a similar procedure as in Sec. 5.1 can be applied to it by taking into account the fact that its opposite, $[-\dot{r}(t)]$, is positive definite for all finite time, see Sec. 3.1.

Let us define the tensor function $g(t)$ as

$$g(t) = -\dot{r}(t), \tag{5.8}$$

and let us consider the associated elasticity problem in which the modulus tensor C has the positive definite value $g(2t)$ everywhere in the body D, the latter being subjected to $\varepsilon_0(t)$–KUBC. The viscoelastic strain field ε^{vel} at time t is still an admissible strain field for this elasticity problem, to which the kinematic elastic minimum theorem can be applied, with $g(t)$ being the kernel of the potential energy functional.

Moreover, let us suppose that, as provided by the assumed existence of a spectral representation (see Sec. 3.1), the relaxation rate tensor is a monotonic increasing function of time t. Then $g(2t)$ is a decreasing monotonic function of t. From Eqs. (4.5), (3.20) and (3.40), one has the following chain of relationships:

$$\begin{aligned} g^{\text{eff el}} \circ \varepsilon_0 \,\square\, \varepsilon_0 = g^{\text{eff el}}(2t) : \varepsilon_0 : \varepsilon_0 &= \langle g(2t) : \varepsilon^{\text{el}}(t) : \varepsilon^{\text{el}}(t) \rangle \\ &\leq \langle g(2t) : \varepsilon^{\text{vel}}(t) : \varepsilon^{\text{vel}}(t) \rangle < \langle g \circ \varepsilon^{\text{vel}} \,\square\, \varepsilon^{\text{vel}} \rangle \\ &= -\langle \dot{r} \circ \varepsilon^{\text{vel}} \,\square\, \varepsilon^{\text{vel}} \rangle = -\dot{r}^{\text{eff}} \circ \varepsilon_0 \,\square\, \varepsilon_0 . \end{aligned} \tag{5.9}$$

As in Sec. 5.1, this corresponds to the following order relationship, in both the tensor algebraic sense and the pseudo-convolutive sense:

$$\dot{r}^{\text{eff}}(2t) < -g^{\text{eff el}}(2t) < 0 \quad \forall\, t \in]0, \infty[\,. \tag{5.10}$$

This is an algebraic upper bound for the effective relaxation rate tensor. It is valid for all finite times. Of course, because of the minus sign, every lower bound to $g^{\text{eff el}}$ provided by the theory of elastic heterogeneous media provides an upper bound for the effective relaxation rate tensor.

6. Bounds for the creep properties

If we apply directly the procedure of Sec. 5.1 to the creep function, one can see that one link is missing. This is so because the creep function is an increasing function of time in place of a decreasing one used above. Nevertheless, from the analyses presented in Secs. 2.1 and 3.5 it can be seen that the local creep rate tensor exhibits the desired properties.

6.1. Lower bounds for the creep rate tensor

In Sec. 2.3, we have shown that the effective tensor associated with the creep rate tensor is the time derivative of the effective creep functions tensor.

Let us assume now that, as provided by the existence of a spectral representation, see Eq. (2.9) at the end of Sec. 2.1, the creep rate tensor is positive definite for all finite times and monotonic decreasing. Thus a chain of arguments similar to the one described above can be used when starting with the associated elastic complementary energy minimum theorem applied to the viscoelastic body in σ_0–SUBC.

Let us define the tensor function $h(t)$ as

$$h(t) = \dot{f}\,(t), \tag{6.1}$$

and let us consider the associated elasticity problem in which the elastic compliance tensor S has the positive definite value $h(2t)$ everywhere in the body D subjected to $\sigma_0(t)$–SUBC. The viscoelastic stress field $\sigma^{\text{vel}}(t)$ at time t is a statically admissible stress field for this elasticity problem, to which the static elastic minimum theorem can be applied, with $h(2t)$ being the kernel of the complementary energy functional.

Under the assumptions made in Sec. 2.1, the tensor function $h(t)$ is a positive definite and monotonic decreasing function of time. Thus one obtains, in a similar manner as above, the following sequence of relationships:

$$\begin{aligned} h^{\text{eff el}} \circ \sigma_0 \,\square\, \sigma_0 = h^{\text{eff el}}(2t) : \sigma_0 : \sigma_0 = \langle h(2t) : \sigma^{\text{el}}(t) : \sigma^{\text{el}}(t) \rangle \\ \leq \langle h(2t) : \sigma^{\text{vel}}(t) : \sigma^{\text{vel}}(t) \rangle < \langle h \circ \sigma^{\text{vel}} \,\square\, \sigma^{\text{vel}} \rangle = \dot{f}^{\text{eff}} \circ \sigma_0 \,\square\, \sigma_0\,. \end{aligned} \tag{6.2}$$

This corresponds to the following order relationship, in the tensor algebraic sense or in the pseudo-convolutive sense:

$$0 < h^{\text{eff el}}(2t) < \dot{f}^{\text{eff}}(2t) \qquad \forall\, t \in]0, \infty[\,. \tag{6.3}$$

Thus we have found an algebraic lower bound for the creep rate tensor. It is valid for all finite times. Since the creep rate is always positive at finite time, this implies that the effective creep function increases at a rate which is always larger than the associated elastic effective rate. This looks consistent with the observation made at the end of Sec. 5.2 above for the relaxation function.

Of course, as in the above sections, every lower bound to the tensor $h^{\text{eff el}}$ provided by the theory of elastic heterogeneous media is a lower bound for the creep rate function tensor $\dot{f}^{\text{eff}}$.

We are now in a position to return to the creep functions tensor itself.

6.2. Lower bounds for the creep functions tensor

Let us write Ineq. (6.3) for time u and integrate it up to t. This yields

$$\int_{0^+}^{t} h^{\text{eff el}}(u)\, du < \int_{0^+}^{t} \dot{f}^{\text{eff}}(u)\, du = f^{\text{eff}}(t) - f^{\text{eff}}(0^+), \tag{6.4}$$

where the definition (2.29) of $\dot{f}^{\text{eff}}$ has again been used.

Thus we get a lower bound for the effective creep function:

$$f^{\text{eff}}(0^+) + \int_{0^+}^{t} h^{\text{eff el}}(t) < f^{\text{eff}}(t) \qquad \forall t \in]0, \infty[\,. \tag{6.5}$$

Of course, the instantaneous compliance $f^{\text{eff}}(0^+)$ is itself an effective elastic compliance tensor, and the inequality becomes an equality at time zero.

Since the L.H.S. of Ineq. (6.5) involves effective elastic quantities only, every lower bound provided to at least one of the latter ones provides also a new lower bound to the effective creep function f^{eff}. In particular, this happens with every lower bound to the tensor $h^{\text{eff el}}$ provided by the theory of elastic heterogeneous media in some finite time interval that can be $(0, t)$ itself.

7. Two-sided bounds

In the elastic case, one-sided bounds for both the effective modulus and compliance tensors provides immediately two-sided bounds by using the fact that they are reciprocals in the algebraic tensor sense. This property is no more valid in

the viscoelastic case since the relaxation function tensor and the creep function tensor are not algebraic reciprocals, but convolutive reciprocals only.

Nevertheless, from the above results can be derived a set of two-sided bounds for both the effective relaxation functions and the effective creep function tensors. They are of two kinds, the first one resulting from the use of the algebraic inequations (2.24) of Sec. 2.2, the second one being due to the results obtained here for the rates.

7.1. Two-sided bounds of the first kind for the creep and relaxation function tensors

Let us first define the tensor function $l(t)$ by the L.H.S. of Ineq. (6.5):

$$l(t) = f^{\text{eff}}(0^+) + \int_{0^+}^{t} h^{\text{eff el}}(u)\, du \qquad \forall t \in]0, \infty[\,. \tag{7.1}$$

Inequalities (2.24) may be written in the form

$$\begin{aligned} f^{\text{eff}}(t) &< \left[r^{\text{eff}}(t)\right]^{-1}, \\ r^{\text{eff}}(t) &< \left[f^{\text{eff}}(t)\right]^{-1}, \end{aligned} \tag{7.2}$$

where the reciprocals are taken in the algebraic sense.

On the other hand, taking the algebraic reciprocals of Eqs. (5.7) and (6.5) we obtain

$$\begin{aligned} \left[r^{\text{eff}}(t)\right]^{-1} &< \left[r^{\text{eff el}}(t)\right]^{-1}, \\ \left[f^{\text{eff}}(t)\right]^{-1} &< [l(t)]^{-1}. \end{aligned} \tag{7.3}$$

Together with Ineqs. (7.2), this gives upper bounds for r^{eff} and f^{eff}. Thus one has finally the following two-sided bounds of the first kind for the effective creep and relaxation function tensors, respectively:

$$\begin{aligned} l(t) < f^{\text{eff}}(t) &< \left[r^{\text{eff el}}(t)\right]^{-1}, \\ r^{\text{eff el}}(t) < r^{\text{eff}}(t) &< [l(t)]^{-1}. \end{aligned} \tag{7.4}$$

7.2. Two-sided bounds of the second kind for the relaxation function tensor

Let us write Ineq. (5.10) for time u and integrate it up to t. This yields:

$$\int_{0^+}^{t} \dot{r}^{\text{eff}}(u)\, du = r^{\text{eff}}(t) - r^{\text{eff}}(0^+) < -\int_{0^+}^{t} g^{\text{eff el}}(u)\, du \quad \forall t \in]0, \infty[\tag{7.5}$$

from which we get immediately an upper bound for r^{eff}:

$$r^{\text{eff}}(t) < r^{\text{eff}}(0^+) - \int_{0^+}^{t} g^{\text{eff el}}(u)\,du \qquad \forall t \in]0,\infty[\,. \tag{7.6}$$

This, together with the previously derived lower bound, Ineq. (5.7), provides us with the following two-sided bounds for the effective relaxation tensor:

$$r^{\text{eff el}}(t) < r^{\text{eff}}(t) < k(t) \qquad \forall t \in]0,\infty[\,, \tag{7.7}$$

with $k(t)$ defined by

$$k(t) = r^{\text{eff}}(0^+) - \int_{0^+}^{t} g^{\text{eff el}}(u)\,du \qquad \forall t \in]0,\infty[\,. \tag{7.8}$$

Since the existence of instantaneous elasticity has been assumed, $r^{\text{eff}}(0^+)$ is finite and the upper bound is not trivial. Of course, $r^{\text{eff}}(0^+)$ is an elasticity moduli tensor that can be calculated or bounded by the elasticity theory. When used in Eq. (7.8), every upper bound to it provides also an upper bound to the effective relaxation functions tensor $r^{\text{eff}}(t)$. And every lower bound to the associated elastic effective tensor $g^{\text{eff el}}(u)$ on some finite interval in $(0,t)$, including the full $(0,t)$ interval, provides also an upper bound to $r^{\text{eff}}(t)$.

In addition to this, it can be shown in a similar fashion that the tensor function $m(t)$ defined by

$$m(t) = r_{\infty}^{\text{eff}} + \int_{t}^{\infty} g^{\text{eff el}}(u)\,du \tag{7.9}$$

is also a lower bound for $r^{\text{eff}}(t)$, but not as good as $r^{\text{eff el}}(t)$ itself.

7.3. Two-sided bounds of the second kind for the creep functions tensor of a viscoelastic solid

For a heterogeneous viscoelastic solid, the effective creep functions tensor grows toward a finite asymptotic value which corresponds to the final equilibrium elastic compliance. Let us denote it by

$$f_{\infty}^{\text{eff}} = f^{\text{eff}}(\infty). \tag{7.10}$$

It can also be obtained from Sec. 2.3 in the form

$$f_{\infty}^{\text{eff}} = f^{\text{eff}}(t) + \int_{t}^{\infty} \dot{f}^{\text{eff}}(u)\,du. \tag{7.11}$$

Thus, taking Ineq. (6.3) into account and integrating it we obtain the following upper bound for f^{eff}:

$$f^{\text{eff}}(t) < j(t), \tag{7.12}$$

the tensor function $j(t)$ being defined by

$$j(t) = f_{\infty}^{\text{eff}} - \int_t^{\infty} h^{\text{eff el}}(u)\, du. \tag{7.13}$$

Combining this with the already derived lower bound yields the following two-sided bounds for the effective creep function tensor of a viscoelastic solid:

$$f^{\text{eff}}(0^+) + \int_{0^+}^{t} h^{\text{eff el}}(u)\, du < f^{\text{eff}}(t) < f_{\infty}^{\text{eff}} - \int_t^{\infty} h^{\text{eff el}}(u) \quad \forall t \in]0, \infty[\,. \tag{7.14}$$

Thus, every lower bound for $h^{\text{eff el}}$ will provide new lower and upper bounds for $f^{\text{eff}}(t)$. On the other hand, every lower bound for $f^{\text{eff}}(0^+)$ will provide new lower bounds, and every upper bound for f_{∞}^{eff} will provide new upper bounds.

7.4. Recapitulation

To summarize, all the bounds obtained in this paper can be, $\forall t \in]0, \infty[$, recapitulated as follows, independently of their order of derivation:

$$\begin{aligned}
&\dot{r}^{\text{eff}}(t) < -g^{\text{eff el}}(t) < 0,\\
&0 < h^{\text{eff el}}(2t) < \dot{f}^{\text{eff}}(2t),\\
&m(t) < r^{\text{eff el}}(t) < r^{\text{eff}}(t) < [f^{\text{eff}}]^{-1} < [l(t)]^{-1},\\
&m(t) < r^{\text{eff el}}(t) < r^{\text{eff}}(t) < k(t),\\
&l(t) < f^{\text{eff}}(t) < [r^{\text{eff}}]^{-1} < [r^{\text{eff el}}(t)]^{-1},\\
&l(t) < f^{\text{eff}}(t) < j(t).
\end{aligned} \tag{7.15}$$

The various auxiliary tensor-valued functions appearing in these inequalities are defined by

$$\begin{aligned}
j(t) &= f_{\infty}^{\text{eff}} - \int_t^{\infty} h^{\text{eff el}}(u)\, du,\\
k(t) &= r_0^{\text{eff}} - \int_{0^+}^{t} g^{\text{eff el}}(u)\, du,
\end{aligned} \tag{7.16}$$

(7.16) [cont.)

$$l(t) = f_0^{\text{eff}} + \int_{0^+}^{t} h^{\text{eff el}}(u)\,du,$$

$$m(t) = r_\infty^{\text{eff}} + \int_{t}^{\infty} g^{\text{eff el}}(u)\,du,$$

where the subscript 0 denotes instantaneous elasticity, and the subscript ∞ – the final equilibrium value when it exists.

It may be thought that some order relationship might be found between $[l(t)]^{-1}$ and $k(t)$ on the one hand, and between $[r^{\text{eff el}}(t)]^{-1}$ and $j(t)$ on the other hand. This is left for further research.

8. Conclusion

In this paper, we have considered the old problem of finding bounds for the creep and relaxation function tensors for heterogeneous, and possibly anisotropic, viscoelastic materials. We have seen that this problem – that resisted several efforts since the sixties – can be given in fact extensive and comprehensive solutions, representing thus at least the beginning of an answer to the 1991 remark of CHRISTENSEN in [26] "*it is not known how to effect the transition from elastic bounds to viscoelastic bounds*". This has been done here without restriction for the wide class of linear viscoelastic materials for which the relaxation and creep function tensors are monotonic tensor function of time with monotonic rates. For these materials, two-sided finite bounds have been obtained for the relaxation functions tensor on the one hand, and for the creep functions tensor on the other hand. The creep and relaxation rates tensors each admit one-sided finite bounds only. These bounds are valid for all finite times. For zero or infinite time values, the bounds reduce to the effective elastic values, or are trivially zero or infinite for cases devoid of instantaneous and/or delayed elasticity. These results hold for any kind of anisotropy and for any kind of heterogeneity, including polycrystalline, spinodal or matrix-inclusions materials, with random or periodic microstructures.

The field of possible applications appears to be very wide since elastic computations only are needed in order to calculate the bounds, and since classical elasticity results or methods (including numerical ones, see for instance AMIEUR *et al.* [49], GUIDOUM *et al.* [50]), may be used to evaluate the bounds or to provide further ones. From these results, it will be also possible to evaluate the maximum error made in using quasi-elastic approximations in place of the real material properties, as suggested for instance in SCHAPERY [17].

It can easily be seen that many other results may be gained from the approach introduced in this paper. In particular, a straightforward application will be new viscoelastic extensions of previous results for the apparent properties of heterogeneous bodies not having the representative volume that we studied in [45] for

the elastic case. This is so because, as shown in this paper, the elastic effective properties are bounded from below and from above by the apparent properties in appropriate uniform boundary conditions. Thus, we will get in this fashion the time domain counterpart of the bounds and hierarchies for the size effects and boundary conditions effects that we derived in [23] for the complex moduli and compliances in the frequency domain, also by an associated elasticity problems method.

It may be expected also that an extension to the class of nonlinear viscoelastic behaviours (for which the pseudo-linear representation recently developed for rubbery materials by our past co-worker ALAOUI in [51] is valid will be possible). This owes to the fact that it makes also use of the Staverman and Schwarzl functionals as in the present paper, but in terms of appropriate nonlinear measures of the strain and stress tensors. One may also expect that these results will find applications in more general nonlinear viscoelastic behaviours, like the ones studied by HUTCHINSON [52] in the steady-state case, or the ones defined through multiple integral expansions for which we developed in HUET [53] a multi-dimensional Carson transform approach. This is so because the inversion formulas that we gave in [54] involve multiple quadratures with kernels associated to the linear part of the nonlinear functional. This is left for further investigation.

On the other hand, despite the fact that many useful results have been obtained through the new approach presented here, there are still classical results of the elasticity theory that cannot be transferred through it to the viscoelastic case, and which still remain open problems. Among them are the important modification or strengthening – theorems derived by HILL [4] and used by WALPOLE [55], and more recently by us in [56]. For this, true viscoelasticity minimum theorems preserving the viscoelastic kernels and valid for heterogeneous materials of any kind are still needed.

Acknowledgements

We gratefully acknowledge partial support for the research related to this paper from the Swiss National Foundation for Science under contract no 20.32206/91. We also thank Prof. OSTOJA–STARZEWSKI, from the Michigan State University, for critical reading and linguistic revision of the manuscript.

References

1. C. HUET and A. ZAOUI [Eds.], *Rheological behaviour and structure of materials*, Presses ENPC, Paris 1981.
2. C. HUET [Ed.], *Micromechanics of concrete and cementitious composites*, Presses Polytechniques et Universitaires Romandes, Lausanne 1993.
3. R. HILL, *The elastic behaviour of a cristalline aggregate*, Proc. Phys. Soc., London, Sect. A, **65**, 349–354, 1952.
4. R. HILL, *Elastic properties of reinforced solids: some theoretical principles*, J. Mech. Phys. Solids, **11**, 357–372, 1963.

5. Z. HASHIN and S. SHTRIKMAN, *A variational approach to the theory of the elastic behaviour of multiphase materials*, J. Mech. Phys. Solids, **11**, 127–140, 1963.
6. L.J. WALPOLE, *On bounds for the overall elastic moduli of inhomogeneous systems I*, J. Mech. Phys. Solids, **14**, 151–162, 1966.
7. L.J. WALPOLE, *On bounds for the overall elastic moduli of inhomogeneous systems II*, J. Mech. Phys. Solids, **14**, 289–301, 1966.
8. E. KRÖNER, *Statistical continuum mechanics*, Springer-Verlag, Berlin 1972.
9. E. KRÖNER, *Ordered versus disordered materials: consequences for the continuum model*, Arch. Mech. Stos., **28**, 3, 455–466, 1977.
10. E. KRÖNER, *Linear properties of random media: the systematic theory*, [in:] Rheological Behaviour and Structure of Materials, C. HUET and A. ZAOUI [Eds.] Presses ENPC, Paris, 15–40, 1981.
11. E. KRÖNER, *Statistical modelling*, [in:] Modelling Small Deformation of Polycrystals, J. ZARKA and J. GITTUS [Eds.], Elsevier, No 1926, 229, 1986.
12. Z. HASHIN, *Viscoelastic behaviour of heterogeneous media*, J. Appl. Mech., **32**, E, 3, 630, 1965.
13. Z. HASHIN, *The inelastic inclusion problem*, Int. J. Engng. Sc., 7, 11–36, 1969.
14. Z. HASHIN, *Complex moduli of viscoelastic composites, I. General theory and application to particulate composites*, Int. J. Solids Struct., 6, 539–552, 1970.
15. R.A. SCHAPERY, *On the time dependence of viscoelastic variational solutions*, Quart. Appl. Math., **22**, 207–216, 1964.
16. R.A. SCHAPERY, *Thermal expansion coefficients of composite materials based on energy principles*, J. Composite Materials, **3**, 2, 380–404, 1968.
17. R.A. SCHAPERY, *Viscoelastic behaviour and analysis of composite materials*, [in:] Mechanics of Composite Materials, G.P. SENDECKYJ [Ed.], Academic Press, 85–168, 1974.
18. J. MINSTER, *Visco-elastic moduli of some heterogeneous materials*, Acta Tech. CSAV, 1, 45–53, 1973.
19. R. ROSCOE, *Bounds for the real and imaginary parts of the dynamic moduli of composite viscoelastic systems*, J. Mech. Phys. Solids, **17**, 17–22, 1969.
20. R. ROSCOE, *Improved bounds for real and imaginary parts of complex moduli of isotropic viscoelastic composites*, J. Mech. Phys. Solids, **20**, 91–99, 1972.
21. S. CHÉTOUI, *Evaluation des propriétés effectives des matériaux hétérogènes anisotropes à constituants viscoélastiques*, Doctoral dissertation, Ecole Nationale des Ponts et Chaussées, Paris 1980.
22. S. CHÉTOUI, P. NAVI and C. HUET, *Recherches sur l'évaluation des propriétés macroscopiques des matériaux hétérogènes viscoélastiques anisotropes*, [in:] Rhéologie des Matériaux Anisotropes, C. HUET, D. BOURGOIN, S. RICHEMOND [Eds.], CEPADUES, Toulouse, 307–326, 1986.
23. C. HUET, *Hierarchies and bounds for size effects in heterogeneous bodies*, [in:] Proc. Sixth Symposium on Continuum Models and Discrete Systems, G. MAUGIN [Ed.], Dijon, 1989; Longmans, London, Vol. **2**, 127–134, 1991.
24. L.V. GIBIANSKY and G.W. MILTON, *On the effective viscoelastic moduli of two-phase media. I. Rigorous bounds on the complex bulk modulus*, Proc. Royal Society London, **440**, 163–168, 1993.
25. R.M. CHRISTENSEN, *Viscoelastic properties of heterogeneous media*, J. Mech. Phys. Solids, **17**, 23–41, 1969.
26. R.M. CHRISTENSEN, *Mechanics of composite materials*, Krieger Publishing Company, Malabar, Florida 1991.
27. M.E. GURTIN, *Variational principles in the linear theory of viscoelasticity*, Arch. Rat. Mech. Anal., **13**, 179–191, 1963.
28. C. HUET, *Minimum theorems for viscoelasticity*, Eur. J. Mech. A/Solids, **11**, 5, 653–684, 1992.
29. C. HUET, *Some basic tools and pending problems in the development of constitutive equations for the delayed behaviour of concrete*, [in:] Creep and Shrinkage of Concrete, Z.P BAZANT and I. CAROL [Eds.], Spon, London, 189–200, 1993.
30. C. HUET, *Bounds for the viscoelastic properties of heterogeneous materials through new variational theorems*, [in:] Xth Int. Congress on Rheology, P.H.T UHLHERR [Ed.], Australian Soc. Rheology, Sydney, **1**, 422–424, 1988.
31. A.J. STAVERMAN and P. SCHWARZL, *Thermodynamics of viscoelastic behaviour*, Proc. Acad. Sc., The Netherlands, **55**, 474–485, 1952.
32. N. LAWS and R. MCLAUGHLIN, *Self-consistent estimates for the viscoelastic creep compliances of viscoelastic materials*, Proc. R. Soc. London, Ser. **A 359**, 251–273, 1978.

33. N. LAWS, *Viscoelastic inclusion problem*, J. Engng. Mech. Div., **EM5**, 915–928, 1980.
34. L. BOLTZMANN, *Zur Theorie der elastischen Nachwirkungen*, Sitzungsberichte, Kaiserliche Akad. Wiss., Wien, Math. Naturwiss. Klasse, 70 (II), 275, 1874.
35. M.A. BIOT, *Theory of stress-strain relations in anisotropic viscoelasticity and relaxation phenomena*, J. Appl. Phys., **13**, 11, 1385–1391, 1954.
36. J. MANDEL, *Cours de mécanique des milieux continus*, T.II, Annexe XXI, Gauthier-Villars, Paris 1966.
37. O. COUSSY, *Matériaux à mémoire et potentiels généralisés*, C.R. Acad. Sci. Paris, 305, Série II, 765–768, 1987.
38. C. HUET, *Etude par une méthode d'impédance du comportement viscoélastique des matériaux hydrocarbonnés*, Annales des Ponts et Chaussées, **6**, 373–429, 1965.
39. C. HUET, *Modelizing the kinetics of thermo-hydro-viscoelastic behaviour of wood in constant climatic conditions*, Proc. Int. Conf. on Timber Engineering, Seattle, USA, 395–401, 1988.
40. N. RUDDOCK, P.W. JAMES and T.E.R. JONES, *Modelling the viscoelasticity of unfilled and carbon black elastomers*, Rheol. Acta, **32**, 286–292, 1993.
41. R.M. CHRISTENSEN, *Theory of viscoelasticity*, Academic Press, New York 1982.
42. L. BRUN, *Sur deux expressions, analogues à la formule de Clapeyron, donnant l'énergie libre et la puissance dissipée pour un corps viscoélastique*, C. R. Acad. Sci. Paris, **261**, 41–44, 1965.
43. L. BRUN, *Méthodes énergétiques dans les systèmes évolutifs linéaires*, J. Méc., **8**, 125–166, 1969.
44. B.N. COLEMAN, *Thermodynamics of materials with memory*, Arch. Rational Mech. Anal., **17**, 1–46, 1964.
45. C. HUET, *Application of variational concepts to size effects in elastic heterogeneous bodies*, J. Mech. and Phys. Solids, **38**, 813–841, 1990.
46. C. HUET, *Universal conditions for assimilation of a heterogeneous material to an effective medium*, Mechanics Research Communications, **9**, 3, 165–170, 1982.
47. C. HUET, *On the definition and experimental determination of effective constitutive equations for heterogeneous materials*, Mechanics Research Communications, **11**, 3, 195–200, 1984.
48. C. HUET, *Some basic principles in the thermorheology of heterogeneous materials*, Progress and Trends in Rheology II, Rheol. Acta, **26**, 1–8, 1988.
49. M. AMIEUR, S. HAZANOV and C. HUET, *Numerical and experimental study of size and boundary conditions effects on the apparent properties of specimens not having the representative volume*, [in:] Micromechanics of Concrete and Cementitious Composites, C. HUET [Ed.], Presses Polytechnique et Universitaire Romandes, Lausanne, 181–202, 1993.
50. A. GUIDOUM, P. NAVI and C. HUET, *Numerical evaluation of microstructural effects on the shrinkage properties of concrete through a three-dimensional model of cohesive granular material*, [in:] Creep and Shrinkage of Concrete, Z.P. BAZANT and I. CAROL [Eds.], Spon, London, 127–132, 1993.
51. A. ALAOUI, *Une méthode énergétique de modélisation de la viscoélasticité non-linéaire en grandes déformations*, Doctoral dissertation, Ecole Nationale des Ponts et Chaussées, Centre d'Enseignement et de Recherche en Analyse des Matériaux, Paris 1993.
52. J.W. HUTCHINSON, *Bounds and self-consistent estimates for the creep of polycrystalline materials*, Proc. R. Soc. London, **A 348**, 101–127, 1976.
53. C. HUET, *Application à la viscoélasticité non linéaire du calcul symbolique à plusieurs variables*, Rheol. Acta, **12**, 279–288, 1973.
54. C. HUET, *Relations between creep and relaxation functions in nonlinear viscoelasticity with or without aging*, J. Rheol., **29**, 3, 245–257, 1985.
55. L.J. WALPOLE, *Strengthening effects in elastic solids*, J. Mech. Phys. Solids, **18**, 343–358, 1970.
56. C. HUET, P. NAVI and P.E. ROELFSTRA, *A homogenization technique based on Hill's modification theorem*, [in:] Continuum Models and Discrete Systems, G.A. MAUGIN [Ed], **2**, 136–143, 1991.

SWISS FEDERAL INSTITUTE OF TECHNOLOGY,
ECOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE,
DEPARTMENT OF MATERIALS, MXG-ECUBLENS, LAUSANNE, SWITZERLAND.

Received December 28, 1994.

Arch. Mech., 47, 6, pp. 1157–1167, Warszawa 1995

Peristaltic transport of a viscoelastic fluid (a long wavelength approximation)

SURIPEDDI SRINIVAS (VIJAYAWADA)
and N. Ch. P. RAMACHARYULU (WARANGAL)

THE AIM of the present investigation is to study the peristaltic transport of a second order viscoelastic fluid (Coleman and Noll-type) in a two-dimensional channel. Assuming that the peristaltic wave has large wavelength compared with the mean half-width of the channel, a solution for the stream function has been obtained as an asymptotic expression in terms of the slope parameter. Expressions for axial pressure gradient and shear stress have been derived up to the second order approximation. The effect of viscoelastic parameter on the stream-line pattern and shear stress, together with the phenomenon of trapping, has been discussed.

1. Introduction

PERISTALTIC PUMPING is a form of fluid transport that occurs when a progressive wave of area contraction or expansion propagate along the length of an extensible tube containing a liquid. It appears to be a major mechanism for urine transport in ureter, food mixing and chyme movement in intenstinal transport of spermatozoa in cervical canal, transport of bile in bile ducts and so on. Technical roller and finger pumps using viscous fluids also operate according to this rule.

Since the first investigation of LATHAM [6], several theoretical and experimental attempts have been made to understand the peristaltic action in different situations. A review of the early literature is presented in the article by JAFFRIN *et al.* [5]. The theoretical investigation of peristaltic motion by BURNS *et al.* [2], BARTON *et al.* [1], FUNG *et al.* [4], CHOW [3], SHAPIRO *et al.* [10] have excited some interest in recent years. However, these studies are confined to Newtonian fluids only. Some of the theoretical and experimental studies considering blood as non-Newtonian fluid are discussed in [7, 8, 9, 11, 12]. RAJU *et al.* [8] studied the peristaltic transport of blood considering blood as a power-law fluid and later extended the problem to a visco-elastic model [9]. They obtained the solution for stream function as a power series in terms of the amplitude of the deformation and evaluated the stream function and velocity components by solving numerically two-point boundary value problems with a singular point at the origin. RADHAKRISHNAMACHARYA *et al.* [7] studied the peristaltic motion of a power fluid, under long-wave length approximation.

In this paper we study the peristaltic transport of a viscoelastic fluid (Coleman–Noll type) in a channel with a sinusoidal wave of small amplitude travelling down its wall. Following [7] we obtain a solution for a stream function in terms of the slope parameter, and the expressions for axial pressure gradient and shear

stress are obtained. The influence of viscoelastic parameter on the stream-line pattern, shear-stress and pressure rise is discussed.

2. Formulation of the problem

We choose non-Newtonian fluid of viscoelastic model which is characterized by the constitutive equation

$$\sigma_{ij} = -pI + \emptyset_1 A^{(1)} + \emptyset_2 A^{(2)} + \emptyset_3 A^{(1)^2}, \tag{2.1}$$

where

$$\begin{aligned} A_{ij}^{(1)} &= u_{i,j} + u_{j,i}\,, \\ A_{ij}^{(2)} &= a_{i,j} + a_{j,i} + 2u_{m,i}u_{m,j}\,, \end{aligned} \tag{2.2}$$

$\emptyset_1$, $\emptyset_2$ and $\emptyset_3$ are material constants, $A^{(1)}$ is the rate of deformation tensor, $A^{(2)}$ – the Coleman–Noll tensor characterizing the second order effects of the fluid, u_i is the component of velocity.

We consider laminar flow of a viscoelastic fluid, characterized by Eq. (2.1) through a two-dimensional channel with flexible boundaries on which are imposed travelling sinusoidal waves of long wavelength. Let us choose a Cartesian coordinate system (x, y) with x-axis coinciding with the central line of the channel. The travelling waves are represented by

$$\eta(x, t) = d + a \sin \frac{2\pi}{\lambda}(x - ct), \tag{2.3}$$

where d is the mean half-width of the channel, a is the amplitude of the wave, λ the wavelength and c is the wave speed.

The equation of momentum and continuity are

$$\begin{aligned} \varrho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y}\,, \\ \varrho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] &= \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}\,, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \end{aligned} \tag{2.4}$$

where u, v are the velocity components along the x and y directions, respectively, and ϱ the density of the fluid. The boundary conditions on the velocity components are

$$\begin{aligned} u &= 0 \qquad \text{at} \quad y = \eta, \\ v &= \frac{\partial \eta}{\partial t} \qquad \text{at} \quad y = \eta, \end{aligned} \tag{2.5}$$

together with the regularity condition

$$v = 0 = \frac{\partial u}{\partial y} \quad \text{at} \quad y = 0. \tag{2.6}$$

Using the transformation

$$\frac{\partial \psi}{\partial y} = u - c, \qquad \frac{\partial \psi}{\partial x} = -v, \qquad x' = \xi - t', \qquad \xi = \frac{x}{\lambda}, \qquad t' = \frac{ct}{\lambda} \tag{2.7}$$

from a stationary to a moving frame of reference, the following non-dimensional quantities are introduced.

$$\begin{aligned} x' &= \xi - t', & y' &= \frac{y}{d}, & \psi' &= \frac{\psi}{cd}, \\ \varepsilon &= \frac{a}{d}, & \alpha &= \frac{d}{\lambda}, & p' &= \frac{pd}{\varrho c^2 \lambda}, \\ R &= \frac{cd}{\nu}, & \beta_* &= \frac{\beta}{d^2}, & \eta &= \frac{\eta}{d}. \end{aligned} \tag{2.8}$$

Equations (2.4) can be writen in non-dimensional form, after dropping the primes, as

$$\frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, y)} = -\frac{1}{R\alpha} \nabla^4 \psi + \beta_* \frac{\partial(\psi, \nabla^4 \psi)}{\partial(x, y)}, \tag{2.9}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial y^2} + \alpha^2 \frac{\partial^2}{\partial x^2}.$$

The corresponding boundary conditions are assumed:

$$\begin{aligned} \frac{\partial \psi}{\partial y} &= -1 && \text{at} \quad y = \eta, \\ \psi &= q && \text{at} \quad y = \eta, \\ \psi &= \frac{\partial^2 \psi}{\partial y^2} = 0 && \text{at} \quad y = 0, \end{aligned} \tag{2.10}$$

where q is half the flux in the negative axial direction in the moving frame of reference. However, by accounting for Galilean transformation, there will be a net positive flux in the stationary frame of reference.

3. Method of solution

Let us assume that the wavelength of the peristaltic wave is large in comparison to the mean half-width of the channel. Solution for the stream function can be obtained in terms of slope parameter α ($\alpha \ll 1$) as

$$\psi = \psi_0 + \alpha \psi_1 + \alpha^2 \psi_2 + \ldots + \ldots . \tag{3.1}$$

Substituting (3.1) in (2.9), (2.10) and collecting terms with coefficients of like powers of α, we get the following set of equations:

ZEROTH ORDER

$$\frac{\partial^4 \psi_0}{\partial y^4} = 0, \tag{3.2}$$

$$\begin{aligned} \frac{\partial \psi_0}{\partial y} &= -1 && \text{at} \quad y = \eta(x), \\ \psi_0 &= q && \text{at} \quad y = \eta(x), \\ \psi_0 &= \frac{\partial^2 \psi_0}{\partial y^2} = 0 && \text{at} \quad y = 0. \end{aligned} \tag{3.3}$$

FIRST ORDER

$$\frac{\partial^4 \psi_1}{\partial y^4} = R\left[\frac{\partial \psi_0}{\partial x}\frac{\partial^3 \psi_0}{\partial y^3} - \frac{\partial \psi_0}{\partial y}\frac{\partial^3 \psi_0}{\partial x \partial y^2}\right], \tag{3.4}$$

$$\begin{aligned} \frac{\partial \psi_1}{\partial y} &= 0 && \text{at} \quad y = \eta(x), \\ \psi_1 &= 0 && \text{at} \quad y = \eta(x), \\ \psi_1 &= \frac{\partial^2 \psi_1}{\partial y^2} = 0 && \text{at} \quad y = 0. \end{aligned} \tag{3.5}$$

SECOND ORDER

$$\begin{aligned} \frac{\partial^4 \psi_2}{\partial y^4} = -2\frac{\partial^4 \psi_0}{\partial x^2 \partial y^2} &+ \beta_* R\left[\frac{\partial \psi_0}{\partial x}\frac{\partial^5 \psi_1}{\partial y^5} - \frac{\partial \psi_0}{\partial y}\frac{\partial^5 \psi_1}{\partial x \partial y^4}\right] \\ &- R\left[\frac{\partial \psi_1}{\partial x}\frac{\partial^3 \psi_0}{\partial y^3} + \frac{\partial \psi_0}{\partial y}\frac{\partial^3 \psi_1}{\partial y^3} - \frac{\partial \psi_1}{\partial y}\frac{\partial^3 \psi_0}{\partial y^2 \partial x} - \frac{\partial \psi_0}{\partial y}\frac{\partial^3 \psi_1}{\partial y^2 \partial x}\right]; \end{aligned} \tag{3.6}$$

$$\begin{aligned} \frac{\partial \psi_2}{\partial y} &= 0 && \text{at} \quad y = \eta(x), \\ \psi_2 &= 0 && \text{at} \quad y = \eta(x), \\ \psi_2 &= \frac{\partial^2 \psi_2}{\partial y^2} = 0 && \text{at} \quad y = 0. \end{aligned} \tag{3.7}$$

Solving Eq. (3.2) subject to the boundary condition (3.3), we get

$$\psi_0 = A_1 y^3 + A_2 y, \tag{3.8}$$

where

$$A_1 = -\frac{(\eta + q)}{2\eta^3}, \qquad A_2 = \frac{3(\eta + q)}{2\eta} - 1.$$

Solving Eq. (3.4) subject to the boundary conditions (3.5) and using (3.8), we get

$$\psi_1 = B_1 y^7 + B_2 y^5 + B_3 y^3 + B_4 y, \tag{3.9}$$

where

$$\begin{aligned} B_1 &= \frac{R}{70} A_1 A_{1x}, \qquad B_2 = \frac{R}{20} A_{12}, \\ B_3 &= \frac{R}{10}\left(\frac{3}{7} A_1 A_{1x}\eta^2 + A_{12}\right)\eta^2, \\ B_4 &= \frac{R}{10}\left(\frac{2}{7} A_1 A_{1x}\eta^2 + \frac{A_{12}}{2}\right)\eta^4, \\ A_{12} &= A_2 A_{1x} - A_1 A_{2x}. \end{aligned}$$

Solution of (3.6), under the boundary conditions (3.7) and using (3.8) and (3.9), can be obtained as

$$\psi_2 = C_1 y^{11} + C_2 y^9 + C_3 y^7 + C_4 y^5 + C_5 y^3 + C_6 y, \tag{3.10}$$

where

$$\begin{aligned} C_1 &= \frac{R}{330}(5A_1 B_{1x} - 7A_{1x} B_1), \\ C_2 &= \frac{R}{504}[420\beta_*(A_{1x} B_1 - A_1 B_{1x}) - 11A_1 B_{2x} - 5A_{1x} B_2 \\ &\qquad -35A_{2x} B_1 - 7A_2 B_{1x}], \\ C_3 &= \frac{R}{70}\Big[10\beta_*(A_{1x} B_2 + 21A_{2x} B_1 - 3A_1 B_{2x} - 7A_{2x} B_{1x}) \\ &\qquad + A_1 B_{3x} - \frac{10}{3} A_2 B_{2x}\Big], \\ C_4 &= -\frac{1}{10} A_{1xx} + \frac{R}{20}(20\beta_* A_{2x} B_2 - 20A_2 B_{2x} - A_1 B_{4x} - A_{2x} B_3 \\ &\qquad - A_{1x} B_4 + A_2 B_{3x}), \\ C_5 &= -\frac{1}{12}\left(\frac{C_1}{132}\eta^6 + \frac{C_2}{63}\eta^4 + \frac{3C_3}{70}\eta^2 + \frac{C_4}{5}\right)\eta^2, \\ C_6 &= \frac{1}{12}\left(\frac{C_1}{165}\eta^6 + \frac{C_2}{84}\eta^4 + \frac{C_3}{35}\eta^2 + \frac{C_4}{10}\right)\eta^4. \end{aligned}$$

The axial pressure gradient can be obtained from $(2.4)_1$ using (3.8), (3.9) and (3.10) as

$$\frac{\partial p}{\partial x} = \left(\frac{\partial p}{\partial x}\right)_0 + \alpha\left(\frac{\partial p}{\partial x}\right)_1 + \alpha^2\left(\frac{\partial p}{\partial x}\right)_2 + \ldots + \ldots . \tag{3.11}$$

Substituting (3.11) in $(2.4)_1$ and collecting the like powers of α, after some rearrangement we get

$$\left(\frac{\partial p}{\partial x}\right)_0 = \frac{6A_1}{R},$$

$$\left(\frac{\partial p}{\partial x}\right)_1 = D_1 y^4 + D_2 y^3 + D_3 y^2 + D_4 y + D_5, \tag{3.12}$$

$$\left(\frac{\partial p}{\partial x}\right)_2 = E_1 y^8 + E_2 y^7 + E_3 y^6 + E_4 y^5 + E_5 y^4 + E_6 y^3 + E_7 y^2 + E_8 y + E_9,$$

where

$$D_1 = 3\left(-A_1 A_{1x} + \frac{70}{R} B_1\right),$$

$$D_2 = 6\beta_* A_1 + A_{1xx},$$

$$D_3 = 3\left(A_1 A_{2x} - A_2 A_{1x} + \frac{20}{R} B_2 + 18\beta_* A_1 A_{1x}\right),$$

$$D_4 = 6\beta_* A_1 A_{1x},$$

$$D_5 = 6\beta_* A_2 A_{1x} - A_2 A_{2x} + \frac{6}{R} B_3,$$

$$E_1 = 3\left(14 A_{1x} B_1 - 5 A_1 B_{1x} + \frac{333}{R} C_1\right),$$

$$E_2 = 6\beta_*(35 A_{1xx} B_1 + A_1 B_{1xx}),$$

$$E_3 = 5 A_{1x} B_2 + 35 A_{2x} B_1 - 9 A_1 B_{2x} - 7 A_2 B_{1x} + \frac{504}{R} C_2 + \beta_*(882 A_1 B_{1x} - 546 A_{1x} B_1),$$

$$E_4 = 6\beta_*(10 A_{1xx} B_2 + 35 A_{2xx} B_1 + A_1 B_{2xx}),$$

$$E_5 = \left[-3 A_{1x} B_3 + 15 A_{2x} B_2 - 3 A_1 B_{3x} - 5 A_2 B_{2x} + \frac{210}{R} C_3 + \beta_*(210 A_2 B_{1x} + 300 A_1 B_{2x} - 840 A_{2x} B_1)\right],$$

$$E_6 = 6\beta_*(10 A_{2xx} B_2 + A_{1xx} B_3 + A_1 B_{3xx}),$$

$$E_7 = 3\left[A_{2x} B_3 + A_1 B_{4x} - A_2 B_{3x} - A_{1x} B_4 + \frac{20}{R} C_4 + \frac{1}{R} A_{1xx} + \beta_*(18 A_1 B_{3x} + 18 A_{1x} B_3 + 20 A_2 B_{2x} - 40 A_{2x} B_2)\right],$$

$$E_8 = 6\beta_*(A_{2xx} B_3 + A_1 B_{4xx}),$$

$$E_9 = 6\beta_*(A_2 B_{3x} + A_{1x} B_4) + \frac{1}{R}(A_{2xx} + 6C_5) - (A_2 B_{4x} + A_{2x} B_4).$$

The pressure change over one wavelength Δp_λ, also called pressure rise, is given

by

$$\Delta p_\lambda = \int_0^1 \frac{\partial p}{\partial x} dx \,. \tag{3.13}$$

This integral is solved numerically and the results are shown graphically (Fig. 3).

Shear stress acting on the wall is defined as

$$\tau = \frac{\sigma_{xy}\left[1 - \left(\frac{d\eta}{dx}\right)^2\right] + (\sigma_{yy} - \sigma_{xx})\frac{d\eta}{dx}}{1 + \left(\frac{d\eta}{dx}\right)^2} \qquad \text{at} \quad y = \eta(x). \tag{3.14}$$

The shear-stress can be obtained from (3.14) in the following form

$$\bar{\tau} = \frac{\tau}{\left[\frac{\phi_1 c}{d}\right]} = \tau_0 + \alpha\tau_1 + \alpha^2\tau_2 + \ldots + \ldots \,, \tag{3.15}$$

where

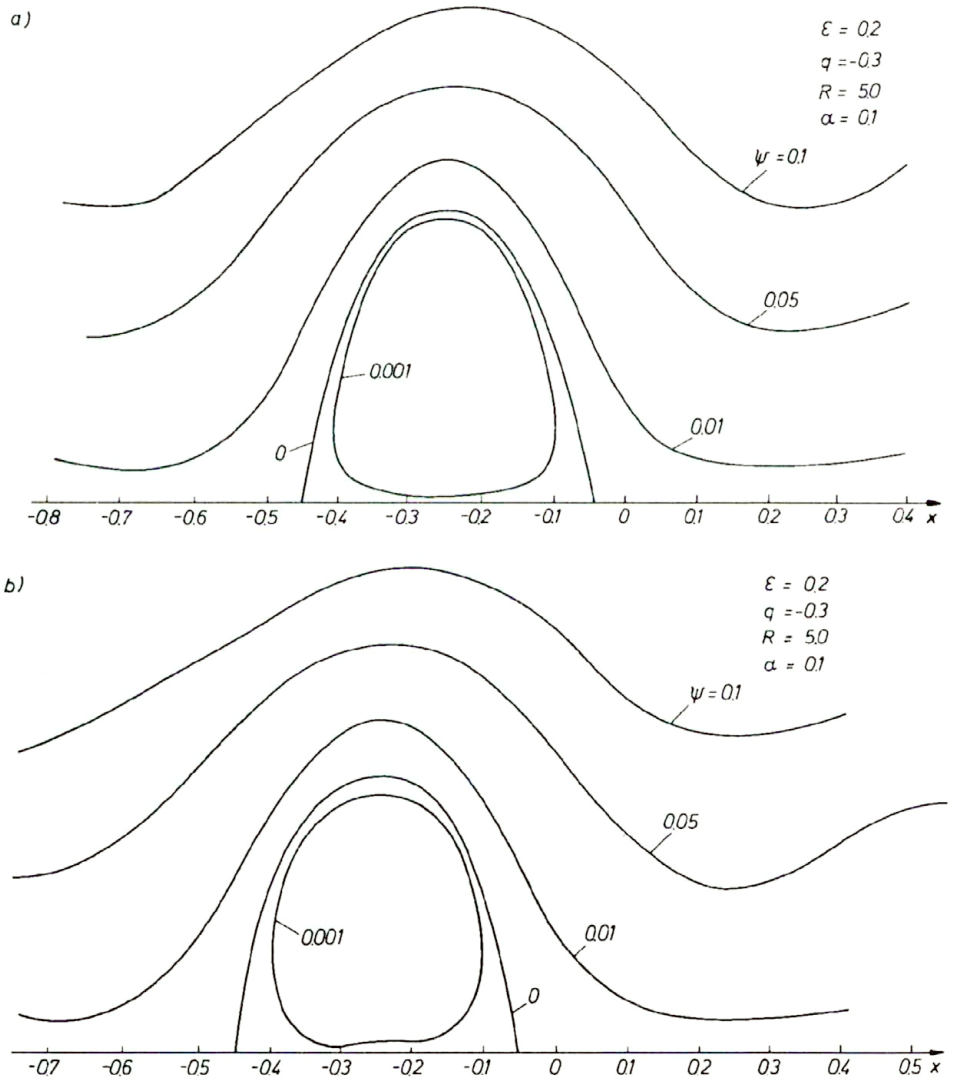
$$\begin{aligned}
\tau_0 &= 72L_2A_1^2\eta^2, \\
\tau_1 &= 6L_1\eta\left[\beta_*(5A_{1xx}\eta^2 + A_1A_{2x} - A_{1x}) + 6A_1\right] + 4L_2\left[R\beta_*(252A_1B_1\eta^5\right. \\
&\qquad \left. +120A_1B_2\eta^4) + 3(12R\beta_*A_1B_3 - A_{1x})\eta^2 + A_{2x}\right], \\
\tau_2 &= 6L_1\eta\left[\beta_*R(7A_{1x}B_1 + 15A_1B_{1x})\eta^6 + \left(7B_1 + \frac{10}{3}B_2 + 11A_1B_{2x}\right.\right. \\
&\qquad \left.\left. + 49A_{2x}B_1\right)\eta^4 + \beta_*R\left(7A_{1x}B_3 + \frac{50}{3}A_{2x}B_2 - \frac{10}{3}B_{2x}\right)\eta^2\right. \\
&\qquad \left. +\beta_*R(2A_{2x}B_3 + 3A_1B_{4x} - B_{3x}) - B_3\right] + 2L_2\left[\beta_*R(660A_1C_1\eta^{10}\right. \\
&\qquad +432A_1C_2\eta^8) + (252R\beta_*A_1C_3 - 14B_{1x}) + (120R\beta_*A_1C_4 - 10B_{2x})\eta^4 \\
&\qquad \left. -(36RB_*A_1C_5 - B_{3x})\eta^2 + 6R\beta_*A_1\eta - 2B_{4x}\right],
\end{aligned} \tag{3.16}$$

where

$$L_1 = \left[1 - \left(\frac{d\eta}{dx}\right)^2\right] \Big/ \left[1 + \left(\frac{d\eta}{dx}\right)^2\right] \qquad \text{and} \qquad L_2 = \left(\frac{d\eta}{dx}\right) \Big/ \left[1 + \left(\frac{d\eta}{dx}\right)^2\right].$$

4. Numerical discussion

The stream-line pattern has been presented for different values of β_* in Fig. 1. The phenomenon of trapping can be noticed in the figure. When trapping occurs, the central stream-line splits and there is a region of recirculating, closed stream-lines and it comprises of the bolus fluid. In the stationary frame of reference the bolus fluid is trapped by the wave and it advances as a whole with the wave speed. Though the stream-line patterns remain the same with β_*, the area of the trapped bolus decreases with the increase of β_*. For $\beta_* = 0$ (Newtonian case) the stream-line pattern is similar to the one obtained by SHAPIRO *et al.*



[FIG. 1 a, b]

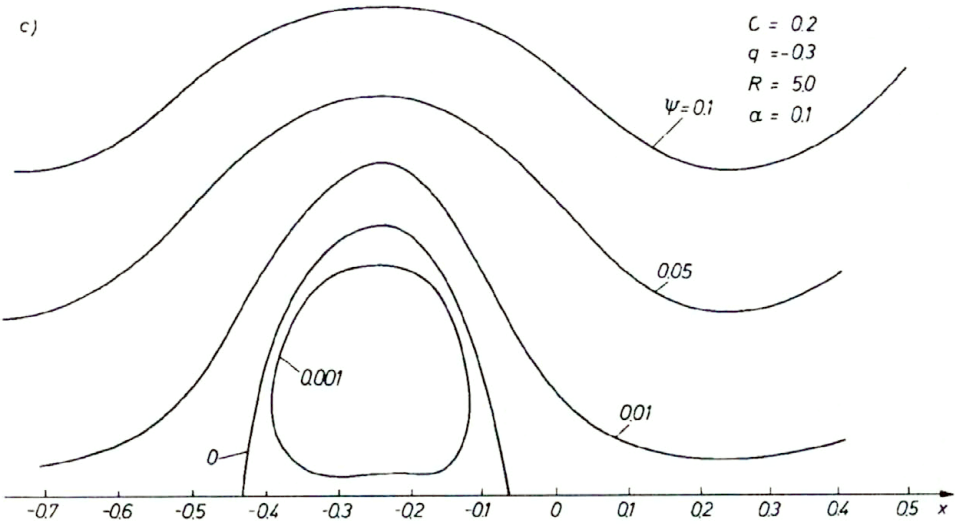


FIG. 1. Stream-lines in wave frame when a trapped bolus exists in a laboratory frame for: a) $\beta_* = 0.0$, b) $\beta_* = 0.2$, c) $\beta_* = 0.4$.

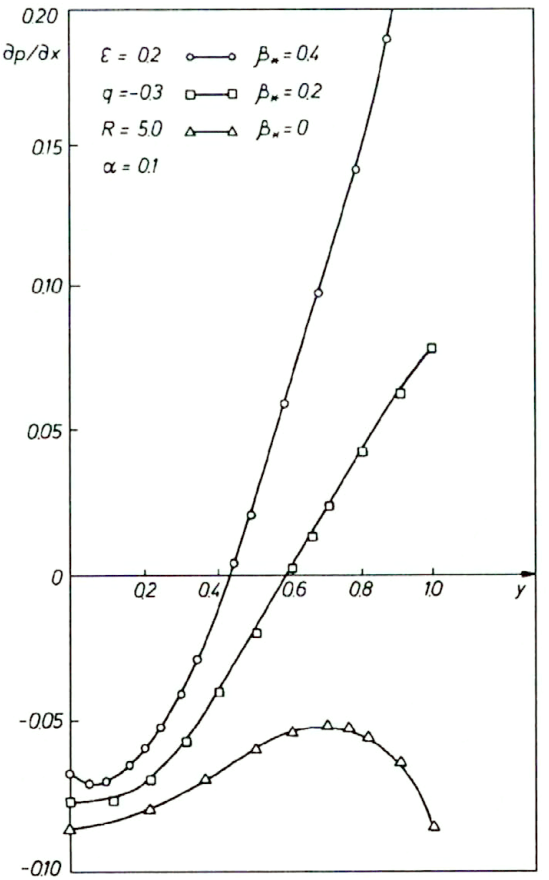


FIG. 2. Variation of pressure gradient versus y.

Figure 2 shows the variation of pressure gradient with radial distance for different β_*. It can be observed that the pressure gradient increases with the increase of β_*.

In Fig. 3 we see the dimensionless pressure rise per wavelength as a function of y and the rheological parameter of the liquid. It can be noticed that the pressure rise decreases with increase of β_*.

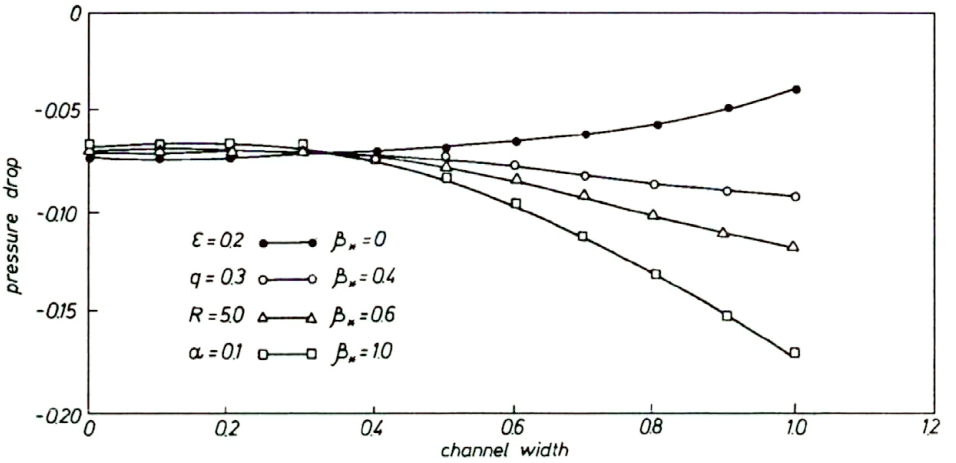


FIG. 3. Pressure rise versus channel width.

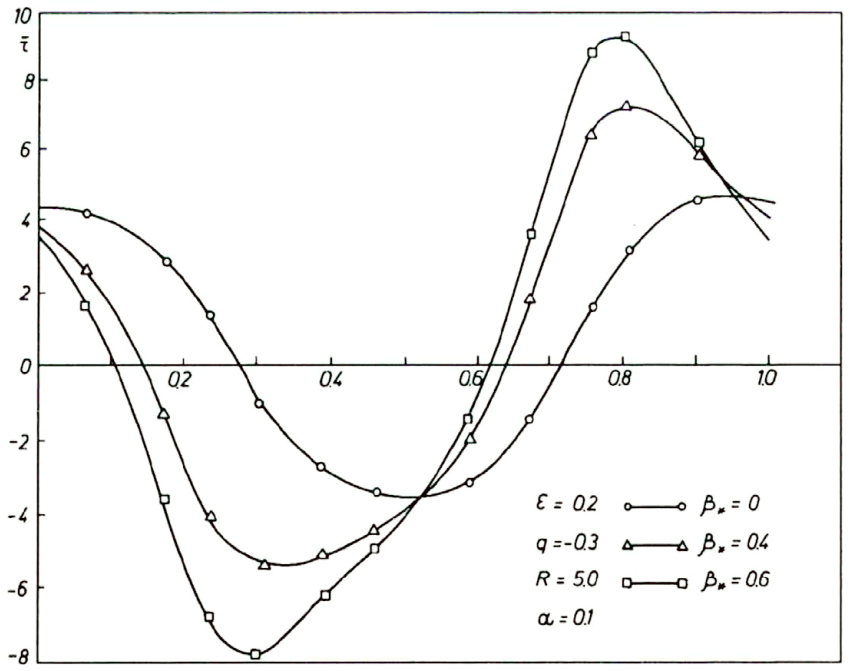


FIG. 4. Variation of shear-stress.

In Fig. 4 the variation of shear stress with x is shown. It can be observed that the amplitude of the shear-stress curve increases with the increase of β_*.

Acknowledgement

Authors acknowledge the financial support from U.G.C., India, under the project No.F.8-1/85(SR-III).

References

1. C. BARTON and S. RAYMOR, *Peristaltic flows in tubes*, Bull. Math. Bio. Phy., **30**, pp. 663–680, 1968.
2. J.C. BURNS and J. PARKES, *Peristaltic motion*, J. F. Mech., **29**, pp. 731–743, 1967.
3. T.S. CHOW, *Peristaltic transport in a circular cylindrical pipe*, J. Appl. Mech., pp. 901–905, 1970.
4. Y.C. FUNG and C.S. YIH, *Peristaltic transport*, J. Appl. Mech., **35**, pp. 669–675, 1968.
5. M.Y. JAFFRIN and A.H. SHAPIRO, *Peristaltic pumping*, Ann. Rev. Fluid Mech., **3**, pp. 13–35, 1971.
6. T.W. LATHAM, *Fluid motion in a peristaltic pump*, M.S. Thesis, M III Cambridge, Mass, 1966.
7. G. RADHAKRISHNAMACHARYA, *Long wavelength approximation to peristaltic motion of power law fluid*, Rheol. Acta, **21**, pp. 30–35, 1982.
8. K.K. RAJU and R. DEVANATHAN, *Peristaltic motion of a non-Netwonian fluid*, Rheol. Acta, **11**, pp. 170–178, 1972.
9. K.K. RAJU and R. DEVANATHAN, *Peristaltic motion of a non-Newtonian fluid. Part II. Viscoelastic fluid*, Rheol. Acta, **13**, pp. 944–948, 1974.
10. A.H. SHAPIRO, M.Y. JAFFRIN and S.L. WEINBERG, *Peristaltic pumping with long wavelength at low Reynolds number*, J. Fluid Mech., **37**, 799–825, 1969.
11. L.M. SRIVASTAVA, *Peristaltic transport of a couple-stress fluid*, Rheol. Acta, **25**, pp. 638–641, 1986.
12. L.M. SRIVASTAVA and V.P. SRIVASTAVA, *Peristaltic transport of power law fluid; Application to ductus efferentes of the reproductive tract*, Rheol. Acta, **27**, pp. 428–433, 1988.

DEPARTMENT OF MATHEMATICS,
V.R. SIDDHARTHA ENGINEERING COLLEGE, VIJAYAWADA
and
DEPARTMENT OF MATHEMATICS AND HUMANITIES,
REGIONAL ENGINEERING COLLEGE, WARANGAL, INDIA.

Received January 9, 1995.

Arch. Mech., 47, 6, pp. 1169–1181, Warszawa 1995

Generalized method of Schwarz and addition theorems in mechanics of materials containing cavities

V. V. MITYUSHEV (SŁUPSK)

THE LINEAR STEADY-STATE problem of materials containing cavities is considered by the generalized method of Schwarz. The convergence of the method has been proved.

1. Introduction

THIS PAPER is devoted to linear steady-state problems of materials containing cavities which are described by self-adjoint elliptic partial differential equations of second order. We shall apply the generalized method of Schwarz [1–3; 4, Chapter VII] which consists in replacing the given problem by a sequence of problems for a domain containing a single cavity. The crucial ingredient in this approach is the repeated solution of the Dirichlet problems for a simple domain. If the shape of the cavity is complex, then the computations can be expected to be tedious. However, in special cases the generalized method of Schwarz leads to a straight-forward solution [3, 7, 10–13]. The method of images [14, 15] can be interpreted as the Schwarz method for spheres. The method of images, when used in combination with addition theorems, allows us to calculate the induced electrostatic moments on a pair of dielectric spheres and the square array of cylinders [16–19].

At the beginning, the generalized method of Schwarz has been proposed by G.M. GOLUSIN [1] and S.G. MIKHLIN [2] for the classical boundary value problems for harmonic functions. In Sec. 3 we treat the method of addition theorems as a discrete interpretation of the method of Schwarz. In [2–4] this method has been developed on equations of fourth order.

There is the question of convergence of this method which is related to the problem of convergence of the method of successive approximations for some linear equation

$$U = AU + f \tag{1.1}$$

in a Banach space with respect to U. Here A is a linear operator, f is a known element. In [1, 2] the additional condition $\|A\| < 1$ implies convergence of the method of Schwarz. This condition may be considered as an additional condition on geometry of the domain [1, 2].

However, there are other types of convergence. Let $\sigma(A)$ be the spectrum of the operator A, $\varrho(A)$ be the resolvent of A. Let λ be a non-zero complex number.

We consider the equation

$$U = \lambda AU + f \tag{1.2}$$

in a Banach space $\mathfrak{B}$. From the definition, the following assertions are equivalent [5]:

i) $\lambda^{-1} \in \varrho(A)$,

ii) $\exists\, (I - \lambda A)$,

iii) Equation (1.2) has a unique solution for each $f \in \mathfrak{B}$.

If we show that the spectral radius

$$r_A = \sup_{\lambda^{-1} \in \sigma(A)} |\lambda^{-1}| = \lim_{n\to\infty} \|A^n\|^{1/n}$$

of A satisfies the inequality

$$r_A < 1, \tag{1.3}$$

i.e. the condition iii) takes place when $|\lambda| \leq 1$, then the method of successive approximations for (1.1) converges in the space $\mathfrak{B}$. It means that the series

$$U = \sum_{m=0}^{\infty} A^m f \tag{1.4}$$

converges in $\mathfrak{B}$. So let us recall the following

THEOREM 1. [5] *If equation* (1.2) *has a unique solution for each* $\lambda \leq 1$, *then the series* (1.4) *converges in* $\mathfrak{B}$.

REMARK. The inequality $\|A\| < 1$ implies absolute convergence of the series (1.4). But the series (1.4) can be convergent in $\mathfrak{B}$ when $\|A\| \geq 1$.

In the present paper we shall study the problem of convergence of the series (1.4) in a Banach space $\mathfrak{B}$ for an elliptic equation of second order. Our results can be considered as not only a suggestion for those who apply the method of Schwarz in computation. Actually, we indicate a modification of the method allowing us to approach a convergent algorithm. Such examples have been carried out in the language of the addition theorems in [8, 19].

Let us consider a self-adjoint equation in $\mathbb{R}^3$ [6]

$$\mathfrak{L}u := \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right) = 0, \tag{1.5}$$

where $x = (x_1, x_2, x_3)$, $a_{ij}(x) = a_{ji}(x)$. We shall consider the Dirichlet problem

$$u(t) = f(t). \tag{1.6}$$

Here, the mutually disjoint domains D_k are bounded by the Lyapunov boundary $\partial D_k \in \mathbb{R}^3$ ($k = 0, 1, ..., n$, $\partial D_k \cap \partial D_m = \emptyset$ for $m \neq k$), $f(t)$ is a given differentiable continuous function ($f \in C^1(\partial D)$). We orientate ∂D_k in the positive direction and assume that $\partial D = -\bigcup_{k=0}^{n} \partial D_k$. Let D_k^- be the complement of $\overline{D}_k$ to $\mathbb{R}^3 \cup \{\infty\}$, $D := (\mathbb{R}^3 \setminus \bigcup_{k=0}^{n} \overline{D}_k) \cup \{\infty\}$. For certain equations, under conditions

$$\lim_{|x| \to \infty} u(x) =: u(\infty) = 0, \tag{1.7}$$

where

$$|x| := (x_1^2 + x_2^2 + x_3^2),$$

the problem (1.5), (1.6) has a unique solution (for instance, the Laplace equation $\Delta u = 0$). For other equations we need an additional condition

$$F_0 u := \lim_{|x| \to \infty} Fu(x) = 0. \tag{1.8}$$

For example, the Dirichlet problem for the Helmholtz equation $\Delta u + k^2 u = 0$ has a unique solution if

$$u = O(|x|^{-1}), \qquad \frac{\partial u}{\partial |x|} + iku = o(|x|^{-1}), \qquad \text{when} \quad |x| \to \infty.$$

The condition (1.8) is a counterpart of the last relations. We shall use the following properties of Eq. (1.5):

a. F is a linear operator, (1.8) implies (1.7).

b. If $\mathfrak{L} u = 0$ in $\mathbb{R}^3$ and if (1.8) holds, then $u(x) \equiv 0$.

c. For each ∂D_k there exist such operators [6]

$$P_k \mu = \iint_{\partial D_k} \frac{\partial U}{\partial n} \mu \, ds, \qquad Q_k \mu = \iint_{\partial D_k} U \mu \, ds,$$

which are counterparts of the potentials of simple and double layers. The following jump relations hold true

$$P_k^+ \mu(t) - P_k^- \mu(t) = \mu(t), \qquad \frac{\partial P_k^+ \mu(t)}{\partial n} = \frac{\partial P_k^- \mu(t)}{\partial n},$$

$$Q_k^+ \mu(t) = Q_k^- \mu(t), \qquad \frac{\partial Q_k^+ \mu(t)}{\partial n} - \frac{\partial Q_k^- \mu(t)}{\partial n} = \mu(t), \qquad t \in \partial D_k.$$

Here, the function $\mu(t)$ is continuously differentiable in ∂D_k. The functions $P_k \mu$ and $Q_k \mu$ satisfy Eq. (1.5) outside ∂D_k and the condition (1.8).

d. The Dirichlet problem $v_k = f$ in ∂D_k with respect to the function v_k satisfying the Eq. (1.5) in D_k^- and continuous in D_k^- when $F_0 v_k(x) = 0$, has a unique solution for each continuous data f. The Dirichlet problem for D_k has a solution. The Dirichlet problem for each ball in $\mathbb{R}^3$ has a unique solution.

e. Let us introduce the Banach space $C(\partial D)$ of functions continuous in ∂D with the norm $\|f\| := \max_{\partial D} |f(t)|$. Let us consider the subspace $\mathfrak{B} \subset C(\partial D)$, where $f \in \mathfrak{B}$ if $\mathfrak{L} f = 0$ in all D_k and f is continuous in all $\overline{D}_k$. Along similar lines define $\mathfrak{B}^- \subset C(\partial D)$, where $f \in \mathfrak{B}^-$ if $\mathfrak{L} f = 0$ in D, $F_0 f(x) = 0$ and f is continuous in $\overline{D}$. We assume that $\mathfrak{B}$ is a closed subspace of $C(\partial D)$, i.e. if $f_n \in \mathfrak{B}$ and $\|f_n - f\| \to 0$, when $n \to \infty$, then $f \in \mathfrak{B}$. Sometimes we shall indicate the domain: $f \in \mathfrak{B}(G)$. Also we shall use the subspace $\mathfrak{B}^1(G)$, when f is continuously differentiable in $\overline{G}$. Let us note that the convergence in $\mathfrak{B}$ means the uniform convergence because the principle of maximum takes place for Eq. (1.5).

The conditions a–e are natural for a wide class of elliptic equations [6]. For instance, b is a counterpart of Liouville's theorem, e is a counterpart of Harnack's theorem.

LEMMA 1. Let the domains G_1 and G_2 constitute the domain $G := G_1 \cup S \cup G_2$, where S is the surface dividing the domains. Let $u_1 \in \mathfrak{B}^1(G_1)$, $u_2 \in \mathfrak{B}^1(G_2)$. Let the relations $u_1 = u_2$, $\dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n}$ hold in S. Then the function

$$u = \begin{cases} u_1 & \text{in} \quad G_1 \cup S, \\ u_2 & \text{in} \quad G_2 \cup S, \end{cases} \qquad \text{belongs to} \quad \mathfrak{B}^1(G).$$

P r o o f. Let us consider the closed ball $\overline{B} \subset G$ with the center at an arbitrary $x \in S$. Let us consider the Dirichlet problem

$$v = \begin{cases} u_1 & \text{in} \quad S_1 := \partial B \cap G_1, \\ u_2 & \text{in} \quad S_2 := \partial B \cap G_2 \end{cases}$$

with respect to v. According to d, this problem has a unique solution satisfying the relations

$$v^+ = v^-, \qquad \frac{\partial v^+}{\partial n} = \frac{\partial v^-}{\partial n} \qquad \text{in} \quad S \cap B,$$

where, for instance, $v^+(t) := \lim\limits_{\substack{x \to t \\ x \in G_1}} v(x)$. Let us consider the following problem

$v_1 = u_1$ in S_1, $v_2 = u_2$ in S_2,

$$v_1 = v_2, \qquad \frac{\partial v_1}{\partial n} = \frac{\partial v_2}{\partial n} \qquad \text{in} \quad S \cap B, \tag{1.9}$$

in B with respect to $v_1 \in \mathfrak{B}^1(G_1)$, $v_2 \in \mathfrak{B}^1(G_2)$. We shall use the following Green's formulas

$$\iiint_G \sum_{i,j=1}^{3} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx = \iint_{\partial G} u\frac{\partial u}{\partial n}ds, \tag{1.10}$$

$$\iint_{\partial G}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)ds = 0,$$

where G is a domain, ∂G is the boundary of G, $u, v \in \mathfrak{B}^1(G)$. Ellipticity of Eq. (1.5) implies the inequality

$$\sum_{i,j=1}^{3} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \geq 0, \qquad \text{when} \quad x \in G. \tag{1.11}$$

Let us show that the homogeneous problem $v = 0$ in ∂B and (1.9) has only the zero solution. From the first Green's formulae (1.10) and conditions (1.9)

$$\iint_{S_1} v_1\frac{\partial v_1}{\partial n}ds + \iint_{S_2} v_2\frac{\partial v_2}{\partial n}ds = \iint_{S\cup S_1} v_1\frac{\partial v_1}{\partial n}ds + \iint_{-S\cup S_2} v_2\frac{\partial v_2}{\partial n}ds$$

$$= \iiint_{B\cap G_1} \sum_{i,j=1}^{3} a_{ij}(x)\frac{\partial v_1}{\partial x_i}\frac{\partial v_1}{\partial x_j}dx + \iiint_{B\cap G_2} \sum_{i,j=1}^{3} a_{ij}(x)\frac{\partial v_2}{\partial x_i}\frac{\partial v_2}{\partial x_j}dx.$$

Since $v_1 = 0$ in S_1 and $v_2 = 0$ in S_2, the integral at the left-hand part is equal to zero. Taking into account (1.11) we obtain $v_{1,2} = 0$ in $G_{1,2}$, respectively. Therefore, a nonhomogeneous problem can have not more than one solution. The function v is a solution of this problem. Hence, $v_1 = v$ in $B \cap G_1$ and $v_2 = v$ in $B \cap G_2$. So we have proved that the function u_1 is continued in each point $x \in S$ into the function u_2 preserving the class $\mathfrak{B}$.

The lemma has been proved.

2. Dirichlet problem

Let us introduce the function $f(x) \in \mathfrak{B}^1$ which satisfies the Dirichlet problems in each domain D_k

$$\lim_{\substack{x\to t \\ x\in D_k}} f(x) = f(t), \qquad t \in \partial D_k, \qquad k = 0, 1, ..., n. \tag{2.1}$$

According to the condition d, the function $f(x)$ exists. According to the same condition there exists such an operator S_k: $C(\partial D_k) \to \mathfrak{B}^-(D_k^-)$ that the function

$S_k v_k(x)$ is a solution of the following boundary value problem:

$$\lim_{\substack{x \to t \\ x \in D_k}} S_k v_k(x) = v_k(t), \qquad t \in \partial D_k,$$
$$F_0 S_k v_k(x) = 0.$$

Let $G_k(x, y)$ be the Green's function of this problem [6]. Then the operator S_k has the form [6, p. 430]

$$S_k v_k(x) = \int\int_{-\partial D_k} v_k(t) \frac{\partial G_k(x,t)}{\partial n_t} ds.$$

If $x \in \partial D_m$ $(m \neq k)$, then the operator S_k: $C(\partial D_k) \to C(\partial D_m)$ is compact as an integral operator with the continuous kernel $\dfrac{\partial G_k(x,t)}{\partial n}$ in $\partial D_m \times \partial D_k$.

LEMMA 2. The problem

$$\frac{\partial u_k}{\partial n}(t) - \frac{\partial S_k^- u_k}{\partial n}(t) = g(t), \qquad t \in \partial D_k, \tag{2.2}$$

with respect to $u_k \in \mathfrak{B}^1(D_k)$ has the unique solution

$$u_k(x) = Q_k g(x), \qquad x \in \overline{D_k}, \tag{2.3}$$

for a given function $g(t) \in C^1(\partial D)$. Here, Q_k is the potential of a single layer.

P r o o f. The function $v_k(x) := S_k u_k(x)$ has to satisfy the conditions

$$u_k(t) = v_k(t), \qquad \frac{\partial u_k}{\partial n}(t) - \frac{\partial v_k}{\partial n}(t) = g(t), \qquad F_0 v_k = 0.$$

The function $u_k(x)$ from (2.3) and $v_k(x) = Q_k g(x)$ satisfy the last problem. Let us prove that the homogeneous problem has only a zero solution. If $g(t) = 0$, then $u_k = v_k \in \mathfrak{B}^-(\mathbb{R}^3)$; it follows from the Lemma 1. Then b implies the equality $u_k(x) \equiv 0$.

The lemma has been proved.

LEMMA 3. The problem

$$\begin{aligned} u(t) &= (1-\lambda) u_k(t), \qquad \frac{\partial u}{\partial n}(t) = \frac{\partial u_k}{\partial n}(t) - \lambda \frac{\partial S_k u_k}{\partial n}(t), \qquad t \in \partial D_i, \\ u(\infty) &= 0, \end{aligned} \tag{2.4}$$

with respect to the complex functions $u \in \mathfrak{B}^-(D) \cap C^1(\overline{D})$, $u_k \in \mathfrak{B}^1(D_k)$ has only a zero solution if $|\lambda| \leq 1$.

P r o o f. Let $u = \psi_1 + i\psi_2$, $u_k = v_k + iw_k$ be the complex solutions of (2.4), and let $\lambda = \nu + i\mu$ be a complex number. Following the book [2], let us multiply the first equality (2.4) on $\overline{u(t)} = (1 - \overline{\lambda})\overline{u_k(t)}$, integrate it and sum from 0 to n. We obtain

$$
\begin{aligned}
-\iint_{\partial D}\left(\psi_1\frac{\partial\psi_1}{\partial n} + \psi_2\frac{\partial\psi_2}{\partial n}\right)ds - i\iint_{\partial D}\left(\psi_1\frac{\partial\psi_2}{\partial n} - \psi_2\frac{\partial\psi_1}{\partial n}\right)ds \\
= (1-\overline{\lambda})\sum_{k=0}^{n}\left(\iint_{\partial D_k}\left(v_k\frac{\partial v_k}{\partial n} + w_k\frac{\partial w_k}{\partial n}\right)ds\right. \\
\left. + i\iint_{\partial D_k}\left(v_k\frac{\partial w_k}{\partial n} - w_k\frac{\partial v_k}{\partial n}\right)ds\right) - \lambda(1-\overline{\lambda})\sum_{k=0}^{n}\left(\iint_{\partial D_k^-}\left(S_k v_k\frac{\partial S_k v_k}{\partial n^-}\right.\right. \\
\left.\left. + S_k w_k\frac{\partial S_k v_k}{\partial n^-}\right)ds + i\iint_{\partial D_k^-}\left(S_k w_k\frac{\partial S_k w_k}{\partial n^-} - S_k v_k\frac{\partial S_k v_k}{\partial n^-}\right)ds\right) \\
= (1-\overline{\lambda})\sum_{k=0}^{n}\iint_{\partial D_k}\left(v_k\frac{\partial w_k}{\partial n} + w_k\frac{\partial v_k}{\partial n}\right)ds \\
-\lambda(1-\overline{\lambda})\sum_{k=0}^{n}\left(\iint_{\partial D_k^-}\left(S_k v_k\frac{\partial S_k v_k}{\partial n^-} + S_k w_k\frac{\partial S_k w_k}{\partial n^-}\right)ds\right),
\end{aligned}
$$

where

$$\frac{\partial}{\partial n^-} := -\frac{\partial}{\partial n}.$$

We have used here the second Green's formulae (1.10).

Let us use the first Green's formulae (1.10)

$$
\begin{aligned}
D\psi &:= \iint_{\partial D}\int \psi\frac{\partial\psi}{\partial n}ds \\
&= \iiint_D\left(\sum_{i,j=1}^{3} a_{ij}(x)\frac{\partial\psi(x)}{\partial x_j}\frac{\partial\psi(x)}{\partial x_i}\right)dx \qquad \text{for} \quad \psi \in \mathfrak{B}^-(D)\cap C^1(\overline{D}), \\
D_k v &:= \iint_{\partial D_k} v\frac{\partial v}{\partial n}ds \\
&= \iiint_{D_k}\left(\sum_{i,j=1}^{3} a_{ij}(x)\frac{\partial v(x)}{\partial x_j}\frac{\partial v(x)}{\partial x_i}\right)dx \qquad \text{for} \quad v \in \mathfrak{B}^1(D_k).
\end{aligned}
$$

Then we have

$$A + (1-\nu)B + (|\lambda|^2 - \nu)C = 0,$$
$$\mu B - \mu C = 0,$$

where all numbers

$$A := D\psi_1 + D\psi_2,$$
$$B := \sum_{k=0}^{n}(D_k v_k + D_k w_k),$$
$$C := \sum_{k=0}^{n}(D_k S_k v_k + D_k S_k w_k)$$

are non-negative. For $\lambda \neq 1$ these equalities are possible only if $A = B = C = 0$. It implies that u = constant and u_k = constant. But $F_0 u = 0$, and hence $u(x) \equiv 0$. If $\lambda = 1$, then from the second equality (2.4) and the Lemma 2 we obtain $u_k(x) \equiv 0$.

The lemma has been proved.

Let the function $u(x)$ be known. Then, using the Lemma 2, we may introduce such function $u_k(x) \in \mathfrak{B}^1(D_k)$ that

$$\frac{\partial u}{\partial n}(t) - \frac{\partial f}{\partial n}(t) = \frac{\partial u_k}{\partial n}(t) - \frac{\partial S_k u_k}{\partial n}(t), \qquad t \in \partial D_k,$$

where $f(x)$ is a solution of the Dirichlet problem (2.1). Let us define the function

$$\Phi(x) = \begin{cases} u_k(x) + \sum\limits_{\substack{m=0 \\ m\neq k}}^{n} S_m u_m(x) + f(x), & x \in D_k, \\ & k = 0, 1, \ldots, n, \\ u(x) + \sum\limits_{m=0}^{n} S_m u_m(x), & x \in D. \end{cases}$$

Let us calculate the values

$$\Phi^+(t) - \Phi^-(t) = u(t) + S_k u_k(t) - u_k(t) - f(t) = 0,$$
$$\frac{\partial \Phi^+}{\partial n}(t) - \frac{\partial \Phi^-}{\partial n}(t) = \frac{\partial u}{\partial n}(t) - \frac{\partial u_k}{\partial n}(t) + \frac{\partial S_k u_k}{\partial n}(t) - \frac{\partial f}{\partial n}(t) = 0, \quad t \in \partial D_k.$$

It means that $\Phi(x)$ satisfies (1.5) in $\mathbb{R}^3$. Checking the condition (1.8) we obtain from b that $\Phi(x) \equiv 0$. Therefore, from the definition of $\Phi(x)$ we have the system of integral equations

$$u_k(x) = -\sum_{\substack{m=0 \\ m\neq k}}^{n} S_m u_m(x) - f(x), \qquad x \in \overline{D}_k, \tag{2.5}$$

and the relation

$$u(x) = -\sum_{m=0}^{n} S_m u_m(x), \qquad x \in \overline{D}. \tag{2.6}$$

LEMMA 4. The system (2.5) for $u_k \in \mathfrak{B}^1(D_k)$ has the unique solution

$$u_k(x) = -f(x) + \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} S_{k_1} f(x) - \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^{n} S_{k_1} S_{k_2} f(x) + \dots . \tag{2.7}$$

The last series converges in the space $\mathfrak{B}(D_k)$.

P r o o f. According to the Theorem 1 it is sufficient to prove that the complex system

$$u_k(x) = -\lambda \sum_{\substack{m=0 \\ m \neq k}}^{n} S_m u_m(x) - f(x), \qquad x \in \overline{D}_k, \tag{2.8}$$

has a unique solution if $|\lambda| \leq 1$. Let us consider the system (2.8) as an integral equation in the space $C(\partial D)$:

$$u_k(t) = -\lambda \sum_{\substack{m=0 \\ m \neq k}}^{n} S_m u_m(t) - f(t), \qquad t \in \partial \overline{D}_k. \tag{2.9}$$

Since S_m is a compact operator, the last equation is of Fredholm type in $C(\partial D)$. If $f \in \mathfrak{B}$, then each solution belongs to $\mathfrak{B}$ too. It follows from the properties of S_m. So if $f \in \mathfrak{B}$, then the system (2.8) has a unique solution if the homogeneous system

$$u_k(x) = -\lambda \sum_{\substack{m=0 \\ m \neq k}}^{n} S_m u_m(x), \qquad x \in \overline{D}_k,$$

has only a zero solution. Study the last system. Let us introduce the function

$$u(x) = -\lambda \sum_{m=0}^{n} S_m u_m(x), \qquad x \in \overline{D},$$

satisfying Eq. (1.5) in $\overline{D}$ and (1.8). The following boundary conditions hold true:

$$u(t) = (1-\lambda) u_k(t), \qquad \frac{\partial u}{\partial n}(t) = \frac{\partial u_k}{\partial n}(t) - \lambda \frac{\partial S_k u_k}{\partial n}(t), \qquad t \in \partial D_k.$$

According to the Lemma 3, the last problem has only a zero solution. This proves the lemma.

THEOREM 2. *The Dirichlet problem* (1.5), (1.6), (1.8) *has the unique solution*

$$u(x) = \sum_{k=0}^{n} S_k f(x) - \sum_{k=0}^{n} \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} S_k S_{k_1} f(x) + \dots . \tag{2.10}$$

The last series converges in the space $\mathfrak{B}^-(\overline{D})$.

P r o o f. If we substitute the formulae (2.7) into (2.6), we obtain the formulae (2.10). The change of the order of summation is proved as in [7].

The series (2.10) corresponds to the generalized method of Schwarz.

3. Addition theorems

Let us sketch the main idea of the method of addition theorems for the Dirichlet problem (1.5), (1.6), (1.8). Let us consider the Hilbert space l^2 of the sequences $\alpha := (\alpha_1, \alpha_2, \dots \alpha_k, \dots)$ with the norm

$$\|\alpha\| := \left(\sum_{k=1}^{\infty} |\alpha_k|^2 \right)^{1/2} .$$

Let the functions $\omega_l^k(z)$ $(l = 1, 2, \dots)$ for fixed numbers $k = 0, 1, \dots, n$ belong to $\mathfrak{B}^-(D_k^-)$ and generate an orthogonal system in the Hilbert space $\mathfrak{L}_k^2$. Here, $f \in \mathfrak{L}_k^2$, if $f \in \mathfrak{B}^-(D_k^-)$ and

$$\|f\| := \left(\int_{\partial D_k} |f(t)|^2 \, dt \right)^{1/2} .$$

Each function $u_m(x)$ from $\mathfrak{L}_m^2$ is represented in the form of the series

$$u_m(x) = \sum_{l=1}^{\infty} X_l^m \omega_l^m(x). \tag{3.1}$$

Since the Hilbert spaces $\mathfrak{L}_k^2$ and l^2 are isometric, the sequence X_l^k belongs to l^2. Let the function $f(x) \in \mathfrak{L}_k^2$, and let the condition (2.1) hold; then

$$f(t) = \sum_{l=1}^{\infty} f_l^k \omega_l^k(t), \qquad t \in \partial D_k , \tag{3.2}$$

where $f_l^k \in l^2$. In view of the representation

$$u(x) = \sum_{k=0}^{n} u_k(x), \qquad x \in \overline{D} \tag{3.3}$$

and (3.1), the unknown function $u(z)$ is represented in the form

$$u(x) = \sum_{m=0}^{n} \sum_{l=1}^{\infty} X_l^m \omega_l^m(x).$$

In each surface ∂D_k from (1.6) and (3.2) we have

$$\sum_{l=1}^{\infty} X_l^k \omega_l^k(t) + \sum_{\substack{m=0 \\ m \neq k}}^{n} \sum_{l=1}^{\infty} X_l^m \omega_l^m(t) = \sum_{l=1}^{\infty} f_l^k \omega_l^k(t), \qquad t \in \partial D_k .$$

The formulae

$$\omega_l^m(t) = \sum_{p=1}^{\infty} \gamma_{pl}^{km} \omega_p^k(t), \qquad t \in \partial D_k , \quad m = 0, 1, \ldots, n, \quad m \neq k, \tag{3.4}$$

where γ_{pl}^{km} are constants, is called an addition theorem. Using the addition theorem we obtain the relations

$$\sum_{l=1}^{\infty} X_l^k \omega_l^k(t) + \sum_{\substack{m=0 \\ m \neq k}}^{n} \sum_{p=1}^{\infty} X_p^m \sum_{l=1}^{\infty} \gamma_{lp}^{km} \omega_l^k(t) = \sum_{l=1}^{\infty} f_l^k \omega_l^k(t).$$

If the second series converges absolutely, then it is possible to change the order of summation

$$\sum_{l=1}^{\infty} \left(X_l^k + \sum_{\substack{m=0 \\ m \neq k}}^{n} \sum_{p=1}^{\infty} \gamma_{lp}^{km} X_p^m \right) \omega_l^k(t) = \sum_{l=1}^{\infty} f_l^k \omega_l^k(t), \qquad k = 0, 1, \ldots, n.$$

Since the system $\omega_l^k(t)$ is orthogonal, we obtain from the last equality the following infinite system of linear algebraic equations

$$X_l^k + \sum_{\substack{m=0 \\ m \neq k}}^{n} \sum_{p=1}^{\infty} \gamma_{lp}^{km} X_p^m = f_l^k , \qquad l = 1, 2, \ldots, \quad k = 0, 1, \ldots, n. \tag{3.5}$$

The method of addition theorems is a discrete variant of the generalized method of Schwarz under the condition of absolute convergence of the corresponding series. Let us consider the method of successive approximations for the system (3.5). As the zero-order approximation we assume $(X_l^k)_0 = f_l^k$. The corresponding function from $\mathfrak{B}^-(D_m^-)$

$$u_{k0}(x) = \sum_{l=1}^{\infty} \left(X_l^k \right)_0 \omega_l^k(x) = \sum_{l=1}^{\infty} f_l^k \omega_l^k(x)$$

coincides with the zero-order approximation of the generalized method of Schwarz. Once the $(s-1)$ approximation $(X_l^k)_{s-1}$ has been constructed, then the s-th approximation will be constructed by the formulae (3.5). This coincides with the construction of the function

$$u_{ks}(x) = -f(x) + \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} S_{k_1} f(x) - \ldots + (-1)^s \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \ldots \sum_{\substack{k_s=0 \\ k_s \neq k_{s-1}}}^{n} S_{k_1} \ldots S_{k_{s2}} f(x)$$

of the generalized method of Schwarz, and $F_0 u_{ks} = 0$ because $F_0 \omega_l^k = 0$. Since convergence in $\mathfrak{B}^-(D_k^-)$ implies convergence in $\mathfrak{k}_k^2 \cong l^2$, convergence of the generalized method of Schwarz automatically implies convergence of the method of successive approximations for the system (3.5):

THEOREM 3. *Let the infinite system* (3.5) *with respect to the sequence* $\{X_l^k\}_{l=1}^{\infty} \in l^2$ $(k = 0, 1, \ldots, n)$ *correspond to the Dirichlet problem* (1.5), (1.6), (1.8). *Then the method of successive approximations always converges in the space* l^2.

The aim of the method of addition theorems is to construct an appropriate system $\omega_l^k(x)$ and to find the relation (3.4), i.e. to calculate γ_{lp}^{km} [8, 9]. We do not study such constructive questions. We can only verify that it is possible to apply addition theorems in such a way that the corresponding infinite system (3.5) can be solved by the method of successive approximations converging in a certain space. There is a concrete example in [8], where the properties of an infinite system concerning the convergence are changed after simple transformations.

Acknowledgement

The paper was supported by the State Committee for Scientific Research through the grant No 3 P404 013 06.

References

1. G.M. GOLUSIN, *Solution of the Dirichlet problem in space for the Laplace equation and for spherical domains* [in Russian], Matem. Sbornik, **41**, 277–283, 1934.
2. S.G. MIKHLIN, *Integral equations*, Pergamon Press, New York 1964.
3. G.P. SENDECKYJ, *Multiply circular inclusion problems in longitudinal shear deformation*, J. Elasticity, 1, 83–86, 1971.
4. G.I. MARCHUK, *Methods of numerical mathematics*, Nauka, Moscow 1989.
5. L.V. KANTOROVICH and G.P. AKILOV, *Functional analysis* [in Russian], Nauka, Moscow 1984.
6. M. KRZYŻAŃSKI, *Partial differential equations of second order*, Vol. 1, PWN, Warszawa 1971.
7. V.V. MITYUSHEV, *Solution of the Hilbert problem for a multiply connected domain*, Słupskie Prace Matem.-Przyrod., 9a, 37–69, 1994.
8. E.A. IVANOV, *Diffraction of the electromagnetic waves on two bodies* [in Russian], Nauka i Technika, Minsk 1968.
9. V.T. EROFEENKO, *Addition theorems* [in Russian], Nauka i Tehnika, Minsk 1988.

10. V.V. MITYUSHEV, *A method of functional equations for boundary value problems of continuous media*, Rep. Math. Phys., **33**, 1-2, 137–147, 1993.
11. V.V. MITYUSHEV, *Plane problem for steady heat conduction of material with circular inclusions*, Arch. Mech., **45**, 211–215, 1993.
12. V.V. MITYUSHEV, *On the solutions of the $\mathbb{R}$-linear boundary value problem (Markushevich's problem) on the torus and cylinder* [in Russian], Izv. Vuzov. Math. Reprint N 7546-B87 in VINITI, 1987.
13. V.V. MITYUSHEV, *On solution of the boundary $\mathbb{R}$-linear problem for certain doubly connected domain* [in Russian], Vestnik BGU, Reprint N 4019-B91, VINITI, 1991.
14. K.J. BINNS and P.J. LAWRENSON, *Analysis and computation of electric and magnetic field problems*, Pergamon Press, Oxford 1973.
15. L. POLADIAN, *General theory of electrical images in sphere pairs*, Q. J .Mech. Appl. Math., **41**, 395–417, 1988.
16. R.C. MCPHERDAN, *Transport properties of cylinder pairs and of the square array of cylinders*, Proc. Roy. Soc. Lond., **A408**, 31–43, 1986.
17. R.C. MCPHERDAN and G.W. MILTON, *Transport properties of touching cylinder pairs and of the square array of touching cylinders*, Proc. Roy. Soc. Lond., **A411**, 236–313, 1987.
18. R.C. MCPHERDAN, L. POLADIAN and G.W. MILTON, *Asymptotic studies of closely spaced, highly conducting cylinders*, Proc. Roy. Soc. Lond., **A415**, 185–196, 1988.
19. N.A. NICOROVICI, R.C. MCPHERDAN and G.W. MILTON, *Transport properties of a three-phase composite material: the square of coated cylinders*, Proc. Roy. Soc. Lond., **A422**, 599–620, 1993.

SŁUPSK UNIVERSITY OF PEDAGOGY
INSTITUTE OF MATHEMATICS, SŁUPSK.

Received March 29, 1994.

Arch. Mech., 47, 6, pp. 1183–1187, Warszawa 1995

BRIEF NOTES

Trapping of waves by horizontal cylinders in a channel containing two-layer fluid

N. KUZNETSOV (ST. PETERSBURG)

A CHANNEL of infinite length and depth is filled by an inviscid, incompressible fluid consisting of two layers of different densities, the upper one being of finite thickness. A rigid cylinder extending across the channel is immersed in one of the layers. Free harmonic oscillations of the fluid are analyzed, special attention being paid to the determination of the so-called trapping modes of finite energy. The integral equations and perturbation method techniques are used for the analysis.

1. Introduction

A HORIZONTAL channel of infinite length and depth and of constant width contains inviscid, incompressible, two-layer fluid under gravity forces. The upper layer has constant finite depth and is occupied by a fluid of constant density ϱ. The lower layer has infinite depth and is occupied by a fluid of constant density $\varrho^* > \varrho$. A parameter $\varepsilon = (\varrho^*/\varrho) - 1$ is assumed to be small. One of the fluids (upper or lower) is bounded internally by an immersed horizontal cylindrical surface S, which extends right across the channel and has its generators normal to the sidewalls. The free, time-harmonic oscillations of fluids having finite kinetic and potential energy (such oscillations are called trapping modes) are investigated. Trapping mode problem for a homogeneous fluid in presence of submerged cylinders or other obstacles is investigated extensively (see EVANS *et al.* [1] and references cited therein for bibliography). Apparently, the first treatment of this problem for the two-layer fluid is given by KUZNETSOV [4]. In the case when a cylinder is immersed in the lower fluid, it was found that, under some restrictions, there exist two finite sets of frequencies of trapping modes. The frequencies in the first set are close to the frequencies of trapping modes for the homogeneous fluid (when $\varrho^* = \varrho$). They correspond to the trapping modes of waves on the free surface of upper fluid. The frequencies in the second set are proportional to ε and correspond to the trapping modes of internal waves on the interface between two fluids.

Here similar results are presented for the case, when a cylinder is immersed in the upper fluid. The general scheme of investigation is the same as in KUZNETSOV [4]. First, the original problem is reduced to the problem in the layer, which

contains the cylinder. Then perturbation technique is applied in combination with URSELL's [6] method of integral operators. This work was stimulated by a remark in FRIIS, GRUE and PALM [3], that long underwater tube bridges are proposed to be constructed across the Norwegian fiords, which are often occupied by two-layer fluid (fresh-salt water).

2. Statement of the problem

The xyz-coordinates are chosen so that the y-axis is directed upwards and the xz-plane coincides with the undisturbed interface between two layers. The depth of upper layer can be assumed to be equal to one without any loss of generality. Using the linear water-wave theory, we consider velocity potentials of the form

$$\exp(-i\omega t)u^*(x,y)\cos kz \quad (\exp(-i\omega t)u(x,y)\cos kz)$$

for the lower (upper) fluid. Here ω is the unknown radian frequency of the trapping mode, and the wavenumber k along the z-axis should be taken so that the impermeability condition holds on the sidewalls. In what follows we suppose k to be prescribed, but its value is an arbitrary positive number.

The pair $\{u, u^*\}$ must be a solution of the following problem

$$u^*_{xx} + u^*_{yy} = k^2 u^* \quad \text{in} \quad W^*, \tag{1}$$

$$\begin{aligned} u_{xx} + u_{yy} &= k^2 u \quad \text{in} \quad W, \\ \partial u/\partial n &= 0 \quad \text{on} \quad S, \\ u_y - \nu u &= 0 \quad \text{when} \quad y = 1, \end{aligned} \tag{2}$$

$$u^*_y = u_y\,, \qquad \varrho^*(u^*_y - \nu u^*) = \varrho(u_y - \nu u) \quad \text{when} \quad y = 0. \tag{3}$$

Here $W(W^*)$ denotes the cross-section of the region, occupied by the fluid of density $\varrho(\varrho^*)$, $\nu = \omega^2/g$ is the spectral parameter to be determined along with u, u^* (g is the acceleration of gravity). For trapped-mode solutions the motion must decay at large distances, i.e the relations

$$u^*,\ |\nabla u^*| \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty, \qquad \text{and} \qquad u,\ |\nabla u| \to 0 \quad \text{as} \quad |x| \to \infty \tag{4}$$

must hold.

3. Perturbation method for spectral problem in the upper fluid

First, with the help of the Fourier transform one can eliminate u^* from (1), (3) and (4). In this way we arrive at the following boundary condition

$$\varepsilon u_y = \nu[(1+\varepsilon)Au - u] \quad \text{when} \quad y = 0. \tag{5}$$

Here

$$(Au)(x,0) = \frac{1}{\pi}\int_{-\infty}^{\infty} K_0(k|x-\xi|)u_y(\xi,0)\,d\xi$$

and K_0 is the MacDonald function. Thus, we have the boundary value problem (2), (5) with the second condition (4) at infinity.

Since the parameter ε is assumed to be small, it is natural to seek eigenvalues and eigenfunctions in the form of expansions

$$\nu = \nu_0 + \varepsilon\nu_1 + \varepsilon^2\nu_2 + \ldots, \qquad u = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \ldots, \tag{6}$$

which is common in the perturbation theory (see e.g. FRIEDRICHS [2]). Substituting (6) into (2) and (5), and equating the coefficients at the same powers of ε, one obtains an infinite system of boundary value problems. The problem of the zero order is

$$\begin{aligned} u^{(0)}_{xx} + u^{(0)}_{yy} &= k^2 u^{(0)} \quad \text{in } W, \qquad \partial u^{(0)}/\partial n = 0 \quad \text{on } S, \\ u^{(0)}_y - \nu_0 u^{(0)} &= 0 \quad \text{when } y = 1, \\ \nu_0(Au^{(0)} - u^{(0)}) &= 0 \quad \text{when } y = 0. \end{aligned} \tag{7}$$

The first order problem has the form

$$u^{(1)}_{xx} + u^{(1)}_{yy} = k^2 u^{(1)} \quad \text{in } W, \qquad \partial u^{(1)}/\partial n = 0 \quad \text{on } S, \tag{8}$$

$$u^{(1)}_y - \nu_0 u^{(1)} = \nu_1 u^{(0)} \quad \text{when } y = 1, \tag{9}$$

$$\nu_0\left(Au^{(1)} - u^{(1)}\right) = u^{(0)}_y - \nu_0 Au^{(0)} - \nu_1\left(Au^{(0)} - u^{(0)}\right) \quad \text{when } y = 0. \tag{10}$$

The problem for $u^{(m)}$ $(m = 2,3,\ldots)$ can be easily written down.

In order to fix an arbitrary factor in the expansion for u, which should be found from the system (7), (8)–(10), it is convenient to use the linear condition $\langle u(\cdot,1); u^{(0)}(\cdot,1)\rangle = 1$, where $\langle\cdot;\cdot\rangle$ is the scalar product in $L_2(\infty,-\infty)$. The last equality combined with the following normalization condition $\langle u^{(0)}(\cdot,1); u^{(0)}(\cdot,1)\rangle = 1$, gives

$$\langle u^{(1)}(\cdot,1); u^{(0)}(\cdot,1)\rangle = \langle u^{(2)}(\cdot,1); u^{(0)}(\cdot,1)\rangle = \ldots = 0. \tag{11}$$

The problem (7) has a finite set of positive point eigenvalues $\{\nu_0^{(+)}\}$, because this problem is another form of the problem on trapping modes above the cylinder immersed in the homogeneous fluid ($\varrho^* = \varrho$). If $\nu_0^{(+)}$ is a positive non-degenerate eigenvalue for (7) and $u_+^{(0)}$ is the corresponding eigenfunction, then (10) takes the form

$$Au_+^{(1)} - u_+^{(1)} = -u_+^{(0)} + \left[\nu_0^{(+)}\right]^{-1} \partial u_+^{(0)}/\partial y \quad \text{when} \quad y = 0. \tag{12}$$

The problem (8), (9), (12) is solvable under the orthogonality condition

$$\nu_1^{(+)} - \langle u_+^{(0)}(\cdot, 0);\, u_+^{(0)}(\cdot, 0)\rangle + \left[\nu_0^{(+)}\right]^{-1} \langle u_+^{(0)}(\cdot, 0);\, (\partial u_+^{(0)}/\partial y)(\cdot, 0)\rangle = 0,$$

which expresses $\nu_1^{(+)}$. Then $u_+^{(1)}$, can be found uniquely in view of (11). Thus, the terms in the expansions (6) can be successively determined, what gives $\nu^{(+)} > 0$ and u_+ with necessary accuracy when ε is small enough.

If $\nu_0^{(0)} = 0$, then $(7)_3$ trivially holds and (10) takes the form

$$u_y^{(0)} = \nu_1^{(0)} \left(Au^{(0)} - u^{(0)}\right) \quad \text{when} \quad y = 0. \tag{13}$$

This boundary condition, complemented by $(7)_1$ and by the homogeneous Neumann condition on $y = 1$ (it follows from $(7)_2$) forms a spectral problem. It differs by the term $Au^{(0)}$ in (13) from the problem on trapped modes above a cylinder immersed in a homogeneous fluid of finite depth.

4. The spectral problem for $\nu_1^{(0)}$

Following URSELL [6] we seek $u_0^{(0)}$ in the form of a single layer Green potential

$$u_0^{(0)}(x, y) = (V\mu)(x, y) = 1/\pi \int_{-\infty}^{\infty} \mu(\xi) g(x, y; \xi, 0)\, d\xi,$$

where $\mu \in L_2(-\infty, \infty)$ and $g(\ldots)$ is Green's function satisfying (8) and the homogeneous Neumann condition on $y = 1$ and on $y = 0$ (except for $z = \xi$ in the last case). This Green function was constructed by URSELL [6]. Since $\partial V\mu/\partial y = -\mu$ when $y = 0$, then (13) yields

$$\mu = \nu_1^{(0)} T\mu = \frac{\nu_1^{(0)}}{\pi} \int_{-\infty}^{\infty} [K_0(k|x - \xi|) + g(x, 0; \xi, 0)]\, \mu(\xi)\, d\xi. \tag{14}$$

In the same way as in KUZNETSOV [4] one can show, that πT differs from the operator $2G$ with the kernel $2g(x,0;\xi,0)$ by an operator, whose norm exponentially decays as $k \to \infty$. On the other hand, URSELL [6] proved that G has a finite set of positive point eigenvalues. Hence, there is a finite set $\{\nu_1^{(0)}\}$ of positive eigenvalues for T, when k is large. Applying the same procedure as in Sec. 3, we arrive at the eigenvalue expansion $\nu^{(0)} = \varepsilon\nu_1^{(0)} + \varepsilon^2\nu_2^{(0)} + \ldots$, which is positive for sufficiently small ε. Then, $\omega_i = (g\nu^{(0)})^{1/2}$ is the frequency of trapped mode of internal waves on the interface.

It is easy to see that $\omega_i/\omega_s \equiv (\varepsilon/2)^{1/2}$, when ε is small enough and k is large enough. Here ω_s is a trapping mode frequency for waves on the free surface of the following finite depth channel. We have to turn over the upper fluid layer with the cylinder and to supply it with the rigid horizontal bottom.

5. Conclusion and discussion

KUZNETSOV [4] demonstrated that there exist trapping modes of internal waves when a cylinder is immersed in the lower infinite depth layer. Here the same is shown to be true when the upper layer contains a cylinder. For both positions of the cylinder the relation $\omega_i/\omega_s \equiv (\varepsilon/2)^{1/2}$ is valid, but with different meaning of ω_s. It should be reminded that in KUZNETSOV [4] ω_s denotes the trapping mode frequency for waves on the surface of the lower fluid, in absence of the upper layer. In Sec. 4 the meaning of ω_s is quite different.

The existence of trapping modes of both the types considered can be demonstrated similarly for any cylinder position in a two-layer fluid of finite depth. The method developed here can also be applied to finding the trapping mode frequencies of internal waves in the case, when a cylinder intersects the free surface of the upper layer. It is interesting to note, that there are no trapping modes of surface waves, if the latter configuration satisfies John's condition. This follows from a result proved by MCIVER ([5], Appendix A) on absence of trapping modes in the homogeneous fluid in presence of such surface-piercing cylinder.

References

1. D.V. EVANS, M. LEVITIN and D. VASSILIEV, J. Fluid Mech., **261**, 21–31, 1994.
2. K.O. FRIEDRICHS, *Perturbation of spectra in Hilbert space*, AMS, Providence, RI, 1965.
3. A. FRIIS, J. GRUE and E. PALM, [in:] *M.P Tulin's Anniversary Volume*, T. MILOH [Ed.], SIAM, 1991.
4. N. KUZNETSOV, J. Fluid Mech., **254**, 113–126, 1993.
5. P. MCIVER, Quart. J. Mech. Appl., **44**, 193–208, 1991.
6. F. URSELL, J. Fluid Mech., **183**, 421–437, 1987.

LABORATORY ON MATHEMATICAL MODELLING IN MECHANICS
RUSSIAN ACADEMY OF SCIENCES, ST. PETERSBURG, RUSSIA.

Received September 29, 1994.

Arch. Mech., 47, 6, pp. 1189–1192, Warszawa 1995

BRIEF NOTES

Comments on instability of disturbed elongational flows of viscoelastic fluids

S. ZAHORSKI (WARSZAWA)

OUR RECENT discussion [1] on the instability or sensitivity problems of simple elongational flows of viscoelastic fluids is extended to the case of non-isothermal spinning processes. It is shown that the flows considered may be unstable even for vanishing external disturbances.

1. Introduction

IN OUR RECENT paper [1] we discussed the instability problem, or rather the problem of sensitivity to external disturbances of steady simple elongational flows of viscoelastic fluids. Such flows are usually realized in the rotary clamp extensometers (cf. [2]). Under the assumption of flows with dominating extension (FDEs), it was shown that the steady elongational flows considered might always be unstable in the presence of external disturbances, either of stationary or oscillatory character.

In the present comments we extend our previous considerations to the case of fibre spinning processes dicussed elsewhere [3, 4]. To this end we repeat only more important relations, retaining the majority of notations used in [1]. It turns out, on the basis of the present results, that non-isothermal steady spinning flows may be unstable also in the case of vanishing external disturbances.

2. Governing equations for steady and unsteady flows

Treating the spinning processes as particular cases of FDEs (cf.[3, 4]), we can use Eqs. (2.1), (2.2) from the paper [1], bearing in mind that the fundamental velocity gradient $V'(z)$ is not constant along the spinline and depends on the axial coordinate z. The boundary conditions at both ends of the thread amount to

$$V(0) = V_0\,, \qquad V(L) = V_L = V'(L)L\,. \tag{2.1}$$

The corresponding incremental constitutive equations (cf.[3, 4]) introduced into the dynamic equations of equilibrium lead to the following nonlinear differential equations:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{1}{2}\frac{k}{V'}\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial r}\right)^2=\frac{C(z)}{\beta}+\frac{\varrho}{\beta}\frac{\partial w}{\partial t}, \tag{2.2}$$

where w denotes the axial component of additional velocity, $C(z)$ is a function of z only, and

$$k=\frac{1}{\beta}\left(\frac{\partial\beta_1}{\partial V'}+\frac{\partial\beta_2}{\partial V'}V'\right)V', \qquad \beta=\frac{1}{3}(\beta_1+\beta_2V'), \tag{2.3}$$

where β_i $(i=1,2)$ are material constants; β is simply related to the extensional viscosity η_E. For instance, under the Newtonian approximation $(k\equiv 0)$ we have the simpler equation:

$$f\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_0}{\partial r}\right)=\frac{C_0(z)}{\beta_0}-\frac{1}{\beta_0}\frac{\partial}{\partial z}(3\beta_0V')+\frac{\varrho}{\beta_0}\frac{\partial w_0}{\partial t}. \tag{2.4}$$

The additional boundary conditions used in the problem are exactly the same as those (3.4), (3.5) in [1]. They express the conservation of volume output in a sample as well as the balance of forces acting on the free surface.

The steady-state solution of Eq. (2.4) can be written as

$$w_0=\frac{D_0(z)}{4}\left(r^2-\frac{R^2}{2}\right), \qquad D_0=\frac{C_0(z)}{\beta_0}-\frac{1}{\beta_0}\frac{\partial}{\partial z}(3\beta_0V'), \tag{2.5}$$

where

$$C_0(z)=6\beta_0\frac{R'}{R}+\frac{\partial}{\partial z}(3\beta_0V'), \tag{2.6}$$

and $R(z)$ denotes the variable radius.

3. Instability or sensitivity problem

For slightly non-Newtonian fluids, i.e. for relatively small parameters k, we seek unsteady solutions in the form of the following series:

$$\begin{aligned} w &= w_0+kw_1+(\tilde{w}_0+k\tilde{w}_1)\exp\lambda(t-t_0),\\ C &= C_0+kC_1+\left(\tilde{C}_0+k\tilde{C}_1\right)\exp\lambda(t-t_0), \end{aligned} \tag{3.1}$$

where λ is related to the first frequency of harmonic oscillations imposed at the moment $t=t_0$.

In the case considered Eq. (2.2) simplifies to the following linearized form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{w}}{\partial r}\right) - \frac{\varrho\lambda}{\beta}\tilde{w} = \frac{\tilde{C}(z)}{\beta} - \frac{1}{\beta}\frac{\partial}{\partial z}(3\beta V'), \tag{3.2}$$

the solution of which amounts to

$$\tilde{w}_0 = \tilde{w}_1 = \tilde{C}_N I_0\left(x\frac{r}{R}\right) - \frac{1}{\varrho\lambda}\left(\tilde{C}_{0,1} - \frac{\partial}{\partial z}(3\beta V')\right), \tag{3.3}$$

or finally to

$$\tilde{w}_{0,1} = \frac{\tilde{C}_{0,1}}{\beta}R^2\left[\frac{1}{2x}\frac{I_0\left(x\frac{r}{R}\right)}{I_1(x)} - \frac{1}{x^2}\right], \tag{3.4}$$

where I_0 and I_1 are the modified Bessel functions of the first kind and of order 0 and 1, respectively.

For time-dependent disturbances the boundary conditions lead to

$$\tilde{C}_i(z) = 2\Delta_i T\frac{R'}{R} + \frac{\partial}{\partial z}(3\beta V'), \tag{3.5}$$

where $\Delta_i T$ $(i = 0, 1)$ denote the increments of normal stress difference, viz.

$$T^{*33} - T^{*11} = 3\beta V' + (\Delta_0 T + k\Delta_1 T)\exp\lambda(t - t_0). \tag{3.6}$$

If the above increments are caused by disturbances of the velocity gradients and the extensional viscosity, we have

$$\Delta_i T = 3\beta\Delta_i V' + 3V'\Delta_i\beta, \qquad i = 0, 1. \tag{3.7}$$

4. Final remarks

It is clearly seen from Eqs. (3.5), (3.7) that unsteady solutions are possible if the corresponding increments are identically equal to zero; then the functions $\tilde{C}_{0,1}(z)$ are fully determined by $\partial/\partial z(3\beta V')$.

Our discussion in [1] concerned with various instability or sensitivity forms remains valid for the case considered. Thus, we may conclude that steady non-isothermal spinning flows realized under the real conditions, i.e. with inertia, drag and surface tension effects, variable viscosity etc., may be unstable also wihout any external disturbances. This is the case, in particular, when the boundary conditions are satisfied exactly.

References

1. S. ZAHORSKI, *Instability of disturbed elongational flows of viscoelastic fluids*, Arch. Mech., **47**, 647–659, 1995.
2. J. MEISSNER and J. HOSTETTLER, *A new elongational rheometer for polymer melts and other highly viscoelastic liquids*, Rheol. Acta, **33**, 1, 1994.
3. S. ZAHORSKI, *An alternative approach to non-isothermal melt spinning with axial and radial viscosity distributions*, J. Non-Newtonian Fluid Mech., **36**, 71, 1990.
4. S. ZAHORSKI, *Fibre spinning processes as viscoelastic flows with dominating extensions*, Arch. Mech., **42**, 233, 1990.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received January 6, 1995.