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An alternative approach to the representation of orthotropic tensor functions in the two-dimensional case

S. JEMIOŁO and J. J. TELEGA (WARSZAWA)

THE AIM of this paper is to derive in a simple fashion the non-polynomial representations of a class of orthotropic functions in the two-dimensional case. Scalar-valued, vector-valued, symmetric and skew-symmetric tensor-valued functions of the second order have been considered.

1. Introduction

STRUCTURES made of anisotropic materials are often used in engineering practice. Constitutive modelling of the behaviour of such materials has been significantly influenced by the theory of invariants and tensor functions, cf. [6, 18, 24]; vice versa, development of the invariant theory has been stimulated by the constitutive modelling. The reader interested in the fundamentals of the theory of invariants and tensor functions and their applications should refer to [6, 13, 21, 22, 23].

The problem of the determination of the general form of a tensor function of specified order and symmetry depending on tensor arguments consists in finding irreducible sets of scalar invariants and tensor generators; to put it simply, in the determination of the so-called canonical form of the tensor function. Though the theory of tensor function representation has been developed for more than three decades [18, 22, 23], yet no comprehensive, systematic and up-to-date study is available in the relevant literature. The book by SMITH [21] is restricted to the presentation of theoretical results elaborated by this author and his coworkers, by employing classical methods of the group representation theory. SMITH [21] has deliberately focussed on polynomial representations only. Many other complementary contributions exist, however, concerning the general representation of practically important isotropic [3, 14–16, 19, 20, 22–28] and anisotropic [1, 2, 4–6, 10, 12, 21, 29, 30] tensor functions.

Irreducibility of a set of invariants may be understood in two ways:

1. If one determines an integrity basis, then none of its elements can be a *polynomial* in the remaining elements, cf. [22].

2. In the case of a functional or non-polynomial basis, none of its elements can be *a function* of the remaining elements.

Similar characterization pertains to the irreducibility of generators appearing in the canonical form of a tensor function, cf. [3, 6, 16]. To find the polynomial representation of a tensor function it suffices to determine the relevant integrity basis, because the generators are obtained by a simple process of differentiation

[6, 22]. The problem of the non-polynomial representation of a tensor function is more complicated, cf. [3, 19, 20, 25–30]. In the paper by the second author [24], a similar approach was suggested for the determination of generators of non-polynomial tensor functions. This method was next developed by KORSGAARD [14, 15] and used in [11, 12].

In general, the determination of functional bases and generators leads to solving complicated algebraic relations. Hence only some classes of tensor functions are known explicitly. Even when the representations of scalar-, vector- and tensor-valued functions are available, alternative methods of their determination are still proposed, cf. [28, 29].

As is well known, two-dimensional problems are often studied in the continuum mechanics. Thus the problem of the representation of isotropic and anisotropic functions in the two-dimensional case is of interest in itself. However, such two-dimensional representations do not necessarily coincide with those derived directly from the corresponding three-dimensional cases.

The aim of this contribution, precisely formulated in the next section, is to propose an alternative derivation of functional bases and generators for orthotropic functions in the two-dimensional case.

2. Formulation of the problem

The objective of our considerations is the determination of the general form of the following functions:

$$
\begin{aligned}
s &= f(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m), \qquad i = 1,\dots,I, \quad p = 1,\dots,P, \quad m = 1,\dots,M,\\
\mathbf{t} &= \mathbf{f}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m),\\
\mathbf{S} &= \mathbf{F}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m), \qquad \mathbf{S} = \mathbf{S}^t,\\
\mathbf{T} &= \mathbf{G}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m), \qquad \mathbf{T} = -\mathbf{T}^t,
\end{aligned}
\tag{2.1}
$$

in the two-dimensional case. Here $s \in \mathbb{R}$, $\mathbf{t}, \mathbf{v}_m \in \mathbb{E}^2$, $\mathbf{S}, \mathbf{A}_i \in T^s$ ($\dim T^s = 3$), $\mathbf{T}, \mathbf{W}_p \in T^a$ ($\dim T^a = 1$), $T = \mathbb{E}^2 \otimes \mathbb{E}^2 = T^s \oplus T^a$ ($\dim T = 4$), $\mathbb{E}^2$ stands for the two-dimensional Euclidean space and $T^s = \{\mathbf{A} \in T \mid \mathbf{A} = \mathbf{A}^t\}$, $T^a = \{\mathbf{W} \in T \mid \mathbf{W} = -\mathbf{W}^t\}$.

In our 2D case, the orthotropy group S satisfies the condition

$$
\forall\ \mathbf{Q} \in S, \quad \mathbf{QMQ}^t = \mathbf{M},
\tag{2.2}
$$

where $\mathbf{M} = \mathbf{e} \otimes \mathbf{e}$ and the unit vector $\mathbf{e}$ characterises orthotropy, see ([6], p. 51). Obviously we have $\operatorname{tr}\mathbf{M} = \operatorname{tr}\mathbf{M}^2 = 1$.

For each $\mathbf{Q} \in S$, the scalar-valued function f, vector-valued function $\mathbf{f}$, symmetric tensor-valued function $\mathbf{F}$ and skew-symmetric tensor-valued function $\mathbf{G}$

satisfy the conditions:

$$
(2.3)\qquad
\begin{aligned}
f(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m) &= f(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^t, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^t, \mathbf{Q}\mathbf{v}_m),\\
\mathbf{Q}\mathbf{f}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m) &= \mathbf{f}(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^t, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^t, \mathbf{Q}\mathbf{v}_m),\\
\mathbf{Q}\mathbf{F}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m)\mathbf{Q}^t &= \mathbf{F}(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^t, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^t, \mathbf{Q}\mathbf{v}_m),\\
\mathbf{Q}\mathbf{G}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m)\mathbf{Q}^t &= \mathbf{G}(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^t, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^t, \mathbf{Q}\mathbf{v}_m).
\end{aligned}
$$

By applying I-SHIH LIU theorem [10] (see also [17]) and taking into account (2.2), the invariance requirement (2.3) may be written in the following way:

$$
(2.4)\qquad
\begin{aligned}
f(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{M}) &= f(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^t, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^t, \mathbf{Q}\mathbf{v}_m, \mathbf{Q}\mathbf{M}\mathbf{Q}^t),\\
\mathbf{Q}\mathbf{f}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{M}) &= \mathbf{f}(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^t, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^t, \mathbf{Q}\mathbf{v}_m, \mathbf{Q}\mathbf{M}\mathbf{Q}^t),\\
\mathbf{Q}\mathbf{F}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{M})\mathbf{Q}^t &= \mathbf{F}(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^t, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^t, \mathbf{Q}\mathbf{v}_m, \mathbf{Q}\mathbf{M}\mathbf{Q}^t),\\
\mathbf{Q}\mathbf{G}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{M})\mathbf{Q}^t &= \mathbf{G}(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^t, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^t, \mathbf{Q}\mathbf{v}_m, \mathbf{Q}\mathbf{M}\mathbf{Q}^t),
\end{aligned}
$$

for each $\mathbf{Q} \in O$, where O denotes the full orthogonal group. Now $\mathbf{M}$ plays the role of a parametric tensor, and the functions f, $\mathbf{f}$, $\mathbf{F}$ and $\mathbf{T}$ depend explicitly on it. We observe that the approach leading to (2.4) has primarily been proposed by BOEHLER [4, 5].

In the sequel we shall derive the functional basis for the scalar function $(2.4)_1$ and generators for the functions $(2.4)_{2-4}$. Our method of determination of the functional basis follows that used by SMITH [19, 20] and KORSGAARD [14, 15] for isotropic functions. Generators will be obtained similarly as in [11, 12, 14, 15], following the idea proposed in the paper by the second author [24].

3. Determination of the orthotropic functional basis

Since the tensor $\mathbf{M}$ appearing in (2.4) is a parametric tensor, the determination of the functional basis is less complicated than in the case of isotropy examined by KORSGAARD [14]. Obviously, in the last case $S = O$, because the invariance with respect to the full orthogonal group has been studied.

To find the functional basis for the orthotropic scalar function $(2.4)_1$, it suffices to consider the following three cases.

CASE 1

In the set of vectors $\{\mathbf{v}_m\}$ $(m = 1, \ldots, M)$ there are vectors non-collinear with the direction of $\mathbf{e}$.

CASE 1.1

At least one vector from the set $\{\mathbf{v}_m\}$, say $\mathbf{v}_1$, is not collinear with $\mathbf{e}$ and $\mathbf{v}_m \neq \mathbf{0}$, $m = 1, \ldots, M$. Then we choose the coordinate system $\{x_\alpha\}$ $(\alpha = 1, 2)$ in such a way that $0x_1$ coincides with $\mathbf{e}$ and $v_1^{(1)} > 0$, $v_2^{(1)} > 0$; here $\mathbf{v}_m = (v_m^{(\alpha)})$. To determine uniquely the representation of the function $(2.4)_1$, it suffices to know the following invariants, since then the components of all arguments are available:

$$
(3.1)\qquad
\begin{aligned}
\mathbf{v}_1\cdot\mathbf{M}\mathbf{v}_1 &= v_1^{(1)}v_1^{(1)} \Rightarrow v_1^{(1)} \qquad (v_1^{(1)}>0),\\
\mathbf{v}_1\cdot\mathbf{v}_1 &= v_1^{(1)}v_1^{(1)} + v_2^{(1)}v_2^{(1)} \Rightarrow v_2^{(1)} \qquad (v_2^{(1)}>0),\\
\mathbf{v}_1\cdot\mathbf{M}\mathbf{v}_m &= v_1^{(1)}v_1^{(m)} \Rightarrow v_1^{(m)},\\
\mathbf{v}_1\cdot\mathbf{v}_m &= v_1^{(1)}v_1^{(m)} + v_2^{(1)}v_2^{(m)} \Rightarrow v_2^{(m)},\\
&\left.\begin{aligned}
\mathbf{v}_1\cdot\mathbf{A}_i\mathbf{v}_1 &= A_{11}^{(i)}v_1^{(1)}v_1^{(1)} + 2A_{12}^{(i)}v_1^{(1)}v_2^{(1)} + A_{22}^{(i)}v_2^{(1)}v_2^{(1)},\\
\mathbf{v}_1\cdot\mathbf{A}_i\mathbf{v}_m &= A_{11}^{(i)}v_1^{(1)}v_1^{(m)} + A_{12}^{(i)}\left(v_1^{(1)}v_2^{(m)} + v_1^{(m)}v_2^{(1)}\right) + A_{22}^{(i)}v_2^{(1)}v_2^{(m)},\\
\mathbf{v}_m\cdot\mathbf{A}_i\mathbf{v}_m &= A_{11}^{(i)}v_1^{(m)}v_1^{(m)} + 2A_{12}^{(i)}v_1^{(m)}v_2^{(m)} + A_{22}^{(i)}v_2^{(m)}v_2^{(m)},
\end{aligned}\right\}\Rightarrow \mathbf{A}_i\\
\mathbf{v}_1\cdot\mathbf{W}_p\mathbf{v}_m &= W_{12}^{(p)}\left(v_1^{(1)}v_2^{(m)} - v_1^{(m)}v_2^{(1)}\right) \Rightarrow W_{12}^{(p)},
\end{aligned}
$$

provided that $v_1^{(1)}v_2^{(m)} - v_1^{(m)}v_2^{(1)} \neq 0$.

CASE 1.2

Only one vector, say $\mathbf{v} = (v_1, v_2) \in \{\mathbf{v}_m\}$ is not collinear with $\mathbf{e}$, whereas the remaining vectors are zero vectors. We choose the coordinate system similarly as before; then $v_1 > 0$ and $v_2 > 0$. The invariants listed below suffice for the determination of the representation of the function $(2.4)_1$:

$$
(3.2)\qquad
\begin{aligned}
\mathbf{v}\cdot\mathbf{M}\mathbf{v} &= v_1v_1 \Rightarrow v_1 \qquad (v_1>0),\\
\mathbf{v}\cdot\mathbf{v} &= v_1^2 + v_2^2 \Rightarrow v_2 \qquad (v_2>0),\\
&\left.\begin{aligned}
\mathbf{v}\cdot\mathbf{A}_i\mathbf{v} &= A_{11}^{(i)}v_1^2 + 2A_{12}^{(i)}v_1v_2 + A_{22}^{(i)}v_2^2,\\
\operatorname{tr}\mathbf{A}_i &= A_{11}^{(i)} + A_{22}^{(i)},\\
\operatorname{tr}\mathbf{MA} &= A_{11}^{(i)},
\end{aligned}\right\}\Rightarrow \mathbf{A}_i\\
\mathbf{v}\cdot\mathbf{M}\mathbf{W}_p\mathbf{v} &= v_1v_2W_{12}^{(p)} \Rightarrow W_{12}^{(p)},
\end{aligned}
$$

where

$$\mathbf{A}_i = \left(A_{\alpha\beta}^{(i)}\right) \qquad (\alpha,\beta = 1,2).$$

Summarizing, we compile Table 1.

Table 1. Functional basis in Case 1.

$\mathbf{v}_m\cdot\mathbf{v}_m$, $\mathbf{v}_m\cdot\mathbf{M}\mathbf{v}_m$, $\mathbf{v}_m\cdot\mathbf{v}_n$, $\mathbf{v}_m\cdot\mathbf{M}\mathbf{v}_n$,	$m = 1,\dots,M,$	$m < n,$
$\mathbf{v}_m\cdot\mathbf{A}_i\mathbf{v}_m$, $\mathbf{v}_m\cdot\mathbf{A}_i\mathbf{v}_n$, $\mathbf{v}_m\cdot\mathbf{W}_p\mathbf{v}_n$, $\mathbf{v}_m\cdot\mathbf{M}\mathbf{W}_p\mathbf{v}_m$,	$i = 1,\dots,I,$	$p = 1,\dots,P.$

CASE 2

We assume that $\mathbf{v}_m = \mathbf{0}$, $m = 1, \ldots, M$. Since $\mathbf{M} = \mathbf{e} \otimes \mathbf{e} \neq \mathbf{0}$, hence the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 0$.

CASE 2.1

Among the tensors $\mathbf{A}_i$ $(i = 1, \ldots, I)$ there is none with non-zero off-diagonal components in the coordinate system $\{x_\alpha\}$, such that $0x_1$ and $0x_2$ coincide with the directions of the eigenvectors of $\mathbf{M}$. Let $\mathbf{W} \in \{\mathbf{W}_p\}$. Then the sense of $0x_1$ is chosen in such a way that $W_{12} > 0$. Now one has to know the following invariants:

$$\begin{aligned}
&\left.\begin{aligned} \operatorname{tr}\mathbf{A}_i &= A_{11}^{(i)} + A_{22}^{(i)}, \\ \operatorname{tr}\mathbf{MA}_i &= A_{11}^{(i)}, \end{aligned}\right\} \Rightarrow A_{11}^{(i)} \text{ and } A_{22}^{(i)}, \\
&\operatorname{tr}\mathbf{W}^2 = -2W_{12}^2 \Rightarrow W_{12} \qquad (W_{12} > 0), \\
&\operatorname{tr}\mathbf{WW}_p = -2W_{12}W_{12}^{(p)} \Rightarrow W_{12}^{(p)}.
\end{aligned} \tag{3.3}$$

CASE 2.2

Let $\mathbf{B} \in \{\mathbf{A}_i\}$ denote a tensor with non-zero off-diagonal components. The positive direction of $0x_1$ is chosen in such a way that $B_{12} > 0$. The set of invariants is:

$$\begin{aligned}
&\left.\begin{aligned} \operatorname{tr}\mathbf{A}_i &= A_{11}^{(i)} + A_{22}^{(i)}, \\ \operatorname{tr}\mathbf{MA}_i &= A_{11}^{(i)}, \end{aligned}\right\} \Rightarrow A_{11}^{(i)} \text{ and } A_{22}^{(i)}, \\
&\left.\begin{aligned} \operatorname{tr}\mathbf{B} &= B_{11} + B_{22}, \\ \operatorname{tr}\mathbf{MB} &= B_{11}, \end{aligned}\right\} \Rightarrow B_{11} \text{ and } B_{22}, \\
&\operatorname{tr}\mathbf{B}^2 = B_{11}^2 + 2B_{12}^2 + B_{22}^2, \Rightarrow B_{12}, \\
&\operatorname{tr}\mathbf{BA} = B_{11}A_{11}^{(i)} + B_{22}A_{22}^{(i)} + 2B_{12}A_{12}^{(i)}, \Rightarrow A_{12}^{(i)}, \\
&\operatorname{tr}\mathbf{MBW}_p = -B_{12}W_{12}^{(p)}, \Rightarrow W_{12}^{(p)}.
\end{aligned} \tag{3.4}$$

By applying formulas (3.3) and (3.4) we construct Table 2.

Table 2. Functional basis in Case 2.

$\operatorname{tr}\mathbf{A}_i$, $\operatorname{tr}\mathbf{A}_i^2$, $\operatorname{tr}\mathbf{MA}_i$, $\operatorname{tr}\mathbf{A}_i\mathbf{A}_j$,	$i, j = 1, \ldots, I$,	$i < j$,
$\operatorname{tr}\mathbf{W}_p^2$, $\operatorname{tr}\mathbf{W}_p\mathbf{W}_q$, $\operatorname{tr}\mathbf{MA}_i\mathbf{W}_p$,	$p, q = 1, \ldots, P$,	$p < q$.

CASE 3

All vectors $\mathbf{v}_m$ $(m = 1, \ldots, M)$ have the form $\mathbf{v}_m = c_m\mathbf{e}$. Let $\mathbf{v} \in \{\mathbf{v}_m\}$, $\mathbf{v} = c\mathbf{e}$, and choose the coordinate system in such way that $c > 0$. Then we have

$$\mathbf{v}\cdot\mathbf{v} = c^2, \Rightarrow c \quad (c > 0), \qquad \mathbf{v}\cdot\mathbf{M}\mathbf{v}_m = cc_m \Rightarrow c_m. \tag{3.5}$$

The remaining invariants are derived similarly as in Case 2.

Summarizing all three cases: 1, 2 and 3, we obtain the orthotropic functional basis for the two-dimensional problem.

The last table coincides with ZHENG's results [29], who has however used a different method.

BOEHLER [4, 5, 6] determined functional bases provided that functions appearing in (1) depend only on symmetric tensors $\mathbf{A}_i$. In the two-dimensional case, Boehler's results correspond to the first row of our Table 3. This author approached the two-dimensional case through the three-dimensional one by using Cayley-Hamilton theorem, cf. also [21]. The method of determination of a functional basis employed in [4, 5, 6] and based on Cayley-Hamilton theorem, proves that the functional basis is also the integrity basis, see also the first row of Table 3.

Table 3. Functional basis for the orthotropic scalar-valued function $(2.4)_1$.

$\operatorname{tr}\mathbf{A}_i$, $\operatorname{tr}\mathbf{A}_i^2$, $\operatorname{tr}\mathbf{MA}_i$, $\operatorname{tr}\mathbf{A}_i\mathbf{A}_j$,	$i,j = 1,\ldots,I,$	$i < j,$
$\operatorname{tr}\mathbf{W}_p^2$, $\operatorname{tr}\mathbf{W}_p\mathbf{W}_q$, $\operatorname{tr}\mathbf{MA}_i\mathbf{W}_p$,	$p,q = 1,\ldots,P,$	$p < q,$
$\mathbf{v}_m\cdot\mathbf{v}_m$, $\mathbf{v}_m\cdot\mathbf{M}\mathbf{v}_m$, $\mathbf{v}_m\cdot\mathbf{v}_n$, $\mathbf{v}_m\cdot\mathbf{M}\mathbf{v}_n$,	$m,n = 1,\ldots,M,$	$m < n,$
$\mathbf{v}_m\cdot\mathbf{A}_i\mathbf{v}_m$, $\mathbf{v}_m\cdot\mathbf{A}_i\mathbf{v}_n$, $\mathbf{v}_m\cdot\mathbf{W}_p\mathbf{v}_n$, $\mathbf{v}_m\cdot\mathbf{MW}_p\mathbf{v}_m$.		

ADKINS [1, 2] determined integrity basis, in the two- and three-dimensional cases, for arbitrary second order tensors, under the condition of linearity of invariants with respect to each argument. Consequently, two-dimensional reduction of the invariants in the case of transverse isotropy characterized by the parametric tensor $\mathbf{M}$ does not yield the invariants listed in the first and second row of Table 3. It is worth noting that the tensor $\mathbf{M}$ describes only one of the five possible cases of 3D transverse isotropy, cf. [29].

Let $\mathbf{D}_i \in T$ $(i = 1,\ldots,I)$ be arbitrary two-dimensional second order tensors, not necessarily symmetric. Assuming that one of the axis of the Cartesian coordinate system coincides with $\mathbf{e}$, Adkins' integrity basis is given by

$$
\begin{aligned}
&D^{(i)}_{11}, \quad D^{(i)}_{\alpha\beta}, \\
&D^{(i)}_{\alpha\beta}D^{(j)}_{\beta\alpha}, \quad D^{(i)}_{1\alpha}D^{(j)}_{\alpha 1}, \qquad \alpha,\beta,\gamma = 1,\ 2, \\
&D^{(i)}_{1\alpha}D^{(k)}_{\alpha\beta}D^{(j)}_{\beta 1}, \qquad i,j,k,l = 1,\ldots,I, \\
&D^{(i)}_{1\alpha}D^{(k)}_{\alpha\beta}D^{(l)}_{\beta\gamma}D^{(j)}_{\gamma 1}, \qquad i > j > k > l,
\end{aligned} \tag{3.6}
$$

where

$$\mathbf{D}_i = D^{(i)}_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta \qquad \text{and} \qquad \mathbf{e}_1 = \mathbf{e}.$$

4. Determination of generators of an orthotropic vector-valued function

In this section we shall derive the general form of the vector-valued function $(2.4)_2$. To this end we consider the scalar function, cf. [11, 14, 24]

$$g = \mathbf{f}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m)\cdot\mathbf{d} = f_\alpha d_\alpha , \tag{4.1}$$

linear in **d**. Thus we may write

$$g(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{d}) = \widehat{g}(I_r, J_s) = \sum_{s=1}^{S} \psi_s(I_r) J_s , \tag{4.2}$$

where I_r are the invariants listed in Table 3, while J_s are the following invariants, linear in **d**:

$$\mathbf{d}\cdot\mathbf{v}_m , \quad \mathbf{d}\cdot\mathbf{M}\mathbf{v}_m , \quad \mathbf{d}\cdot\mathbf{A}_i\mathbf{v}_m , \quad \mathbf{d}\cdot\mathbf{W}_p\mathbf{v}_m . \tag{4.3}$$

They are obtained by using the procedure outlined in the previous section. In fact, since in (4.1) a vector **d** appears, therefore we do not consider Case 2. In Case 1 the invariants $\mathbf{d}\cdot\mathbf{v}_m$, $\mathbf{d}\cdot\mathbf{M}\mathbf{v}_m$, permit us to determine **d** uniquely. Considering Case 3, since

$$\mathbf{d}\cdot\mathbf{v}_m = d_1 c_m \Rightarrow d_1 ,$$

we must additionally examine the following two situations.

CASE 3.1

At least one of the tensors, say $\mathbf{A} \in \{\mathbf{A}_i\}$, is not singular, that is it has two different eigenvalues. Then the two invariants: $\mathbf{d}\cdot\mathbf{v}_m$, $\mathbf{d}\cdot\mathbf{A}\mathbf{v}_m$ determine the components d_α $(\alpha = 1, 2)$ of **d** uniquely.

CASE 3.2

At least one of the tensors, say $\mathbf{W} \in \{\mathbf{W}_p\}$, is such that the corresponding axial vector [22] is not collinear with **e**. Then

$$\mathbf{d}\cdot\mathbf{v}_m , \quad \mathbf{d}\cdot\mathbf{W}\mathbf{v}_m \Rightarrow d_1 \ \text{ and } \ d_2 ,$$

and **d** is determined uniquely.

We observe that if in Case 3 the situations covered by Cases 3. 1 and 3. 2 do not occur, then it suffices to know the invariant $\mathbf{d}\cdot\mathbf{v}_m = d_1 c_m$, because the vector-valued function has the form $\mathbf{f} = \phi\mathbf{e}$, where ϕ stands for an invariant.

The canonical form of the vector-valued function $(2.4)_2$ is given by

$$\mathbf{f}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m) = \frac{\partial \widehat{g}}{\partial \mathbf{d}} = \sum_{s=1}^{S} \psi_s(I_r)\frac{\partial J_s}{\partial \mathbf{d}} = \sum_{s=1}^{S} \psi_s(I_r)\mathbf{g}_s . \tag{4.4}$$

The generators $\mathbf{g}_s$ are listed in Table 4 and coincide with the results due to ZHENG [29].

Table 4. Generators of the orthotropic vector-valued function $(2.4)_2$.

$\mathbf{v}_m$,	$\mathbf{M}\mathbf{v}_m$,	$\mathbf{A}_i\mathbf{v}_m$,	$\mathbf{W}_p\mathbf{v}_m$,	$m = 1,\ldots,M$,	$i = 1,\ldots,I$,	$p = 1,\ldots,P$.

5. Determination of generators of the orthotropic symmetric tensor-valued function

Proceeding similarly as in the previous section we take

$$h = \operatorname{tr}\mathbf{FC}, \tag{5.1}$$

where $\mathbf{C}$ is a symmetric second-order tensor. The scalar-valued function h has now the form

$$h(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{C}) = \widehat{h}(I_r, J_s) = \sum_{s=1}^{S} \phi_s(I_r)J_s\,, \tag{5.2}$$

where I_r are the invariants listed in Table 3, and J_s are linear in $\mathbf{C}$:

$$\operatorname{tr}\mathbf{C},\quad \operatorname{tr}\mathbf{MC},\quad \operatorname{tr}\mathbf{CA}_i\,,\quad \operatorname{tr}\mathbf{CMW}_p\,,\quad \mathbf{v}_m\cdot\mathbf{C}\mathbf{v}_m\,,\quad \mathbf{v}_m\cdot\mathbf{C}\mathbf{v}_n\,. \tag{5.3}$$

To justify (5.3) one has to consider the following three cases.

CASE 1.1

Let $\mathbf{v}_1$, $\mathbf{v}_2 \in \{\mathbf{v}_m\}$ be such that $\det[v_\alpha^{(1)} v_\beta^{(2)}] \neq 0$. Then by using the invariants $\mathbf{v}_1\cdot\mathbf{C}\mathbf{v}_1$, $\mathbf{v}_2\cdot\mathbf{C}\mathbf{v}_2$ and $\mathbf{v}_1\cdot\mathbf{C}\mathbf{v}_2$ we determine $\mathbf{C}$ uniquely. In Case 1.2 one can also calculate these invariants, because $\mathbf{v}_1$ and $\mathbf{e}$ are not collinear.

CASE 2.1

Knowing the invariants: $\operatorname{tr}\mathbf{C}$, $\operatorname{tr}\mathbf{MC}$, $\operatorname{tr}\mathbf{CMW}$ one determines $\mathbf{C}$ uniquely.

If in Case 2.1 all skew-symmetric tensors disappear or their axial vectors are collinear with $\mathbf{e}$, then it suffices to know the invariants: $\operatorname{tr}\mathbf{C}$, $\operatorname{tr}\mathbf{MC}$, because $\mathbf{F}$ has diagonal form.

CASE 2.2

Since the off-diagonal components of the tensor $\mathbf{B}$ are non-zero, it suffices to know the invariants: $\operatorname{tr}\mathbf{C}$, $\operatorname{tr}\mathbf{MC}$ and $\operatorname{tr}\mathbf{CB}$.

All in all, to satisfy the cases considered, the set of invariants linear in $\mathbf{C}$ has to be specified by (5.3).

The canonical form of the tensor-valued function $(2.4)_3$ is given by

$$\mathbf{F}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m) = \frac{1}{2}\left(\frac{\partial h}{\partial \mathbf{C}} + \frac{\partial h}{\partial \mathbf{C}^T}\right) = \frac{\partial h}{\partial \mathbf{C}} = \sum_{s=1}^{S} \phi_s(I_r)\frac{\partial J_s}{\partial \mathbf{C}} = \sum_{s=1}^{S} \overline{\phi}_s(I_r)\mathbf{F}_s\,. \tag{5.4}$$

The results are summarized in Table 5. The generators $\mathbf{F}_s$ are the same as those obtained by ZHENG [29]. The case considered by BOEHLER [4, 6] is covered by the first row of Table 5.

Table 5. Generators of the orthotropic, symmetric tensor-valued function $(2.4)_3$.

$\mathbf{I},\ \mathbf{M},\ \mathbf{A}_i,$	$i = 1, \ldots, I,$	
$\mathbf{M}\mathbf{W}_p - \mathbf{W}_p\mathbf{M},$	$p = 1, \ldots, P,$	
$\mathbf{v}_m \otimes \mathbf{v}_m,\quad \mathbf{v}_m \otimes \mathbf{v}_n + \mathbf{v}_n \otimes \mathbf{v}_m\,,$	$m, n = 1, \ldots, M,$	$m < n.$

6. Determination of generators of the orthotropic skew-symmetric tensor-valued function

We begin by constructing the scalar function [14, 24]

$$k = \operatorname{tr}\mathbf{T}\mathbf{X}, \tag{6.1}$$

where $\mathbf{X}$ is a skew-symmetric tensor. Hence we may write

$$k(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{X}) = \widehat{k}(I_r, K_s) = \sum_{s=1}^{S} \phi_s(I_r) K_s\,, \tag{6.2}$$

where K_s are the invariants, linear in $\mathbf{X}$:

$$\operatorname{tr}\mathbf{M}\mathbf{A}_i\mathbf{X},\quad \operatorname{tr}\mathbf{X}\mathbf{W}_p\,,\quad \mathbf{v}_m\cdot\mathbf{M}\mathbf{X}\mathbf{v}_m\,,\quad \mathbf{v}_m\cdot\mathbf{X}\mathbf{v}_n\,. \tag{6.3}$$

To justify (6.3) we have to examine the following cases.

CASE 1.1

$$\mathbf{v}_m\cdot\mathbf{X}\mathbf{v}_n = X_{12}\left(v_1^{(m)}v_2^{(n)} - v_1^{(n)}v_2^{(m)}\right), \Rightarrow X_{12}\,.$$

CASE 1.2

$$\mathbf{v}_m\cdot\mathbf{M}\mathbf{X}\mathbf{v}_n = X_{12}v_1^{(m)}v_2^{(m)}, \Rightarrow X_{12}\,.$$

CASE 2.1

$$\operatorname{tr}\mathbf{X}\mathbf{W}_p \Rightarrow X_{12}^{(p)}\,.$$

CASE 2.2

$$\operatorname{tr}\mathbf{M}\mathbf{B}\mathbf{X} = -B_{12}X_{12}\,,\qquad B_{12} > 0, \Rightarrow X_{12}\,.$$

Case 3 is treated similarly as Cases 2.1 and 2.2.

The canonical form of the function $\mathbf{T}$ is given by

$$\begin{aligned}\mathbf{T}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m) &= \frac{1}{2}\left(\frac{\partial k}{\partial \mathbf{X}} - \frac{\partial k}{\partial \mathbf{X}^T}\right)\\ &= \frac{1}{2}\sum_{s=1}^{S}\phi_s(I_r)\left(\frac{\partial K_s}{\partial \mathbf{X}} - \frac{\partial K_s}{\partial \mathbf{X}^T}\right) = \sum_{s=1}^{S}\widetilde{\phi}_s(I_r)\mathbf{T}_s\,.\end{aligned} \tag{6.4}$$

The generators of $\mathbf{T}_s$ are listed in Table 6. They coincide with those obtained by ZHENG [29].

Table 6. Generators of the orthotropic, skew-symmetric tensor-valued function $(2.4)_4$.

$\mathbf{MA}_i - \mathbf{A}_i\mathbf{M}$, $\mathbf{W}_p$,	$i = 1, \ldots, I$,	$p = 1, \ldots, P$,
$\mathbf{v}_m \otimes \mathbf{Mv}_m - \mathbf{Mv}_m \otimes \mathbf{v}_m$, $\mathbf{v}_m \otimes \mathbf{v}_n - \mathbf{v}_n \otimes \mathbf{v}_m$,	$m, n = 1, \ldots, M$,	$m < n$.

7. Equivalent functional bases and sets of generators

ZHENG [30] determined an alternative form of the functional basis and generators in comparison with the results of his first paper [29]. In [30] the representations of functions (2.1) corresponding to all anisotropy groups have been investigated. Then orthotropy group is the group C_{2v} (cf. also [21]) and the parametric tensor $\mathbf{K}$ has the form

$$\mathbf{K} = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 . \tag{7.1}$$

Here $\mathbf{e}_\alpha$ ($\alpha = 1, 2$) are unit vectors specifying the directions of orthotropy. By setting $\mathbf{e}_1 = \mathbf{e}$, we readily obtain

$$\mathbf{K} = 2\mathbf{M} - \mathbf{I}. \tag{7.2}$$

This relation enables the passage from our results to those due to ZHENG [30] in the two-dimensional case of orthotropy.

The results obtained by ZHENG [29, 30] and in this contribution can be applied to the determination of representations of the following functions:

$$\begin{aligned} \widetilde{s} &= \widetilde{f}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{H}), \quad i = 1, \ldots, I, \quad p = 1, \ldots, P, \quad m = 1, \ldots, M, \\ \widetilde{\mathbf{t}} &= \widetilde{\mathbf{f}}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{H}), \\ \widetilde{\mathbf{S}} &= \widetilde{\mathbf{F}}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{H}), \qquad \widetilde{\mathbf{S}} = \widetilde{\mathbf{S}}^t, \\ \widetilde{\mathbf{T}} &= \widetilde{\mathbf{G}}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{H}), \qquad \widetilde{\mathbf{T}} = -\widetilde{\mathbf{T}}^t, \end{aligned} \tag{7.3}$$

where $\mathbf{H}$ is a symmetric, positive definite tensor. Its eigenvalues are denoted by H_1 and H_2, $H_1 > H_2$. Now we have

$$\begin{aligned} \mathbf{H} &= H_1\mathbf{e}_1 \otimes \mathbf{e}_1 + H_2\mathbf{e}_2 \otimes \mathbf{e}_2 , \\ \mathbf{H} &= H_1\mathbf{M} + H_2(\mathbf{I} - \mathbf{M}). \end{aligned} \tag{7.4}$$

Consequently one can easily determine the representations of the functions appearing in (7.3).

The last case is important for applications if $\mathbf{H}$ plays the role of a fabric tensor, cf. [7, 8, 9]. This tensor is sometimes used to model the mechanical behaviour of materials as different as soils [6] and bones [7–9].

In the case when $H_1 = H_2$, $\mathbf{H}$ is a spherical tensor and the representations of functions (7.3) coincide with those derived by KORSGAARD [14]; then the tensor $\mathbf{H}$ does not appear in these functions.

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WARSAW UNIVERSITY OF TECHNOLOGY
CIVIL ENGINEERING FACULTY
INSTITUTE OF STRUCTURAL MECHANICS
and
POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

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Friction relations for the many-sphere Oseen hydrodynamic interactions

I. PIEŃKOWSKA (WARSZAWA)

THE PAPER concerns weak inertia effects arising in the many-sphere hydrodynamic interactions. Rigid spheres are held fixed in an incompressible fluid flowing with uniform velocity U at infinity. The friction relations, up to the contributions of the order $O(\mathrm{Re})$, where Re is the Reynolds number, are considered on the basis of the Oseen equations.

1. Introduction

THE MOTIVATION for this work is to analyse the effects of weak convective inertia on the hydrodynamic interactions of a finite number of spheres, immersed in an incompressible, unbounded fluid. The present paper is a continuation of earlier publications on the low Reynolds number hydrodynamic interactions [1]. The $O(\mathrm{Re})$ convective inertia effects, where Re is the Reynolds number of the sphere (based on the radius a of the spheres, the kinematic viscosity ν of the fluid and the uniform velocity $\mathbf{U}$ of the fluid at infinity), are considered on the basis of the Oseen equations [2]. In particular, we will consider the $O(\mathrm{Re})$ contributions, appearing in the friction tensors, describing the dependence of the forces $\mathbf{F}_j$ and torques $\mathbf{T}_j$, $j = 1, \ldots, N$, exerted on the spheres by the fluid, on the uniform velocity of the fluid. The study of the friction relations enables an insight into the hydrodynamic interactions between the spheres.

To quote the literature, concerning the study of the uniform flow past a single sphere at low Re, we recall the paper by DENNIS and WALKER [3], and by DENNIS, INGHAM and SINGH [4]. The authors have compared the calculated drag force exerted by the fluid on the sphere, with the results of previous investigations and with the experimental data. In author's opinion, the approach to $\mathrm{Re} \to 0$ is via CHESTER and BREACH [5] drag, rather than via the Oseen drag, as suggested by the experimental results of MAXWORTHY [6]. To calculate the drag force up to terms of the order of $O(\mathrm{Re}^3)$, Chester and Breach used the method of matched asymptotic expansions. Recently, an arbitrary time-dependent motion of a rigid particle in a time-dependent flow of a fluid has been examined by LOVALENTI and BRADY [7]. The authors have calculated the hydrodynamic force acting on the particle, including the contributions up to the terms of $O(\mathrm{Re})$.

Referring to the examination of the small inertia effects appearing in the many-sphere hydrodynamic interactions, we recall the early experimental results of JAYAWERRA, MASON and SLACK [16]. The authors have analysed the behaviour of clusters of spheres, falling in a viscous fluid. Their observations have been discussed in the theoretical paper by HOCKING [17]. He has pointed out that some hydrodynamic phenomena, observed by the authors of the paper [16], are not explicable by the Stokes slow motion hydrodynamics. Subsequently, the influence of small nonlinear effects on the hydrodynamic interactions of spheres has been discussed by HAPPEL and BRENNER [18]. For the particular case of two falling spheres, these effects have been observed experimentally for the cases of $\mathrm{Re} > 0.25$.

Recently, the effects of weak inertia on the motion of particles in a viscous fluid have been reported in a review paper by LEAL [19]. He has argued that even small departures from the Stokes flows can have a strong influence on the positions or orientations of the particle. Problems of the motion of a few particles in the presence of the bounding walls at moderate Re have been treated by means of a numerical package that simulates two-phase Navier–Stokes flows [8]. The authors of that package have taken into account full nonlinearity and the fluid–solid coupling. The papers concerns, however, two-dimensional flows. KIM, ELGHOBASHI and SIRIGNANO [9] have performed a three-dimensional numerical simulation of a steady uniform flow past two fixed spheres, at Re reaching up to 150.

In the present paper we regard small inertial effects appearing in the steady uniform flow past N fixed spheres, at $\mathrm{Re} < 1$. The problem is considered on the basis of the Oseen equations. To deduce the friction relations, the velocity field of the fluid is expressed in terms of the integral equation, involving the Green tensor acting on the forces $\mathbf{f}_j$, distributed on the surfaces of the spheres [1]. The properties of the Green tensor have been recently discussed by GALDI [2]. The dependence of the Green tensor on $|\mathbf{U}|/\nu$ leads to a nonlinear dependence of the hydrodynamic interactions between the particles on the value of Re. However, for the particular case of the hydrodynamic interactions characterized by $\mathrm{Re}_m < 1$, where $\mathrm{Re}_m = R|\mathbf{U}|/\nu$, R – typical distance between the centres of the spheres, we are, qualitatively speaking, close to the Stokes hydrodynamics. For that régime, we confine our considerations to the $O(\mathrm{Re})$ convective effects. The hydrodynamic interactions are presented to be due to the multiple scattering processes. In terms of the multiple scattering events, such properties of the hydrodynamic interactions as non-locality and non-addivity can be conveniently discussed. Knowing the dependence of the hydrodynamic interactions on Re and on the spatial configuration of the particles, we obtain the $O(\mathrm{Re})$-friction relations. These relations describe the convective corrections to the respective Stokes friction relations. The obtained relations have the form of series expansions with respect to σ, where $\sigma = a/R$, $\sigma < 1/2$.

As an example, we consider the drag and side forces exerted on three spheres, placed in the transversal and longitudinal directions with respect to the uniform flow of the fluid at infinity. The dependence of the associated hydrodynamic interactions on σ is taken into account up to the terms of order $O(\sigma)$.

2. Multiple scattering representation of the hydrodynamic interactions

We adopt the idea of induced forces $\mathbf{f}_j$, $j = 1,\ldots,N$, distributed on the surfaces of the spheres [10], to describe the presence of the spheres in the flow. The dependence of the induced forces on the uniform velocity of the fluid $\mathbf{U}$ can be expressed in terms of the following set of integral equations [1]:

$$
\begin{aligned}
\dot{\mathbf{R}}_j(\Omega_j) = \mathbf{v}^0(\mathbf{R}_j(\Omega_j)) + \int d\Omega'_j \mathbf{T}[\mathbf{R}_j(\Omega_j) - \mathbf{R}'_j(\Omega'_j)]\cdot\mathbf{f}'_j(\Omega'_j) \\
+ \sum_{k\neq j}^{N} \int d\Omega_k \mathbf{T}[\mathbf{R}_j(\Omega_j) - \mathbf{R}_k(\Omega_k)]\cdot\mathbf{f}_k(\Omega_k);
\end{aligned}
\tag{2.1}
$$

$$
\mathbf{V}_j(\Omega_j) = \dot{\mathbf{R}}_j(\Omega_j) - \mathbf{v}^0(R_j(\Omega_j)) = -\mathbf{U}, \qquad \dot{\mathbf{R}}_j \equiv 0,
$$

where $\mathbf{V}_j$ is the relative velocity of the j-th sphere with respect to the fluid, $\mathbf{R}_j$ – position coordinates of the points on the surface of the j-th sphere, $\dot{\mathbf{R}}_j$ – velocity of the j-th sphere, $\mathbf{T}(\mathbf{R}_j - \mathbf{R}_k)$ – Green tensor of the problem considered. The first integral on the r.h.s. accounts for the interaction of the j-th sphere with the fluid, the second integral is due to the hydrodynamic interactions between the spheres.

The convolution form of Eq. (2.1) reflects the non-local character of hydrodynamic interactions. For the present purposes it is convenient to work with the Fourier transform of the Green tensor [11]:

$$
\mathbf{T}(\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\exp(i\mathbf{k}\cdot\mathbf{r})}{\mu(k^2 + i\nu^{-1}\mathbf{U}\cdot\mathbf{k})}(\mathbf{1} - \widehat{\mathbf{k}}\widehat{\mathbf{k}}),
\tag{2.2}
$$

where $\widehat{k} = \mathbf{k}/|\mathbf{k}|$, $k = |\mathbf{k}|$, μ – the dynamic viscosity of the fluid.

The dependence of the Green tensor on $|\mathbf{U}|/\nu$ leads to the nonlinear relations between the induced forces $\mathbf{f}_j$ and Re. These relations can be expressed in terms of admissible sequences of the hydrodynamic interactions. To deduce the multiple scatttering representation of the respective interactions, we follow the procedure used by YOSHIZAKI and YAMAKAWA for the case of the Stokes hydrodynamics [12].

After the two steps:

(i) expansions of $\mathbf{V}_j$, $\mathbf{f}_j$ in terms of the normalized spherical harmonics,

(ii) integrations with respect to the angular variables Ω_j,

we arrive at the set of algebraic equations, relating the expansion coefficients $\mathbf{f}_{j,l_1m_1}$ of the induced forces to the expansion coefficients $\mathbf{V}_{j,l_1m_1}$ of the relative velocities:

$$
\mathbf{V}_{j,l_1m_1} = \sum_{l_2m_2} \mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{O}_j)\cdot\mathbf{f}_{j,l_2m_2} + \sum_{k\neq j}^{N} \sum_{l_2m_2} \mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{R}_{jk})\cdot\mathbf{f}_{k,l_2m_2},
\tag{2.3}
$$

where $\mathbf{r}_j = \mathbf{R}_j - \mathbf{R}_j^0$, $\mathbf{r}_j = \mathbf{r}_j(a, \Omega_j)$, $\mathbf{R}_{jk} = \mathbf{R}_k^0 - \mathbf{R}_j^0$, and R_j^0 – position of the centre of the j-th sphere,

$$\mathbf{V}_{j,lm} = \left\{ \begin{array}{ll} -\mathbf{U}, & l = 0 \\ 0, & l \geq 1 \end{array} \right\}. \tag{2.4}$$

Tensors $\mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{O}_j)$ are called the self-interaction tensors, representing the particular type (specified by the indices l_1m_1, l_2m_2) of influence of a single sphere on the surrounding fluid; tensors $\mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{R}_{jk})$ denote the mutual interaction tensors, describing the interaction between the j-th and k-th spheres, respectively. Dependence of the above tensors on the spatial configuration of the spheres can be presented in the following form:

$$\mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{R}_{jk}) = \sum_{l_3m_3} \mathbf{T}_{l_1m_1,l_3m_3}^{l_2m_2}(|\mathbf{R}_{jk}|) Y_{l_3}^{m_3}(\Omega_{jk}), \tag{2.5}$$

where spherical polar coordinates $\mathbf{R}_{jk}(|\mathbf{R}_{jk}|, \Omega_{jk})$ are used.

Next, we formally solve the basic set of the algebraic equations by iteration,

$$\begin{aligned} \mathbf{f}_{j,l_1m_1} &= \sum_{l_2m_2} \widetilde{\mathbf{T}}_{l_1m_1}^{l_2m_2}(\mathbf{O}_j) \cdot \mathbf{V}_{j,l_2m_2} \\ &- \sum_{k \neq j}^{N} \sum_{l_2m_2} \sum_{l_3m_3} \sum_{l_4m_4} \widetilde{\mathbf{T}}_{l_1m_1}^{l_2m_2}(\mathbf{O}_j) \cdot \mathbf{T}_{l_2m_2}^{l_3m_3}(\mathbf{R}_{jk}) \cdot \widetilde{\mathbf{T}}_{l_3m_3}^{l_4m_4}(\mathbf{O}_k) \cdot \mathbf{V}_{k,l_4m_4} + \ldots, \end{aligned} \tag{2.6}$$

where $\widetilde{\mathbf{T}}_{l_1m_1}^{l_2m_2}$ is the inverse self-interaction tensor. The inverse tensors can be expressed by the following approximate formula [1]:

$$\widetilde{\mathbf{T}} = \widetilde{\mathbf{T}}_d - \widetilde{\mathbf{T}}_d \cdot \mathbf{T}_{od} \cdot \widetilde{\mathbf{T}}_d + \widetilde{\mathbf{T}}_d \cdot \mathbf{T}_{od} \cdot \widetilde{\mathbf{T}}_d \cdot \mathbf{T}_{od} \cdot \widetilde{\mathbf{T}}_d - \ldots, \tag{2.7}$$

where $\widetilde{\mathbf{T}}_d$ are diagonal, and $\mathbf{T}_{od}$ – off-diagonal in l (it means, they are of the form $\widetilde{\mathbf{T}}_{l_1m_1}^{l_1m_2}(\mathbf{O}_j)$ and $\widetilde{\mathbf{T}}_{l_1m_1}^{l_2m_2}(\mathbf{O}_j)$, where $l_1 \neq l_2$, respectively). Thus the expansion coefficients of the induced forces are expressed in terms of admissible sequences of the hydrodynamic interactions. These sequences, depending on the properties of the interaction tensors involved, present the allowed types of coupling of the spheres to the fluid.

3. Weak inertial effects

Considering the weak inertial effects, we focus our attention on the low Re properties of the hydrodynamic interaction tensors. In paper [1], the dependence of the tensors on Re is expressed in terms of the modified Bessel functions $I_{l_1+1/2}$

and $K_{l_1+1/2}$. From the properties of the Bessel functions for $\mathrm{Re} \to 0$ it follows that the Stokes self-interaction tensors are equal to

$$\mathbf{T}^{l_1 m_2}_{l_1 m_1}(\mathbf{O}_j) = \frac{1}{4\sqrt{\pi}\, a\mu} \frac{1}{(l_1 + 1/2)} \mathbf{K}^{l_1 m_2}_{l_1 m_1, 00}, \tag{3.1}$$

where

$$\mathbf{K}^{l_2 m_2}_{l_1 m_1, l_3 m_3} = i^{l_1 - l_2 - l_3} \int d\widehat{\mathbf{k}} (\mathbf{1} - \widehat{\mathbf{k}}\widehat{\mathbf{k}}) Y^{-m_1}_{l_1} Y^{m_2}_{l_2} Y^{-m_3}_{l_3}.$$

It was shown in the paper [12], that $\mathbf{K}^{l_2 m_2}_{l_1 m_1, l_3 m_3} \neq 0$ for the following sets of indices

$$l_1 + l_2 - l_3 \geq -2, \qquad l_1 + l_2 + l_3 = 2n, \qquad n = 0, 1, 2, \ldots . \tag{3.2}$$

We note that the Stokes self-interaction tensors are diagonal in l.

The Stokes mutual interaction tensors, under the assumption of $\mathrm{Re}_m \to 0$, can be obtained in the following form:

$$\begin{aligned} \mathbf{T}^{l_2 m_2}_{l_1 m_1, l_3 m_3} &= \sum_{m=0}^{\infty} \mathbf{T}^{l_2 m_2, m}_{l_1 m_1, l_3 m_3} = \frac{\sqrt{\pi}}{4a\mu\Gamma(l_1 + 3/2)\Gamma(l_2 + 3/2)} \left(\frac{a}{R_{jk}}\right)^{l_1 + l_2 + 1} \\ &\cdot \mathbf{K}^{l_2 m_2}_{l_1 m_1, l_3 m_3} \sum_{m=0} \frac{(l_1 + l_2 + 2m + 1/2)\Gamma(l_1 + l_2 + m + 1/2)}{m!} \frac{\Gamma(\zeta)}{\Gamma(Z+1)} \\ &\cdot F_4 \left[-m, l_1 + l_2 + m + \frac{1}{2};\ l_1 + \frac{3}{2},\ l_2 + \frac{3}{2};\ \left(\frac{a}{R_{jk}}\right)^2, \left(\frac{a}{R_{jk}}\right)^2 \right], \end{aligned} \tag{3.3}$$

where $|l_1 + l_2 + 2m - l_3| = 0$,

$$Z = \max\left(l_1 + l_2 + 2m + \frac{1}{2},\ l_3 + \frac{1}{2}\right), \qquad \zeta = \min\left(l_1 + l_2 + 2m + \frac{1}{2},\ l_3 + \frac{1}{2}\right),$$

and F_4 is the hypergeometric series.

The allowed values of the respective indices read:

$$\begin{aligned} &\text{(i)} \qquad m = 0, \qquad l_1 + l_2 = l_3, \\ &\text{(ii)} \qquad m = 1, \qquad l_1 + l_2 + 2 = l_3. \end{aligned} \tag{3.4}$$

From (3.3) we obtain the following leading order dependence of the interaction tensors on R_{jk}:

$$\mathbf{T}^{l_2 m_2}_{l_1 m_1, l_3 m_3} \sim \left(\frac{a}{R_{jk}}\right)^{l_1 + l_2 + 1}. \tag{3.5}$$

This property is characteristic for the Stokes flow régime [12].

The next step is to consider the quadratic dependence of $\mathbf{F}_j$ and $\mathbf{T}_j$ on $\mathbf{U}$. To that end, we have to take into account the $O(\mathrm{Re})$ contributions to the interaction tensors. First, we have the self-interaction tensors, being of the leading order of $O(\mathrm{Re})$,

$$\mathbf{T}^{l_2m_2}_{l_2+1m_1}(\mathbf{O}_j) = \frac{\mathrm{Re}}{8\sqrt{3}\sqrt{\pi}\,a\mu(l_2+1/2)(l_2+3/2)}\Bigg[\widehat{U}_z\mathbf{K}^{l_2m_2}_{l_2+1m_1,10}$$

$$+\frac{1}{\sqrt{2}}\left(\widehat{U}_x - i\widehat{U}_y\right)\mathbf{K}^{l_2m_2}_{l_2+1m_1,1-1} + \frac{1}{\sqrt{2}}\left(\widehat{U}_x + i\widehat{U}_y\right)\mathbf{K}^{l_2m_2}_{l_2+1m_1,11}\Bigg] + \dots, \tag{3.6}$$

$$\mathbf{T}^{l_2m_2}_{l_2+1m_1}(\mathbf{O}_j) = -\mathbf{T}^{l_2+1-m_1}_{l_2-m_2}(\mathbf{O}_j).$$

They are off-diagonal in l. Then, the $O(\mathrm{Re})$ contributions appear in a series expansion of the tensor $\mathbf{T}^{00}_{00}$ with respect to Re [1].

The mutual interaction tensors, being of the leading order of Re, under the assumption $\mathrm{Re}_m < 1$, read:

$$\mathbf{T}^{l_2m_2}_{l_1m_1,l_3m_3} = \sum_{m=0}^{\infty}\mathbf{T}^{l_2m_2,m}_{l_1m_1,,l_3m_3} = -\frac{i^{1-l_3}\mathrm{Re}\sqrt{2\pi(2l_3+1)}}{16a\mu\Gamma(l_1+3/2)\Gamma(l_2+3/2)}\beta(-m_3) \tag{3.7}$$

$$\cdot\Bigg\{\sum_{rs} i^r\sqrt{2r+1}\,\mathbf{K}^{l_2m_2}_{l_1m_1,rs}(-1)^s\beta(-s)\begin{pmatrix} l_3 & 1 & r\\ 0 & 0 & 0\end{pmatrix}\Bigg[\sqrt{2}\widehat{U}_z\begin{pmatrix} l_3 & 1 & r\\ -m_3 & 0 & s\end{pmatrix}$$

$$-\left(\widehat{U}_x - i\widehat{U}_y\right)\begin{pmatrix} l_3 & 1 & r\\ -m_3 & 1 & s\end{pmatrix} + \left(\widehat{U}_x + i\widehat{U}_y\right)\begin{pmatrix} l_3 & 1 & r\\ -m_3 & -1 & s\end{pmatrix}\Bigg]\Bigg\}$$

$$\cdot\left(\frac{a}{R_{jk}}\right)^{l_1+l_2}\sum_{m=0}^{\infty}\frac{(l_1+l_2+2m+1/2)\Gamma(l_1+l_2+m+1/2)}{m!}\frac{\Gamma(\zeta)}{\Gamma(Z+1)}$$

$$\cdot F_4\left[-m, l_1+l_2+m+\frac{1}{2};\ l_1+\frac{3}{2},\ l_2+\frac{3}{2};\ \left(\frac{a}{R_{jk}}\right)^2, \left(\frac{a}{R_{jk}}\right)^2\right] + \dots,$$

where $\left(\begin{smallmatrix}\dots\\ \dots\end{smallmatrix}\right)$ is the Wigner $3-j$ symbol [15],

$$|l_1 + l_2 + 2m - l_3| = 1;$$

$$r = 1 \quad \text{for} \quad l_3 = 0; \quad r = (l_3-1, l_3+1) \quad \text{for} \quad l_3 \geq 1;$$

$$\beta(s) = (-1)^{3s+|s|)/2}.$$

In contrast to the Stokes régime, we have here the following sets of admissible indices:

$$\begin{array}{llll} \text{(i)} & m = 0, & l_1+l_2-l_3 = 1, & \text{and}\quad l_1+l_2-l_3 = -1,\\ \text{(ii)} & m = 1, & l_1+l_2-l_3 = -1, & \text{and}\quad l_1+l_2-l_3 = -3,\\ \text{(iii)} & m = 2, & l_1+l_2-l_3 = -3. & \end{array} \tag{3.8}$$

The above tensors exhibit the following leading order dependence on R_{jk}:

$$\mathbf{T}^{l_2 m_2}_{l_1 m_1, l_3 m_3} \sim \left(\frac{a}{R_{jk}}\right)^{l_1+l_2} . \tag{3.9}$$

We have also the second source of the contributions to the mutual interaction tensors, being of the order of $O(\mathrm{Re})$: a series expansion of the tensor $\mathbf{T}^{00}_{00}(\mathbf{R}_{jk})$ with respect to Re_m [1].

4. Friction relations

In the present section we will examine the friction relations which express the forces and the torques, experienced by the spheres, as quadratic functions of the fluid velocity **U**. To that end, the forces and torques are presented in terms of the respective expansion coefficients of the induced forces:

$$\begin{aligned} \mathbf{F}_j &= -\mathbf{f}_{j,00} , \\ \mathbf{T}_j &= \boldsymbol{\varepsilon} : \sum_{m=-1}^{1} (\mathbf{r}_j)_{-m} \mathbf{f}_{j,1m} , \end{aligned} \tag{4.1}$$

where

$$((\mathbf{r}_j)_m)_k = \frac{a}{\sqrt{3}} \left[\delta_{k1}(\delta_{m1} + \delta_{m-1})\frac{1}{\sqrt{2}} + \delta_{k2}(-\delta_{m1} + \delta_{m-1})\frac{i}{\sqrt{2}} + \delta_{k3}\delta_{m0}\right] .$$

Using the result (2.6) for the expansion coefficients, the friction relations can be written in the following form:

$$\begin{aligned} \mathbf{F}_j &= \sum_{k=1}^{N} \boldsymbol{\xi}^{TV}_{jk} \cdot \mathbf{U}, \\ \mathbf{T}_j &= \sum_{k=1}^{N} \xi^{RV}_{jk} \cdot \mathbf{U}, \end{aligned} \tag{4.2}$$

where $\boldsymbol{\xi}^{\cdots}_{jj}$ denote self-, and $\boldsymbol{\xi}^{\cdots}_{jk}$, $j \neq k$, the mutual friction tensors.

The above relations are of a structure similar to that of the respective Stokes friction relations, the difference consisting in the properties of the friction tensors. The tensors involved can be presented as a sum of the Stokes contributions and the corrections due to weak convection. That sum accounts for the influence of the spatial configuration of the spheres on their hydrodynamic interactions.

The translational self-friction tensors are equal to

$$(4.3)\qquad \boldsymbol{\xi}_{jj}^{TV} = \widetilde{\mathbf{T}}_j + \sum_{l\neq j} \widetilde{\mathbf{T}}_j \cdot \mathbf{T}_{jl} \cdot \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{lj} \cdot \widetilde{\mathbf{T}}_j + \widetilde{\mathbf{T}}_j^1$$
$$+\widetilde{\mathbf{T}}_j \cdot \sum_{l\neq j}\left[\mathbf{T}_{jl}^1 \cdot \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{lj} + \mathbf{T}_{jl} \cdot \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{lj}^1\right] \cdot \widetilde{\mathbf{T}}_j + \widetilde{\mathbf{T}}_j^1 \cdot \sum_{l\neq j} \mathbf{T}_{jl} \cdot \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{lj} \cdot \widetilde{\mathbf{T}}_j$$
$$+\widetilde{\mathbf{T}}_j \cdot \sum_{l\neq j} \mathbf{T}_{jl} \cdot \left[\widetilde{\mathbf{T}}_l^1 \cdot \mathbf{T}_{lj} \cdot \widetilde{\mathbf{T}}_j + \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{lj} \cdot \mathbf{T}_j^1\right]$$
$$-\widetilde{\mathbf{T}}_j \cdot \sum_{l\neq j}\sum_{n\neq l}\sum_{n\neq j}\left[\mathbf{T}_{jl} \cdot \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{ln}^1 \cdot \widetilde{\mathbf{T}}_n \cdot \mathbf{T}_{nj}\right.$$
$$\left.+\mathbf{T}_{jl}^1 \cdot \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{ln} \cdot \widetilde{\mathbf{T}}_n \cdot \mathbf{T}_{nj} + \mathbf{T}_{jl} \cdot \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{ln} \cdot \widetilde{\mathbf{T}}_n \cdot \mathbf{T}_{nj}^1\right] \cdot \widetilde{\mathbf{T}}_j + \ldots .$$

The Stokes dependence of the friction tensor on σ is described by the first two terms, being of $O(\sigma^0)$ and $O(\sigma^2)$, respectively. The Stokes hydrodynamic interactions between two spheres contribute to the above tensors. The remaining terms express the inertial corrections, being of $O(\sigma^0)$, $O(\sigma^1)$ and $O(\sigma^2)$, respectively. They are due to two and three-sphere interactions. The non-additivity of the interactions appears in the inertial corrections through the terms of order $O(\sigma^2)$. The self- and mutual interaction tensors entering the expression for $\boldsymbol{\xi}_{jj}^{TV}$ are equal to

(i) the self-interaction tensors

$$(4.4)\qquad \begin{aligned} \widetilde{\mathbf{T}}_j &= 6\pi\mu a\mathbf{I}, \\ \widetilde{\mathbf{T}}_j^1 &= 6\pi\mu a\left[\frac{3}{16}\left(3\mathbf{I} - \widehat{\mathbf{U}}\widehat{\mathbf{U}}\right)\right]\mathrm{Re}, \end{aligned}$$

(ii) the mutual interaction tensors:

$$\mathbf{T}_{jk} = \frac{1}{6\pi\mu R_{jk}}\left[\frac{3}{4}(\mathbf{1} + \widehat{\mathbf{e}}_{jk}\widehat{\mathbf{e}}_{jk})\right], \qquad \widehat{\mathbf{e}}_{jk} = \mathbf{R}_{jk}/|\mathbf{R}_{jk}|,$$

$$(4.5)\qquad \mathbf{T}_{jk}^1 = -\frac{1}{6\pi\mu a}\left[\frac{3}{16}\left(3\mathbf{I} - \widehat{\mathbf{U}}\widehat{\mathbf{U}}\right) + \sqrt{\pi}\frac{3}{4}\sqrt{\frac{1}{5}}\sum_{m=-1}^{1}\varepsilon(-m)\mathbf{L}_1(m)Y_1^m(\Omega_{jk})\right.$$
$$\left.-\sqrt{\pi}\frac{1}{16}\sqrt{\frac{1}{5}}\sum_{m=-3}^{3}\varepsilon(-m)\mathbf{L}_3(m)Y_3^m(\Omega_{jk})\right]\mathrm{Re},$$

where

$$\mathbf{L}_q(m) = \sum_{rs} i^r\sqrt{2r+1}\,\mathbf{K}_s(-1)^s\varepsilon(-s)\begin{pmatrix} q & 1 & r \\ 0 & 0 & 0\end{pmatrix}\left[\sqrt{2}\widehat{U}_z\begin{pmatrix} q & 1 & r \\ -m & 0 & s\end{pmatrix}\right.$$
$$\left.-\left(\widehat{U}_x - i\widehat{U}_y\right)\begin{pmatrix} q & 1 & r \\ -m & 1 & s\end{pmatrix} + \left(\widehat{U}_x + i\widehat{U}_y\right)\begin{pmatrix} q & 1 & r \\ -m & -1 & s\end{pmatrix}\right],$$

and tensors $\mathbf{K}_s$ are given by YOSHIZAKI, and YAMAKAWA [12]:

$$\begin{aligned}
\mathbf{K}_0 &= \sqrt{\frac{2}{3}}(-\mathbf{e}_x\mathbf{e}_x - \mathbf{e}_y\mathbf{e}_y + 2\mathbf{e}_z\mathbf{e}_z),\\
\mathbf{K}_1 &= \mathbf{e}_x\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_x - i\mathbf{e}_y\mathbf{e}_z - i\mathbf{e}_z\mathbf{e}_y\,,\\
\mathbf{K}_2 &= \mathbf{e}_x\mathbf{e}_x - \mathbf{e}_y\mathbf{e}_y - i\mathbf{e}_x\mathbf{e}_y - i\mathbf{e}_y\mathbf{e}_x\,,\\
\mathbf{K}_{-m} &= \mathbf{K}_m^*\,,
\end{aligned}$$

where the complex conjugate is denoted by an asterisk, and $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ is an external Cartesian coordinate system.

The first two tensors describe, for the particular case of a single sphere, the Stokes and Oseen contributions to the drag force, exerted by the fluid on the sphere.

The translational mutual friction tensors read:

$$\begin{aligned}
(4.6)\qquad \boldsymbol{\xi}_{jk}^{TV} = &-\widetilde{\mathbf{T}}_j\cdot\mathbf{T}_{jk}\cdot\widetilde{\mathbf{T}}_k + \sum_{l\neq k}\sum_{l\neq j}\widetilde{\mathbf{T}}_j\cdot\mathbf{T}_{jl}\cdot\widetilde{\mathbf{T}}_l\cdot\mathbf{T}_{lk}\cdot\widetilde{\mathbf{T}}_k\\
&-\widetilde{\mathbf{T}}_j\cdot\mathbf{T}_{jk}^1\cdot\widetilde{\mathbf{T}}_k - \widetilde{\mathbf{T}}_j\cdot\mathbf{T}_{jk}\cdot\widetilde{\mathbf{T}}_k^1 - \widetilde{\mathbf{T}}_j^1\cdot\mathbf{T}_{jk}\cdot\widetilde{\mathbf{T}}_k - \widetilde{\mathbf{T}}_j\cdot\mathbf{T}_{jk}^2\cdot\widetilde{\mathbf{T}}_k\\
&-\sum_m\widetilde{\mathbf{T}}_{00}^{1m}(\mathbf{O}_j)\cdot\mathbf{T}_{1m}^{00}(\mathbf{R}_{jk})\cdot\widetilde{\mathbf{T}}_k - \widetilde{\mathbf{T}}_j\cdot\sum_m\mathbf{T}_{00}^{1m}(\mathbf{R}_{jk})\cdot\widetilde{\mathbf{T}}_{1m}^{00}(\mathbf{O}_k)\\
&+\widetilde{\mathbf{T}}_j\cdot\sum_{l\neq k}\sum_{l\neq j}\left[\mathbf{T}_{jl}^1\cdot\widetilde{\mathbf{T}}_l\cdot\mathbf{T}_{lk} + \mathbf{T}_{jl}\cdot\widetilde{\mathbf{T}}_l\cdot\mathbf{T}_{lk}^1\right]\cdot\widetilde{\mathbf{T}}_k + \widetilde{\mathbf{T}}_j^1\cdot\sum_{l\neq k}\sum_{l\neq j}\mathbf{T}_{jl}\cdot\widetilde{\mathbf{T}}_l\cdot\mathbf{T}_{lk}\cdot\widetilde{\mathbf{T}}_k\\
&+\widetilde{\mathbf{T}}_j\cdot\sum_{l\neq k}\sum_{l\neq j}\mathbf{T}_{jl}\cdot\left[\widetilde{\mathbf{T}}_l^1\cdot\mathbf{T}_{lk}\cdot\widetilde{\mathbf{T}}_k + \widetilde{\mathbf{T}}_l\cdot\mathbf{T}_{lk}\cdot\widetilde{\mathbf{T}}_k^1\right]\\
&-\widetilde{\mathbf{T}}_j\cdot\sum_{l\neq k}\sum_{n\neq l}\sum_{n\neq j}\left[\mathbf{T}_{jl}\cdot\widetilde{\mathbf{T}}_l\cdot\mathbf{T}_{ln}^1\cdot\widetilde{\mathbf{T}}_n\cdot\mathbf{T}_{nk}\right.\\
&\left.+\mathbf{T}_{jl}^1\cdot\widetilde{\mathbf{T}}_l\cdot\mathbf{T}_{ln}\cdot\widetilde{\mathbf{T}}_n\cdot\mathbf{T}_{nk} + \mathbf{T}_{jl}\cdot\widetilde{\mathbf{T}}_l\cdot\widetilde{\mathbf{T}}_{ln}\cdot\widetilde{\mathbf{T}}_n\cdot\mathbf{T}_{nk}^1\right]\cdot\widetilde{\mathbf{T}}_k + \ldots\,.
\end{aligned}$$

The Stokes contributions to the friction tensors are described by the first two terms, being of $O(\sigma)$ and $O(\sigma^2)$, respectively. For that régime, the hydrodynamic interactions of two and three spheres occur. The remaining terms are due to the inertial effects. They contain two, three, and four-sphere contributions. In the Stokes régime, the property of non-additivity appears starting from the terms of $O(\sigma^2)$, whereas in the $O(\mathrm{Re})$ régime, the three-body effect enters at $O(\sigma^1)$.

The mutual friction tensors $\boldsymbol{\xi}_{jk}^{TV}$ are built up of the following interaction tensors:

(i) the self-interaction tensors:

$\widetilde{\mathbf{T}}_j$ and $\widetilde{\mathbf{T}}_j^1$, given by the expressions (4.4),

$$\begin{aligned}
(4.7)\qquad \widetilde{\mathbf{T}}_{00}^{1m}(\mathbf{O}_j) &= \sqrt{6}\pi\mu a\mathrm{Re}\left[\sqrt{2}\widehat{U}_z\delta_{m0} + \left(\widehat{U}_x - i\widehat{U}_y\right)\delta_{m(-1)} + \left(\widehat{U}_x + i\widehat{U}_y\right)\delta_{m(1)}\right]\mathbf{I},\\
\widetilde{\mathbf{T}}_{1m}^{00}(\mathbf{O}_j) &= -\sqrt{6}\pi\mu a\mathrm{Re}\left[\sqrt{2}\widehat{U}_z\delta_{m0} + \left(\widehat{U}_x + i\widehat{U}_y\right)\delta_{m(-1)} + \left(\widehat{U}_x - i\widehat{U}_y\right)\delta_{m(1)}\right]\mathbf{I};
\end{aligned}$$

(ii) the mutual interactions tensors:

$\mathbf{T}_{jk}$ and $\mathbf{T}^1_{jk}$, given by the expressions (4.5),

$$\mathbf{T}^{1m}_{00}(\mathbf{R}_{jk}) = \sum_{m_3} \mathbf{T}^{1m}_{00,1m_3}(|\mathbf{R}_{jk}|)Y_1^{m_3}(\Omega_{jk}) + \sum_{m_3} \mathbf{T}^{1m}_{00,3m_3}(|\mathbf{R}_{jk}|)Y_3^{m_3}(\Omega_{jk}),$$

where

$$\mathbf{T}^{1m}_{00,1m_3} = \frac{1}{3a\mu}\left(\frac{a}{R_{jk}}\right)^2 \mathbf{K}^{1m}_{00,1m_3} + \ldots,$$

$$\mathbf{T}^{1m}_{00,3m_3} = \frac{1}{2a\mu}\left(\frac{a}{R_{jk}}\right)^2 \mathbf{K}^{1m}_{00,3m_3} + \ldots,$$

$$\mathbf{T}^{00}_{1m}(\mathbf{R}_{jk}) = -\mathbf{T}^{1-m}_{00}(\mathbf{R}_{jk}), \tag{4.8}$$

$$\mathbf{T}^2_{jk} = \frac{\mathrm{Re}}{6\pi\mu a}\left(\frac{a}{R_{jk}}\right)^2 \left[\frac{\sqrt{\pi}}{2\sqrt{5}} \sum_{m=-1}^{1} \varepsilon(-m)\mathbf{L}_1(m)Y_1^m(\Omega_{jk}) - \frac{1}{4}\sqrt{\frac{21}{5}} \sum_{m=-3}^{3} \varepsilon(-m)\mathbf{L}_3(m)Y_3^m(\Omega_{jk})\right].$$

In view of the properties of the considered hydrodynamic interaction tensors, the $O(\mathrm{Re})$ contributions to the friction tensors $\boldsymbol{\xi}^{TV}_{jk}$ do not obey the symmetry relations

$$\left(\boldsymbol{\xi}^{TV}_{jk}\right)_{pq} = \left(\xi^{TV}_{kj}\right)_{qp}, \tag{4.9}$$

characteristic for the Stokes contributions [13].

The rotational self-friction tensors are of the following form:

$$\boldsymbol{\xi}^{RV}_{jj} = -\boldsymbol{\varepsilon}\colon \sum_{m=-1}^{1} (\mathbf{r}_j)_{-m} \left[\widetilde{\mathbf{T}}^{00}_{1m}(\mathbf{O}_j)\cdot \sum_{l\neq j} \mathbf{T}_{jl}\cdot\widetilde{\mathbf{T}}_l\cdot\mathbf{T}_{lj}\cdot\widetilde{\mathbf{T}}_j + \sum_{m_1} \widetilde{\mathbf{T}}^{1m_1}_{1m}(\mathbf{O}_j)\cdot \sum_{l\neq j} \mathbf{T}^{00}_{1m_1}(\mathbf{R}_{jl})\cdot\widetilde{\mathbf{T}}_l\cdot\mathbf{T}^1_{lj}\cdot\widetilde{\mathbf{T}}_j\right] + \ldots, \tag{4.10}$$

where

$$\widetilde{\mathbf{T}}^{1m_1}_{1m}(\mathbf{O}_j) = 6\sqrt{\pi}\mu a\widetilde{\mathbf{K}}^{1m_1}_{1m,00} + \ldots .$$

There is no Stokes contribution to the approximation considered. The inertial contributions consist of two terms of order $O(\sigma^2)$, due to two-sphere interactions.

It is seen from the properties of the tensors $\tilde{\mathbf{T}}^{00}_{1m}(\mathbf{O}_j)$ that

$$\text{(4.11)} \qquad \boldsymbol{\varepsilon}\colon \sum_{m=-1}^{1} (\mathbf{r}_j)_{-m} \tilde{\mathbf{T}}^{00}_{1m}(\mathbf{O}_j) \cdot \mathbf{V}_{j,00} \equiv 0.$$

Hence we have recovered the well known result that, due to the symmetry of the problem, the torque acting on a single sphere vanishes.

The rotational mutual friction tensors read:

$$\text{(4.12)} \qquad \boldsymbol{\xi}^{RV}_{jk} = -\boldsymbol{\varepsilon}\colon \sum_{m=-1}^{1} (\mathbf{r}_j)_{-m} \Bigg[-\sum_{m_1} \tilde{\mathbf{T}}^{1m_1}_{1m}(\mathbf{O}_j) \cdot \mathbf{T}^{00}_{1m_1}(\mathbf{R}_{jk}) \cdot \tilde{\mathbf{T}}_k - \tilde{\mathbf{T}}^{00}_{1m}(\mathbf{O}_j) \cdot \mathbf{T}_{jk} \cdot \tilde{\mathbf{T}}_k - \sum_{m_1} \tilde{\mathbf{T}}^{1m_1}_{1m}(\mathbf{O}_j) \cdot \mathbf{T}^{00}_{1m_1}(\mathbf{R}_{jk}) \cdot \tilde{\mathbf{T}}^1_k + \tilde{\mathbf{T}}^{00}_{1m}(\mathbf{O}_j) \cdot \sum_{l\neq k} \sum_{l\neq j} \mathbf{T}_{jl} \cdot \tilde{\mathbf{T}}_l \cdot \mathbf{T}_{lk} \cdot \tilde{\mathbf{T}}_k + \sum_{m_1} \tilde{\mathbf{T}}^{1m_1}_{1m}(\mathbf{O}_j) \cdot \sum_{l\neq k} \sum_{l\neq j} \mathbf{T}^{00}_{1m_1}(\mathbf{R}_{jl}) \cdot \tilde{\mathbf{T}}_l \cdot \mathbf{T}^1_{lk} \cdot \tilde{\mathbf{T}}_k \Bigg] + \ldots .$$

The Stokes contributions to the friction tensors, due to two-sphere interactions, are given by the first term, being of $O(\sigma^2)$. The remaining four terms describe the convective inertia effects, being of $O(\sigma^1)$ and $O(\sigma^2)$, respectively. They contain two- and three-sphere contributions. The non-additivity comes in at $O(\sigma^2)$.

To conclude: the weak convection effects enhance the hydrodynamic coupling of the spheres to the fluid. In the approximation considered, this enhancement consists in the following effects:

(i) The Stokes interactions involve not more than three spheres, the $O(\mathrm{Re})$ interactions – four spheres;

(ii) the non-additivity effects appear at $O(\sigma^2)$ for the Stokes régime, but at $O(\sigma^1)$ for the Oseen régime;

(iii) the tensors $\boldsymbol{\xi}^{RV}_{jk}$, vanishing for the Stokes flows, occur in the Oseen flows;

(iv) the contributions to $\boldsymbol{\xi}^{TV}_{jk}$, dependent on the angular, but not on the radial variables, absent in the case of Stokes interactions, are generated in case of the Oseen interactions.

5. Three-sphere effects

As an example, the forces acting on three rigidly held spheres are calculated for two particular configurations of the spheres. Consider first three spheres with the centreline perpendicular to the flow direction ($|\mathbf{R}_{12}| = |\mathbf{R}_{23}| = R$, $\mathbf{U} = (O, O, U)$). Up to the terms of the order of $O(\sigma)$, the hydrodynamic interaction tensors required read:

(i) the self-interaction tensors

$$(5.1)\qquad \begin{aligned} \widetilde{\mathbf{T}}_j &= 6\pi\mu a \mathbf{1},\\ \widetilde{\mathbf{T}}_j^1 &= 6\pi\mu a\left[\frac{3}{16}\mathrm{Re}(3\mathbf{e}_x\mathbf{e}_x + 3\mathbf{e}_y\mathbf{e}_y + 2\mathbf{e}_z\mathbf{e}_z)\right], \end{aligned}$$

(ii) the mutual interaction tensors ($\sigma \ll 1/2$, $\mathrm{Re}_m < 1$)

$$\mathbf{T}_{jk} = \frac{1}{6\pi\mu a}\sigma\frac{3}{4}\left[2\mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z\right], \quad j,k = 1,2,3,$$

$$(5.2)\qquad \begin{aligned} \mathbf{T}_{12}^1 = \mathbf{T}_{13}^1 = \mathbf{T}_{23}^1 &= -\frac{1}{6\pi\mu a}\frac{3}{16}\mathrm{Re}\left[3\mathbf{e}_x\mathbf{e}_x - \mathbf{e}_x\mathbf{e}_z + 3\mathbf{e}_y\mathbf{e}_y - \mathbf{e}_z\mathbf{e}_x + 2\mathbf{e}_z\mathbf{e}_z\right],\\ \mathbf{T}_{21}^1 = \mathbf{T}_{31}^1 = \mathbf{T}_{32}^1 &= -\frac{1}{6\pi\mu a}\frac{3}{16}\mathrm{Re}\left[3\mathbf{e}_x\mathbf{e}_x + \mathbf{e}_x\mathbf{e}_z + 3\mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_x + 2\mathbf{e}_z\mathbf{e}_z)\right], \end{aligned}$$

in the external Cartesian coordinate system $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$.

The obtained drag forces, exerted by the fluid on the spheres, are given by the following formulae:

(5.3)
(i) for the side spheres:

$$(F_1)_z = (F_3)_z = 6\pi\mu a U\left[1 + \frac{3}{8}\mathrm{Re} - \frac{9}{8}\sigma + \frac{3}{4}\mathrm{Re}\left(1 - \frac{57}{16}\sigma\right) + \ldots\right],$$

(ii) for the central sphere:

$$(F_2)_z = 6\pi\mu a U\left[1 + \frac{3}{8}\mathrm{Re} - \frac{3}{2}\sigma + \frac{3}{4}\mathrm{Re}\left(1 - \frac{33}{8}\sigma\right) + \ldots\right].$$

The inertial contributions to $(F_1)_z$ and $(F_3)_z$ are due to the following types of interactions:

(i) self-interaction of a single sphere: $3/8\mathrm{Re}$,
(ii) pair-wise interactions, independent of R: $3/4\mathrm{Re}$,
(iii) R-dependent pair-wise interactions: $-(108/64)\mathrm{Re}\sigma$,
(iv) non-additive interactions: $-(63/64)\mathrm{Re}\sigma$.

For the vector component $(F_2)_z$, the respective terms are qualitatively similar,

$$\frac{3}{8}\mathrm{Re}, \quad \frac{3}{4}\mathrm{Re}, \quad -\left(\frac{72}{32}\right)\mathrm{Re}\sigma \quad \text{and} \quad -\left(\frac{27}{32}\right)\mathrm{Re}\sigma.$$

In the expression (5.3), the first two terms describe the drag force, experienced by a single sphere; the remaining terms describe the decrease of the drag forces due to the hydrodynamic interactions between the spheres. The vector components $(F_i)_x$, $(F_i)_y$, $i = 1,2,3$, representing the side forces, read

$$(5.4)\qquad \begin{aligned} (F_1)_x &= -(F_3)_x = -6\pi\mu a U\left[\frac{3}{8}\mathrm{Re}\left(1 - \frac{15}{16}\sigma\right) + \ldots\right],\\ (F_2)_x &= 0,\\ (F_i)_y &= 0, \qquad i = 1,2,3. \end{aligned}$$

We note that the two side spheres are repelled. The side forces contain pair-wise contributions equal to $-3/8\mathrm{Re}$ ($3/8\mathrm{Re}$, respectively), and non-additive contributions, equal to $(45/128)\mathrm{Re}\sigma$ ($-(45/128)\mathrm{Re}\sigma$, respectively).

Let us now consider three spheres in line with the flow direction ($|\mathbf{R}_{12}| = |\mathbf{R}_{23}| = R$, $\mathbf{U} = (O, O, U)$).

Here the respective interaction tensors read:

(i) self-interaction tensors are given by the formulae (5.1),

(ii) mutual interaction tensors ($\sigma \ll 1/2$, $\mathrm{Re}_m < 1$):

$$\tag{5.5} \begin{aligned} \mathbf{T}_{jk} &= \frac{1}{6\pi\mu a}\sigma\frac{3}{4}\left[\mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y + 2\mathbf{e}_z\mathbf{e}_z\right], \\ \mathbf{T}^1_{12} &= \mathbf{T}^1_{13} = \mathbf{T}^1_{23} = -\frac{1}{6\pi\mu a}\frac{3}{8}\mathrm{Re}(3\mathbf{e}_x\mathbf{e}_x + 3\mathbf{e}_y\mathbf{e}_y + 2\mathbf{e}_z\mathbf{e}_z), \\ \mathbf{T}^1_{21} &= \mathbf{T}^1_{31} = \mathbf{T}^1_{32} = 0. \end{aligned}$$

The obtained drag forces are given by the formulae,

(5.6)

(i) for the leading sphere:

$$(F_1)_z = 6\pi\mu a U\left[1 + \frac{3}{8}\mathrm{Re} - \frac{9}{4}\sigma + \frac{3}{2}\mathrm{Re}\left(1 - \frac{9}{2}\sigma\right) + \ldots\right],$$

(ii) for the central sphere:

$$(F_2)_z = 6\pi\mu a U\left[1 + \frac{3}{8}\mathrm{Re} - 3\sigma + \frac{3}{4}\mathrm{Re}\left(1 - \frac{33}{4}\sigma\right) + \ldots\right],$$

(iii) for the rear sphere:

$$(F_3)_z = 6\pi\mu a U\left[1 + \frac{3}{8}\mathrm{Re} - \frac{9}{4}\sigma - \frac{63}{16}\mathrm{Re}\sigma + \ldots\right].$$

The inertial contributions to the drag forces, quadratic in the fluid velocity, are generated by:

(i) self-interactions of the respective spheres: $3/8\mathrm{Re}$,

(ii) pair-wise interactions, independent of R: $3/2\mathrm{Re}$, $3/4\mathrm{Re}$, 0, respectively,

(iii) R-dependent pair-wise interactions: $-27/8\mathrm{Re}\sigma$, $-9/2\mathrm{Re}\sigma$, $-27/8\mathrm{Re}\sigma$, respectively,

(iv) non-additive interactions: $-27/8\mathrm{Re}\sigma$, $-27/16\mathrm{Re}\sigma$, $-9/16\mathrm{Re}\sigma$, respectively.

Let us note the differentiation of the drag forces, exerted by the fluid on the spheres. The side forces are equal to zero, due to the symmetry of the considered spatial distribution.

The above examples illustrate the properties of the friction tensors $\boldsymbol{\xi}^{TV}_{jj}$ and $\boldsymbol{\xi}^{TV}_{jk}$ up to $O(\sigma^1)$. We note that the multisphere interactions, giving rise to the drag and side forces, cannot be described in a pair-wise additive scheme. The approximations in the Oseen equations can in principle be refined, by using the results presented in a series of papers by FINN [14], for a particular class of flows.

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Stiffness loss of laminates with aligned intralaminar cracks Part I. Macroscopic constitutive relations

T. LEWIŃSKI and J.J. TELEGA (WARSZAWA)

THE PAPER deals with analysis of reduction of the in-plane effective elastic moduli of the $[0^\circ_m/90^\circ_n]_s$ laminates weakened by aligned cross-cracks in the 90°-layer. A regular crack pattern is assumed. The case of dense crack distribution is modelled by (h_0, l_0) approach, while the case of arbitrary crack density c_d is described by a more accurate model (h_0, l). Both models have been derived in our paper [10]. Closed-form formulae describing decaying curves $E_1(c_d)$, $E_2(c_d)$, $\nu_{12}(c_d)$, $\nu_{21}(c_d)$, $G_{12}(c_d)$ are found by solution of the local problems for both models.

1. Introduction

CROSS-PLY LAMINATES of the $[0^\circ_m/90^\circ_n]_s$ type incur matrix cracking, interlaminar delamination and fibre breakage. The matrix cracks observed are straight or curved, cf. GROVES *et al.* [3]. The aim of the present paper is to assess the loss of effective elastic characteristics of the laminates with the straight matrix cracks going transversely through the whole thickness of the 90°-plies. The influence of crack curving as well as the onset of delamination is neglected. The cracks are assumed to be aligned. The present paper is mainly concerned with the case when these cracks are equally spaced. The assumption seems to be non-restrictive, since matrix cracks form usually regular patterns, cf. GARRETT and BAILEY [2], HIGHSMITH and REIFSNIDER [6], GROVES [4] and GROVES *et al.* [3]. The method (h_0, l) to be used has been proposed by us in [9, 10] and mathematically justified in TELEGA and LEWIŃSKI [15]. This method makes it possible to evaluate reduction of all components of the stiffness matrix of the three-layer balanced (transversely symmetric) laminates with transverse cracks in the internal layer.

The aim of this part of the paper is to find closed-form formulae interrelating effective Young's moduli $E^c{}_1$, $E^c{}_2$, effective Kirchhoff's modulus $G^c{}_{12}$ and Poisson's ratios $\nu^c{}_{\alpha\beta}$ with crack density c_d. The second part of the paper [11] is devoted to placing these results into the available literature of the subject as well as to compare the theoretical predictions of the (h_0, l) model with experimental data.

The following conventions are employed: small Greek indices (except for ε) run over 1, 2, while Latin ones (except for h) take values 1, 2, 3; h labels quantities of the homogenized description. Summation convention concerns repeated indices at different levels. Sometimes the same letter denotes an index and a parameter (e.g. α, β, γ, δ, etc. defined in the Appendix), which should not lead to ambiguities. The system of notations is compatible with that employed in LEWIŃSKI and TELEGA [9, 10]. Some auxiliary quantities are defined in the Appendix.

2. A laminate composed of orthotropic plies. The case of short cracks parallel to the axis x_2

The aim of this section is to exhibit simplifications in the homogenized description of in-plane deformations of the cracked laminate considered in [10] which take place when:

i) the plies are orthotropic, and

ii) the cracks weakening the internal layer are aligned.

Assume that the axes x_1, x_2 are axes of orthotropy. Cracks are parallel to the axis x_2, cf. Fig. 1. In view of the orthotropy assumption, the only non-zero components of the stiffness are

$$\begin{gathered} A_v^{\alpha\alpha\beta\beta},\ A_v^{\alpha\beta\alpha\beta};\ A_{vw}^{\alpha\alpha};\ A_{vu}^{\alpha\alpha\beta\beta},\ A_{vu}^{\alpha\beta\alpha\beta}, \\ A_{uw}^{\alpha\alpha},\ A_u^{\alpha\alpha\beta\beta},\ A_u^{\alpha\beta\alpha\beta};\ A_w; \qquad \alpha,\beta = 1,\ 2. \end{gathered} \tag{2.1}$$

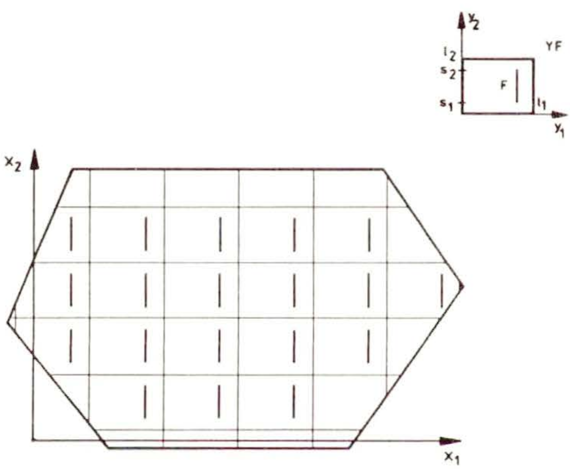

FIG. 1. Laminate with short aligned cracks.

Consequently, the only non-vanishing components of the tensors $A_\sigma^{\alpha\beta\lambda\mu}$, cf.([10], Eqs. (4.9)), are

$$A_1^{\alpha\alpha\beta\beta},\ A_2^{\alpha\alpha\beta\beta},\ A_1^{\alpha\beta\alpha\beta} = A_v^{\alpha\beta\alpha\beta},\ A_2^{\alpha\beta\alpha\beta} = A_{vu}^{\alpha\beta\alpha\beta}. \tag{2.2}$$

Note that $A_1^{1122} = A_1^{2211}$ but $A_2^{1122} \neq A_2^{2211}$.

The components of the vectors **N**, **T** are $(0,1)$ and $(1,0)$, respectively, (cf. [10], Fig. 1). According to the definition (4.7) given in [10] of the tensor of crack deformation measures $\boldsymbol{\varepsilon}^F$, one finds:

$$\varepsilon_{1\lambda}^F = \frac{1}{\lambda l_1 l_2}\int_{s_1}^{s_2} [\![u_\lambda^1]\!]\,dy_2\,, \qquad \varepsilon_{22}^F = 0, \qquad \lambda = 1,\ 2. \tag{2.3}$$

Thus, regardless of the type of the scaling, the homogenized constitutive relations have the form (cf. [10], Eqs. (4.5), (5.36))

$$
\begin{aligned}
N_h^{11} &= A_v^{1111}\left(\alpha_{11}\varepsilon_{11}^h + \alpha_{12}\varepsilon_{22}^h - \beta_{11}\varepsilon_{11}^F\right),\\
N_h^{22} &= A_v^{1111}\left(\alpha_{12}\varepsilon_{11}^h + \alpha_{22}\varepsilon_{22}^h - \beta_{21}\varepsilon_{11}^F\right),\\
N_h^{12} &= 2A_v^{1212}\left(\varepsilon_{12}^h - \widehat{\alpha}\varepsilon_{12}^F\right),
\end{aligned}
\tag{2.4}
$$

where $\varepsilon_{\alpha\beta}^F = \varepsilon_{\alpha\beta}^F(\varepsilon_{11}^h, \varepsilon_{22}^h, \varepsilon_{12}^h)$; the coefficients involved in (2.4) are defined by Eqs. (A.1). We cannot expect that in general $\varepsilon_{\alpha\alpha}^F = \varepsilon_{\alpha\alpha}^F(\varepsilon_{11}^h, \varepsilon_{22}^h)$ and $\varepsilon_{12}^F = \varepsilon_{12}^F(\varepsilon_{12}^h)$, since the cracks considered are of a unilateral type.

3. Parallel cracks: effective characteristics according to the (h_0, l_0) approach

From now onward we shall deal with a laminate composed of orthotropic plies and weakened by straight-line cracks in the internal layer, lying at equal distances l. The crack lines coincide with $x_1 = nl$ lines ($n = 1, 2, \ldots$), cf. Fig. 2. The aim of this section is to find effective stiffnesses of the laminate considered resulting from the (h_0, l_0) method discussed in ref. [10], Sec. 4. This method follows from the in-plane scaling: $h \to h$, $l_\alpha \to \varepsilon l_\alpha$, and hence it will also be called the in-plane scaling approach. Results of this model apply for laminates with cracks of high density. Predictions of the model will be independent of the value of the l/h ratio.

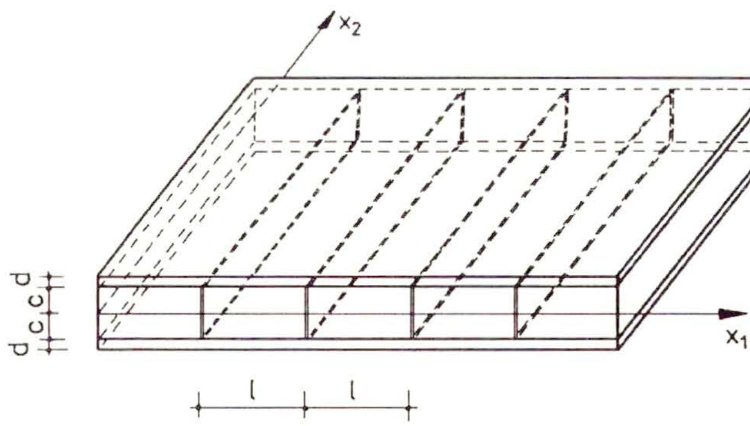

FIG. 2. Laminate with infinite aligned cracks.

The key to homogenization is to find solutions to the local problem (P_{loc}°) formulated in Ref. [10], Sec. 4. Geometry of the periodicity cell is here simple, cf. Fig. 3. For the sake of simplicity, the crack F is located so that it divides the cell into two equal parts.

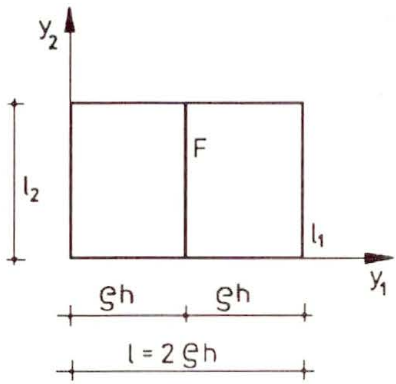

FIG. 3. Basic cell of periodicity.

Since the crack F lies along the axis y_2, one can guess that the solution to the basic cell problem (P°_{loc}) of Sec. 4 in [10] does not depend on y_2. Similar problem has been solved by LEWIŃSKI and TELEGA [8], hence only the outline of the derivation will be given here.

The unknown fields of (P°_{loc}) are $v^1_1(y_1)$, $u^1_1(y_1)$ and $v^1_2(y_1)$, $u^1_2(y_1)$. It turns out that these two pairs of functions are solutions of the independent (decoupled) stretching and shearing problems.

3.1. Solution of the local stretching problem. Stiffnesses $A_c^{\alpha\alpha\beta\beta}$

The unknown fields are $v^1_1(y_1)$ and $u^1_1(y_1)$. Let $\xi = y_1/h$ be a non-dimensional variable; $\xi \in (0, 2\varrho)$; $2\varrho = l/h$. Analysis of the local equations of (P°_{loc}) shows that both unknown fields are piece-wise linear in ξ, i.e.

$$v^1_1 = \begin{cases} c_1\xi + c_2, \\ D_1\xi + D_2, \end{cases} \qquad u^1_1 = \begin{cases} E_1\xi + E_2, & \xi \in (0, \varrho), \\ F_1\xi + F_2, & \xi \in (\varrho, 2\varrho). \end{cases} \tag{3.1}$$

The stress resultants (cf. [10], Eqs. (4.11)) are given by

$$\begin{aligned} N^{11}_0 &= \frac{1}{h}\left[A^{1111}_1 \frac{dv^1_1}{d\xi} + A^{1111}_2 \frac{du^1_1}{d\xi}\right] + n^{11}_0, \\ L^{11}_0 &= \frac{1}{h}\left[A^{1111}_2 \frac{dv^1_1}{d\xi} + A^{1111}_4 \frac{du^1_1}{d\xi}\right] + l^{11}_0, \end{aligned} \tag{3.2}$$

where n^{11}_0, l^{11}_0 are defined by Eqs. (4.14) in Ref. [10].

The constants c_α, D_α, E_α, F_α are interrelated according to:

- periodicity conditions

$$\begin{aligned} v^1_1(0) &= v^1_1(2\varrho), & u^1_1(0) &= u^1_1(2\varrho), \\ N^{11}_0(0) &= N^{11}_0(2\varrho), & L^{11}_0(0) &= L^{11}_0(2\varrho); \end{aligned} \tag{3.3}$$

• switching and contact conditions

$$v_1^1(\varrho - 0) = v_1^1(\varrho + 0), \qquad N_0^{11}(\varrho - 0) = N_0^{11}(\varrho + 0),$$

(3.4)

$$L_0^{11}(\varrho - 0) = L_0^{11}(\varrho + 0) \leq 0, \qquad L_0^{11}(\varrho - 0)[\![u_1^1]\!] = 0,$$

(3.5)
$$[\![u_1^1]\!] = u_1^1(\varrho + 0) - u_1^1(\varrho - 0) \geq 0.$$

Analysis of the above conditions leads to

$$\varepsilon_{11}^y(\mathbf{v}^1) = 0, \qquad \langle \gamma_{11}^y(\mathbf{u}^1) \rangle = \begin{cases} 0 & \text{for } l_0^{11} \leq 0, \\ -\dfrac{l_0^{11}}{A_4^{1111}} & \text{for } l_0^{11} > 0. \end{cases}$$

Here $\langle \cdot \rangle = \dfrac{1}{l} \displaystyle\int_0^l (\cdot) dy_1$. Since $\mathbf{u}^1$ is periodic, cf. Eq. (3.3)$_2$, one can make use of the relation: $\langle \gamma_{11}^y(\mathbf{u}^1) \rangle = -[\![u_1^1]\!]/l$.

Hence we find a non-zero component of the crack deformation tensor (2.3)

(3.6)
$$\varepsilon_{11}^F := \frac{[\![u_1^1]\!]}{l} = \begin{cases} 0, & \text{if } E_h \leq 0 \ \text{(the crack is closed)}, \\ F_{11}^0 E_h, & \text{if } E_h > 0 \ \text{(the crack is open)}, \end{cases}$$

where, cf. Eq. (4.14)$_2$ in Ref. [10]

(3.7)
$$F_{11}^0 = A_v^{1111}/A_4^{1111}, \qquad E_h = l_0^{11}/A_v^{1111},$$
$$l_0^{11} = A_3^{1111}\varepsilon_{11}^h + A_3^{1122}\varepsilon_{22}^h,$$

or, using relations (Ref. [10], Eq. (4.12)$_2$, (4.12)$_1$), (A.1) we can write

(3.8)
$$F_{11}^0 = [\gamma - (\lambda_1)^2/\mu]^{-1}, \qquad E_h = \beta_{11}\varepsilon_{11}^h + \beta_{21}\varepsilon_{22}^h.$$

According to the formula (4.5) in Ref. [10] and Eqs. (2.4), we arrive at the homogenized constitutive relationships for axial stress resultants

(3.9)
$$N_h^{\alpha\alpha} = \begin{cases} A_1^{\alpha\alpha 11}\varepsilon_{11}^h + A_1^{\alpha\alpha 22}\varepsilon_{22}^h, & \text{for } E_h \leq 0, \\ A_c^{\alpha\alpha 11}\varepsilon_{11}^h + A_c^{\alpha\alpha 22}\varepsilon_{22}^h, & \text{for } E_h > 0, \end{cases}$$

where

(3.10)
$$A_c^{\alpha\alpha\beta\beta} = A_1^{\alpha\alpha\beta\beta} - A_2^{\alpha\alpha 11}A_3^{11\beta\beta}(A_4^{1111})^{-1}.$$

Relations (3.9) are continuous along the line $E_h = 0$.

By virtue of the symmetry relations (cf. [10], Eqs. $(4.9)_1$ and $(4.12)_1$)

$$A_1^{\alpha\beta\lambda\mu} = A_1^{\lambda\mu\alpha\beta}, \qquad A_2^{\alpha\beta\lambda\mu} = A_3^{\lambda\mu\alpha\beta}, \tag{3.11}$$

we have

$$A_c^{\alpha\alpha\beta\beta} = A_c^{\beta\beta\alpha\alpha}, \tag{3.12}$$

hence the symmetry $A_c^{2211} = A_c^{1122}$ holds also when the crack is open.

It turns out to be helpful to write the components $A_c^{\alpha\alpha\beta\beta}$ in the form

$$A_c^{\lambda\lambda\mu\mu}/A_v^{1111} = \alpha_{\lambda\mu} - \beta_{\lambda 1}\beta_{\mu 1}F_{11}^0, \tag{3.13}$$

the non-dimensional coefficients $\alpha_{\lambda\mu}$ and $\beta_{\lambda\mu}$ being defined in the Appendix.

In the "technical" notation relationships (3.9) should be rewritten as follows:

- in the case of closed cracks ($E_h \le 0$)

$$\begin{bmatrix} N_h^{11} \\ N_h^{22} \end{bmatrix} = \frac{2h}{1-\nu_{12}\nu_{21}} \begin{bmatrix} E_1 & \nu_{12}E_1 \\ \nu_{21}E_2 & E_2 \end{bmatrix} \begin{bmatrix} \varepsilon_{11}^h \\ \varepsilon_{22}^h \end{bmatrix}, \tag{3.14}$$

- in the case of open cracks ($E_h > 0$)

$$\begin{bmatrix} N_h^{11} \\ N_h^{22} \end{bmatrix} = \frac{2h}{1-\nu_{12}^c\nu_{21}^c} \begin{bmatrix} E_1^c & \nu_{12}^c E_1^c \\ \nu_{21}^c E_2^c & E_2^c \end{bmatrix} \begin{bmatrix} \varepsilon_{11}^h \\ \varepsilon_{22}^h \end{bmatrix}. \tag{3.15}$$

The orthotropic constants for the case of closed cracks are given by

$$\nu_{12} = \frac{A_1^{1122}}{A_1^{1111}}, \qquad \nu_{21} = \frac{A_1^{1122}}{A_1^{2222}}, \qquad E_\alpha = (1-\nu_{12}\nu_{21})\frac{A_1^{\alpha\alpha\alpha\alpha}}{2h}. \tag{3.16}$$

The components ν_{12}^c, ν_{21}^c and E_α^c are defined in terms of $A_c^{\alpha\alpha\beta\beta}$ in a similar manner.

3.2. Solution of the shearing local problem. Stiffness A_c^{1212}

It turns out that the unknown fields v_2^1, u_2^1 are piecewise linear functions:

$$v_2^1 = \begin{cases} c_1\xi + c_2, \\ D_1\xi + D_2, \end{cases} \qquad u_2^1 = \begin{cases} E_1\xi + E_2, & \xi \in (0,\varrho), \\ F_1\xi + F_2, & \xi \in (\varrho, 2\varrho). \end{cases} \tag{3.17}$$

The stress resultants, cf. ([10], Eqs. (4.11)–(4.14))

$$\begin{aligned} N_0^{21} &= \frac{1}{h}\left[A_1^{2121}\frac{dv_2^1}{d\xi} + A_2^{2121}\frac{du_2^1}{d\xi}\right] + n_0^{21}, \\ L_0^{21} &= \frac{1}{h}\left[A_2^{2121}\frac{dv_2^1}{d\xi} + A_4^{2121}\frac{du_2^1}{d\xi}\right] + l_0^{21}, \end{aligned} \tag{3.18}$$

are piecewise constant. The periodicity conditions read:

$$(3.19)\qquad \begin{aligned} v_2^1(0) &= v_2^1(2\varrho), & u_2^1(0) &= u_2^1(2\varrho),\\ N_0^{21}(0) &= N_0^{21}(2\varrho), & L_0^{21}(0) &= L_0^{21}(2\varrho), \end{aligned}$$

while the switching conditions are

$$(3.20)\qquad \begin{aligned} v_2^1(\varrho-0) &= v_2^1(\varrho+0),\\ N_0^{21}(\varrho-0) &= N_0^{21}(\varrho+0), \qquad L_0^{21}(\varrho-0) = L_0^{21}(\varrho+0). \end{aligned}$$

Using (3.17)–(3.20) one finds

$$\varepsilon_{21}^{y}(\mathbf{v}^1) = 0, \qquad \langle \gamma_{21}^{y}(\mathbf{u}^1)\rangle = -\frac{l_0^{21}}{2A_4^{2121}}\,.$$

Due to the orthotropy we have $A_4^{2121} = A_2^{2121}$. By virtue of the relation $l_0^{21} = 2A_2^{2121}\varepsilon_{12}^{h}$ and Eq. (4.6) in [10], one finds

$$(3.21)\qquad \varepsilon_{12}^{F} = \varepsilon_{12}^{h}\,.$$

Therefore the homogenized constitutive relation $(2.4)_3$ assumes the form

$$(3.22)\qquad N_h^{12} = 2A_c^{1212}\varepsilon_{12}^{h}\,, \qquad A_c^{1212}/A_v^{1212} = 1-\widehat{\alpha}\,,$$

where $\widehat{\alpha}$ is defined by $(A.1)_5$. According to (A.2) we have $1-\widehat{\alpha} = d/h$. The effective Kirchhoff moduli of the cracked and uncracked laminate are

$$(3.23)\qquad G_{12} = A_v^{1212}/2h, \qquad G_{12}^{c} = A_c^{1212}/2h.$$

3.3. The homogenized potential

Having found relationships (3.9) and (3.22), one can express the homogenized constitutive relations in the hyperelastic form, cf. [10], Eq. (4.18)

$$(3.24)\qquad N_h^{\alpha\beta} = \frac{\partial \mathcal{V}_h}{\partial \varepsilon_{\alpha\beta}^{h}}\,,$$

the potential $\mathcal{V}_h$ being defined by

$$(3.25)\qquad \mathcal{V}_h = \begin{cases} \mathcal{V}_h^0 & \text{for} \quad E_h \le 0,\\ \mathcal{V}_h^c & \text{for} \quad E_h \ge 0, \end{cases}$$

where, cf. [10], Eq. (4.22)

$$(3.26)\qquad \begin{aligned} 2\mathcal{V}_h^0 &= \sum_{\alpha,\beta} A_1^{\alpha\alpha\beta\beta}\varepsilon_{\alpha\alpha}^{h}\varepsilon_{\beta\beta}^{h} + 2A_c^{1212}\left[(\varepsilon_{12}^{h})^2 + (\varepsilon_{21}^{h})^2\right],\\ 2\mathcal{V}_h^c &= \sum_{\alpha,\beta} A_c^{\alpha\alpha\beta\beta}\varepsilon_{\alpha\alpha}^{h}\varepsilon_{\beta\beta}^{h} + 2A_c^{1212}\left[(\varepsilon_{12}^{h})^2 + (\varepsilon_{21}^{h})^2\right]. \end{aligned}$$

The formula $(3.26)_2$ can be rewritten as follows

$$\mathcal{V}_h^c = \mathcal{V}_h^0 - \frac{1}{2} A_v^{1111} F_{11}^0 (E_h)^2. \tag{3.27}$$

By virtue of (3.27) one can readily prove that $\mathcal{V}_h$ is of class C^1 (not C^2), the result already known from Sec. 4 of Ref. [10]. We see that the line $E_h = 0$, cf. $(3.8)_2$,

$$\beta_{11}\varepsilon_{11}^h + \beta_{21}\varepsilon_{22}^h = 0 \tag{3.28}$$

is a line of discontinuity of the second order derivatives of the potential $\mathcal{V}_h$, cf. Fig. 4a. This figure characterizes the $[0°/90_3°]_s$ glass/epoxy laminate examined in detail in Sec. 3.1 of the second part of the present paper [11].

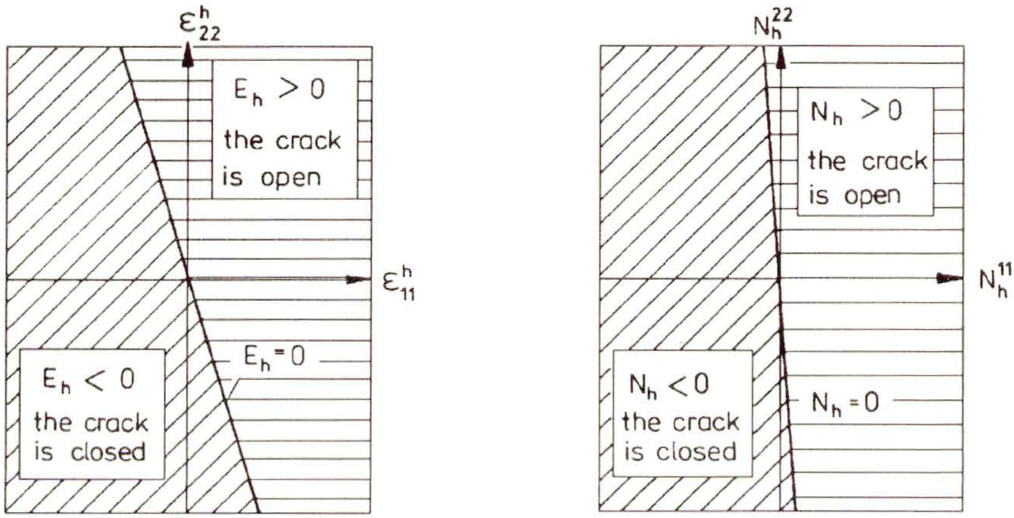

FIG. 4. a. $(\varepsilon_{11}^h, \varepsilon_{22}^h)$-plane; condition $E_h > 0$ for the crack opening in the glass/epoxy $[0°, 90_3°]_s$ laminate tested by HIGHSMITH and REIFSNIDER [6]; $E_h = 0.4298\varepsilon_{11}^h + 0.1286\varepsilon_{22}^h$. b. (N_h^{11}, N_h^{22})-plane. Condition $N_h > 0$ for the crack opening; the same laminate, $N_h = 0.632N_h^{11} + 0.0408N_h^{22}$.

3.4. Inverted form of the homogenized constitutive relations

The constitutive relations (3.24) can be inverted. We shall now find this inverse form. The main problem reduces to inverting relations (3.9). For this purpose we introduce here matrix notation.

Let us define the following vectors and matrices

$$\mathbf{k} = [\beta_{11}, \beta_{21}], \qquad \mathbf{k}_\perp = [-\beta_{21}, \beta_{11}], \qquad \mathbf{C} = \begin{bmatrix} \mathbf{k} \\ \mathbf{k}_\perp \end{bmatrix},$$

$$\mathbf{A}_m = \begin{bmatrix} A_m^{1111} & A_m^{1122} \\ A_m^{1122} & A_m^{2222} \end{bmatrix}, \qquad m = 1 \quad \text{or} \quad c, \tag{3.29}$$

$$\boldsymbol{\varepsilon} = \left[\varepsilon_{11}^h, \varepsilon_{22}^h\right]^T, \qquad \mathbf{N} = \left[N_h^{11}, N_h^{22}\right]^T.$$

The constitutive relations (3.9) can be written as follows

$$\mathbf{N} = \begin{cases} \mathbf{A}_1 \boldsymbol{\varepsilon}, & \text{for} \quad \mathbf{k} \cdot \boldsymbol{\varepsilon} \le 0, \\ \mathbf{A}_c \boldsymbol{\varepsilon}, & \text{for} \quad \mathbf{k} \cdot \boldsymbol{\varepsilon} \ge 0. \end{cases} \tag{3.30}$$

This relation is continuous because $\mathcal{V}_h$ is of class C^1; hence

$$\mathbf{A}_1 \mathbf{k}_\perp^T = \mathbf{A}_c \mathbf{k}_\perp^T . \tag{3.31}$$

Let us set $\mathbf{e} = \mathbf{C}\boldsymbol{\varepsilon}$, $\mathbf{e} = (e_1, e_2)$. The inverse relation reads

$$\boldsymbol{\varepsilon} = \mathbf{C}^{-1}\mathbf{e}, \qquad \mathbf{C}^{-1} = \frac{1}{\det \mathbf{C}} \mathbf{C}^T. \tag{3.32}$$

Hence (3.30) assumes the form

$$\mathbf{N} = \begin{cases} \mathbf{B}_1 \mathbf{e}, & \text{for} \quad e_1 \le 0, \\ \mathbf{B}_c \mathbf{e}, & \text{for} \quad e_1 \ge 0, \end{cases} \tag{3.33}$$

where

$$\mathbf{B}_1 = \mathbf{A}_1 \mathbf{C}^{-1}, \qquad \mathbf{B}_c = \mathbf{A}_c \mathbf{C}^{-1}. \tag{3.34}$$

Consequently

$$\mathbf{e} = \begin{cases} \mathbf{D}_1 \mathbf{N}, & \text{for} \quad e_1 \le 0, \\ \mathbf{D}_c \mathbf{N}, & \text{for} \quad e_1 \ge 0, \end{cases} \tag{3.35}$$

where

$$\mathbf{D}_1 = \mathbf{C}\mathbf{A}_1^{-1}, \qquad \mathbf{D}_c = \mathbf{C}\mathbf{A}_c^{-1}. \tag{3.36}$$

Our aim is to express conditions $e_1 < 0$ or $e_1 > 0$ in terms of $\mathbf{N}$. To this end let us define a new vector $\mathcal{E} = \mathbf{D}_1\mathbf{N}$. This definition does not depend on the sign of e_1. We express (3.35) in terms of $\mathcal{E}$:

$$\mathbf{e} = \begin{cases} \mathcal{E}, & \text{for} \quad e_1 \le 0, \\ \mathbf{P}\mathcal{E}, & \text{for} \quad e_1 \ge 0, \end{cases} \tag{3.37}$$

where

$$\mathbf{P} = \mathbf{D}_c \mathbf{D}_1^{-1} \qquad \text{or} \qquad \mathbf{P} = \frac{1}{\det \mathbf{C}} \mathbf{C}\mathbf{A}_c^{-1}\mathbf{A}_1\mathbf{C}^T. \tag{3.38}$$

One can prove that

$$P_{11} = \frac{\det \mathbf{A}_1}{\det \mathbf{A}_c} \qquad \text{and} \qquad P_{12} = 0. \tag{3.39}$$

The last equality is crucial here. It is a consequence of continuity requirements (3.31). The relation $(3.39)_1$ implies $P_{11} > 0$. Hence we conclude that

$$e_1 = \begin{cases} \mathcal{E}_1, & \text{for} \quad e_1 \le 0, \\ P_{11}\mathcal{E}_1, & \text{for} \quad e_1 \ge 0, \quad P_{11} > 0. \end{cases} \tag{3.40}$$

The relations given above show that sign e_1 = sign $\mathcal{E}_1$, which makes it possible to rewrite (3.37) in the form

$$\mathbf{C}\boldsymbol{\varepsilon} = \begin{cases} \mathbf{D}_1\mathbf{N}, & \text{for} \quad \mathcal{E}_1 \le 0, \\ \mathbf{D}_c\mathbf{N}, & \text{for} \quad \mathcal{E}_1 \ge 0, \end{cases} \tag{3.41}$$

and, finally, to find

$$\boldsymbol{\varepsilon} = \begin{cases} \mathbf{A}_1^{-1}\mathbf{N}, & \text{for} \quad \mathcal{E}_1 \le 0, \\ \mathbf{A}_c^{-1}\mathbf{N}, & \text{for} \quad \mathcal{E}_1 \ge 0. \end{cases} \tag{3.42}$$

The condition $\mathcal{E}_1 \le 0$ can be expressed as $N_h \le 0$, where

$$N_h = (\beta_{11}\alpha_{22} - \beta_{21}\alpha_{21})N_h^{11} + (\alpha_{11}\beta_{21} - \beta_{11}\alpha_{12})N_h^{22}, \tag{3.43}$$

and sign N_h = sign E_h.

The inverted form of the homogenized relations (3.9) is

$$\varepsilon^h_{\alpha\alpha} = \begin{cases} \dfrac{1}{2hE_\alpha}(N_h^{\alpha\alpha} - \nu_{\beta\alpha}N_h^{\beta\beta}), & \text{for} \quad N_h \le 0, \\ \dfrac{1}{2hE^c_\alpha}(N_h^{\alpha\alpha} - \nu^c_{\beta\alpha}N_h^{\beta\beta}), & \text{for} \quad N_h \ge 0, \end{cases} \tag{3.44}$$

and $\beta = 3 - \alpha$; do not sum over α and β!

Recalling relations (4.22) of Ref. [10] and (3.22) one can easily express $\mathcal{V}_h$ in terms of $N_h^{\alpha\beta}$. Its line of discontinuity of its second order derivatives is $N_h = 0$, cf. Fig. 4b. The data for this figure were taken for the laminate considered in Sec. 3.1 of Ref. [11].

REMARK 3.1

The considerations of Section 3 may be viewed as a practical procedure for finding the complementary or dual effective potential $\mathcal{V}_h^*$. Detailed study of duality is provided by our mathematical paper (TELEGA and LEWIŃSKI [15]). Nevertheless it is worth noting that $\mathcal{V}_h^*$ may be determined as the Fenchel conjugate of $\mathcal{V}_h$, i.e.

$$\mathcal{V}_h^*(\mathbf{E}^*) = \sup\left\{E^{*\alpha\beta}E_{\alpha\beta} - \mathcal{V}_h(\mathbf{E}) \mid \mathbf{E} \in \mathbb{E}^2_s\right\}, \qquad \mathbf{E}^* \in \mathbb{E}^2_s. \tag{3.45}$$

The complementary potential $\mathcal{V}_h^*$ is strictly convex, of class C^1 and

$$\boldsymbol{\varepsilon}^h = \frac{\partial \mathcal{V}_h^*}{\partial \mathbf{N}^h}, \qquad \boldsymbol{\varepsilon}^h \in \mathbb{E}^2_s, \qquad \mathbf{N}^h \in \mathbb{E}^2_s. \tag{3.46}$$

4. Parallel cracks: the space-scaling homogenization approach – model (h_0, l)

The aim of this section is to find effective characteristics of the laminate of Fig. 2 according to the (h_0, l) model of Sec. 5, Ref. [10]. The predictions of the loss of effective stiffnesses found in this section involve the l/h ratio and apply for arbitrary values of this ratio.

Similarly as in the in-plane scaling method, the local problem (P^2_{loc}) (formulated in Sec. 5.2 of Ref. [10]) splits up into two: stretching and shearing problems. The unknown functions depend solely on $y_1 = h\xi$.

4.1. Solution of the stretching local problem

The unknown functions are v^1_1, u^1_1 and w^2. The non-vanishing stress resultants are given by Eqs. (5.20) of Ref. [10]; they assume the form

$$
\begin{aligned}
N^{11}_0 &= A^{1111}_v(v' + \alpha u' + \beta w) + n^{11}_0,\\
N^{22}_0 &= A^{1111}_v(\beta_1 v' + \beta_4 u' + \beta_2 w) + n^{22}_0,\\
L^{11}_0 &= A^{1111}_v(\alpha v' + \gamma u' + \lambda_1 w) + l^{11}_0,\\
L^{22}_0 &= A^{1111}_v(\beta_4 v' + \gamma_3 u' + \beta_3 w) + l^{22}_0,\\
R_0 &= \frac{1}{h^2} A^{1111}_v(\beta v' + \lambda_1 u' + \mu w),\\
Q^1_0 &= \frac{\delta}{h} A^{1111}_v(u - w'), \qquad (\cdot)' = d(\cdot)/d\xi\,,
\end{aligned}
\tag{4.1}
$$

where new unknowns have been introduced

$$
v = v^1_1/h\,, \qquad u = u^1_1/h\,, \tag{4.2}
$$

$$
w = w^2/h^2 + w_0\,, \qquad w_0 = \frac{1}{\mu}\left(\beta\varepsilon^h_{11} + \beta_2\varepsilon^h_{22}\right). \tag{4.3}
$$

The quantities $n^{\alpha\alpha}_0$, $l^{\alpha\alpha}_0$ are defined by Eqs. (4.14) of Ref. [10]. The new coefficients involved in (4.1)–(4.3) are defined by (A.1).

The equilibrium equations reduce to the form

$$
\frac{dN^{11}_0}{d\xi} = 0, \qquad \frac{dL^{11}_0}{d\xi} = hQ^1_0, \qquad \frac{dQ^1_0}{d\xi} = -hR_0\,. \tag{4.4}
$$

On expressing the equilibrium equations (4.4) in terms of the unknowns (v, u, w), one arrives at the following system of differential equations

$$
\begin{aligned}
v'' + \alpha u'' + \beta w' &= 0,\\
\alpha v'' + (\gamma u'' - \delta u) + \lambda w' &= 0,\\
-\beta v' - \lambda u' + (\delta w'' - \mu w) &= 0.
\end{aligned}
\tag{4.5}
$$

The strong formulation of the local problem amounts here to finding the fields (v, u, w) defined on the interval $[0, 2\varrho]$ such that:

- the equations (4.5) are satisfied for each $\xi \in (0, \varrho) \cup (\varrho, 2\varrho)$;
- the periodicity conditions

$$
\begin{aligned}
v(0) &= v(2\varrho), & u(0) &= u(2\varrho), & w(0) &= w(2\varrho),\\
N_0^{11}(0) &= N_0^{11}(2\varrho), & L_0^{11}(0) &= L_0^{11}(2\varrho), & Q_0^1(0) &= Q_0^1(2\varrho)
\end{aligned}
\tag{4.6}
$$

are satisfied;

- the switching conditions are fulfilled at $\xi = \varrho$

$$
\begin{aligned}
v(\varrho - 0) &= v(\varrho + 0), & w(\varrho - 0) &= w(\varrho + 0),\\
N_0^{11}(\varrho - 0) &= N_0^{11}(\varrho + 0), & Q_0^1(\varrho - 0) &= Q_0^1(\varrho + 0);
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
&L = L_0^{11}(\varrho - 0) = L_0^{11}(\varrho + 0) \leq 0,\\
&L[\![u]\!] = 0, \qquad [\![u]\!] = u(\varrho + 0) - u(\varrho - 0) \geq 0.
\end{aligned}
\tag{4.8}
$$

A detailed solution to the problem stated above will be given a little later. Suppose now that this solution is known. Similarly as in Sec. 3, the problem can be reduced to finding the field ε_{11}^F given by $(2.3)_1$. In the case considered here $s_1 = 0$, $s_2 = l_2$, $|Y| = l_1 l_2 = l l_2$; hence

$$
\tilde{\varepsilon}_{11}^F := \frac{[\![u_1^1]\!]}{l} = \frac{[\![u]\!]}{2\varrho}\,. \tag{4.9}
$$

The tilde over ε_{11}^F indicates that this quantity is evaluated by the (h_0, l) approach. Thus the only unknown which is really needed for assessing the loss of stiffnesses is the jump $[\![u]\!]$.

Let us proceed now to the analysis of the local problem. One can note first that the unknown w can be eliminated from Eqs. (4.5). One finds

$$
\begin{aligned}
\mu_{11}v'' + (\mu_{12}u'' + \mu_{13}u) &= 0,\\
(\mu_{21}v'' + \mu_{22}v) + (\mu_{23}u'' + \mu_{24}u) &= c_1\xi + c_2\,,
\end{aligned}
\tag{4.10}
$$

where $\mu_{\alpha k}$ are defined in the Appendix and c_α are arbitrary constants; $u'' = d^2u/d\xi^2$. The fields u and v satisfy the following uncoupled equations

$$
Lu = 0, \qquad Lv = \mu_{13}(c_1\xi + c_2), \tag{4.11}
$$

where

$$
L = a_1\frac{d^4}{d\xi^4} + a_2\frac{d^2}{d\xi^2} + a_3; \tag{4.12}
$$

the coefficients a_1, a_2 and a_3 are defined in the Appendix. Let $\pm\sigma$, $\pm\omega$ be the roots of the characteristic equation

$$a_1 x^4 + a_2 x^2 + a_3 = 0. \tag{4.13}$$

In general, σ and ω can assume real or complex values. In the latter case $\sigma = \overline{\omega}$; the bar denotes the complex conjugate.

Symmetries characterizing the problem imply that the fields (u, v) are antisymmetric with respect to the point $\xi = \varrho$. Thus we can write

$$u = \begin{cases} u_{\mathrm{I}}, \\ u_{\mathrm{II}}, \end{cases} \qquad v = \begin{cases} v_{\mathrm{I}}, & \xi \in (0, \varrho), \\ v_{\mathrm{II}}, & \xi \in (\varrho, 2\varrho), \end{cases} \tag{4.14}$$

where

$$\begin{aligned} u_{\mathrm{I}} &= B_1 e^{-\sigma\xi} + B_2 e^{-\sigma(\varrho-\xi)} + B_3 e^{-\omega\xi} + B_4 e^{-\omega(\varrho-\xi)}, \\ u_{\mathrm{II}} &= -B_1 e^{-\sigma(2\varrho-\xi)} - B_2 e^{-\sigma(\xi-\varrho)} - B_3 e^{-\omega(2\varrho-\xi)} - B_4 e^{-\omega(\xi-\varrho)}; \end{aligned} \tag{4.15}$$

$$\begin{aligned} v_{\mathrm{I}} &= D_1\xi + D_2 + G_1 e^{-\sigma\xi} + G_2 e^{-\sigma(\varrho-\xi)} + G_3 e^{-\omega\xi} + G_4 e^{-\omega(\varrho-\xi)}, \\ v_{\mathrm{II}} &= F_1\xi + F_2 - G_1 e^{-\sigma(2\varrho-\xi)} - G_2 e^{-\sigma(\xi-\varrho)} - G_3 e^{-\omega(2\varrho-\xi)} - G_4 e^{-\omega(\xi-\varrho)}, \end{aligned} \tag{4.16}$$

here B_i, G_i, D_α and F_α are unknown constants. The first equilibrium equation $(4.5)_1$ makes it possible to determine the function w, being equal to w_{I} for $\xi \in (0, \varrho)$ and w_{II} for $\xi \in (\varrho, 2\varrho)$

$$\begin{aligned} w_{\mathrm{I}} = K_1 &+ \frac{\sigma}{\beta}(\alpha B_1 + G_1)e^{-\sigma\xi} - \frac{\sigma}{\beta}(\alpha B_2 + G_2)e^{-\sigma(\varrho-\xi)} \\ &+ \frac{\omega}{\beta}(\alpha B_3 + G_3)e^{-\omega\xi} - \frac{\omega}{\beta}(\alpha B_4 + G_4)e^{-\omega(\varrho-\xi)}, \\ w_{\mathrm{II}} = L_1 &+ \frac{\sigma}{\beta}(\alpha B_1 + G_1)e^{-\sigma(2\varrho-\xi)} - \frac{\sigma}{\beta}(\alpha B_2 + G_2)e^{-\sigma(\xi-\varrho)} \\ &+ \frac{\omega}{\beta}(\alpha B_3 + G_3)e^{-\omega(2\varrho-\xi)} - \frac{\omega}{\beta}(\alpha B_4 + G_4)e^{-\omega(\xi-\varrho)}. \end{aligned} \tag{4.17}$$

Having found the formulae (4.14)–(4.17) one can express the stress resultants N_0^{11}, L_0^{11}, Q_0^1 in terms of the functions involved in (4.15)–(4.17) and unknown constants D_α, F_α, K_1, L_1, B_i, G_i; $i = 1, 2, 3, 4$; $\alpha = 1, 2$. We shall omit details of the evaluation of these constants and report only the final results. The relative opening of the crack $\tilde{\varepsilon}_{11}^F$ defined by Eq. (4.9) depends on $\operatorname{sign} l_0^{11} = \operatorname{sign} E_h$:

$$\tilde{\varepsilon}_{11}^F = \begin{cases} 0, & \text{for} \quad E_h \le 0, \\ F_{11}(\varrho)E_h, & \text{for} \quad E_h > 0, \end{cases} \tag{4.18}$$

where E_h has been defined by Eq. $(3.8)_2$ and the function $F_{11}(\varrho)$ has the form

$$F_{11}(\varrho) = f_{11}\left[\frac{\beta_{11}}{\sigma^2\omega^2}g_1(\sigma,\omega) + g_2(\sigma,\omega)F(\varrho;\omega,\sigma)\right]^{-1}, \tag{4.19}$$

where

$$g_\alpha(\sigma,\omega) = \gamma_{\alpha 1} + \gamma_{\alpha 2}\sigma^2\omega^2 + \gamma_{\alpha 3}(\sigma^2+\omega^2) \tag{4.20}$$

and

$$F(\varrho;\omega,\sigma) = \frac{\varrho}{\sigma^2-\omega^2}\left(\frac{\operatorname{cth}\omega\varrho}{\omega} - \frac{\operatorname{cth}\sigma\varrho}{\sigma}\right). \tag{4.21}$$

Parameters $\gamma_{\alpha k}$, f_{11} and β_{11}, depending on the geometry and material properties of the laminate, are defined in the Appendix.

Note that the function $F_{11}(\varrho)$ preserves its form after the change: $\sigma \to \omega$, $\omega \to \sigma$; moreover, g_α do not depend on whether σ and ω are real- or complex-valued. In fact, in view of (4.13)

$$\sigma^2+\omega^2 = -a_2/a_1\,, \qquad \sigma^2\omega^2 = a_3/a_1\,. \tag{4.22}$$

If $\sigma = p - iq$, $\omega = \overline{\sigma} = p + iq$ $(p, q \in \mathbb{R})$ we change the definition

$$F(\varrho;\omega,\sigma) = F_0(\varrho;p,q). \tag{4.23}$$

After appropriate manipulations we find

$$F_0(\varrho;p,q) = \frac{f(p\varrho,q\varrho)}{2pq(p^2+q^2)}, \tag{4.24}$$

where the function f is defined by

$$f(x,y) = \frac{y\operatorname{sh}2x + x\sin 2y}{\operatorname{ch}2x - \cos 2y}. \tag{4.25}$$

4.2. Assessing loss of the $\widetilde{A}_c^{\alpha\alpha\beta\beta}$ stiffnesses

Having found the relation $\tilde{\varepsilon}_{11}^F(\boldsymbol{\varepsilon}^h)$ one can determine the homogenized constitutive relationships via Eqs. (2.4)

$$N_h^{\alpha\alpha} = \begin{cases} A_1^{\alpha\alpha 11}\varepsilon_{11}^h + A_1^{\alpha\alpha 22}\varepsilon_{22}^h, & \text{for} \quad E_h \le 0, \\ \widetilde{A}_c^{\alpha\alpha 11}\varepsilon_{11}^h + \widetilde{A}_c^{\alpha\alpha 22}\varepsilon_{22}^h, & \text{for} \quad E_h > 0, \end{cases} \tag{4.26}$$

where the reduced stiffnesses can be expressed by a single formula

$$\widetilde{A}_c^{\lambda\lambda\mu\mu}/A_v^{1111} = \alpha_{\lambda\mu} - \beta_{\lambda 1}\beta_{\mu 1}F_{11}(\varrho), \tag{4.27}$$

and the coefficients $\alpha_{\lambda\mu}$ and $\beta_{\lambda\mu}$ are defined by Eqs. (A.1). The relations (4.26) are continuous along the line $E_h = 0$.

The constitutive relationship (4.26) can be expressed in terms of orthotropic constants. For the case of closed cracks ($E_h \leq 0$) these relations have the form (3.14), and for the case of open cracks ($E_h > 0$) they assume the form

$$\begin{bmatrix} N_h^{11} \\ N_h^{22} \end{bmatrix} = \frac{2h}{1 - \widetilde{\nu}_{12}^c \widetilde{\nu}_{21}^c} \begin{bmatrix} \widetilde{E}_1^c & \widetilde{\nu}_{12}^c \widetilde{E}_1^c \\ \widetilde{\nu}_{21}^c \widetilde{E}_2^c & \widetilde{E}_2^c \end{bmatrix} \begin{bmatrix} \varepsilon_{11}^h \\ \varepsilon_{22}^h \end{bmatrix}, \tag{4.28}$$

where

$$\begin{aligned} \widetilde{\nu}_{12}^c &= \frac{\widetilde{A}_c^{1122}}{\widetilde{A}_c^{1111}}, \qquad \widetilde{\nu}_{21}^c = \frac{\widetilde{A}_c^{1122}}{\widetilde{A}_c^{2222}}, \\ \widetilde{E}_\alpha^c &= (1 - \widetilde{\nu}_{12}^c \widetilde{\nu}_{21}^c) \frac{\widetilde{A}_c^{\alpha\alpha\alpha\alpha}}{2h}. \end{aligned} \tag{4.29}$$

The formula for $\widetilde{E}_1^c(\varrho)$ does not coincide with the analogous formula found by HASHIN [5], although one can note a similarity between the formulae (2.40) and (2.46) of HASHIN [5] and formulae (4.27) for $\lambda = \mu = 1$, (4.19) and (4.24) derived above.

REMARK 4.1

The constitutive relations (4.26) can be inverted to the form similar to (3.44), where instead of E_α^c, $\nu_{\alpha\beta}^c$ one should put $\widetilde{E}_\alpha^c$, $\widetilde{\nu}_{\alpha\beta}^c$. The condition $N_h < 0$ or $N_h > 0$ remains unchanged.

4.3. Solution of the shearing local problem

The dimensionless fields

$$\widehat{u} = u_2^1/h, \qquad \widehat{v} = v_2^1/h, \tag{4.30}$$

will play the role of basic unknowns. The stress resultants that intervene in the shearing deformation are, cf. ([10], Eq. (5.20))

$$\begin{aligned} N_0^{21} &= A_v^{2121} \left(\frac{d\widehat{v}}{d\xi} + \widehat{\alpha} \frac{d\widehat{u}}{d\xi} + 2\varepsilon_{21}^h \right), \\ L_0^{21} &= \widehat{\alpha} A_v^{2121} \left(\frac{d\widehat{v}}{d\xi} + \frac{d\widehat{u}}{d\xi} + 2\varepsilon_{21}^h \right), \\ Q_0^2 &= \frac{\widehat{\delta}}{h} A_v^{2121} \widehat{u}. \end{aligned} \tag{4.31}$$

The equilibrium equations

$$\frac{dN_0^{21}}{d\xi} = 0, \qquad \frac{dL_0^{21}}{d\xi} = hQ_0^2, \tag{4.32}$$

expressed in terms of the unknowns (4.30) assume the form

$$\frac{d^2\widehat{v}}{d\xi^2} + \widehat{\alpha}\frac{d^2\widehat{u}}{d\xi^2} = 0, \qquad \widehat{\alpha}\frac{d^2\widehat{v}}{d\xi^2} + \left(\widehat{\alpha}\frac{d^2}{d\xi^2} - \widehat{\delta}\right)\widehat{u} = 0. \tag{4.33}$$

Further analysis will be confined to the case when the matrix

$$\begin{bmatrix} A_v^{2121} & A_{vu}^{2121} \\ A_{vu}^{2121} & A_u^{2121} \end{bmatrix} \tag{4.34}$$

is positive definite, which for real laminates is not a restriction. This means that

$$0 < \widehat{\alpha} < 1, \tag{4.35}$$

which is readily satisfied since $\widehat{\alpha} = c/h$, cf. (A.2). Let us pass to the strong formulation of the local problem considered. Our goal is to find the fields $(\widehat{v}, \widehat{u})$ defined on $[0, 2\varrho]$ and satisfying:

- the equations (4.33) for $\xi \in (0, \varrho)$ and $\xi \in (\varrho, 2\varrho)$,
- the conditions of periodicity

$$\begin{aligned} \widehat{v}(0) &= \widehat{v}(2\varrho), & \widehat{u}(0) &= \widehat{u}(2\varrho), \\ N_0^{21}(0) &= N_0^{21}(2\varrho), & L_0^{21}(0) &= L_0^{21}(2\varrho), \end{aligned} \tag{4.36}$$

- the switching conditions at $\xi = \varrho$

$$\begin{aligned} &\widehat{v}(\varrho - 0) = \widehat{v}(\varrho + 0), \qquad N_0^{21}(\varrho - 0) = N_0^{21}(\varrho + 0), \\ &L_0^{21}(\varrho - 0) = L_0^{21}(\varrho + 0) = 0. \end{aligned} \tag{4.37}$$

Prior to solving the local problem formulated above let us recall that the only field we need for assessing the loss of G_{12} is the quantity ε_{12}^F, cf. Eq. (2.3). Here

$$2\tilde{\varepsilon}_{12}^F := \frac{[\![u_2^1]\!]}{l} = \frac{[\![\widehat{u}]\!]}{2\varrho}. \tag{4.38}$$

The tilde indicates that we use the space-scaling (h_0, l) method. The homogenized constitutive relation has the form $(2.4)_3$ with ε_{12}^F defined by Eq. (4.38).

Bearing in mind that we are now interested only in finding the field $\tilde{\varepsilon}_{12}^F$, we proceed to analyze the local problem. Equations (4.33) yields the governing equations of the form

$$L\widehat{v} = 0, \qquad L\widehat{u} = 0, \qquad L = \frac{d^4}{d\xi^4} - (\widehat{\lambda})^2\frac{d^2}{d\xi^2}. \tag{4.39}$$

The parameter

$$\widehat{\lambda} = \left[\frac{\widehat{\delta}}{\widehat{\alpha}(1-\widehat{\alpha})}\right]^{1/2}, \tag{4.40}$$

is positive, cf. Eq. (4.35). Taking into account (A.2) one can express $\widehat{\lambda}$ by the formula

$$\widehat{\lambda} = \frac{h}{c}\left[\frac{3((c/d)+1)}{(dG_A/cG_T)+1}\right]^{1/2}. \tag{4.41}$$

Thus the fields $(\widehat{u}, \widehat{v})$ are spanned over the basis $\{1, \xi, \exp(\widehat{\lambda}\xi), \exp(-\widehat{\lambda}\xi)\}$ on both subintervals $(0, \varrho)$ and $(\varrho, 2\varrho)$. For the sake of brevity we omit the derivation and report only the final result:

$$\begin{aligned} \widetilde{\varepsilon}^F_{12} &= \frac{h}{c} F_{12}(\widehat{\lambda}\varrho)\varepsilon^h_{12}, \\ F_{12}(x) &= \left(1 + \frac{d}{c} x \operatorname{cth} x\right)^{-1}. \end{aligned} \tag{4.42}$$

4.4. Assessing the loss of the Kirchhoff modulus

According to the definition $(2.4)_3$ combined with (4.42), one finds

$$N^{12}_h = 2\widetilde{A}^{1212}_c \varepsilon^h_{12}, \qquad \widetilde{A}^{1212}_c / A^{1212}_v = 1 - F_{12}(\widehat{\lambda}\varrho), \tag{4.43}$$

$$\widetilde{G}^c_{12} = \widetilde{A}^{1212}_c / 2h, \tag{4.44}$$

where $\widetilde{G}^c_{12}$ is the reduced Kirchhoff modulus of the laminate. One can prove that formulae (4.43) and (4.44) coincide with those of HASHIN [5, Eq. (3.22)], TAN and NUISMER [14] and TSAI and DANIEL [16].

4.5. Homogenized potential

Having derived the homogenized constitutive relations (4.26) and (4.43) we can combine them to form the hyperelastic law, cf. Eq. (5.30) in Ref. [10]

$$N^{\alpha\beta}_h = \frac{\partial W_h}{\partial \varepsilon^h_{\alpha\beta}}. \tag{4.45}$$

The hyperelastic potential is given by

$$W_h = \begin{cases} W^0_h, & \text{for} \quad E_h \le 0, \\ W^c_h, & \text{for} \quad E_h \ge 0, \end{cases} \tag{4.46}$$

where

$$(4.47)\qquad \begin{aligned} 2W_h^0 &= \sum_{\alpha,\beta} A_1^{\alpha\alpha\beta\beta}\varepsilon_{\alpha\alpha}^h\varepsilon_{\beta\beta}^h + 2\widetilde{A}_c^{1212}\left[(\varepsilon_{12}^h)^2 + (\varepsilon_{21}^h)^2\right],\\ 2W_h^c &= \sum_{\alpha,\beta} \widetilde{A}_c^{\alpha\alpha\beta\beta}\varepsilon_{\alpha\alpha}^h\varepsilon_{\beta\beta}^h + 2\widetilde{A}_c^{1212}\left[(\varepsilon_{12}^h)^2 + (\varepsilon_{21}^h)^2\right]. \end{aligned}$$

The potential W_h^2 for the case of open cracks can be expressed as follows

$$(4.48)\qquad W_h^c = W_h^0 - \frac{1}{2}A_v^{1111}F_{11}(\varrho)(E_h)^2.$$

By virtue of the above expression one easily verifies that the potential W_h is of class C^1, $E_h = 0$ being its line of non-smoothness of its first order derivatives. The complementary effective potential $W_h^*(N_h^{\alpha\beta})$ can be calculated by using the Fenchel transformation of W_h, cf. Remarks 3.1 and 4.1. The potential W_h^*, defined on the space $\mathbb{E}_s^2$ remains smooth, the equation $N_h = 0$ (cf. Eq. (3.43)) determines the line of the non-smoothness of its first order derivatives, cf. Fig. 4b. We recall that $\mathbb{E}_s^2$ is the space of symmetric 2×2 matrices, here identified with its dual.

5. Final remarks

Accuracy of the formulae for effective moduli of the cracked laminates found in this work is examined in the second part of the paper [11]. There we refer to other known analytical models concerning aligned, regularly distributed cracks as well as to available experimental data. We show that for the case of aligned cracks the predictions of the model (h_0, l) lie closely to results of McCARTNEY [12, 13]. In their principles, however, these models are completely different, see Introductions to Refs. [9, 10].

Possible generalization of the formulae found in this paper to the case of other damage modes and, in general, to the case of angle-ply laminates would be of vital interest. For instance one can choose a different way: use the homogenization scheme of Caillerie – Kohn – Vogelius (cf. Ref. [7]) and then apply the finite element method to solve the local problems. The recent paper of ADOLFSSON and GUDMUNDSON [1] goes in this direction, yet in the manner that circumvents the homogenization formalism of the passage from the original problem to the effective macroscopic problem and the underlying local analysis.

Appendix

The following non-dimensional parameters depending on the quantities defined in Ref. [10] are used in the present paper:

$$\alpha_{\lambda\mu} = A_1^{\lambda\lambda\mu\mu}/A_v^{1111}, \qquad \beta_{\lambda\mu} = A_2^{\lambda\lambda\mu\mu}A_v^{1111},$$

$$
\text{(A.1)}\qquad
\begin{aligned}
(\alpha,\beta,\gamma,\delta) &= (A_{vu}^{1111}, h^2A_{vw}^{11}, A_{u}^{1111}, h^2H^{11})/A_{v}^{1111},\\
(\lambda_1,\mu) &= (h^2A_{uw}^{11}, h^4A_w)/A_v^{1111}, \qquad \lambda=\delta+\lambda_1,\\
(\beta_1,\beta_2,\beta_3,\beta_4,\gamma_3) &= (A_v^{1122}, h^2A_{vw}^{22}, h^2A_{uw}^{22}, A_{vu}^{1122}, A_u^{1122})/A_v^{1111},\\
(\widehat{\alpha},\widehat{\gamma},\widehat{\delta}) &= (A_{vu}^{2121}, A_u^{2121}, hH^{22})/A_v^{2121}.
\end{aligned}
$$

Since $D_{1212}^{NL} = -D_{1212}^{N}$ we have $A_{vu}^{1212} = A_{u}^{1212}$. Hence one can prove that

$$A_v^{1212} = 2hG_A$$

and

$$
\text{(A.2)}\qquad
\begin{aligned}
&\widehat{\alpha} = \widehat{\gamma} = \frac{c}{h}, \qquad \widehat{\delta} = 3\left[\frac{dG_A}{hG_T} + \frac{c}{h}\right]^{-1},\\
&\widehat{\lambda} = h(\widehat{\delta}/cd)^{1/2}.
\end{aligned}
$$

The parameters appearing in Eqs. (4.10), (4.13), (4.19) and (4.20) are defined by

$$
\text{(A.3)}\qquad
\begin{aligned}
\mu_{11} &= 1-\alpha\beta/\lambda, \qquad \mu_{12} = \alpha-\beta\gamma/\lambda, \qquad \mu_{13} = \delta\beta/\lambda,\\
\mu_{21} &= \alpha\mu_{13}, \qquad \mu_{22} = \beta^2-\mu, \qquad \mu_{23} = \gamma\mu_{13},\\
\mu_{24} &= \beta\lambda-\delta\mu_{13}-\mu\alpha, \qquad \mu_{44} = \mu_{13}/\mu_{11};\\
a_1 &= \mu_{12}\mu_{21}-\mu_{11}\mu_{23}, \qquad a_3 = \mu_{13}\mu_{22},\\
a_2 &= \mu_{22}\mu_{12}+\mu_{13}\mu_{21}-\mu_{24}\mu_{11};\\
f_{11} &= \alpha-\mu_{12}/\mu_{11},\\
\gamma_{11} &= \mu_{44}(\beta-\mu_{44}), \qquad \gamma_{12} = \frac{\mu_{12}}{\mu_{11}}f_{11},\\
\gamma_{22} &= \frac{\delta}{\beta}(f_{11})^2, \qquad \gamma_{13} = \mu_{44}f_{11},\\
\gamma_{21} &= \frac{\delta}{\beta}(\beta-\mu_{44})^2, \qquad \gamma_{23} = \frac{\delta}{\beta}f_{11}(\beta-\mu_{44}).
\end{aligned}
$$

The pameters $\alpha_{\lambda\mu}$, $\beta_{\lambda\mu}$ defined by $(\text{A.1})_{1,2}$ can be expressed in terms of other parameters as follows

$$
\text{(A.4)}\qquad
\begin{aligned}
\alpha_{11} &= 1-\beta^2/\mu, \qquad \alpha_{12} = \beta_1-\beta\beta_2/\mu, \qquad \alpha_{22} = \gamma_1-(\beta_2)^2/\mu,\\
\beta_{11} &= \alpha-\beta\lambda_1/\mu, \qquad \beta_{12} = \beta_4-\beta\beta_3/\mu,\\
\beta_{21} &= \beta_4-\beta_2\lambda_1/\mu, \qquad \beta_{22} = \gamma_2-\beta_2\beta_3/\mu.
\end{aligned}
$$

Note that $\alpha_{12} = \alpha_{21}$ but $\beta_{12} \neq \beta_{21}$.

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WARSAW UNIVERSITY OF TECHNOLOGY
CIVIL ENGINEERING FACULTY
INSTITUTE OF STRUCTURAL MECHANICS
and
POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

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Stiffness loss of laminates with aligned intralaminar cracks Part II. Comparisons

T. LEWIŃSKI and J.J. TELEGA (WARSZAWA)

THE EFFECTIVE MODELS (h_0, l_0) and (h_0, l) [7] describing reduction of the in-plane effective moduli of the $[0^\circ_m/90^\circ_n]_s$ cross-ply composites cracked in the internal layer and subjected to in-plane boundary forces are applied to the description of the degradation of the effective Young, Kirchhoff and Poisson moduli of the $[0^\circ/90^\circ]_s$ and $[0^\circ/90_3]_s$ glass/epoxy and $[0^\circ/90_2]_s$ graphite/epoxy laminates. It is shown that the graphs of $E_1(c_d)$ (c_d represents crack density) lie slightly above the Hashin's curves, while $G_{12}(c_d)$ predictions coincide with the curves of Hashin. Evaluation of the off-diagonal terms, i.e. $\nu_{12}(c_d)$, $\nu_{21}(c_d)$ are incorporated in the algorithm. In all comparisons with the experimental results of Groves, Ogin, Highsmith and Reifsnider the predictions of $E_1(c_d)$ according to the model (h_0, l) provide lower bounds, slightly better than the bounds of Hashin. Some predictions of the model (h_0, l) are proved to be similar to McCartney's "generalized-plane-strain" results.

1. Introduction

THE FIRST DAMAGE mode observed in the in-plane loaded, three-layer, balanced cross-ply laminates is usually transverse cracking along the fibres of the outer or inner layers. When stretched along the fibres of the outer layers or sheared in its plane, samples of the balanced $[0^\circ_m/90^\circ_n]_s$ laminates undergo transverse cracking in the 90° layer, with values of crack density c_d determined by magnitude of the in-plane loads applied. Such cracks lead to degradation of effective elastic characteristics of the laminate. A unified model of such degradation has recently been proposed in Refs. [I.9, I.10] (Roman numeral I refers to bibliography of the first part of this paper, [7]) and in Ref. [7], where the case of aligned cracks is dealt with in detail. The aim of this paper is twofold. First we show that the (h_0, l_0) model (Ref. [I.10], Sec. 4) concerns the case of infinitely dense distribution of cracks. Consequently, this model provides the asymptotes for the curves of decay of the effective moduli with respect to the crack density. Then we check the accuracy of the (h_0, l) model proposed in Ref. [I.10]. To assess its accuracy with respect to the experimental results published in the available literature, we analyze the decay of:

- the effective Young modulus E_1^c of laminates of the $[0^\circ_m/90^\circ_n]_s$ type. Accuracy of the (h_0, l) predictions is examined for the laminates tested by GROVES [I.4] (cf. LEE *et al.* [6]), HIGHSMITH and REIFSNIDER [I.6] and by OGIN *et al.* [9];
- the effective moduli of Kirchhoff (G_{12}^c), Young (E_2^c) and Poisson ($\nu_{\alpha\beta}^c$) for the $[0^\circ/90^\circ_3]_s$ laminate tested by HIGHSMITH and REIFSNIDER [I.6].

The results concerning E_1^c show that the (h_0, l) method leads to lower estimates of the experimental data, providing the curves lying closer to the test data than the curves produced by the method of HASHIN [I.5] and almost coinciding with recent MCCARTNEY's [I.12, I.13] GPS (generalized plane strain model) – predictions.

The formula for G_{12}^c coincides with that found by HASHIN [I.5] and rederived later by TAN and NUISMER [I.14] and TSAI and DANIEL [I.16]. According to the experimental results published in the last paper, concerning the graphite/epoxy $[0^\circ/90_2^\circ]_s$ and $[0^\circ/90_4^\circ]_s$ laminates, the accuracy of this formula is satisfactory. On the other hand, the experiments concerning graphite/epoxy AS/3502 $(0_2^\circ/90_2^\circ)$ laminates performed by HAN and HAHN [4] do not confirm its utility, cf. their paper and the discussion by MOTOGI and FUKUDA [8].

Our analysis shows that predictions of the (h_0, l_0) model proposed in Ref. [I.10] are comparable with the ply-discount method.

The (h_0, l) method predicts a small decay of the E_2^c modulus of the $[0^\circ/90_3^\circ]_s$ glass/epoxy laminates, very similar to that predicted by the GPS model of MCCARTNEY [I.13]. Other methods known to the present authors do not describe the decay of E_2^c or keep an open mind on the subject.

The (h_0, l) method provides a unified algorithm for predicting decay of all components of the stiffness matrix. In particular, the method makes it possible to evaluate the decay of Poisson ratios. In the present paper the curves of the decay of these ratios for the glass/epoxy $[0^\circ/90_3^\circ]_s$ laminate are given and compared with GPS-predictions of MCCARTNEY [I.13]. A very close juxtaposition of these predictions are noted. For the laminate analyzed no relevant experimental results were available to us. The only experimental results available to the present authors, concerning reduction of Poisson ratios of other types of laminates, are given in SMITH and WOOD [10]. A comparison of these results with (h_0, l) predictions will be published separately. The present paper concerns only the case of cracking in the internal layer. A generalization to the case of the simultaneous cracking in external and internal layers requires a reformulation of the original model of Sec. 2 proposed by LEWIŃSKI and TELEGA [I.9], which could probably be done by adopting the assumptions put forward by HASHIN [5] and TSAI and DANIEL [I.16].

The system of notations is compatible with that employed in Part I of the present paper, namely in Ref. [7]. For the sake of brevity, Roman numeral I refers also to equations or sections of Part I.

2. Parallel cracks. Comparison of (h_0, l_0) and (h_0, l) predictions

The subject of consideration will be the same as in Ref. [7], Secs. 3, 4. We examine a three-layer laminate of thickness $2h$ weakened by regularly distributed transverse cracks in the internal layer, and subject to in-plane loading; the crack spacing equals l, cf. Fig. I.2. These cracks result in the degradation of effective moduli. The aim of this section is to prove that decaying curves of moduli degradation predicted by the (h_0, l) model presented in Ref. [7], Sec. 4, tend to crack density-independent values of the effective moduli, predicted by the (h_0, l_0) model proposed in Sec. 3 of [7], if the number of cracks tends to infinity.

2.1. Stiffnesses $\widetilde{A}_c^{\alpha\alpha\beta\beta}$ versus $A_c^{\alpha\alpha\beta\beta}$

Let us compare formulae (I.3.9) with (I.4.26) and (I.3.13) with (I.4.27). Note that the line of non-smoothness of the constitutive relations: $E_h = 0$ is common for both approaches, which makes the results of both approaches similar. However, Eq. (I.3.10) is independent of $\varrho = l/2h$. Let us examine the $\widetilde{A}_c^{\alpha\alpha\beta\beta}(\varrho)$ curves.

If ϱ tends to infinity, $F_{11}(\varrho)$ tends to zero. Hence

$$\lim_{\varrho\to\infty} \widetilde{A}_c^{\alpha\alpha\beta\beta} = A_1^{\alpha\alpha\beta\beta}. \tag{2.1}$$

Thus if the crack spacing is much greater than h, the loss of stiffness will not be observed. This effect is also observed in experiments, which will be discussed in Sec. 3. According to the (h_0, l_0) model, the loss of stiffness is ϱ-independent provided that ϱ is small, cf. comments in Ref. [I.10].

Consider the case when the number of cracks increases to infinity; then $\varrho \to 0$. One can prove that

$$\begin{aligned} &\lim_{\varrho\to 0} F_0(\varrho;\omega,\sigma) = \frac{1}{\sigma^2\omega^2}, \\ &\lim_{\varrho\to 0} F_0(\varrho;\omega,\sigma) = \frac{1}{(p^2+q^2)^2}. \end{aligned} \tag{2.2}$$

Since $\sigma^2\omega^2 = (p^2+q^2)^2$, we see that both limits coincide, irrespective of whether the roots of polynomial (I.4.13) are real or complex. Hence we have

$$\lim_{\varrho\to 0} (\widetilde{\varepsilon}_{11}^F/E_h) = F_{11}(0), \tag{2.3}$$

for the case $E_h > 0$, where $F_{11}(0) = \lim\limits_{\varrho\to 0} F_{11}(\varrho)$ is given by

$$F_{11}(0) = a_3 f_{11}[(\beta_{11}\gamma_{11}+\gamma_{21})a_1 - (\beta_{11}\gamma_{13}+\gamma_{23})a_2 + (\beta_{11}\gamma_{12}+\gamma_{22})a_3]^{-1}. \tag{2.4}$$

On the other hand, according to the (h_0, l_0) approach, for the case $E_h > 0$ one finds

$$\varepsilon_{11}^F/E_h = F_{11}^0, \tag{2.5}$$

F_{11}^0 being defined by Eq. $(\mathrm{I}.3.8)_1$. By using the relations between constants summarized in the Appendix of [7], after lengthy algebraic calculations one can prove that $F_{11}(0) = F_{11}^0$, which confirms the thesis of Sec. 5.6 of Ref. [I.10]: the (h_0, l_0) model provides asymptotes for the curves predicted by the (h_0, l) model, namely

$$\lim_{\varrho\to 0} \widetilde{A}_c^{\alpha\alpha\beta\beta} = A_c^{\alpha\alpha\beta\beta}. \tag{2.6}$$

2.2. Stiffness $\widetilde{A}_c^{1212}$ versus A_c^{1212}

One can prove that

$$\lim_{\varrho\to 0} F_{12}(\widehat{\lambda}\varrho) = \frac{c}{h}, \qquad \lim_{\varrho\to\infty} F_{12}(\widehat{\lambda}\varrho) = 0, \tag{2.7}$$

and hence, cf. Eq. (I.3.21)

$$\lim_{\varrho\to 0} \widetilde{\varepsilon}_{12}^F = \varepsilon_{12}^F, \qquad \lim_{\varrho\to 0} \widetilde{A}_c^{1212} = A_c^{1212}, \tag{2.8}$$

$$\lim_{\varrho\to\infty} \widetilde{A}_c^{1212} = A_v^{1212}. \tag{2.9}$$

Thus, if the crack density ϱ^{-1} tends to zero, the stiffness $\widetilde{A}_c^{1212}$ tends to the stiffness A_v^{1212} of the uncracked laminate. If the crack density tends to infinity, the (h_0, l) predictions tend to (h_0, l_0) predictions (I.3.21) – (I.3.23). In particular, a constant line $G_{12}^c = A_c^{1212}/2h$ is an asymptote for the $\widetilde{G}_{12}^c$ curve describing the decays of the effective Kirchhoff modulus.

We observe that (2.6) and (2.9) imply the relation between hyperelastic potentials

$$\mathcal{V}_h = \lim_{\varrho\to 0} W_h(\varrho). \tag{2.10}$$

The line of non-smoothness of both potentials $E_h = 0$ remains ϱ-independent.

3. Degradation of effective stiffnesses of laminates $[0_n^\circ/90_m^\circ]_s$. Comparison with experimental results and with other analytical predictions

In this section we shall verify the (h_0, l_0) and (h_0, l) models predictions for:

i) $[0_n^\circ/90_m^\circ]_s$ glass/epoxy laminates tested by HIGHSMITH and REIFSNIDER [I.6] and by OGIN *et al.* [9].

ii) $[0^\circ/90_2^\circ]_s$ graphite/epoxy laminates tested by GROVES [I.4] (this paper was not available for the present authors; Groves' results are reported here after LEE *et al.* [6]).

The results of Sec. I.4 will be compared with theoretical predictions of ABOUDI [1], HASHIN [I.5] and LEE *et al.* [6].

3.1. $[0^\circ/90_3^\circ]_s$ glass/epoxy laminate

We start with the laminate first examined by HIGHSMITH and REIFSNIDER [I.6] and then often referred to in the relevant literature. The complete characteristics of this laminate have been recorded by HASHIN ([I.5], Sec. 4). We repeat them to make our paper self-contained. The external plies are 0°-plies, their thickness d being equal to 0.203 mm; the internal layer composed of 90°-plies has thickness

$2c$, $c = 3d$, cf. Fig. I.2. The compliances $D^{\mathbf{n}}_{ijkl}$ $(\mathbf{n} = m, f)$ cf. Eq. (2.3) of Ref. [I.9] are defined by

$$
\begin{aligned}
& D^f_{1111} = D^m_{2222} = \frac{1}{E_A}, \qquad D^m_{1111} = D^f_{2222} = D^f_{3333} = D^m_{3333} = \frac{1}{E_T},\\
& D^f_{1122} = D^m_{1122} = D^f_{1133} = D^m_{2233} = -\frac{\nu_A}{E_A},\\
& D^m_{1133} = D^f_{2233} = -\frac{\nu_T}{E_T},\\
& D^f_{1212} = D^m_{1212} = D^m_{2323} = \frac{1}{4G_A},\\
& D^f_{1313} = \frac{1}{4G_A}, \qquad D^m_{1313} = D^f_{2323} = \frac{1}{4G_T},
\end{aligned}
\tag{3.1}
$$

where, according to Table 1 of HASHIN [I.5],

$$
\begin{aligned}
& E_A = 41.7\,\mathrm{GPa}, \qquad E_T = 13.0\,\mathrm{GPa}, \qquad G_A = 3.4\,\mathrm{GPa},\\
& G_T = 4.58\,\mathrm{GPa}, \qquad \nu_A = 0.30, \qquad \nu_T = 0.42.
\end{aligned}
\tag{3.2}
$$

The index A labels the fibre direction, while T indicates the direction transverse to the fibres.

Now we can determine the generalized compliances $\mathbf{D}^N$, $\mathbf{D}^L$, $\mathbf{D}^{NL}$, $\mathbf{D}^Q$, $\mathbf{D}^{RL}$, $\mathbf{D}^{RN}$, D^R by Eqs. (2.17) of Ref. [I.9]. Then we invert the constitutive matrices of Eqs. (2.19) and (2.20) of Ref. [I.9] and find the stiffness matrices of the primal constitutive relationships (2.24) and (2.25) of [I.9]. We can calculate the stiffnesses (4.9) of Ref. [I.10] and then the effective moduli (I.3.16) of the uncracked laminate. We obtain

$$
\begin{aligned}
& E_1 = 20.30\,\mathrm{GPa}, \qquad E_2 = 34.75\,\mathrm{GPa}, \qquad G_{12} = 3.40\,\mathrm{GPa},\\
& \nu_{12} = 0.193, \qquad \nu_{21} = 0.113.
\end{aligned}
\tag{3.3}
$$

The first three results coincide with the data reported by HASHIN [I.5], while results concerning E_α and $\nu_{\alpha\beta}$ coincide with those obtained by MCCARTNEY ([I.12], Appendix A).

According to the in-plane scaling ((h_0, l_0) approach), the reduced moduli are crack density independent. Using formulae (I.3.13) and (I.3.16) for the case of $\nu^c_{\alpha\beta}$, E^c_α, and $(\mathrm{I.3.23})_2$, $(\mathrm{I.3.22})_2$ we find

$$
\begin{aligned}
& E^c_1 = 10.70\,\mathrm{GPa}, \qquad E^c_2 = 34.53\,\mathrm{GPa}, \qquad G^c_{12} = 0.85\,\mathrm{GPa},\\
& \nu^c_{12} = 0.0943, \qquad \nu^c_{21} = 0.0292.
\end{aligned}
\tag{3.4}
$$

According to the experimental data of HIGHSMITH and REIFSNIDER [I.6], the minimum value of E^c_1 achieved for 0.75 cracks/mm equals 11.0 GPa while $E_1 =$

21.0 GPa. However, it is not sure whether the values measured in the paper cited above are viewed as the effective Young moduli or have been defined by means of the longitudinal stiffnesses

$$\overline{E}_1 = A_1^{1111}/2h, \qquad \overline{E}_1^c = A_c^{1111}/2h.$$

It is worth noting that the results $\overline{E}_1 = 20.76$ GPa, $\overline{E}_1^c = 10.73$ GPa lie closer to the values found experimentally than the quantities E_1, E_1^c which are Young moduli by the definition.

The experiments show that the reduction of the effective characteristics depends upon the crack density. The space-scaling (h_0, l) approach accounts for such a dependence. Having found the matrices involved in Eqs. (3.16) of [I.10] one can calculate the parameters defined by Eqs. (I.A.1) – (I.A.4) and then the coefficients of Eq. (I.4.13). The roots of this characteristic equation turn out to be complex $(p, \pm q)$, where $p = 1.98025$ and $q = 0.8934$, hence the function $F_{11}(\varrho)$ is defined by means of $F = F_0$, cf. Eqs. (I.4.23) and (I.4.24). The decay of stiffnesses is defined by (I.4.27) and $(\mathrm{I}.4.43)_2$. The effective Young, Poisson and Kirchhoff moduli are given by Eqs. (I.4.29) and (I.4.44).

As it has been emphasized by LEE *et al.* [6], the decay of the stiffnesses should rather be displayed versus the crack density defined by $2c/l$ (crack depth/crack spacing). However, to compare our results with the theoretical predictions of HASHIN [I.5] and with experimental data of HIGHSMITH and REIFSNIDER [I.6], we quote them in some of our figures also as functions of the crack density c_d defined as 1 mm/l.

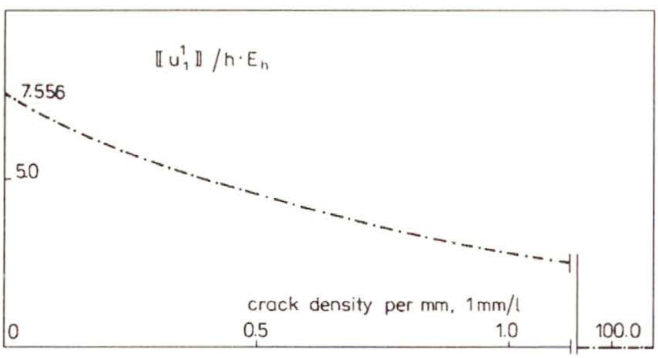

FIG. 1. $[0°/90°_3]_s$ glass/epoxy laminate tested by HIGHSMITH and REIFSNIDER [I.6]. The crack opening $[\![u_1^\varepsilon/h_\varepsilon]\!] = [\![u_1^1/h]\!]$ (normalized with respect to E_h) versus crack density.

The crack opening $[\![u_1^\varepsilon/h_\varepsilon]\!] = [\![u_1^1/h]\!] + 0(\varepsilon)$ decays to zero if c_d tends to infinity, cf. Fig. 1. The longitudinal crack deformation $\tilde{\varepsilon}_{11}^F$ behaves quite differently. The curve $\tilde{\varepsilon}_{11}^F(c_d)$ starts from zero and tends asymptotically to the ε_{11}^F value predicted by the in-plane scaling method (h_0, l_0), cf. Fig. 2. The shear crack deformation $\tilde{\varepsilon}_{12}^F$ behaves similarly, cf. Fig. 3. For sufficiently large values of c_d the crack deformations $\tilde{\varepsilon}^F_{11}$, $\tilde{\varepsilon}^F_{12}$ become practically independent of the crack density. This insensibility to large values of c_d corresponds to the saturation of cracks observed in experiments, cf. GARRETT and BAILEY [I.2].

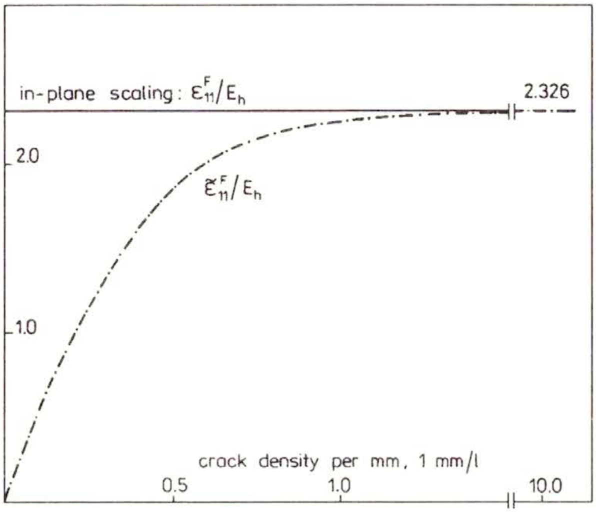

FIG. 2. The same laminate. Longitudinal crack deformation versus crack density. The in-plane scaling prediction: $\varepsilon^F_{11} \approx \varepsilon^h_{11} + 0.299\varepsilon^h_{22}$ is an asymptote for the space scale prediction $\tilde{\varepsilon}^F_{11}$.

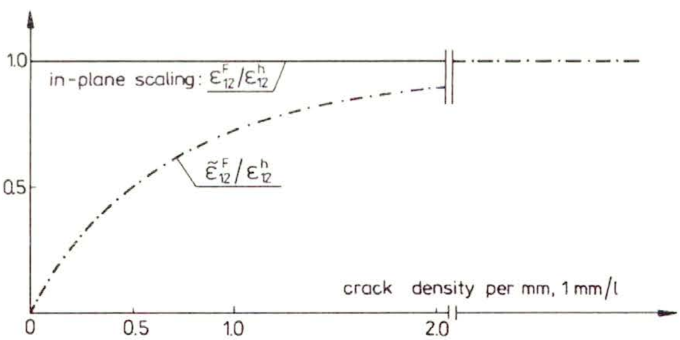

FIG. 3. The same laminate. Shear crack deformation versus crack density. The in-plane scaling prediction $\varepsilon^F_{12} = \varepsilon^h_{12}$ is an asymptote for the space scaling prediction $\tilde{\varepsilon}^F_{12}$.

The decay of the effective Young modulus $\widetilde{E}^c_1$ observed in experiments by HIGHSMITH and REIFSNIDER [I.6] and predicted by the method of HASHIN [I.5], the GPS method of MCCARTNEY [I.12], the method of LEE *et al.* [6], cf. ALLEN *et al.* [2], and by the space-scaling based (h_0, l) method is presented in Fig. 4. The Hashin's curve has not been repeated after Fig. 3 in HASHIN [I.5] but has been independently plotted by the present authors. The experimental data are placed according to Fig. 14 in HIGHSMITH and REIFSNIDER [I.6] and Fig. 1a in LEE

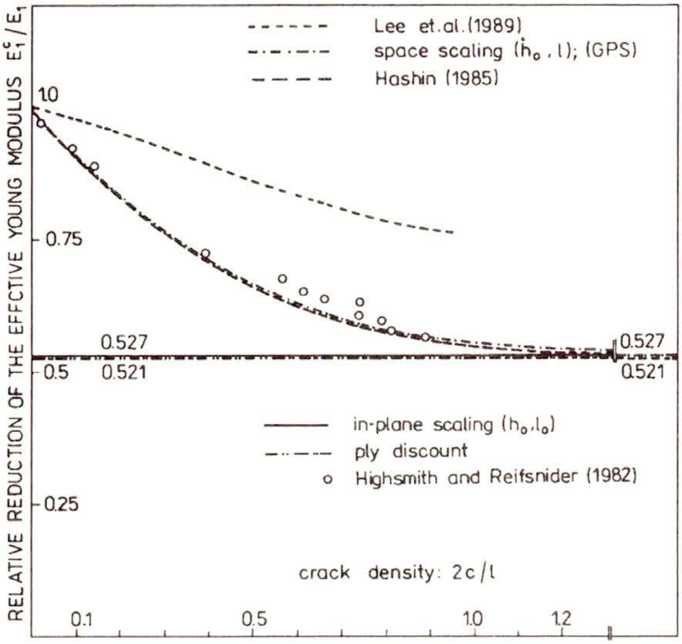

FIG. 4. The same laminate. Assessing the loss of the effective Young modulus E_1 versus crack density. Experimental results of HIGHSMITH and REIFSNIDER [I.6] are denoted by circles.

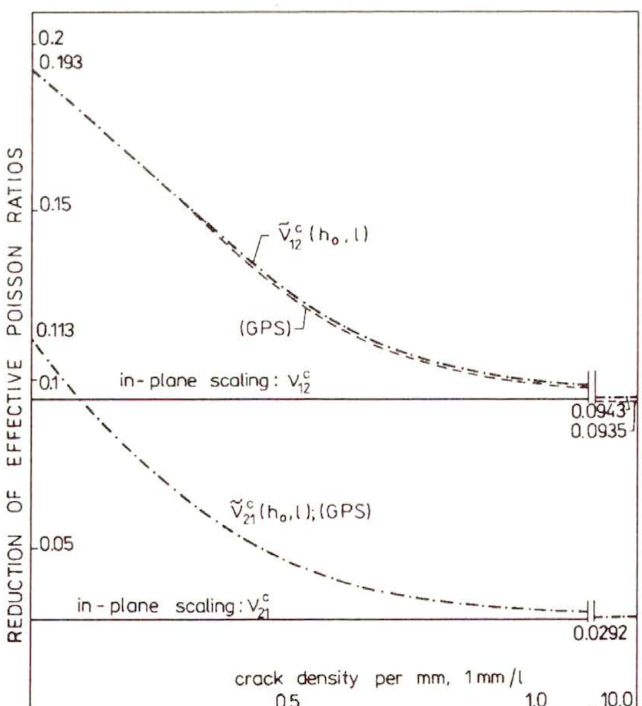

FIG. 5. The same laminate. Effective Poisson ratios as functions of the crack density.

et al. [6]. The Hashin's curve lies slightly below the curves predicted by the GPS model and by the (h_0, l) space-scaling method, the juxtaposition of the last two curves being too close to be noticeable in Fig. 4. The curves mentioned above provide lower bounds for the experimental results. Small differences in these results can be read off from Table 1a. On the other hand, the predictions of LEE *et al.* [6] are upper bounds for the experimental data. The in-plane scaling method ((h_0, l_0) approach) determines a horizontal asymptote for the $\widetilde{E}_1^c/E_1$ curve; the conventional ply-discount assessment lies a little below and is an asymptote for the Hashin's curve.

Table 1. Decay of E_α^c/E_α, $\nu_{\alpha\beta}^c$ as function of crack density $2c/l$ for the $(0°/90_3°)_s$ glass/epoxy laminate tested by HIGHSMITH and REIFSNIDER (1982). Comparison of predictions by (h_0, l) model proposed with results due to HASHIN (1985) (case E_1^c/E_1) and model (GPS) of MCCARTNEY (1992, 1993).

(a)

$2c/l$	E_1^c/E_1		
	Hashin (1985)	McCartney (1992) GPS	Lewiński and Telega (h_0, l)
0.1	0.9069	0.90918	0.90914
0.5	0.6609	0.66638	0.66628
1.0	0.54782	0.55347	0.55341
100.	0.52127	0.52683	0.52681

(c)

$2c/l$	ν_{21}^c	
	McCartney (1992) GPS	Lewiński and Telega (h_0, l)
0.1	0.09684	0.09688
0.5	0.05375	0.05390
1.0	0.03371	0.03393
100.	0.0290	0.02922

(b)

$2c/l$	ν_{12}^c	
	McCartney (1992) GPS	Lewiński and Telega (h_0, l)
0.1	0.18215	0.18223
0.5	0.13753	0.1380
1.0	0.10362	0.10432
100.	0.09353	0.09432

(d)

$2c/l$	E_2^c/E_2	
	McCartney (1992) GPS	Lewiński and Telega (h_0, l)
0.1	0.99929	0.99930
0.5	0.99644	0.99650
1.0	0.99428	0.99437
100.	0.99364	0.99374

The decaying character of the graphs $\widetilde{\nu}_{12}^c$, $\widetilde{\nu}_{21}^c$ is reported in Fig. 5. The in-plane scaling predictions are constants lines – the asymptotes of more realistic space-scaling results. The GPS and (h_0, l) predictions turn out to be very similar, see Tables 1b, 1c.

A very slight decay of E_2^c is predicted by GPS as well as by the (h_0, l) method, cf. Fig. 6. Both models mentioned lead to very similar results, see Table 1d. The decay of E_2^c as well as of $\nu_{\alpha\beta}^c$ cannot be described within the framework of HASHIN'S [I.5] approach, hence the lack of comparisons.

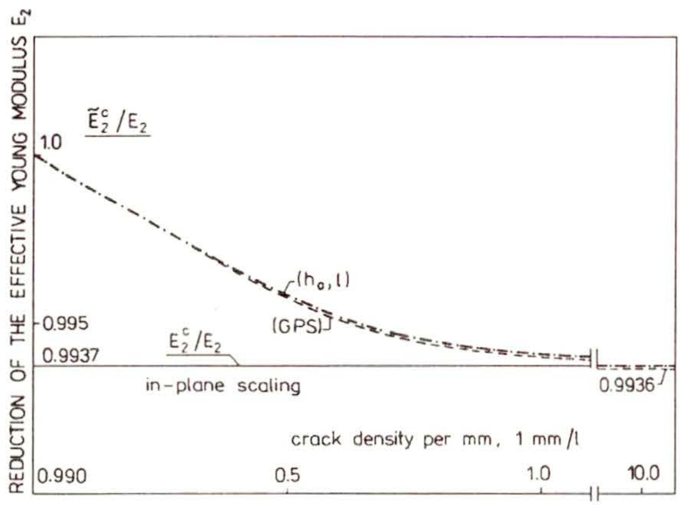

FIG. 6. The same laminate. Decay of the effective Young modulus E_2.

The method of HASHIN [I.5] and the (h_0, l) method lead to the same formula describing the decay of the Kirchhoff modulus, cf. Fig. 7. Recently TSAI and DANIEL [I.16] have confirmed that this formula predicts values of G_{12}^{c} comparing favourably with experimental data concerning graphite/epoxy laminates, cf. Fig. 5

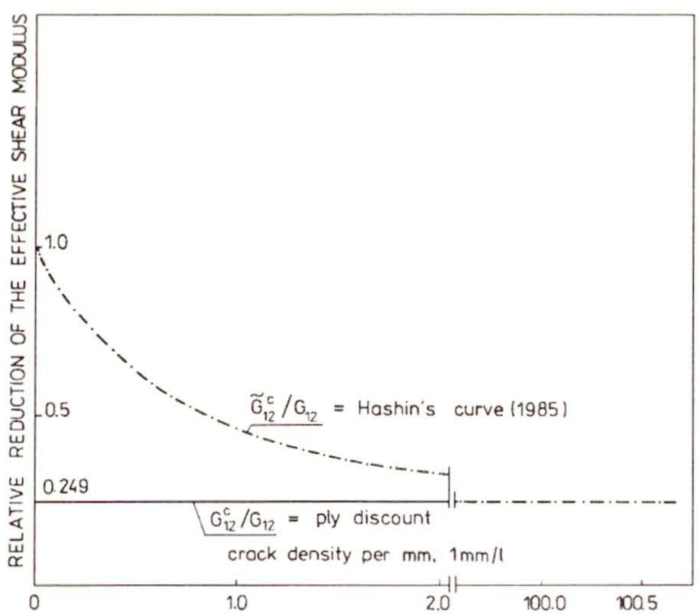

FIG. 7. The same laminate. Decay of the effective Kirchhoff modulus. The HASHIN's [I.5] and (h_0, l) predictions coincide. Predictions based on the in-plane scaling coincide with ply-discount result.

in the cited paper. On the other hand, the experimental results due to HAN and HAHN [4] concerning the GFRP $[0^\circ_2, 90^\circ_2]_s$ laminates lie far away from the Hashin's curve. Experimental data concerning G^c_{12} for the laminate considered here were not available to the present authors.

3.2. $[0^\circ/90^\circ]_s$ glass/epoxy laminate

Consider the $[0^\circ/90^\circ]_s$ glass/epoxy laminate tested by OGIN *et al.* [9] for which

$$
\begin{aligned}
&c = d = 0.125\,\text{mm}, \qquad E_A = 40\,\text{GPa}, \qquad E_T = 11\,\text{GPa},\\
&G_A = 5\,\text{GPa}, \qquad G_T = 3.87\,\text{GPa}, \qquad \nu_A = 0.3, \qquad \nu_T = 0.42.
\end{aligned}
\tag{3.5}
$$

These data, except for the last two which are assumed here, are taken from ABOUDI [1].

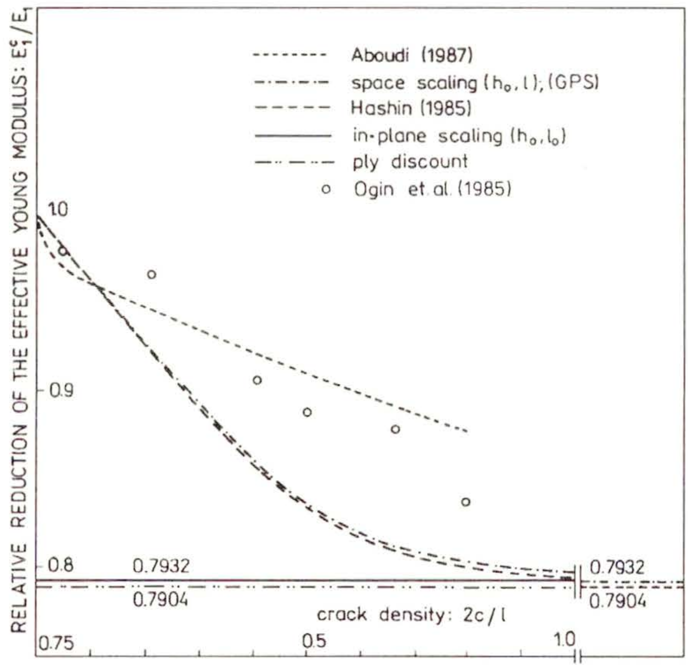

FIG. 8. The $[0^\circ/90^\circ]_s$ glass/epoxy laminate tested by OGIN *et al.* [9] (circles). Assessing the loss of the effective Young modulus E_1.

The Hashin's curve as well as the almost coinciding curves provided by the GPS model of McCARTNEY [I.12] and by the space-scaling (h_0, l) method yield lower bounds for the experimental results of OGIN *et al.* [9], cf. Fig. 8. The accuracy, however, is not so satisfactory as for the laminate considered previously. Better results are provided by the displacement-based method of ABOUDI [1]. His method, however, similarly to that of Hashin is based on comparing energies and hence is uncapable of assessing off-diagonal terms of the effective stiffness matrix.

The precise values of E^c_α / E_α and $\nu^c_{\alpha\beta}$ predicted by the GPS model of MCCARTNEY [I.12, 13] and by the (h_0, l) model proposed in the present paper are set up in Tables 2a–2d. It is seen that both models produce almost identical results. In particular, these differences could not be displayed in Fig. 8 concerning E^c_1 / E_1.

Table 2. Decay of E^c_α / E_α, $\nu^c_{\alpha\beta}$ as function of crack density $2c/l$ for the $(0°/90°)_s$ glass/epoxy laminate tested by OGIN *et al.* (1985). Comparison of predictions by (h_0, l) model proposed and GPS approach of MCCARTNEY (1992, 1993).

(a)

$2c/l$	E^c_1 / E_1	
	McCartney (1992) GPS	Lewiński and Telega (h_0, l)
0.1	0.96195	0.96194
0.5	0.83782	0.83780
1.0	0.79856	0.79846
100.	0.79333	0.79322

(c)

$2c/l$	ν^c_{21}	
	McCartney (1992) GPS	Lewiński and Telega (h_0, l)
0.1	0.11773	0.11789
0.5	0.07964	0.08030
1.0	0.06759	0.06839
100.	0.06599	0.06680

(b)

$2c/l$	ν^c_{12}	
	McCartney (1992) GPS	Lewiński and Telega (h_0, l)
0.1	0.12224	0.12241
0.5	0.09448	0.09529
1.0	0.08397	0.08501
100.	0.08250	0.08356

(d)

$2c/l$	E^c_2 / E_2	
	McCartney (1992) GPS	Lewiński and Telega (h_0, l)
0.1	0.99876	0.99881
0.5	0.99394	0.99421
1.0	0.99212	0.99247
100.	0.99186	0.99223

3.3. $[0°/90°_2]_s$ graphite/epoxy laminate

Let us consider the loss of Young modulus of the $[0°, 90°_2]_s$ graphite/epoxy laminate with the following characteristics

$$
\begin{aligned}
&d = 0.127\,\text{mm}, \qquad c = 2d, \qquad E_A = 144.8\,\text{GPa}, \qquad E_T = 9.6\,\text{GPa},\\
&G_A = 4.8\,\text{GPa}, \qquad G_T = 3.29\,\text{GPa}, \qquad \nu_A = 0.31, \qquad \nu_T = 0.46.
\end{aligned}
\tag{3.6}
$$

The experimental results of GROVES [I.4] lie between the curve of LEE *et al.* [6] and the curve of HASHIN [I.5]; the curves (almost coinciding) provided by the GPS model and the space-scaling (h_0, l) approach lie slightly over the latter one, but all three curves are so close to each other that practically they overlap, cf. Fig. 9 and Table 3a. As in other cases, the in-plane scaling method leads to a line $E^c_1 = 0.8842$ being an asymptote for the space – scaling curve. The Hashin's curve tends to the value 0.8840.

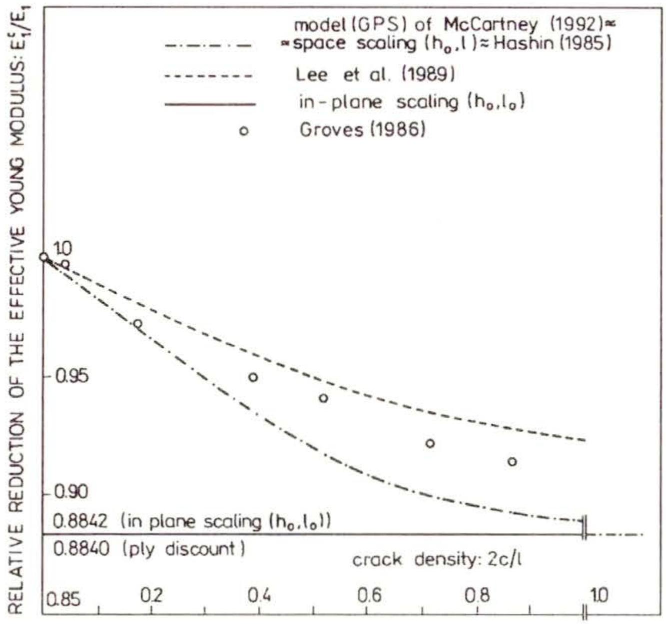

FIG. 9. The $[0°/90°_2]_s$ graphite/epoxy laminate tested by GROVES [I.4] (circles). Assessing the loss of the effective Young modulus E_1.

Table 3. Decay of E^c_α/E_α, $\nu^c_{\alpha\beta}$ as function of crack density $2c/l$ for the $(0°/90°_2)_s$ glass/epoxy laminate tested by GROVES *et al.* (1986). Comparison of predictions by (h_0, l) model proposed and GPS approach of MCCARTNEY (1992, 1993).

(a)

$2c/l$	E^c_1/E_1	
	McCartney (1992) GPS	Lewiński and Telega (h_0, l)
0.1	0.98269	0.98270
0.5	0.91971	0.91973
1.0	0.88964	0.88964
100.	0.88418	0.88418

(c)

$2c/l$	v^c_{21}	
	McCartney (1992) GPS	Lewiński and Telega (h_0, l)
0.1	0.02688	0.02689
0.5	0.01609	0.01614
1.0	0.01094	0.01101
100.	0.01001	0.01001

(b)

$2c/l$	v^c_{12}	
	McCartney (1992) GPS	Lewiński and Telega (h_0, l)
0.1	0.04986	0.04988
0.5	0.03182	0.03192
1.0	0.02234	0.02248
100.	0.02055	0.02069

(d)

$2c/l$	E^c_2/E_2	
	McCartney (1992) GPS	Lewiński and Telega (h_0, l)
0.1	0.99936	0.99936
0.5	0.99683	0.99685
1.0	0.99550	0.99553
100.	0.99525	0.99528

The precise values of E^c_α / E_α and $\nu^c_{\alpha\beta}$ predicted by the GPS model of McCARTNEY [I.12, I.13] and by the (h_0, l) one are given in Tables 3a–3d. The results are almost identical.

4. Final remarks

The analysis of the response of the cracked $[0^\circ_m, 90^\circ_n]$ laminates did not encompass a stress analysis. A detailed stress analysis will be published separately. We put only some relevant remarks concerning relations between (h_0, l) stress predictions and those found in HASHIN [I.5].

Within the framework of the (h_0, l) approach, the stresses in periodicity cells are expressed in terms of macrodeformations $\varepsilon^h_{\alpha\beta}$, cf. Eqs. (2.7)–(2.9) of [I.9] and (5.19)–(5.21) of [I.10]. On the other hand, in HASHIN [I.5] the stresses are determined by the density of the boundary shearing τ and tensile σ loading. To bridge a gap between both approaches let us introduce the following interpretations of τ and σ in terms of macro-stress resultants of the (h_0, l) model:

$$\tau = \tau_h = N_h^{12}/2h, \qquad \sigma = \sigma_h = N_h^{11}/2h\,. \tag{4.1}$$

Let us focus our attention on the stresses arising at shear. According to (I.4.43) one finds

$$\tau_h = 2G_A\left[1 - F_{12}(\widehat{\lambda}\varrho)\right]\varepsilon^h_{12}\,. \tag{4.2}$$

Note that within the interpretation suggested by (4.1), τ_h becomes crack-density dependent: $\tau_h = \tau_h(\varrho)$. A direct relation links τ_h and ε^h_{12}, owing to which one can compare formulae for $\sigma^{12}_m = \sigma^{12}(x, z)$, $|z| < c$, due to HASHIN [I.5] with those resulting from the (h_0, l) model.

Using Eqs. (2.7) of [I.9] and (I.4.31) one finds

$$\sigma^{12}_m/\tau_0 = S^m_{12}(\widehat{\lambda}\varrho, \widehat{\lambda}\xi), \tag{4.3}$$

where $\tau_0 = 2G_A\varepsilon^h_{12}$ stands for the shear stress in the uncracked laminate and

$$S^m_{12}(x, y) = \frac{x(\operatorname{ch} x - \operatorname{ch} y)}{\dfrac{c}{d}\operatorname{sh} x + x \operatorname{ch} x}\,. \tag{4.4}$$

Note that τ_0 does not explicitly depend upon the crack density.

HASHIN [I.5] obtained the following relation

$$\sigma^{12}_m/\tau = 1 - \frac{\operatorname{ch}\widehat{\lambda}\xi}{\operatorname{ch}\widehat{\lambda}\varrho}\,. \tag{4.5}$$

Taking into account $(\mathrm{I}.4.42)_2$ and (4.1) one can readily prove that formulae (4.3) and (4.5) coincide. Similarly, one can show that other components of the state of stress appearing when the laminate is subjected to shearing are predicted in the same manner by both models, inasmuch as a "bridging" relation (4.1) is acceptable.

Comparison of the formulae for stresses related to tension is less clear, since in general $N_h^{22} \neq 0$ while in HASHIN [I.5] only the case $N_h^{22} = 0$ (according to our interpretation) is considered. On imposing $N_h^{22} = 0$ one can derive a formula for $\sigma_m^{11} = \sigma^{11}(x, z)$, $|z| < c$:

$$\sigma_m^{11}/\sigma_h = f(\varrho, \xi), \tag{4.6}$$

where $\sigma_h = \sigma_h(\varrho)$, cf. $(4.1)_2$. HASHIN [I.5] normalized the stress σ_m^{11} with respect to the averaged stress in the middle layer. This formula does not coincide with (4.6) even if the latter is appropriately rearranged. For the laminate considered in Sec. 3.1, formula (4.6) produces results somewhat greater than its counterpart found by HASHIN [I.5], but the differences are measured in promilles.

The formulae found in the present paper for the decay of the effective stiffnesses and possible to find (but not displayed) formulae for stresses due to tension are more complicated than those found by HASHIN [I.5] and MCCARTNEY [I.12, model GPS]. This is a consequence of treating the stress resultants $N^{\alpha\beta}$ as independent unknown variables and completion of the model with displacements v_α relevant to them. Note, however, that an independent treatment of $N^{\alpha\beta}$ is in general indispensable when the shapes of the laminate is arbitrary and $N^{\alpha\beta}$ cannot be determined directly by the boundary loading.

Thus the present paper does not present any set of formulae for the analysis of cracked laminates, but forms a consistent and well-posed laminate model (h_0, l) from which such formulae can be inferred. This model makes it possible to approximate boundary value problems for a relatively large class. It seems that the model constitutes a reasonable starting point to the construction of a damage model that would take into account:

i) damage induced anisotropy, and ii) unilateral effect of damage.

According to CHABOCHE [3], none of hitherto existing theories of damage of laminates satisfies both the conditions simultaneously.

Acknowledgement

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WARSAW UNIVERSITY OF TECHNOLOGY
CIVIL ENGINEERING FACULTY
INSTITUTE OF STRUCTURAL MECHANICS
and
POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

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On finite deformation dynamic analysis of saturated soils

M. T. MANZARI (WASHINGTON)

A GENERAL FORMULATION is proposed to treat the dynamic response of saturated soils in finite deformation regime. Considering the soil as a saturated porous medium, the formulation for finite deformation analysis was established by extending Biot's classical theory to incorporate finite deformation effects. Particular attention was given to the flow of water through the soil while the soil skeleton undergoes a finite deformation. The derived formulation constitutes the theoretical basis for analysis of the liquefaction induced flow failure in soil embankments. Due to the integral form of the governing equations, they are specially suitable for application of numerical methods such as the finite element method.

Notations

$({}^{t}\beta), ({}^{t+\Delta t}\beta)$	body configurations at time t and $t + \Delta t$, respectively,
δ_{ij}	Kronecker delta,
${}^{0}\varrho, {}^{t}\varrho, {}^{t+\Delta t}\varrho$	mass densities of soil per unit volume in the configurations at time 0, t, $t + \Delta t$, respectively,
ϱ_s, ϱ_f	mass density of solid particles and pore water, respectively,
${}^{t+\Delta t}\sigma_{ij}$	Cartesian components of the Cauchy total stress tensor measured at time $t + \Delta t$,
$\sigma_{ij}, \overline{\sigma}_{ij}$	total stress and effective stress tensors, respectively,
$\overset{\nabla}{\sigma}, \overset{\nabla}{\overline{\sigma}}$	corotational rates of the total stress and effective stress tensors, respectively,
Ω	material spin tensor,
${}_{t}\Lambda_{ijkl}$	finite deformation tensor of tangent stiffness moduli,
${}^{t+\Delta t}b_i$	i-th component of body force per unit mass measured at time $t + \Delta t$,
${}^{t+\Delta t}_{t}b_i$	body force in the configuration at time $t + \Delta t$ $({}^{t+\Delta t}\beta)$ and measured in the configuration at time t $({}^{t}\beta)$,
${}^{0}dV, {}^{t+\Delta t}dV, {}^{t}dV$	volume of an infinitesimal element in the configuration at time 0, t, $t + \Delta t$,
${}^{t+\Delta t}e_{ij}$	Cartesian components of infinitesimal strain tensor measured at time $t + \Delta t$,
${}^{t+\Delta t}_{t}\varepsilon_{ij}$	the Green–Lagrange strain tensor,
${}^{t+\Delta t}f_i^B, {}^{t+\Delta t}f_i^S$	components of the applied body forces and surface traction, respectively, measured at time $t + \Delta t$,
${}^{t+\Delta t}_{t}F_i^S$	surface traction in the configuration at time $t + \Delta t$ $({}^{t+\Delta t}\beta)$ and measured in the configuration at time t $({}^{t}\beta)$,
$\overset{*}{k}_{ij}$	permeability tensor,
K_S, K_f	bulk moduli for solid particles and pore fluid, respectively,
n	porosity,
p	pore water pressure,

${}^{t+\Delta t}S_u, {}^{t+\Delta t}S_T,$ ${}^{t+\Delta t}S_p, {}^{t+\Delta t}S_q$	different portions of body surface, respectively related to prescribed displacement, traction, pore pressure, and flow, measured at time $t + \Delta t$,
${}^{t+\Delta t}_{t}S_{ij}$	the second Piola–Kirchhoff stress tensor,
u_i	components of incremental displacement at time t,
${}^{t+\Delta t}u_i, {}^{t}u_i$	components of displacements at time $t + \Delta t$, and t, respectively,
$\dot{U}_i$	components of the absolute velocity of pore water in the direction of x_i,
$\dot{w}_i$	relative surface velocity of pore water with respect to the soil skeleton,
$({}^{t}x_1, {}^{t}x_2, {}^{t}x_3)$	coordinates of a generic particle of the body in Cartesian coordinate system at time t.

1. Introduction

ANALYSIS OF SOIL liquefaction and its consequences, such as permanent deformations in constructed facilities or earthen structures, requires a rational analytical procedure. Such a procedure should be based on a proper understanding of the physics and mechanics of soil as a particulate medium composed of three phases, i.e. solid particles, water, and air. Due to discontinuous nature of granular soils, it appears that the best approach to study the mechanics of soil is a *micro-mechanical* approach. In principle, if the behaviour of saturated granular soils on the microscopic scale was known, it would be possible to calculate the behaviour of granular soils on the macroscopic scale by applying appropriate statistical methods. In practice, however, such calculations are extremely difficult and, at the present time, limited to some simple cases. On the other hand, our knowledge of mechanical behaviour of soils is mainly based on observations and experimental studies of the samples of soils whose dimensions are large compared to those of an individual particle. In particular, most of the experimental results available in the field of soil mechanics are expressed in terms of the overall macroscopic quantities, such as confining pressure, axial stress, axial strain, etc., which indicate a wide acceptance of *continuum* approach in the study of soil behaviour. In a continuum approach, the particulate nature of soil is ignored and it is assumed that material is uniformly distributed throughout the regions of space. For dry soils or in the case of drainage processes for saturated soils, the regular equations of continuum mechanics may be used to formulate the problem. But in the case of saturated soils which are subjected to disturbances of transient nature, the effect of pore water pressure should be considered by a proper regularization of soil as a two-phase medium [4, 5] or a mixture of two different materials [23, 24, 43].

Both of the aforementioned approaches, i.e. the micro-mechanical and continuum approach, have received much attention during the past three decades. Micro-mechanical approaches have been continuously used to study some of the important features of granular soils, such as dilatancy, shear strength, and anisotropy. However, their application to boundary value problems has been started only recently by introduction of the distinct element method [10, 11, 12].

The distinct element method considers an assembly of large number of particles representing the soil mass and solves the dynamic equilibrium equations for each particle, subject to body forces and boundary interaction forces. The method can potentially handle nonlinearities which may arise from large displacements, rotation, slip, separation and material behaviour, but its performance is highly dependent upon the constitutive laws selected to represent the inter-particle forces. In addition to application of the distinct element method to the dry soils [2, 11], a few attempts have been reported [42] to utilize the method in a simulation of soil liquefaction. However, these developments are in the initial stages and the micro-mechanical approach is far from application to the real boundary value problems.

In contrast to micro-mechanical approach, the continuum approach has been successfully used in the analysis of geotechnical problems during the past few decades. Following the introduction of a coupled stress-flow formulation for dynamics of porous media by BIOT [4, 5], many investigators employed the new formulation to solve some practically significant boundary value problems using the finite element method [38, 48, 18, 19, 21, 40, 36, 37, 49, 50]. A historical review of such applications for liquefaction analysis is given in [33]. Recently ADVANI, *et al.* [1] have used a generalized form of the Biot's formulation for hygrothermo-mechanical evaluation of porous media under finite deformation regime. CHOPRA and DARGUSH [9] have also utilized the Biot's formulation for large deformation analysis of time-dependent problems.

In this paper, a generalized form of Biot's formulation for dynamics of porous media [5, 50] is derived by taking into account the finite deformation effects. The developed formulation serves as the basis for the numerical procedure proposed in a companion paper on the analysis of soil liquefaction and deformations in a finite deformation regime.

2. Statement of the problem

For a saturated earthen structure which occupies an initial volume of ${}^0V$ with the boundary surface ${}^0S$ at time 0, we seek to establish the governing field equations necessary to evaluate its equilibrium positions and entire time histories of responses during a quasi-static or transient process of deformation.

It is assumed that specified displacements, surface traction, pore water pressure, or water flow boundary conditions are defined on different portions of the boundary surface ${}^{t+\Delta t}S$ at a generic time $t + \Delta t$. These portions of the boundary surface are named ${}^{t+\Delta t}S_u$, ${}^{t+\Delta t}S_T$, ${}^{t+\Delta t}S_p$, and ${}^{t+\Delta t}S_q$, respectively. It is attempted to establish the governing equation without imposing any restriction on the magnitude of strains and displacements which the soil body may experience in the course of deformation. In order to deal with nonlinearities involved in the problem, an incremental analysis is adopted and the equilibrium position at time $t + \Delta t$ is searched for, assuming that the solutions for all time steps from time 0 to time t are known.

We adopt a Lagrangian (material) formulation and follow the material points in their motion. Therefore, in a generic time step from time t to time $t + \Delta t$, it is assumed that the initial configuration of the soil body (${}^0\beta$) and the configuration at time t (${}^t\beta$) are known and we are searching for the configuration of structure at time $t + \Delta t$ (${}^{t+\Delta t}\beta$). In the following development, an updated Lagrangian formulation is followed.

3. The principle of virtual work

Let us consider the motion of a generic point P of a saturated earth structure (Fig. 1). In the process of deformation from the initial configuration at time 0 to the configuration at time t, its coordinates with respect to a fixed Cartesian coordinate system change from $({}^0x_1, {}^0x_2, {}^0x_3)$ to $({}^tx_1, {}^tx_2, {}^tx_3)$, where the left-hand

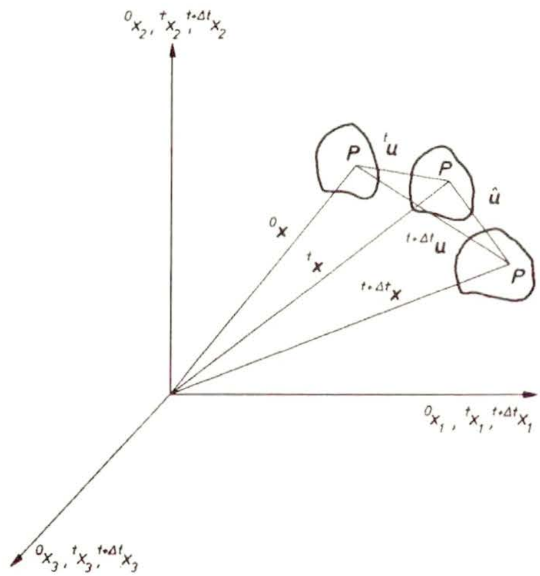

FIG. 1. Three different configuration of the soil body during its motion.

superscripts refer to the configuration of body, and the subscripts refer to different axes of the Cartesian coordinate system. In our analysis, we seek to find the position of each material point in the next configuration, i.e. at time $t + \Delta t$. Let us suppose that the soil body, in the configuration at time $t + \Delta t$, is subjected to a virtual displacement field $\delta\mathbf{u}$ which satisfies all the boundary conditions (Sec. 6). The principle of virtual work requires that the virtual work performed, when the soil body undergoes a virtual displacement $\delta\mathbf{u}$, is equal to the external work done by the body forces and surface traction, i.e.

$$(3.1)\qquad {}^{t+\Delta t}W_v^{\text{int}} = \int\limits_{{}^{t+\Delta t}V} {}^{t+\Delta t}\sigma_{ij}\delta\, {}_{t+\Delta t}e_{ij}\, {}^{t+\Delta t}dV$$

$$= {}^{t+\Delta t}W_v^{\text{ext}} = \int\limits_{{}^{t+\Delta t}V} {}^{t+\Delta t}f_i^B \delta u_i\, {}^{t+\Delta t}dV + \int\limits_{{}^{t+\Delta t}S_T} {}^{t+\Delta t}f_i^S \delta u_i\, {}^{t+\Delta t}dS,$$

where the ${}^{t+\Delta t}\sigma_{ij}$ are Cartesian components of the Cauchy total stress tensor, the ${}^{t+\Delta t}e_{ij}$ are Cartesian components of infinitesimal strain tensor, ${}^{t+\Delta t}f_i^B$ and ${}^{t+\Delta t}f_i^S$ are the components of the applied body forces and surface traction, respectively, and δu_i represents the components of virtual displacement field in the direction of axis i of the Cartesian coordinate system. The ${}^{t+\Delta t}S_T$ is a part of soil body surface on which a specified traction ${}^{t+\Delta t}f_i^S$ is applied. The $\delta\, {}_{t+\Delta t}e_{ij}$ is the variation in the small strain tensor defined as follows:

$$(3.2)\qquad \delta_{t+\Delta t}e_{ij} = \delta\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial^{t+\Delta t}x_j} + \frac{\partial u_j}{\partial^{t+\Delta t}x_i}\right)\right] = \frac{1}{2}\left(\frac{\partial(\delta u_i)}{\partial^{t+\Delta t}x_j} + \frac{\partial(\delta u_j)}{\partial^{t+\Delta t}x_i}\right),$$

where u_i is the incremental displacement at time t defined as

$$u_i = {}^{t+\Delta t}u_i - {}^{t}u_i$$

in which ${}^{t+\Delta t}u_i$ and ${}^{t}u_i$ denote the displacements at time $t + \Delta t$ and t, respectively. Note that the first term on the right-hand side of Eq. (3.2) implies the partial derivative of the variation u_i with respect to ${}^{t+\Delta t}x_j$.

In a dynamic loading of saturated soil systems, there are three contributions to the body forces ${}^{t+\Delta t}f_i^B$ in Eq. (3.1):

${}^{t+\Delta t}(\varrho b_i)$ body force due to gravity or centrifugal acceleration, where ${}^{t+\Delta t}\varrho$ is the mass density of the soil and ${}^{t+\Delta t}b_i$ is the i-th component of body force per unit mass, both measured at time $t + \Delta t$,

${}^{t+\Delta t}(\varrho \ddot{u}_i)$ body force due to acceleration of the soil skeleton ${}^{t+\Delta t}\ddot{u}_i$; negative sign is used because this force is in opposite direction to ${}^{t+\Delta t}\ddot{u}_i$,

${}^{t+\Delta t}f_i^{Bw}$ body force due to relative acceleration of the pore water with respect to the soil skeleton.

The first two terms are common in any structural dynamics problem, but the third term ${}^{t+\Delta t}f_i^{Bw}$ is due to the presence of water and its relative motion with respect to the soil skeleton. To account for ${}^{t+\Delta t}f_i^{Bw}$, we note that in a differential volume ${}^{t+\Delta t}dV$ of the soil with porosity n, only $(n\, {}^{t+\Delta t}dV)$ is occupied by the pore water, therefore $\varrho_f(n\, {}^{t+\Delta t}dV)$ is mass of the pore water available in the differential volume of the soil. Here ϱ_f is the mass density of pore water and the following relation holds:

$$\varrho = n\varrho_f + (1-n)\varrho_s, \tag{3.3}$$

where ϱ_s is the mass density of solid particles.

Now if we define a relative average or superficial displacement, w_i, so that $\dot{w}_i$ is the relative superficial velocity (1) of the pore water with respect to soil skeleton (in the direction of axis i, $i = 1,2,3$), the actual displacement of water in the pores is w_i/n. The body force due to the relative acceleration of pore water with respect to the soil skeleton is expressed by

$$ {}^{t+\Delta t}f_i^{Bw} = -\frac{\left[\varrho_f\left(n\,{}^{t+\Delta t}dV\right)\right]\left(\dfrac{1}{n}\dfrac{D\dot{w}_i}{Dt}\right)}{{}^{t+\Delta t}dV} = -\varrho_f\frac{D\dot{w}_i}{Dt}, \tag{3.4}$$

where D/Dt is the symbol of total derivative with respect to time (2). Here we must use a total time derivative, because $\dot{w}_i$ is measured with respect to the soil skeleton that itself is moving and makes a moving coordinate system for measuring $\dot{w}_i$. The negative sign in Eq. (3.4) is used because the ${}^{t+\Delta t}f_i^{Bw}$ applies in the opposite direction of water flow. It must also be noted that the effect of change of porosity has been ignored in the acceleration term in Eq. (3.4). This effect will be very small during a usual time step.

Considering the above mentioned contributions to the body force ${}^{t+\Delta t}f_i^{Bw}$, we can now write Eq. (3.1) as

$$\begin{aligned} {}^{t+\Delta t}W_v^{\text{int}} &= \int\limits_{{}^{t+\Delta t}V} {}^{t+\Delta t}\sigma_{ij}\,\delta\,{}_{t+\Delta t}e_{ij}\,{}^{t+\Delta t}dV \\ &= {}^{t+\Delta t}W_v^{\text{ext}} = \int\limits_{{}^{t+\Delta t}S_T} {}^{t+\Delta t}f_i^S\,\delta u_i^S\,{}^{t+\Delta t}dS \\ &+ \int\limits_{{}^{t+\Delta t}V}\left[{}^{t+\Delta t}\varrho\,{}^{t+\Delta t}b_i - {}^{t+\Delta t}\varrho\,{}^{t+\Delta t}\ddot{u}_i - {}^{t+\Delta t}\varrho_f\,{}^{t+\Delta t}\left(\frac{D\dot{w}_i}{Dt}\right)\right]\delta u_i\,{}^{t+\Delta t}dV. \end{aligned} \tag{3.5}$$

There are two major difficulties in application of Eq. (3.5) to a finite deformation problem involving saturated soils. First, the configuration at time $t+\Delta t$ is

(1) This is the superficial velocity of water used in Darcy's law for seepage of water through a porous medium, i.e. $\dot{w}_i = v_i = k_{ij}(\partial h/\partial x_j)$, where k_{ij} is the hydraulic conductivity of the soil in the direction i due to a unit flow in the direction j and h is the hydraulic potential at the point of interest.

(2) The material time derivative or the rate of a quantity, $A = A(x(t), t)$ is defined as

$$\frac{DA}{Dt} = \dot{A} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x_i}\dot{x}_i,$$

where A is a scalar quantity and it is a function of time and space.

unknown and the integration over the ${}^{t+\Delta t}V$ and ${}^{t+\Delta t}S_T$ cannot be performed before calculating the equilibrium position at time $t + \Delta t$. The second difficulty is the presence of total stress tensor, ${}^{t+\Delta t}\sigma_{ij}$ in Eq. (3.5) which does not have any direct influence on the mechanical behaviour of the soil and cannot be used in a realistic constitutive equation relating a proper measure of stress to a measure of strain. To resolve the first difficulty, we can rewrite Eq. (3.5) by referring the applied forces, stresses, and strains to a known equilibrium configuration, such as the initial configuration at time 0 (Total Lagrangian Formulation) or the configuration at time t (Updated Lagrangian Formulation). The second problem can be resolved by applying the principle of effective stress and introducing effective stresses in Eq. (3.5). The aforementioned measures are adopted in the following sections.

4. The principle of effective stress

Terzaghi's principle of effective stress can be written in the following form:

$$\sigma_{ij} = \overline{\sigma}_{ij} - p\delta_{ij}, \tag{4.1}$$

where σ_{ij} and $\overline{\sigma}_{ij}$ are the total stress and effective stress tensors, respectively, and p stands for the pore water pressure. The δ_{ij} is the Kronecker delta defined as

$$\begin{aligned} \delta_{ij} &= 1 \qquad \text{for} \quad i = j, \\ \delta_{ij} &= 0 \qquad \text{for} \quad i \neq j. \end{aligned}$$

In direct notation, Eq. (4.1) can be written as

$$\sigma = \overline{\sigma} - p\mathbf{1}, \tag{4.2}$$

where $\mathbf{1}$ is the symbolic form of the Kronecker delta.

Here the conventional sign convention of solid mechanics is used which considers tensile stresses as positive values and compressive stresses as negative values. The negative sign of p in Eqs. (4.1) or (4.2) is associated with the fact that pore pressure is considered as a compressive stress.

Since the effective stress principle is defined in terms of the Cauchy stress tensor which is not an objective measure of stress, it is important to establish a suitable rate form for Eq. (4.2). Taking the time derivative of Eq. (4.2), we find

$$\frac{D}{Dt}(\boldsymbol{\sigma}) = \frac{D}{Dt}(\overline{\boldsymbol{\sigma}}) - \frac{D}{Dt}(p\mathbf{1}) \tag{4.3}$$

or

$$\begin{aligned} \overset{\triangledown}{\boldsymbol{\sigma}} + \sigma_{ij}(\dot{\mathbf{e}}_i \otimes \mathbf{e}_j) + \sigma_{ij}(\mathbf{e}_i \otimes \dot{\mathbf{e}}_j) = \overset{\triangledown}{\overline{\boldsymbol{\sigma}}} &+ \overline{\sigma}_{ij}(\dot{\mathbf{e}}_i \otimes \mathbf{e}_j) + \overline{\sigma}_{ij}(\mathbf{e}_i \otimes \dot{\mathbf{e}}_j) \\ &- \dot{p}\,\mathbf{1} - p\delta_{ij}(\dot{\mathbf{e}}_i \otimes \mathbf{e}_j) - p\delta_{ij}(\mathbf{e}_i \otimes \dot{\mathbf{e}}_j), \end{aligned} \tag{4.4}$$

where $\mathbf{e}_i$ and $\mathbf{e}_j$ are the unit vectors in a Cartesian coordinate system. The $\overset{\nabla}{\boldsymbol{\sigma}}$ and $\overset{\nabla}{\bar{\boldsymbol{\sigma}}}$ are corotational rates of the total stress and effective stress tensors, respectively. Using Eq. (4.2), we can write the above equation as

$$\overset{\nabla}{\boldsymbol{\sigma}} = \overset{\nabla}{\bar{\boldsymbol{\sigma}}} - \dot{p}\mathbf{1}. \tag{4.5}$$

Equation (4.5) is of paramount importance in our subsequent developments. We will use this equation in development of the incremental equations governing the dynamics of saturated soils.

As previously mentioned, Eqs. (4.2) and (4.5) enable us to formulate the governing equations of motion (Eq. $(3.5)_1$) in terms of effective stresses. However, substitution of Eq. (4.2) in Eq. (3.5) leads to the appearance of a pore pressure related term which prevents a direct application of Eq. (3.5) as a sole field equation in the solution of boundary value problems in soil dynamics. The additional unknown, p, requires an additional field equation which governs the flow of water through the soil. Derivation of this equation is the subject of the next section.

5. Equations governing the flow of water through a saturated soil

In Sec. 3, we derived an integral equation governing the motion of the soil mass by making use of the principle of virtual work for the bulk mass of the soil body. In this section, we consider the equations of motion and mass balance for the pore fluid (water) alone in order to establish a complementary equation to Eq. (3.5). To this end, let us consider a unit volume of the soil in the current configuration at time $t + \Delta t$ as a control volume for the flow of the pore water. We assume that the coordinate system is attached to the soil skeleton and is moving with it. The flow of water in this control volume is affected by inertial forces and by a viscous (velocity-dependent) drag force due to interaction of the pore water and solid grains. In the following consideration, it is assumed that the viscous drag force can be determined by application of Darcy's law. In a quasi-static flow of the pore water, Darcy's equation is written as

$$\dot{w}_i = -k_{ij}\frac{\partial p}{\partial x_j} \tag{5.1}$$

in which

$$k_{ij} = \frac{1}{\gamma_w}\overset{*}{k}_{ij}, \tag{5.2}$$

where $\overset{*}{k}_{ij}$ is a component of the permeability tensor. The $\dot{w}_i$ in Eq. (5.1) is the superficial velocity of water, i.e. the volume of water flowing per unit time and per unit gross area through the face of the control volume perpendicular to the

x_i axis. The negative sign in Eq. (5.1) emphasizes that the water flow occurs in the direction of decreasing potential.

Now if we define the resistivity tensor r_{ij} as the inverse of the specific permeability tensor,

$$r_{ij}k_{jk} = \delta_{ik}\,, \tag{5.3}$$

Eq. (5.1) can be written as:

$$\frac{\partial p}{\partial x_i} = -r_{ij}\,\dot{w}_j = \Re_i\,, \tag{5.4}$$

where $\Re_i$ is the viscous drag force in the direction of x_i axis applied to the pore water flowing through a unit control volume of the soil. Considering the effects of the inertial and body forces (Fig. 2), Eq. (5.4) can be generalized,

$$-\frac{\partial p}{\partial x_i} - r_{ij}\,\dot{w}_j + \varrho_f\left(b_i - \ddot{u}_i - \frac{D\,\dot{w}_i}{Dt}\right) = 0, \tag{5.5}$$

where

$$\ddot{u}_i + \frac{D\,\dot{w}_i}{Dt} = \ddot{u}_i^{\text{tot}}$$

represents the total acceleration of pore water.

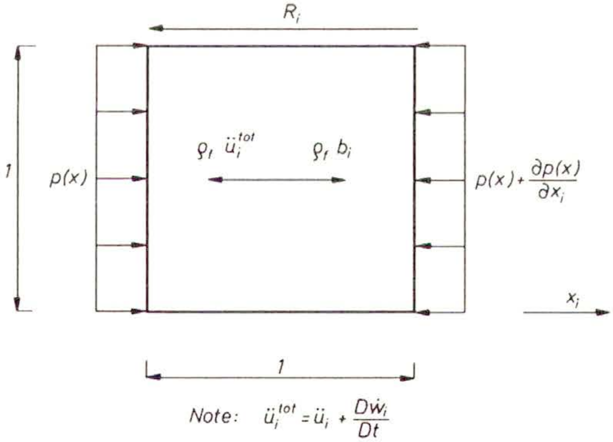

FIG. 2. Free body diagram for the pore fluid in a control volume.

In order to reduce Eq. (5.5) to a form containing only the displacements of soil skeleton (**u**) and pore water pressure (p), we first use the axiom of mass balance to establish a relationship between the rate of change of pore pressure $\dot{p}$ and the rates of volumetric strains for the pore water $\dot{w}_{i,i}$ and the soil skeleton $\dot{u}_{i,i}$. Such a relationship can be used to remove the relative displacement of the pore water w from Eq. (5.5).

Let us consider a unit volume of the soil in which the masses of the pore water and solid grains are respectively $n\varrho_f$ and $(1-n)\varrho_S$. The axiom of mass balance requires that in the process of flow of the water through the soil, these two masses must be conserved, i.e.

$$\frac{D}{Dt}\int_V (n\varrho_f)\, dV = 0, \tag{5.6}$$

$$\frac{D}{Dt}\left[\int_V (1-n)\varrho_S\, dV\right] = 0. \tag{5.7}$$

Equations (5.6) and (5.7) lead to:

$$\dot{n}\varrho_f + n\dot{\varrho}_f + (n\varrho_f)\dot{U}_{i,i} = 0, \tag{5.8}$$

$$-\dot{n}\varrho_S + (1-n)\dot{\varrho}_S + (1-n)\varrho_S\,\dot{u}_{i,i} = 0, \tag{5.9}$$

where $\dot{U}_i$ is the component of the absolute velocity of pore water in the direction of x_i axis, i.e.

$$\dot{w}_i = n(\dot{U}_i - \dot{u}_i). \tag{5.10}$$

Dividing Eqs. (5.8) and (5.9) by ϱ_f and ϱ_S, respectively, and adding up these two equations, we find:

$$\frac{n\dot{\varrho}_f}{\varrho_f} + (1-n)\frac{\dot{\varrho}_S}{\varrho_S} + \left[n(\dot{U}_{i,i} - \dot{u}_{i,i})\right] + \dot{u}_{i,i} = 0, \tag{5.11}$$

or by using Eq. (5.10), we have:

$$\frac{n\dot{\varrho}_f}{\varrho_f} + (1-n)\frac{\dot{\varrho}_S}{\varrho_S} + \dot{w}_{i,i} + \dot{u}_{i,i} = 0. \tag{5.12}$$

The first term in the above equation represents the compressibility of the pore fluid (water) which is of cardinal importance in dynamic analysis of saturated soils. In order to stress the importance of this term, it suffices to mention that the compressibility of pore water (fluid) is highly dependent on the degree of saturation of the soil, and a small fraction of percentage of air in the pore water may significantly increase its compressibility [31]. The second term in Eq. (5.12) accounts for the compressibility of solid grains and, in general, is much smaller than the first term. In the following considerations, we seek to substitute the first two terms in Eq. (5.12) by means of simple constitutive equations. To this end, we note that a change of effective stress will result in a change of volume of solid particles, while a pore pressure change will induce a change of volume in both the solid particles and the pore water. Thus

$$
(5.13)\qquad
\begin{aligned}
(\dot{\varrho}_f) &= \frac{\partial(\varrho_f)}{\partial p}\,\dot{p}\,,\\
(\dot{\varrho}_S) &= \frac{\partial(\varrho_S)}{\partial p}\,\dot{p} + \frac{\partial(\varrho_S)}{\partial(\overline{\sigma}_{ii})}\,\dot{\overline{\sigma}}_{ii}\,.
\end{aligned}
$$

In practice, the $\dot{\varrho}_S$ is very small and negligible as compared to the $\dot{\varrho}_f$. Thus it can be ignored in the subsequent procedure. However, it is kept in the formulation for the comparison purposes. It is noted that the constitutive law representing the change of ϱ_S is similar for the change of hydrostatic pressure or the change of pore water pressure. Therefore, the terms on the right-hand side in Eq. $(5.13)_2$ can be described in terms of the change of hydrostatic total stress (σ_{ii}), i.e.

$$
(5.13)_3\qquad (\dot{\varrho}_S) = \frac{\partial(\varrho_S)}{\partial\sigma_{ii}}\,\dot{\sigma}_{ii}\,.
$$

It is also assumed that the following linear relationships exist between the change of pore water pressure (or any hydrostatic pressure) and the changes of volumes of the pore water and solid grains:

$$
(5.14)\qquad \frac{\dfrac{\partial V_S}{\partial p}}{V_S} = \frac{1}{K_S}\,,
$$

$$
(5.15)\qquad \frac{\dfrac{\partial V_w}{\partial p}}{V_w} = \frac{1}{K_f}\,,
$$

where V_S and V_w are the volumes of solid grains and the pore water in a unit volume of the soil mixture, respectively, while K_S and K_f indicate the compressibility of the above constituents. In general, K_S is by several orders of magnitude larger than K_f. Considering $V_S = (1-n)\varrho_S$ and $V_w = n\varrho_f$, and ignoring the change of soil porosity due to the change of p, we can rewrite Eqs. (5.14) and (5.15) as

$$
(5.16)\qquad \frac{\dfrac{\partial \varrho_S}{\partial p}}{\varrho_S} = \frac{1}{K_S}\,,
$$

$$
(5.17)\qquad \frac{\dfrac{\partial \varrho_f}{\partial p}}{\varrho_f} = \frac{1}{K_f}\,.
$$

Substituting Eqs. (5.16) and (5.17) in Eq. (5.12), we finally find the equation of mass balance in a desired form:

$$\left(\frac{n}{K_f} + \frac{1-n}{K_S}\right)\dot{p} + \dot{w}_{i,i} + \dot{u}_{i,i} = 0. \tag{5.18}$$

Denoting:

$$\frac{1}{\Gamma} = \frac{n}{K_f} + \frac{1-n}{K_S} \tag{5.19}$$

Eq. (5.18) is written as

$$\frac{1}{\Gamma}\dot{p} + \dot{w}_{i,i} + \dot{u}_{i,i} = 0. \tag{5.20}$$

Equations (5.20) and (5.5) yield the following relations:

$$\dot{w}_{i,i} = \frac{\partial}{\partial x_i}\left\{k_{ij}\left[-\frac{\partial p}{\partial x_j} + \varrho_f\left(b_j - \ddot{u}_j - \frac{D\dot{w}_j}{Dt}\right)\right]\right\} = -\frac{1}{\Gamma}\dot{p} - \dot{u}_{i,i} \tag{5.21}$$

or

$$\frac{1}{\Gamma}\dot{p} + \dot{u}_{i,i} - \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial p}{\partial x_j}\right) + \frac{\partial}{\partial x_i}(k_{ij}\varrho_f b_j) - \frac{\partial}{\partial x_i}\left[k_{ij}\varrho_f\left(\ddot{u}_j + \frac{D\dot{w}_j}{Dt}\right)\right] = 0. \tag{5.22}$$

This is the final equation governing the flow of the pore fluid (water) through the soil and combines the axiom of mass conservation and equation of motion for the pore fluid. Presence of the term $D\dot{w}_j/Dt$ in the above equation is still an undesirable feature which inhibits a direct coupling of Eq. (5.22) with Eq. (3.5) in order to get a coupled systems of equations in terms of $\mathbf{u}$ and p. However, it has been shown [49] that for the range of frequencies encountered in the earthquake loading, the relative acceleration of the pore water with respect to the soil skeleton is negligible. Therefore by ignoring $D\dot{w}_j/Dt$ in Eq. (5.22), we find:

$$\frac{1}{\Gamma}\dot{p} + \dot{u}_{i,i} - \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial p}{\partial x_j}\right) + \frac{\partial}{\partial x_i}(k_{ij}\varrho_f b_j) - \frac{\partial}{\partial x_i}(k_{ij}\varrho_f\,\ddot{u}_j) = 0. \tag{5.23}$$

Equation (5.23) is written in terms of u and p and is suitable to be solved in combination with Eq. (3.5), for which we also neglect the $D\dot{w}_j/Dt$ term for the foregoing reasons.

6. Boundary conditions

As it was mentioned in Sec. 2, we assume that four types of boundary conditions are specified on different portions of the boundary surface ${}^{t+\Delta t}S$ of the soil body at a generic time $t + \Delta t$. These boundary conditions are defined in the following sub-sections.

6.1. Displacement boundary condition

It is assumed that on a portion of the boundary surface ${}^{t+\Delta t}S$ of the soil body, displacements of soil skeleton are specified as follows:

$$(6.1) \qquad {}^{t+\Delta t}u_i = {}^{t+\Delta t}\overline{u}_i \qquad \text{on} \quad {}^{t+\Delta t}S_u ,$$

where ${}^{t+\Delta t}\overline{u}_i$ is the specified value of displacement on the boundary surface ${}^{t+\Delta t}S_u$ at time $t + \Delta t$.

6.2. Pore pressure boundary condition

The pore water pressure boundary condition is defined on ${}^{t+\Delta t}S_p$ as follows:

$$(6.2) \qquad {}^{t+\Delta t}p = {}^{t+\Delta t}\overline{p} \qquad \text{on} \quad {}^{t+\Delta t}S_p ,$$

where ${}^{t+\Delta t}\overline{p}$ is the specified pressure on the surface ${}^{t+\Delta t}S_p$ at time $t + \Delta t$.

6.3. Traction boundary condition

We assume that on a portion of the boundary surface, there is a specified traction which must be in equilibrium with the internal total stresses, i.e.

$$(6.3) \qquad {}^{t+\Delta t}\sigma_{ij} n_j = {}^{t+\Delta t}f_i^S \qquad \text{on} \quad {}^{t+\Delta t}S_T ,$$

where the ${}^{t+\Delta t}f_i^S$ is the specified traction on the surface ${}^{t+\Delta t}S_T$ with a unit normal of $\mathbf{n}$, and ${}^{t+\Delta t}\sigma_{ij}$ is the total Cauchy stress tensor acting on the neighborhood of the ${}^{t+\Delta t}f_i^S$.

6.4. Water flow boundary condition

It is assumed that on some portion of the boundary surface, the water flow boundary conditions are specified. One of the typical examples of such boundary conditions is the impervious boundary. The water flow boundary condition follows from Eq. (5.8) and is expressed as a flux condition, i.e.

$$(6.4) \qquad \dot{w}_i n_i = \left[-k_{ij}\frac{\partial p}{\partial x_j} + k_{ij}\varrho_f b_j - k_{ij}\varrho_f \left(\ddot{u}_j + \frac{D\dot{w}_j}{Dt} \right) \right] n_i = {}^{t+\Delta t}\overline{q}_S \quad \text{on} \quad {}^{t+\Delta t}S_q ,$$

where n_i denotes the i-th component of the outward unit normal to the surface ${}^{t+\Delta t}S_q$, and ${}^{t+\Delta t}\overline{q}_S$ is the prescribed fluid flow on the ${}^{t+\Delta t}S_q$.

7. Constitutive equations for the soil skeleton

The governing field equations developed in Sec. 3 (Eq. (3.5)) and in Sec. 5 (Eq. (5.23)) along with the boundary conditions defined in Sec. 6 are not sufficient to solve a boundary value problem in soil dynamics. For ten unknowns (3 displacements of soil skeleton, pore pressure, and six components of stress tensor) in a boundary value problem, we have established only four governing equations. Thus six constitutive equations are necessary to make the problem well-posed. Due to nonlinearity of soil behaviour, it is desirable to define the constitutive equations in a rate form relating an appropriate measure of stress to the rate of deformation. In a finite deformation analysis, an objective stress rate must be used to ensure that the effects of rigid body rotation are correctly considered. This criterion, however, does not determine completely which stress rate should be used. There are different forms of stress rates which satisfy the objectivity requirement. The most commonly used objective stress rate is the JAUMANN [27, 28] corotational rate of the Cauchy stress tensor, $\overset{\nabla}{\sigma}_{ij}$, defined as follows:

$$\overset{\nabla}{\sigma}_{ij} = \dot{\sigma}_{ij} + \sigma_{ik}\Omega_{kj} + \sigma_{jk}\Omega_{ki}\,, \tag{7.1}$$

where $\dot{\sigma}_{ij}$ is a Cartesian component of the material (time) derivative of the Cauchy stress tensor, and Ω_{ij} is a Cartesian component of the spin tensor, i.e.

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} - \frac{\partial \dot{u}_j}{\partial x_i}\right). \tag{7.2}$$

Numerous application of the Jaumann stress rate have been reported in the finite deformation analysis of crystalline solids in the crystal plasticity context (e.g. 22, 35). In a crystal plasticity application, the material spin tensor $\mathbf{\Omega}$ is replaced by the rate of rotation or spin of the crystal lattice. However for non-crystalline solids, a proper choice of the spin tensor is not clear. Previous study by NAGTEGAAL and DE JOND [34] has shown that a direct application of Eq. (7.2) in the large strain simple shear analysis of a material obeying a Mises-type kinematic hardening plasticity results in an oscillatory response during monotonic shearing. Such an unrealistic result has motivated several investigators (e.g. reference [13]) to explore the possibility of removing the stress oscillation by using different spin tensors. Later the original suggestion by MANDEL [32] and KRATOCHVIL [29] for a decomposition of the spin tensor to an "elastic" or "rigid" part and a plastic part, and Mandel's concept of material underlying substructure, motivated DAFALIAS [13, 14, 15] and LORET [30] to propose some constitutive equations for the plastic spin in the case of anisotropic materials. These studies suggested that the "elastic" part of the spin tensor must be used in a Jaumann-type corotational rate.

The concept of plastic spin has received increasing attention in the recent years and many investigators have studied the effect of plastic spin on the large deformation of solid materials (e.g. [44, 45]). One interesting point shown in the closed form analytical solutions presented by Dafalias [13, 14, 15] is that unless strong initial anisotropy preexists, the difference in the material response between using the substructure and material spin for a material which is initially isotropic becomes important only after very large strains (of the order of 100%) are developed.

In the light of the above discussion and due to the lack of experimental data necessary for calibration of the constitutive equations for the plastic spin, we will use a corotational stress rate without restricting the formulation to particular choices of the spin tensor.

Assuming an inelastic behaviour for the soil skeleton, we choose the following incremental form to relate the corotational rate of the effective stress tensor $\overset{\nabla}{\bar{\sigma}}_{ij}$ to the rate of deformation tensor $d_{kl} = 1/2(\partial \dot{u}_k/\partial x_l + \partial \dot{u}_l/\partial x_k)$,

$$\overset{\nabla}{\bar{\sigma}}_{ij} = D_{ijkl} d_{kl}, \tag{7.3}$$

where **D** is the *tangential stiffness tensor* which may be a function of the current state of effective stresses, strains and some internal variables.

The specific form of the tangential stiffness tensor will depend upon the type of mathematical framework (e.g., elasticity, plasticity, viscoplasticity, etc.) that we choose to model the behaviour of the soil skeleton. Equation (7.3) is general enough to enclose a wide variety of existing frameworks for the soil constitutive modeling.

8. Expression of the virtual work equation in terms of the coordinates of the configuration at time t

As mentioned in Sec. 3, all the integrals appeared in Eq. (3.5) must be written in terms of a known configuration, such as the initial configuration of the soil body (${}^0\beta$) or its converged equilibrium position at the end of the previous time step (${}^t\beta$). Here we choose the latter option and our aim in this section is to rewrite Eq. (3.5) in terms of the coordinates of the configuration at time t.

Let us consider an infinitesimal cubic element of the soil body (Fig. 3) whose volume in the configuration at time t can be expressed as ${}^t dV = \prod_{i=1}^{3} dx_i$. During the motion of soil from time t to time $t + \Delta t$, the material enclosed in the cubic element ${}^t dV$ will occupy a new volume of ${}^{t+\Delta t}dV$ and the initial shape of the element will be distorted. Considering the axiom of mass balance, we can relate ${}^{t+\Delta t}dV$ to ${}^t dV$ by the following equation:

$${}^0\varrho\, {}^0dV = {}^t\varrho\, {}^t dV = {}^{t+\Delta t}\varrho\, {}^{t+\Delta t}dV, \tag{8.1}$$

where ${}^{0}\varrho$, ${}^{t}\varrho$, and ${}^{t+\Delta t}\varrho$ are the mass densities per unit volume in the configurations at time 0, t, $t+\Delta t$, respectively. The ${}^{0}dV$ is the volume of the infinitesimal element in the initial configuration at time 0 (${}^{0}\beta$).

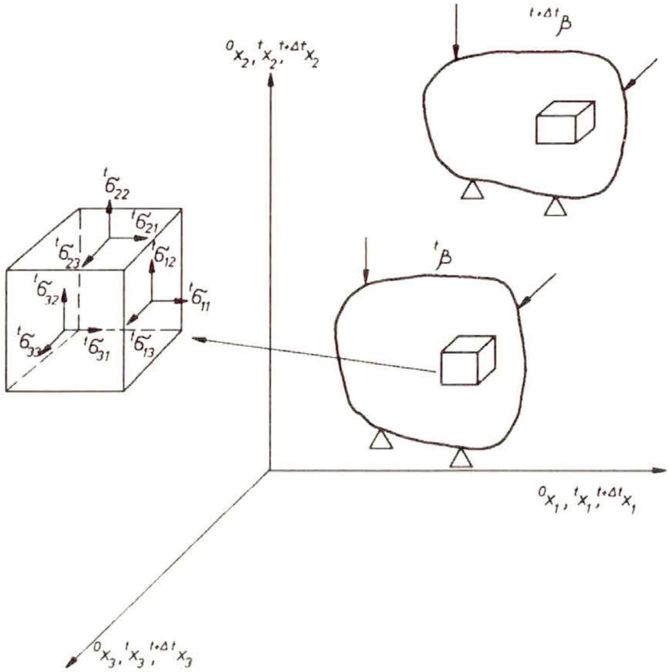

FIG. 3. The soil body at two subsequent configurations.

In general, the external loading, such as surface traction, external water pressure, gravitational and centrifugal loading are deformation-dependent. However, in most geotechnical structures, the aforementioned loading does not induce such a large displacement, large strain, or large rotation which would require a finite deformation analysis. Therefore, it is reasonable to assume that the magnitude and direction of surface force and body forces are independent of the current configuration of the soil body, i.e. [3]

(8.2)
$$\begin{aligned} {}^{t+\Delta t}b_i &= {}^{t+\Delta t}_{\ \ t}b_i\,, \\ {}^{t+\Delta t}f_i^S\ {}^{t+\Delta t}dS &= {}^{t+\Delta t}_{\ \ t}f_i^S\ {}^{t}dS\,, \end{aligned}$$

where ${}^{t+\Delta t}_{\ \ t}b_i$ and ${}^{t+\Delta t}_{\ \ t}f_i^S$ are respectively the body force and surface traction in the configuration at time $t+\Delta t$ (${}^{t+\Delta t}\beta$), and measured in the configuration at time t (${}^{t}\beta$). Combining Eqs. (8.1) and $(8.2)_1$, we have:

(8.3)
$$ {}^{t+\Delta t}\varrho\ {}^{t+\Delta t}b_i\ {}^{t+\Delta t}dV = {}^{t}\varrho\ {}^{t+\Delta t}_{\ \ t}b_i\ {}^{t}dV\,. $$

If we further assume that the effect of the pore water relative acceleration $D\dot{w}_j/Dt$ with respect to the soil skeleton is negligible as compared to the inertial effect of the soil bulk mass, we can rewrite Eq. (3.5) by using Eqs. (8.1), (8.2) and (8.3):

$$(8.4)\qquad {}^{t+\Delta t}W_{\nu}^{\text{ext}} = \int_{{}^{t}V} {}^{t}\varrho \; {}^{t+\Delta t}_{t}b_i \; \delta u_i \; {}^{t}dV - \int_{{}^{0}V} {}^{0}\varrho \; {}^{t+\Delta t}\ddot{u}_i \; \delta u_i \; {}^{0}dV + \int_{{}^{t}S_T} {}^{t+\Delta t}_{t}f_i^S \; \delta u_i^S \; {}^{t}dS.$$

The second integral on the r.h.s. in Eq. (8.4) is evaluated using the initial configuration (${}^{0}\beta$) and hence its contribution can be calculated prior to the incremental step-by-step analysis.

As to the internal virtual work (Eq. $(3.5)_1$), we first use the principle of effective stress (Eq. (4.1)) to rewrite $(3.5)_1$ in terms of effective stresses. Thus, substituting Eq. (4.1) in Eq. $(3.5)_1$ leads to

$$\begin{aligned} {}^{t+\Delta t}W_{\nu}^{\text{int}} &= \int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\sigma_{ij} \; \delta \; {}_{t+\Delta t}e_{ij} \; {}^{t+\Delta t}dV \\ &= \int_{{}^{t+\Delta t}V} \left({}^{t+\Delta t}\overline{\sigma}_{ij} - {}^{t+\Delta t}p \; \delta_{ij} \right) \delta \; {}_{t+\Delta t}e_{ij} \; {}^{t+\Delta t}dV \end{aligned}$$

or

$$(8.5)\qquad {}^{t+\Delta t}W_{\nu}^{\text{int}} = \int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\overline{\sigma}_{ij} \; \delta \; {}_{t+\Delta t}e_{ij} \; {}^{t+\Delta t}dV - \int_{{}^{t+\Delta t}V} {}^{t+\Delta t}p \; \delta_{ij}\delta \; {}_{t+\Delta t}e_{ij} \; {}^{t+\Delta t}dV.$$

We now need to refer the Cauchy effective stress tensor ${}^{t+\Delta t}\overline{\sigma}_{ij}$ and the infinitesimal strain tensor ${}_{t+\Delta t}e_{ij}$ to the configuration at time t (${}^{t}\beta$). It is well known that the *second Piola–Kirchhoff stress tensor* ${}^{t+\Delta t}_{t}S_{ij}$ and the *Green–Lagrange strain tensor* ${}^{t+\Delta t}_{t}\varepsilon_{ij}$ are a work-conjugate pair of stress and strain measures which relate the ${}^{t+\Delta t}\overline{\sigma}_{ij}$ and ${}_{t+\Delta t}e_{ij}$ to the configuration at time t. The second Piola–Kirchhoff stress tensor ${}^{t+\Delta t}_{t}S_{ij}$ is defined as [8]:

$$(8.6)\qquad {}^{t+\Delta t}_{t}S_{ij} = \frac{{}^{t}\varrho}{{}^{t+\Delta t}\varrho} \frac{\partial^{t} x_i}{\partial^{t+\Delta t} x_m} \overline{\sigma}_{mn} \frac{\partial^{t} x_j}{\partial^{t+\Delta t} x_n}\,.$$

The Green–Lagrange strain tensor can be defined by considering the deformation of a generic line segment of the soil body whose lengths are denoted by ${}^{t}ds$ and ${}^{t+\Delta t}ds$ in the configurations at time t and $t+\Delta t$, respectively. Without giving the details of this derivation, we find [8]:

$$(8.7)\qquad {}^{t+\Delta t}_{t}\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial^{t} x_j} + \frac{\partial u_j}{\partial^{t} x_i} + \frac{\partial u_k}{\partial^{t} x_i}\frac{\partial u_k}{\partial^{t} x_j} \right).$$

Taking a variation of the both sides of Eq. (8.7), we have

$$\text{(8.8)} \qquad \delta\, {}^{t+\Delta t}_{t}\varepsilon = \frac{1}{2}\left[\frac{\partial(\delta u_i)}{\partial^t x_j} + \frac{\partial(\delta u_j)}{\partial^t x_i} + \frac{\partial(\delta u_k)}{\partial^t x_i}\frac{\partial u_k}{\partial^t x_j} + \frac{\partial u_k}{\partial^t x_i}\frac{\partial(\delta u_k)}{\partial^t x_j}\right],$$

where δu_i is the variation (virtual displacement) in the displacement ${}^{t+\Delta t}u_i$. We also note that:

$$\text{(8.9)} \qquad \frac{\partial^{t+\Delta t}x_m}{\partial^t x_i} = \delta_{mi} + \frac{\partial u_m}{\partial^t x_i}\,.$$

Combining Eq. (3.2) and the above equation, we can relate the variation $\delta\, {}^{t+\Delta t}_{t}\varepsilon_{ij}$ to $\delta\, {}_{t+\Delta t}e_{mn}$ in the following manner:

$$\text{(8.10)} \qquad \delta\, {}^{t+\Delta t}_{t}\varepsilon_{ij} = \frac{\partial^{t+\Delta t}x_m}{\partial^t x_i}\,\delta\, {}_{t+\Delta t}e_{mn}\,\frac{\partial^{t+\Delta t}x_n}{\partial^t x_j}\,.$$

Finally by using Eqs. (8.1), (8.6), and (8.10), we can write Eq. (8.5) as

$$\text{(8.11)} \qquad {}^{t+\Delta t}W_v^{\text{int}} = \int_V {}^{t+\Delta t}_{t}S_{ij}\;\delta\, {}^{t+\Delta t}_{t}\varepsilon_{ij}\; {}^t dV - \int_{{}^tV} \left({}^{t+\Delta t}_{t}h_{ij}\right)\delta\, {}^{t+\Delta t}_{t}\varepsilon_{ij}\; {}^t dV,$$

where

$$\text{(8.12)} \qquad {}^{t+\Delta t}_{t}h_{mn} = \frac{{}^t\varrho}{{}^{t+\Delta t}\varrho}\,\frac{\partial^t x_i}{\partial^{t+\Delta t}x_m}\,{}^{t+\Delta t}(p\delta_{ij})\,\frac{\partial^t x_j}{\partial^{t+\Delta t}x_n}\,.$$

Equation (8.11) together with Eq. (8.12) complete the virtual work expression in terms of the coordinates of the configuration at time t (${}^t\boldsymbol{\beta}$). However, in order to use this equation in an incremental analysis, it is necessary to establish its equivalent incremental form. Derivation of such incremental form will be discussed in the next section.

9. Incremental form of the virtual work equation

An incremental form of the internal virtual work equation (8.11) can be established by introducing truncated Taylor series expansions of the second Piola-Kirchhoff stress tensor and the **h** tensor, i.e.

$$\text{(9.1)} \qquad \begin{aligned} {}^{t+\Delta t}_{t}S_{ij} &= {}^t_tS_{ij} + \left[\frac{d}{dt}(S_{ij})\right]_t \Delta t + \text{higher order terms},\\ {}^{t+\Delta t}_{t}h_{ij} &= {}^t_th_{ij} + \left[\frac{d}{dt}(h_{ij})\right]_t \Delta t + \text{higher order terms},\end{aligned}$$

where

$$(9.2)\qquad {}^t_tS_{ij} = {}^t\overline{\sigma}_{ij} ,$$

$$(9.3)\qquad {}^t_th_{ij} = {}^tp\delta_{ij} .$$

Ignoring the higher order terms in Eqs. (9.1) and using Eqs. (9.2) and (9.3), we have

$$(9.4)\qquad \begin{aligned} {}^{t+\Delta t}_{\ \ \ t}S_{ij} &= {}^t\overline{\sigma}_{ij} + \left[\frac{d}{dt}(S_{ij})\right]_t \Delta t , \\ {}^{t+\Delta t}_{\ \ \ t}h_{ij} &= {}^tp\delta_{ij} + \left[\frac{d}{dt}(h_{ij})\right]_t \Delta t. \end{aligned}$$

In order to evaluate the second terms on the r.h.s in Eqs. (9.4), we make use of the following kinematic relationships [8]

$$(9.5)\qquad \frac{d}{dt}\left(\frac{{}^t\varrho}{{}^{t+\Delta t}\varrho}\right) = \frac{{}^t\varrho}{{}^{t+\Delta t}\varrho}\,\frac{\partial^{t+\Delta t}\nu_i}{\partial^{t+\Delta t}x_i} ,$$

$$(9.6)\qquad \frac{d}{dt}\left(\frac{\partial^t x_i}{\partial^{t+\Delta t}x_j}\right) = -\frac{\partial^{t+\Delta t}\nu_k}{\partial^{t+\Delta t}x_j}\,\frac{\partial^t x_i}{\partial^{t+\Delta t}x_k} ,$$

where ${}^{t+\Delta t}\nu_k$ denotes the velocity of the soil mass in the direction of axis k. Utilizing Eqs. (9.5), (9.6) and (8.6), we find:

$$(9.7)\qquad \frac{d}{dt}(S_{ij}) = \frac{{}^t\varrho}{{}^{t+\Delta t}\varrho}\,\frac{\partial^t x_i}{\partial^{t+\Delta t}x_k}\,{}^{t+\Delta t}\dot{\overline{\sigma}}{}^T_{kl}\,\frac{\partial^t x_j}{\partial^{t+\Delta t}x_l} ,$$

in which ${}^{t+\Delta t}\dot{\overline{\sigma}}{}^T_{kl}$ is the Truesdell rate of the effective stress tensor $\overline{\sigma}_{kl}$ and defined as

$$(9.8)\qquad \begin{aligned} {}^{t+\Delta t}\dot{\overline{\sigma}}{}^T_{kl} = {}^{t+\Delta t}\dot{\overline{\sigma}}_{kl} + {}^{t+\Delta t}\nu_{m,m}\,{}^{t+\Delta t}\overline{\sigma}_{kl} - {}^{t+\Delta t}\nu_{l,m}\,{}^{t+\Delta t}\overline{\sigma}_{km} \\ - {}^{t+\Delta t}\nu_{k,m}\,{}^{t+\Delta t}\overline{\sigma}_{ml} . \end{aligned}$$

Since we seek to find $\left[\frac{d}{dt}(S_{ij})\right]_t$, Eq. (9.7) should be evaluated at time t, i.e.

$$(9.9)\qquad \left[\frac{d}{dt}(S_{ij})\right]_t = {}^t\dot{\overline{\sigma}}{}^T_{ij} = {}^t\dot{\overline{\sigma}}_{ij} + {}^t\nu_{m,m}\,{}^t\overline{\sigma}_{ij} - {}^t\nu_{j,m}\,{}^t\overline{\sigma}_{im} - {}^t\nu_{i,m}\,{}^t\overline{\sigma}_{mj} .$$

The Truesdell stress rate appearing in Eq. (9.9) can be related to the Jaumann stress rate by decomposing the velocity gradient $\nu_{i,m}$ to the sum of the rate of deformation tensor d_{im} and the spin tensor Ω_{im}, i.e.

$$(9.10)\qquad \nu_{i,m} = d_{im} + \Omega_{im} .$$

Substituting (9.10) in Eq. (9.9), leads to

$$\left[\frac{d}{dt}(S_{ij})\right]_t = {}^t\dot{\overline{\sigma}}_{ij} + {}^t\nu_{m,m}\,{}^t\overline{\sigma}_{ij} - \left({}^td_{jm} + {}^t\Omega_{jm}\right){}^t\overline{\sigma}_{im} - \left({}^td_{im} + {}^t\Omega_{im}\right){}^t\overline{\sigma}_{mj}\,, \tag{9.11}$$

or by using Eq. (7.1), we find:

$$\left[\frac{d}{dt}(S_{ij})\right]_t = {}^t\overset{\triangledown}{\overline{\sigma}}_{ij} + {}^t\nu_{m,m}\,{}^t\overline{\sigma}_{ij} - {}^td_{jm}\,{}^t\overline{\sigma}_{im} - {}^td_{im}\,{}^t\overline{\sigma}_{mj}\,, \tag{9.12}$$

where ${}^t\overset{\triangledown}{\overline{\sigma}}_{ij}$ is the Jaumann rate of the effective stress tensor.

Considering the general form of the constitutive equation (7.3) applied to ${}^t\overset{\triangledown}{\overline{\sigma}}_{ij}$ and substituting (9.12) in Eq. (9.4), we have:

$${}^{t+\Delta t}_{\ \ t}S_{ij} = {}^t\overline{\sigma}_{ij} + \Delta t\left({}_tD_{ijkl}\,{}^td_{kl} + {}^t\nu_{m,m}\,{}^t\overline{\sigma}_{ij} - {}^td_{jm}\,{}^t\overline{\sigma}_{im} - {}^td_{im}\,{}^t\overline{\sigma}_{mj}\right). \tag{9.13}$$

It must be noted that the ${}_tD_{ijkl}$ appearing from now on in the subsequent equations is the one which relates the rate of deformation tensor to the Jaumann rate of effective stress. However, if the initial formulation of the constitutive law calls for the use of a corotational rate with respect to a different spin than ${}^t\Omega_{ij}$, then one must perform a subsequent transformation to a Jaumann rate for the effective stress with simultaneous change of the constitutive moduli which will be again defined by ${}_tD_{ijkl}$ after the transformation.

Equation (9.13) can be written in a compact form by using the following relations:

$$\begin{aligned}\Delta t\,{}^td_{kl} &= \frac{1}{2}\Delta t\left(\frac{\partial\,{}^t\dot{u}_k}{\partial^t x_l} + \frac{\partial\,{}^t\dot{u}_l}{\partial^t x_k}\right) = \frac{1}{2}\left[\frac{\partial(\Delta u_k)}{\partial^t x_l} + \frac{\partial(\Delta u_l)}{\partial^t x_k}\right] = {}_te_{kl}\,,\\ \Delta t\,{}^t\nu_{m,m} &= \frac{\partial(\Delta u_m)}{\partial^t x_m} = {}_te_{mm}\,,\end{aligned} \tag{9.14}$$

where Δu_m is the m-th component of the incremental displacement at a generic point of the soil body. Thus Eq. (9.13) can be written as:

$${}^{t+\Delta t}_{\ \ t}S_{ij} = {}^t\overline{\sigma}_{ij} + {}_t\Lambda_{ijkl}\,{}_te_{kl}\,, \tag{9.15}$$

where

$${}_t\Lambda_{ijkl} = {}_tD_{ijkl} + {}^t\overline{\sigma}_{ij}\delta_{kl} - {}^t\overline{\sigma}_{il}\delta_{kj} - \delta_{il}\,{}^t\overline{\sigma}_{kj}\,. \tag{9.16}$$

The ${}_t\Lambda_{ijkl}$ is the finite deformation tensor of tangent stiffness moduli and it includes the regular tangent stiffness moduli tensor and the effect of stresses at the beginning of the step. It must be noted that if the components of the effective stress tensor are of the same order of magnitude as the ${}_tD_{ijkl}$, contribution of the initial stresses to the ${}_t\Lambda_{ijkl}$ tensor can be significant.

Similarly to Eq. (9.9) for the rate of the second Piola - Kirchhoff stress tensor, one can write the following equation for the rate of the h_{mn},

$$\left[\frac{d}{dt}(h_{mn})\right]_t = \frac{d}{dt}(p\delta_{mn}) + {}^t\nu_{k,k}\,{}^tp\,\delta_{mn} - {}^t\nu_{n,m}\,{}^tp - {}^t\nu_{m,n}\,{}^tp \tag{9.17}$$

recalling that

$${}^td_{mn} = \frac{1}{2}\left({}^t\nu_{m,n} + {}^t\nu_{n,m}\right), \tag{9.18}$$

Eq. (9.17) can be written as

$$\left[\frac{d}{dt}(h_{mn})\right]_t = {}^t\dot{p}\,\delta_{mn} + {}^t\nu_{k,k}\,{}^tp\,\delta_{mn} - 2\,{}^tp\,{}^td_{mn}\,. \tag{9.19}$$

Using Eqs. (9.20) and (9.14), we can now write $(9.1)_2$ as

$${}^{t+\Delta t}_{\ \ t}h_{mn} = {}^tp\,\delta_{mn} + ({}^t\Delta p\,\delta_{mn}) + {}_te_{kk}\,{}^tp\,\delta_{mn} - 2\,{}^tp\,{}_te_{mn}\,, \tag{9.20}$$

where

$${}^t(\Delta p) = {}^t\dot{p}\,\Delta t. \tag{9.21}$$

Regarding the variation in the Green - Lagrange strain tensor, i.e. $\delta\,{}^{t+\Delta t}_{\ \ t}\varepsilon_{ij}$, we note that:

$$\delta\,{}^{t+\Delta t}_{\ \ t}\varepsilon_{ij} = \delta\,{}_te_{ij} + \delta\,{}_t\eta_{ij}\,, \tag{9.22}$$

where

$${}_te_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial^t x_j} + \frac{\partial u_j}{\partial^t x_i}\right) \tag{9.23}$$

and

$${}_t\eta_{ij} = \frac{1}{2}\left(\frac{\partial u_k}{\partial^t x_i}\,\frac{\partial u_k}{\partial^t x_j}\right) \tag{9.24}$$

in which u_k is the incremental displacement.

Substituting Eqs. (9.15), (9.20), and (9.22) in Eq. (8.13), we find

$$\begin{aligned}{}^{t+\Delta t}W_v^{\text{int}} = &\int\limits_{{}^tV}\left({}^t\overline{\sigma}_{ij} + {}_t\Lambda_{ijkl}\,{}_te_{kl}\right)\left(\delta\,{}_te_{ij} + \delta\,{}_t\eta_{ij}\right){}^tdV \\ &- \int\limits_{{}^tV}\left({}^tp\delta_{mn} + {}^t\Delta p\delta_{mn} + {}_te_{kk}\,{}^tp\delta_{mn} - 2\,{}^tp\,{}_te_{mn}\right)\left(\delta\,{}_te_{mn} + \delta\,{}_t\eta_{mn}\right){}^tdV.\end{aligned} \tag{9.25}$$

This is the final form of the internal virtual work expression in terms of the coordinates of the configuration at time t. A more compact and computatonally useful form of Eq. (9.25) can be obtained by utilizing Eq. (4.1) and combining the effective stress-related terms implicit in the ${}_t\Lambda_{ijkl}$ tensor (Eq. (9.16)) with the pore pressure-related terms in Eq. (9.25). We finally find:

$$(9.26) \qquad {}^{t+\Delta t}W_v^{\text{int}} = \int_{{}^tV} \left({}^t\sigma_{ij} + {}_tL_{ijkl}\,{}_te_{kl}\right)\left(\delta\,{}_te_{ij} + \delta\,{}_t\eta_{ij}\right){}^tdV - \int_{{}^tV} \left({}^t\Delta p\right)\left(\delta\,{}_te_{ii} + \delta\,{}_t\eta_{ii}\right){}^tdV,$$

where

$$(9.27) \qquad {}_tL_{ijkl} = {}_tD_{ijkl} + {}^t\sigma_{ij}\,\delta_{kl} - {}^t\sigma_{il}\,\delta_{kj} - {}^t\delta_{il}\,{}^t\sigma_{kj}$$

or by using Eqs. (3.5), we have:

$$(9.28) \qquad \int_{{}^tV} {}_tL_{ijkl}\,{}_te_{kl}\,\delta\,{}_te_{ij}\,{}^tdV - \int_{{}^tV} {}^t\Delta p\,\delta\,{}_te_{ii}\,{}^tdV + \int_{{}^tV} {}_tL_{ijkl}\,{}_te_{kl}\,\delta\,{}_t\eta_{ij}\,{}^tdV + \int_{{}^tV} {}^t\sigma_{ij}\,\delta\,{}_t\eta_{ij}\,{}^tdV - \int_{{}^tV} {}^t(\Delta p)\,\delta\,{}_t\eta_{ii}\,{}^tdV = {}^{t+\Delta t}W_v^{\text{ext}} - \int_{{}^tV} {}^t\sigma_{ij}\,\delta\,{}_te_{ij}\,{}^tdV.$$

The last three terms on the left-hand side of Eq. (9.28) are due to finite deformation effects, and they may be omitted in a small deformation analysis. In the case of infinitesimal strains and small rotations, the ${}_tL_{ijkl}$ tensor will also reduce to the ${}_tD_{ijkl}$, tensor of tangent stiffness moduli. It should be mentioned that in an incremental numerical solution, Eq. (9.28) is normally linearized by ignoring the third and fifth terms on the left-hand side in this equation. This linearization is justified due to small effects of these higher order terms in a regular earthquake engineering problem, where the time steps are generally small if a plasticity-based constitutive model is to be used.

Here it must be noted that Eqs. (9.25) and (9.28) are incremental approximations of the internal virtual work at time $t + \Delta t$. These equations, along with equations governing the motion of the pore water (Eq. (5.23)), are used to calculate an incremental displacement and pore water pressure. The calculated incremental values are then used to evaluate approximations to the displacements of soil skeleton, strains, stresses, and pore water pressure at time $t + \Delta t$. The calculated values of displacements can be employed to establish an approximation to the configuration at time $t + \Delta t$ (${}^{t+\Delta t}\beta$, ${}^{t+\Delta t}V$, ${}^{t+\Delta t}S$). Therefore it is possible to calculate the difference between the internal virtual work evaluated with the calculated static and kinematic quantities at time $t + \Delta t$, and the external virtual work. In general, linearization of Eq. (9.28) introduces some errors and the aforementioned difference may not be negligible. Thus, in order to reduce the difference between the estimated internal work and the external work, an iterative solution strategy is necessary. Different schemes may be used for an iterative analysis. A full Newton-Raphson iteration scheme leads to the following form:

$$\begin{aligned}(9.29)\qquad &\int\limits_{^tV} {}_tL_{ijkl}{}^{(m)}\Delta\, {}_te_{kl}{}^{(m)}\,\delta\, {}_te_{ij}{}^{(m)}\, {}^tdV - \int\limits_{^tV} {}^t\Delta p^{(m)}\,\delta\, {}_te_{ii}{}^{(m)}\, {}^tdV \\ &+ \int\limits_{^tV} {}_tL_{ijkl}{}^{(m)}\,\Delta_t e_{kl}{}^{(m)}\,\delta\, {}_t\eta_{ij}{}^{(m)}\, {}^tdV + \int\limits_{^tV} {}^t\sigma_{ij}\,\delta\Delta\, {}_t\eta_{ij}{}^{(m)}\, {}^tdV \\ &- \int\limits_{^tV} {}^t(\Delta p)^{(m)}\,\delta\, {}_t\eta_{ii}{}^{(m)}\, {}^tdV = {}^{t+\Delta t}W_v^{\text{ext}} - \int\limits_{^tV^{(m-1)}} {}^t\sigma_{ij}{}^{(m-1)}\,\delta\, {}_te_{ij}{}^{(m-1)}\, {}^tdV,\end{aligned}$$

where m is the iteration number and the first iteration ($m = 1$) corresponds to Eq. (9.28). The $\Delta_t e_{kl}{}^{(m)}$ in Eq. (9.28) is a component of the incremental strain tensor for iteration m, i.e.

$$(9.30)\qquad \Delta\, {}_te_{kl}{}^{(m)} = \frac{1}{2}\left(\frac{\partial(\Delta u_k{}^{(m)})}{\partial^t x_l} + \frac{\partial(\Delta u_l{}^{(m)})}{\partial^t x_k}\right).$$

Similarly, the $\delta\Delta\, {}_t\eta_{ij}{}^{(m)}$ is defined as

$$(9.31)\qquad \Delta\, {}_t\eta_{kl}{}^{(m)} = \frac{1}{2}\left(\frac{\partial(\Delta u_i{}^{(m)})}{\partial^t x_k}\cdot\frac{\partial(\Delta u_i{}^{(m)})}{\partial^t x_l}\right).$$

Iterations are repeated until the r.h.s. in Eq. (9.29) is negligible within a certain convergence tolerance. After each iteration, the displacements and pore water pressure are updated.

The full Newton scheme adopted in Eq. (9.29) is obviously expensive due to the necessity of evaluation of the constitutive tensor ${}_tL_{ijkl}$ at each iteration. A modified Newton scheme can be achieved by keeping the constitutive tensor ${}_tL_{ijkl}$ constant during each step of incremental solution, i.e.

$$\begin{aligned}(9.32)\qquad &\int\limits_{^tV} {}_tL_{ijkl}\,\Delta\, {}_te_{kl}{}^{(m)}\,\delta\, {}_te_{ij}{}^{(m)}\, {}^tdV - \int\limits_{^tV} {}^t\Delta p^{(m)}\,\delta\, {}_te_{ii}{}^{(m)}\, {}^tdV \\ &+ \int\limits_{^tV} {}_tL_{ijkl}\Delta_t e_{kl}{}^{(m)}\,\delta\, {}_t\eta_{ij}{}^{(m)}\, {}^tdV + \int\limits_{^tV} {}^t\sigma_{ij}\,\delta\Delta\, {}_t\eta_{ij}{}^{(m)}\, {}^tdV \\ &- \int\limits_{^tV} {}^t(\Delta p)^{(m)}\,\delta\, {}_t\eta_{ii}{}^{(m)}\, {}^tdV = {}^{t+\Delta t}W_v^{\text{ext}} - \int\limits_{^tV^{(m-1)}} {}^t\sigma_{ij}{}^{(m-1)}\,\delta\, {}_te_{ij}{}^{(m-1)}\, {}^tdV.\end{aligned}$$

Such a solution strategy has been successfully used in some of the applications reported in [33].

As a final note in this section, it should be mentioned that the case of a deformation-dependent external loading can be conveniently handled by applying an iterative incremental procedure as described for Eq. (9.29). For example, in the case of centrifugal loading, the body force applied to an infinitesimal volume of the soil is a function of its current position, i.e.

$$ {}^{t+\Delta t}b_i = {}^{t+\Delta t}b_i({}^{t+\Delta t}\mathbf{x}). \tag{9.33} $$

In such a case, the corresponding term in Eq. $(2.5)_1$ is approximated as follows:

$$ \int\limits_{{}^{t+\Delta t}V} {}^{t+\Delta t}\varrho \, {}^{t+\Delta t}b_i \, \delta u_i \, {}^{t+\Delta t}dV \approx \int\limits_{{}^{t+\Delta t}V^{(m-1)}} {}^{t+\Delta t}\varrho^{(m-1)} \, {}^{t+\Delta t}b_i \left({}^{t+\Delta t}\mathbf{x}^{(m-1)}\right) \delta u_i \, {}^{t+\Delta t}dV, \tag{9.34} $$

where

$$ {}^{t+\Delta t}\varrho^{(m-1)} = {}^{t}\varrho \det \left| \frac{\partial \, {}^{t+\Delta t}x_i{}^{(m-1)}}{\partial \, {}^{t}x_j} \right|. \tag{9.35} $$

The approximation introduced in Eq. (9.35) is only accurate for a small load increment. Evidently, a better approximation for the finite load increments can be achieved by linearizing ${}^{t+\Delta t}b_i$, as it was done for the second Piola–Kirchhoff stress tensor and the Green–Lagrange strain tensor. Such a linearization, however, introduces a new contribution to the stiffness matrix and reduces the computational efficiency of the formulation, as mentioned in [3].

10. Integral form of the equation governing the flow of the pore water

In Sec. 5 we have established a differential equation (Eq. (5.23)) governing the flow of the pore water through the soil. For the purpose of numerical solutions, however, it is appropriate to develop an integral form of this equation which complements the virtual work equation developed in the previous section.

In order to establish an integral form of Eq. (5.23), we recall that this equation is basically an expression of the axiom of mass balance implying that a tendency of volumetric strain in the soil skeleton (first term in Eq. (5.23)) is counteracted by a change in pore pressure (second term), and by the flow of the pore water through the soil (the last three terms). Therefore, a weak form of Eq. (5.23) can be generated by using the Galerkin weighted residual method and recognizing that the pore water pressure is the appropriate weighting function on the volumetric strain rate, i.e.

$$(10.1)\qquad \int\limits_{^{t+\Delta t}V} \left\{ \frac{1}{\Gamma}\, {}^{t+\Delta t}\dot{p} + {}^{t+\Delta t}\dot{u}_{i,i} - \frac{\partial}{\partial\, {}^{t+\Delta t}x_i} \left({}^{t+\Delta t}k_{ij} \frac{\partial\, {}^{t+\Delta t}p}{\partial\, {}^{t+\Delta t}x_j} \right) + \frac{\partial}{\partial\, {}^{t+\Delta t}x_i} \left({}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}b_j \right) - \frac{\partial}{\partial\, {}^{t+\Delta t}x_i} \left({}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}\ddot{u}_j \right) \right\} \delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dV = 0,$$

where $\delta\, {}^{t+\Delta t}p$ is a virtual pore water pressure analogous to the virtual displacement $\delta\mathbf{u}$ previously used in the virtual work expression (Eq. (5.1)).

An expanded form of Eq. (10.1) can be written as

$$(10.2)\qquad \int\limits_{^{t+\Delta t}V} {}^{t+\Delta t}\dot{u}_{i,i}\,\delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dV + \int\limits_{^{t+\Delta t}V} \frac{1}{\Gamma}\, {}^{t+\Delta t}\dot{p}\,\delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dV + \int\limits_{^{t+\Delta t}V} \left[\frac{\partial}{\partial\, {}^{t+\Delta t}x_i} \left(-\, {}^{t+\Delta t}k_{ij} \frac{\partial\, {}^{t+\Delta t}p}{\partial\, {}^{t+\Delta t}x_j} + {}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}b_j - {}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}\ddot{u}_j \right) \right] \delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dV = 0.$$

Applying the Green's theorem to the last integral, we have:

$$(10.3)\qquad \int\limits_{^{t+\Delta t}V} {}^{t+\Delta t}\dot{u}_{i,i}\,\delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dV + \int\limits_{^{t+\Delta t}V} \frac{1}{\Gamma}\, {}^{t+\Delta t}\dot{p}\,\delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dV + \int\limits_{^{t+\Delta t}V} \left({}^{t+\Delta t}k_{ij} \frac{\partial\, {}^{t+\Delta t}p}{\partial\, {}^{t+\Delta t}x_j} - {}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}b_j + {}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}\ddot{u}_j \right) \frac{\partial}{\partial\, {}^{t+\Delta t}x_i} (\delta\, {}^{t+\Delta t}p)\, {}^{t+\Delta t}dV + \int\limits_{^{t+\Delta t}S} \left(-\, {}^{t+\Delta t}k_{ij} \frac{\partial\, {}^{t+\Delta t}p}{\partial\, {}^{t+\Delta t}x_j} + {}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}b_j - {}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}\ddot{u}_j \right) . n_i\, \delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dS = 0,$$

where n_i is the component of outward unit normal vector to the surface ${}^{t+\Delta t}S$ in the direction of x_i axis.

Utilizing the water flow boundary condition (Eq. (6.4)), we can write Eq. (10.3) in the form

$$(10.4)\qquad \int\limits_{t+\Delta t_V} {}^{t+\Delta t}\dot{u}_{i,i}\,\delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dV + \int\limits_{t+\Delta t_V} \frac{1}{\Gamma}\, {}^{t+\Delta t}\dot{p}\,\delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dV$$

$$+ \int\limits_{t+\Delta t_V} \left({}^{t+\Delta t}k_{ij} \frac{\partial\, {}^{t+\Delta t}p}{\partial\, {}^{t+\Delta t}x_j} - {}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}b_j \right.$$

$$\left. + {}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}\ddot{u}_j \right) \frac{\partial}{\partial\, {}^{t+\Delta t}x_i} (\delta\, {}^{t+\Delta t}p)\, {}^{t+\Delta t}dV$$

$$+ \int\limits_{t+\Delta t_{S_q}} {}^{t+\Delta t}\bar{q}_S\, \delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dS = 0.$$

Equation (10.4) is written in terms of the coordinates of the current configuration ${}^{t+\Delta t}\boldsymbol{\beta}$, whose equilibrium position is to be calculated while proceeding from time t to $t + \Delta t$ in the incremental solution. By applying the chain rule and using Eq. (8.1), we can write Eq. (10.2) in terms of the coordinates of the configuration at time t, ${}^{t}\boldsymbol{\beta}$, i.e.

$$(10.5)\qquad \int\limits_{{}^tV} \frac{{}^t\varrho}{{}^{t+\Delta t}\varrho} \left(\frac{\partial\, {}^{t+\Delta t}\dot{u}_i}{\partial\, {}^tx_j} \delta\, {}^{t+\Delta t}p \right) \frac{\partial\, {}^tx_j}{\partial\, {}^{t+\Delta t}x_i}\, {}^tdV + \int\limits_{t+\Delta t_V} \frac{{}^t\varrho}{{}^{t+\Delta t}\varrho} \left(\frac{{}^{t+\Delta t}\dot{p}}{\Gamma} \delta\, {}^{t+\Delta t}p \right) {}^tdV$$

$$+ \int\limits_{{}^tV} \frac{{}^t\varrho}{{}^{t+\Delta t}\varrho} \frac{\partial\, {}^tx_k}{\partial\, {}^{t+\Delta t}x_j} \left[{}^{t+\Delta t}k_{ij} \frac{\partial\, {}^{t+\Delta t}p}{\partial\, {}^tx_k} \frac{\partial(\delta\, {}^{t+\Delta t}p)}{\partial\, {}^tx_l} \right] \frac{\partial\, {}^tx_l}{\partial\, {}^{t+\Delta t}x_i}\, {}^tdV$$

$$- \int\limits_{{}^tV} \frac{{}^t\varrho}{{}^{t+\Delta t}\varrho} \left[{}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}b_j \frac{\partial(\delta\, {}^{t+\Delta t}p)}{\partial\, {}^tx_l} \right] \frac{\partial\, {}^tx_l}{\partial\, {}^{t+\Delta t}x_i}\, {}^tdV$$

$$+ \int\limits_{t+\Delta t_V} \frac{{}^t\varrho}{{}^{t+\Delta t}\varrho} \left[{}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}\ddot{u}_j \frac{\partial(\delta\, {}^{t+\Delta t}p)}{\partial\, {}^tx_l} \right] \frac{\partial\, {}^tx_l}{\partial\, {}^{t+\Delta t}x_i}\, {}^tdV$$

$$+ \int\limits_{{}^tS_q} {}^{t+\Delta t}_{\ t}\bar{q}_S\, \delta\, {}^{t+\Delta t}p\, {}^tdS = 0,$$

where ${}^{t+\Delta t}_{\ t}\bar{q}_S$ was assumed to be a deformation-independent flow on ${}^{t+\Delta t}S_q$, so that

$$(10.6)\qquad {}^{t+\Delta t}\bar{q}_S\, {}^{t+\Delta t}dS = {}^{t+\Delta t}_{\ t}\bar{q}_S\, {}^tdS.$$

It is noted that due to the presence of ${}^{t+\Delta t}\varrho_f$, ${}^{t+\Delta t}k_{ij}$, and the inverse of the deformation gradient tensor, $(\partial\, {}^tx_l)/(\partial\, {}^{t+\Delta t}x_i)$, in Eq. (10.5), most of the integrals in this equation cannot be evaluated without further simplifying assumptions. Similar to the procedure adopted in Sec. 8, one may utilize linearization technique to reduce Eq. (10.5) to a suitable form for incremental iterative analysis. Linearization of Eq. (10.5), however, leads to a very complicated equation which significantly reduces the computational efficiency of the formulation. Therefore, in order to avoid such a difficulty, it is suggested to use some reasonable approximation for the aforementioned redundant terms (${}^{t+\Delta t}\varrho_f$, ${}^{t+\Delta t}k_{ij}$, ...). For example, in the first iteration, one may utilize the values obtained from the previous time step. Corrections to the results of the first iteration can be achieved by establishing approximate position of the configuration at time $t + \Delta t$. The new configuration can be used in iterative solution of Eq. (10.4) written in the following form:

$$\begin{aligned}(10.7)\qquad & \int\limits_{{}^{t+\Delta t}V^{(m-1)}} \frac{\partial\, {}^{t+\Delta t}\dot{u}_i}{\partial\, {}^{t+\Delta t}x_i^{(m-1)}}\, \delta\, {}^{t+\Delta t}p\; {}^{t+\Delta t}dV + \int\limits_{{}^{t+\Delta t}V^{(m-1)}} \frac{1}{\Gamma}\, {}^{t+\Delta t}\dot{p}\; \delta\, {}^{t+\Delta t}p\; {}^{t+\Delta t}dV \\ & + \int\limits_{{}^{t+\Delta t}V^{(m-1)}} {}^{t+\Delta t}k_{ij}^{(m-1)} \left[\frac{\partial\, {}^{t+\Delta t}p}{\partial\, {}^{t+\Delta t}x_j^{(m-1)}} - {}^{t+\Delta t}\varrho_f^{(m-1)}\, {}^{t+\Delta t}b_j^{(m-1)} \right. \\ & \left. + {}^{t+\Delta t}\varrho_f^{(m-1)}\, {}^{t+\Delta t}\ddot{u}_j \right] \frac{\partial(\delta\, {}^{t+\Delta t}p)}{\partial\, {}^{t+\Delta t}x_i^{(m-1)}}\, {}^{t+\Delta t}dV \\ & + \int\limits_{{}^{t+\Delta t}S_q^{(m-1)}} {}^{t+\Delta t}\overline{q}_S\; \delta\, {}^{t+\Delta t}p\; {}^{t+\Delta t}dS = 0,\end{aligned}$$

where the right superscript $(m-1)$ refers to the iteration number $(m-1)$, and the case $m = 1$ is defined as

$$(10.8)\qquad {}^{t+\Delta t}(\;)^{(0)} = {}^{t}(\;),$$

and ${}^{t+\Delta t}\varrho_f^{(m-1)}$ is calculated by using the following equation:

$$(10.9)\qquad J^{(m-1)} = \left| \frac{\partial\, {}^{t+\Delta t}\mathbf{x}^{(m-1)}}{\partial\, {}^{t}\mathbf{x}} \right| = \frac{{}^{t+\Delta t}\varrho^{(m-1)}}{{}^{t}\varrho} = \frac{(1-n)\, {}^{t+\Delta t}\varrho_f^{(m-1)} + n\varrho_S}{(1-n)\, {}^{t}\varrho_f + n\varrho_S}.$$

In Eq. (10.9), we assumed that the change of soil porosity during the time increment was negligible. This assumption is used to prevent the need for iteration over porosity.

Equation (10.7) has a number of special characteristics which distinguish it from the virtual work expression, i.e. Eq. (9.25). First, the dependence of the acceleration term on ${}^{t+\Delta t}\varrho_f$ and ${}^{t+\Delta t}k_{ij}$ requires that the corresponding "mass" matrix in a discretized solution procedure should be calculated in every iteration. Similar situation renders the body force contribution in Eq. (10.7) a deformation-dependent loading. It is also noted that components of the effective permeability tensor are variable quantities which may change due to the change of fabric in the soil skeleton. Unfortunately, experimental data in order to characterize such a change in the soil fabric is very limited.

In so far as the permeability tensor is concerned, it is important to note that in a finite deformation regime, a generic element of the soil system may undergo large rotations. Therefore, special care is necessary to define the coefficients of the permeability tensor in terms of the coordinates of the Cartesian reference system used in the Lagrangian formulation. Assuming that the permeability coefficients are intrinsic to the soil element, it can easily be shown [7] that the matrix of the permeability coefficients obeys the following transformation:

$$\mathbf{k} = \mathbf{R}^T \mathbf{k}_0 \mathbf{R}, \tag{10.10}$$

where $\mathbf{k}_0$ is the matrix of permeability coefficients in the initial position and $\mathbf{R}$ is the matrix characterizing the rotation of the soil element with respect to the reference Cartesian coordinates.

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THE GEORGE WASHINGTON UNIVERSITY, WASHINGTON DC, USA.

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Stokes flow past a composite porous spherical shell with a solid core

B.S. PADMAVATHI and T. AMARANATH (HYDERABAD)

A GENERAL SOLUTION of the Brinkman equations in the form of an infinite series is presented. A representation for the solution of Brinkman's equations is also proposed and its equivalence to the infinite series is established. The usefulness of the representation is demonstrated by applying it to design a general method of solving an arbitrary Stokes flow past a composite porous spherical shell with a rigid core. Some physical properties, such as the drag and torque exerted on the composite sphere are calculated. Several illustrative examples are discussed.

1. Introduction

IN THE STUDY of flow and heat transfer problems in porous media, two models which have been extensively used are those due to DARCY [1] and BRINKMAN [2]. However, the Brinkman model seems to be favoured in some problems in porous media, owing to the limitations of Darcy's law. The inadequacy of Darcy's law in the formulation of problems in bounded porous media is primarily due to the order of Darcy's equations being lower than the second order Navier–Stokes equations. A variety of flow and heat transfer problems in porous media were solved using the Brinkman's equations. In this paper, we give a general solution of the Brinkman equations in the form of an infinite series by using a procedure followed by LAMB [3] in the case of Stokes equations. We also propose a representation for the solution of Brinkman equations in terms of two scalar functions and establish its equivalence to the series solution. We shall use this representation to study the problem of an arbitrary Stokes flow of an incompressible, viscous fluid past a composite porous sphere with a rigid core, using the Brinkman model in the porous region. The results obtained by MASLIYAH *et al.* [4] who considered a uniform flow past a composite porous sphere with a rigid core can be recovered as a special case. Some illustrative examples are discussed.

2. Structure of the general solution of Brinkman's equations

We consider Brinkman's equations

$$-\nabla p + \mu\nabla^2\mathbf{V} = \frac{\mu}{k}\mathbf{V}, \tag{2.1}$$

and the equation of continuity

$$\nabla\cdot\mathbf{V} = 0, \tag{2.2}$$

where $\mathbf{V}$ is the velocity, p is the pressure, μ is the coefficient of dynamic viscosity, and $k > 0$ is the permeability coefficient of the porous medium. Equation (2.1) can be rewritten as

$$\mu(\nabla^2 - \lambda^2)\mathbf{V} = \nabla p, \tag{2.3}$$

where $\lambda^2 = 1/k$.

The general solution of the equation

$$(\nabla^2 - \lambda^2)\Psi = 0, \tag{2.4}$$

is as follows:

$$\Psi = \sum_{-\infty}^{\infty}(X_n F_n(\lambda r) + Y_n H_n(\lambda r))\chi_n\,, \tag{2.5}$$

where X_n, Y_n are arbitrary constants, $\chi_n = r^n S_n(\theta, \phi)$ is a solid harmonic of degree n, and

$$S_n(\theta,\phi) = \sum_{m=0}^{n} P_n^m(\zeta)(A_{nm}\cos m\phi + B_{nm}\sin m\phi), \qquad \zeta = \cos\theta.$$

The functions $F_n(z)$ and $H_n(z)$ $(z = \lambda r)$ are defined as follows,

$$z^n F_n(z) = \sqrt{\frac{\pi}{2z}} I_{n+\frac{1}{2}}(z), \qquad z^n H_n(z) = \sqrt{\frac{\pi}{2z}} K_{n+\frac{1}{2}}(z),$$

where $\sqrt{\frac{\pi}{2z}} I_{n+\frac{1}{2}}(z)$ and $\sqrt{\frac{\pi}{2z}} K_{n+\frac{1}{2}}(z)$ are the modified spherical Bessel functions which are finite at the origin and infinity, respectively. The functions $F_n(\lambda r)$ or $H_n(\lambda r)$ are retained in the solution depending on whether the motion is finite at the origin or at infinity, respectively. Suppose we assume the condition of finiteness of the motion at the origin $r = 0$, then the general solution of Eqs. (2.2) and (2.3) is

$$p = \sum_{-\infty}^{\infty} p_n,$$

$$\mathbf{V} = \sum_{-\infty}^{\infty}\Bigg(\Big[(n+1)F_{n-1}(\lambda r) + nF_{n+1}(\lambda r)\lambda^2 r^2\Big]\nabla\phi_n - n(2n+1)F_{n+1}(\lambda r)\lambda^2\mathbf{r}\phi_n - F_n(\lambda r)\nabla\times(\mathbf{r}\chi_n) - \frac{1}{\lambda^2\mu}\nabla p_n\Bigg), \tag{2.6}$$

where χ_n, ϕ_n and p_n are solid harmonics of positive degree n. When the condition of finiteness at the origin is not imposed, we have an additional system of solutions in which the functions $F_n(\lambda r)$ are replaced by $H_n(\lambda r)$.

3. A representation for the solution of Brinkman's equations

We now propose a representation for the velocity and pressure in Brinkman's equations (2.2) and (2.3) in terms of two scalar functions A and B and establish its equivalence to the series solution given in (2.6). We assume the following form for the velocity $\mathbf{V}$,

$$\begin{aligned} \mathbf{V} &= \operatorname{curl}\operatorname{curl}(\mathbf{r}A) + \operatorname{curl}(\mathbf{r}B), \\ &= \operatorname{grad}\operatorname{div}(\mathbf{r}A) - \nabla^2(\mathbf{r}A) + \operatorname{curl}(\mathbf{r}B). \end{aligned} \tag{3.1}$$

Equation (2.2) is satisfied identically and substitution of $(3.1)_2$ in Eq. (2.3) results in

$$\begin{aligned} &\operatorname{grad}\left(p - \mu\frac{\partial}{\partial r}\left[r(\nabla^2 - \lambda^2)A\right]\right) \\ &= \mu\left(-\widehat{e}_r r(\nabla^4 - \lambda^2\nabla^2)A + \widehat{e}_\theta \csc\theta\frac{\partial}{\partial\phi}(\nabla^2 - \lambda^2)B - \widehat{e}_\phi\frac{\partial}{\partial\theta}(\nabla^2 - \lambda^2)B\right), \end{aligned} \tag{3.2}$$

where $\widehat{e}_r$, $\widehat{e}_\theta$ and $\widehat{e}_\phi$ are the unit vectors along the radial, transverse and azimuthal directions, respectively. Equations (2.2) and (2.3) are satisfied if

$$\begin{aligned} p &= p_0 + \mu\frac{\partial}{\partial r}[r(\nabla^2 - \lambda^2)A], \\ \nabla^2(\nabla^2 - \lambda^2)A &= 0, \\ (\nabla^2 - \lambda^2)B &= 0. \end{aligned} \tag{3.3}$$

A general solution of $(3.3)_2$ is given by $A = A_1 + A_2$, where A_1 and A_2 are, respectively, the solutions of

$$\begin{aligned} \nabla^2 A_1 &= 0, \\ (\nabla^2 - \lambda^2)A_2 &= 0. \end{aligned} \tag{3.4}$$

Equation $(3.1)_1$ can also be written as

$$\mathbf{V} = 2\operatorname{grad} A + r\frac{\partial}{\partial r}\operatorname{grad} A - \mathbf{r}\nabla^2 A + \operatorname{curl}(\mathbf{r}B). \tag{3.5}$$

From the above equation, we recover the solution given in Eqs. (2.6) by assuming

$$\begin{aligned} B &= -\sum_{-\infty}^{\infty} F_n(\lambda r)\chi_n, \\ A_1 &= -\sum_{-\infty}^{\infty} \frac{1}{\lambda^2\mu}\frac{p_n}{(n+1)}, \\ A_2 &= \sum_{-\infty}^{\infty}(2n+1)F_n(\lambda r)\phi_n. \end{aligned} \tag{3.6}$$

It is observed that such B, A_1 and A_2 satisfy Eqs. $(3.3)_3$ and (3.4), respectively. It may be noted that when the condition of finiteness at the origin is not imposed, the functions $H_n(\lambda r)$ also have to be considered along with the functions $F_n(\lambda r)$. Thus $(3.1)_1$ and $(3.3)_1$ give a general solution of the Brinkman's equations. Similar representations have been considered earlier in the literature and, more recently, in connection with the solution of Stokes equations by PALANIAPPAN *et al.* [5]. However, the application of the representation proposed here to the Brinkman's equations is new and this representation lends itself to useful applications in problems of flows through porous media; in particular, in problems involving spherical boundaries, owing to the simplicity of its form. This fact is exemplified in the next section in the discussion of a general, non-axisymmetric Stokes flow past a composite porous spherical shell with a rigid core, using the Brinkman model in the porous region.

4. Stokes flow over a composite sphere: Solid core with a porous shell

Consider a stationary, solid, impermeable sphere of radius b surrounded by a porous shell of permeability k and thickness $(a - b)$. We shall consider a non-axisymmetric, Stokes flow of an incompressible, viscous fluid over the composite sphere. The Stokes equations are

$$\mu\nabla^2\mathbf{V} = \nabla p, \qquad \nabla\cdot\mathbf{V} = 0. \tag{4.1}$$

We find it advantageous to use the representation, proposed by PALANIAPPAN *et al.* [5] for the solution of the Stokes equations (4.1), given below in the form

$$\mathbf{V} = \operatorname{curl}\operatorname{curl}(\mathbf{r}A) + \operatorname{curl}(\mathbf{r}B), \qquad p = p_0 + \mu\frac{\partial}{\partial r}\left[r\nabla^2 A\right], \tag{4.2}$$

where

$$\nabla^4 A = 0, \qquad \nabla^2 B = 0. \tag{4.3}$$

Suppose now that the basic, unperturbed velocity is given by

$$\mathbf{V}_0 = \operatorname{curl}\operatorname{curl}(\mathbf{r}A_0) + \operatorname{curl}(\mathbf{r}B_0), \tag{4.4}$$

where

$$A_0 = \sum_{n=1}^{\infty}\left(\alpha_n r^n + \alpha'_n r^{n+2}\right) S_n(\theta,\phi), \qquad B_0 = \sum_{n=1}^{\infty}\xi_n r^n T_n(\theta,\phi), \tag{4.5}$$

where

$$
(4.6)\qquad
\begin{aligned}
S_n(\theta,\phi) &= \sum_{m=0}^{n} P_n^m(\zeta)(A_{nm}\cos m\phi + B_{nm}\sin m\phi), \qquad \zeta = \cos\theta,\\
T_n(\theta,\phi) &= \sum_{m=0}^{n} P_n^m(\zeta)(C_{nm}\cos m\phi + D_{nm}\sin m\phi),
\end{aligned}
$$

$\alpha_n, \alpha'_n, \xi_n, A_{nm}, B_{nm}, C_{nm}$ and D_{nm} are known constants and $P_n^m(\zeta)$ is the Legendre polynomial. For the flow quantities in the region $a < r < \infty$ we shall use the superscript e. Therefore in the presence of the sphere, we shall assume the modified flow in this region to be given by $(\mathbf{V}^e, p^e)$ in terms of two scalar functions A^e and B^e, where

$$
(4.7)\qquad
\begin{aligned}
\nabla^4 A^e &= 0,\\
\nabla^2 B^e &= 0.
\end{aligned}
$$

The equations which describe the flow field in the porous region $b < r < a$ are assumed to be the Brinkman equations (2.1) and (2.2). We make use of the representation $(3.1)_1$ and $(3.3)_1$ proposed for the Brinkman's equations, to find the modified flow $(\mathbf{V}^i, p^i)$ in this region in terms of two scalar functions A^i and B^i, where

$$
(4.8)\qquad
\begin{aligned}
\nabla^2(\nabla^2 - \lambda^2)A^i &= 0,\\
(\nabla^2 - \lambda^2)B^i &= 0.
\end{aligned}
$$

We assume the following forms for these scalar functions as

$$
(4.9)\qquad
\begin{aligned}
A^e(r,\theta,\phi) &= \sum_{n=1}^{\infty}\left(\alpha_n r^n + \alpha'_n r^{n+2} + \frac{\beta_n}{r^{n+1}} + \frac{\beta'_n}{r^{n-1}}\right) S_n(\theta,\phi),\\
B^e(r,\theta,\phi) &= \sum_{n=1}^{\infty}\left(\xi_n r^n + \frac{\sigma_n}{r^{n+1}}\right) T_n(\theta,\phi),\\
A^i(r,\theta,\phi) &= A_1^i(r,\theta,\phi) + A_2^i(r,\theta,\phi),\\
B^i(r,\theta,\phi) &= \sum_{n=1}^{\infty}\left(\gamma_n f_n(\lambda r) + \gamma'_n g_n(\lambda r)\right) T_n(\theta,\phi),
\end{aligned}
$$

where

$$
(4.9')\qquad
\begin{aligned}
A_1^i(r,\theta,\phi) &= \sum_{n=1}^{\infty}\left(\varepsilon_n r^n + \frac{\varepsilon'_n}{r^{n+1}}\right) S_n(\theta,\phi),\\
A_2^i(r,\theta,\phi) &= \sum_{n=1}^{\infty}\left(\delta_n f_n(\lambda r) + \delta'_n g_n(\lambda r)\right) S_n(\theta,\phi),
\end{aligned}
$$

where $f_n(z) = \sqrt{\frac{\pi}{2z}} I_{n+\frac{1}{2}}(z)$ and $g_n(z) = \sqrt{\frac{\pi}{2z}} K_{n+\frac{1}{2}}(z)$. The boundary conditions to be satisfied at $r = a$ and $r = b$ are

1) continuity of velocity components on the surface $r = a$

$$
\begin{aligned}
q_r^e(a,\theta,\phi) &= q_r^i(a,\theta,\phi),\\
q_\theta^e(a,\theta,\phi) &= q_\theta^i(a,\theta,\phi),\\
q_\phi^e(a,\theta,\phi) &= q_\phi^i(a,\theta,\phi);
\end{aligned}
\tag{4.10}
$$

2) continuity of stresses on the surface $r = a$

$$
\begin{aligned}
T_{rr}^e(a,\theta,\phi) &= T_{rr}^i(a,\theta,\phi),\\
T_{r\theta}^e(a,\theta,\phi) &= T_{r\theta}^i(a,\theta,\phi),\\
T_{r\phi}^e(a,\theta,\phi) &= T_{r\phi}^i(a,\theta,\phi);
\end{aligned}
\tag{4.11}
$$

3) no-slip conditions on the surface $r = b$

$$
\begin{aligned}
q_r^i(b,\theta,\phi) &= 0,\\
q_\theta^i(b,\theta,\phi) &= 0,\\
q_\phi^i(b,\theta,\phi) &= 0,
\end{aligned}
\tag{4.12}
$$

where q_r^i, q_θ^i and q_ϕ^i are the radial, transverse and azimuthal velocities, T_{rr}^i is the normal stress and $T_{r\theta}^i$ and $T_{r\phi}^i$ are the tangential stresses in the region $b < r < a$. The corresponding velocities and stresses in the region $a < r < \infty$ are defined in a similar manner using the superscript e.

In terms of the scalar functions which appear in (4.8)–(4.9′), the boundary conditions (4.10)–(4.12) can be restated as follows

$$
\begin{aligned}
A^e(a,\theta,\phi) &= A^i(a,\theta,\phi),\\
A_r^e(a,\theta,\phi) &= A_r^i(a,\theta,\phi),\\
A_{rr}^e(a,\theta,\phi) &= A_{rr}^i(a,\theta,\phi),\\
a(A_{rrr}^e(a,\theta,\phi) - A_{rrr}^i(a,\theta,\phi)) &= -\lambda^2 \frac{\partial}{\partial r}(rA^i)(a,\theta,\phi),\\
B^e(a,\theta,\phi) &= B^i(a,\theta,\phi),\\
B_r^e(a,\theta,\phi) &= B_r^i(a,\theta,\phi),\\
A^i(b,\theta,\phi) &= 0,\\
A_r^i(b,\theta,\phi) &= 0,\\
B^i(b,\theta,\phi) &= 0.
\end{aligned}
\tag{4.13}
$$

The functions A^e, B^e, A^i and B^i which correspond to the modified flow can be determined by determining the nine unknown constants $\beta_n, \beta'_n, \sigma_n, \varepsilon_n, \varepsilon'_n, \delta_n, \delta'_n$,

γ_n, and γ'_n from the nine equations (4.13) in terms of $\alpha_n, \alpha'_n, \xi_n, A_{nm}, B_{nm}, C_{nm}$ and D_{nm}. The nine unknown constants are determined to be as follows:

$$
\begin{aligned}
\beta_n &= \frac{\text{numb}}{\text{deno}}, \\
\beta'_n &= \frac{\text{numb}'}{\text{deno}}, \\
\varepsilon_n &= \frac{\text{nume}}{\text{deno}}, \\
\varepsilon'_n &= \frac{\text{nume}'}{\text{deno}}, \\
\delta_n &= \frac{\text{numd}}{\text{deno}}, \\
\delta'_n &= \frac{\text{numd}'}{\text{deno}}, \\
\sigma_n &= -\left[a + \frac{s_n(1+2n)}{\lambda t_n}\right] a^{2n}\xi_n, \\
\gamma_n &= \frac{(1+2n)a^{n-1}g_n(\lambda b)}{\lambda t_n}\xi_n, \\
\gamma'_n &= -\frac{(1+2n)a^{n-1}f_n(\lambda b)}{\lambda t_n}\xi_n,
\end{aligned}
\tag{4.14}
$$

where

$$
\begin{aligned}
\text{deno} = 2a^{n+5}b^{n+1}\lambda^3\{&[(1+n)a^{2n+3}b\lambda^2 + na^2b^{2+2n}\lambda^2 - n(1-4n^2)a^{1+2n}b]a_n \\
&+(1-2n)\lambda[(1+n)a^{2+2n}b + nab^{2+2n}]b_n \\
&+n(1-4n^2)a^n b^{2+n}c_n + n(1-4n^2)a^{n+2}b^n r_n \\
&-n(1+2n)\lambda a^2 b^{1+2n}s_n - n(1-4n^2)ab^{1+2n}t_n\},
\end{aligned}
$$

$$
\begin{aligned}
\text{numb} = a^{3n+6}b^{n+1}\lambda^3(2n-1)\{&[a^{2n+3}b\lambda^2(1+n) + 2a^{2n+1}b(1+n)(1+2n) \\
&+a^2b^{2n+2}\lambda^2 n]a_n + 2ab\lambda[a^{2n+1}(1+n) + b^{2n+1}n]b_n \\
&-2a^n b^{n+2}(1+n)(1+2n)c_n + 2a^{2+n}b^n n(1+2n)r_n \\
&-a^2b^{1+2n}\lambda n(1+2n)s_n - 2ab^{1+2n}n(1+2n)t_n\}\alpha_n \\
+a^{3n+6}b^{n+1}\lambda(1+2n)\{&ab[a^{2n+4}\lambda^4(1+n) \\
&+a^3b^{2n+1}\lambda^4 n + 4a^{2n+2}\lambda^2(2n+5) \\
&-4a^{2n}(1-4n^2)(2n+3) + 4\lambda^2 ab^{2n+1}(2n+3)]a_n \\
&+ab\lambda[4a^{2n+3}\lambda^2(1+n) + 4a^2b^{2n+1}\lambda^2 n \\
&-4a^{2n+1}(1-2n)(2n+3) + 4b^{2n+1}(1-2n)(2n+3)]b_n \\
&+2a^n b^{n+2}(1+2n)[2(1-2n)(2n+3) - 3\lambda^2 a^2]c_n
\end{aligned}
$$

[cont.]

$$-2a^{n+2}b^{n}(1+2n)[3a^2\lambda^2-2(1-2n)(2n+3)]r_n$$
$$-a^2b^{2n+1}\lambda(1+2n)[a^2\lambda^2 n+4(2n+3)]s_n$$
$$-4ab^{2n+1}(1+2n)[a^2\lambda^2 n+(1-2n)(2n+3)]t_n\}\alpha_n',$$

$$\begin{aligned}\text{numb}' = &-a^{3n+6}b^{n+2}(1+2n)\lambda^4\{[a^{2n+1}(1+n)+b^{2n+1}n]\lambda a_n\\ &-b^{2n}n(1+2n)s_n\}\alpha_n\\ &-a^{3n+6}b^{n+1}(3+2n)\lambda^3\{[a^{2n+3}b\lambda^2(1+n)+a^2b^{2n+2}\lambda^2 n\\ &+2a^{2n+1}b(1+n)(1+2n)]a_n+2\lambda[a^{2n+2}b(1+n)+ab^{2n+2}n]b_n\\ &+2a^nb^{n+2}n(1+2n)c_n-2a^{n+2}b^n(1+n)(1+2n)r_n\\ &-a^2b^{2n+1}\lambda n(1+2n)s_n-2ab^{2n+1}n(1+2n)t_n\}\alpha_n',\end{aligned}$$

$$\begin{aligned}\text{numd} = &\,2a^{2n+5}b^{n+1}\lambda^2(1-4n^2)\{a^{n+1}b^nn(1+2n)g_n(\lambda a)\\ &-\lambda[b^{2n+2}n+a^{2n+1}b(1+n)]g_{n-1}(\lambda b)-b^{2n+1}n(1+2n)g_n(\lambda b)\}\alpha_n\\ &+2a^{2n+6}b^{n+1}\lambda(1+2n)(3+2n)\{[a^{2n+2}b\lambda^2(1+n)\\ &+\lambda^2ab^{2n+2}n-2a^{2n}b(1-4n^2)]g_{n-1}(\lambda b)\\ &-a^{n+2}b^n\lambda(n-2)(1+2n)g_n(\lambda a)+2a^{n+1}b^n(1-4n^2)g_{n-1}(\lambda a)\\ &+ab^{2n+1}\lambda n(1+2n)g_n(\lambda b)\}\alpha_n',\end{aligned}$$

$$\begin{aligned}\text{numd}' = &-2\lambda^2(1-4n^2)a^{2n+5}b^{n+1}\{a^{n+1}b^nn(1+2n)f_n(\lambda a)\\ &+\lambda(b^{2n+2}n+a^{2n+1}b(1+n))f_{n-1}(\lambda b)\\ &-b^{2n+1}n(1+2n)f_n(\lambda b)\}\alpha_n\\ &+2\lambda(1+2n)(3+2n)a^{2n+6}b^{n+1}\{a^{2n+2}b\lambda^2(1+n)f_{n-1}(\lambda b)\\ &+a^{n+2}b^n\lambda(n-2)(1+2n)f_n(\lambda a)\\ &+2a^{n+1}b^n(1-4n^2)f_{n-1}(\lambda a)+ab^{2n+2}n\lambda^2f_{n-1}(\lambda b)\\ &-ab^{2n+1}n(1+2n)\lambda f_n(\lambda b)-2a^{2n}b(1-4n^2)f_{n-1}(\lambda b)\}\alpha_n',\end{aligned}$$

$$\begin{aligned}\text{nume} = &-2\lambda^3a^{2n+5}b^{n+2}n(1-4n^2)[a^{n+1}a_n-b^{n+1}c_n]\alpha_n\\ &+2\lambda^2a^{2n+7}b^{n+2}(1+2n)(3+2n)\{a^{n+1}(n-2)\lambda a_n\\ &-2a^n(1-2n)b_n-b^{n+1}n\lambda c_n\}\alpha_n',\end{aligned}$$

$$\begin{aligned}\text{nume}' = &\,2\lambda^2a^{3n+6}b^{2n+2}(1-4n^2)\{\lambda b^{n+1}na_n-b^nn(1+2n)s_n\\ &+\lambda a^nb(1+n)c_n\}\alpha_n\\ &+\lambda a^{3n+6}b^{2n+2}(1+2n)(3+2n)\{-2\lambda^2a^2b^{n+1}(n-2)a_n\\ &+4\lambda ab^{n+1}(1-2n)b_n+(4a^nb(1-4n^2)-2\lambda^2a^{n+2}b(1+n))c_n\\ &+2\lambda a^2b^n(n-2)(1+2n)s_n+4ab^n(4n^2-1)t_n\}\alpha_n',\end{aligned}$$

and

$$\begin{aligned} a_n &= g_n(\lambda a)f_{n-1}(\lambda b) + f_n(\lambda a)g_{n-1}(\lambda b),\\ b_n &= g_{n-1}(\lambda a)f_{n-1}(\lambda b) - f_{n-1}(\lambda a)g_{n-1}(\lambda b),\\ c_n &= g_n(\lambda b)f_{n-1}(\lambda b) + f_n(\lambda b)g_{n-1}(\lambda b),\\ r_n &= g_n(\lambda a)f_{n-1}(\lambda a) + f_n(\lambda a)g_{n-1}(\lambda a),\\ s_n &= g_n(\lambda a)f_n(\lambda b) - f_n(\lambda a)g_n(\lambda b),\\ t_n &= g_n(\lambda b)f_{n-1}(\lambda a) + f_n(\lambda b)g_{n-1}(\lambda a). \end{aligned}$$

5. Drag and torque

The force exerted by the fluid on the composite sphere is given by

$$\mathbf{D} = \mathbf{X}/Y, \tag{5.1}$$

where

$$\begin{aligned} \mathbf{X} = \Big\{&12\pi\mu\lambda a^2\{(2a^3+b^3)\lambda a_1 - 3b^2 s_1\}\alpha_1\\ &+20\pi\mu a^3\{(2a^4\lambda^2 + ab^3\lambda^2 + 12a^2)a_1 + 2\lambda(2a^3+b^3)b_1\\ &+6b^2c_1 - 12a^2r_1 - 3ab^2\lambda s_1 - 6b^2t_1\}\alpha_1'\Big\}(A_{11}\hat{i} + B_{11}\hat{j} + A_{10}\hat{k}),\\ = {}&6\pi\mu\lambda a^2\{(2a^3+b^3)\lambda a_1 - 3b^2s_1\}[\mathbf{V}_0]_0\\ &+\pi\mu a^3\{(2a^4\lambda^2 + ab^3\lambda^2 + 12a^2)a_1 + 2\lambda(2a^3+b^3)b_1\\ &+6b^2c_1 - 12a^2r_1 - 3ab^2\lambda s_1 - 6b^2t_1\}[\nabla^2\mathbf{V}_0]_0, \end{aligned} \tag{5.2}$$

(see Appendix)

$$\begin{aligned} Y = \{&(2a^4\lambda^2 + ab^3\lambda^2 + 3a^2)a_1 - \lambda(2a^3+b^3)b_1\\ &-3b^2c_1 - 3a^2r_1 - 3ab^2\lambda s_1 + 3b^2t_1\}, \end{aligned} \tag{5.3}$$

and where $\mathbf{V}_0$ is the velocity corresponding to the basic flow, and $[\ \ ]_0$ denotes the evaluation at the origin $r = 0$.

Similarly, the torque $\mathbf{T}$ is given by

$$\begin{aligned} \mathbf{T} &= 8\pi\mu\left\{a^3 + \frac{3a^2 s_1}{\lambda t_1}\right\}\xi_1(C_{11}\hat{i} + D_{11}\hat{j} + C_{10}\hat{k}),\\ &= 4\pi\mu\left\{a^3 + \frac{3a^2 s_1}{\lambda t_1}\right\}[\nabla\times\mathbf{V}_0]_0, \end{aligned} \tag{5.4}$$

(see Appendix).

It is found that when $a = b$, in the limit $k \to 0$, i.e., $\lambda \to \infty$, we recover the well known Faxen's laws [6] for drag and torque acting on a rigid sphere of radius a, i.e.,

$$
\begin{aligned}
\mathbf{D} &= 6\pi\mu a[\mathbf{V}_0]_0 + \pi\mu a^3[\nabla^2\mathbf{V}_0]_0, \\
\mathbf{T} &= 4\pi\mu a^3[\nabla \times \mathbf{V}_0]_0.
\end{aligned} \tag{5.5}
$$

Similarly, when $b = 0$, we recover the expressions for drag and torque obtained by PADMAVATHI and AMARANATH [7] for the Stokes flow past a porous sphere, i.e.,

$$
\begin{aligned}
\mathbf{D} = {} & \frac{12\pi\mu a^3\lambda^2 f_1(\lambda a)[\mathbf{V}_0]_0}{((2a^2\lambda^2+3)f_1(\lambda a) + 2a\lambda f_0(\lambda a))} \\
& + \frac{2\pi\mu[(a^5\lambda^2 + 6a^3)f_1(\lambda a) - 2a^4\lambda f_0(\lambda a)][\nabla^2\mathbf{V}_0]_0}{((2a^2\lambda^2+3)f_1(\lambda a) + 2a\lambda f_0(\lambda a))}, \\
\mathbf{T} = {} & 4\pi\mu\left(\frac{a^3\lambda f_0(\lambda a) - 3a^2 f_1(\lambda a)}{\lambda f_0(\lambda a)}\right)[\nabla \times \mathbf{V}_0]_0.
\end{aligned} \tag{5.6}
$$

6. Effective viscosity

The effective viscosity μ^* of a dilute suspension of composite porous spheres with rigid cores, each of outer radius a is found (as in [7]) to be

$$
\mu^* = \mu\left\{1 + \frac{5R}{2S}\Phi\right\}, \tag{6.1}
$$

where

$$
\begin{aligned}
R &= a\lambda[(3a^5 + 2b^5)\lambda a_2 - 10b^4 s_2], \\
S &= 2[(a\lambda^2(3a^5 + 2b^5) + 30a^4)a_2 - 3\lambda(3a^5 + 2b^5)b_2 \\
&\qquad - 10ab(3b^2c_2 + 3a^2r_2 + \lambda b^3 s_2) + 30b^4 t_2],
\end{aligned} \tag{6.2}
$$

where Φ denotes the concentration by volume of the fluid containing the spheres.

When $a = b$, in the limit $k \to 0$, we obtain the well known formula due to EINSTEIN [8] for the effective viscosity of a dilute suspension of rigid spheres

$$
\mu^* = \mu\left\{1 + \frac{5}{2}\Phi\right\}. \tag{6.3}
$$

When $b \to 0$, we recover the formula obtained by PADMAVATHI and AMARANATH [7] for a dilute suspension of porous spheres of radius a

$$
\mu^* = \mu\left\{1 + \frac{5(a^3\lambda^3 f_0(\lambda a) - 3a^2\lambda^2 f_1(\lambda a))}{2[(a^3\lambda^3 + 10a\lambda)f_0(\lambda a) - 30 f_1(\lambda a)]}\Phi\right\}. \tag{6.4}
$$

7. Examples

7.1. Stokeslet

Consider a Stokeslet of strength $F_1/8\pi\mu$ located at $(0,0,c)$, $c > a$, its axis extending along the positive direction of the x-axis. The corresponding expressions for A_0 and B_0 due to the Stokeslet are [5]

$$(7.1)\qquad \begin{aligned} A_0(r,\theta,\phi) &= \frac{F_1 R_1}{8\pi\mu c}(r\cos\theta - c + R_1)\frac{\cos\phi}{r\sin\theta}, \\ B_0(r,\theta,\phi) &= \frac{F_1}{4\pi\mu c}(r\cos\theta - c + R_1)\frac{\sin\phi}{r\sin\theta}, \end{aligned}$$

where

$$(7.2)\qquad R_1^2 = r^2 + c^2 - 2cr\cos\theta.$$

For $r < c$,

$$(7.3)\qquad \begin{aligned} A_0(r,\theta,\phi) &= \frac{F_1}{8\pi\mu}\sum_{n=1}^{\infty}\left[\frac{r^{n+2}}{(n+1)(2n+3)c^{n+2}} - \frac{(n-2)r^n}{n(n+1)(2n-1)c^n}\right] P_n^1(\zeta)\cos\phi, \\ B_0(r,\theta,\phi) &= \frac{F_1}{4\pi\mu}\sum_{n=1}^{\infty}\left[\frac{r^n}{n(n+1)c^{n+1}}\right] P_n^1(\zeta)\sin\phi. \end{aligned}$$

The drag **D** and torque **T** are given by

$$(7.4)\qquad \mathbf{D} = \frac{M}{N}F_1\hat{i},$$

where

$$(7.5)\qquad \begin{aligned} M &= \Big(3\lambda a^2c^2\{(2a^3 + b^3)\lambda a_1 - 3b^2 s_1\} \\ &\quad + a^3\{(2a^4\lambda^2 + ab^3\lambda^2 + 12a^2)a_1 + 2\lambda(2a^3 + b^3)b_1 \\ &\quad + 6b^2c_1 - 12a^2r_1 - 3ab^2\lambda s_1 - 6b^2t_1\}\Big), \\ N &= 4c^3\{(2a^4\lambda^2 + ab^3\lambda^2 + 3a^2)a_1 - \lambda(2a^3 + b^3)b_1 \\ &\quad - 3b^2c_1 - 3a^2r_1 - 3ab^2\lambda s_1 + 3b^2t_1\}, \\ \mathbf{T} &= \left(a^3 + \frac{3a^2 s_1}{\lambda t_1}\right)\frac{F_1}{c^2}\hat{j}. \end{aligned}$$

As before, the results for the rigid case [5] are recovered by putting $k \to 0$ i.e., $\lambda \to \infty$ and $a = b$.

$$(7.6)\qquad \begin{aligned} \mathbf{D} &= \left(\frac{3a}{4c} + \frac{a^3}{4c^3}\right) F_1 \hat{\imath}, \\ \mathbf{T} &= \frac{a^3}{c^2} F_1 \hat{\jmath}. \end{aligned}$$

Similarly when $b \to 0$, we recover the results obtained for the case of a porous sphere [7]

$$(7.7)\qquad \begin{aligned} \mathbf{D} &= \left(\frac{(3a^3c^2\lambda^2 + a^5\lambda^2 + 6a^3) f_1(\lambda a) - 2a^4\lambda f_0(\lambda a)}{2c^3[(2a^2\lambda^2 + 3) f_1(\lambda a) + 2a\lambda f_0(\lambda a)]}\right) F_1 \hat{\imath}, \\ \mathbf{T} &= \frac{a^3\lambda f_0(\lambda a) - 3a^2 f_1(\lambda a)}{c^2\lambda f_0(\lambda a)} F_1 \hat{\jmath}. \end{aligned}$$

7.2. Uniform flow

The basic, undisturbed flow is given by

$$(7.8)\qquad \begin{aligned} A_0 &= \frac{U}{2} r \cos\theta, \\ B_0 &= 0, \\ \mathbf{D} &= \frac{6\pi\mu\lambda a^2\{(2a^3 + b^3)\lambda a_1 - 3b^2 s_1\} U}{\{Za_1 - \lambda(2a^3 + b^3) b_1 - 3b^2 c_1 - 3a^2 r_1 - 3ab^2\lambda s_1 + 3b^2 t_1\}} \hat{k}, \\ \mathbf{T} &= 0, \end{aligned}$$

where

$$Z = 2a^4\lambda^2 + ab^3\lambda^2 + 3a^2.$$

This result agrees with that of MASLIYAH *et al.* [4] who solved the uniform flow past a composite porous sphere with a rigid core.

8. Conclusions

An infinite series solution and a representation for the solution of Brinkman's equations are presented. They are shown to be equivalent. It is found that this representation is very useful for discussing an arbitrary Stokes flow past a composite porous sphere with a rigid core, and a general method is suggested for finding the solution. The formulae to calculate drag and torque are given. The effective viscosity of a dilute suspension of composite porous spheres with rigid cores is calculated. The previous results pertaining to Stokes flow past rigid and porous spheres are recovered as special cases. It may be noted that the method suggested in this paper can also be used effectively to discuss the problem of Stokes flow past a porous spherical shell, where the rigid core in the present problem is replaced by a region filled with a viscous fluid.

Appendix

$$[\mathbf{V}_0]_0 = [2\,\mathrm{grad}\,A_0]_0 = 2\alpha_1[A_{11}\hat{i} + B_{11}\hat{j} + A_{10}\hat{k}],$$
$$[\nabla^2\mathbf{V}_0]_0 = 20\alpha_1'[A_{11}\hat{i} + B_{11}\hat{j} + A_{10}\hat{k}],$$
$$[\nabla \times \mathbf{V}_0]_0 = 2\xi_1[C_{11}\hat{i} + D_{11}\hat{j} + C_{10}\hat{k}].$$

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DEPARTMENT OF MATHEMATICS AND STATISTICS, SCHOOL OF M.C.I.S.,
UNIVERSITY OF HYDERABAD, HYDERABAD, INDIA.

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Two-dimensional Hooke's tensors – isotropic decomposition, effective symmetry criteria

A. BLINOWSKI, J. OSTROWSKA–MACIEJEWSKA
and J. RYCHLEWSKI (WARSZAWA)

ANY FOURTH RANK plane tensor **H** obeying the "Hooke's" symmetries ($H_{ijkl} = H_{jikl} = H_{klij}$) can be split into three parts, behaving differently under the two-dimensional space rotation and belonging to the three different, mutually orthogonal, two-dimensional subspaces remaining stable under the rotation. Such representation leads to a convenient set of functionally independent invariants, vanishing of some of these invariants demarcating the transitions of the tensor to the higher symmetry class. A non-trivial effective condition of orthotropy has been obtained. Some problems concerning the necessary and complete set of measurements of the elastic properties are also encountered.

1. Introduction

LARGE VARIETY of engineering problems of structural mechanics concerning the applications of natural or man-made anisotropic composite materials can be effectively analyzed with the use of the plane stress and/or strain state concepts. Thus the convenient description of the plane elasticity and limit criteria is not only of theoretical, but also of practical interest. In some recent papers [8, 9], it was shown that some problems, which, due to their discouraging complexity, look rather boring and demanding time-consuming analysis in general (three-dimensional) case (cf. [5]), can be, with moderate efforts, effectively solved in the plane case.

In the present paper the authors will demonstrate an effective description of the properties of Hooke's tensor making easier both the better comprehension of the matter and the practical applications of the results. Almost all the considerations can be applied without change to elastic stiffness and/or compliance tensors as well as to the quadratic limit condition tensor. The results, together with the earlier obtained results presented in [8, 9] exhaust most of the practical aspects of the description of anisotropy of the plane, linearly elastic and quadratic limit properties (1).

2. Hooke's tensors

Our subject are plane tensors of the fourth rank $\mathbf{H} \in \mathcal{T}_4$, having the following internal symmetries:

$$H_{ijkl} = H_{jikl} = H_{ijlk} = H_{klij}\,. \tag{2.1}$$

(1) Some interesting but purely theoretical problems, like the polynomial integrity basis, remain out of the sphere of our interest in the present paper.

The most important tensors of this kind are the *stiffness tensors* and the *compliance tensors* of the theory of plane elasticity, thus in [3] it was proposed to call them *Hooke's tensors*. Among the applications of Hooke's tensors one can mention their role as linear operators $\boldsymbol{\alpha} \to \mathbf{H}\cdot\boldsymbol{\alpha}$, bilinear forms $(\boldsymbol{\alpha},\boldsymbol{\beta}) \to \boldsymbol{\alpha}\cdot\mathbf{H}\cdot\boldsymbol{\beta}$ or as quadratic functionals $\boldsymbol{\alpha} \to \boldsymbol{\alpha}\cdot\mathbf{H}\cdot\boldsymbol{\alpha}$, e.g. functionals of energy or the limit stress intensity [1]. A Hooke's tensor $\mathbf{H}$ can play a role of the stiffness or compliance tensor only if $\boldsymbol{\alpha}\cdot\mathbf{H}\cdot\boldsymbol{\alpha} \geq 0$ for every $\boldsymbol{\alpha}$.

In the present section we shall present the important decompositions of the Hooke's tensors, useful for the analysis of the symmetries and the invariance. It would be convenient to begin with recalling the notions and the notation for the second rank tensors.

2.1. Second rank plane tensors

All the *isotropic orthogonal decompositions* of the plane second rank tensors are included in the following formula:

$$\mathcal{T}_2 = \mathcal{S} + \mathcal{A} = \mathcal{P} + \mathcal{D} + \mathcal{A}, \qquad 2^2 = 3 + 1 = 1 + 2 + 1, \tag{2.2}$$

where $\mathcal{S}$ is the three-dimensional space of plane *symmetric* tensors $\boldsymbol{\alpha}^{\mathsf{T}} = \boldsymbol{\alpha}$, $\mathcal{A}$ is the one-dimensional space of *skew-symmetric* tensors $\boldsymbol{\alpha}^{\mathsf{T}} = -\boldsymbol{\alpha}$, $\mathcal{P}$ is one-dimensional space of *isotropic* tensors $u\mathbf{1}$ and $\mathcal{D}$ is the plane of the two-dimensional deviators: $\boldsymbol{\alpha}^{\mathsf{T}} = \boldsymbol{\alpha}$, $\operatorname{tr}\boldsymbol{\alpha} = 0$. These decompositions are orthogonal, $\mathcal{S} \perp \mathcal{A}$, $\mathcal{P} \perp \mathcal{D}$. To these decompositions correspond the following *orthogonal decompositions of unity* $\mathcal{I}$ of $\mathsf{E} \otimes \mathsf{E}$ (see (A.5))

$$\mathcal{I} = \mathcal{I}_{\mathcal{S}} + \mathcal{I}_{\mathcal{A}} = \mathcal{I}_{\mathcal{P}} + \mathcal{I}_{\mathcal{D}} + \mathcal{I}_{\mathcal{A}}\,, \tag{2.3}$$

where

$$\begin{aligned} (\mathcal{I}_{\mathcal{S}})_{ijkl} &= \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \\ (\mathcal{I}_{\mathcal{P}})_{ijkl} &= \frac{1}{2}\delta_{ij}\delta_{kl}\,. \end{aligned}$$

The unity $\mathcal{I}_{\mathcal{S}}$ of the space $\mathcal{S}$ (see (A.5)) acting on the second rank tensors, $\boldsymbol{\alpha} \to \mathcal{I}_{\mathcal{S}}\cdot\boldsymbol{\alpha}$, performs an orthogonal projection of the space $\mathcal{T}_2$ onto the $\mathcal{S}$ space, hence $\mathcal{I}_{\mathcal{S}}\cdot\boldsymbol{\alpha} = \boldsymbol{\alpha}$ iff $\boldsymbol{\alpha} \in \mathcal{S}$. The other unities $\mathcal{I}_{\mathcal{D}}$, $\mathcal{I}_{\mathcal{P}}$, $\mathcal{I}_{\mathcal{A}}$ act in a similar way.

In the forthcoming considerations the one-dimensional space $\mathcal{A}$ and its unity $\mathcal{I}_{\mathcal{A}}$ will remain out of the scope of our interest.

Taking an arbitrary Cartesian basis $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$ in $\mathcal{S}$ and an arbitrary Cartesian basis $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ in $\mathcal{D}$, one can write

$$\begin{aligned} \mathcal{I}_{\mathcal{S}} &= \boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \otimes \boldsymbol{\omega}_2 + \boldsymbol{\omega}_3 \otimes \boldsymbol{\omega}_3\,, \\ \mathcal{I}_{\mathcal{D}} &= \boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 \otimes \boldsymbol{\tau}_2\,, \\ \mathcal{I}_{\mathcal{P}} &= \frac{1}{2}\mathbf{1} \otimes \mathbf{1}. \end{aligned} \tag{2.4}$$

For every tensor $\boldsymbol{\omega} \in \mathcal{S}$ takes place the well-known *spectral decomposition*

$$\boldsymbol{\omega} = \omega_1 \mathbf{w}_1 \otimes \mathbf{w}_1 + \omega_2 \mathbf{w}_2 \otimes \mathbf{w}_2 \,, \tag{2.5}$$

where $\mathbf{w}_1$, $\mathbf{w}_2$ is the Cartesian basis in the *physical plane* $\boldsymbol{\omega}\mathbf{w}_1 = \omega_1 \mathbf{w}_1$, $\boldsymbol{\omega}\mathbf{w}_2 = \omega_2 \mathbf{w}_1$. Thus every deviator has the following canonical form:

$$\boldsymbol{\tau} = \mathbf{d} \otimes \mathbf{d} - \mathbf{d}^{\perp} \otimes \mathbf{d}^{\perp} = 2\mathbf{d} \otimes \mathbf{d} - |\mathbf{d}|^2 \, \mathbf{1}, \tag{2.6}$$

where $\mathbf{d}\cdot\mathbf{d}^{\perp} = 0$, $|\mathbf{d}| = |\mathbf{d}^{\perp}|$. It can be also represented as $\boldsymbol{\tau} = \mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}$, $\mathbf{n}\cdot\mathbf{m} = 0$. The interpretation of the deviators as stresses is shown in Fig. 1; thus we shall further call them *pure shears*.

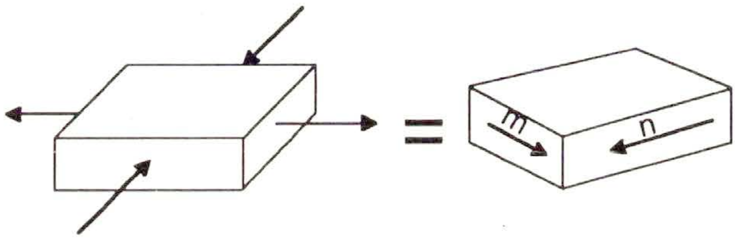

FIG. 1.

The following expression, uniquely representing a tensor $\boldsymbol{\omega} \in \mathcal{S}$ as the orthogonal sum of the isotropic tensor and the pure shear:

$$\boldsymbol{\omega} = \pi \mathbf{1} + \boldsymbol{\tau} = \pi \mathbf{1} + \mathbf{d} \otimes \mathbf{d} - \mathbf{d}^{\perp} \otimes \mathbf{d}^{\perp} \tag{2.7}$$

we shall call *the isotropic decomposition of a second rank tensor*.

The *rotations* $\mathbf{R}$ (and the *mirror reflection* $\mathbf{M}$) of the physical plane act in $\mathcal{T}_2$ according to the rule: $\boldsymbol{\alpha} \to \mathbf{R} * \boldsymbol{\alpha}$, where

$$\mathbf{R} * \boldsymbol{\alpha} \equiv \mathbf{R}\boldsymbol{\alpha}\mathbf{R}^{\top}. \tag{2.8}$$

One-dimensional subspaces $\mathcal{P}, \mathcal{A}$ are the axes of every rotation $\mathbf{R}*$. In the plane of deviators $\mathcal{D}$, a rotation $\mathbf{R}(\varphi)$ of the physical plane by the angle φ acts as a rotation $\mathbf{R}(\varphi)*$ by the double angle 2φ (Fig. 2). Indeed, since

$$\mathbf{R}(\varphi)\mathbf{d} = \cos\varphi\mathbf{d} + \sin\varphi\mathbf{d}^{\perp}, \qquad \mathbf{R}(\varphi)\mathbf{d}^{\perp} = -\sin\varphi\mathbf{d} + \cos\varphi\mathbf{d}^{\perp} \tag{2.9}$$

thus

$$\boldsymbol{\tau}\cdot(\mathbf{R} * \boldsymbol{\tau}) = |\boldsymbol{\tau}|^2 \cos 2\varphi. \tag{2.10}$$

The action of the mirror reflections is similar.

If a Cartesian basis $(\mathbf{n}_1, \mathbf{n}_2)$ in the physical plane is chosen, then the vectors $\mathbf{x}$ are represented by the pairs of numbers (x_1, x_2) and the tensors by the

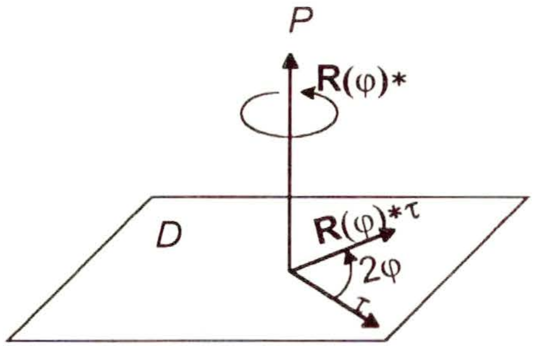

FIG. 2.

number-valued matrices 2×2 (α_{ij}). Then

$$
(2.11) \qquad \begin{gathered}
\mathbf{1} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{R}(\varphi) \sim \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}, \\
\mathbf{R} * \boldsymbol{\alpha} \sim \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}.
\end{gathered}
$$

2.2. Decompositions of Hooke's tensors

During the last decade the spectral decomposition of the elasticity tensors (which can be traced back to Lord Kelvin and has been recalled by J. RYCHLEWSKI in early eighties [2]), is becoming almost canonical and even finds its way to textbooks [10]. In the plane case, such a decomposition of the two-dimensional Hooke's tensor has the form

$$
(2.12) \qquad \mathbf{H} = \chi_{\mathrm{I}} \boldsymbol{\omega}_{\mathrm{I}} \otimes \boldsymbol{\omega}_{\mathrm{I}} + \chi_{\mathrm{II}} \boldsymbol{\omega}_{\mathrm{II}} \otimes \boldsymbol{\omega}_{\mathrm{II}} + \chi_{\mathrm{III}} \boldsymbol{\omega}_{\mathrm{III}} \otimes \boldsymbol{\omega}_{\mathrm{III}},
$$

where the tensors $\boldsymbol{\omega}_K$ ($K = \mathrm{I, II, III}$), called *the proper states* – the eigenelements of the symmetric linear operator $\boldsymbol{\alpha} \to \mathbf{H} \cdot \boldsymbol{\alpha}$ constitute an orthonormal basis

$$
(2.13) \qquad \boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_L = \delta_{KL},
$$

χ_K being the corresponding eigenvalues (2).

It is not difficult to observe that, if one of the proper states is a pure shear, then the other two should be mutually coaxial. Indeed, if, say, $\boldsymbol{\omega}_{\mathrm{III}}$ is a deviator,

(2) For the case of the elastic stiffness tensor S J. RYCHLEWSKI proposed [2] to call these eigenvalues denoted by λ_K *the Kelvin moduli*, their reciprocals $1/\lambda_K$ are the eigenvalues of the elastic compliance tensor C, which has the same *elastic proper states* as S, while the independent parameters, defining the elastic proper states $\boldsymbol{\kappa}_I$ he proposed to call *the stiffness distributors*.

then there exists such a basis in the two-dimensional space that

$$\boldsymbol{\omega}_{\mathrm{III}} \sim \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}. \tag{2.14}$$

Thus, due to the orthogonality condition, the other two proper states should have in the same basis the following diagonal representations:

$$\boldsymbol{\omega}_{\mathrm{I}} \sim \begin{bmatrix} \cos\alpha & 0 \\ 0 & \sin\alpha \end{bmatrix}, \qquad \boldsymbol{\omega}_{\mathrm{II}} \sim \begin{bmatrix} -\sin\alpha & 0 \\ 0 & \cos\alpha \end{bmatrix} \tag{2.15}$$

(we recall here that α has nothing in common with any rotation, it is merely a convenient parameter).

Obviously, such a set of proper states describes the orthotropic material – the reflections with respect to any of the two common proper axes of $\boldsymbol{\omega}_{\mathrm{I}}$ and $\boldsymbol{\omega}_{\mathrm{II}}$ merely changes the sign of $\boldsymbol{\omega}_{\mathrm{III}}$ leaving $\mathbf{C}$ unchanged. Moreover – vanishing of the trace of at least one proper state is the *necessary condition* of orthotropy. If the pure shear $\boldsymbol{\tau}$ along the orthotropy axis were not a proper state, then it would give rise to non-vanishing diagonal terms in $\mathbf{H}\cdot\boldsymbol{\tau}$ tensor. The reflection would change the sign of $\boldsymbol{\tau}$ while $\mathbf{H}\cdot\boldsymbol{\tau}$ would change according to a different rule (diagonal terms are insensitive to such a transformation) i.e., against the assumption, the reflection would not preserve the shape of $\mathbf{H}$.

According to (2.6), the rotation by $\pi/2$ interchanging the vectors $\mathbf{d}$, $\mathbf{d}^{\perp}$ (the change of sign is insignificant) transforms arbitrary traceless tensor $\boldsymbol{\tau}$ into $-\boldsymbol{\tau}$, hence if in (2.15) $\alpha = \pi/4$, then we are dealing with the tetragonal symmetry (the symmetry of the square). Observe that in such a case the hydrostatic state, (proportional to the unit tensor) must be a proper elastic state.

At last, if the two Kelvin moduli, corresponding to the two pure shear proper states, are equal – one obtains the case of isotropic material. We shall prove in the forthcoming consideration, that no other symmetries of the plane Hooke's tensors are possible.

The spectral decomposition (2.12) is an exact counterpart of the spectral decomposition (2.5). Let us find a counterpart of the isotropic decomposition (2.7).

The rotations of the physical plane $\mathbf{R}$ act on the fourth-order tensors according to the rule $\mathbf{A} \to \mathbf{R} * \mathbf{A}$ (see (A.2)). It is evident that every Hooke's tensor $\mathbf{H}$, being rotated preserves its "Hookean nature", any linear combination of Hooke's tensors produces again a Hooke's tensor. Thus the set of all Hooke's tensors is the *tensorial space* (see (A.2)) $\mathcal{H} \subset \mathcal{T}_4$. For further considerations only this space will be of our interest; it is evident that $\dim \mathcal{H} = 6$. We have to find an isotropic decomposition of the space $\mathcal{H}$.

The earlier introduced unities $\mathcal{I}_{\mathcal{S}}, \mathcal{I}_{\mathcal{P}}, \mathcal{I}_{\mathcal{D}}$ are Hooke's tensors. Moreover, every isotropic Hooke's tensor is a linear combination of the two arbitrarily chosen tensors out of this threesome. In such a way we obtain the tensorial plane $\mathcal{J} \subset \mathcal{H}$ consisting of the *isotropic Hooke's tensors*. The pair $\mathcal{I}_\mathcal{P}, \mathcal{I}_\mathcal{D}$ is an orthonormal basis in $\mathcal{J}$, thus every isotropic Hooke's tensor has a unique orthogonal decomposition

$$\mathcal{I} = \lambda_\mathcal{P}\mathcal{I}_\mathcal{P} + \lambda_\mathcal{D}\mathcal{I}_\mathcal{D}, \qquad \mathcal{I}_\mathcal{P}\cdot\mathcal{I}_\mathcal{D} = 0. \tag{2.16}$$

The orthogonal complement of this plane $\mathcal{J}^\perp$ is a four-dimensional tensorial space. Its possible isotropic decomposition can be only of the following form: $\mathcal{J}^\perp = \mathcal{A} + \mathcal{B}$, $4 = 2 + 2$. Indeed, all one-dimensional tensorial subspaces in $\mathcal{H}$ belong to $\mathcal{J}$. The conditions of orthogonality of the tensor $\mathbf{H}$ to $\mathcal{J}$, $\mathcal{I}_\mathcal{P}\cdot\mathbf{H} = 0$, $\mathcal{I}_\mathcal{D}\cdot\mathbf{H} = 0$ are of the following form

$$\mathbf{1}\cdot\mathbf{H}\cdot\mathbf{1} = H_{iijj} = 0, \qquad \operatorname{Tr}\mathbf{H} \equiv H_{ijij} = 0. \tag{2.17}$$

These conditions meet e.g. all the tensors from the set $\mathcal{A}$ of the following form:

$$\mathbf{1} \otimes \boldsymbol{\tau} + \boldsymbol{\tau} \otimes \mathbf{1}, \qquad \boldsymbol{\tau} \in \mathcal{D}. \tag{2.18}$$

Since for every rotation $\mathbf{R}*$ the tensor $\mathbf{R}*(\mathbf{1}\otimes\boldsymbol{\tau} + \boldsymbol{\tau}\otimes\mathbf{1}) = \mathbf{1}\otimes(\mathbf{R}*\boldsymbol{\tau}) + (\mathbf{R}*\boldsymbol{\tau})\otimes\mathbf{1}$ remains in $\mathcal{A}$ and every linear combination of the tensors from $\mathcal{A}$ belongs to $\mathcal{A}$, thus $\mathcal{A}$ is one of the two tensorial planes in $\mathcal{J}^\perp$, $\dim\mathcal{A} = \dim\mathcal{D} = 2$.

The last component of the isotropic decomposition of the space $\mathcal{H}$ is the orthogonal complement $\mathcal{B}$ of the space $\mathcal{A}$ in $\mathcal{J}^\perp$. Let us find the general form of the tensors $\mathbf{D} \in \mathcal{B}$. From the orthogonality condition $\mathbf{D} \perp \mathcal{A}$ we have $(\mathbf{1}\cdot\mathbf{D})\cdot\boldsymbol{\tau} = 0$ for every $\boldsymbol{\tau} \in \mathcal{D}$. Combining this with the condition $\mathbf{1}\cdot\mathbf{D}\cdot\mathbf{1} = 0$ one can see that $(\mathbf{1}\cdot\mathbf{D})\cdot\boldsymbol{\alpha} = 0$ for every $\boldsymbol{\alpha} \in \mathcal{S}$, therefore $\mathbf{1}\cdot\mathbf{D} = \mathbf{0} \in \mathcal{S}$. Making use of the spectral decomposition

$$\mathbf{D} = \lambda_\mathrm{I}\boldsymbol{\omega}_\mathrm{I} \otimes \boldsymbol{\omega}_\mathrm{I} + \lambda_\mathrm{II}\boldsymbol{\omega}_\mathrm{II} \otimes \boldsymbol{\omega}_\mathrm{II} + \lambda_\mathrm{III}\boldsymbol{\omega}_\mathrm{III} \otimes \boldsymbol{\omega}_\mathrm{III} \tag{2.19}$$

from the conditions $\mathbf{1}\cdot\mathbf{D} = \mathbf{0}$, $\operatorname{Tr}\mathbf{D} = 0$, one obtains readily

$$\begin{aligned} \lambda_\mathrm{I}\operatorname{tr}\boldsymbol{\omega}_\mathrm{I} = \lambda_\mathrm{II}\operatorname{tr}\boldsymbol{\omega}_\mathrm{II} = \lambda_\mathrm{III}\operatorname{tr}\boldsymbol{\omega}_\mathrm{III} &= 0, \\ \lambda_\mathrm{I} + \lambda_\mathrm{II} + \lambda_\mathrm{III} &= 0. \end{aligned} \tag{2.20}$$

The only solution, other than $\mathbf{D} = \mathbf{0}$, is the following one

$$\lambda_\mathrm{III} = -\lambda_\mathrm{I}, \qquad \lambda_\mathrm{II} = 0, \qquad \operatorname{tr}\boldsymbol{\omega}_\mathrm{I} = \operatorname{tr}\boldsymbol{\omega}_\mathrm{III} = 0. \tag{2.21}$$

Thus every tensor $\mathbf{D} \in \mathcal{B}$ can be uniquely expressed in the following form

$$\mathbf{D} = \boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}^\perp \otimes \boldsymbol{\tau}^\perp = 2\boldsymbol{\tau} \otimes \boldsymbol{\tau} - |\boldsymbol{\tau}|^2\,\mathcal{I}_\mathcal{D}, \tag{2.22}$$

where $\boldsymbol{\tau}\cdot\boldsymbol{\tau}^\perp = 0$, $|\boldsymbol{\tau}| = |\boldsymbol{\tau}^\perp|$. It is not difficult to check that the tensor $\mathbf{D}$ is totally symmetric and traceless, i.e.

$$D_{\sigma(i)\sigma(j)\sigma(k)\sigma(l)} = D_{ijkl}, \qquad D_{iikl} = 0, \tag{2.23}$$

where σ is an arbitrary permutation. Thus the plane $\mathcal{B}$ consists of the *plane fourth rank deviators* [3] ([3]). Thus, concluding:

Isotropic complete decomposition of the space of plane Hooke's tensors has the following form

$$\mathcal{H} = \mathcal{J} + \mathcal{A} + \mathcal{B}, \qquad 6 = (1+1) + 2 + 2, \tag{2.24}$$

where $\mathcal{J}$ is the plane of space of isotropic Hooke's tensors, $\mathcal{A}$ is the plane of the tensors: $\mathbf{1} \otimes \boldsymbol{\tau} + \boldsymbol{\tau} \otimes \mathbf{1}$, $\boldsymbol{\tau}$ denoting pure shear, $\mathcal{B}$ is the plane of the fourth-rank plane deviators. In other words: the isotropic decomposition of every plane Hooke's tensor has the following form:

$$\mathbf{H} = \lambda_{\mathcal{P}} \mathcal{I}_{\mathcal{P}} + \lambda_{\mathcal{D}} \mathcal{I}_{\mathcal{D}} + (\mathbf{1} \otimes \boldsymbol{\tau} + \boldsymbol{\tau} \otimes \mathbf{1}) + \mathbf{D}, \tag{2.25}$$

the four components defined above being mutually orthogonal; invariants $\lambda_{\mathcal{P}}$ and $\lambda_{\mathcal{D}}$ and the deviators $\boldsymbol{\tau}$, $\mathbf{D}$ are the linear isotropic functions of Hooke's tensor. ([4])

The rotation $\mathbf{R}(\varphi)*$ is a rotation of the six-dimensional space $\mathcal{H}$ around the fixed plane $\mathcal{J}$. It is evident that the tensorial plane $\mathcal{A}$ rotates by the double angle 2φ. The deviatoric plane $\mathcal{B}$ rotates by the quadruple angle 4φ, because, according to the formulae

$$\begin{aligned} \mathbf{R}(\varphi) * \boldsymbol{\tau} &= \cos 2\varphi\, \boldsymbol{\tau} + \sin 2\varphi\, \boldsymbol{\tau}^{\perp}, \\ \mathbf{R}(\varphi) * \boldsymbol{\tau}^{\perp} &= -\sin 2\varphi\, \boldsymbol{\tau} + \cos 2\varphi\, \boldsymbol{\tau}^{\perp}, \end{aligned} \tag{2.26}$$

taking $\mathbf{D} = \boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}^{\perp} \otimes \boldsymbol{\tau}^{\perp}$, one obtains

$$\mathbf{D} \cdot [\mathbf{R}(\varphi) * \mathbf{D}] = |\mathbf{D}|^2 \cos 4\varphi\,. \tag{2.27}$$

3. Hooke's tensors as the second rank tensors

The intriguing similarity between the canonical forms of the pure shears $\boldsymbol{\tau}$ (2.6) and the Hooke's deviators $\mathbf{D}$ (2.22) can be noticed. The explanation of this fact is simple and leading to the well known techniques of handling stiffness tensors, commonly used in the engineering applications.

The space $\mathcal{T}_4$ can be considered, if it is convenient, as any tensorial product $\mathcal{T}_i \otimes \mathcal{T}_j$, $i + j = 4$. The representation $\mathcal{T}_4 = \mathcal{T}_2 \otimes \mathcal{T}_2$ turns out to be especially useful; it means that the tensors of the *fourth rank* are considered as the "*second*" rank tensors from the sequence $\otimes^p \mathcal{T}_2$, $p = 1, 2, \ldots$. This is particularly useful in the case of the Hooke's tensors.

([3]) The last expression wll be called the *canonical form* of the deviator $\mathbf{D}$; (in [3] the canonical form of the plane deviator of arbitrary rank has been shown).

([4])
$$2\lambda_{\mathcal{P}} = \mathbf{1} \cdot \mathbf{H} \cdot \mathbf{1}, \qquad 2\lambda_{\mathcal{D}} = \mathrm{Tr}\mathbf{H} - \lambda_{\mathcal{P}}, \qquad 2\boldsymbol{\tau} = \mathbf{H} \cdot \mathbf{1} - \lambda_{\mathcal{P}} \mathbf{1},$$
$$\mathbf{D} = \mathbf{H} + \lambda_{\mathcal{P}} \mathcal{I}_{\mathcal{P}} - \lambda_{\mathcal{D}} \mathcal{I}_{\mathcal{D}} - \frac{1}{2}[(\mathbf{H} \cdot \mathbf{1}) \otimes \mathbf{1} + \mathbf{1} \otimes (\mathbf{H} \cdot \mathbf{1})].$$

The space of the plane symmetric second-rank tensors is the symmetrised tensorial square of the physical plane E,

$$\mathcal{S} = \operatorname{sym} \mathsf{E} \otimes \mathsf{E}, \qquad \dim \mathcal{S} = 3. \tag{3.1}$$

Quite similarly, it is convenient to consider the space of the Hooke's tensors as the symmetrised tensorial square of the $\mathcal{S}$ space,

$$\mathcal{H} = \operatorname{sym} \mathcal{S} \otimes \mathcal{S}, \qquad \dim \mathcal{H} = 6. \tag{3.2}$$

In other words: Hooke's tensors can be considered as the symmetric "second rank" tensors, generated by the tensors $\alpha \in \mathcal{S}$, exactly in the same manner as the tensors $\alpha \in \mathcal{S}$ are generated by the vectors $\mathbf{x} \in \mathsf{E}$. Such a viewpoint is correct and useful, under the following important condition, however: the orthogonal group $\mathcal{O}(\mathcal{S})$ of the transformations of the Euclidean three-dimensional space $\mathcal{S}$ contains such rotation and mirror reflections, which are not generated by the rotations and the reflections of the physical plane E, for example, the rotation transforming the isotropic tensor $\mathbf{1} \in \mathcal{P}$ into the pure shear $\boldsymbol{\tau} \in \mathcal{D}$, $|\boldsymbol{\tau}| = |\mathbf{1}| = \sqrt{3}$. Such rotations and reflections remain out of the scope of our interests.

If $\{\boldsymbol{\nu}_K\}$, $K = \mathrm{I}, \mathrm{II}, \mathrm{III}$ is a Cartesian basis in $\mathcal{S}$, then $\{\boldsymbol{\nu}_K \otimes \boldsymbol{\nu}_L\}$ is a Cartesian basis in $\mathcal{S} \otimes \mathcal{S}$, thus we can write

$$\mathbf{H} = H_{KL}(\boldsymbol{\nu}_K \otimes \boldsymbol{\nu}_L). \tag{3.3}$$

Symmetrising dyads $\boldsymbol{\nu}_K \otimes \boldsymbol{\nu}_L$ one obtains a basis in the space of the Hooke's tensors.

The usefulness of the description of the Hooke's tensors as the "second rank" tensors can be demonstrated using the three following examples:

1. Taking in the last relation the proper states $\boldsymbol{\omega}_K$ of the tensor $\mathbf{H}$ as the base elements $\boldsymbol{\nu}_K$, one obtains the spectral decomposition of the tensor $\mathbf{H}$ (2.12).

2. According to the new view on $\mathcal{H}$, we shall express the rotations $\alpha \to \mathbf{R} * \alpha$ and $\mathbf{H} \to \mathbf{R} * \mathbf{H}$ in the following form

$$\mathbf{R} * \boldsymbol{\alpha} = \mathcal{R} \cdot \boldsymbol{\alpha}, \qquad \mathbf{R} * \mathbf{H} = \mathcal{R} \circ \mathbf{H} \circ \mathcal{R}^{\mathsf{T}}, \tag{3.4}$$

where $\mathcal{R} \in \mathcal{S} \otimes \mathcal{S}$, $\mathcal{R}^{\mathsf{T}} \circ \mathcal{R} = \mathcal{I}_{\mathcal{S}}$.

Since $\boldsymbol{\alpha} \to \mathbf{R} * \boldsymbol{\alpha}$ is the rotation of the three-dimensional space $\mathcal{S}$ around the unit base vector $\mathbf{1}/\sqrt{2}$ by the double angle (Fig. 3), therefore ([5])

$$\mathcal{R} = \mathcal{I}_{\mathcal{P}} + \cos 2\varphi\, \mathcal{I}_{\mathcal{D}} + \sin 2\varphi\, \mathbf{E}_{\mathcal{D}}\,, \tag{3.5}$$

([5]) This is a generalization of the rotation in the three-dimensional Euclidean vector space around the unit vector $\mathbf{n}$ by the angle φ, $\mathbf{R} = \mathbf{n} \otimes \mathbf{n} + \cos\varphi\,(\mathbf{1} - \mathbf{n} \otimes \mathbf{n}) + \sin\varphi\,\mathbf{E}$, where $\mathbf{E} \equiv \mathbf{n}_1 \wedge \mathbf{n}_2 \equiv \mathbf{n}_1 \otimes \mathbf{n}_2 - \mathbf{n}_2 \otimes \mathbf{n}_1$ and $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}\}$ is an orthonormal basis.

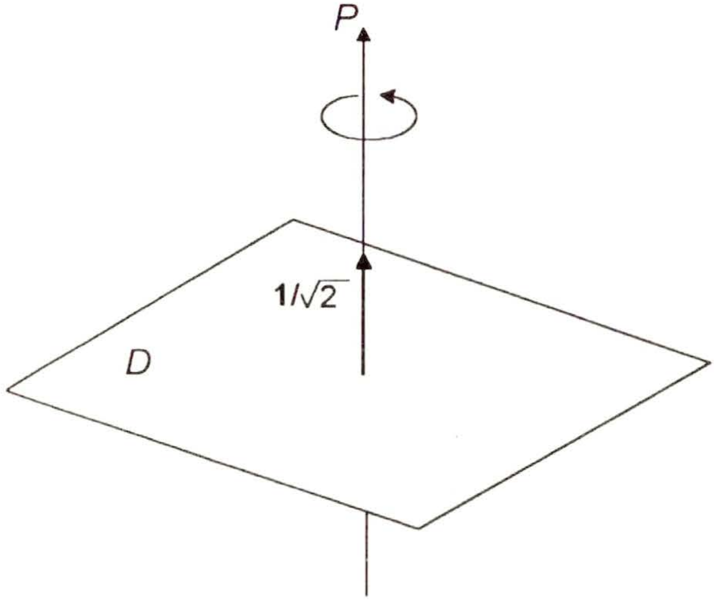

Fig. 3.

where $\mathbf{E}_{\mathcal{D}}$ is the tensor of orientation of the deviatoric plane $\mathcal{D}$, i.e.

$$\mathbf{E}_{\mathcal{D}} \equiv \boldsymbol{\alpha} \otimes \boldsymbol{\beta} - \boldsymbol{\beta} \otimes \boldsymbol{\alpha}\,, \tag{3.6}$$

where $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ is an arbitrary basis in $\mathcal{D}$ left-oriented in the orientation of $\mathbf{E}_{\mathcal{D}}$.

3. Let us adopt in $\mathcal{S}$ an orthogonal basis, generated by the isotropic state $\mathbf{1} \in \mathcal{P}$ and the pure shears $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{D}$, $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0$, $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}| = \sqrt{2}$. Symmetrised and normalised tensor products of these tensors generate the following Cartesian basis in $\mathcal{H}$:

$$\begin{aligned} &\mathbf{B}_1 = \mathcal{I}_{\mathcal{P}}, && \mathbf{B}_2 = \frac{1}{\sqrt{2}}\mathcal{I}_{\mathcal{D}}, \\ &\mathbf{B}_3 = \frac{1}{2\sqrt{2}}(\mathbf{1} \otimes \boldsymbol{\alpha} + \boldsymbol{\alpha} \otimes \mathbf{1}), && \mathbf{B}_4 = \frac{1}{2\sqrt{2}}(\mathbf{1} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \mathbf{1}), \\ &\mathbf{B}_5 = \frac{1}{2\sqrt{2}}(\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} - \boldsymbol{\beta} \otimes \boldsymbol{\beta}), && \mathbf{B}_6 = \frac{1}{2\sqrt{2}}(\boldsymbol{\alpha} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \boldsymbol{\alpha}). \end{aligned} \tag{3.7}$$

Clearly the pairs $(\mathbf{B}_1, \mathbf{B}_2)$, $(\mathbf{B}_3, \mathbf{B}_4)$, $(\mathbf{B}_5, \mathbf{B}_6)$ are the bases in the corresponding tensorial spaces $\mathcal{J}$, $\mathcal{A}$, $\mathcal{B}$. Hence the matrix of rotation $\mathbf{R}*$ has in the basis

$\mathbf{B}_1, \ldots, \mathbf{B}_6$ the following form:

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \cos 2\varphi & -\sin 2\varphi & & \\ & & \sin 2\varphi & \cos 2\varphi & & \\ & & & & \cos 4\varphi & -\sin 4\varphi \\ & & & & \sin 4\varphi & \cos 4\varphi \end{bmatrix}. \tag{3.8}$$

The decomposition

$$\mathbf{H} = H_1\mathbf{B}_1 + \ldots + H_6\mathbf{B}_6 \tag{3.9}$$

can be reduced by denoting

$$\boldsymbol{\tau} = \frac{1}{2\sqrt{2}}(H_3\boldsymbol{\alpha} + H_4\boldsymbol{\beta}), \qquad \mathbf{D} = H_5\mathbf{B}_5 + H_6\mathbf{B}_6 \tag{3.10}$$

to the isotropic decomposition (2.25).

The last example leads to some interesting relations, which will turn out to be useful in the forthcoming considerations. Introducing the following notation:

$$\begin{aligned} R_1 &\equiv \sqrt{H_3^2 + H_4^2}, & R_2 &\equiv \sqrt{H_5^2 + H_6^2}, \\ \cos\beta &\equiv \frac{H_3}{R_1}, & \sin\beta &\equiv \frac{H_4}{R_1}, \\ \cos\gamma &\equiv \frac{H_5}{R_2}, & \sin\gamma &\equiv \frac{H_6}{R_2}, \end{aligned} \tag{3.11}$$

one can write

$$\mathbf{H} = H_1\mathbf{B}_1 + H_2\mathbf{B}_2 + R_1(\cos\beta\mathbf{B}_3 + \sin\beta\mathbf{B}_4) + R_2(\cos\gamma\mathbf{B}_5 + \sin\gamma\mathbf{B}_6). \tag{3.12}$$

The angles β and γ are not merely the handy parameters, they change under the rotation of the physical plane. Using the representation (3.8), one can write

$$\begin{aligned} \mathbf{R} * \mathbf{H} = H_1\mathbf{B}_1 + H_2\mathbf{B}_2 &+ R_1[\cos(\beta + 2\varphi)\mathbf{B}_3 + \sin(\beta + 2\varphi)\mathbf{B}_4] \\ &+ R_2[\cos(\gamma + 4\varphi)\mathbf{B}_5 + \sin(\gamma + 4\varphi)\mathbf{B}_6]. \end{aligned} \tag{3.13}$$

This relation clearly discloses the geometric interpretation of the angles β and γ (see e.g. Fig. 4).

Let us establish the way of choice of bases in $\mathcal{S}$ and in $\mathcal{S} \otimes \mathcal{S}$. Let $\{\mathbf{n}_1, \mathbf{n}_2\}$ be a Cartesian basis in the physical plane E; we shall adopt the following Cartesian basis in $\mathcal{S}$ (6)

$$\boldsymbol{\nu}_{\mathrm{I}} = \mathbf{n}_1 \otimes \mathbf{n}_1, \qquad \boldsymbol{\nu}_{\mathrm{II}} = \mathbf{n}_2 \otimes \mathbf{n}_2, \qquad \boldsymbol{\nu}_{\mathrm{III}} = \frac{1}{\sqrt{2}}(\mathbf{n}_1 \otimes \mathbf{n}_2 + \mathbf{n}_2 \otimes \mathbf{n}_1). \tag{3.14}$$

Note that $\boldsymbol{\nu}_I + \boldsymbol{\nu}_{\mathrm{II}} = \mathbf{1} \in \mathcal{P}$, $\boldsymbol{\nu}_1 - \boldsymbol{\nu}_{\mathrm{II}} \in \mathcal{D}$, $\boldsymbol{\nu}_{\mathrm{III}} \in \mathcal{D}$.

(6) The coefficient $1/\sqrt{2}$ in the expression for $\boldsymbol{\nu}_{\mathrm{III}}$ is essential. Taking instead the symmetric part $(1/2)(\mathbf{n}_1 \otimes \mathbf{n}_2 + \mathbf{n}_2 \otimes \mathbf{n}_1)$, we would not obtain the Cartesian basis, compare [12].

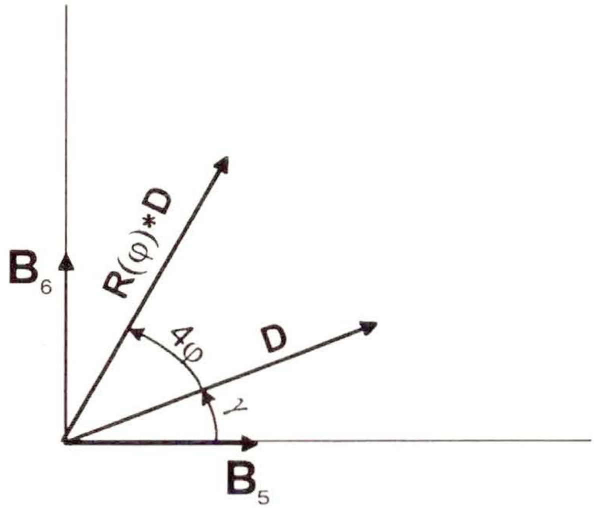

FIG. 4.

In $\mathcal{S} \otimes \mathcal{S}$ we shall take the Cartesian basis $\boldsymbol{\nu}_K \otimes \boldsymbol{\nu}_L$, $K, L =$ I, II, III. The expressions for the tensors $\boldsymbol{\alpha} \in \mathcal{S}$ and $\mathbf{A} \in \mathcal{S} \otimes \mathcal{S}$,

$$(3.15) \qquad \boldsymbol{\alpha} = \alpha_{\mathrm{I}} \boldsymbol{\nu}_{\mathrm{I}}, \qquad \mathbf{A} = A_{IJ}(\boldsymbol{\nu}_I \otimes \boldsymbol{\nu}_J)$$

in the fixed basis $\{\mathbf{n}_1, \mathbf{n}_2\}$ are determined by the mutually unique relations:

$$(3.16) \qquad \boldsymbol{\alpha} \sim \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \alpha_{11} \\ \alpha_{22} \\ \sqrt{2}\alpha_{12} \end{bmatrix},$$

$$(3.17) \qquad \mathbf{A} \sim \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{1111} & A_{1122} & \sqrt{2}A_{1112} \\ A_{2211} & A_{2222} & \sqrt{2}A_{2212} \\ \sqrt{2}A_{1211} & \sqrt{2}A_{1222} & 2A_{1212} \end{bmatrix},$$

$$(3.18) \qquad \alpha_K \beta_K = \alpha_{ij}\beta_{ij} = \boldsymbol{\alpha} \cdot \boldsymbol{\beta},$$

$$(3.19) \qquad A_{IJ}B_{IJ} = A_{ijkl}B_{ijkl} = \mathbf{A} \cdot \mathbf{B};$$

moreover:

$$(3.20) \qquad \varepsilon_I = C_{IJ}\alpha_J \Leftrightarrow \varepsilon_{ij} = C_{ijkl}\alpha_{kl} \Leftrightarrow \boldsymbol{\beta} = \mathbf{C} \cdot \boldsymbol{\alpha},$$

$$(3.21) \qquad A_{IJ} = B_{IK}C_{KJ} \Leftrightarrow A_{ijkl} = B_{ijpq}C_{pqkl} \Leftrightarrow \mathbf{A} = \mathbf{B} \circ \mathbf{C}.$$

The representations of some important tensors have the following form:

$$(3.22)\qquad \begin{aligned} &\mathcal{I} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, && \mathcal{I}_{\mathcal{P}} \sim \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ &\mathcal{I}_{\mathcal{D}} \sim \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, && \mathbf{E}_{\mathcal{D}} \sim \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \end{aligned}$$

while, for the base tensors $\mathbf{B}_K$, one obtains

$$(3.23)\qquad \begin{aligned} &\mathbf{B}_1 \sim \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, && \mathbf{B}_2 \sim \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \\ &\mathbf{B}_3 \sim \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, && \mathbf{B}_4 \sim \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \\ &\mathbf{B}_5 \sim \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, && \mathbf{B}_6 \sim \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}. \end{aligned}$$

Under such a choice, the components H_I of the Hooke's tensor $\mathbf{H}$ in the base $\mathbf{B}_K$ and the components of the "second rank" representation of the same tensor H_{ij} are related by the following equalities:

$$(3.24)\qquad \begin{aligned} &H_1 = \frac{H_{11}+H_{22}+2H_{12}}{2}, && H_2 = \frac{H_{11}+H_{22}-2H_{12}+2H_{33}}{2\sqrt{2}}, \\ &H_3 = \frac{H_{11}-H_{22}}{\sqrt{2}}, && H_4 = (H_{13}+H_{23}), \\ &H_5 = \frac{H_{11}+H_{22}-2H_{12}-2H_{33}}{2\sqrt{2}}, && H_6 = (H_{13}-H_{23}). \end{aligned}$$

At last, the representation of the rotation tensor $\mathcal{R}$ given by (3.4), (3.5) has the following form:

$$(3.25)\qquad \mathcal{R}_{KL} \sim \frac{1}{2}\begin{bmatrix} 1+\cos 2\varphi & 1-\cos 2\varphi & -\sqrt{2}\sin 2\varphi \\ 1-\cos 2\varphi & 1+\cos 2\varphi & \sqrt{2}\sin 2\varphi \\ \sqrt{2}\sin 2\varphi & -\sqrt{2}\sin 2\varphi & 2\cos 2\varphi \end{bmatrix}.$$

4. Invariants and symmetries – effective formulae

4.1. Symmetries

The problems of the symmetries of plane Hooke's tensor has already been discussed in the previous section in terms of proper elastic states and Kelvin moduli. In the case of the "second rank" representations, the matter is also not difficult if only an axis of the presumed symmetry is known. In such a case, taking one base vector, say $\mathbf{n}_1$ along this axis, one can determine the convenient "second rank" representation H_{KL} of the Hooke's tensor (7). Inspecting the shape of this representation and recalling some considerations from the Subsec. 2.2, particularly the expressions (2.15), (216) and the two subsequent paragraphs of text, one can easily tell, what kind of symmetry we really observe, depending on the shape of the representation of the Hooke's tensor, namely:

full symmetry (isotropy),

$$H_{KL} \sim \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & (a-b) \end{bmatrix},$$

symmetry of the square, (tetragonal),

$$H_{KL} \sim \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix},$$

symmetry of rectangle (orthotropy)

$$H_{KL} \sim \begin{bmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & d \end{bmatrix}.$$

The problem arises if we find H_{13} and/or H_{23} different from zero: it is difficult to say, in this case, if there is no symmetry at all or, maybe, we have chosen a wrong axis. We must check up in this case if there exists such a rotation by the angle φ which annihilates the terms containing $\mathbf{B}_4$ and $\mathbf{B}_6$ in the expression (3.13). To this end the following two relations must hold true:

$$\begin{aligned} \sin(\beta + 2\varphi) &\equiv \sin\beta\cos 2\varphi + \cos\beta\sin 2\varphi = 0, \\ \sin(\gamma + 4\varphi) &\equiv \sin\gamma\cos 4\varphi + \cos\gamma\sin 4\varphi = 0, \end{aligned} \tag{4.1}$$

or

$$\tan 2\varphi = -\tan\beta, \qquad \tan 4\varphi = -\tan\gamma. \tag{4.2}$$

(7) The corresponding measurement rules will be discussed in the last subsection of this section.

These two equations can be fulfilled simultaneously if the following relation holds true:

$$\tan\gamma = \frac{2\tan\beta}{1-\tan^2\beta} \equiv \tan 2\beta. \tag{4.3}$$

Using (3.11) one can rewrite this condition in terms of the representation of the Hooke's tensor in the basis $\{\mathbf{B}_K\}$ obtaining the following effective symmetry criterion for the Hooke's tensor $\mathbf{H}$:

Hooke's plane tensor $\mathbf{H}$ *obeys at least orthotropic symmetry if and only if the components of its representation in the basis* $\{\mathbf{B}_K\}$ *fulfil the following relation:*

$$J_5 \equiv (H_3^2 - H_4^2)H_6 - 2H_3H_4H_5 = 0. \tag{4.4}$$

We shall prove in the next subsection that J_5 is invariant under rotation (but not under reflections). The condition (4.4) is trivially fulfilled, if R_1 and/or R_2 vanish. Looking at the relation (3.13) one can readily observe that:

$R_1 = 0$ *yields symmetry of the square, while simultaneous vanishing of* R_1 *and* R_2 *give rise to the isotropy of the Hooke's tensor.*

As it has already been shown, the presence of the plane of symmetry bears orthotropy. We shall prove now that

The only possible non-trivial (i.e. different from the total isotropy) rotational symmetry of the plane Hooke's tensor is the invariance under the rotation by $\pi/2$ *– the tetragonal one.*

Indeed, in virtue of the uniqueness of the tensor decomposition in given orthonormal basis and the functional independence of sin(·) and cos(·), to preserve the plane Hooke's tensor under the two-dimensional rotation by the angle $2\pi/n$ one has to fulfil the following two conditions:

$$\beta + \frac{4\pi}{n} = \beta + 2\pi m \qquad \text{or} \quad R_1 = 0, \tag{4.5}$$

$$\gamma + \frac{8\pi}{n} = \gamma + 2\pi k \qquad \text{or} \quad R_2 = 0, \tag{4.6}$$

where n, m and k are arbitrary integers. The only (non-trivial) solution of (4.5) and (4.6) is: $R_1 = 0,\ n = 4,\ k = 1$, what proves our assertion ([8]).

([8]) One may ask, why by cutting off a slice perpendicularly to the axis of the trigonal symmetry of the three-dimensional body we are gaining additional rotational symmetry? A closer inspection of the case shows that the trigonal symmetry of the three-dimensional body is connected with shearing in the planes orthogonal to the axis of the trigonal symmetry. This shearing stiffness is immaterial in the case of a plane state.

4.2. Invariants

Looking at the relation (3.13) one can tell at once that the following four quantities:

$$
(4.7)\qquad
\begin{aligned}
J_1 &\equiv H_1 = \frac{1}{2}(H_{11}+H_{22}+2H_{12}),\\
J_2 &\equiv H_2 = \frac{1}{2\sqrt{2}}(H_{11}+H_{22}-2H_{12}+2H_{33}),\\
J_3 &\equiv R_1^2 = H_3^2+H_4^2 = \frac{1}{2}(H_{11}-H_{22})^2+(H_{13}+H_{23})^2,\\
J_4 &\equiv R_2^2 = H_5^2+H_6^2 = \frac{1}{8}(H_{11}+H_{22}-2H_{12}-2H_{13})^2+(H_{13}-H_{23})^2
\end{aligned}
$$

are invariants of the *proper orthogonal group* (the group of rotations). The plane Hooke's tensor, however, has in general six independent components, while the proper plane orthogonal group is one-parametric, thus one can expect five functionally independent invariants.

Let us denote:

$$(4.8)\qquad \hat{\beta} \equiv \beta + 2\varphi, \qquad \hat{\gamma} \equiv \gamma + 4\varphi.$$

Certainly the quantity

$$(4.9)\qquad \psi = \hat{\gamma} - 2\hat{\beta} = \gamma - 2\beta$$

is invariant with respect to the proper orthogonal group, and, moreover, it is (modulo 2π) uniquely determined by the components of Hooke's tensor in an arbitrary basis. On the other side, if the values of the previous four invariants as well as ψ are known, then the relation (3.13) determines the Hooke's tensor to within the accuracy of an arbitrary rotation. Thus these five invariants constitute *a complete functionally independent set of invariants with respect to the proper orthogonal group (complete irreducible hemitropic function basis).*

Tracing the derivation of the orthotropy condition (4.4) one can observe that its left-hand term can be expressed by ψ (9) and the condition (4.4) can be rewritten as follows:

$$(4.10)\qquad J_5 = R_1^2 R_2 \sin\psi = 0.$$

Any reflection tensor in two-dimensional space can be represented as the superposition of the axes exchange (reflection) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and some rotation, the axes exchange merely changes the sign of the terms containing $\mathbf{B}_3$ and $\mathbf{B}_6$ changing β into $\pi-\beta$ and γ into $-\gamma$, i.e. ψ (taken modulo 2π) changes its sign; thus only $\cos\psi$ but not ψ itself or $\sin\psi$ is invariant with respect to the *complete orthogonal group* (i.e. containing both the rotation and the mirror reflections), while the previous

(9) The square, or absolute value of this term can be considered as an invariant measure of deviation from the orthotropy.

four rotationally-invariant terms are the invariants of the complete orthogonal group as well.

These considerations lead to the following important conclusion:

No rotation in the plane of the stress (strain) of the plane elastic state is able to change the sign of ψ, thus the class of the materials of the lowest symmetry can be subdivided into two classes of "left" and "right" materials, depending on the sign of ψ.

The "left" materials can be changed into "right" ones by the off-plane turning them upside-down. This means that the sheets of such a material have two distinct sides, which should be specially marked in order to make the information on the elastic properties meaningful.

For completeness we shall express the obtained invariants in terms of the four-index representation of the Hooke's tensors.

Using relation (3.17) to express the first two invariants (4.7) by the components H_{ijkl} it is not difficult to observe that the following two identities hold true:

$$H_1 = \frac{1}{2}(H_{1111} + H_{1122} + H_{2211} + H_{2222}) = \frac{1}{2}H_{iijj} = \lambda_{\mathcal{P}}, \tag{4.11}$$

$$\begin{aligned} H_2 &= \frac{1}{2\sqrt{2}}(H_{1111} - H_{1122} - H_{2211} + H_{2222} + 2H_{1212} + 2H_{2121}) \\ &= \frac{1}{2\sqrt{2}}(2H_{ijij} - H_{iijj}) = \sqrt{2}\lambda_{\mathcal{D}}. \end{aligned} \tag{4.12}$$

The expressions for the remaining invariants are not straightforward. Observe that they depend only on the traceless part $\mathbf{H}'$ of the Hooke's tensor $\mathbf{H}$ (3.12):

$$\mathbf{H}' = R_1(\cos\beta\mathbf{B}_3 + \sin\beta\mathbf{B}_4) + R_2(\cos\gamma\mathbf{B}_5 + \sin\gamma\mathbf{B}_6), \tag{4.13}$$

or

$$\begin{gathered} H'_{ijkl} = H_{ijkl} - \frac{1}{6}[H_{pprr}(2\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}) + H_{prpr}(2\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl})], \\ (H'_{iill} = 0, \qquad H'_{ilil} = 0). \end{gathered} \tag{4.14}$$

Substituting relations (3.7), (3.11) and (3.23) into (4.13) one obtains the following representation of the plane second rank tensor $\mathbf{H}'\cdot\mathbf{1}$:

$$[H'_{ijkk}] = \frac{R_1}{\sqrt{2}}\begin{bmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{bmatrix}. \tag{4.15}$$

Thus

$$H'_{ijkk}H'_{ijll} = R_1^2. \tag{4.16}$$

Due to the orthonormality of the base tensors $\{\mathbf{B}_K\}$, directly from (4.13) it follows that

$$\mathbf{H}'\cdot\mathbf{H}' \equiv H'_{ijkl}H'_{ijkl} = R_1^2 + R_2^2, \tag{4.17}$$

hence:

$$R_2^2 = H'_{ijkl}H'_{ijkl} - H'_{ijpp}H'_{ijqq}. \tag{4.18}$$

The most time-consuming is the derivation of the last relation – the one describing the "shape of deviatoric part", i.e. expressing the functions of ψ in terms of polynomial invariants of the Hooke's tensor. Omitting the tedious calculations ([10]) we present the following result:

$$H'_{ijkl}H'_{klmn}H'_{mnij} = \frac{3}{2\sqrt{2}}R_1^2 R_2 \cos\psi. \tag{4.19}$$

It is not difficult (however it can be fairly boring) to show that our set of invariants: H_1, H_2, R_1^2, R_2^2 and $\mathrm{Tr}(\mathbf{H}'^3)$ is equivalent to the set of invariants obtained by ZHENG [9], who proved that they constitute the complete irreducible isotropic function basis.

One cannot expect to find an expression of such a kind for $\sin\psi$. There is a simple reason for this: all the polynomial scalar expressions obtained by the contraction are *invariant with respect to complete orthogonal group* while $\sin\psi$, as we have already shown, is *the hemitropic function of the plane Hooke's tensor.*

The last question, concerning the invariants of the Hooke's tensors, which has to be discussed are the conditions of positive definiteness

$$\boldsymbol{\alpha}\cdot\mathbf{H}\cdot\boldsymbol{\alpha} \geq 0 \tag{4.20}$$

for every $\boldsymbol{\alpha} \in \mathcal{S}$, which are required for most applications of the Hooke's tensors. In the case of the spectral decomposition of the stiffness (compliance) tensor the problem reduces to the trivial conditions of non-negativeness of the three Kelvin moduli, which are equivalent to the conditions:

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &\geq 0, \\ \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 &\geq 0, \\ \lambda_1\lambda_2\lambda_3 &\geq 0. \end{aligned} \tag{4.21}$$

Recalling that in the basis of proper states the representation of the Hooke's tensor is diagonal, and taking into account that all three expressions (4.21) are invariant with respect to *any orthogonal transformation in* $\mathcal{S}\otimes\mathcal{S}$ (including those,

([10]) The following interesting relations can make this boring procedure slightly simpler: $\mathbf{B}_3^2 = \frac{1}{4}[\mathbf{B}_1 + \sqrt{2}(\mathbf{B}_2 + \mathbf{B}_5)]$, $\mathbf{B}_4^2 = \frac{1}{2}[\mathbf{B}_1 + \sqrt{2}(\mathbf{B}_2 - \mathbf{B}_5)]$, $\mathbf{B}_5^2 = \mathbf{B}_2$, $\mathbf{B}_6^2 = \mathbf{B}_2$, $\mathrm{sym}(\mathbf{B}_3\mathbf{B}_4) = \frac{1}{2\sqrt{2}}\mathbf{B}_6$, $\mathrm{sym}(\mathbf{B}_3\mathbf{B}_5) = \frac{1}{2\sqrt{2}}\mathbf{B}_3$, $\mathrm{sym}(\mathbf{B}_3\mathbf{B}_6) = \frac{1}{2\sqrt{2}}\mathbf{B}_4$, $\mathrm{sym}(\mathbf{B}_4\mathbf{B}_5) = -\frac{1}{2\sqrt{2}}\mathbf{B}_4$, $\mathrm{sym}(\mathbf{B}_4\mathbf{B}_6) = \frac{1}{2\sqrt{2}}\mathbf{B}_3$, $\mathrm{sym}(\mathbf{B}_5\mathbf{B}_6) = 0$.

which do not correspond to any rotation of the physical plane), after some rearrangements one can express these conditions by the polynomial invariants of **H** as the fourth-rank tensor

$$
\begin{aligned}
&\operatorname{Tr}\mathbf{H} \geq 0,\\
&(\operatorname{Tr}\mathbf{H})^2 - \operatorname{Tr}\mathbf{H}^2 \geq 0,\\
&\frac{1}{3}\operatorname{Tr}\mathbf{H}^3 - \frac{1}{2}\operatorname{Tr}\mathbf{H}\operatorname{Tr}\mathbf{H}^2 + \frac{1}{6}(\operatorname{Tr}\mathbf{H})^3 \geq 0.
\end{aligned}
\tag{4.22}
$$

The same relations can be expressed in the language of the invariants generated by the isotropic decomposition as follows:

$$
\begin{aligned}
H_1 + \sqrt{2}H_2 &\geq 0,\\
H_2^2 + 2\sqrt{2}H_1H_2 - R_1^2 - R_2^2 &\geq 0,\\
\sqrt{2}H_1(H_2^2 - R_2^2) - R_1^2(H_2 - R_2\cos\psi) &\geq 0.
\end{aligned}
\tag{4.23}
$$

It is not difficult to notice that no restriction on the sign of ψ has been imposed by the "thermodynamic" condition of the positive definiteness of the Hooke's tensor. Thus both "left" and "right" materials are thermodynamically admissible.

4.3. The rules of the measurements

The procedures of measurements of the elastic properties in the case of materials supplied in the form of sheets and foils very seldom include direct measurements of the shear moduli ([11]); not only the standard, but even more sophisticated laboratory equipment is usually rather inappropriate for such measurements. Usually the Young moduli and Poisson ratio in the chosen directions are measured and then, if needed, the other elastic constants are calculated.

Let us denote the direction of uniaxial tension by x_1 and let **C** denote the elastic compliance tensor, then the stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\varepsilon}$ have the following representations:

$$
\boldsymbol{\sigma} \sim \begin{bmatrix} \sigma \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{\varepsilon} \sim \begin{bmatrix} \sigma C_{11} \\ \sigma C_{21} \\ \sigma C_{31} \end{bmatrix}.
\tag{4.24}
$$

Consequently, by the definitions of the Young modulus E and the Poisson ratio ν, one can write:

$$
\frac{1}{E} = C_{11}, \qquad \frac{\nu}{E} = -C_{21}.
\tag{4.25}
$$

In general we have to determine six unknown elastic constants; to this end we should take at least three specimens oriented at three different angles φ_i ($i = 1, 2, 3$) with respect to some fixed material basis. Performing measurements we

([11]) We shall leave aside in this paper the acoustic measurement techniques.

would obtain then six quantities: E_i, ν_i ($i = 1, 2, 3$). Making use of the Eq. (3.13), and substituting expressions (3.23) for the base tensors $\mathbf{B}_K$, one can write relations (4.25) in the following form:

$$\frac{1}{E_i} = \frac{1}{2}C_1 + \frac{1}{2\sqrt{2}}C_2 + \frac{1}{\sqrt{2}}R_1\cos(\beta + 2\varphi_i) + \frac{1}{2\sqrt{2}}R_2\cos(\gamma + 4\varphi_i), \tag{4.26}$$

$$-\frac{\nu_i}{E_i} = \frac{1}{2}C_1 - \frac{1}{2\sqrt{2}}C_2 - \frac{1}{2\sqrt{2}}R_2\cos(\gamma + 4\varphi_i). \tag{4.27}$$

Using (3.11) one can rewrite Eq. (4.26) and (4.27) in the following form:

$$\frac{1}{2}C_1 + \frac{1}{2\sqrt{2}}C_2 + \frac{\cos 2\varphi_i}{\sqrt{2}}C_3 - \frac{\sin 2\varphi_i}{\sqrt{2}}C_4 + \frac{\cos 4\varphi_i}{2\sqrt{2}}C_5 - \frac{\sin 4\varphi_i}{2\sqrt{2}}C_6 = \frac{1}{E_i}, \tag{4.28}$$

$$\frac{1}{2}C_1 - \frac{1}{2\sqrt{2}}C_2 - \frac{\cos 4\varphi_i}{2\sqrt{2}}C_5 + \frac{\sin 4\varphi_i}{2\sqrt{2}}C_6 = -\frac{\nu_i}{E_i}. \tag{4.29}$$

Taking $i = 1, 2, 3$ we obtain the system of the three pairs of equations for six unknown constants C_K. The determinant Δ of this system can be expressed as follows:

$$\Delta = 2\sqrt{2}\sin^2(\varphi_1 - \varphi_2)\sin^2(\varphi_2 - \varphi_3)\sin^2(\varphi_3 - \varphi_1) \times \cos(\varphi_1 - \varphi_2)\cos(\varphi_2 - \varphi_3)\cos(\varphi_3 - \varphi_1). \tag{4.30}$$

Hence the following rule of the measurements should be observed:

For the determination of the plane Hooke's tensor for the material of no (or unknown) symmetry, using the uniaxial tension tests one should take at least three specimens whose axes are neither parallel nor orthogonal to each other.([12])

It is not difficult to show that if the axes of orthotropy are known, only two specimens are necessary (the one along an orthotropy axis and the other under the angle of $\pi/4$ being particularly convenient). In the case of the isotropy recognized in advance, only one specimen is necessary.

Appendix

A.1. Plane tensors

Two-dimensional Euclidean plane E consisting of the elements $\mathbf{x}, \mathbf{y}, \ldots$ with the scalar product $\mathbf{x}\cdot\mathbf{y}$ we shall call the *physical plane* (it can be e.g. the plane tangent

([12]) This result is not quite unexpected: it is not difficult to observe (compare (4.27)) that for the orthogonal directions $\nu_i/E_i \equiv \nu_j/E_j$.

to the median surface of the shell at an arbitrary point). The plane E generates the *plane Euclidean tensors* as the elements of the tensorial powers $\mathcal{T}_p \equiv \bigotimes^p \mathsf{E}$, $p = 1, 2, \ldots$. Every tensor $\mathbf{A} \in \mathcal{T}_p$ is a finite linear combination of the simple tensors $\mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_p$.

A.2. The rotations and mirror reflections of the tensors

Every *rotation* $\mathbf{x} \to \mathbf{Rx}$ of the physical plane, $\mathbf{R} \in \mathsf{E} \bigotimes \mathsf{E}$, rotates all the tensors, $\mathbf{A} \to \mathbf{R} * \mathbf{A}$. The operators $\mathbf{R}*$ are linear and defined on the simple tensors as follows: $\mathbf{R}*(\mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_p) = \mathbf{Rx}_1 \otimes \ldots \otimes \mathbf{Rx}_p$. Similarly act the mirror reflections $\mathbf{x} \to \mathbf{Mx}$, $\mathbf{M} \in \mathsf{E} \bigotimes \mathsf{E}$.

A.3. Tensorial spaces

Every linear subspace $\mathcal{U} \subset \mathcal{T}_p$ invariant under the rotations and the mirror reflections of the physical plane, $\mathbf{R}*\mathcal{U} = \mathcal{U}$, $\mathbf{M}*\mathcal{U} = \mathcal{U}$, we call the *tensorial space*. The representation of the tensorial space $\mathcal{U}$ (as well as the whole space $\mathcal{T}_p$) in the form of direct sum of the tensorial spaces $\mathcal{U} = \mathcal{U}_1 + \ldots + \mathcal{U}_k$ we call the *isotropic decomposition* of this space. The linear operators mapping $\mathcal{U}$ onto itself, particularly the rotations and the reflections, can be considered as the tensors from $\mathcal{U} \bigotimes \mathcal{U}$.

A.4. The tensors of the second and fourth rank

In the present paper we use the *second* rank tensors denoted (except for $\mathbf{1}, \mathbf{R}, \mathbf{M}$) as $\boldsymbol{\alpha}, \boldsymbol{\beta}, \ldots$ and the *fourth* rank tensors denoted as $\mathbf{A}, \mathbf{B}, \ldots$. The tensorial operations which we use can be expressed in the well-known language of the Cartesian representations as follows:

$$
\text{(A.1)} \qquad \begin{aligned}
&\mathbf{x}\cdot\mathbf{y} \leftrightarrow x_i y_i\,, && \boldsymbol{\alpha}\mathbf{x} \leftrightarrow \alpha_{ij} x_j\,,\\
&\operatorname{tr}\boldsymbol{\alpha} = \alpha_{ii}\,, && \boldsymbol{\alpha} \otimes \boldsymbol{\beta} \leftrightarrow \alpha_{ij}\beta_{pq}\,,\\
&\mathbf{A}\cdot\boldsymbol{\alpha} \leftrightarrow A_{ijpq}\alpha_{pq}\,, && \operatorname{Tr}\mathbf{A} = A_{pqpq}\,,\\
&\mathbf{A}\cdot\mathbf{B} = A_{pqrs}B_{pqrs}\,, && \mathbf{A} \circ \mathbf{B} \leftrightarrow A_{ijpq}B_{pqkl}\,.
\end{aligned}
$$

The following relations hold true

$$
\text{(A.2)} \qquad \begin{aligned}
&(\mathbf{x} \otimes \mathbf{y})\mathbf{z} = (\mathbf{y}\cdot\mathbf{z})\mathbf{x}, \qquad (\boldsymbol{\alpha} \otimes \boldsymbol{\beta})\cdot\boldsymbol{\tau} = (\boldsymbol{\beta}\cdot\boldsymbol{\tau})\boldsymbol{\alpha},\\
&(\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) \circ (\boldsymbol{\tau} \otimes \boldsymbol{\nu}) = (\boldsymbol{\beta}\cdot\boldsymbol{\tau})\boldsymbol{\alpha} \otimes \boldsymbol{\nu},\\
&R * (\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) = (R * \boldsymbol{\alpha}) \otimes (R * \boldsymbol{\beta}).
\end{aligned}
$$

A.5. The tensorial unities

The tensorial *unity* of the plane E we shall denote by $\mathbf{1}$, while the *unity* of the space $\mathsf{E} \bigotimes \mathsf{E}$ – by $\mathcal{I}$, thus $\mathbf{1x} = \mathbf{x}$, $\mathcal{I}\boldsymbol{\alpha} = \boldsymbol{\alpha}$ for all $\mathbf{x} \in \mathsf{E}$, $\boldsymbol{\alpha} \in (\mathsf{E} \bigotimes \mathsf{E})$. In a similar way one can introduce the tensorial *unity* $\mathcal{I}_{\mathcal{U}} \in \mathcal{U} \otimes \mathcal{U}$ into every tensorial space $\mathcal{U}$. In the language of Cartesian representations:

$$(\mathbf{1})_{ij} = \delta_{ij}\,, \qquad (\mathcal{I})_{ijkl} = \delta_{ik}\delta_{jl}\,. \tag{A.3}$$

A.6. Euclidean tensor spaces

In every tensor space $\mathcal{T}_p$ a scalar product $\mathbf{A}\cdot\mathbf{B}$, defined for the simple tensors: $(\mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_p)\cdot(\mathbf{y}_1 \otimes \ldots \otimes \mathbf{y}_p) = (\mathbf{x}_1\mathbf{y}_1)\ldots(\mathbf{x}_p\mathbf{y}_p)$ can be introduced, yielding *2^p-dimensional Euclidean space*. Every orthonormal basis in $\mathcal{T}_p$ we shall call *Cartesian*. Only such rotations of the Euclidean spaces $\mathcal{T}_p$ remain in the scope of our interest, which are generated by rotations of the physical space, as described in A.2.

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Stress tensors associated with deformation tensors via duality

P. HAUPT (KASSEL) and CH. TSAKMAKIS (KARLSRUHE)

THE CONCEPT of dual variables, initially introduced by HAUPT and TSAKMAKIS [3], enables us to relate to each other strain and stress tensors, as well as associated rates, independently of particular material properties. Generally, it is different than the method of conjugate variables, as defined e.g. by MAC VEAN [2] or HILL [4–6]. The duality concept postulated by HAUPT and TSAKMAKIS [3] deals only with two classes of dual stress and strain tensors. The second Piola–Kirchhoff stress tensor and the Green strain tensor, as well as the negative convected stress tensor and the Piola strain tensor, are respectively the Lagrangean stress and strain tensors included in the two classes of dual stress and strain tensors. However, there are further (infinitely many) Lagrangean stress and strain tensors, which may be taken into consideration. The aim of the present paper is to develop further the concept of dual variables to take into account the whole set of Lagrangean stress and strain tensors. Doing this, we obtain a specific mathematical structure in the sets of all strain and stress tensors, which makes it possible to relate strain and stress tensors, as well as associated rates, independently of the particular material properties.

1. Introduction

IT IS WELL-KNOWN that in the theory of finite deformations, several stress and strain tensors can be introduced in various ways. These stress and strain tensors are not *a priori* related to each other, raising the question of whether or not there exists a method to associate with each stress tensor a strain tensor independently of specific material properties. The stress power is usually the convenient framework for answering this question.

According to ZIEGLER and MAC VEAN [1, 2], a stress tensor is assigned to a given strain tensor, if the stress power can be represented by this stress tensor and an appropriate rate of the given strain tensor. We call stress and strain tensors related in this way conjugate in the sense of Ziegler and MacVean. Note in passing that this definition of conjugancy was also adopted by HAUPT and TSAKMAKIS [3]. However, in HAUPT and TSAKMAKIS [3], it was also shown that the above definition brings out the difficulty that arises because the stress and strain tensors associated in such a manner are not unique. For example, consider the strain tensor $\underline{\mathbf{K}} = \frac{1}{2}\left(\mathbf{1} - \mathbf{F}^{-1}\mathbf{F}^{T-1}\right)$. $\underline{\mathbf{K}}$ is conjugate in the sense of Ziegler and MacVean, on the one hand, to the stress tensor $\overline{\mathbf{T}} = (\det \mathbf{F})\mathbf{F}^T\mathbf{T}\mathbf{F}$, with respect to the material time derivative $\dot{\underline{\mathbf{K}}}$, and on the other hand, to the stress tensor $\overline{\mathbf{S}} = (\det \mathbf{F})\mathbf{R}^T\mathbf{T}\mathbf{R}$, with respect to the rate $\overset{\triangle}{\underline{\mathbf{K}}} = \dot{\underline{\mathbf{K}}} + (\dot{\mathbf{U}}\mathbf{U}^{-1})\underline{\mathbf{K}} + \underline{\mathbf{K}}(\dot{\mathbf{U}}\mathbf{U}^{-1})$,

$$W = \overline{\mathbf{T}} \cdot \dot{\underline{\mathbf{K}}} = \overline{\mathbf{S}} \cdot \overset{\triangle}{\underline{\mathbf{K}}}\ .$$

In these relations (¹), $\mathbf{F}$ denotes the deformation gradient tensor, with polar decomposition $\mathbf{F} = \mathbf{RU}$, $\mathbf{T}$ is the Cauchy stress tensor, and W the stress power per unit volume of the reference configuration.

Another concept used to relate stress and strain tensors within the framework of the stress power is due to Hill (see e.g. HILL [4–6] as well as HAVNER [7], OGDEN [8, Sec. 3.5.2] and WANG and TRUESDELL [9, Secs. 3.8 and 3.9]). According to this concept, a stress tensor $\mathbf{t}$ is postulated to be conjugate (in the sequel called conjugate in Hill's sense) to a given strain tensor $\mathbf{e}$ if the inner product of $\mathbf{t}$ with the material time derivative of $\mathbf{e}$ yields the stress power W, i.e., if

$$W = \mathbf{t} \cdot \dot{\mathbf{e}} \, .$$

Clearly, all pairs of stress and strain variables conjugate in Hill's sense are also conjugate in the sense of Ziegler and MacVean, but the converse is generally not true.

Hill's concept of conjugancy has the characteristic feature that there exist stress tensors which do not necessarily have any conjugate strain tensor associated with them having an integrable strain rate. Strain rate tensors are called integrable (²) (not-integrable) if they are expressible (not-expressible) as material time derivatives of some strain tensors, which are defined as functions of the deformation. It is well-known that the strain rate $\mathbf{D}$, representing the symmetrical part of the velocity gradient tensor $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, is a non-integrable rate in general. Thus the weighted Cauchy stress tensor $\mathbf{S} = (\det \mathbf{F})\mathbf{T}$, having the property $W = \mathbf{S} \cdot \mathbf{D}$, is e.g. not conjugate in Hill's sense to a strain tensor which possesses an integrable rate. The same is also true for the stress tensor $\overline{\mathbf{S}}$. On the other hand, if a strain tensor is given, it must not necessarily have a conjugate stress tensor associated with it. As an example of strain tensors to which no stress tensor conjugate in Hill's sense exists, we mention the Almansi strain tensor $\mathbf{A} = \frac{1}{2}(\mathbf{1} - \mathbf{F}^{T-1}\mathbf{F}^{-1})$. These issues have also been discussed e.g. by OGDEN [8, p. 159].

A further possibility for associating stress and strain tensors within the framework of the stress power has been proposed by HAUPT and TSAKMAKIS [3], and referred to as the concept of dual variables (³). Several mathematical aspects from a local differential geometric point of view were discussed by SVENDSEN and TSAKMAKIS [11]. The relation between stress and strain tensors within the duality concept of HAUPT and TSAKMAKIS [3] is unique; in fact, this constitutes the

(¹) The nomenclature is introduced in the Secs. 2 and 3.

(²) The term integrable (not-integrable) strain rate is adopted from PALGEN and DRUCKER [10].

(³) We take this opportunity to correct some misleading and erroneous statements in HAUPT and TSAKMAKIS [3]. The notion of conjugancy used in this reference should be understood in the sense of Ziegler and MacVean, even though in some places this notion was attributed to Hill. Further, on page 184 in HAUPT and TSAKMAKIS [3], the interpretation of the term "direct flux" in Hill's expression "$\mathbf{R}^T\mathbf{DR}$ is not a direct flux", as the specification of a strain tensor with the associated rate $\mathbf{R}^T\mathbf{DR}$, is not correct. Indeed, the term "direct flux" as used by Hill must be interpreted to mean the material time derivative. Furthermore, the statement on p. 174 that $\mathbf{\Psi}$, which is not necessarily assumed to be the gradient of a vector field, induces a system of spatial coordinates, is not true in general.

differences to the conjugancy concept according to Ziegler and MacVean. In addition, concerning dual pairs of variables, use is made not only of the material time derivative, but also e.g. of the so-called objective derivatives. This clarifies the differences compared with the conjugancy concept due to Hill. In the present work, the concept of duality will appropriately be generalized, to include the generalized Lagrangean strain tensors, which are introduced in Sec. 5.1. To be definite, the duality concept postulated in HAUPT and TSAKMAKIS [3] deals only with two classes of dual stress and strain tensors, called family 1 and 2. Representative (Lagrangean) strain tensors are the Green strain tensor $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1})$ (family 1) and the Piola strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{F}^{-1}\mathbf{F}^{T-1} - \mathbf{1})$ (family 2). The purpose of the present paper is to complete the duality concept of HAUPT and TSAKMAKIS [3] by introducing further classes of dual strain and stress tensors, which include the whole set of generalized Lagrangean strain tensors.

After introducing the notation and some background relations in Secs. 2 and 3 we show in Sec. 4 how various so-called objective time derivatives can be assigned to the Cauchy stress tensor. To each of these objective time derivatives of the Cauchy stress tensor corresponds a Lagrangean stress tensor. It turns out that, among all these derivatives, only two possess the structure of generalized Oldroyd time derivatives (the term "generalized" Oldroyd time derivative is specified in Chapter 3). In other words, among all Lagrangean stress tensors, only two are associated to the Cauchy stress tensor with respect to the definition of the generalized Oldroyd time derivatives. This result motivates in Sec. 5 the introduction of a set of generalized strain and stress tensors respectively. Considering various scalar quantities, which are required to be form-invariant with respect to the chosen configuration, the above sets can be partitioned into equivalence classes of generalized strains and associated generalized dual stress tensors, respectively. The concept of duality used here is a generalization of that in HAUPT and TSAKMAKIS [3]. Furthermore, to each strain and stress tensor, a time derivative can be associated, having the form of "generalized" Oldroyd time derivative. This way, we obtain a specific mathematical structure in the sets of all strain and stress tensors, which enables us to relate strain and stress tensors, as well as the associated rates, independently of particular material properties. Some examples formulated using strain and associated dual stress tensors, are briefly discussed in Sec. 6. Finally, in Sec. 7, the duality concept is appropriately extended to take into account two-point tensor fields, as well.

2. Preliminaries

We denote by $\mathbb{R}$ and $\mathbb{N}$ the sets of real and natural numbers, respectively. The absolute value of $c \in \mathbb{R}$ is $|c|$. We use the letter t for the time variable. If φ is a function of t we write $\dot{\varphi}$ or $d\varphi/dt$ for its material time derivative. For the

n-th material time derivative of φ we write also $d^n\varphi/dt^n$, where $n \in \mathbb{N}$, $n \geq 0$. If x is a scalar variable other than t and $f(x)$ is a function of x, then we denote the derivative of $f(x)$ with respect to x by $f'(x)$. In particular, we write $f'(a)$, $a \in \mathbb{R}$, instead of $f'(x)|_{x=a}$. Commonly the same symbol is used to designate a function and the value of that function at a point. However, if we deal with different representations of the same function, then use will be made of different symbols.

Given two sets A and B, the Cartesian product of A and B is denoted by $A \times B$. In particular, we write

$$A^n = \underbrace{A \times A \times \cdots \times A}_{n-\text{times}},$$

$n \in \mathbb{N}$, $n \geq 1$. Let $\mathbf{a}$ and $\mathbf{b}$ be elements of a three-dimensional Euclidean vector space $\mathbb{V}$. By $\mathbf{a} \otimes \mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{b}$ we denote the tensor product, the vector product and the inner product, respectively. The magnitude of $\mathbf{a}$ is denoted by $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. In this work, we identify the vector space $\mathbb{V}$ with its dual space $\mathbb{V}^*$, the identification being specified with the help of the metric tensor induced by the inner product in $\mathbb{V}$. Thus, any n-order tensor $\mathbf{T}$ on $\mathbb{V}$ is regarded as an n-linear function from $\mathbb{V}^n$ to $\mathbb{R}$, denoted by $\mathbf{T} \in \mathsf{L}(\mathbb{V}^n, \mathsf{R})$. In the following, second-order tensors (like vectors) are denoted by boldface letters, whereas for fourth-order tensors we use script letters. For example, $\mathbf{A}, \mathbf{B}, \ldots$ denote second-order tensors, whereas $\mathcal{A}, \mathcal{B}, \ldots$ denote fourth-order tensors, respectively.

Let $\mathbf{A}$, $\mathbf{B}$ be second-order tensors. We write $\operatorname{tr}\mathbf{A}$, $\det\mathbf{A}$ and $\mathbf{A}^T$ for the trace, the determinant and the transpose of $\mathbf{A}$, respectively, while $\mathbf{A} \cdot \mathbf{B} = \operatorname{tr}(\mathbf{A}\mathbf{B}^T)$ is the inner product between $\mathbf{A}$ and $\mathbf{B}$. We write $\mathbf{1} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ for the identity tensor of second order, where δ_{ij} is the Kronecker delta symbol and $\{\mathbf{e}_i\}$, $i = 1, 2, 3$, is an orthonormal basis in $\mathbb{V}$. Further, we use the notations $\mathbf{AB} = A_{ij}B_{jk}\mathbf{e}_i \otimes \mathbf{e}_k$ and $\mathbf{A}^{T-1} = (\mathbf{A}^{-1})^T$, provided $\mathbf{A}^{-1}$ exists. In these relations the convention of summation over repeated indices is employed.

If $\mathbf{A}$ is a symmetric and positive definite second-order tensor having eigenvalues λ_i and corresponding eigenvectors $\mathbf{a}_i$, then the spectral decomposition (see e.g. GURTIN [12, Ch. I.2])

$$\mathbf{A} = \sum_{i=1}^{3} \lambda_i \mathbf{a}_i \otimes \mathbf{a}_i$$

applies. In this case, we denote by $\mathbf{A}^m$, $m \in \mathsf{R}$, the second-order tensor

$$\mathbf{A}^m = \sum_{i=1}^{3} \lambda_i^m \mathbf{a}_i \otimes \mathbf{a}_i \,.$$

Let $\mathcal{K}$, $\mathcal{P}$ be two fourth-order tensors, i.e., linear transformations from the space of second-order tensors into itself. With respect to the orthonormal basis $\{\mathbf{e}_i\}$, the following rules apply: if $\mathcal{K}$, $\mathcal{P}$ and $\mathbf{A}$ are represented by $\mathcal{K} = K_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$, $\mathcal{P} = P_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$, and $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, respectively, the relations

$$\begin{aligned}\mathcal{KP} &= K_{ijmn}P_{mnkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l\,,\\ \mathcal{K}^T &= K_{ijkl}\mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_i \otimes \mathbf{e}_j\,,\\ \mathcal{K}[\mathbf{A}] &= K_{ijmn}A_{mn}\mathbf{e}_i \otimes \mathbf{e}_j\end{aligned}$$

hold. In addition, if $\mathbf{B}$ is a second-order tensor, we have $\mathbf{A} \cdot \mathcal{K}[\mathbf{B}] = \mathbf{B} \cdot \mathcal{K}^T[\mathbf{A}]$. We write $\mathcal{I}$ for the fourth-order identity tensor,

$$\mathcal{I} = \delta_{ij}\delta_{mn}\mathbf{e}_i \otimes \mathbf{e}_m \otimes \mathbf{e}_j \otimes \mathbf{e}_n\,.$$

The tensor $\mathcal{I}$ can be decomposed in the form

$$\mathcal{I} = \mathcal{E} + \mathcal{J}\,,$$

where

$$\mathcal{E} = \frac{1}{2}\left(\delta_{ij}\delta_{mn} + \delta_{in}\delta_{mj}\right)\mathbf{e}_i \otimes \mathbf{e}_m \otimes \mathbf{e}_j \otimes \mathbf{e}_n$$

and

$$\mathcal{J} = \frac{1}{2}\left(\delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj}\right)\mathbf{e}_i \otimes \mathbf{e}_m \otimes \mathbf{e}_j \otimes \mathbf{e}_n\,.$$

This implies $\mathcal{E}[\mathbf{A}] = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$, $\mathcal{J}[\mathbf{A}] = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$, and $\mathcal{I}[\mathbf{A}] = \mathbf{A}$.

3. Background relations

Consider a material body B which occupies the region R_0 in the three-dimensional Euclidean point space E in some reference configuration. Choosing a fixed point (origin) in E, we identify each particle of B by the position vector $\mathbf{X}$ to the place X in R_0 occupied by the considered particle. We write $\mathbf{x}$ for the position vector to the place x occupied by the same material particle in the (current) configuration at time t. In this configuration, the body B occupies the region R_t in E.

A motion of B in E, i.e., an one-parameter family of configurations parameterized by the time t, is a mapping

$$\bar{\mathbf{x}} : (\mathbf{X}, t) \longmapsto \mathbf{x} = \bar{\mathbf{x}}(\mathbf{X}, t), \tag{3.1}$$

which has an inverse $\mathbf{X} = \overline{\mathbf{X}}(\mathbf{x}, t)$ for fixed time t. In what follows, it is assumed that all functions possess continuous derivatives up to any desired order with respect to the space variables and the time t.

The deformation gradient tensor corresponding to (3.1) is denoted by

$$\mathbf{F} = \frac{\partial \bar{\mathbf{x}}}{\partial \mathbf{X}} = \mathrm{GRAD}\,\bar{\mathbf{x}}\,. \tag{3.2}$$

We distinguish between GRAD and grad, representing the gradient operator with respect to $\mathbf{X}$ and $\mathbf{x}$, respectively. Furthermore, $\det \mathbf{F} > 0$ is assumed.

The right Cauchy-Green tensor $\mathbf{C}$ and the left Cauchy-Green tensor $\mathbf{B}$ are given by

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 \,, \tag{3.3}$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2 \,, \tag{3.4}$$

in which $\mathbf{U}$ and $\mathbf{V}$ are the right and the left stretch tensor, respectively, appearing in the polar decomposition of $\mathbf{F}$:

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} \,. \tag{3.5}$$

Here, $\mathbf{R}$ represents a proper orthogonal second-order tensor. Since $\mathbf{U}$ and $\mathbf{V}$ are symmetric and positive definite, they possess the spectral decompositions

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{M}_i \otimes \mathbf{M}_i \,, \tag{3.6}$$

and

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \, \boldsymbol{\mu}_i \otimes \boldsymbol{\mu}_i \tag{3.7}$$

respectively, with

$$\boldsymbol{\mu}_i = \mathbf{R}\,\mathbf{M}_i \,. \tag{3.8}$$

λ_i $(i = 1, 2, 3)$ are positive eigenvalues and $\mathbf{M}_i$, as well as $\boldsymbol{\mu}_i$ are the corresponding unit eigenvectors. It is common (see e.g. OGDEN [8, Sec. 2.2.5]) to call $\mathbf{M}_i$ and $\boldsymbol{\mu}_i$ the Lagrangean and Eulerian principal axes, respectively. Note that the spectral decomposition (3.6) implies

$$\mathbf{U}^{-1} = \sum_{i=1}^{3} \frac{1}{\lambda_i} \mathbf{M}_i \otimes \mathbf{M}_i \,. \tag{3.9}$$

Let X be the place of a material particle in R_0 and denote by y the place of the same material particle in an arbitrary configuration, in which B occupies the region M. Further, we denote by $T_y\mathsf{M}$ the tangent space of M at y. Note that M does not need to be an Euclidean manifold. This is for example the case for the non-Euclidean intermediate configuration in plasticity. An n-order tensor $\mathbf{A}$ is called a tensor at $y \in \mathsf{M}$ if $\mathbf{A} \in \mathsf{L}((T_y\mathsf{M})^n, \mathbb{R})$. If $\mathsf{M} = \mathsf{R}_0$, $\mathbf{A}$ is called a Lagrangean tensor at $y \in \mathsf{R}_0$. In the case when M is different than R_0, the tensor $\mathbf{A}$ is called a spatial tensor at $y \in \mathsf{M}$. In particular, if $\mathsf{M} = \mathsf{R}_t$, then $\mathbf{A}$ is called an Eulerian tensor ([4]) at $y \in \mathsf{R}_t$. In the following, we denote by

$$\boldsymbol{\Psi} = \widehat{\boldsymbol{\Psi}}(\mathbf{X}, t) \in \mathrm{Lin}^+ \tag{3.10}$$

([4]) The definition on spatial tensors given here is not standard. The definitions on Lagrangean and Eulerian tensors are taken from OGDEN [8, Sec. 2.4.1].

a space- and time-dependent linear transformation (second-order two-point tensor field [5]) mapping vectors from $T_X \mathsf{R}_0$ onto $T_y\mathsf{M}$ ($\mathbf{\Psi} \in \mathsf{L}(T_X \mathsf{R}_0 \times T_y \mathsf{M}, \mathbb{R})$) and having a positive determinant.

Let X^k ($k = 1, 2, 3$) be a system of material coordinates, and let

$$\mathbf{X} = \widetilde{\mathbf{X}}(X^k) \tag{3.11}$$

be the position vector of a material particle in the reference configuration. The coordinate system induces the local basis of tangent vectors $\{\mathbf{G}_k\}$,

$$\mathbf{G}_k = \frac{\partial \widetilde{\mathbf{X}}}{\partial X^k}\,, \tag{3.12}$$

and the gradient vectors $\{\mathbf{G}^k\}$,

$$\mathbf{G}^k = \mathrm{GRAD}\, \widetilde{X}^k(\mathbf{X}), \tag{3.13}$$

being the reciprocal basis of the tangent vectors $\{\mathbf{G}_k\}$, where

$$X^k = \widetilde{X}^k(\mathbf{X}) \tag{3.14}$$

are the relations inverse to (3.11). With respect to (3.10), (3.12) and (3.13), various bases $\{\mathbf{g}_k^{(\Psi)}\}$ in $T_y\mathsf{M}$, with reciprocal basis $\{\mathbf{g}^{(\Psi)k}\}$, can be defined by

$$\mathbf{g}_k^{(\Psi)} := \mathbf{\Psi}\mathbf{G}_k\,, \tag{3.15}$$

$$\mathbf{g}^{(\Psi)k} := \mathbf{\Psi}^{T-1}\mathbf{G}^k\,. \tag{3.16}$$

Note that the special case $\mathbf{\Psi} = \mathbf{F}$ defines the so-called convected coordinate systems. From (3.15), (3.16),

$$\dot{\mathbf{g}}_k^{(\Psi)} := \dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1}\mathbf{g}_k^{(\Psi)}\,, \tag{3.17}$$

$$\dot{\mathbf{g}}^{(\Psi)k} = -(\dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1})^T\mathbf{g}^{(\Psi)k}\,. \tag{3.18}$$

Next consider the spatial, time-dependent tensor field $\mathbf{u}$, having the representation

$$\mathbf{u} = u^k\mathbf{g}_k^{(\Psi)} = u_k\,\mathbf{g}^{(\Psi)k}\,. \tag{3.19}$$

The relations

$$\frac{\delta^{(\cdot)}}{\delta t}\mathbf{u} := \dot{u}^k\,\mathbf{g}_k^{(\Psi)}\,, \tag{3.20}$$

$$\frac{\delta_{(\cdot)}}{\delta t}\mathbf{u} := \dot{u}_k\,\mathbf{g}^{(\Psi)k}\,, \tag{3.21}$$

[5] $\mathbf{\Psi}$ can be interpreted to be related with a local deformation process.

define time derivatives which are called generalized Oldroyd time derivatives of $\mathbf{u}$. Clearly, from (3.17)–(3.21),

$$\frac{\delta^{(\cdot)}}{\delta t}\mathbf{u} = \dot{\mathbf{u}} - \dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1}\mathbf{u}, \tag{3.22}$$

$$\frac{\delta_{(\cdot)}}{\delta t}\mathbf{u} = \dot{\mathbf{u}} + (\dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1})^T\mathbf{u}. \tag{3.23}$$

Note that the time derivatives $\delta_{(\cdot)}\mathbf{u}/\delta t$ and $\delta^{(\cdot)}\mathbf{u}/\delta t$ are related to the material time derivative of the Lagrangean vectors $\mathbf{u}^{(\mathrm{L})}$, $\mathbf{u}_{(\mathrm{L})}$,

$$\mathbf{u}^{(\mathrm{L})} := \boldsymbol{\Psi}^{-1}\mathbf{u}, \tag{3.24}$$

$$\mathbf{u}_{(\mathrm{L})} := \boldsymbol{\Psi}^{T}\mathbf{u}, \tag{3.25}$$

through

$$\dot{\mathbf{u}}^{(\mathrm{L})} = \boldsymbol{\Psi}^{-1}\frac{\delta^{(\cdot)}}{\delta t}\mathbf{u}, \tag{3.26}$$

$$\dot{\mathbf{u}}_{(\mathrm{L})} := \boldsymbol{\Psi}^{T}\frac{\delta_{(\cdot)}}{\delta t}\mathbf{u}, \tag{3.27}$$

respectively. These definitions of generalized Oldroyd time derivatives for vector fields can easily be extended to introduce generalized Oldroyd time derivatives for tensor fields. For example, for a spatial symmetric second-order tensor

$$\mathbf{A} = A^{kl}\mathbf{g}_k^{(\Psi)} \otimes \mathbf{g}_l^{(\Psi)} = A_{kl}\mathbf{g}^{(\Psi)k} \otimes \mathbf{g}^{(\Psi)l}, \tag{3.28}$$

the corresponding symmetric generalized Oldroyd rates are defined by

$$\frac{\delta^{(\cdot\cdot)}}{\delta t}\mathbf{A} = \dot{A}^{kl}\mathbf{g}_k^{(\Psi)} \otimes \mathbf{g}_l^{(\Psi)}, \tag{3.29}$$

$$\frac{\delta_{(\cdot\cdot)}}{\delta t}\mathbf{A} = \dot{A}_{kl}\mathbf{g}^{(\Psi)k} \otimes \mathbf{g}^{(\Psi)l}. \tag{3.30}$$

It follows that

$$\frac{\delta^{(\cdot\cdot)}}{\delta t}\mathbf{A} = \dot{\mathbf{A}} - \dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1}\mathbf{A} - \mathbf{A}(\dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1})^T, \tag{3.31}$$

$$\frac{\delta_{(\cdot\cdot)}}{\delta t}\mathbf{A} = \dot{\mathbf{A}} + (\dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1})^T\mathbf{A} + \mathbf{A}\dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1}, \tag{3.32}$$

and that

$$\dot{\mathbf{A}}^{(\mathrm{L})} = \boldsymbol{\Psi}^{-1}\left(\frac{\delta_{(\cdot\cdot)}}{\delta t}\mathbf{A}\right)\boldsymbol{\Psi}^{T-1}, \tag{3.33}$$

$$\dot{\mathbf{A}}_{(\mathrm{L})} = \boldsymbol{\Psi}^{T}\left(\frac{\delta^{(\cdot\cdot)}}{\delta t}\mathbf{A}\right)\boldsymbol{\Psi}, \tag{3.34}$$

where

$$\mathbf{A}^{(\mathrm{L})} := \mathbf{\Psi}^{-1}\mathbf{A}\mathbf{\Psi}^{T-1}, \tag{3.35}$$

$$\mathbf{A}_{(\mathrm{L})} := \mathbf{\Psi}\mathbf{A}\mathbf{\Psi}^{T}. \tag{3.36}$$

Next we note that with respect to the basis $\{\mathbf{M}_i\}$, various strain tensors can be defined. In order to obtain the Lagrangean strain tensors introduced by Hill (6), we consider monotonic scalar functions $g: (0,\infty) \to \mathbb{R}$, such that

$$g(1) = 0, \qquad g'(1) = 1. \tag{3.37}$$

Then, the symmetric tensors $\mathbf{G}_{(g)}$, defined by means of the isotropic tensor-valued function $\mathbf{g}(\cdot)$,

$$\mathbf{g}: \mathbf{U} \quad \longmapsto \quad \mathbf{G}_{(g)} = \mathbf{g}(\mathbf{U}) := \sum_{i=1}^{3} g(\lambda_i)\,\mathbf{M}_i \otimes \mathbf{M}_i\,, \tag{3.38}$$

represent Lagrangean measures of strain, referred to as Hill's Lagrangean strain tensors. Examples of such functions are given by ($m \in \mathbb{R}$)

$$g_{(m)}(\lambda_i) := \begin{cases} \dfrac{1}{m}(\lambda_i^m - 1) & \text{if} \quad m \neq 0, \\ \ln \lambda_i & \text{if} \quad m = 0, \end{cases} \tag{3.39}$$

inducing the strain tensors (7)

$$\mathbf{G}_{(m)} := \mathbf{G}_{(g_{(m)})} := \begin{cases} \displaystyle\sum_{i=1}^{3} \frac{1}{m}(\lambda_i^m - 1)\mathbf{M}_i \otimes \mathbf{M}_i = \frac{1}{m}(\mathbf{U}^m - \mathbf{1}) & \text{if } m \in (\mathbb{R}\backslash 0), \\ \displaystyle\sum_{i=1}^{3} (\ln \lambda_i)\mathbf{M}_i \otimes \mathbf{M}_i = \ln \mathbf{U} & \text{if } m = 0. \end{cases} \tag{3.40}$$

In relating stress tensors to the given strain tensors, we will employ the stress power per unit volume of the reference configuration W, which can also be written in the form

$$W = \mathbf{T}_R \cdot \dot{\mathbf{F}}\,. \tag{3.41}$$

In this formula, $\mathbf{T}_R$ stands for the first Piola-Kirchhoff stress tensor, i.e.,

$$\mathbf{T}_R = (\det \mathbf{F})\mathbf{T}\mathbf{F}^{T-1} = \mathbf{S}\mathbf{F}^{T-1}, \tag{3.42}$$

(6) The treatment of Hill's Lagrangean strain tensors given here is taken from Ogden [8, Sec. 2.2.7] as well as Wang and Truesdell [9, Sec. 3.8].

(7) These Lagrangean strain tensors were introduced for the first time by Doyle and Ericksen [13, Ch. 4].

where $\mathbf{T}$ and $\mathbf{S} = (\det \mathbf{F})\mathbf{T}$ are the Cauchy and the weighted Cauchy (or Kirchhoff) stress tensor ([8]), respectively. Furthermore, we have

$$W = \mathbf{S} \cdot \mathbf{D}, \tag{3.43}$$

where $\mathbf{D}$ represents the symmetric part of the velocity gradient tensor $\mathbf{L}$ (the antisymmetric part of $\mathbf{L}$ being $\mathbf{W}$):

$$\mathbf{L} = \operatorname{grad} \dot{\mathbf{x}} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{D} + \mathbf{W}, \tag{3.44}$$

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \tag{3.45}$$

$$\mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T). \tag{3.46}$$

4. Objective rates for S

In this section, we shall consider the Lagrangean stress tensors conjugate (in Hill's sense) to the strain tensors (3.37)–(3.40). As a first step towards the development of a general duality concept for associating strain and stress tensors, we will derive the relations between these Lagrangean stress tensors and the weighted Cauchy stress tensor $\mathbf{S}$. These relations are expressed in terms of linear transformations and using the same transformations, we shall establish various so-called objective rates for $\mathbf{S}$. It turns out that, among all the transformations corresponding to arbitrary m, only those for $m = \pm 2$ lead to objective rates for $\mathbf{S}$ having the structure of a generalized Oldroyd time derivative.

In order to derive this result, we turn to the strain tensors $\mathbf{G}_{(g)}$ defined by (3.38), where $g(\lambda_i)$ may be specified by (3.29). First of all, the stress power W is rewritten as

$$W = \mathbf{T}_{(\mathrm{BS})} \cdot \dot{\mathbf{U}}, \tag{4.1}$$

where

$$\mathbf{T}_{(\mathrm{BS})} := \frac{1}{2}\left(\mathbf{T}_R^T\mathbf{R} + \mathbf{R}^T\mathbf{T}_R\right) \tag{4.2}$$

is the symmetric part of the Biot stress tensor

$$\mathbf{T}_{(\mathrm{B})} := \mathbf{R}^T\mathbf{T}_R \tag{4.3}$$

(see OGDEN [8, Sec. 3.5.2]). The definition of the stress tensors $\mathbf{T}_{(g)}$, conjugate in Hill's sense to $\mathbf{G}_{(g)}$, should be based on the identity

$$\mathbf{T}_{(\mathrm{BS})} \cdot \dot{\mathbf{U}} = \mathbf{T}_{(g)} \cdot \dot{\mathbf{G}}_{(g)}. \tag{4.4}$$

([8]) We are concerned here only with nonpolar materials, so that T, and therefore S, is symmetric.

In view of $\mathbf{G}_{(g)} = \mathbf{g}(\mathbf{U})$, by (3.38), we obtain

$$\dot{\mathbf{G}}_{(g)} = \mathcal{U}_{(g)}[\dot{\mathbf{U}}] \tag{4.5}$$

where

$$\mathcal{U}_{(g)} = \frac{\partial \mathbf{g}}{\partial \mathbf{U}} \tag{4.6}$$

and

$$\mathcal{U}_{(g)} = \mathcal{U}_{(g)}^{T}, \tag{4.7}$$

i.e., $\mathcal{U}_{(g)}$ is symmetric. Furthermore, there exists a symmetric fourth-order tensor $\mathcal{T}_{(g)}$ satisfying the relation

$$\mathcal{U}_{(g)}\mathcal{T}_{(g)} = \mathcal{T}_{(g)}\mathcal{U}_{(g)} = \mathcal{E}. \tag{4.8}$$

Thus,

$$\mathbf{T}_{(\mathrm{BS})} \cdot \dot{\mathbf{U}} = \mathbf{T}_{(g)} \cdot \mathcal{U}_{(g)}[\dot{\mathbf{U}}] = \mathcal{U}_{(g)}[\mathbf{T}_{(g)}] \cdot \dot{\mathbf{U}} \tag{4.9}$$

and

$$\mathbf{T}_{(g)} = \mathcal{T}_{(g)}[\mathbf{T}_{(\mathrm{BS})}]. \tag{4.10}$$

With respect to the basis $\{\mathbf{M}_i\}$, the following representations hold:

$$\begin{aligned} \mathcal{U}_{(g)} = {} & \sum_{i=1}^{3} g'(\lambda_i)\mathbf{M}_i \otimes \mathbf{M}_i \otimes \mathbf{M}_i \otimes \mathbf{M}_i \\ & + \sum_{i \neq j} \ell_{(g)ij} \left(\mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_i \otimes \mathbf{M}_j + \mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_j \otimes \mathbf{M}_i\right), \end{aligned} \tag{4.11}$$

$$\begin{aligned} \mathcal{T}_{(g)} = {} & \sum_{i=1}^{3} \frac{1}{g'(\lambda_i)}\mathbf{M}_i \otimes \mathbf{M}_i \otimes \mathbf{M}_i \otimes \mathbf{M}_i \\ & + \frac{1}{4}\sum_{i \neq j} \frac{1}{\ell_{(g)ij}} \left(\mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_i \otimes \mathbf{M}_j + \mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_j \otimes \mathbf{M}_i\right), \end{aligned} \tag{4.12}$$

where

$$\ell_{(g)ij} := \begin{cases} \dfrac{1}{2}\dfrac{g(\lambda_j) - g(\lambda_i)}{\lambda_j - \lambda_i} & \text{if} \quad \lambda_i \neq \lambda_j, \quad i \neq j, \\ \dfrac{1}{2}g'(\lambda_i) & \text{if} \quad \lambda_i = \lambda_j, \quad i \neq j. \end{cases} \tag{4.13}$$

(For a more detailed derivation of the relations (4.1)–(4.13) see OGDEN [8, Sec. 3.5.2]).

In order to express the dependence on the weighted Cauchy stress tensor $\mathbf{S}$, we note that in view of (4.2) and $(3.42)_2$, (3.5), the equation

$$\mathbf{T}_{(\mathrm{BS})} = \frac{1}{2}\left(\mathbf{U}^{-1}\mathbf{R}^{T}\mathbf{S}\mathbf{R} + \mathbf{R}^{T}\mathbf{S}\mathbf{R}\mathbf{U}^{-1}\right) =: \mathcal{K}[\mathbf{S}] \tag{4.14}$$

applies. Taking into account the relations (3.6)–(3.9), it is not difficult to derive for $\mathcal{K}$ the representation

$$(4.15)\qquad \mathcal{K} = \frac{1}{4}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\lambda_i+\lambda_j}{\lambda_i\lambda_j}\left(\mathbf{M}_i\otimes\mathbf{M}_j\otimes\boldsymbol{\mu}_i\otimes\boldsymbol{\mu}_j+\mathbf{M}_i\otimes\mathbf{M}_j\otimes\boldsymbol{\mu}_j\otimes\boldsymbol{\mu}_i\right).$$

Inserting $(4.14)_2$ in (4.10) yields

$$(4.16)\qquad \mathbf{T}_{(g)} = \mathcal{T}_{(g)}\mathcal{K}[\mathbf{S}] = \mathcal{A}_{(g)}[\mathbf{S}]$$

with

$$(4.17)\qquad \mathcal{A}_{(g)} := \mathcal{T}_{(g)}\mathcal{K}\,.$$

From (4.12) and (4.15) we obtain

$$(4.18)\qquad \mathcal{A}_{(g)} = \sum_{i=1}^{3}\frac{1}{\lambda_i g'(\lambda_i)}\mathbf{M}_i\otimes\mathbf{M}_i\otimes\boldsymbol{\mu}_i\otimes\boldsymbol{\mu}_i + \sum_{i\neq j}\alpha_{(g)ij}\left(\mathbf{M}_i\otimes\mathbf{M}_j\otimes\boldsymbol{\mu}_i\otimes\boldsymbol{\mu}_j+\mathbf{M}_i\otimes\mathbf{M}_j\otimes\boldsymbol{\mu}_j\otimes\boldsymbol{\mu}_i\right),$$

where (no summation over $i, j, i\neq j$)

$$(4.19)\qquad \alpha_{(g)ij} := \begin{cases} \dfrac{1}{4}\dfrac{\lambda_i^2-\lambda_j^2}{\lambda_j\lambda_i(g(\lambda_i)-g(\lambda_j))} & \text{if}\quad \lambda_i\neq\lambda_j,\quad i\neq j,\\[2ex] \dfrac{1}{2}\dfrac{1}{\lambda_i g'(\lambda_i)} & \text{if}\quad \lambda_i=\lambda_j,\quad i\neq j. \end{cases}$$

Introducing the fourth-order tensor $\mathcal{P}_{(g)}$ by

$$(4.20)\qquad \mathcal{A}_{(g)}\mathcal{P}_{(g)} = \mathcal{P}_{(g)}\mathcal{A}_{(g)} = \mathcal{E}\,,$$

where

$$(4.21)\qquad \mathcal{P}_{(g)} = \sum_{i=1}^{3}\lambda_i g'(\lambda_i)\boldsymbol{\mu}_i\otimes\boldsymbol{\mu}_i\otimes\mathbf{M}_i\otimes\mathbf{M}_i + \frac{1}{4}\sum_{i\neq j}\frac{1}{\alpha_{(g)ij}}\left(\boldsymbol{\mu}_i\otimes\boldsymbol{\mu}_j\otimes\mathbf{M}_i\otimes\mathbf{M}_j+\boldsymbol{\mu}_i\otimes\boldsymbol{\mu}_j\otimes\mathbf{M}_j\otimes\mathbf{M}_i\right),$$

in view of (4.18), we deduce from $(4.16)_2$ that

$$(4.22)\qquad \mathbf{S} = \mathcal{P}_{(g)}[\mathbf{T}_{(g)}].$$

The tensors $\mathcal{P}_{(g)}$ and $\mathcal{A}_{(g)}$ induce transformations relating the stress tensors $\mathbf{T}_{(g)}$ and $\mathbf{S}$, respectively. This enables us to associate with each function $g(\cdot)$ an objective rate of $\mathbf{S}$, defined by application of the same transformation $\mathcal{P}_{(g)}$ to the material time derivative of the Lagrangean stress tensor $\mathbf{T}_{(g)}$:

$$\frac{D_{(g)}}{Dt}\mathbf{S} := \mathcal{P}_{(g)}[\dot{\mathbf{T}}_{(g)}] \quad \Longleftrightarrow \quad \dot{\mathbf{T}}_{(g)} = \mathcal{A}_{(g)}\left[\frac{D_{(g)}}{Dt}\mathbf{S}\right]. \tag{4.23}$$

From this, as well as from $(4.16)_2$, we obtain

$$\frac{D_{(g)}}{Dt}\mathbf{S} = \mathcal{P}_{(g)}\left[\left(\mathcal{A}_{(g)}[\mathbf{S}]\right)^{\bullet}\right] = \mathcal{P}_{(g)}\left[\mathcal{A}_{(g)}[\dot{\mathbf{S}}] + \dot{\mathcal{A}}_{(g)}[\mathbf{S}]\right]. \tag{4.24}$$

Inserting herein the relation

$$\mathcal{P}_{(g)}\dot{\mathcal{A}}_{(g)} = -\dot{\mathcal{P}}_{(g)}\mathcal{A}_{(g)}, \tag{4.25}$$

which follows from (4.20), and taking into account (4.20), we see that

$$\frac{D_{(g)}}{Dt}\mathbf{S} = \dot{\mathbf{S}} - \dot{\mathcal{P}}_{(g)}\mathcal{A}_{(g)}[\mathbf{S}]. \tag{4.26}$$

It is verified in Appendix A that the rate $D_{(g)}\mathbf{S}/Dt$ constitutes an objective Eulerian tensor.

Next, we discuss the requirement that the objective stress rate $D_{(g)}\mathbf{S}/Dt$ should fit into the structure of a generalized Oldroyd time derivative. We see, that this requirement implies a special structure of the fourth-order tensor $\mathcal{A}_{(g)}$, namely the property

$$\mathcal{A}_{(g)}[\mathbf{S}] = \mathbf{\Psi}^T_{(g)}\mathbf{S}\mathbf{\Psi}_{(g)}, \tag{4.27}$$

valid for all Eulerian second-order tensors $\mathbf{S}$, where $\mathbf{\Psi}_{(g)} \in \mathrm{Lin}^+$. Indeed, if this relation is true, $(4.16)_2$ reads

$$\mathbf{T}_{(g)} = \mathcal{A}_{(g)}[\mathbf{S}] = \mathbf{\Psi}^T_{(g)}\mathbf{S}\mathbf{\Psi}_{(g)}, \tag{4.28}$$

and $(4.23)_2$ implies

$$\dot{\mathbf{T}}_{(g)} = \mathcal{A}_{(g)}\left[\frac{D_{(g)}}{Dt}\mathbf{S}\right] = \mathbf{\Psi}^T_{(g)}\left(\frac{D_{(g)}}{Dt}\mathbf{S}\right)\mathbf{\Psi}_{(g)}, \tag{4.29}$$

with

$$\frac{D_{(g)}}{Dt}\mathbf{S} = \dot{\mathbf{S}} + \left(\dot{\mathbf{\Psi}}_{(g)}\mathbf{\Psi}^{-1}_{(g)}\right)^T\mathbf{S} + \mathbf{S}\dot{\mathbf{\Psi}}_{(g)}\mathbf{\Psi}^{-1}_{(g)}. \tag{4.30}$$

Using the representation

$$\mathbf{\Psi}_{(g)} = \Psi_{(g)ij}\boldsymbol{\mu}_i \otimes \mathbf{M}_j, \tag{4.31}$$

we conclude from (4.28) that

$$\mathcal{A}_{(g)} = \frac{1}{2}\Psi_{(g)pq}\Psi_{(g)mn}\left(\mathbf{M}_q \otimes \mathbf{M}_n \otimes \boldsymbol{\mu}_p \otimes \boldsymbol{\mu}_m + \mathbf{M}_q \otimes \mathbf{M}_n \otimes \boldsymbol{\mu}_m \otimes \boldsymbol{\mu}_p\right). \tag{4.32}$$

Comparing this with (4.18), (4.19) yields

$$\boldsymbol{\Psi}_{(g)} = \sum_{i=1}^{3} \Psi_{(g)ii}\,\boldsymbol{\mu}_i \otimes \mathbf{M}_i\,, \tag{4.33}$$

and therefore

$$\begin{aligned}\mathcal{A}_{(g)} = &\sum_{i=1}^{3}(\Psi_{(g)ii})^2\mathbf{M}_i \otimes \mathbf{M}_i \otimes \boldsymbol{\mu}_i \otimes \boldsymbol{\mu}_i \\ &+\frac{1}{2}\sum_{i\neq j}\Psi_{(g)ii}\Psi_{(g)jj}\left(\mathbf{M}_i \otimes \mathbf{M}_j \otimes \boldsymbol{\mu}_i \otimes \boldsymbol{\mu}_j + \mathbf{M}_i \otimes \mathbf{M}_j \otimes \boldsymbol{\mu}_j \otimes \boldsymbol{\mu}_i\right).\end{aligned} \tag{4.34}$$

Hence, through (4.34) and (4.18), (4.19), it follows (no summation over i, j)

$$(\Psi_{(g)ii})^2 = \frac{1}{\lambda_i g'(\lambda_i)}, \tag{4.35}$$

$$\Psi_{(g)ii}\Psi_{(g)jj} = \frac{1}{2}\,\frac{\lambda_i^2 - \lambda_j^2}{\lambda_i\lambda_j(g(\lambda_i) - g(\lambda_j))} \qquad (\lambda_i \neq \lambda_j, \quad i \neq j). \tag{4.36}$$

If $i \neq j$ and $\lambda_i = \lambda_j$, only (4.35) applies, so that it suffices to concentrate on $\lambda_i \neq \lambda_j$. Since $\{(\lambda_i^2 - \lambda_j^2)/(g(\lambda_i) - g(\lambda_j))\} > 0$, from (4.35), (4.36), we have

$$\frac{1}{\sqrt{\lambda_i\lambda_j g'(\lambda_i)g'(\lambda_j)}} = \frac{1}{2}\,\frac{\lambda_i^2 - \lambda_j^2}{\lambda_i\lambda_j(g(\lambda_i) - g(\lambda_j))}. \tag{4.37}$$

We recall that λ_i, being eigenvalues of the positive definite second-order tensor $\mathbf{U}$, are positive. Thus, if (4.29) holds, the function $g(\cdot)$ has to satisfy the relation (4.37) for all positive λ_i, λ_j.

Now, suppose $g(\cdot)$ belonging to the one-parameter set of functions $g_{(m)}(\cdot)$, defined by (3.39). It is readily seen that for $m = 0$, equation (4.37) cannot be satisfied. For $m \neq 0$, on use of $(3.39)_1$, we obtain from (4.37), after some algebraic manipulations,

$$g_{(m)}(\lambda_i) - g_{(m)}(\lambda_j) = \frac{1}{2}\left(\lambda_i^{\frac{m}{2}+1}\lambda_j^{\frac{m}{2}-1} - \lambda_i^{\frac{m}{2}-1}\lambda_j^{\frac{m}{2}+1}\right). \tag{4.38}$$

On taking the derivative with respect to λ_i and then to λ_j, (4.38) reduces to

$$\left(\frac{m}{2}+1\right)\left(\frac{m}{2}-1\right)\left(\lambda_i^{\frac{m}{2}}\lambda_j^{\frac{m}{2}-2} - \lambda_i^{\frac{m}{2}-2}\lambda_j^{\frac{m}{2}}\right) = 0. \tag{4.39}$$

This formula must be satisfied for all $\lambda_i, \lambda_j > 0$ with $\lambda_i \neq \lambda_j$, which is possible if and only if

(4.40) $$m = 2 \quad \text{or} \quad m = -2.$$

For $m = 2$, on the basis of $(3.40)_1$, we have

(4.41) $$\mathbf{G}_{(2)} := \mathbf{E} = \frac{1}{2}\left(\mathbf{U}^2 - \mathbf{1}\right),$$

which is called the Green strain tensor, while for $m = -2$ we have

(4.42) $$\mathbf{G}_{(-2)} := -\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{U}^{-2} - \mathbf{1}\right).$$

$\boldsymbol{\varepsilon}$ is called the Piola strain tensor. The corresponding conjugate stress tensors in the sense of Hill are given by

(4.43) $$\mathbf{T}_{(2)} := \widetilde{\mathbf{T}} = \mathbf{F}^{-1}\mathbf{S}\mathbf{F}^{T-1}$$

and

(4.44) $$\mathbf{T}_{(-2)} := \overline{\mathbf{T}} = \mathbf{F}^{T}\mathbf{S}\mathbf{F},$$

referred to as the second Piola - Kirchhoff stress tensor and the convected stress tensor, respectively. Clearly, along with $\mathbf{G}_{(-2)}$ and $\mathbf{T}_{(-2)}$, the variables $\boldsymbol{\varepsilon}$ and $-\overline{\mathbf{T}}$ form also a pair of conjugate (in Hill's sense) strain and stress tensors.

This general result suggests the following restriction on the choice of Lagrangean strain tensors: If we define the associated Lagrangean stress tensors which are conjugate in the sense of Hill, various objective time derivatives can be assigned to $\mathbf{S}$. If we require from these derivatives the structure of generalized Oldroyd time derivatives, then only two strain tensors are left, namely $\mathbf{E}$ and $\boldsymbol{\varepsilon}$. We remark that the Lagrangean variables $(\mathbf{E}, \widetilde{\mathbf{T}})$ and $(\boldsymbol{\varepsilon}, \widetilde{\boldsymbol{\tau}})$, where

(4.45) $$\widetilde{\boldsymbol{\tau}} := -\overline{\mathbf{T}} = \mathbf{F}^{T}\boldsymbol{\varsigma}\mathbf{F}$$

and

(4.46) $$\boldsymbol{\varsigma} := -\mathbf{S},$$

were chosen in HAUPT and TSAKMAKIS [3] as basic pairs for introducing, by means of linear transformations, two different classes of pairs of spatial strain and stress tensors, referred to as family 1 and family 2, respectively. Strain and stress measures forming a pair belonging to one of the two classes were called dual variables. As it will be seen in what follows, the pairs of Lagrangean variables $(\mathbf{G}_{(m)}, \mathbf{T}_{(m)})$ if $m > 0$, or $(-\mathbf{G}_{(m)}, -\mathbf{T}_{(m)})$ if $m < 0$, are representatives of related classes, which can be interpreted as classes of generalized dual variables. Moreover, similar to the cases $m = \pm 2$, each of the Lagrangean stress tensors introduces a specific "generalized" Oldroyd time derivative for each of the stress tensors belonging to the same class. In fact, such a concept is established in the next section and essentially, it can be conceived as a generalization of the concept developed in HAUPT and TSAKMAKIS [3].

5. The concept of generalized dual variables

5.1. Generalized Lagrangean strain tensors

We remark that the set of strain tensors defined by (3.40) includes for each $m \neq 0$ the strain tensor $\mathbf{G}_{(m)}$ as well as its counterpart $\mathbf{G}_{(-m)}$. However, if $m = 0$, there is no such counterpart for $\ln \mathbf{U}$. This motivates the definition of a set of generalized Lagrangean strain tensors, slightly different from that introduced in (3.40), as follows.

The two-parameter set of functions

$$g_{(qm)}(\lambda_i) := \frac{1}{m}\,(\lambda_i^{qm} - 1)\,, \tag{5.1}$$

where

$$q \in \{-1, 1\} \qquad \text{and} \qquad m \in (0, \infty), \tag{5.2}$$

introduces the strain tensors

$$\boldsymbol{\epsilon}_{(qm)} = \sum_{i=1}^{3} \frac{1}{m}\,(\lambda_i^{qm} - 1)\,\mathbf{M}_i \otimes \mathbf{M}_i\,. \tag{5.3}$$

Note that the functions $g_{(qm)}$, in contrast to (3.37), are monotonic but not necessarily increasing with

$$g_{(qm)}(1) = 0, \qquad |g'_{(qm)}(1)| = 1. \tag{5.4}$$

Since q is equal either to $+1$ or to -1, we have

$$\boldsymbol{\epsilon}_{(qm)}|_{q=-1} = \boldsymbol{\epsilon}_{(-m)}\,, \tag{5.5}$$

$$\boldsymbol{\epsilon}_{(qm)}|_{q=1} = \boldsymbol{\epsilon}_{(m)}\,. \tag{5.6}$$

Notice that, by taking the limit for $m \to 0$, we arrive at the strain tensors

$$\lim_{m\to 0} \boldsymbol{\epsilon}_{(qm)} = q \ln \mathbf{U}\,, \tag{5.7}$$

which is equivalent to

$$\lim_{m\to 0} \boldsymbol{\epsilon}_{(qm)} = \begin{cases} \ln \mathbf{U} & \text{if} \quad q = 1, \\ \ln \mathbf{U}^{-1} & \text{if} \quad q = -1. \end{cases} \tag{5.8}$$

We call the set of all strain tensors defined by (5.2), (5.3), together with the strain tensors $\ln \mathbf{U}$ and $\ln \mathbf{U}^{-1}$, the set of generalized Lagrangean strain (deformation) tensors and denote it by $\mathrm{D_L}$:

$$\mathrm{D_L} := \left\{\boldsymbol{\epsilon}_{(qm)}/q \in \{-1, 1\},\ \ m > 0\right\} \cup \left\{\ln \mathbf{U},\ \ \ln \mathbf{U}^{-1}\right\}. \tag{5.9}$$

In order to give a geometrical interpretation to the Lagrangean strain tensors included in D_L, it is convenient to introduce the generalized Green strain tensors $\mathbf{E}_{(m)}$, and the generalized Piola strain tensors $\boldsymbol{\epsilon}_{(m)}$, defined by

$$\mathbf{E}_{(m)} := \begin{cases} \boldsymbol{\epsilon}_{(m)} = \dfrac{1}{m}(\mathbf{U}^m - \mathbf{1}) & \text{if} \quad m > 0, \\ \ln \mathbf{U} & \text{if} \quad m = 0, \end{cases} \tag{5.10}$$

$$\boldsymbol{\varepsilon}_{(m)} := \begin{cases} \boldsymbol{\epsilon}_{(-m)} = \dfrac{1}{m}(\mathbf{U}^{-m} - \mathbf{1}) & \text{if} \quad m > 0, \\ \ln \mathbf{U}^{-1} & \text{if} \quad m = 0. \end{cases} \tag{5.11}$$

Further, we denote the set of all $\mathbf{E}_{(m)}$ by D_{LG} and the set of all $\boldsymbol{\varepsilon}_{(m)}$ by D_{LP},

$$D_{LG} := \left\{\mathbf{E}_{(m)}/m \geq 0\right\}, \tag{5.12}$$

$$D_{LP} := \left\{\boldsymbol{\varepsilon}_{(m)}/m \geq 0\right\}. \tag{5.13}$$

Clearly,

$$D_L = D_{LG} \cup D_{LP} \tag{5.14}$$

and

$$D_{LG} \cap D_{LP} = \emptyset\,. \tag{5.15}$$

Next, we give geometric interpretations for the Green strain tensor $\mathbf{E} \equiv \mathbf{E}_{(2)}$, defined by (4.41), and the Piola strain tensor $\boldsymbol{\varepsilon} \equiv \boldsymbol{\varepsilon}_{(2)}$, defined by (4.42). As we shall see below, the geometric interpretation of the generalized strains $\mathbf{E}_{(m)}$ and $\boldsymbol{\varepsilon}_{(m)}$ is similar to that for $\mathbf{E}$ and $\boldsymbol{\varepsilon}$, respectively.

Let $d\mathbf{X}$ be a material line element in the reference configuration, which is transformed, under the deformation, into the material line element $d\mathbf{x}$ in the current configuration, i.e.,

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}\,. \tag{5.16}$$

Then we have the well-known formula

$$\Delta := \frac{1}{2}\left(d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{X} \cdot d\mathbf{X}\right) = d\mathbf{X} \cdot \mathbf{E}d\mathbf{X}\,. \tag{5.17}$$

To obtain a geometric interpretation for the Piola strain tensor $\boldsymbol{\varepsilon}$, we consider a material surface $\Phi(\mathbf{X}) = C = \text{const}$ in the reference configuration. In the current configuration this surface has the time-dependent form $\varphi(\mathbf{x}, t) = C$, where $\varphi(\bar{\mathbf{x}}(\mathbf{X}, t), t) = \Phi(\mathbf{X})$ holds for all $\mathbf{X}$ satisfying $\Phi(\mathbf{X}) = C$. It follows that

$$\boldsymbol{\xi} = \mathbf{F}^{T-1}\boldsymbol{\Xi}\,, \tag{5.18}$$

where

$$\boldsymbol{\xi} = \operatorname{grad} \varphi(\mathbf{x}, t), \tag{5.19}$$

$$\boldsymbol{\Xi} = \operatorname{GRAD} \Phi(\mathbf{X}) \tag{5.20}$$

and

$$\delta := \frac{1}{2}(\boldsymbol{\xi}\cdot\boldsymbol{\xi} - \boldsymbol{\Xi}\cdot\boldsymbol{\Xi}) = \boldsymbol{\Xi}\cdot\boldsymbol{\varepsilon}\boldsymbol{\Xi}. \tag{5.21}$$

Thus, the Green strain tensor $\mathbf{E}$ is used to refer to the reference configuration the difference Δ between material line elements in the current and the reference configuration. Analogously, one can make use of the Piola strain tensor $\boldsymbol{\varepsilon}$ in order to refer to the reference configuration the difference δ between normals to material surfaces in the current and the reference configuration.

Now, consider linear transformations described by $\mathbf{F}_{(m)}$, $\det \mathbf{F}_{(m)} > 0$, $m > 0$, where $\mathbf{F}_{(m)}$ is constructed as follows. From the polar decomposition theorem we have

$$\mathbf{F}_{(m)} = \mathbf{R}_{(m)}\mathbf{U}_{(m)} = \mathbf{V}_{(m)}\mathbf{R}_{(m)}, \tag{5.22}$$

$$\mathbf{U}^2_{(m)} = \mathbf{C}_{(m)} = \mathbf{F}^T_{(m)}\mathbf{F}_{(m)}, \tag{5.23}$$

$$\mathbf{V}^2_{(m)} = \mathbf{B}_{(m)} = \mathbf{F}_{(m)}\mathbf{F}^T_{(m)}, \tag{5.24}$$

where $\mathbf{R}_{(m)}$ denotes a proper orthogonal tensor. If we define

$$\mathbf{U}_{(m)} := \mathbf{U}^{m/2} = \sum_{i=1}^{3} \lambda_i^{m/2}\mathbf{M}_i \otimes \mathbf{M}_i, \tag{5.25}$$

($m > 0$), then it follows that $\mathbf{U}_{(m)}$ describes a class of right stretch tensors. Furthermore, defining $\mathbf{R}_{(2)} = \mathbf{R}$, we have $\mathbf{F}_{(2)} = \mathbf{F}$. Clearly $\mathbf{F}$, and so $\mathbf{U}$, must satisfy the compatibility conditions [9] in order to form a deformation gradient tensor derived from a deformation function. Although $\mathbf{U}$ and $\mathbf{F}$ satisfy the appropriate compatibility conditions, $\mathbf{U}_{(m)}$ and $\mathbf{F}_{(m)}$ in general do not.

Proceeding to complete the definition of $\mathbf{F}_{(m)}$, we note that all $\mathbf{U}_{(m)}$ possess the same principal vectors $\mathbf{M}_i$. This motivates to define all the corresponding left stretch tensors $\mathbf{V}_{(m)}$ to have the same principal vectors. Since the principal vectors of $\mathbf{V}_{(2)} = \mathbf{V}$ are $\boldsymbol{\mu}_{(i)}$ (see Eqs. (3.7), (3.8)), we have

$$\mathbf{V}_{(m)} := \mathbf{V}^{m/2} = \sum_{i=1}^{3} \lambda_i^{m/2}\boldsymbol{\mu}_i \otimes \boldsymbol{\mu}_i \tag{5.26}$$

and

$$\mathbf{R}_{(m)} := \mathbf{R}. \tag{5.27}$$

Notice that $\mathbf{F}_{(m)}$ can be interpreted as a two-point tensor field which maps tangent spaces of material points in the reference configuration onto the corresponding tangent spaces of the same material points in configurations at time t. This fact

([9]) A detailed discussion on the compatibility conditions concerning $\mathbf{F}$, as well as $\mathbf{U}$ and $\mathbf{R}$, is given by NAGHDI and VONGSARNPIGOON [14].

follows by virtue of **U** and **V** (and therefore $\mathbf{U}_{(m)}$ and $\mathbf{V}_{(m)}$ too) operating within the tangent spaces of material points in the reference configuration and tangent spaces of material points in configurations at time t, respectively, and **R** mapping tangent spaces of material points in the reference configuration to the corresponding tangent spaces of the same material points in configurations at time t (see e.g. MARSDEN and HUGHES [15, pp. 51–52]).

Thus, analogous to **F** in (5.16), $\mathbf{F}_{(m)}$ transforms line elements $d\mathbf{X}$ in the reference configuration to vectors $d\mathbf{x}_{(m)}$ in configurations at time t:

(5.28) $$d\mathbf{x}_{(m)} = \mathbf{F}_{(m)}\, d\mathbf{X}.$$

If we define

(5.29) $$\Delta_{(m)} := \frac{1}{m}\left(d\mathbf{x}_{(m)} \cdot d\mathbf{x}_{(m)} - d\mathbf{X} \cdot d\mathbf{X}\right),$$

then we have, in view of the transformation rule (5.28), as well as the relations $(5.22)_1$, $(5.10)_1$ and (5.25),

(5.30) $$\Delta_{(m)} := \frac{1}{m} d\mathbf{X} \cdot (\mathbf{U}^m - \mathbf{1})\, d\mathbf{X} = d\mathbf{X} \cdot \mathbf{E}_{(m)}\mathbf{X},$$

with the property $d\mathbf{x}_{(2)} \equiv d\mathbf{x}$ and $\Delta_{(2)} \equiv \Delta$. On the other hand, $\mathbf{F}_{(m)}$ can be interpreted to transform normals $\boldsymbol{\Xi}$ on material surfaces in the reference configuration to vectors $\boldsymbol{\xi}_{(m)}$,

(5.31) $$\boldsymbol{\xi}_{(m)} = \mathbf{F}_{(m)}^{T-1}\boldsymbol{\Xi},$$

in configurations at time t, which generalizes the transformation formula (5.18). On defining

(5.32) $$\delta_{(m)} := \frac{1}{m}\left(\boldsymbol{\xi}_{(m)} \cdot \boldsymbol{\xi}_{(m)} - \boldsymbol{\Xi} \cdot \boldsymbol{\Xi}\right),$$

we obtain, by virtue of $(5.22)_1$, (5.11) and (5.25),

(5.33) $$\delta_{(m)} := \frac{1}{m}\boldsymbol{\Xi} \cdot (\mathbf{U}^{-m} - \mathbf{1})\boldsymbol{\Xi} = \boldsymbol{\Xi} \cdot \boldsymbol{\varepsilon}_{(m)}\boldsymbol{\Xi}.$$

Obviously, we have $\boldsymbol{\xi}_{(2)} \equiv \boldsymbol{\xi}$ and $\delta_{(2)} \equiv \delta$. This completes the geometrical interpretation of $\mathbf{E}_{(m)}$ and $\boldsymbol{\varepsilon}_{(m)}$. For arbitrary $m > 0$, these strain tensors represent the differences $\Delta_{(m)}$ and $\delta_{(m)}$ with respect to the reference configuration. We may extend the result to the limit case $m = 0$, by defining

(5.34) $$\Delta_{(0)} := \lim_{m \to 0} \Delta_{(m)} = d\mathbf{X} \cdot \mathbf{E}_{(0)}\mathbf{X}$$

and

(5.35) $$\delta_{(0)} := \lim_{m \to 0} \delta_{(m)} = \boldsymbol{\Xi} \cdot \boldsymbol{\varepsilon}_{(0)}\boldsymbol{\Xi}.$$

5.2. Generalized strain tensors and associated rates

Let $d\mathbf{X}$ and $\mathbf{\Xi}$ be material line elements and normals on material surfaces in the reference configuration, which are mapped by the linear transformation $\mathbf{\Psi}$, to vector fields $d\mathbf{x}^{(\Psi)}$ and $\mathbf{\Xi}^{(\Psi)}$, respectively (cf. (5.16) and (5.18) in Sec. 5.1):

$$d\mathbf{x}^{(\Psi)} := \mathbf{\Psi} d\mathbf{X}, \tag{5.36}$$

$$\boldsymbol{\xi}^{(\Psi)} := \mathbf{\Psi}^{T-1}\mathbf{\Xi}. \tag{5.37}$$

Next, consider for arbitrary but fixed $\mathbf{\Psi} \in \mathrm{Lin}^+$ and $m \geq 0$, the differences $\Delta_{(m)}$ and $\delta_{(m)}$. Requiring the derivatives $d^n\Delta_{(m)}/dt^n$ and $d^n\delta_{(m)}/dt^n$ ($n \in \mathbb{N}$, $n \geq 0$) to be form-invariant with respect to the chosen configuration, various symmetric strain tensors $\mathbf{\Pi}^{(\Psi)}_{(m)}$ and $\boldsymbol{\pi}^{(\Psi)}_{(m)}$, as well as the associated time derivatives (rates) $\overset{\triangle}{D}{}^n\mathbf{\Pi}^{(\Psi)}_{(m)}/Dt^n$ and $\overset{\triangle}{D}{}^n\,\boldsymbol{\pi}^{(\Psi)}_{(m)}/Dt^n$ can be defined ([10]):

$$\frac{d^n}{dt^n}\Delta_{(m)} = d\mathbf{x}^{(\Psi)} \cdot \left(\frac{\overset{\triangle}{D}{}^n}{Dt^n}\mathbf{\Pi}^{(\Psi)}_{(m)}\right) d\mathbf{x}^{(\Psi)}, \tag{5.38}$$

$$\frac{d^n}{dt^n}\delta_{(m)} = \boldsymbol{\xi}^{(\Psi)} \cdot \left(\frac{\overset{\triangle}{D}{}^n}{Dt^n}\boldsymbol{\pi}^{(\Psi)}_{(m)}\right) \boldsymbol{\xi}^{(\Psi)}. \tag{5.39}$$

These definitions imply

$$\frac{\overset{\triangle}{D}{}^n}{Dt^n}\mathbf{E}_{(m)} \equiv \frac{d^n}{dt^n}\mathbf{E}_{(m)}, \tag{5.40}$$

$$\frac{\overset{\triangle}{D}{}^n}{Dt^n}\boldsymbol{\varepsilon}_{(m)} \equiv \frac{d^n}{dt^n}\boldsymbol{\varepsilon}_{(m)}, \tag{5.41}$$

as well as

$$\frac{\overset{\triangle}{D}{}^n}{Dt^n}\mathbf{\Pi}^{(\Psi)}_{(m)} = \mathcal{L}_{(\psi)}\left[\frac{d^n}{dt^n}\mathbf{E}_{(m)}\right], \tag{5.42}$$

$$\frac{\overset{\triangle}{D}{}^n}{Dt^n}\boldsymbol{\pi}^{(\Psi)}_{(m)} = \mathcal{M}_{(\Psi)}\left[\frac{d^n}{dt^n}\boldsymbol{\varepsilon}_{(m)}\right], \tag{5.43}$$

where $\mathcal{L}_{(\Psi)}$ and $\mathcal{M}_{(\Psi)}$ are fourth-order tensors operating on the set of all Lagrangean symmetric second-order tensors $\tilde{\mathbf{S}}$:

$$\mathcal{L}_{(\Psi)}: \quad \tilde{\mathbf{S}} \longmapsto \mathcal{L}_{(\Psi)}[\tilde{\mathbf{S}}] = \mathbf{\Psi}^{T-1}\tilde{\mathbf{S}}\mathbf{\Psi}^{-1}, \tag{5.44}$$

$$\mathcal{M}_{(\Psi)}: \quad \tilde{\mathbf{S}} \longmapsto \mathcal{M}_{(\Psi)}[\tilde{\mathbf{S}}] = \mathbf{\Psi}\tilde{\mathbf{S}}\mathbf{\Psi}^{T}. \tag{5.45}$$

([10]) Here, and in what follows, symbol $\triangle$ denotes the associated time derivative for the strain tensor considered. In other words, $\triangle$ defines different time derivatives depending on the kind of the strain tensor considered.

In particular, for $n = 0$ we have

$$\mathbf{\Pi}_{(m)}^{(\Psi)} = \mathcal{L}_{(\Psi)}[\mathbf{E}_{(m)}] = \mathbf{\Psi}^{T-1}\mathbf{E}_{(m)}\mathbf{\Psi}^{-1}, \tag{5.46}$$

$$\boldsymbol{\pi}_{(m)}^{(\Psi)} = \mathcal{M}_{(\Psi)}[\boldsymbol{\varepsilon}_{(m)}] = \mathbf{\Psi}\boldsymbol{\varepsilon}_{(m)}\mathbf{\Psi}^{T}. \tag{5.47}$$

It is readily seen that

$$\begin{aligned} \overset{\triangle}{\mathbf{\Pi}}{}_{(m)}^{(\Psi)} &:= \frac{\overset{\triangle}{D}}{Dt}\mathbf{\Pi}_{(m)}^{(\Psi)} = \dot{\mathbf{\Pi}}{}_{(m)}^{(\Psi)} + (\dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1})^T\mathbf{\Pi}_{(m)}^{(\Psi)} + \mathbf{\Pi}_{(m)}^{(\Psi)}\dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1}, \\ \overset{\triangle\triangle}{\mathbf{\Pi}}{}_{(m)}^{(\Psi)} &:= \frac{\overset{\triangle}{D}{}^2}{Dt^2}\mathbf{\Pi}_{(m)}^{(\Psi)} = \left(\overset{\triangle}{\mathbf{\Pi}}{}_{(m)}^{(\Psi)}\right)^{\bullet} + (\dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1})^T\,\overset{\triangle}{\mathbf{\Pi}}{}_{(m)}^{(\Psi)} + \overset{\triangle}{\mathbf{\Pi}}{}_{(m)}^{(\Psi)}\,\dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1}, \\ &\;\;\vdots \end{aligned} \tag{5.48}$$

as well as

$$\begin{aligned} \overset{\triangle}{\boldsymbol{\pi}}{}_{(m)}^{(\Psi)} &:= \frac{\overset{\triangle}{D}}{Dt}\boldsymbol{\pi}_{(m)}^{(\Psi)} = \dot{\boldsymbol{\pi}}{}_{(m)}^{(\Psi)} - \dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1}\boldsymbol{\pi}_{(m)}^{(\Psi)} - \boldsymbol{\pi}_{(m)}^{(\Psi)}(\dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1})^T, \\ \overset{\triangle\triangle}{\boldsymbol{\pi}}{}_{(m)}^{(\Psi)} &:= \frac{\overset{\triangle}{D}{}^2}{Dt^2}\boldsymbol{\pi}_{(m)}^{(\Psi)} = \left(\overset{\triangle}{\boldsymbol{\pi}}{}_{(m)}^{(\Psi)}\right)^{\bullet} - \dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1}\,\overset{\triangle}{\boldsymbol{\pi}}{}_{(m)}^{(\Psi)} - \overset{\triangle}{\boldsymbol{\pi}}{}_{(m)}^{(\Psi)}\,(\dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1})^T, \\ &\;\;\vdots \end{aligned} \tag{5.49}$$

Further relations are obtained if we represent the various strain tensors with respect to the bases $\{\mathbf{g}_k^{(\Psi)}\}$ and $\{\mathbf{g}_k^{(\Psi)}\}$. From

$$\mathbf{E}_{(m)} = E_{(m)kl}\mathbf{G}^k \otimes \mathbf{G}^l, \tag{5.50}$$

$$\mathbf{\Pi}_{(m)}^{(\Psi)} = \Pi_{(m)kl}^{(\Psi)}\mathbf{g}^{(\Psi)k} \otimes \mathbf{g}^{(\Psi)l}, \tag{5.51}$$

as well as

$$\boldsymbol{\varepsilon}_{(m)} = \varepsilon_{(m)}^{kl}\mathbf{G}_k \otimes \mathbf{G}_l, \tag{5.52}$$

$$\boldsymbol{\pi}_{(m)}^{(\Psi)} = \pi_{(m)}^{(\Psi)kl}\mathbf{g}_k^{(\Psi)} \otimes \mathbf{g}_l^{(\Psi)}, \tag{5.53}$$

we infer that

$$E_{(m)kl} = \Pi_{(m)kl}^{(\Psi)} \tag{5.54}$$

and

$$\varepsilon_{(m)}^{kl} = \pi_{(m)}^{(\Psi)kl}, \tag{5.55}$$

respectively. In addition, it holds

$$(5.56)\qquad \frac{\overset{\triangle}{D}{}^n}{Dt^n}\mathbf{\Pi}^{(\Psi)}_{(m)} = \left(\frac{d^n}{dt^n}\Pi^{(\Psi)}_{(m)kl}\right)\mathbf{g}^{(\Psi)k}\otimes\mathbf{g}^{(\Psi)l}$$

and

$$(5.57)\qquad \frac{\overset{\triangle}{D}{}^n}{Dt^n}\boldsymbol{\pi}^{(\Psi)}_{(m)} = \left(\frac{d^n}{dt^n}\pi^{(\Psi)kl}_{(m)}\right)\mathbf{g}^{(\Psi)}_k\otimes\mathbf{g}^{(\Psi)}_l\,,$$

which indicates, that the operators $\overset{\triangle}{D}{}^n(\cdot)/Dt^n$ induce generalized Oldroyd time derivatives. We call the strain tensors $\mathbf{\Pi}^{(\Psi)}_{(m)}$ and $\boldsymbol{\pi}^{(\Psi)}_{(m)}$ generalized strain tensors. The set of all generalized strain tensors is denoted by D,

$$(5.58)\qquad \mathrm{D} := \left\{\mathbf{\Pi}^{(\Psi)}_{(m)}, \boldsymbol{\pi}^{(\Psi)}_{(m)} \;/\; m\geq 0,\quad \mathbf{\Psi}\in\mathrm{Lin}^+\right\}.$$

Obviously, for arbitrary but fixed $m\geq 0$, the sets of all generalized strain tensors related to the differences $\Delta_{(m)}$ and $\delta_{(m)}$ constitute equivalence classes in D. We denote these equivalence classes by $\Theta^{(\Pi)}_{(m)}$ and $\Theta^{(\pi)}_{(m)}$, respectively,

$$(5.59)\qquad \Theta^{(\Pi)}_{(m)} := \left\{\mathbf{\Pi}^{(\Psi)}_{(m)} \;/\; \mathbf{\Psi}\in\mathrm{Lin}^+\right\},$$

$$(5.60)\qquad \Theta^{(\pi)}_{(m)} := \left\{\boldsymbol{\pi}^{(\Psi)}_{(m)} \;/\; \mathbf{\Psi}\in\mathrm{Lin}^+\right\}.$$

Then, for the system Ω_{D} of all equivalence classes in D,

$$(5.61)\qquad \Omega_{\mathrm{D}} := \left\{\Theta^{(\Pi)}_{(m)}, \Theta^{(\pi)}_{(m)} \;/\; m\geq 0\right\},$$

the equality holds

$$(5.62)\qquad \mathrm{D} = \bigcup_{\Theta\in\Omega_{\mathrm{D}}}\Theta\,.$$

5.3. Generalized stress tensors and associated rates

For defining the generalized strain tensors and their associated rates, use is made of the scalar quantities $d^n\Delta_{(m)}/dt^n$ and $d^n\delta_{(m)}/dt^n$. These scalars were required to be form-invariant with respect to the chosen configuration. In the following we consider the stress power as well as the material time derivatives d^nW/dt^n and require from these scalar quantities to be form-invariant with respect to the chosen configuration. This leads to the introduction of generalized stress tensors and the associated rates.

Proceeding to define generalized stress tensors, we draw attention to symmetric stress tensors only and assign to the generalized Lagrangean strain tensors

$\mathbf{E}_{(m)}$ and $\boldsymbol{\varepsilon}_{(m)}$, the symmetric generalized Lagrangean stress tensors $\widetilde{\mathbf{T}}_{(m)}$ and $\widetilde{\boldsymbol{\tau}}_{(m)}$, respectively, so that

(5.63) $$W = \widetilde{\mathbf{T}}_{(m)} \cdot \dot{\mathbf{E}}_{(m)} = \widetilde{\boldsymbol{\tau}}_{(m)} \cdot \dot{\boldsymbol{\varepsilon}}_{(m)},$$

for each $m \geq 0$. The set of all generalized Lagrangean stress tensors is denoted by $\mathrm{S_L}$

(5.64) $$\mathrm{S_L} := \Big\{ \widetilde{\mathbf{T}}_{(m)}, \widetilde{\boldsymbol{\tau}}_{(m)} \ / \ \widetilde{\mathbf{T}}_{(m)}, \widetilde{\boldsymbol{\tau}}_{(m)} : \quad \text{symmetric}, \quad W = \widetilde{\mathbf{T}}_{(m)} \cdot \dot{\mathbf{E}}_{(m)} = \widetilde{\boldsymbol{\tau}}_{(m)} \cdot \boldsymbol{\varepsilon}_{(m)}, \quad m \geq 0 \Big\}.$$

The set $\mathrm{S_L}$ for the stress tensors is the counterpart of the set $\mathrm{D_L}$ for the strain tensors, while the sets

(5.65) $$\Big\{ \widetilde{\mathbf{T}}_{(m)} \ / \ m \geq 0 \Big\}$$

and

(5.66) $$\Big\{ \widetilde{\boldsymbol{\tau}}_{(m)} \ / \ m \geq 0 \Big\},$$

are the counterparts of the sets $\mathrm{D_{LG}}$ and $\mathrm{D_{LP}}$, respectively. Moreover, to the generalized strain tensors $\boldsymbol{\Pi}^{(\Psi)}_{(m)}$ and $\boldsymbol{\pi}^{(\Psi)}_{(m)}$ the symmetric generalized stress tensors

(5.67) $$\boldsymbol{\Sigma}^{(\Psi)}_{(m)} = \mathcal{M}_{(\Psi)}[\widetilde{\mathbf{T}}_{(m)}] = \boldsymbol{\Psi} \widetilde{\mathbf{T}}_{(m)} \boldsymbol{\Psi}^T$$

and

(5.68) $$\boldsymbol{\sigma}^{(\Psi)}_{(m)} = \mathcal{L}_{(\Psi)}[\widetilde{\boldsymbol{\tau}}_{(m)}] = \boldsymbol{\Psi}^{T-1} \widetilde{\boldsymbol{\tau}}_{(m)} \boldsymbol{\Psi}^{-1},$$

can be assigned, respectively, so that

(5.69) $$W = \boldsymbol{\Sigma}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\boldsymbol{\Pi}}{}^{(\Psi)}_{(m)} = \boldsymbol{\sigma}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\boldsymbol{\pi}}{}^{(\Psi)}_{(m)},$$

for each $m \geq 0$ and $\boldsymbol{\Psi} \in \mathrm{Lin}^+$.

Notice that $(\mathbf{E}_{(m)}, \widetilde{\mathbf{T}}_{(m)})$, as well as $(\boldsymbol{\varepsilon}_{(m)}, \widetilde{\boldsymbol{\tau}}_{(m)})$, are pairs of variables which are conjugate in the sense of Hill. However, this is in general not true for the pairs of variables

(5.70) $$\Big(\boldsymbol{\Pi}^{(\Psi)}_{(m)}, \boldsymbol{\Sigma}^{(\Psi)}_{(m)} \Big) \qquad \text{and} \qquad \Big(\boldsymbol{\pi}^{(\Psi)}_{(m)}, \boldsymbol{\sigma}^{(\Psi)}_{(m)} \Big).$$

For arbitrary $m \geq 0$ and $\boldsymbol{\Psi} \in \mathrm{Lin}^+$, the pairs of variables (5.70) are called *pairs of generalized dual variables*, or simply *dual variables*. Equivalently, the generalized stress tensors $\boldsymbol{\Sigma}^{(\Psi)}_{(m)}$ and $\boldsymbol{\sigma}^{(\Psi)}_{(m)}$ are said to be dual to the generalized strain tensors $\boldsymbol{\Pi}^{(\Psi)}_{(m)}$ and $\boldsymbol{\pi}^{(\Psi)}_{(m)}$, respectively, and *vice versa* ([11]).

([11]) This notation of generalized dual variables is just a generalization of the duality notation introduced in HAUPT and TSAKMAKIS [3].

If we write S for the set of all stress tensors $\boldsymbol{\Sigma}_{(m)}^{(\Psi)}$ and $\boldsymbol{\sigma}_{(m)}^{(\Psi)}$,

$$\mathrm{S} := \left\{ \boldsymbol{\Sigma}_{(m)}^{(\Psi)}, \boldsymbol{\sigma}_{(m)}^{(\Psi)} \;\; / \;\; m \geq 0, \quad \boldsymbol{\Psi} \in \mathrm{Lin}^{+} \right\} \tag{5.71}$$

then S can be partitioned into the equivalent classes

$$\Theta_{(m)}^{(\Sigma)} := \left\{ \boldsymbol{\Sigma}_{(m)}^{(\Psi)} \;\; / \;\; \boldsymbol{\Psi} \in \mathrm{Lin}^{+} \right\} \tag{5.72}$$

and

$$\Theta_{(m)}^{(\sigma)} := \left\{ \boldsymbol{\sigma}_{(m)}^{(\Psi)} \;\; / \;\; \boldsymbol{\Psi} \in \mathrm{Lin}^{+} \right\}, \tag{5.73}$$

which for $m \geq 0$ cover S. Note that the counterpart of the sets D, $\Theta_{(m)}^{(\Pi)}$ and $\Theta_{(m)}^{(\pi)}$ for the strain tensors are the sets S, $\Theta_{(m)}^{(\Sigma)}$ and $\Theta_{(m)}^{(\sigma)}$ for the stress tensors, respectively.

To determine the time derivatives which are associated with the generalized stress tensors $\boldsymbol{\Sigma}_{(m)}^{(\Psi)}$ and $\boldsymbol{\sigma}_{(m)}^{(\Psi)}$, we next consider the quantity $\dot{W}$, which like W is required to be form-invariant with respect to the chosen configuration. On taking the material time derivative of (5.63), we obtain

$$\begin{aligned} \dot{W} &= \dot{\tilde{\mathbf{T}}}_{(m)} \cdot \dot{\mathbf{E}}_{(m)} + \tilde{\mathbf{T}}_{(m)} \cdot \ddot{\mathbf{E}}_{(m)} \\ &= \dot{\tilde{\boldsymbol{\tau}}}_{(m)} \cdot \dot{\boldsymbol{\varepsilon}}_{(m)} + \tilde{\boldsymbol{\tau}}_{(m)} \cdot \ddot{\boldsymbol{\varepsilon}}_{(m)} \; . \end{aligned} \tag{5.74}$$

Using the stress and strain tensors included in the equivalence classes $\Theta_{(m)}^{(\Pi)}$, $\Theta_{(m)}^{(\Sigma)}$ and $\Theta_{(m)}^{(\pi)}$, $\Theta_{(m)}^{(\sigma)}$, respectively, as well as the associated strain rates defined by (5.48) and (5.49), the terms $\tilde{\mathbf{T}}_{(m)} \cdot \ddot{\mathbf{E}}_{(m)}$ and $\tilde{\boldsymbol{\tau}}_{(m)} \cdot \ddot{\boldsymbol{\varepsilon}}_{(m)}$ can be rewritten in the form

$$\tilde{\mathbf{T}}_{(m)} \cdot \ddot{\mathbf{E}}_{(m)} = \boldsymbol{\Sigma}_{(m)}^{(\Psi)} \cdot \overset{\triangle\triangle}{\boldsymbol{\Pi}}{}_{(m)}^{(\Psi)}, \tag{5.75}$$

$$\tilde{\boldsymbol{\tau}}_{(m)} \cdot \ddot{\boldsymbol{\varepsilon}}_{(m)} = \boldsymbol{\sigma}_{(m)}^{(\Psi)} \cdot \overset{\triangle\triangle}{\boldsymbol{\pi}}{}_{(m)}^{(\Psi)}. \tag{5.76}$$

Thus, the quantities $\tilde{\mathbf{T}}_{(m)} \cdot \ddot{\mathbf{E}}_{(m)}$ and $\tilde{\boldsymbol{\tau}}_{(m)} \cdot \ddot{\boldsymbol{\varepsilon}}_{(m)}$ represent, for arbitrary but fixed $m \geq 0$, scalars which are form-invariant with respect to the chosen configuration. Consequently, the terms

$$W_{(m)}^{\mathrm{incr}} := \dot{\tilde{\mathbf{T}}}_{(m)} \cdot \dot{\mathbf{E}}_{(m)} \tag{5.77}$$

and

$$w_{(m)}^{\mathrm{incr}} := \dot{\tilde{\boldsymbol{\tau}}}_{(m)} \cdot \dot{\boldsymbol{\varepsilon}}_{(m)}, \tag{5.78}$$

which are called the incremental stress powers $W^{\text{incr}}_{(m)}$ and $w^{\text{incr}}_{(m)}$, respectively, must also be scalars which are form-invariant with respect to the chosen configuration. Indeed, we have

$$W^{\text{incr}}_{(m)} = \overset{\triangledown}{\boldsymbol{\Sigma}}{}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\boldsymbol{\Pi}}{}^{(\Psi)}_{(m)} \tag{5.79}$$

and

$$w^{\text{incr}}_{(m)} = \overset{\triangledown}{\boldsymbol{\sigma}}{}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\boldsymbol{\pi}}{}^{(\Psi)}_{(m)}, \tag{5.80}$$

where use is made of the definitions ([12])

$$\begin{aligned}\overset{\triangledown}{\boldsymbol{\Sigma}}{}^{(\Psi)}_{(m)} := \frac{\overset{\triangledown}{D}}{Dt}\boldsymbol{\Sigma}^{(\Psi)}_{(m)} &= \boldsymbol{\Psi}\left(\boldsymbol{\Psi}^{-1}\boldsymbol{\Sigma}^{(\Psi)}_{(m)}\boldsymbol{\Psi}^{T-1}\right)^{\bullet}\boldsymbol{\Psi}^{T} \\ &= \dot{\boldsymbol{\Sigma}}{}^{(\Psi)}_{(m)} - \dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1}\boldsymbol{\Sigma}^{(\Psi)}_{(m)} - \boldsymbol{\Sigma}^{(\Psi)}_{(m)}(\dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1})^{T},\end{aligned} \tag{5.81}$$

$$\begin{aligned}\overset{\triangledown}{\boldsymbol{\sigma}}{}^{(\Psi)}_{(m)} := \frac{\overset{\triangledown}{D}}{Dt}\boldsymbol{\sigma}^{(\Psi)}_{(m)} &= \boldsymbol{\Psi}^{T-1}\left(\boldsymbol{\Psi}^{T}\boldsymbol{\sigma}^{(\Psi)}_{(m)}\boldsymbol{\Psi}\right)^{\bullet}\boldsymbol{\Psi}^{-1} \\ &= \dot{\boldsymbol{\sigma}}{}^{(\Psi)}_{(m)} + (\dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1})^{T}\boldsymbol{\sigma}^{(\Psi)}_{(m)} + \boldsymbol{\Sigma}^{(\Psi)}_{(m)}\dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1}.\end{aligned} \tag{5.82}$$

This way, by considering form-invariant scalar quantities, we can associate with each stress tensor $\boldsymbol{\Sigma}^{(\Psi)}_{(m)}$ and $\boldsymbol{\sigma}^{(\Psi)}_{(m)}$ a time derivative of the form (5.81) and (5.82), respectively. Similarly, by considering higher time derivatives d^nW/dt^n, associated time derivatives of higher order $\overset{\triangledown}{D}{}^n\,\boldsymbol{\Sigma}^{(\Psi)}_{(m)}/Dt^n$ and $\overset{\triangledown}{D}{}^n\,\boldsymbol{\sigma}^{(\Psi)}_{(m)}/Dt^n$ can be introduced in a natural way. In particular, we have

$$\frac{\overset{\triangledown}{D}{}^n}{Dt^n}\tilde{\mathbf{T}}_{(m)} \equiv \frac{d^n}{dt^n}\tilde{\mathbf{T}}_{(m)} \tag{5.83}$$

and

$$\frac{\overset{\triangledown}{D}{}^n}{Dt^n}\tilde{\boldsymbol{\tau}}_{(m)} \equiv \frac{d^n}{dt^n}\tilde{\boldsymbol{\tau}}_{(m)}, \tag{5.84}$$

as well as

$$\frac{\overset{\triangle}{D}{}^n}{Dt^n}\boldsymbol{\Sigma}^{(\Psi)}_{(m)} = \mathcal{M}_{(\psi)}\left[\frac{d^n}{dt^n}\tilde{\mathbf{T}}_{(m)}\right] = \boldsymbol{\Psi}^{T-1}\left(\frac{d^n}{dt^n}\tilde{\mathbf{T}}_{(m)}\right)\boldsymbol{\Psi}^{-1} \tag{5.85}$$

and

$$\frac{\overset{\triangle}{D}{}^n}{Dt^n}\boldsymbol{\sigma}^{(\Psi)}_{(m)} = \mathcal{L}_{(\Psi)}\left[\frac{d^n}{dt^n}\tilde{\boldsymbol{\tau}}_{(m)}\right] = \boldsymbol{\Psi}\left(\frac{d^n}{dt^n}\tilde{\boldsymbol{\tau}}_{(m)}\right)\boldsymbol{\Psi}^{T}. \tag{5.86}$$

([12]) Similar to the notation of the symbol $\triangle$ for the strain tensors (see footnote 11), symbol $\triangledown$ denotes the associated time derivative for the stress tensor considered.

5.4. Properties of dual variables

Using the bases $\{\mathbf{G}_k\}$, $\{\mathbf{g}_k^{(\Psi)}\}$ and their reciprocal bases $\{\mathbf{G}^k\}$, $\{\mathbf{g}^{(\Psi)k}\}$, as well as the representations

$$\tilde{\mathbf{T}}_{(m)} = \tilde{T}^{kl}_{(m)}\mathbf{G}_k \otimes \mathbf{G}_l\,, \tag{5.87}$$

$$\boldsymbol{\Sigma}^{(\Psi)}_{(m)} = \Sigma^{(\Psi)kl}_{(m)}\mathbf{g}^{(\Psi)}_k \otimes \mathbf{g}^{(\Psi)}_l\,, \tag{5.88}$$

and

$$\tilde{\boldsymbol{\tau}}_{(m)} = \tilde{\tau}_{(m)kl}\mathbf{G}^k \otimes \mathbf{G}^l\,, \tag{5.89}$$

$$\boldsymbol{\sigma}^{(\Psi)}_{(m)} = \sigma^{(\Psi)}_{(m)kl}\mathbf{g}^{(\Psi)k} \otimes \mathbf{g}^{(\Psi)l}\,, \tag{5.90}$$

we readily obtain

$$\tilde{T}^{kl}_{(m)} = \Sigma^{(\Psi)kl}_{(m)}\,, \tag{5.91}$$

$$\tilde{\tau}_{(m)kl} = \sigma^{(\Psi)}_{(m)kl} \tag{5.92}$$

and

$$\frac{\overset{\triangledown}{D}{}^n}{Dt^n}\boldsymbol{\Sigma}^{(\Psi)}_{(m)} = \left(\frac{d^n}{dt^n}\Sigma^{(\Psi)kl}_{(m)}\right)\mathbf{g}^{(\Psi)}_k \otimes \mathbf{g}^{(\Psi)}_l\,, \tag{5.93}$$

$$\frac{\overset{\triangledown}{D}{}^n}{Dt^n}\boldsymbol{\sigma}^{(\Psi)}_{(m)} = \left(\frac{d^n}{dt^n}\sigma^{(\Psi)}_{(m)kl}\right)\mathbf{g}^{(\Psi)k} \otimes \mathbf{g}^{(\Psi)l}\,. \tag{5.94}$$

The relations (5.93), (5.94) together with (5.85) and (5.86) indicate that, similarly to the case of the generalized strain tensors, the operators $\overset{\triangledown}{D}{}^n(\cdot)/Dt^n$ induce generalized Oldroyd time derivatives.

We now compare the relations (5.56), (5.57), which concern the generalized strain tensors, and the relations (5.93), (5.94), which concern the generalized stress tensors. It turns out that $\boldsymbol{\Pi}^{(\Psi)}_{(m)}$ and $\boldsymbol{\Sigma}^{(\Psi)}_{(m)}$ or $\boldsymbol{\pi}^{(\Psi)}_{(m)}$ and $\boldsymbol{\sigma}^{(\Psi)}_{(m)}$, as well as the associated time derivatives, display their physical and geometrical properties in the context of a representation relative to a basis and the corresponding reciprocal (dual) basis, respectively. Moreover, the duality concept can also be verified by means of the following scalar products, which are form-invariant with respect to the chosen configuration:

$$I^{NM}_{(m)} := \left(\frac{\overset{\triangledown}{D}{}^N}{Dt^N}\boldsymbol{\Sigma}^{(\Psi)}_{(m)}\right)\cdot\left(\frac{\overset{\vartriangle}{D}{}^M}{Dt^M}\boldsymbol{\Pi}^{(\Psi)}_{(m)}\right), \tag{5.95}$$

$$i^{NM}_{(m)} := \left(\frac{\overset{\triangledown}{D}{}^N}{Dt^N}\boldsymbol{\sigma}^{(\Psi)}_{(m)}\right)\cdot\left(\frac{\overset{\vartriangle}{D}{}^M}{Dt^M}\boldsymbol{\pi}^{(\Psi)}_{(m)}\right), \tag{5.96}$$

where $m \geq 0$, and $N, M \in \mathbb{N}$ with $N, M \geq 0$. Some particular cases of (5.95) and (5.96) are:

1

$$I^{00}_{(m)} = \tilde{\mathbf{T}}_{(m)} \cdot \mathbf{E}_{(m)} = \boldsymbol{\Sigma}^{(\Psi)}_{(m)} \cdot \boldsymbol{\Pi}^{(\Psi)}_{(m)}, \tag{5.97}$$

$$i^{00}_{(m)} = \tilde{\boldsymbol{\tau}}_{(m)} \cdot \boldsymbol{\varepsilon}_{(m)} = \boldsymbol{\sigma}^{(\Psi)}_{(m)} \cdot \boldsymbol{\pi}^{(\Psi)}_{(m)}. \tag{5.98}$$

(Scalar product of dual stress and strain tensors).

2

$$I^{01}_{(m)} \equiv W = \tilde{\mathbf{T}}_{(m)} \cdot \dot{\mathbf{E}}_{(m)} = \boldsymbol{\Sigma}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\boldsymbol{\Pi}}{}^{(\Psi)}_{(m)}, \tag{5.99}$$

$$i^{01}_{(m)} \equiv W = \tilde{\boldsymbol{\tau}}_{(m)} \cdot \dot{\boldsymbol{\varepsilon}}_{(m)} = \boldsymbol{\sigma}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\boldsymbol{\pi}}{}^{(\Psi)}_{(m)}. \tag{5.100}$$

(Stress power per unit volume of the reference configuration).

3

$$I^{10}_{(m)} = \dot{\tilde{\mathbf{T}}}_{(m)} \cdot \mathbf{E}_{(m)} = \overset{\triangledown}{\boldsymbol{\Sigma}}{}^{(\Psi)}_{(m)} \cdot \boldsymbol{\Pi}^{(\Psi)}_{(m)}, \tag{5.101}$$

$$i^{10}_{(m)} = \dot{\tilde{\boldsymbol{\tau}}}_{(m)} \cdot \boldsymbol{\varepsilon}_{(m)} = \overset{\triangledown}{\boldsymbol{\sigma}}{}^{(\Psi)}_{(m)} \cdot \boldsymbol{\pi}^{(\Psi)}_{(m)}. \tag{5.102}$$

(Complementary stress powers).

4

$$I^{11}_{(m)} \equiv W^{\text{incr}}_{(m)} = \dot{\tilde{\mathbf{T}}}_{(m)} \cdot \dot{\mathbf{E}}_{(m)} = \overset{\triangledown}{\boldsymbol{\Sigma}}{}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\boldsymbol{\Pi}}{}^{(\Psi)}_{(m)}, \tag{5.103}$$

$$i^{11}_{(m)} \equiv w^{\text{incr}}_{(m)} = \dot{\tilde{\boldsymbol{\tau}}}_{(m)} \cdot \dot{\boldsymbol{\varepsilon}}_{(m)} = \overset{\triangledown}{\boldsymbol{\sigma}}{}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\boldsymbol{\pi}}{}^{(\Psi)}_{(m)}. \tag{5.104}$$

(Incremental stress powers).

6. Some examples

In most applications, m is chosen equal to 2. In such a case, the equivalence classes $\Theta^{(\Pi)}_{(2)}$ and $\Theta^{(\Sigma)}_{(2)}$ ($\Theta^{(\pi)}_{(2)}$ and $\Theta^{(\sigma)}_{(2)}$) are denoted as family 1 of strain tensors and family 1 of stress tensors (family 2 of strain tensors and family 2 of stress tensors), respectively. Some examples for particular choices of $\boldsymbol{\Psi}$ are given ([13]) in Tables 1 and 2 (the orthogonal second-order tensor $\mathbf{P}$ is given by $\dot{\mathbf{P}} = \mathbf{WP}$). Possible physical interpretations for the stress tensors $\boldsymbol{\Sigma}^{(\Psi)}_{(2)}$ and $\boldsymbol{\sigma}^{(\Psi)}_{(2)}$ are given in Appendix B.

([13]) For more details see HAUPT and TSAKMAKIS [3].

Table 1. Dual variables and associated derivatives: family 1.

Ψ	$\Pi^{(\Psi)}_{(2)} = \Psi^{T-1}E\Psi^{-1}$	$\overset{\triangle}{\Pi}{}^{(\Psi)}_{(2)} = \Psi^{T-1}\dot{E}\Psi^{-1}$	$\Sigma^{(\Psi)}_{(2)} = \Psi\tilde{T}\Psi^T$	$\overset{\nabla}{\Sigma}{}^{(\Psi)}_{(2)} = \Psi\dot{\tilde{T}}\Psi^T$
1	$E = \frac{1}{2}(C - 1)$	$\overset{\triangle}{E} \equiv \dot{E}$	$\tilde{T} = F^{-1}SF^{T-1}$	$\dot{\tilde{T}}$
F	$A = \frac{1}{2}(1 - B^{-1})$	$\overset{\triangle}{A} = \dot{A} + L^T A + AL = D$	$S = (\det F)T$	$\overset{\nabla}{S} = \dot{S} - LS - SL^T$
R	$\overline{K} = \frac{1}{2}(B - 1)$	$\overset{\triangle}{\overline{K}} = \dot{\overline{K}} - \dot{R}R^T\overline{K} + \overline{K}\dot{R}R^T$	$\underline{S} = R\tilde{T}R^T$	$\overset{\nabla}{\underline{S}} = \underline{S} - \dot{R}R^T\underline{S} - \underline{S}\dot{R}R^T$
U	$\underline{K} = \frac{1}{2}(1 - C^{-1})$	$\overset{\triangle}{\underline{K}} = \dot{\underline{K}} + (\dot{U}U^{-1})^T\underline{K} + \underline{K}\dot{U}U^{-1}$	$\overline{S} = U\tilde{T}U = R^T SR$	$\overset{\nabla}{\overline{S}} = \dot{\overline{S}} - \dot{U}U^{-1}\overline{S} - \overline{S}(\dot{U}U^{-1})^T$
P	$\Pi_W = PEP^T$	$\overset{\triangle}{\Pi}_W = \dot{\Pi}_W - W\Pi_W + \Pi_W W$	$\Sigma_W = P\tilde{T}P^T$	$\overset{\nabla}{\Sigma}_W = \dot{\Sigma}_W - W\Sigma_W + \Sigma_W W$

Table 2. Dual variables and associated derivatives: family 2.

Ψ	$\pi^{(\Psi)}_{(2)} = \Psi\varepsilon\Psi^T$	$\overset{\triangle}{\pi}{}^{(\Psi)}_{(2)} = \Psi\dot{\varepsilon}\Psi^T$	$\sigma^{(\Psi)}_{(2)} = \Psi^{T-1}\tilde{\tau}\Psi^{-1}$	$\overset{\nabla}{\sigma}{}^{(\Psi)}_{(2)} = \Psi^{T-1}\dot{\tilde{\tau}}\Psi^{-1}$
1	$\varepsilon = \frac{1}{2}(C^{-1} - 1)$	$\overset{\triangle}{\varepsilon} \equiv \dot{\varepsilon}$	$\tilde{\tau} = F^T\varsigma F$	$\dot{\tilde{\tau}}$
F	$\alpha = \frac{1}{2}(1 - B)$	$\overset{\triangle}{\alpha} = \dot{\alpha} - L\alpha - \alpha L^T = -D$	$\varsigma = -(\det F)T$	$\overset{\nabla}{\varsigma} = \dot{\varsigma} + L^T\varsigma + \varsigma L$
R	$\overline{k} = \frac{1}{2}(B^{-1} - 1)$	$\overset{\triangle}{\overline{k}} = \dot{\overline{k}} - \dot{R}R^T\overline{k} + \overline{k}\dot{R}R^T$	$\underline{\varsigma} = R\tilde{\tau}R^T$	$\overset{\nabla}{\underline{\varsigma}} = \underline{\varsigma} - \dot{R}R^T\underline{\varsigma} + \underline{\varsigma}\dot{R}R^T$
U	$\underline{k} = \frac{1}{2}(1 - C)$	$\overset{\triangle}{\underline{k}} = \dot{\underline{k}} - \dot{U}U^{-1}\underline{k} - \underline{k}(\dot{U}U^{-1})^T$	$\overline{\varsigma} = U^{-1}\tilde{\tau}U^{-1} = R^T\varsigma R$	$\overset{\nabla}{\overline{\varsigma}} = \dot{\overline{\varsigma}} + (\dot{U}U^{-1})^T\overline{\varsigma} + \overline{\varsigma}\dot{U}U^{-1}$
P	$\pi_W = P\varepsilon P^T$	$\overset{\triangle}{\pi}_W = \dot{\pi}_W - W\pi_W + \pi_W W$	$\sigma_W = P\tilde{\tau}P^T$	$\overset{\nabla}{\sigma}_W = \dot{\sigma}_W - W\sigma_W + \sigma_W W$

Next, we give the equivalent representations of hyperelastic constitutive equations using generalized dual variables. By definition, an elastic material is hyperelastic if and only if the work done by the the actual surface tractions in every closed homogeneous deformation process is non-negative (see e.g. TRUESDELL and NOLL [16, Sect. 82 & 83]). This is equivalent to the existence of scalar-valued functions $H_{(m)}$ and $h_{(m)}$,

$$H_{(m)} = \overline{H}_{(m)}(\mathbf{E}_{(m)}), \tag{6.1}$$

$$h_{(m)} = \overline{h}_{(m)}(\boldsymbol{\varepsilon}_{(m)}), \tag{6.2}$$

satisfying the relations

$$W = \dot{H}_{(m)} = \dot{h}_{(m)} \tag{6.3}$$

and therefore

$$\widetilde{\mathbf{T}}_{(m)} = \frac{\partial \overline{H}_{(m)}}{\partial \mathbf{E}_{(m)}}, \qquad \widetilde{\boldsymbol{\tau}}_{(m)} = \frac{\partial \overline{h}_{(m)}}{\partial \boldsymbol{\varepsilon}_{(m)}}, \tag{6.4}$$

respectively. Taking into account the relations (5.44)–(5.45), $H_{(m)}$ and $h_{(m)}$ can also be written in the form

$$H_{(m)} = \overline{H}_{(m)}(\mathbf{E}_{(m)}) = \overline{H}_{(m)}(\boldsymbol{\Psi}^T \boldsymbol{\Pi}^{(\Psi)}_{(m)} \boldsymbol{\Psi}) =: \widehat{H}_{(m)}(\boldsymbol{\Pi}^{(\Psi)}_{(m)}, \boldsymbol{\Psi}), \tag{6.5}$$

$$h_{(m)} = \overline{h}_{(m)}(\boldsymbol{\varepsilon}_{(m)}) = \overline{h}_{(m)}(\boldsymbol{\Psi}^{-1} \boldsymbol{\pi}^{(\Psi)}_{(m)} \boldsymbol{\Psi}^{T-1}) =: \widehat{h}_{(m)}(\boldsymbol{\pi}^{(\Psi)}_{(m)}, \boldsymbol{\Psi}), \tag{6.6}$$

respectively. From these equations, the stress relations $(6.4)_{1,2}$ as well as the transformation formulas (5.67) and (5.68), we conclude that

$$\boldsymbol{\Sigma}^{(\Psi)}_{(m)} = \frac{\partial \widehat{H}_{(m)}}{\partial \boldsymbol{\Pi}^{(\Psi)}_{(m)}}, \tag{6.7}$$

$$\boldsymbol{\sigma}^{(\Psi)}_{(m)} = \frac{\partial \widehat{h}_{(m)}}{\partial \boldsymbol{\pi}^{(\Psi)}_{(m)}}, \tag{6.8}$$

which are the spatial counterparts of $(6.4)_1$ and $(6.4)_2$, respectively. In view of (5.50)–(5.57), also the representations

$$\boldsymbol{\Sigma}^{(\Psi)}_{(m)} = \frac{\partial \overline{\overline{H}}_{(m)}}{\partial E_{(m)kl}} \mathbf{g}^{(\Psi)}_k \otimes \mathbf{g}^{(\Psi)}_l, \tag{6.9}$$

$$\boldsymbol{\sigma}^{(\Psi)}_{(m)} = \frac{\partial \overline{\overline{h}}_{(m)}}{\partial \varepsilon^{kl}_{(m)}} \mathbf{g}^{(\Psi)k} \otimes \mathbf{g}^{(\Psi)l}, \tag{6.10}$$

apply, where the functions $\overline{\overline{H}}_{(m)}$ and $\overline{\overline{h}}_{(m)}$ are given by

$$\overline{H}_{(m)}(\mathbf{E}_{(m)}) = \overline{H}_{(m)}(E_{(m)kl}\mathbf{G}^k \otimes \mathbf{G}^l) =: \overline{\overline{H}}_{(m)}(E_{(m)kl}), \tag{6.11}$$

$$\overline{h}_{(m)}(\boldsymbol{\varepsilon}_{(m)}) = \overline{h}_{(m)}(\varepsilon^{kl}_{(m)}\mathbf{G}_k \otimes \mathbf{G}_l) =: \overline{\overline{h}}_{(m)}(\varepsilon^{kl}_{(m)}), \tag{6.12}$$

respectively.

For $m = 2$ we write $E_{(2)kl} \equiv E_{kl}$ and $\varepsilon^{kl}_{(2)} \equiv \varepsilon^{kl}$. Then, for $\mathbf{\Psi} = \mathbf{F}$,

$$\overline{H}_{(2)}(\mathbf{E}) = \widehat{H}_{(2)}(\mathbf{A}, \mathbf{F}) = \overline{\overline{H}}_{(2)}(E_{kl}), \tag{6.13}$$

$$\overline{h}_{(2)}(\boldsymbol{\varepsilon}) = \widehat{h}_{(2)}\boldsymbol{\alpha}, \mathbf{F}) = \overline{\overline{h}}(\varepsilon^{kl}). \tag{6.14}$$

In this case, (6.7)–(6.10) reduce to

$$\mathbf{S} = \frac{\partial \widehat{H}_{(2)}}{\partial \mathbf{A}} = \frac{\partial \overline{\overline{H}}_{(2)}}{\partial E_{kl}} \mathbf{g}^{(F)}_k \otimes \mathbf{g}^{(F)}_l, \tag{6.15}$$

$$\boldsymbol{\varsigma} = \frac{\partial \widehat{h}_{(2)}}{\partial \boldsymbol{\alpha}} = \frac{\partial \overline{\overline{h}}_{(2)}}{\partial \varepsilon_{kl}} \mathbf{g}^{(F)k} \otimes \mathbf{g}^{(F)l}, \tag{6.16}$$

respectively. Furthermore, setting $G_{ij} := \mathbf{G}_i \cdot \mathbf{G}_j$, $\gamma_{ij} := \mathbf{g}^{(F)}_i \cdot \mathbf{g}^{(F)}_j$, as well as $G^{ij} := \mathbf{G}^i \cdot \mathbf{G}^j$, $\gamma^{ij} := \mathbf{g}^{(F)i} \cdot \mathbf{g}^{(F)j}$, we arrive at the identities

$$E_{kl} = \frac{1}{2}(\gamma_{kl} - G_{kl}), \tag{6.17}$$

$$\varepsilon_{kl} = \frac{1}{2}(\gamma^{kl} - G^{kl}). \tag{6.18}$$

Hence,

$$\mathbf{S} = 2\frac{\partial \widetilde{H}_{(2)}}{\partial \gamma_{kl}} \mathbf{g}^{(F)}_k \otimes \mathbf{g}^{(F)}_l \tag{6.19}$$

and

$$\boldsymbol{\varsigma} = 2\frac{\partial \widetilde{h}_{(2)}}{\partial \gamma^{kl}} \mathbf{g}^{(F)k} \otimes \mathbf{g}^{(F)l}, \tag{6.20}$$

where the functions $\widetilde{H}_{(2)}(\cdot)$ and $\widetilde{h}_{(2)}(\cdot)$ are defined by

$$H_{(2)} = \overline{\overline{H}}_{(2)}(E_{kl}) = \overline{\overline{H}}_{(2)}\left(\frac{1}{2}(\gamma_{kl} - G_{kl})\right) =: \widetilde{H}_{(2)}(\gamma_{kl}), \tag{6.21}$$

$$h_{(2)} = \overline{\overline{h}}_{(2)}(\varepsilon^{kl}) = \overline{\overline{h}}_{(2)}\left(\frac{1}{2}(\gamma^{kl} - G^{kl})\right) =: \widetilde{h}_{(2)}(\gamma^{kl}). \tag{6.22}$$

Equation (6.19) corresponds to the well-known Doyle - Ericksen formula (see Doyle and Ericksen [13]).

Further examples for the application of dual variables and their associated rates in Continuum Mechanics are provided in Haupt and Tsakmakis [3].

7. Duality for two-point tensors

The concept of dual variables developed can be extended to two-point tensors as well. For example, formula (3.41) shows that $\mathbf{F}$ is conjugate in Hill's sense to $\mathbf{T}_R$. However, $\dot{\mathbf{F}}$ does not indicate the same properties as $\mathbf{F}$ under an observer transformation. Proceeding to define an associated rate $\overset{\triangle}{\mathbf{F}}$ for $\mathbf{F}$ which behaves like $\mathbf{F}$ under the observer transformations, we consider a skew-symmetric tensor $\boldsymbol{\Omega}$, so that (3.41) is rewritten as

$$(7.1) \qquad W = \mathbf{T}_R \cdot (\dot{\mathbf{F}} - \boldsymbol{\Omega}\mathbf{F}).$$

This is possible, since $\mathbf{T}_R\mathbf{F}^T$ is symmetric. Note that by the polar decomposition $\mathbf{F}$ is related to the Lagrangean tensor $\mathbf{U}$ by means of $(3.5)_1$. Therefore, it appears natural to define $\overset{\triangle}{\mathbf{F}}$ in such a way, that $\overset{\triangle}{\mathbf{F}}$ is related to $\dot{\mathbf{U}}$ in the same manner as $\mathbf{F}$ to $\mathbf{U}$:

$$(7.2) \qquad \frac{\overset{\triangle}{D}}{Dt}\mathbf{F} \equiv \overset{\triangle}{\mathbf{F}} = \mathbf{R}\dot{\mathbf{U}}\,.$$

From this, as well as $(3.5)_1$ and (7.1), it follows that

$$(7.3) \qquad \boldsymbol{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T\,,$$

and therefore

$$(7.4) \qquad \overset{\triangle}{\mathbf{F}} = \dot{\mathbf{F}} - \dot{\mathbf{R}}\mathbf{R}^T\mathbf{F}\,.$$

It is not difficult now to show that $\mathbf{F}$ and $\overset{\triangle}{\mathbf{F}}$ behave similarly if the observer transformations are regarded.

Following steps similar as in the case of symmetric tensors, we define higher associated derivatives of $\mathbf{F}$ by

$$(7.5) \qquad \frac{\overset{\triangle}{D^n}}{Dt^n}\mathbf{F} = \left(\frac{\overset{\triangle}{D^{n-1}}}{Dt^{n-1}}\mathbf{F}\right)^{\bullet} - \dot{\mathbf{R}}\mathbf{R}^T\left(\frac{\overset{\triangle}{D^{n-1}}}{Dt^{n-1}}\mathbf{F}\right) = \mathbf{R}\frac{d^n}{dt^n}\mathbf{U}\,.$$

Next, we note that

$$(7.6) \qquad \mathbf{T}_R \cdot \mathbf{F} = \widetilde{\mathbf{T}}_{(1)} \cdot \mathbf{U}\,,$$

where

$$(7.7) \qquad \widetilde{\mathbf{T}}_{(1)} = \mathbf{T}_{(\mathrm{BS})} = \frac{1}{2}\left(\mathbf{T}_R^T\mathbf{R} + \mathbf{R}^T\mathbf{T}_R\right)$$

was referred to in (4.2) as the symmetrized Biot stress tensor ([14]). This motivates us to define the associated time derivatives for $\mathbf{T}_R$ in the form

$$\text{(7.8)} \qquad \frac{\overset{\triangledown}{D}{}^n}{Dt^n}\mathbf{T}_R = \left(\frac{\overset{\triangledown}{D}{}^{n-1}}{Dt^{n-1}}\mathbf{T}_R\right)^{\bullet} - \dot{\mathbf{R}}\mathbf{R}^T\left(\frac{\overset{\triangledown}{D}{}^{n-1}}{Dt^{n-1}}\mathbf{T}_R\right),$$

having the property

$$\text{(7.9)} \qquad \frac{d^n}{dt^n}\widetilde{\mathbf{T}}_{(1)} = \frac{1}{2}\left(\left(\frac{\overset{\triangledown}{D}{}^n}{Dt^n}\mathbf{T}_R\right)^T\mathbf{R} + \mathbf{R}^T\left(\frac{\overset{\triangledown}{D}{}^n}{Dt^n}\mathbf{T}_R\right)\right).$$

As a result, we have then ($\overset{\triangledown}{\mathbf{T}}_R := \overset{\triangledown}{D}\,\mathbf{T}_R/Dt$):

$$\text{(7.10)} \qquad (\mathbf{T}_R \cdot \mathbf{F})^{\bullet} = \overset{\triangledown}{\mathbf{T}}_R \cdot \mathbf{F} + \mathbf{T}_R \cdot \overset{\triangle}{\mathbf{F}}\,.$$

The results derived above, concerning the pair $(\mathbf{F}, \mathbf{T}_R)$, can be extended, in exactly the same way, to the pair $(\mathbf{F}^{T-1}, \boldsymbol{\tau}_R)$, where

$$\text{(7.11)} \qquad \boldsymbol{\tau}_R := \boldsymbol{\varsigma}\mathbf{F} = -(\det\mathbf{F})\mathbf{T}\mathbf{F}.$$

We recall, from the polar decomposition $(3.5)_1$, that

$$\text{(7.12)} \qquad \mathbf{F}^{T-1} = \mathbf{R}\mathbf{U}^{-1}.$$

This motivates us to define the associated time derivatives of $\mathbf{F}^{T-1}$ as follows:

$$\text{(7.13)} \qquad \frac{\overset{\triangle}{D}{}^n}{Dt^n}\mathbf{F}^{T-1} = \left(\frac{\overset{\triangle}{D}{}^{n-1}}{Dt^{n-1}}\mathbf{F}^{T-1}\right)^{\bullet} - \dot{\mathbf{R}}\mathbf{R}^T\left(\frac{\overset{\triangle}{D}{}^{n-1}}{Dt^{n-1}}\mathbf{F}^{T-1}\right) = \mathbf{R}\left(\frac{d^n}{dt^n}\mathbf{U}^{-1}\right).$$

Thus, the stress power W becomes

$$\text{(7.14)} \qquad W = \boldsymbol{\tau}_R \cdot \left(\mathbf{F}^{T-1}\right)^{\bullet} = \boldsymbol{\tau}_R \cdot \left(\mathbf{F}^{T-1}\right)^{\triangle},$$

where

$$\text{(7.15)} \qquad \left(\mathbf{F}^{T-1}\right)^{\triangle} = \frac{\overset{\triangle}{D}}{Dt}\mathbf{F}^{T-1}.$$

([14]) The analysis in the present paper is based on the relation between $\mathbf{T}_R$ and $\mathbf{T}_{(BS)}$. However, the results remain valid, if the analysis is referred to the relation between $\mathbf{T}_R$ and the Biot stress tensor $\mathbf{T}_{(B)} = \mathbf{R}^T\mathbf{T}_R$, defined in (4.3).

Furthermore, the identity

$$\boldsymbol{\tau}_R \cdot \mathbf{F}^{T-1} = \tilde{\boldsymbol{\tau}}_{(1)} \cdot \mathbf{U}^{-1} \tag{7.16}$$

holds, where

$$\tilde{\boldsymbol{\tau}}_{(1)} = \frac{1}{2}\left(\boldsymbol{\tau}_R^T \mathbf{R} + \mathbf{R}^T \boldsymbol{\tau}_R^T\right). \tag{7.17}$$

The last relation motivates us to define the associated time derivatives of $\boldsymbol{\tau}_R$ by

$$\frac{\overset{\triangledown}{D}{}^n}{Dt^n}\boldsymbol{\tau}_R = \left(\frac{\overset{\triangledown}{D}{}^{n-1}}{Dt^{n-1}}\boldsymbol{\tau}_R\right)^{\bullet} - \dot{\mathbf{R}}\mathbf{R}^T\left(\frac{\overset{\triangledown}{D}{}^{n-1}}{Dt^{n-1}}\boldsymbol{\tau}_R\right), \tag{7.18}$$

satisfying the relation

$$\frac{\overset{\triangledown}{d}{}^n}{dt^n}\tilde{\boldsymbol{\tau}}_{(1)} = \frac{1}{2}\left(\left(\frac{\overset{\triangledown}{D}{}^n}{Dt^n}\boldsymbol{\tau}_R\right)^T \mathbf{R} + \mathbf{R}^T\left(\frac{\overset{\triangledown}{D}{}^n}{Dt^n}\boldsymbol{\tau}_R\right)\right). \tag{7.19}$$

Again, a relation of the form

$$\left(\boldsymbol{\tau}_R \cdot \mathbf{F}^{T-1}\right)^{\bullet} = \overset{\triangledown}{\boldsymbol{\tau}}_R \cdot \mathbf{F}^{T-1} + \boldsymbol{\tau}_R \cdot \left(\mathbf{F}^{T-1}\right)^{\triangle} \tag{7.20}$$

holds, where

$$\overset{\triangledown}{\boldsymbol{\tau}}_R := \frac{\overset{\triangledown}{D}}{Dt}\boldsymbol{\tau}_R. \tag{7.21}$$

Traditionally, in formulating constitutive equations, we assume $m = 2$. However, if we deal e.g. with problems concerning uniqueness or constitutive inequalities, further pairs of dual variables may be convenient in formulating the theory. As an example, discussing intrinsic stability of the material, Hill (see HILL [4–6]) proposed a class of constitutive inequalities, which must be satisfied for some domain of deformation spaces. In the nomenclature of the present work, Hill's inequalities correspond either to

$$W^{\text{incr}}_{(m)} = \overset{\triangledown}{\boldsymbol{\Sigma}}{}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\boldsymbol{\Pi}}{}^{(\Psi)}_{(m)} > 0, \tag{7.22}$$

or to

$$w^{\text{incr}}_{(m)} = \overset{\triangledown}{\boldsymbol{\Sigma}}{}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\boldsymbol{\pi}}{}^{(\Psi)}_{(m)} > 0. \tag{7.23}$$

As a consequence, for $m = 1$, Eqs. (7.22) and (7.23) reduce to

$$W^{\text{incr}}_{(1)} = \dot{\tilde{\mathbf{T}}}_{(1)} \cdot \dot{\mathbf{E}}_{(1)} = \overset{\triangledown}{\mathbf{T}}_R \cdot \overset{\triangle}{\mathbf{F}} > 0, \tag{7.24}$$

and

(7.25) $$w_{(1)}^{\text{incr}} = \dot{\tilde{\boldsymbol{\tau}}}_{(1)} \cdot \dot{\boldsymbol{\varepsilon}}_{(1)} = \overset{\nabla}{\boldsymbol{\tau}}_R \cdot (\mathbf{F}^{T-1})^{\triangle} > 0\,,$$

respectively. These relations demonstrate that dual variables, in combination with associated time derivatives, are appropriate terms for formulating objective constitutive inequalities, even in the case of two-point stress and strain tensors (in this context see also OGDEN [8, p. 407]).

Appendix A

Let

(A.1) $$\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}, \qquad t^* = t - a$$

describe an observer transformation in E, where $\mathbf{c}(t)$ denotes some vector-valued function of time and $a \in \mathbb{R}$. For our purposes, it suffices to assume $\mathbf{Q}(t)$ to be a proper orthogonal second-order tensor.

Assuming the reference configuration to be independent of the observer, the observer transformation (A.1) implies for the motion (3.1)

(A.2) $$\bar{\mathbf{x}}(\mathbf{X}, t) = \mathbf{c}(t) + \mathbf{Q}(t)\bar{\mathbf{x}}(\mathbf{X}, t), \qquad t^* = t - a\,.$$

Well-known results obtainable from (A.2) are the transformation rules

(A.3) $$\mathbf{F}^* = \mathbf{QF}, \qquad \mathbf{R}^* = \mathbf{QR}, \qquad \mathbf{U}^* = \mathbf{U}, \qquad \mathbf{V}^* = \mathbf{QVQ}^T\,.$$

An Eulerian second-order tensor $\mathbf{A}$ is said to be objective if it satisfies the transformation rule

(A.4) $$\mathbf{A}^* = \mathbf{QAQ}^T$$

under the observer transformation (A.1). Commonly, it is assumed that the stress tensor $\mathbf{S}$ is objective, i.e.,

(A.5) $$\mathbf{S}^* = \mathbf{QSQ}^T\,.$$

Now, let $\mathbf{S}$ be represented by

(A.6) $$\mathbf{S} = S_{kl}\boldsymbol{\mu}_k \otimes \boldsymbol{\mu}_l\,,$$

so that

(A.7) $$\mathbf{T}_{(g)} = \sum_{i=1}^{3} \frac{S_{ij}}{\lambda_i g'(\lambda_i)}\mathbf{M}_i \otimes \mathbf{M}_i + 2\sum_{i \neq j} \alpha_{(g)ij} S_{ij} \mathbf{M}_i \otimes \mathbf{M}_j\,,$$

by $(4.16)_2$, (4.18). On using the relations (A.3), it is a straightforward matter to derive the transformation rules $(i, j = 1, 2, 3)$

(A.8) $$\begin{aligned} \boldsymbol{\mu}_i^* &= \mathbf{Q}\boldsymbol{\mu}_i\,, & \mathbf{M}_i^* &= \mathbf{M}_i\,, & & \\ \lambda_i^* &= \lambda_i\,, & g(\lambda_i^*) &= g(\lambda_i), & g'(\lambda_i^*) &= g'(\lambda_i), \\ \ell_{(g)ij}^* &= \ell_{(g)ij}\,, & \alpha_{(g)ij}^* &= \alpha_{(g)ij}\,, & S_{ij}^* &= S_{ij}\,. \end{aligned}$$

Hence,

(A.9) $$\mathbf{T}^*_{(g)} = \mathbf{T}_{(g)},$$

from (A.7). Thus, we have

(A.10) $$\left(\mathbf{T}^*_{(g)}\right)^{\bullet} = \dot{\mathbf{T}}_{(g)}$$

and therefore $(i, j = 1, 2, 3)$

(A.11) $$\mathbf{M}^*_i \cdot \left(\mathbf{T}^*_{(g)}\right)^{\bullet} \mathbf{M}^*_j = \mathbf{M}_i \cdot \dot{\mathbf{T}}_{(g)} \mathbf{M}_j .$$

Next, we discuss how $D_{(g)}\mathbf{S}/Dt$ is affected under the observer transformation (A.1). To this end, using (4.21), we rewrite $(4.23)_1$ in the form

(A.12) $$\begin{aligned} \frac{D_{(g)}}{Dt}\mathbf{S} &= \mathcal{P}_{(g)}[\dot{\mathbf{T}}_{(g)}] \\ &= \sum_{i=1}^{3} \lambda_i g'(\lambda_i) \left(\mathbf{M}_i \cdot \dot{\mathbf{T}}_{(g)} \mathbf{M}_i\right) \boldsymbol{\mu}_i \otimes \boldsymbol{\mu}_i + \frac{1}{2} \sum_{i \neq j} \frac{1}{\alpha^{(g)}_{ij}} \left(\mathbf{M}_i \cdot \dot{\mathbf{T}}_{(g)} \mathbf{M}_j\right) \boldsymbol{\mu}_i \otimes \boldsymbol{\mu}_j . \end{aligned}$$

From this result, as well as from (A.8) and (A.11), we conclude that

$$\frac{D_{(g)}}{Dt^*}\mathbf{S}^* = \mathbf{Q} \left(\frac{D_{(g)}}{Dt}\mathbf{S}\right) \mathbf{Q}^T ,$$

which shows that $D_{(g)}\mathbf{S}/Dt$ represents an objective Eulerian second-order tensor.

Appendix B

In this Appendix we give possible physical interpretations for the stress tensors $\boldsymbol{\Sigma}^{(\Psi)}_{(2)}$ and $\boldsymbol{\sigma}^{(\Psi)}_{(2)}$, which confessedly are somewhat artificial.

By Cauchy's theorem, we have

(B.1) $$\mathbf{t} = \mathbf{T}\mathbf{n} = \mathbf{S}\left[(\det \mathbf{F})^{-1}\mathbf{n}\right] = \boldsymbol{\varsigma}\left[(\det \mathbf{F})^{-1}\mathbf{m}\right],$$

where $\mathbf{t}$ represents the stress vector acting on a surface element in the current configuration

(B.2) $$d\mathbf{a} = \mathbf{n}\, da ,$$

oriented by a unit normal $\mathbf{n}$, and

(B.3) $$\mathbf{m} := -\mathbf{n} .$$

Let now $d\mathbf{a}$ be represented by

(B.4) $$d\mathbf{a} = d\mathbf{x}_{[1]} \times d\mathbf{x}_{[2]} ,$$

where $d\mathbf{x}_{[i]}, i = 1, 2$, are non-collinear line elements in the current configuration. For the corresponding surface element

(B.5) $$d\mathbf{A}_0 = \mathbf{N}_0 \, dA_0$$

$(\mathbf{N}_0 \cdot \mathbf{N}_0 = 1)$ in the reference configuration, the well-known formula

(B.6) $$d\mathbf{a} = (\det \mathbf{F})\mathbf{F}^{T-1} d\mathbf{A}_0$$

holds, with

(B.7) $$d\mathbf{A}_0 = d\mathbf{X}_{[1]} \times d\mathbf{X}_{[2]}$$

and

(B.8) $$d\mathbf{X}_{[i]} = \mathbf{F}^{-1} d\mathbf{x}_{[i]},$$

by (5.16). Furthermore, assuming that the transformation rule (5.16) applies also to the vector $\mathrm{t}\, da$, we can introduce a transformed "force" $d\widetilde{\mathrm{Q}}$ in the reference configuration by

(B.9) $$\mathrm{t}\, da = \mathbf{F}\, d\widetilde{\mathrm{Q}}\,.$$

Analogously, further transformed "forces" $d\widetilde{\mathrm{Q}}^{(\Psi)}$ are given by

(B.10) $$d\mathrm{Q}^{(\Psi)} := \mathbf{\Psi}\, d\widetilde{\mathrm{Q}}\,,$$

with $d\mathrm{Q}^{(F)} = \mathrm{t}\, da$. In addition, we define the "stress vectors"

(B.11) $$\mathrm{t}^{(\Psi)} := \frac{d\mathrm{Q}^{(\Psi)}}{dA^{(\Psi)}},$$

where $dA^{(\Psi)}$ is given by the relation

(B.12) $$d\mathbf{A}^{(\Psi)} = \mathbf{N}^{(\Psi)}\, dA^{(\Psi)} = (\det \mathbf{\Psi})\mathbf{\Psi}^{T-1} d\mathbf{A}_0$$

$(\mathbf{N}^{(\Psi)} \cdot \mathbf{N}^{(\Psi)} = 1)$, which is analogous to (B.6). Finally, on the basis of $(\mathrm{B.1})_2$, it is not difficult to derive the relation

(B.13) $$\mathrm{t}^{(\Psi)} = \mathbf{\Sigma}^{(\Psi)}_{(2)} \left[(\det \mathbf{\Psi})^{-1}\mathbf{N}^{(\Psi)}\right]$$

with $\left(\mathrm{t}^{(F)}, \mathbf{\Sigma}^{(F)}_{(2)}, \mathbf{N}^{(F)}\right) = (\mathrm{t}, \mathbf{S}, \mathbf{n})$. Thus, the stress tensor $\mathbf{\Sigma}^{(\Psi)}_{(2)}$ acting on the "weighted normal" $(\det \mathbf{\Psi})^{-1}\mathbf{N}^{(\Psi)}$ gives the "stress vector" $\mathrm{t}^{(\Psi)}$.

The physical interpretation of $\boldsymbol{\sigma}^{(F)}_{(2)}$ is similar. We start by considering again the surface element $d\mathbf{a}$ (see Eqs. (B.2) and (B.4)). Besides (B.6), the surface element $d\mathbf{a}$ can be mapped on the reference configuration as follows. Let $d\mathbf{Y}_{[i]}$ be vectors in the reference configuration, which are related to $d\mathbf{x}_{[i]}$ by means of (5.18),

(B.14) $$d\mathbf{Y}_{[i]} = \mathbf{F}^{T}\, d\mathbf{x}_{[i]}\,.$$

We define the transformed "surface element" in the reference configuration $d\mathbf{a}_0$ by

$$d\mathbf{a}_0 = \mathbf{n}_0 da_0 = d\mathbf{Y}_{[1]} \times d\mathbf{Y}_{[2]}, \tag{B.15}$$

with $\mathbf{n}_0 \cdot \mathbf{n}_0 = 1$. It is readily shown that $d\mathbf{a}$ is related to $d\mathbf{a}_0$ through

$$d\mathbf{a} = (\det \mathbf{F})^{-1} \mathbf{F}\, d\mathbf{a}_0 . \tag{B.16}$$

Next, assuming the transformation formula (5.18) (or (B.14)) to apply also to the vector $\mathsf{t}\, da$, we can introduce a transformed "force" in the reference configuration $d\tilde{\mathsf{q}}$ by

$$\mathsf{t}\, da = \mathbf{F}^{T-1}\, d\tilde{\mathsf{q}} . \tag{B.17}$$

Analogously, further transformed "forces" $d\mathsf{q}^{(\Psi)}$ are defined through

$$d\mathsf{q}^{(\Psi)} := \boldsymbol{\Psi}^{T-1}\, d\tilde{\mathsf{q}} . \tag{B.18}$$

Finally, we introduce the "stress vectors"

$$\mathsf{t}^{(\Psi)} := \frac{d\mathsf{q}^{(\Psi)}}{da^{(\Psi)}}, \tag{B.19}$$

where $da^{(\Psi)}$ is given by the relation

$$d\mathbf{a}^{(\Psi)} = \mathbf{n}^{(\Psi)}\, da^{(\Psi)} = (\det \boldsymbol{\Psi})^{-1} \boldsymbol{\Psi}\, d\mathbf{a}_0 , \tag{B.20}$$

$(\mathbf{n}^{(\Psi)} \cdot \mathbf{n}^{(\Psi)} = 1)$, which is analogous to (B.16). Then, on the basis of $(B.1)_3$, it can be seen that

$$\mathsf{t}^{(\Psi)} = \boldsymbol{\sigma}^{(\Psi)}_{(2)} \left[\frac{(\det \boldsymbol{\Psi})}{(\det \mathbf{F})^2} \mathbf{m}^{(\Psi)} \right], \tag{B.21}$$

where

$$\mathbf{m}^{(\Psi)} := -\mathbf{n}^{(\Psi)} \tag{B.22}$$

and $\left(\mathsf{t}^{(F)}, \boldsymbol{\sigma}^{(F)}_{(2)}, \mathbf{m}^{(F)}\right) = (\mathsf{t}, \varsigma, \mathbf{m})$. That is, the stress tensor $\boldsymbol{\sigma}^{(\Psi)}_{(2)}$ acting on the "weighted normal" $\left((\det \boldsymbol{\Psi})/(\det \mathbf{F})^2\right) \mathbf{m}^{(\Psi)}$ gives the "stress vector" $\mathsf{t}^{(\Psi)}$.

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INSTITUT FÜR MECHANIK
UNIVERSITÄT GESAMTHOCHSCHULE KASSEL, KASSEL
and
FORSCHUNGSZENTRUM KARLSRUHE, TECHNIK UND UMWELT,
INSTITUT FÜR MATERIALFORSCHUNG II, KARLSRUHE, GERMANY.

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On the existence of solutions for two-dimensional Stokes flows past rigid obstacles

M. KOHR-ILE (CLUJ-NAPOCA)

IN THIS PAPER we obtain some existence and uniqueness properties for the solution corresponding to the problem of the plane unbounded Stokes flow past rigid obstacles. The stream function of the flow is represented in the form of simple layer potentials.

1. Introduction

IN SOME PREVIOUSLY published papers [5, 6, 7], the authors treated the problem of an unbounded two-dimensional viscous flow past an arbitrary obstacle, using the method of matched inner and outer expansions of the corresponding solution. These results were then generalized to the three-dimensional case.

The purpose of this paper is to present a method for studying the problem of the Stokes flow past some rigid two-dimensional obstacles, using the properties of simple layer potentials.

Let $N \geq 2$ be the number of obstacles denoted by Ω_i, $i = \overline{1, N}$, Ω denoting the region exterior to these obstacles. The flow is described by the velocity $\mathbf{u}$ and the pressure p. We suppose that $\mathbf{u} \to U\mathbf{i}$, $p \to p_\infty$ as $|x| \to \infty$, where $x = x_1\mathbf{i} + x_2\mathbf{j}$, and U, p are prescribed constants. Using the dimensionless variables: $x' = x/l$, $\mathbf{u}' = \mathbf{u}/U$, $p' = l(p - p_\infty)/\mu U$ and the Reynolds number $\mathrm{Re} = \varrho l U/\mu$, where l is a characteristic length, μ the dynamic viscosity, and ϱ the fluid density, then $\mathbf{u}'$ and p' are solutions of the Navier–Stokes problem (disregarding the primes over u and p)

$$
\begin{aligned}
\Delta\mathbf{u} - \nabla p &= \mathrm{Re}(\mathbf{u}\cdot\nabla)\mathbf{u} \qquad \text{in } \Omega,\\
\nabla\cdot\mathbf{u} &= 0,\\
\mathbf{u} &= \mathbf{f}^i \qquad \text{on } C^i = \partial\Omega_i, \quad i = \overline{1, N},\\
\mathbf{u} &\to \mathbf{i}, \qquad p \to 0, \qquad \text{as } |x| \to \infty.
\end{aligned}
\tag{1.1}
$$

Here Δ and ∇ denote the two-dimensional Laplacean and the gradient operator, respectively. We require the given velocities $\mathbf{f}^i$, $i = \overline{1, N}$ to satisfy the zero outflow conditions:

$$
\int_{C^i} \mathbf{f}^i\cdot\mathbf{n}^i\, ds = 0,
\tag{1.2}
$$

where $\mathbf{n}^i$ is the exterior vector normal to Ω_i, $i = \overline{1, N}$.

We suppose that the Reynolds number defined above is sufficiently small.

The Navier-Stokes problem (1.1), for the case $N = 1$, is singular in the sense that the linearized Stokes form:

$$\begin{aligned} \Delta \mathbf{u} - \nabla p &= 0, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \tag{1.3}$$

together with the same conditions as in $(1.1)_{3,4}$, has no solution in view of the Stokes paradox. But, in this case, it is possible to obtain a solution, if the condition at infinity is replaced by:

$$\mathbf{u} = \mathbf{A} \ln|x| + \mathcal{O}(1), \qquad \text{as} \quad |x| \to \infty, \tag{1.4}$$

for any given constant vector **A** [6, 7]. Also, in the case of $N \geq 2$, we prove that there exists a constant vector **A** such that the problem (1.3) has a solution, if the condition at infinity is replaced with (1.4).

2. Integral equation of the first kind

The equation of continuity $\nabla \cdot \mathbf{u} = 0$ implies the existence of a stream function ψ such that

$$\mathbf{u} = (\nabla \psi)^{\perp}, \tag{2.1}$$

where $\mathbf{v}^{\perp}$ denotes the vector obtained by rotating the vector $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j}$ by $\pi/2$ counterclockwise, so that $\mathbf{v}^{\perp} = -v_2 \mathbf{i} + v_1 \mathbf{j}$. Because the domain Ω is not simply connected, the condition (1.4) is only local, i.e. ψ might not be a single-valued function. But the following arguments prove that ψ is necessarily a single-valued function.

Let C be any closed curve bounding the domain $\Omega^0 \subset \Omega$ and $\Omega^* = (\Omega \backslash \Omega^0) \cap B_R$, where B_R is a large disk of radius R. Applying the Green's formula, we obtain:

$$0 = \int_{\Omega^*} \operatorname{div} \mathbf{u} \, dx = \sum_{i=1}^{N} \int_{C^i} \mathbf{u} \cdot \mathbf{n} \, ds + \int_{C} \mathbf{u} \cdot \mathbf{n} \, ds - \int_{\partial B_R} \mathbf{u} \cdot \mathbf{n} \, ds . \tag{2.2}$$

From $(1.1)_3$ and (1.2), it results that $\int_{C^i} \mathbf{u} \cdot \mathbf{n} \, ds = 0$, $i = \overline{1, N}$.

From Green's formula in $\Omega_R = \Omega \cap B_R$, we have:

$$0 = \int_{\Omega_R} \operatorname{div} \mathbf{u} \, dx = \sum_{i=1}^{N} \int_{C^i} \mathbf{u} \cdot \mathbf{n} \, ds + \int_{\partial B_R} \mathbf{u} \cdot \mathbf{n} \, ds . \tag{2.3}$$

Hence (2.3) implies that $\int_{\partial B_R} \mathbf{u}\cdot\mathbf{n}\, ds = 0$. The above arguments show that

$$\int_C \mathbf{u}\cdot\mathbf{n}\, ds = 0, \qquad \text{so} \qquad \int_C \mathbf{u}^{\perp}\cdot d\mathbf{s} = 0.$$

Then we express ψ in the form

$$\psi(x) = -\int_{x_0}^{x} \mathbf{u}^{\perp}\cdot d\mathbf{s}, \qquad x \in \Omega, \tag{2.4}$$

where x_0 is a fixed point in Ω, x is an arbitrary point in Ω, and the integral is evaluated along an arbitrary polygonal line between x_0 and x. Also, it is easy to establish the condition (2.1).

Using (1.3) and (2.1), we obtain the Stokes problem for stream function ψ:

$$\begin{aligned} \Delta^2\psi &= 0 \quad \text{in } \Omega, \\ \nabla\psi(x) &= \mathbf{j} - \mathbf{f}^{i\perp}(x), \qquad x \in C^i, \quad i = \overline{1,N}. \end{aligned} \tag{2.5}$$

We shall prove that there exists a real constant vector $\mathbf{A}$ such that

$$\nabla\psi(x) = \mathbf{A}\ln|x| + \mathcal{O}(1), \qquad \text{as} \quad |x| \to \infty, \tag{2.6}$$

and that the problem (2.5)–(2.6) has a solution.

For these purposes, we represent the stream function ψ in the form:

$$\psi(x) = \sum_{i=1}^{N} \int_{C^i} \nabla_y F(x,y)\cdot\boldsymbol{\phi}^i(y)\, ds_y^i, \qquad x \in \Omega \cup \left(\bigcup_{i=1}^{N} C^i\right), \tag{2.7}$$

where s_y^i denotes the arc length measured along C^i, $i = \overline{1,N}$ and F is the fundamental solution of biharmonic equation:

$$F(x,y) = \frac{1}{8\pi}|x-y|^2[\ln|x-y| - 1]. \tag{2.8}$$

It is easy to show that ψ given by (2.7), satisfies the equation $(2.5)_1$ and will be a solution of the boundary conditions $(2.5)_2$, if the density function $\widetilde{\phi}$, with $\widetilde{\phi}(x) = \boldsymbol{\phi}^i(x)$, $x \in C^i$, $i = \overline{1,N}$, satisfies the following system of integral equations of the first kind:

$$\sum_{i=1}^{N} \int_{C^i} \nabla_x\nabla_y F(x^k,y)\boldsymbol{\phi}^i(y)\, ds_y^i = \mathbf{g}^k(x^k), \qquad x^k \in C^k, \qquad k = \overline{1,N}, \tag{2.9}$$

where

$$\mathbf{g}^k := \mathbf{j} - \mathbf{f}^{k\perp}. \tag{2.10}$$

The integral operator V^i defined by

$$V^i \boldsymbol{\phi}^i(x) := \int\limits_{C^i} \nabla_x \nabla_y F(x,y) \boldsymbol{\phi}^i(y)\, ds_y^i, \qquad x \in C^i$$

has a kernel with logarithmic singularity.

Differentiating (2.9) with respect to the arc length s_x^k, $k = \overline{1,N}$, we obtain the set of integral equations with a Cauchy singularity:

$$\sum_{i=1}^{N} \int\limits_{C^i} \frac{\partial}{\partial s_x^k} \nabla_x \nabla_y F(x^k, y) \cdot \boldsymbol{\phi}^i(y)\, ds_y^i = \frac{d}{ds_x^k} \mathbf{g}^k(x^k), \qquad k = \overline{1,N}. \tag{2.11}$$

Because F is a function of $|x-y|$ only, it is seen that the adjoint homogeneous system of (2.11) has the form:

$$\sum_{i=1}^{N} \int\limits_{C^i} \frac{\partial}{\partial s_x} \nabla_x \nabla_y F(x, y^k) \cdot \mathbf{S}^i(x)\, ds_x = 0, \qquad y^k \in C^k, \qquad k = \overline{1,N}. \tag{2.12}$$

We remark that the functions $\widetilde{S}^i \colon \bigcup\limits_{j=1}^{N} C^j \to \mathbb{R}^2$, given by

$$\widetilde{S}^i(x) = a_j^i x + \mathbf{b}_j^i, \qquad x \in C^j, \qquad j = \overline{1,N}, \tag{2.13}$$

with a_j^i, $\mathbf{b}_j^i$ denoting constants, are the solutions of the system (2.12). These functions determine a linear space with $3N$ dimensions, which implies that the dimension of to solution space corresponding to the homogeneous system (2.11) is at least $3N$. We use here the fact that the homogeneous system (2.11) and the adjoint system (2.12) have the same number of linearly independent solutions (see [10]).

THEOREM 1. *There exist at most $3N$ linearly independent solutions of the homogeneous system* (2.11).

P r o o f. The functions

$$\widetilde{\tau}^i \ : \ \bigcup_{j=1}^{N} C^j \to \mathbb{R}^2, \qquad \widetilde{\tau}^i(x) = \begin{cases} 0, & x \in C^j, \quad j \neq i, \\ \boldsymbol{\tau}^i(x), & x \in C^i, \end{cases}$$

$i = \overline{1,N}$, where $\boldsymbol{\tau}^i(x)$ denotes the unit tangent vector in the point $x \in C^i$, are N linearly independent solutions of the homogeneous system (2.11).

Let $\widetilde{\varphi}^i \colon \bigcup_{j=1}^{N} C^j \to \mathbb{R}^2$ be any $2N+1$ solutions of the homogeneous system (2.11), and $\psi^i = \psi^i(\widetilde{\varphi}^i)$, $i = \overline{1,\ 2N+1}$, denote the corresponding stream functions, as in (2.7). Then functions ψ^i satisfy the equations

$$
\begin{aligned}
&\Delta^2\psi^i = 0 \qquad \text{in } \Omega,\\
(2.14)\qquad &\nabla\psi^i\Big|_{C^j} = \mathbf{C}_i^j, \qquad j = \overline{1,N},\\
&\nabla\psi^i(x) = \mathbf{A}^i \ln|x| + \mathcal{O}(1), \qquad \text{as } |x| \to \infty,
\end{aligned}
$$

where $\mathbf{C}_i^j$ is a constant vector and

$$\mathbf{A}^i = \frac{1}{4\pi}\sum_{j=1}^{N}\int_{C^j} \boldsymbol{\varphi}_j^i(y)\, ds_y^j .$$

We define the function $\widetilde{\varphi}^i$ as $\widetilde{\varphi}^i(x) = \boldsymbol{\varphi}_j^i(x)$ for $x \in C^j$, $j = \overline{1,N}$.

We can choose real constants $\alpha_1, \ldots, \alpha_{2N+1}$, not all equal to zero, and the vector $\mathbf{c}(c_1, c_2)$, such that:

$$
\begin{aligned}
&\sum_{i=1}^{2N+1} \alpha_i \mathbf{c}_i^j - \mathbf{c} = 0, \qquad j = \overline{1,N},\\
(2.15)\qquad &\sum_{i=1}^{N} \alpha_i \mathbf{A}^i = 0
\end{aligned}
$$

because we have here $2N+2$ homogeneous equations with $2N+3$ unknowns.

Let the functions ψ_0 and $\widetilde{\phi}_0$ be defined by:

$$(2.16)\qquad \psi_0 = \sum_{i=1}^{2N+1} \alpha_i \psi^i, \qquad \widetilde{\phi}_0 = \sum_{i=1}^{N} \alpha_i \widetilde{\varphi}^i .$$

Then ψ_0 satisfies the equation

$$
\begin{aligned}
&\Delta^2\psi_0 = 0 \qquad \text{in } \Omega,\\
(2.17)\qquad &\nabla\psi_0(x) = \mathbf{C}, \qquad x \in C^j, \quad j = \overline{1,N},\\
&\nabla\psi_0(x) = \mathcal{O}(1), \qquad \text{as } |x| \to \infty.
\end{aligned}
$$

The problem (2.17) has a solution of linear form $\psi_0(x) = \mathbf{c}\cdot\mathbf{x}$. From the uniqueness theorem of the solution corresponding to the exterior Stokes problem (see Theorem 3), we deduce that ψ_0 is the unique solution of (2.17). The function ψ_0 given by (2.16) is also biharmonic in each domain Ω_i and is continuous together with its first derivatives on C^i, $i = \overline{1,N}$. Using the uniqueness result of the inner Stokes problem, we conclude that ψ_0 has also a linear form in Ω_i, $i = \overline{1,N}$.

Using [5], it is easy to prove that on each contour C^j, $j = \overline{1,N}$, the stream function ψ given by (2.7) has the properties:

$$(2.18)\qquad \begin{aligned} (\Delta\psi)^+ - (\Delta\psi)^- &= \lambda^j \mathbf{n}^j \cdot \boldsymbol{\phi}^j, \\ \left(\frac{\partial}{\partial n^j}\Delta\psi\right)^+ - \left(\frac{\partial}{\partial n^j}\Delta\psi\right)^- &= \lambda^j \frac{d}{ds^j}\left(\boldsymbol{\tau}^j \cdot \boldsymbol{\phi}^j\right), \end{aligned}$$

where the symbols $+$, $-$ denote the limits in Ω and Ω_j, respectively, and $\partial/\partial n^j$ is the normal derivative on C^j, $j = \overline{1,N}$.

Since ψ_0 has a linear form in Ω and Ω_j, respectively, from (2.18) we obtain that there exists a constant β^j such that:

$$(2.19)\qquad \boldsymbol{\phi}_0^j(x) = \beta^j \boldsymbol{\tau}^j(x), \qquad x \in C^j, \quad j = \overline{1,N},$$

where the function $\boldsymbol{\phi}_0^j$ is defined by $\widetilde{\phi}_0(x) = \boldsymbol{\phi}_0^j(x)$, $x \in C^j$, $j = \overline{1,N}$.

Hence we deduce that

$$(2.20)\qquad \widetilde{\phi}_0(x) - \sum_{j=1}^{N} \beta^j \widetilde{\tau}^j(x) = 0, \qquad x \in \bigcup_{j=1}^{N} C^j$$

or

$$(2.21)\qquad \sum_{i=1}^{2N+1} \alpha_i \widetilde{\varphi}^i(x) - \sum_{j=1}^{N} \beta^j \widetilde{\tau}^j(x) = 0, \qquad x \in \bigcup_{j=1}^{N} C^j,$$

with the functions $\widetilde{\tau}^j$ defined above. It results that the functions $\widetilde{\varphi}^i$, $\widetilde{\tau}^j$, $i = \overline{1,\ 2N+1}$, $j = \overline{1,N}$, are linearly dependent.

So, we have proved that the dimension of the solutions space of the homogeneous system (2.12) equals exactly $3N$, and each solution $\widetilde{S}$ has the form:

$$(2.22)\qquad \widetilde{S}(x) = a^i x + \mathbf{b}^i, \qquad x \in C^i, \qquad i = \overline{1,N},$$

where a^i, $\mathbf{b}^i$ are constants.

Using the theory of singular integral equations (the Fredholm alternative, [10]), the system (2.11) has solutions if and only if

$$(2.23)\qquad \sum_{i=1}^{N} \int_{C^i} \frac{d}{ds_x^i}\mathbf{g}^i(x) \cdot \mathbf{S}^i(x)\, ds_x^i = 0,$$

where $\widetilde{S}$, with $\widetilde{S}(x) = \mathbf{S}^i(x)$, $x \in C^i$, $i = \overline{1,N}$, is a solution of the adjoint system (2.12).

From (2.22), (2.10) and (1.2) the conditions (2.23) follow immediately.

Let $\widetilde{\phi}^0$ be a solution of the system (2.11), with $\widetilde{\phi}^0\Big|_{C^j} = \boldsymbol{\phi}_j^0$, $j = \overline{1,N}$. The corresponding stream function $\psi^0 = \psi^0(\widetilde{\phi}^0)$ satisfies:

$$
\begin{aligned}
\Delta^2 \psi^0 &= 0 \qquad \text{in } \Omega, \\
\nabla \psi^0(x) &= \mathbf{g}^i(x) + \mathbf{k}^i, \qquad x \in C^i, \quad i = \overline{1,N}, \\
\nabla \psi^0(x) &= \mathbf{A}^0 \ln|x| + \mathcal{O}(1), \qquad \text{as } |x| \to \infty,
\end{aligned}
\tag{2.24}
$$

where

$$
\mathbf{A}^0 = \frac{1}{4\pi} \sum_{j=1}^{N} \int_{C^j} \boldsymbol{\phi}_j^0(x)\, ds_x^j ,
$$

and $\mathbf{k}^i$, $i = \overline{1,N}$ are constant vectors. Let $\widetilde{k}^0$: $\bigcup\limits_{j=1}^{N} C^j \to \mathbb{R}^2$ be defined by $\widetilde{k}^0\Big|_{C^j} = \mathbf{k}^j$, $j = \overline{1,N}$.

Also let $\widetilde{\varphi}^i$, $i = \overline{1,2N}$ and $\widetilde{\tau}^j$, $j = \overline{1,N}$, be the $3N$ linearly independent solutions of the homogeneous system (2.11). Then the stream functions $\psi^i = \psi^i(\widetilde{\varphi}^i)$, $i = \overline{1,2N}$ satisfy the equations

$$
\begin{aligned}
\Delta^2 \psi^i &= 0 \qquad \text{in } \Omega, \\
\nabla \psi^i(x) &= \mathbf{k}_j^i, \qquad x \in C^j, \quad j = \overline{1,N}, \\
\nabla \psi^i(x) &= \mathbf{A}^i \ln|x| + \mathcal{O}(1), \qquad \text{as } |x| \to \infty,
\end{aligned}
\tag{2.25}
$$

with

$$
\mathbf{A}^i = \frac{1}{4\pi} \sum_{j=1}^{N} \int_{C^j} \boldsymbol{\varphi}_j^i(x)\, ds_x^j , \qquad \widetilde{\varphi}^i\Big|_{C^j} = \boldsymbol{\varphi}_j^i, \quad j = \overline{1,N} \quad \text{and} \quad \mathbf{k}_j^i, \quad j = \overline{1,N},
$$

are the constant vectors, $i = \overline{1,2N}$. Let $\widetilde{k}^i$: $\bigcup\limits_{j=1}^{N} C^j \to \mathbb{R}^2$, be given by $\widetilde{k}^i\Big|_{C^j} = \mathbf{k}_j^i$, $j = \overline{1,N}$, $i = \overline{1,2N}$.

Let V be the set defined by:

$$
V = \left\{ \widetilde{k} : \bigcup_{j=1}^{N} C^j \to \mathbb{R}^2 \mid \widetilde{k}(x) = \mathbf{k}^j, \;\; x \in C^j, \;\; \mathbf{k}^j \text{ a constant vector}, \;\; j = \overline{1,N} \right\}.
$$

V is a linear space with $\dim V = 2N$, and the functions $\widetilde{k}^0$, $\widetilde{k}^i$, $i = \overline{1,2N}$ belong to V. Hence, there exist the real constants $\alpha_1, \ldots, \alpha_{2N}$ with the property:

$$
\sum_{i=1}^{2N} \alpha_i \widetilde{k}^i(x) + \widetilde{k}^0(x) = 0, \qquad x \in \bigcup_{j=1}^{2N} C^j,
\tag{2.26}
$$

if we suppose that the functions $\widetilde{\varphi}^i$, $i = \overline{1, 2N}$, satisfy:

$$\mathbf{A}^i = 0, \qquad i = \overline{1, 2N}, \tag{2.27}$$

since $\widetilde{k}^i$ are linearly independent functions.

Using (2.24), (2.25) and (2.26), we deduce that the function

$$\psi = \psi^0 + \sum_{i=1}^{2N} \alpha_i \psi^i$$

is a solution of the Stokes problem (2.5). At infinity ψ satisfies the condition:

$$\psi(x) = \mathbf{A}^0 \ln|x| + \mathcal{O}(1), \qquad \text{as } |x| \to \infty, \tag{2.28}$$

where $\mathbf{A}^0$ is defined in (2.24).

So, we obtain the following result:

THEOREM 2. *If the functions* $\mathbf{f}^i \colon C^i \to \mathbb{R}^2$, $i = \overline{1, N}$ *satisfy the conditions* (1.2), *then in the hypothesis* (2.27), *there exists a constant vector* $\mathbf{A}$ *such that the problem* (2.5) *with the condition* (2.28) *at infinity, has a solution* ψ.

In the proof of the Theorem 1, we used the uniqueness property of solution for the exterior Stokes problem. This result is given by:

THEOREM 3. *The Stokes problem* (2.5) *has at most one solution (up to an additive constant), under the condition that*

$$\psi(x) = \mathcal{O}\left(|x|^{-1}\right), \qquad D^m\psi(x) = \mathcal{O}\left(|x|^{-2}\right), \quad m \geq 1, \quad \text{as } |x| \to \infty, \tag{2.29}$$

and

$$\int_{C^i} \frac{\partial \omega}{\partial n}(x)\, ds_x^i = 0, \qquad i = \overline{1, N}, \tag{2.30}$$

where $\omega = \Delta\psi$.

P r o o f. We suppose that there exist two solutions ψ^1 and ψ^2 of the problem (2.5). If we consider the difference $\psi = \psi^1 - \psi^2$, then ψ satisfies the equation

$$\begin{aligned} \nabla\psi &= 0 \qquad \text{in } \Omega, \\ \nabla\psi\Big|_{C^i} &= 0, \qquad i = \overline{1, N}, \end{aligned} \tag{2.31}$$

with the additional conditions (2.29) and (2.30).

Let $\Omega_R = \Omega \cap B_R$, where B_R is a large disk of radius R. From Green's formula we obtain:

$$\tag{2.32} \int\limits_{\Omega_R} \left[\psi(x)\Delta^2\psi(x) - (\Delta\psi(x))^2\right] dx$$
$$= \sum_{i=1}^{N} \int\limits_{C^i} \left[\psi(x)\frac{\partial\omega}{\partial n}(x) - \omega(x)\frac{\partial\psi}{\partial n}(x)\right] ds_x^i + \int\limits_{\partial B_R} \left[\psi(x)\frac{\partial\omega}{\partial n}(x) - \omega(x)\frac{\partial\psi}{\partial n}(x)\right] ds_x\,,$$

where ∂B_R denotes the boundary of the disk B_R.

From (2.32), it results that the integrals taken along ∂B_R are zero, for $R \to \infty$. From the homogeneous conditions $(3.31)_2$ we have

$$\int\limits_{C^i} \omega(x)\frac{\partial\psi}{\partial n}(x)\, ds_x^i = 0, \qquad i = \overline{1,N}.$$

Also $\psi(x) = c_i$, for $x \in C^i$, where c_i is a real constant, $i = \overline{1,N}$.

Now, if we use the conditions (2.30), we deduce:

$$\sum_{i=1}^{N} \int\limits_{C^i} \psi(x)\frac{\partial\omega}{\partial n}(x)\, ds_x^i = \sum_{i=1}^{N} c_i \int\limits_{C^i} \frac{\partial\omega}{\partial n}(x)\, ds_x^i = 0.$$

Hence the above identity (2.32) implies $\Delta\psi = 0$ in Ω.

Applying again the Green's formula, we obtain:

$$\tag{2.33} 0 = \int\limits_{\Omega_R} \psi(x)\Delta\psi(x)\, dx$$
$$= \int\limits_{\partial B_R} \psi(x)\frac{\partial\psi}{\partial n}(x)\, ds_x + \sum_{i=1}^{N} \int\limits_{C^j} \psi(x)\frac{\partial\psi}{\partial n}(x)\, ds_x^i - \int\limits_{\Omega_R} (\nabla\psi(x))^2\, dx.$$

Using the conditions (2.29), (2.30), $(2.31)_2$ we obtain $\nabla\psi = 0$ in Ω, hence ψ is a constant in Ω and $\psi_1 = \psi_2$ (up to an additive constant).

Remark. Since we determine the stream function ψ in the form (2.7), the conditions (2.30) are easily obtained as a consequence of Green's identity.

Using the stream function ψ determined above, we obtain the velocity $\mathbf{u} = (\nabla\psi)^{\perp}$, and the pressure p as the harmonic conjugate of $\omega = \Delta\psi$, but only locally, because the domain Ω is not simply connected.

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"BABES-BOLYAI" UNIVERSITY
FACULTY OF MATHEMATICS CLUJ-NAPOCA, ROMANIA.

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The development of a nonstationary separation and coherent structures in a two-dimensional viscous incompressible flow around a body

M.N. ZAKHARENKOV (MOSCOW)

Dedicated to the memory of Vladimir M. Galkin killed in an aircraft crash in September 1994

TWO-DIMENSIONAL VISCOUS incompressible flows around a circular cylinder and a 12% Zhukovsky airfoil are considered. Numerous examples of complex separated flows around these bodies with coherent structures and detached separation generation, as well as examples of flow stabilization and separation destruction are obtained. Numerical experiment technology based on parametrization of the far-field boundary conditions and effect of sequential exclusion of the scheme parameters and problem statement disadvantages is proposed.

1. Problem statement

THE STATEMENT of the problem and solution procedure are detailed in [1, 2]. Two-dimensional $N-S$ equations are written in terms of the stream function Ψ and vorticity Ω, which are defined by relations $u = \partial\Psi/\partial y$, $v = -\partial\Psi/\partial x$, $\Omega = \partial u/\partial y - \partial v/\partial x$:

$$\Delta\Psi = H^2\Omega, \tag{1.1}$$

$$H^2 d\Omega/dt = \mathrm{Re}^{-1}\Delta\Omega, \tag{1.2}$$

where H^2 is the Jacobian of transformation of Cartesian coordinates x, y to curvilinear orthogonal coordinates ξ, η. A grid of "0"-type obtained by a conformal mapping of an airfoil onto a circle is used. $\mathrm{Re} = U_\infty b/\nu$ is the Reynolds number, where U_∞ is a free stream velocity, b is a characteristic length, ν is coefficient of kinematic viscosity. Dimensionless time t is defined by the relation $t_{\text{phys}} = tb/U_\infty$, where b is eiter the chord of airfoil or $b = R$ is radius of the cylinder.

Boundary condition. On a solid body surface S the following no-slip conditions are defined:

$$v_\xi = -H^{-1}\partial\Psi/\partial\eta\Big|_S = \varphi(\eta), \qquad v_\eta = H^{-1}\partial\Psi/\partial\xi\Big|_S = f(\eta).$$

The condition $\partial\Psi/\partial\xi\Big|_S = Hf(\eta)$ is transformed into a boundary condition for a vorticity Ω_S [1, 3] by using a two-parameter approximating formula; this permits us to eliminate the approximation effect on the solution accuracy and to maximize the iterative solution process rate by employing the procedure described in [1].

Over the far boundary S_∞ (being about 10 chords away from the body), the following boundary conditions are specified:

$$\partial\Omega/\partial\xi = 0, \tag{1.3}$$

$$\partial\Psi/\partial\xi = v_\infty\left(\frac{\partial x}{\partial\xi}\sin\alpha - \frac{\partial y}{\partial\xi}\cos\alpha\right) - \frac{v_\infty}{R_\infty}\left[D_x\sin(\eta-\alpha) + D_y\cos(\eta-\alpha)\right] - \Gamma/2\pi, \tag{1.4}$$

where D_x, D_y and Γ are parameters.

The flow starts from the state when the body and fluid are at rest.

2. Method of solution

A solution of decoupled equations of the system is used. Equation (1.1) is solved directly by expansion into a Fourier series in terms of the cyclic coordinate η.

Equation (1.2) is solved by the ADI method. Central differences for second derivatives and one-sided upwind differences for nonlinear terms in (1.2) are used.

At each time step an iterative process is employed. A zonal approach used in [4] is applied.

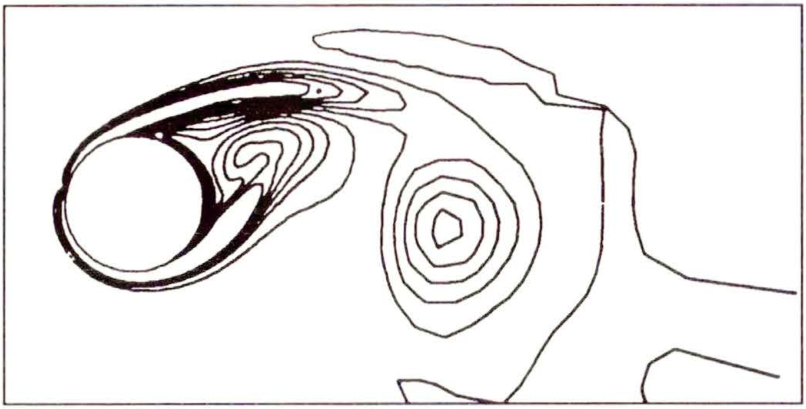
3. Flow past a circular cylinder

Two problems are considered.

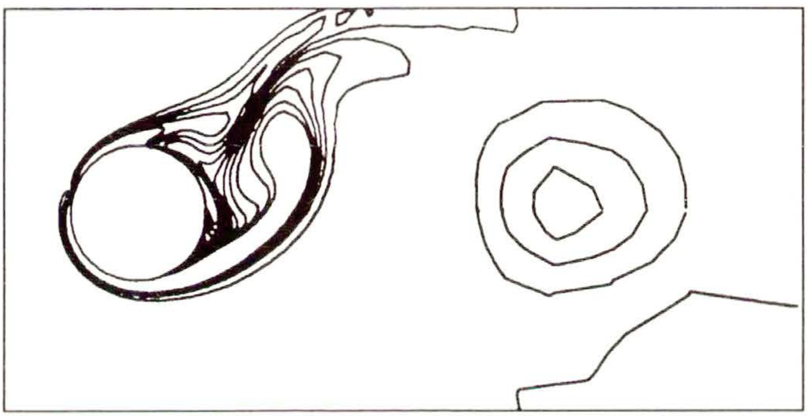
3.1.

Uniform flow around a circular cylinder in a viscous incompressible fluid that is preliminarily spun can serve as an interesting example. An initially steady flow around a cylinder rotating at a constant angular velocity W in a uniform viscous flow has been obtained by calculation for $\mathrm{Re} = U_\infty R/\nu = 200$ (R denotes the radius of the cylinder) and Rossby number $\mathrm{Ro} = WR/U_\infty = 2$. In this case the boundary conditions on S_∞ include the circulation term, as in [5, 6, 7]. When the flow becomes steady, the cylinder is suddenly stopped. If we apply in this case the widely used argument that the velocity over S_∞ will change when cylinder-induced vortical disturbances carried by the flow reach this boundary, we conclude that after a long time (comparable with the distance between S and S_∞), the presence of the vortex term in the asymptotic on S_∞ will be retained.

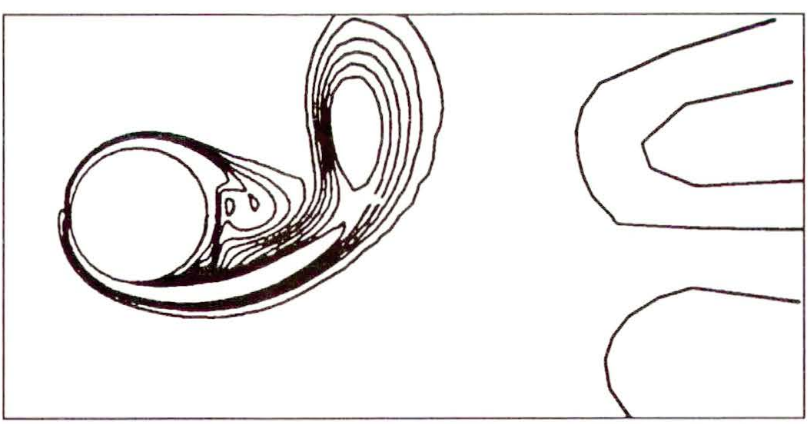
The equi-vorticity lines are shown in Fig. 1 for the solution to $N-S$ equations when the problem statement includes (i) no-slip boundary condition over



a) $t = 16$

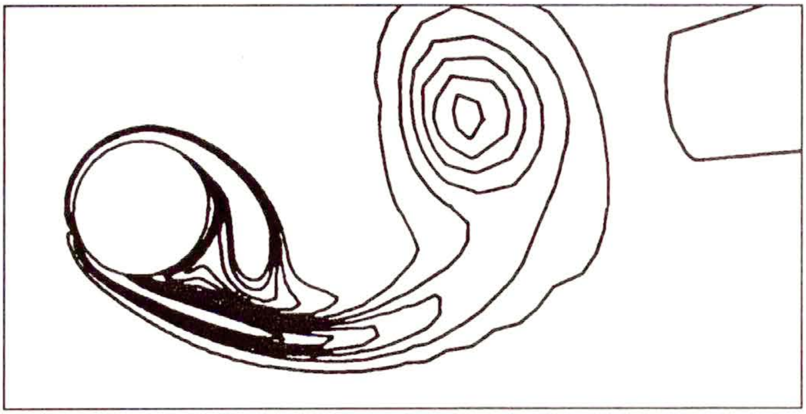


b) $t = 18$

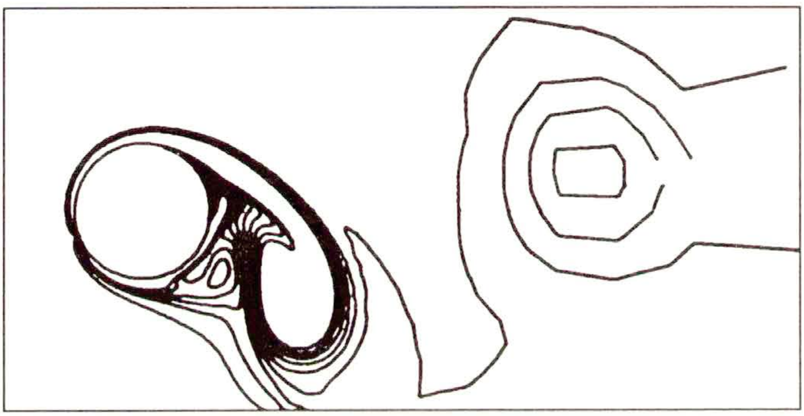


c) $t = 20$

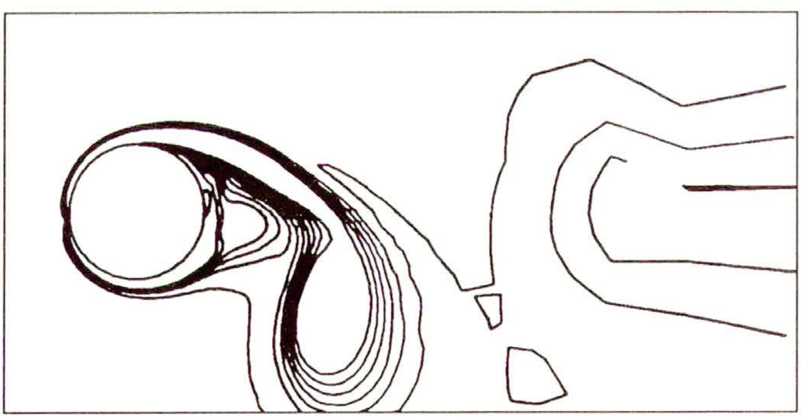
[Fig. 1 a, b, c]



d) $t = 22$



e) $t = 24$



f) $t = 25$

[Fig. 1 d, e, f]

Fig. 1.

S for immovable cylinder, and (ii) uniform flow with circulation term over S_∞. Reversed Kármán street is observed: a vortex A, leaving the upper side of the cylinder, gets down and is then found under the vortex B which has departed from the lower side of the cylinder; and simultaneously, all vortex street is carried downwards due to the flow spinning effect (or upwards, if the flow has been spun in the opposite direction). The first vortices of Golubev street [8] are obtained. It follows that a thrust is generated.

The computation domain size is limited to 20 radii of the cylinder. Therefore the vortex street development is computed over a time interval $\Delta t = 15$. In this case it is clear that changes over S_∞ will occur earlier than the wake will reach it. It is obvious that, after the initial vortex street reversal, some time is necessary for the Kármán street to be restored.

To study this phenomenon, the computation domain size should be expanded and the problem of proper boundary conditions for S_∞ should be solved. In view of technical problems, a more powerful computer than MICROVAX-2 is desirable.

3.2.

Detached separation in flow around a circular cylinder which performs angular oscillations about its axis in a free stream has been studied previously in [4]. The law of oscillations is as follows:

$$W = \frac{1}{2} A \sin(\omega(t - t_0)), \qquad \omega = 2\pi K, \qquad K = R/U_\infty T.$$

Figure 2 a presents the equi-vorticity lines at Re = 35. The oscillation amplitude $A = 45°$, reduced frequency $K = R/U_\infty T = 3$ (T is an oscillation period); Fig. 2 b presents the streamlines. One can see a symmetrical separation region that is separated from the cylinder by the circular layer in which the flow is essentially unsteady.

At Reynolds numbers as high as 200, the flow topology presented in Fig. 2 is conserved [4]. Effects of scheme parameters were studied by diminishing the mesh steps in both space and time.

A further study of the problem is concerned with the opportunity of flow stabilization of the previously developed separated flow. The unsteady flow with a Kármán street (Reynolds number Re = 200) past a circular cylinder was taken as an initial state. Attempts to attain flow stabilization were made with the help of angular oscillations of the cylinder about its axis with the reduced frequency $k = 3$ at $A = 45°$. Survey of the equi-vorticity lines in Fig. 3 raises the question about an intermediate separation, when detached separation with asymmetrical flow pattern alternates with the attached (conventional) separation. The vortex is detached from the cylinder surface by a liquid layer and the inflowing liquid particles do not reach the cylinder [4]. To compare, one can refer to Fig. 1 drawn

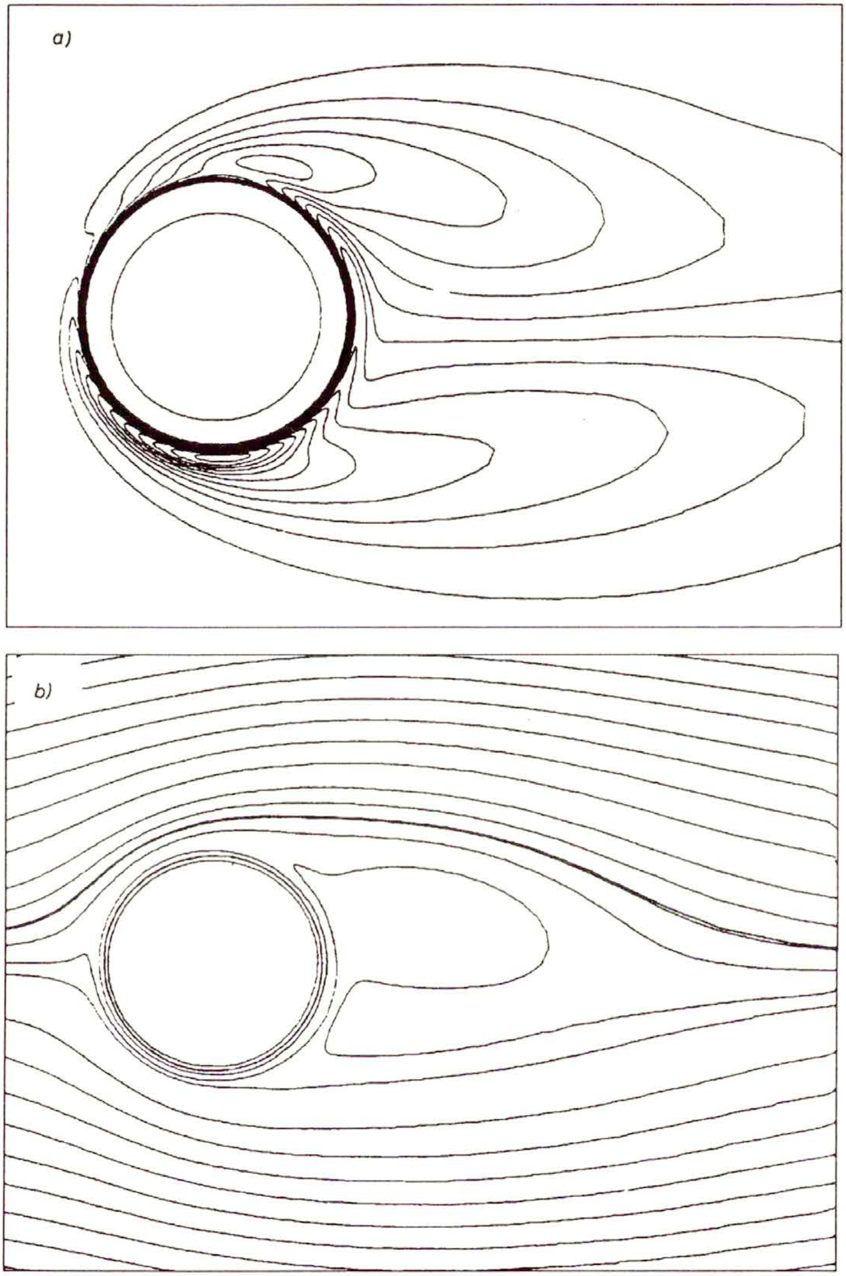
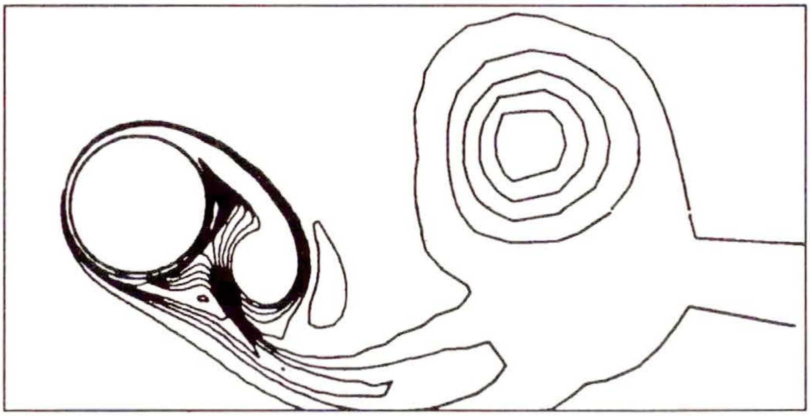
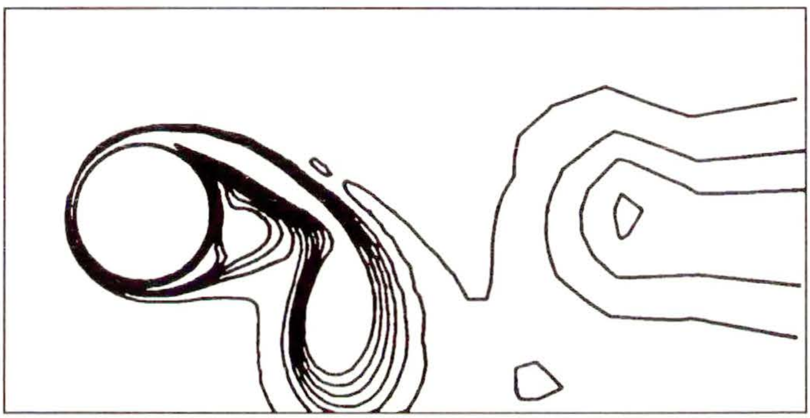


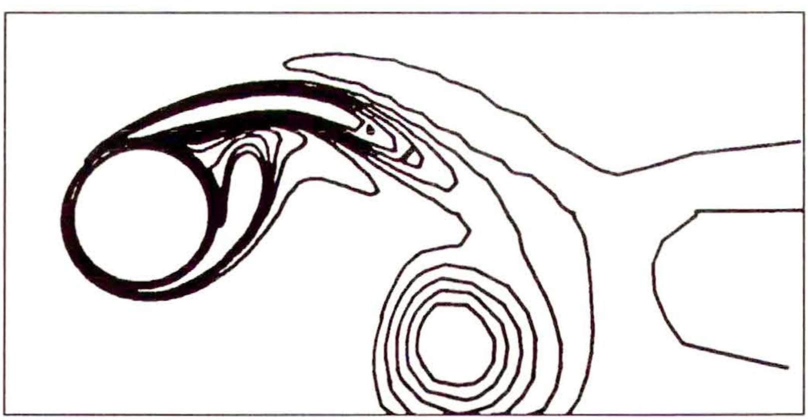
FIG. 2.



a) $t = 23$

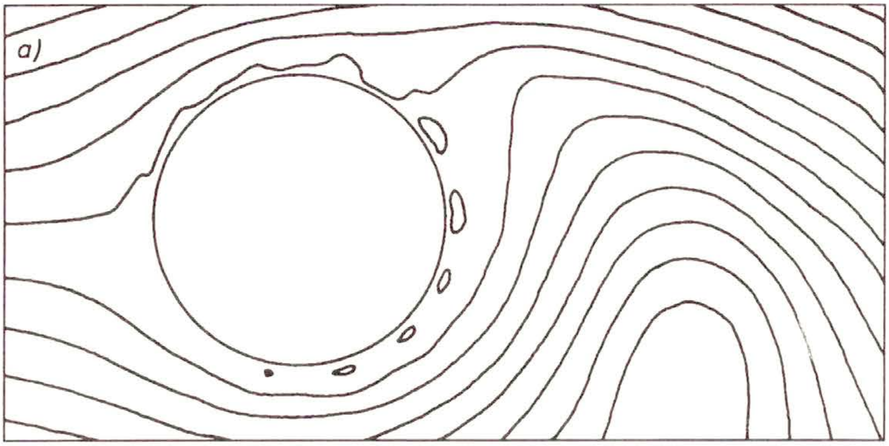


b) $t = 25$

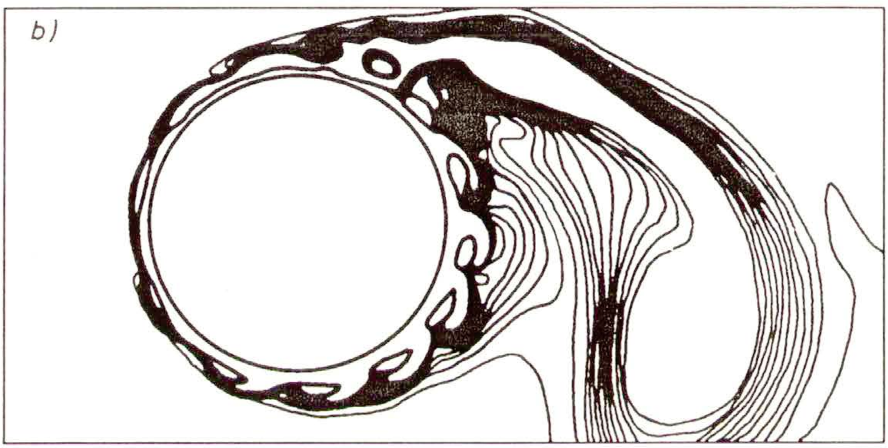


c) $t = 27$

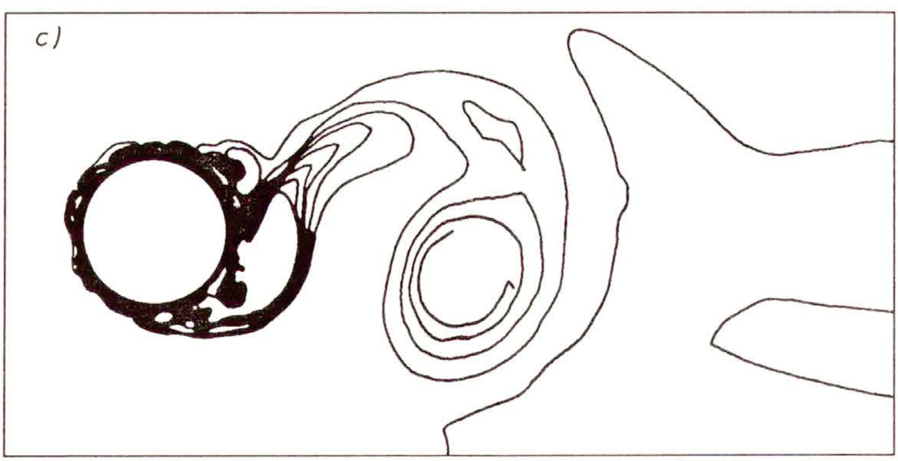
FIG. 3.



$t = 23$



$t = 23$



$t = 26.025$

[FIG. 4 a, b, c]

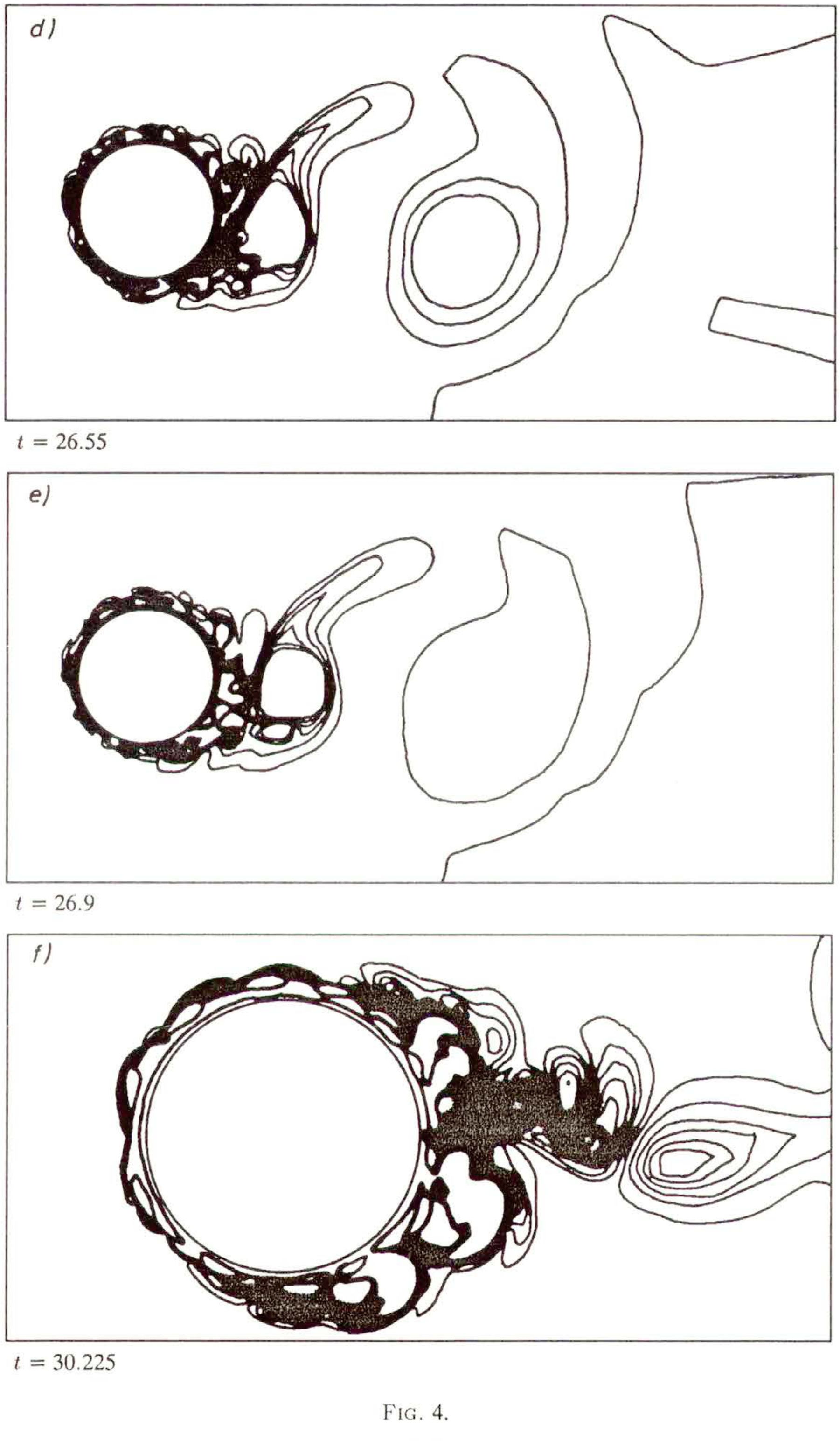


FIG. 4.

for a cylinder at the same Reynolds number, but without angular oscillations. No differences in generation and motion of vortices are seen. Differences in geometry of vortices and hence in their intensity are also not observed.

However, changing the scheme parameters results in an unexpected new type of flow presented in Fig. 4. In this case the calculations are stable and, with parameters of the finite-difference scheme being fixed, the results converge precisely to the revealed solution. Different solutions at different parameters of a finite-difference scheme means the lack of convergence in a strong mathematical sense. So some new aspects of the computational fluid dynamics theory must be developed.

Small vortices generation presented in Fig. 4 reduces the intensity of primarily separated vortices, which move away from the cylinder, and even eliminates the generation of large vortices that are known as a vortex street of the Kármán street type. A completely different topology is realized (Fig. 4).

Note that the time interval from $t = 23.0$ to $t = 30.225$ when such changes have taken place, is quite short and comparable with the specific period of vortex generation in the Kármán street.

At $\mathrm{Re} = 35$ small vortices are not generated, with any finite-difference scheme parameters. A considerable growth of errors in the region of the reversal wake flow is observed at a time step greater than a certain value. In such a way, at this Reynolds number the solution converges only to the unique flow pattern, which is identified as the detached separation, see Fig. 2.

The examples presented have raised the problem of estimation of adequacy of a numerical solution to physical reality.

4. Numerical experiment for a flow past an airfoil

Numerical experiment technique was designed for the problem of flow past an airfoil. Primary effect of specifying the circulation term for a velocity over S_∞ on the solution was studied. Flow past the 12% Zhukovsky airfoil with a finite trailing edge angle at $\mathrm{Re} = 10^4$ and angle of attack $\alpha = 5°$ was estimated. For the boundary condition (1.4) $D_x = D_y = 0$ was specified. There exists the range of values $\Gamma = (0 : -0.21)$ where the condition of pressure uniqueness over the trailing edge is satisfied [9, 10]. The pressure coefficient $C_p = (p - p_\infty)/\frac{1}{2}\varrho U_\infty^2$ is presented in Fig. 5, where a) $\Gamma = 0$; b) $\Gamma = -0.21$; c) $\Gamma = -0.40$. The vortex within the domain limited by S_∞ is placed rather arbitrarily. For example, when the centre of the vortex with intensity $\Gamma = -0.21$ lies on the positive $0X$ axis downstream the airfoil at $X_\gamma = 0.5$ or $X_\gamma = 2$, we obtain the coefficient C_p presented in Fig. 5 d or 5 e, respectively. Pressure coefficient C_p in both cases is the same and close to that occurring in the case with $X_\gamma = 0$ (Fig. 5 b).

When an asymptote of far field flow with two vortices is specified over S_∞, we conclude the following: if the second vortex is outside the domain bounded

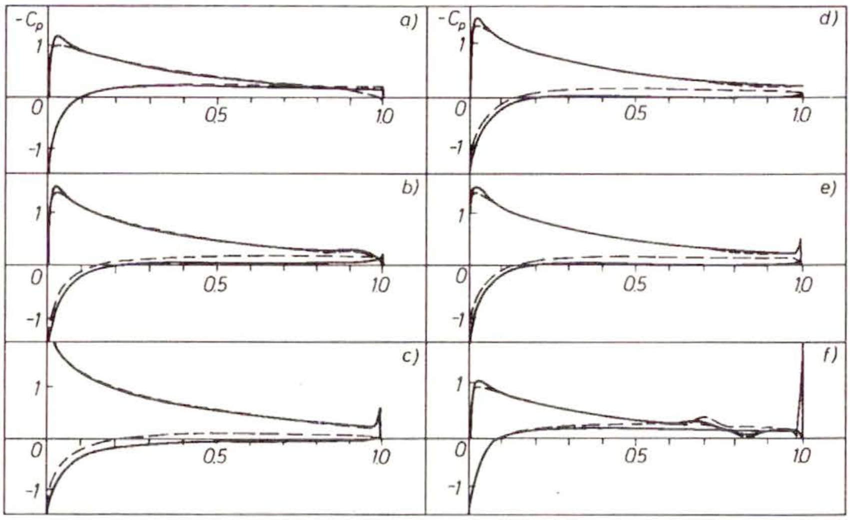


FIG. 5.

by S_∞, then it insignificantly affects the integral characteristics and C_p (Fig. 5 f, $X_\gamma = 12$, $\Gamma = -0.21$). In this case the first vortex can be located inside the computational region (bounded by S_∞) arbitrarily, retaining the integral characteristics unchanged. Such a dependence of Γ on S_∞ and C_p emphasizes the connection of this asymptotics with the lifting capability of the airfoil and makes it very suitable for modelling these phenomena.

The dipole term effect was studied. For example, Fig. 6 presents the streamlines and the equi-vorticity lines (Fig. 6 e) in the vicinity of the 1/4 chord of the 12% Zhukovsky airfoil; Reynolds number Re = 10^4, angle of attack $\alpha = 7.25°$. Parameters in (1.4) are as follows: $D_x = D_y = \Gamma = 0$ (Fig. 6 a-f) and $D_x = -4$, $D_y = 4$, $\Gamma = -0.20$ (Fig. 6 g-l). Development of coherent vortex structures in the vicinity of the trailing edge was obtained. This study is discussed in detail in [2].

The next step of the investigation is to study the flow with an increasing Reynolds number. Figure 7 presents the streamlines (7 a–7 f) and equi-vorticity lines (7 g–7 l) for the 12% Zhukovsky airfoil at $\alpha = 5°$, $D_x = D_y = 0$ and $\Gamma = -0.21$, when the Reynolds numbers are the following: a), g) Re = 1.5×10^4; b), h) Re = 2×10^4; c) i) Re = 2.5×10^4; d), j) Re = 3×10^4; e), k) Re = 3.5×10^4; f), l) Re = 3.75×10^4.

Considerable development of separation over the leeward side of the airfoil is observed. Reynolds number of 37500 is the highest value at which the computation convergence in the framework of laminar flow is obtained (at $\alpha = 5°$). In this experiment the computations are performed with successively increasing Re and the flow for previous Re is the initial condition for computations of flow at a next Re. The fact that distributions C_p over S (Fig. 8 a for Re = 15000), obtained by integrating (along the different paths but with the same method of integration)

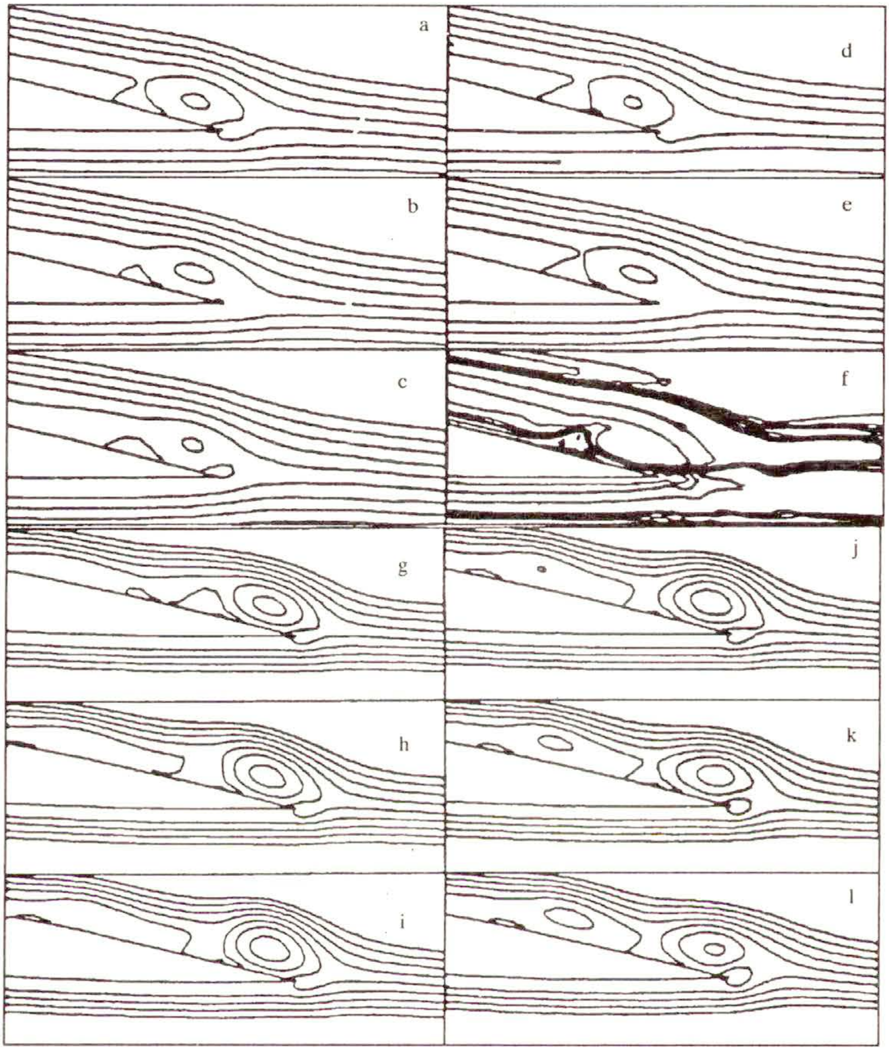


FIG. 6.

the equation of motion, do not coincide, indicates that the solution is not quite correct [11]. The study in [12] for a circular cylinder rotating in viscous flow shows that asymptotic condition for far field flow may be properly stated so as to eliminate the pressure nonuniqueness. This is one of the goals of our study of the viscous flow around a body with boundary conditions (1.3) and (1.4), where the number of terms in the asymptotic expansion may be increased.

The effect of scheme factors and parameters of the mathematical model on the problem solution is studied. It has been found that, within the investigated angles-of-attack and Reynolds numbers ranges, the flow turbulization cannot yet

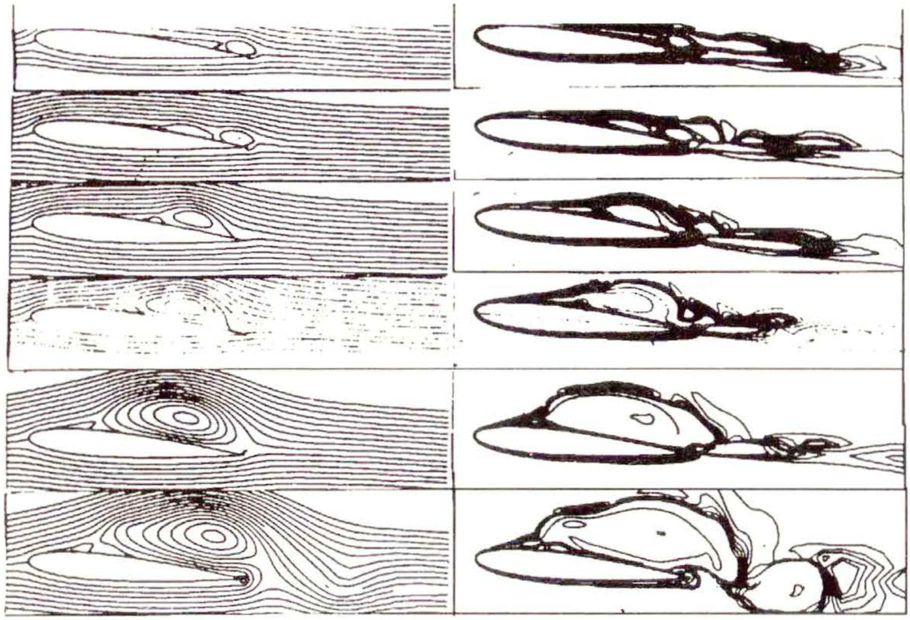


FIG. 7.

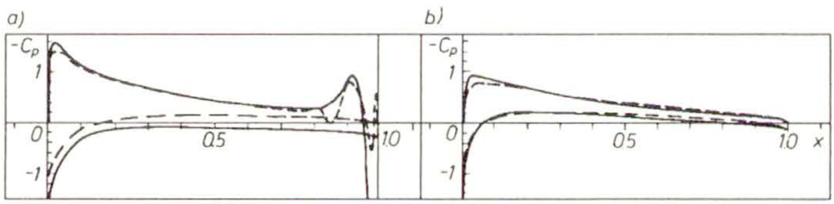


FIG. 8.

reduce the separation region, presented in Fig. 7. The Boldwin-Lomax model was used.

The search of explanations of the causes of the presented topology of flow around an airfoil results in construction of a flow model including a laminar wake [13]. In this case the theory suggests the solution in which the flow in the wake conserves (and convects) momentum losses of two types. The first one defines the drag acting on an airfoil due to viscous friction. The second one corresponds to lift variation. Those losses must occur within the near-wall layer and then be convected by the wake. In the framework of this theory, construction of distributed sources and sinks is necessary.

We will synthesize the above-mentioned theory [13] with the Lighthill construction [14], where the near-wall layer momentum losses are simulated by distributed sources and sinks with their extension into the wake. The latter combines the suggestion of [13] and the model of [14].

However, practical application of the part of this theory which deals with specification of distributed wake singularities is a difficult problem. It has been found that application of this model doesn't resolve yet all the difficulties which we have in numerical solution for the flow presented in Fig. 7. Thus, generation of a developed separated flow is connected with construction of a general asymptotic behaviour of the solution for flow around an airfoil.

A detailed study by including additional sources, dipoles and vortex terms in the asymptotic expansion for velocity over S_∞ was conducted. It is obtained that at least one of possible representations includes the additional asymptotics of two vortices with opposite signs of circulation and their centres are located inside the airfoil.

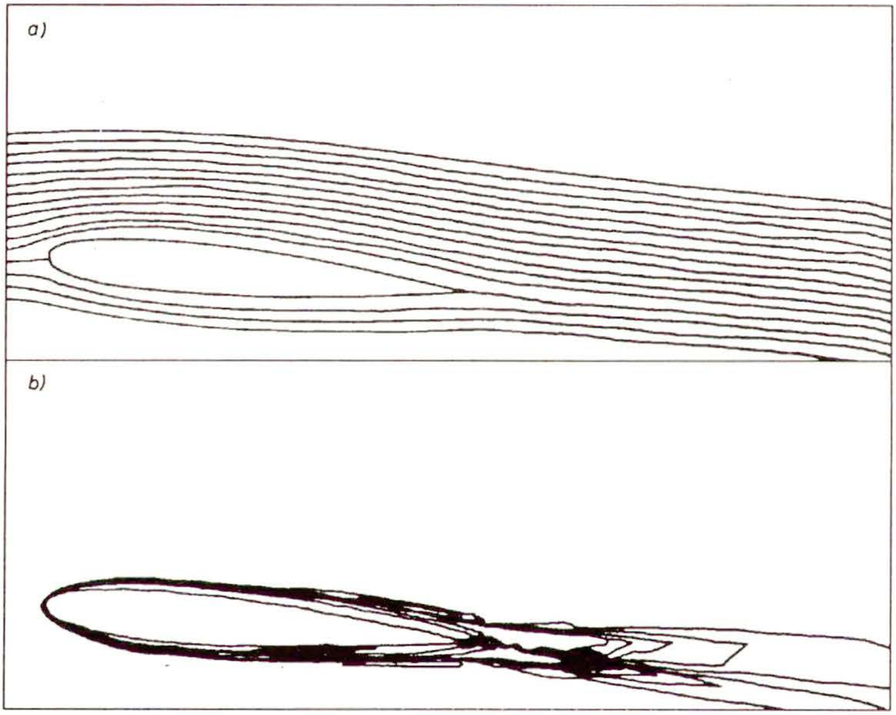


FIG. 9.

Figure 9 a, b present the topology for Re = 1.5×10^4, $\alpha = 5°$, and Fig. 8 b, the pressure distribution when the mentioned singularities are added to the initial state shown in Fig. 7 a, g. As we see, the flow obtained is similar to that studied before at Re = 10^4 and $\alpha = 5°$ and is in agreement with our knowledge of the full-scale experiment. Note that the separation region disappears, what is especially clear in comparison with flow in Fig. 7. The agreement between C_p and C_p^0 obtained by different ways of integration [11] indicates mathematical accuracy of the solution obtained.

After the same construction, the elimination of massive separation in the flow at $\mathrm{Re} = 3.5 \times 10^4$ is obtained too. It should be noted that the flow topology singularities that were presented in Fig. 7, are still of interest to investigators because similar vortex structures are realized obviously at high angles of attack.

At last, Fig. 10 shows a distribution of circulation γ along a coordinate line $\xi = \mathrm{const}$ as a function of the coordinate x at which this line intersects the positive $0x$ axis; these data indicate that, in accordance with the model of [13], solutions can be realized when value of γ is bounded by Γ given at S_∞ from below, see Fig. 10 a ($\alpha = 5°$, $\mathrm{Re} = 1.5 \times 10^4$, $\Gamma = -0.21$, $D_x = D_y = 0$) and flows where $|\gamma| < \Gamma|$ (at the same Re, D_x, D_y, but the above mentioned two vortices with opposite signs of circulation are included in (1.4)). The latter means that in the airfoil wall region there occur momentum losses resulting in a decrease of C_y in comparison with that C_y defined by Γ in the framework of potential flow theory for an ideal fluid.

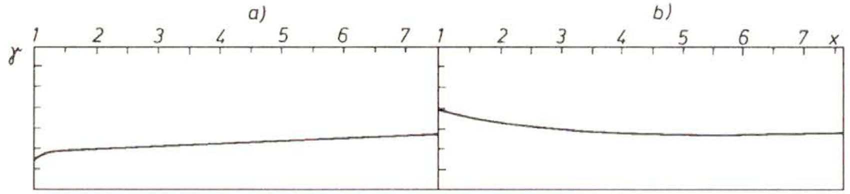


FIG. 10.

It can be noted briefly that inclusion of additional vortices into the asymptotic for S_∞ affects significantly the aerodynamic moment that is induced by the potential part of the solution. This follows from the Chaplygin – Blasius theorem for an ideal fluid. The results obtained for an ideal fluid are not related directly to viscous flows, but mechanical meaning of inclusion of the mentioned singularities – to change the aerodynamic moment – is certainly the same.

The examples presented have shown that the application of the technology of the numerical experiment allows us not only to reveal disadvantages of the problem statement but also to eliminate the difficulties. The results concerning the flow past the circular cylinder present new problems that couldn't be studied earlier in the framework of numerical experiments.

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CENTRAL AERO-HYDRODYNAMIC INSTITUTE, MOSCOW, RUSSIA.

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Wave propagation in anisotropic layered media

M. ROMEO (GENOVA)

THE PROPAGATION of time-harmonic waves in a continuously stratified anisotropic, viscoelastic layer bounded by two homogeneous anisotropic solid half-spaces, is studied analytically. A plane wave is assumed to impinge on the boundary of the layer, and the resulting field, inside and outside of the layer, is described according to the causality principle and formal wave-splitting. Reflection and transmission coefficients are derived for arbitrary angle of incidence, together with a formal expression of the wave field within the layer. A local reflectivity is defined as a function of the depth and used to obtain up and down-going modes in the layer. Reduction of the model to particular material symmetries allows for scalar fields whose properties generalize known results concerning the isotropic media. Numerical results are given to illustrate the method in the scalar case.

1. Introduction

WAVE PROPAGATION in stratified layers has been extensively investigated in connection with a wide range of constitutive and geometric models which are mainly motivated by geophysical applications. Beside the frequent approaches based on homogeneous waves in elastic isotropic materials (see for ex. [1, 2]), inhomogeneous waves have also been exploited in order to account for dissipative effects [3, 4], and anisotropic materials have been considered in the multilayered case [5, 6]. However, in these last works each layer is assumed to be homogeneous, thus allowing for an effective use of the propagator matrix.

The aim of the present paper is to investigate wave propagation across a continuously stratified (and hence inhomogeneous) viscoelastic solid layer with arbitrary material symmetry. A time-harmonic inhomogeneous plane wave, coming from a homogeneous anisotropic half-space, is assumed to impinge on the outset of the layer, giving rise to a reflected field. A transmitted wave field propagates along the edge of the layer within a second homogeneous anisotropic half-space. For arbitrary angle of incidence, three reflected modes and three transmitted modes are, in general, possible within the homogeneous half-spaces. Although forward and backward plane waves are allowed in the first solid half-space, the causality reasons imply that only forward waves propagate in the second solid half-space. Transmitted modes are then exploited to infer a formal wave-splitting within the layer, where the wave field is described by three independent components whose amplitudes and polarizations are functions of the depth. Continuity requirements imposed on the displacement and on the traction are used to obtain the reflection and the transmission matrices, and to get boundary conditions in order to integrate the differential equation for the displacement. A wave-splitting is then introduced for each wave component in the layer. To this end, a reflectivity matrix is defined which satisfies suitable conditions at the boundaries. As a result, the wave field within the layer is given as the superposition of three pairs of up and down-going modes. A notable simplification of the present model is achieved by considering special material symmetries. In Sec. 6 of this paper it is shown that some crystal systems such as orthorhombic, tetragonal, cubic and hexagonal systems allow for a decoupling of the governing differential equation, which splits into a scalar equation for horizontally polarized waves and a vector equation for vertically polarized waves. The first one is analyzed in detail to stress the comparison with the known results on isotropic layers [7]. In particular, the reflectivity is shown to satisfy a Riccati equation as occurs in scalar theories of wave propagation in isotropic layered media [8, 9]. The scalar problem for horizontally polarized waves is also solved numerically to explicitly obtain the split wave-field. Two examples are considered of the dependence of the constitutive properties on the depth.

2. Stratified anisotropic layers

We are here concerned with an inhomogeneous anisotropic solid layer $\mathcal{L}$ bounded by two plane parallel surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. A Cartesian coordinate system is chosen in such a way that $\mathcal{S}_1$ and $\mathcal{S}_2$ correspond to the planes $z = 0$ and $z = d$, where d is the thickness of the layer. Inhomogeneity in $\mathcal{L}$ is assumed to consist of a continuously stratified structure along the z direction. Two homogeneous anisotropic solid media $\mathcal{B}_1$ and $\mathcal{B}_2$ occupy, respectively, the half-spaces $z < 0$ and $z > d$. All the media $\mathcal{B}_1$, $\mathcal{L}$, $\mathcal{B}_2$ are supposed to behave as viscoelastic materials where the Cauchy stress $\mathbf{T}$ has a linear dependence on the strain history and an arbitrary dependence on the space coordinates. More precisely, denoting by $\widehat{\mathbf{e}} = \mathrm{Sym}(\nabla\widehat{\mathbf{u}})$ the infinitesimal strain tensor, $\widehat{\mathbf{u}}$ being the displacement, we assume (cf. [10])

$$\mathbf{T}(\mathbf{x},t) = \widetilde{\mathbf{G}}(z,0)\widehat{\mathbf{e}}(\mathbf{x},t) + \int_0^\infty \widetilde{\mathbf{G}}_s(z,s)\,\widehat{\mathbf{e}}(\mathbf{x},t-s)\,ds, \tag{2.1}$$

where $\widetilde{\mathbf{G}} : \mathbf{R}\times\mathbf{R}^+ \to \mathrm{Lin(Sym)}$ is the relaxation function and $\widetilde{\mathbf{G}}_s = \partial\widetilde{\mathbf{G}}/\partial s$. It is convenient, in elastic theories of anisotropic solids, to adopt a double-indices notation for the relaxation function (see [11]), introducing the indicial correspondence $(ij) \to \alpha$ $(i,j = 1,2,3;\ \alpha = 1,...,6)$ given by $(11) \to 1$, $(22) \to 2$, $(33) \to 3$, $(23) \to 4$, $(13) \to 5$, $(12) \to 6$. The corresponding six-dimensional relaxation matrix $\Gamma(z,s)$ is assumed to be non-singular for any $z \in \mathbf{R}$ and $s \in \mathbf{R}^+$. In the following we shall assume that the displacement $\widehat{\mathbf{u}}$ has a time-harmonic dependence

$$\widehat{\mathbf{u}}(\mathbf{x},t) = \overline{\mathbf{u}}(\mathbf{x})\exp(-i\omega t),$$

with $\omega \in \mathbf{R}^{++}$. Hence, assuming $\widehat{\mathbf{e}}(\mathbf{x},-\infty) = 0$, integration by parts reduces Eq. (2.1) to

$$\overline{\mathbf{T}}(\mathbf{x},\omega) = \mathbf{G}(z,\omega)\,\overline{\mathbf{e}}(\mathbf{x},\omega), \tag{2.2}$$

where $\overline{\mathbf{T}} = \widehat{\mathbf{T}} \exp(i\omega t)$ and $\mathbf{G}(z,\omega) = -i\omega \int_0^\infty \widetilde{\mathbf{G}}(z,s) \exp(i\omega s)\, ds$. Accounting for the model we are dealing with, the constitutive parameters take the form

$$\Gamma_{\alpha\beta} = \begin{cases} \Gamma^{(1)}_{\alpha\beta}(\omega) & \text{for } z < 0, \\ \Gamma_{\alpha\beta}(z,\omega) & \text{for } z \in [0,d], \\ \Gamma^{(2)}_{\alpha\beta}(\omega) & \text{for } z > d, \end{cases} \tag{2.3}$$

where $\Gamma_{\alpha\beta} = -i\omega \int_0^\infty \widetilde{\Gamma}_{\alpha\beta} \exp(i\omega s)\, ds$. We also assume that $\Gamma_{\alpha\beta}$ are continuous throughout z and sufficiently smooth in $[0,d]$. Additional restrictions hold if particular material symmetries are allowed for the solid media. A classification of such symmetries can be achieved by the determination of the planes of reflective symmetry (see [12]) and of the consequent non-vanishing elastic constants. For the future purposes we observe that most of the crystal systems (such as orthorhombic, tetragonal, cubic and hexagonal) can be characterized by the nine non-vanishing parameters

$$\begin{aligned} &\Gamma_{\alpha\beta} \quad \text{with} \quad \alpha,\beta = 1,2,3, \\ &\Gamma_{\gamma\gamma} \quad \text{with} \quad \gamma = 4,5,6. \end{aligned} \tag{2.4}$$

These, in turn, may reduce to a lower number of independent entries for particular crystal classes (see for ex. [13]).

3. The governing differential equation

Accounting for layer's inhomogeneities along the z-axis, we assume that the displacement $\overline{\mathbf{u}}(\mathbf{x})$ has a plane-wave-like dependence on x and y, that is

$$\overline{\mathbf{u}}(x,y,z) = \mathbf{u}(z) \exp[i(k_x x + k_y y)], \tag{3.1}$$

where k_x and k_y are complex-valued wave-numbers and where $\mathbf{u} \in \mathbf{C}^3$. Avoiding inessential formal difficulties, we can choose the x-axis in such a way that the real part of k_x vanishes. This can be accomplished by a suitable orthogonal transformation applied to the constitutive tensor $\mathbf{G}$ (see e.g. [5]). We also neglect the imaginary part of k_x. This amounts to assume that the incident wave-number bivector lies on the (y,z) plane. Hence, putting $k_y = k$, Eq. (3.1) takes the form

$$\overline{\mathbf{u}}(y,z) = \mathbf{u}(z) \exp(iky), \tag{3.2}$$

which, according to the Snell's law, holds at any point in $\mathcal{B}_1$, $\mathcal{L}$, and $\mathcal{B}_2$. In view of the time-harmonic dependence, the equation of motion for $\overline{\mathbf{u}}$ reads

$$\nabla \cdot \overline{\mathbf{T}} + \varrho\omega^2 \overline{\mathbf{u}} = 0,$$

where ϱ is the mass density. By exploiting Eqs. (2.2) and (3.2) and accounting for the description in terms of Γ, we arrive at

$$(3.3) \qquad [\mathbf{C} + \varrho\omega^2\mathbf{I}]\mathbf{u} = 0,$$

where $\mathbf{C}$ is a linear symmetric differential operator whose entries are expressed by

$$(3.4) \qquad \begin{aligned} C_{11} &= \partial_z(\Gamma_{55}\partial_z) + ik(\Gamma_{56,z} + 2\Gamma_{56}\partial_z) - k^2\Gamma_{66}, \\ C_{12} &= \partial_z(\Gamma_{45}\partial_z) + ik(\Gamma_{25,z} + 2\Gamma_{46}\partial_z) - k^2\Gamma_{26}, \\ C_{13} &= \partial_z(\Gamma_{35}\partial_z) + ik(\Gamma_{45,z} + 2\Gamma_{36}\partial_z) - k^2\Gamma_{46}, \\ C_{22} &= \partial_z(\Gamma_{44}\partial_z) + ik(\Gamma_{24,z} + 2\Gamma_{24}\partial_z) - k^2\Gamma_{22}, \\ C_{23} &= \partial_z(\Gamma_{34}\partial_z) + ik(\Gamma_{44,z} + 2\Gamma_{23}\partial_z) - k^2\Gamma_{24}, \\ C_{33} &= \partial_z(\Gamma_{33}\partial_z) + ik(\Gamma_{34,z} + 2\Gamma_{34}\partial_z) - k^2\Gamma_{44}. \end{aligned}$$

Equation (3.3) is a second order homogeneous linear differential equation for $\mathbf{u}$. More explicitly, it can be written as

$$(3.5) \qquad \mathbf{L}\mathbf{u}'' + (\mathbf{L}' + 2ik\mathbf{M}_1)\mathbf{u}' + (ik\mathbf{M}_2' - k^2\mathbf{Q} + \varrho\omega^2\mathbf{I})\mathbf{u} = 0,$$

where prime denotes differentiation with respect to z, and where

$$\mathbf{L} = \begin{pmatrix} \Gamma_{55} & \Gamma_{45} & \Gamma_{35} \\ \Gamma_{45} & \Gamma_{44} & \Gamma_{34} \\ \Gamma_{35} & \Gamma_{34} & \Gamma_{33} \end{pmatrix}, \qquad \mathbf{M}_1 = \begin{pmatrix} \Gamma_{56} & \Gamma_{46} & \Gamma_{36} \\ \Gamma_{46} & \Gamma_{24} & \Gamma_{23} \\ \Gamma_{36} & \Gamma_{23} & \Gamma_{34} \end{pmatrix},$$

$$\mathbf{M}_2 = \begin{pmatrix} \Gamma_{56} & \Gamma_{25} & \Gamma_{45} \\ \Gamma_{25} & \Gamma_{24} & \Gamma_{44} \\ \Gamma_{45} & \Gamma_{44} & \Gamma_{34} \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} \Gamma_{66} & \Gamma_{26} & \Gamma_{46} \\ \Gamma_{26} & \Gamma_{22} & \Gamma_{24} \\ \Gamma_{46} & \Gamma_{24} & \Gamma_{44} \end{pmatrix}.$$

Since Γ is non-singular, the operator $\mathbf{L}(z)$ is invertible for any $z \in \mathbf{R}$, hence Eq. (3.5) may be rewritten in the following form

$$(3.6) \qquad \mathbf{u}'' + \mathbf{A}\mathbf{u}' + \mathbf{B}\mathbf{u} = 0,$$

where

$$(3.7) \qquad \mathbf{A} = \mathbf{L}^{-1}(\mathbf{L}' + 2ik\mathbf{M}_1), \qquad \mathbf{B} = \mathbf{L}^{-1}(ik\mathbf{M}_2' - k^2\mathbf{Q} + \varrho\omega^2\mathbf{I}).$$

Before developing a procedure to obtain a representation of the displacement within the layer $\mathcal{L}$, we look for solutions of Eq. (3.6) in the homogeneous regions. In $\mathcal{B}_1$ and $\mathcal{B}_2$ the tensor Γ is taken to be independent of z, whence

$$(3.8) \qquad \begin{aligned} \mathbf{A}^{(1,2)} &= 2ik(\mathbf{L}^{(1,2)})^{-1}\mathbf{M}_1^{(1,2)}, \\ \mathbf{B}^{(1,2)} &= (\mathbf{L}^{(1,2)})^{-1}[\varrho\omega^2\mathbf{I} - k^2\mathbf{Q}^{(1,2)}], \end{aligned}$$

with obvious meaning of the superscripts 1, 2. In view of a formal splitting of the wave field into elementary modes, we first look for non-dissipative, normal incident waves, such that $k = 0$. Owing to (3.8) we have $\mathbf{A}^{(1,2)} = 0$, and the solutions of Eq. (3.6) take the form

$$\mathbf{u} = \sum_{h=1}^{3}[\mathbf{u}_{h+}\exp(i\zeta_h z) + \mathbf{u}_{h-}\exp(-i\zeta_h z)], \tag{3.9}$$

where ζ_h $(h = 1, 2, 3)$ are those solutions of the bi-cubic secular equation

$$\det[\varrho\omega^2\mathbf{L}^{-1} - \zeta^2\mathbf{I}] = 0, \tag{3.10}$$

which have positive real parts. From Eq. (3.9) the displacement of normal incident waves in homogeneous regions consists of three pairs of up and down-going modes. For an arbitrary incidence and possible dissipation ($k \neq 0$), Eqs. (3.9) and (3.10) must be replaced by

$$\mathbf{u} = \sum_{h=1}^{6}\mathbf{u}_h\exp(i\zeta_h z), \tag{3.11}$$

$$\det[\varrho\omega^2\mathbf{I} - k^2\mathbf{Q} - 2k\zeta\mathbf{M}_1 - \zeta^2\mathbf{L}] = 0. \tag{3.12}$$

The left-hand side of Eq. (3.12) is a sixth-degree polynomial with constant, complex-valued coefficients, parametrized by k. Its zeroes ζ_h $(h = 1, ..., 6)$ appear in the representation (3.11). If the solutions $\pm\zeta_h, (h = 1, 2, 3)$ of Eq. (3.10) are distinct, there will be a neighbourhood C_k of $k = 0$ in the complex k-plane where each solution of Eq. (3.12) has a one-to-one correspondence with each value $\pm\zeta_h$ and keeps its own sign. Then, assuming $k \in C_k$, solutions of Eq. (3.12) may be represented by the set

$$\{\zeta_{1+}, \zeta_{1-}, \zeta_{2+}, \zeta_{2-}, \zeta_{3+}, \zeta_{3-}\},$$

and Eq. (3.11) becomes

$$\mathbf{u} = \sum_{h=1}^{3}[\mathbf{u}_{h+}\exp(i\zeta_{h+}z) + \mathbf{u}_{h-}\exp(i\zeta_{h-}z)], \tag{3.13}$$

thus yelding three couples of "up" and "down-going" modes. In the following we shall assume that the eigenvalue problem (3.12) and the corresponding eigenvector problem have been solved in $\mathcal{B}_1$ and in $\mathcal{B}_2$ so that the constant amplitudes $\mathbf{u}_{h\pm}$ are known. In view of further developments, we represent these vectors in the form

$$\mathbf{u}_{h\pm} = \alpha_h^{\pm}\begin{pmatrix} 1 \\ p_{h\pm} \\ q_{h\pm} \end{pmatrix}. \tag{3.14}$$

4. Formal wave-splitting and the field within the layer

A plane harmonic wave, coming from the homogeneous region $\mathcal{B}_1$, is supposed to impinge on the boundary $\mathcal{S}_1$ of the inhomogeneous layer ($z = 0$). Owing to the superposition principle, we can restrict our attention to one of the possible up-going modes, labelled by $l+$, ($l = 1, 2, 3$) and study the reflected and transmitted modes at the respective surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. Each impinging mode allows for a superposition of all the possible down-going reflected modes in $\mathcal{B}_1$, and all the possible up-going transmitted modes in $\mathcal{B}_2$, according to (3.13). The causality principle implies that no down-going modes arise in $\mathcal{B}_2$, that is $\mathbf{u}_{h-}^{(2)} = 0$ for $h = 1, 2, 3$. Hence the wave fields in $\mathcal{B}_1$ and $\mathcal{B}_2$ can be expressed, respectively, as

$$\text{(4.1)} \qquad \mathbf{u}^{(1)} = \begin{pmatrix} 1 \\ p_{l+}^{(1)} \\ q_{l+}^{(1)} \end{pmatrix} \exp(i\zeta_{l+}^{(1)} z) + \sum_{h=1}^{3} V_{lh} \begin{pmatrix} 1 \\ p_{h-}^{(1)} \\ q_{h-}^{(1)} \end{pmatrix} \exp(i\zeta_{h-}^{(1)} z) \qquad \text{for} \quad z < 0,$$

$$\text{(4.2)} \qquad \mathbf{u}^{(2)} = \sum_{h=1}^{3} W_{lh} \begin{pmatrix} 1 \\ p_{h+}^{(2)} \\ q_{h+}^{(2)} \end{pmatrix} \exp(i\zeta_{h+}^{(2)} z) \qquad \text{for} \quad z > d,$$

for any impinging wave ($l = 1, 2, 3$). Equations (4.1) and (4.2) can also be viewed as a definition of the complex-valued reflection and transmission coefficients V_{lh} and W_{lh}. Compatibly with the causality principle, each component mode of the transmitted field (4.2) may be thought of as being originated by a corresponding field within the layer. Specifically, we decompose the field in $\mathcal{L}$ as

$$\text{(4.3)} \qquad \mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$

and impose continuity requirements on $\mathcal{S}_2$, pertinent to each component separately. To this end we observe that, in view of (2.2), the traction $\mathbf{t} = \mathbf{T}\mathbf{e}_3$ ($\mathbf{e}_3$ being the unit vector along the z-direction) is given by

$$\text{(4.4)} \qquad \mathbf{t} = ik\mathbf{P}\mathbf{u} + \mathbf{L}\mathbf{u}',$$

where

$$\mathbf{P} = \begin{pmatrix} \Gamma_{56} & \Gamma_{25} & \Gamma_{45} \\ \Gamma_{46} & \Gamma_{24} & \Gamma_{44} \\ \Gamma_{36} & \Gamma_{23} & \Gamma_{34} \end{pmatrix},$$

and hence

$$\text{(4.5)} \qquad \mathbf{u}' = \mathbf{L}^{-1}(\mathbf{t} - ik\mathbf{P}\mathbf{u}).$$

According to the regularity conditions on Γ at the boundaries, the continuity of the displacement and of the traction across $\mathcal{S}_1$ and $\mathcal{S}_2$ implies continuity of the derivative of $\mathbf{u}$ at layer's boundaries. As a consequence, from (4.2) and (4.3) we have, at $z = d$,

$$\mathbf{u}_j(d) = W_{lj} \begin{pmatrix} 1 \\ p_{j+}^{(2)} \\ q_{j+}^{(2)} \end{pmatrix} \exp(i\zeta_{j+}^{(2)} d) \qquad (j = 1, 2, 3), \tag{4.6}$$

$$\mathbf{u}'_j(d) = i\zeta_{j+}^{(2)} W_{lj} \begin{pmatrix} 1 \\ p_{j+}^{(2)} \\ q_{j+}^{(2)} \end{pmatrix} \exp(i\zeta_{j+}^{(2)} d) \qquad (j = 1, 2, 3), \tag{4.7}$$

for any $l = 1, 2, 3$. Now we introduce a triad of second-rank matrices $\mathbf{N}^{[j]}$ ($j = 1, 2, 3$) such that

$$\mathbf{u}'_j = i\mathbf{N}^{[j]}\mathbf{u}_j \qquad (j = 1, 2, 3). \tag{4.8}$$

Substituting this into the governing differential equation (3.6) we obtain the following first-order Riccati-type differential equations for the matrices $\mathbf{N}^{[j]}$

$$(\mathbf{N}^{[j]})' = i\mathbf{B} - \mathbf{A}\mathbf{N}^{[j]} - i\mathbf{N}^{[j]}\mathbf{N}^{[j]}. \tag{4.9}$$

Boundary conditions, in order to integrate (4.9), may be obtained from (4.6)–(4.8) as

$$\mathbf{N}^{[j]}(d) = \zeta_{j+}^{(2)}\mathbf{I} \qquad (j = 1, 2, 3). \tag{4.10}$$

According to (3.14) we assume

$$\mathbf{u}_j(z) = \alpha_j(z) \begin{pmatrix} 1 \\ p_j(z) \\ q_j(z) \end{pmatrix} \qquad (j = 1, 2, 3), \tag{4.11}$$

where $\alpha_j(z)$ are scalar, complex-valued amplitudes and $p_j(z)$, $q_j(z)$ characterize the polarization of the field. Substitution of (4.11) into (4.8) yields a first-order differential equation for a two-dimensional polarization vector, and the expression of the scalar amplitudes in terms of the entries of the matrices $\mathbf{N}^{[j]}$. Explicitly

$$\begin{pmatrix} p_j \\ q_j \end{pmatrix}' = i \begin{pmatrix} N_{21}^{[j]} \\ N_{31}^{[j]} \end{pmatrix} + i \begin{pmatrix} N_{22}^{[j]} - N_{11}^{[j]} & N_{23}^{[j]} \\ N_{32}^{[j]} & N_{33}^{[j]} - N_{11}^{[j]} \end{pmatrix} \begin{pmatrix} p_j \\ q_j \end{pmatrix} - i \begin{pmatrix} p_j \\ q_j \end{pmatrix} \left(N_{12}^{[j]},\ N_{13}^{[j]} \right) \begin{pmatrix} p_j \\ q_j \end{pmatrix}, \tag{4.12}$$

$$(4.13)\qquad \alpha_j(z) = \alpha_j(0)\exp\left[i\int_0^z \left(N_{11}^{[j]} + N_{12}^{[j]}p_j + N_{13}^{[j]}q_j\right)d\tau\right],$$

with $j = 1,2,3$. Equation (4.12) has the form of a Riccati equation. Boundary conditions are obtained from (4.6) and (4.11) as

$$(4.14)\qquad \begin{pmatrix} p_j(d) \\ q_j(d) \end{pmatrix} = \begin{pmatrix} p_{j+}^{(2)} \\ q_{j+}^{(2)} \end{pmatrix}.$$

Consequently, integration of equations (4.9) and (4.12), together with Eq. (4.13) allows us to obtain the field in the layer. In order to complete the picture, we have to determine the constants of integration $\alpha_j(0)$ in Eq. (4.13). This can be performed by imposing the continuity of **u** and **t** at the surface $\mathcal{S}_1$ ($z = 0$). As a result, we also obtain the reflection and transmission matrices V_{lh} and W_{lh}, ($h, l = 1,2,3$). Just like the previous conditions at $\mathcal{S}_2$, we require the continuity of **u** and **u**′ at $\mathcal{S}_1$. According to (4.11), we obtain, for any impinging mode l,

$$(4.15)\qquad \begin{aligned} 1 + \sum_{j=1}^{3} V_{lj} &= \sum_{j=1}^{3} \alpha_j(0), \\ p_{l+}^{(1)} + \sum_{j=1}^{3} p_{j-}^{(1)} V_{lj} &= \sum_{j=1}^{3} p_j(0)\alpha_j(0), \\ q_{l+}^{(1)} + \sum_{j=1}^{3} q_{j-}^{(1)} V_{lj} &= \sum_{j=1}^{3} q_j(0)\alpha_j(0); \end{aligned}$$

$$(4.16)\qquad \begin{aligned} \zeta_{l+}^{(1)} + \sum_{j=1}^{3} \zeta_{j-}^{(1)} V_{lj} &= \sum_{j=1}^{3} \Omega_1^{[j]}(0)\alpha_j(0), \\ \zeta_{l+}^{(1)}p_{l+}^{(1)} + \sum_{j=1}^{3} \zeta_{j-}^{(1)}p_{j-}^{(1)} V_{lj} &= \sum_{j=1}^{3} \Omega_2^{[j]}(0)\alpha_j(0), \\ \zeta_{l+}^{(1)}q_{l+}^{(1)} + \sum_{j=1}^{3} \zeta_{j-}^{(1)}q_{j-}^{(1)} V_{lj} &= \sum_{j=1}^{3} \Omega_3^{[j]}(0)\alpha_j(0), \end{aligned}$$

where Eq. (4.12) has been used in working out the last two of Eqs. (4.16), and where

$$(4.17)\qquad \Omega_k^{[j]}(z) = N_{k1}^{[j]}(z) + N_{k2}^{[j]}(z)p_j(z) + N_{k3}^{[j]}(z)q_j(z) \qquad (k = 1,2,3),$$

for any $j = 1,2,3$. From Eq. (4.15) we have, for any l,

$$(4.18)\qquad \alpha_j(0) = \nu_{jl}^{+}(0) + \sum_{h=1}^{3} V_{lh}\nu_{jh}^{-}(0) \qquad (j = 1,2,3),$$

with

$$\nu_{jm}^{\pm} = \frac{1}{r}\left[q_{m\pm}^{(1)}(p_{j+1}-p_{j+2}) + q_{j+1}(p_{j+2}-p_{m\pm}^{(1)}) + q_{j+2}(p_{m\pm}^{(1)}-p_{j+1})\right]$$

$$(j, m = 1, 2, 3),$$

$$r = \sum_{j=1}^{3} q_j(p_{j+1}-p_{j+2}),$$

and where a cyclic permutation of the indices j is understood. Substitution of Eq. (4.18) into (4.16) yields, after some manipulations, the reflection matrix as

(4.19)
$$\mathbf{V} = -\mathbf{H}_{+}^{-1}\mathbf{H}_{-}\,,$$

where $\mathbf{H}_{+}$ and $\mathbf{H}_{-}$ are matrices whose entries are given by

(4.20)
$$\begin{aligned}(H_{\pm})_{1h} &= \zeta_{h\mp}^{(1)} - \sum_{j=1}^{3}\nu_{jh}^{\mp}(0)\Omega_1^{[j]}(0),\\ (H_{\pm})_{2h} &= \zeta_{h\mp}^{(1)}p_{h\mp}^{(1)} - \sum_{j=1}^{3}\nu_{jh}^{\mp}(0)\Omega_2^{[j]}(0),\\ (H_{\pm})_{3h} &= \zeta_{h\mp}^{(1)}q_{h\mp}^{(1)} - \sum_{j=1}^{3}\nu_{jh}^{\mp}(0)\Omega_3^{[j]}(0),\end{aligned}$$

with $h = 1, 2, 3$. The transmission coefficients may be obtained from the reflection matrix $\mathbf{V}$ by simply observing that Eq. (4.13) can be also written as

$$\alpha_j(z) = W_{lj}\exp\left[-i\int_z^d \Omega_1^{[j]}(\tau)\,d\tau\right]\exp(i\zeta_{j+}^{(2)}d).$$

Hence we obtain

(4.21)
$$\mathbf{W} = \mathbf{K}_{+}\mathbf{V} + \mathbf{K}_{-},$$

where

$$(K_{\pm})_{jh} = \nu_{jh}^{\mp}(0)\exp\left[i\int_0^d\left[\Omega_1^{[j]}(\tau) - \zeta_{j+}^{(2)}\right]d\tau\right] \qquad (j, k = 1, 2, 3).$$

5. Local reflectivity and couples of opposite modes

The aim of the present section is to give a representation of the displacement within the layer $\mathcal{L}$, as a set of pairs of up-going and down-going modes. Accounting for the formal splitting (4.3), we write

$$\mathbf{u}_j = \alpha_j \begin{pmatrix} 1 \\ p_j \\ q_j \end{pmatrix} = \alpha_j^+ \begin{pmatrix} 1 \\ p_j^+ \\ q_j^+ \end{pmatrix} + \alpha_j^- \begin{pmatrix} 1 \\ p_j^- \\ q_j^- \end{pmatrix} \qquad (j = 1, 2, 3), \tag{5.1}$$

where the dependence on z of the amplitudes and polarizations is understood. Let us introduce the local reflectivity matrix $\mathbf{R}(z)$ as

$$(\alpha_1^-, \alpha_2^-, \alpha_3^-) = \mathbf{R}\,(\alpha_1^+, \alpha_2^+, \alpha_3^+). \tag{5.2}$$

From Eqs. (5.1) and (5.2) we can express the amplitudes $\alpha_j^\pm$ in terms of the amplitudes α_j, which have been derived in the previous section. We get

$$(\alpha_1^+, \alpha_2^+, \alpha_3^+) = (\mathbf{I} + \mathbf{R})^{-1}(\alpha_1, \alpha_2, \alpha_3), \tag{5.3}$$

$$(\alpha_1^-, \alpha_2^-, \alpha_3^-) = \mathbf{R}(\mathbf{I} + \mathbf{R})^{-1}(\alpha_1, \alpha_2, \alpha_3). \tag{5.4}$$

In order to match the wave-splitting given by (5.1) and (5.2) with the solutions of Eq. (3.6) in $\mathcal{B}_1$ and $\mathcal{B}_2$, according to the analysis of the previous section, we must impose the following conditions at the boundaries

$$(\alpha_1^-, \alpha_2^-, \alpha_3^-)|_{z=0} = \mathbf{V}\,(\alpha_1^+, \alpha_2^+, \alpha_3^+)|_{z=0}, \tag{5.5}$$

$$(\alpha_1^-, \alpha_2^-, \alpha_3^-)|_{z=d} = 0. \tag{5.6}$$

In view of Eqs. (5.2) and (5.4), this implies that the matrix function $\mathbf{R}(z)$ must satisfy the conditions

$$\mathbf{R}(0) = \mathbf{V}, \tag{5.7}$$

$$\mathbf{R}(d) = \mathbf{0}. \tag{5.8}$$

Let us consider the matrix function

$$\mathbf{R}(z) = -\mathbf{H}_+^{-1}(z)\,\mathbf{H}_-(z), \tag{5.9}$$

where $\mathbf{H}_\pm(z)$ are given by (4.20) being $\nu_{jh}^\mp$ and $\Omega_k^{[j]}$ evaluated at the depth z in the layer. It is a simple matter to show that (5.9) satisfies conditions (5.7) and (5.8). In fact, Eq. (5.7) is the obvious consequence of (4.19) and (5.8) follows from the fact that $\mathbf{H}_-(d) = 0$, in view of (4.20), (4.17) and (4.10). Hence Eqs. (5.3), (5.4) and (5.9) yield the appropriate representation of the split field within the layer. However, this description is not the only one, since other representations are possible for different matrix functions which satisfy Eqs. (5.7) and (5.8). As to the polarizations $p_j^{\pm}(z)$, $q_j^{\pm}(z)$, we can apply the previous analysis in view of the formulae

$$(\alpha_1^- p_1^-, \alpha_2^- p_2^-, \alpha_3^- p_3^-) = \mathbf{R}(\alpha_1^+ p_1^+, \alpha_2^+ p_2^+, \alpha_3^+ p_3^+),$$
$$(\alpha_1^- q_1^-, \alpha_2^- q_2^-, \alpha_3^- q_3^-) = \mathbf{R}(\alpha_1^+ q_1^+, \alpha_2^+ q_2^+, \alpha_3^+ q_3^+),$$
$$\alpha_j^+ p_j^+ + \alpha_j^- p_j^- = \alpha_j p_j, \qquad \alpha_j^+ q_j^+ + \alpha_j^- q_j^- = \alpha_j q_j,$$

with $j = 1, 2, 3$.

6. Horizontally polarized waves for particular symmetries

According to Eq. (2.6), orthorhombic, tetragonal, cubic and hexagonal systems are characterizerd by the following restrictions

$$\begin{aligned} &\Gamma_{14} = \Gamma_{15} = \Gamma_{16} = \Gamma_{24} = \Gamma_{25} = \Gamma_{26} = 0, \\ &\Gamma_{34} = \Gamma_{35} = \Gamma_{36} = \Gamma_{45} = \Gamma_{46} = \Gamma_{56} = 0. \end{aligned} \tag{6.1}$$

For waves incident on the plane (y, z), Eq. (3.5) splits into

$$\Gamma_{55} u_1'' + \Gamma_{55}' u_1' - (k^2 \Gamma_{66} - \varrho\omega^2) u_1 = 0, \tag{6.2}$$

$$\begin{aligned} &\begin{pmatrix} \Gamma_{44} & 0 \\ 0 & \Gamma_{33} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}'' + \left[\begin{pmatrix} \Gamma_{44}' & 0 \\ 0 & \Gamma_{33}' \end{pmatrix} + 2ik \begin{pmatrix} 0 & \Gamma_{23} \\ \Gamma_{23} & 0 \end{pmatrix} \right] \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}' \\ &\quad + \left[ik \begin{pmatrix} 0 & \Gamma_{44}' \\ \Gamma_{44}' & 0 \end{pmatrix} - k^2 \begin{pmatrix} \Gamma_{22} & 0 \\ 0 & \Gamma_{44} \end{pmatrix} + \varrho\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = 0. \end{aligned} \tag{6.3}$$

Equation (6.2) is the governing equation for waves polarized along the x-direction, i.e. horizontally polarized waves, and Eq. (6.3) accounts for waves whose amplitude lies on the vertical propagation plane, i.e. vertically polarized waves. The analysis of Sec. 4 may be applied separately to Eq. (6.2) and Eq. (6.3). Here we remark some peculiar features of horizontally polarized waves. Let us note that, according to (6.1)

$$t_1 = \Gamma_{55} u_1',$$

hence continuity of t_1 at the boundaries of the layer is equivalent to continuity of u_1'. The continuity requirements reduce to

$$\begin{aligned} u_1(0) &= 1 + V, \\ u_1'(0) &= i(\zeta_+^{(1)} + \zeta_-^{(1)} V), \\ u_1(d) &= W \exp(i\zeta_+^{(2)} d), \\ u_1'(d) &= i\zeta_+^{(2)} W \exp(i\zeta_+^{(2)} d). \end{aligned} \tag{6.4}$$

Introducing the complex-valued function $\sigma(z)$ such that

$$u_1' = i\sigma u_1, \tag{6.5}$$

Eq. (6.2) yields

$$\sigma' = \frac{i}{\Gamma_{55}}(\varrho\omega^2 - k^2\Gamma_{66}) - \frac{\Gamma_{55}'}{\Gamma_{55}}\sigma - i\sigma^2. \tag{6.6}$$

In addition, from (6.4) we get

$$\sigma(d) = \zeta_+^{(2)}. \tag{6.7}$$

If Eq. (6.6) is solved together with the boundary condition (6.7), the horizontal displacement $u_1(z)$ may be given in the form

$$u_1(z) = \frac{\zeta_-^{(1)} - \zeta_+^{(1)}}{\zeta_-^{(1)} - \sigma(0)} \exp\left[i\int_0^z \sigma(\tau)\,d\tau\right]. \tag{6.8}$$

As to the reflection coefficient V, Eqs. (6.4) yield

$$V = -\frac{\zeta_+^{(1)} - \sigma(0)}{\zeta_-^{(1)} - \sigma(0)}. \tag{6.9}$$

The scattering problem has been reduced to the solution of the first-order Riccati equation (6.6) for the function $\sigma(z)$.

Consider now the splitting of horizontally polarized waves and introduce the up and down-going modes $u_1^+(z)$, $u_1^-(z)$ and a local reflectivity $R(z)$ such that

$$u_1 = u_1^+ + u_1^-, \qquad u_1^- = Ru_1^+. \tag{6.10}$$

$$R(0) = V, \qquad R(d) = 0. \tag{6.11}$$

It is easy to show that the function

$$R(z) = -\frac{\zeta_+(z) - \sigma(z)}{\zeta_-(z) - \sigma(z)}, \tag{6.12}$$

where the functions $\zeta_\pm(z)$ are defined according to

$$\begin{aligned} \zeta_+(z) + \zeta_-(z) &= i\frac{\Gamma_{55}'(z)}{\Gamma_{55}(z)}, \\ \zeta_+(z)\zeta_-(z) &= -\frac{1}{\Gamma_{55}(z)}\left[\varrho\omega^2 - k^2\Gamma_{66}(z)\right], \end{aligned} \tag{6.13}$$

satisfies restrictions (6.11). This fact is a direct consequence of Eqs. (6.9) and (6.7). We finally show that, in this case, the reflectivity $R(z)$ satisfies a first-order Riccati differential equation. To this end we observe that, owing to (6.13), Eq. (6.6) can be rewritten as

$$\sigma' = -i(\sigma - \zeta_+)(\sigma - \zeta_-). \tag{6.14}$$

Then, differentiating Eq. (6.12) and accounting for Eq. (6.14), we obtain

$$R' = \frac{\zeta'_+}{\zeta_+ - \zeta_-} - \left[i(\zeta_+ - \zeta_-) - \frac{\zeta'_+ + \zeta'_-}{\zeta_+ - \zeta_-} \right] R + \frac{\zeta'_-}{\zeta_+ - \zeta_-} R^2. \tag{6.15}$$

Integration of Eq. (6.15) with the boundary condition $(6.11)_2$ turns out to be an alternative approach in deriving the reflection coefficient. The result (6.15) is a generalization of recent results on isotropic layers [7]. More generally, a Riccati-type equation for the reflectivity is a common feature of scalar theories in layered media (see for ex. [9]).

7. Numerical examples

In order to varify the method previously outlined, we give a numerical solution for the wave-field inside a solid layer with known constitutive properties. We restrict our computations to the scalar problem developed in Sec. 6; extension to the more general case may be performed without qualitative changes. Two different examples are considered for the dependence of the constitutive parameters on the depth within the layer. In each instance, the quantities Γ_{55}, Γ_{66}, ϱ have the same dependence on z and, according to the present model, are C^1 throughout z. The first example accounts for a monotone increasing dependence on z as

$$(\varrho, \Gamma_{55}, \Gamma_{66}) = (\varrho_0, \Gamma^0_{55}, \Gamma^0_{66})[1 + Q(1 - \cos(\pi Z))], \qquad Z \in [0, 1], \tag{7.1}$$

where ϱ_0, Γ^0_{55}, Γ^0_{66} are constant quantities pertaining to $\mathcal{B}_1$, $2Q$ is the ratio between the maximum and the minimum value of the constitutive parameters and where the dimensionless variable $Z = z/d$ has been introduced. In the second example a symmetric layer is considered, with

$$(\varrho, \Gamma_{55}, \Gamma_{66}) = (\varrho_0, \Gamma^0_{55}, \Gamma^0_{66})[1 + Q(1 - \cos(2\pi Z))], \qquad Z \in [0, 1], \tag{7.2}$$

so that $\mathcal{B}_1$ and $\mathcal{B}_2$ are mechanically equivalent.

Effective wave propagation within the layer requires a non-zero real part of $\zeta_\pm$. According to (6.13), this implies

$$\varrho\omega^2 - \Gamma_{66}k^2 > \frac{{\Gamma'_{55}}^2}{4\Gamma_{55}}, \qquad \forall\, Z. \tag{7.3}$$

In view of (7.1) and (7.2), the inequality (7.3) amounts to the following restriction on ω and k,

$$\varrho\omega^2 - \Gamma^0_{66}k^2 > c\Gamma^0_{55}\frac{\pi^2 Q^2}{1+2Q}, \tag{7.4}$$

with $c = 1/4$ or $c = 1$ depending on the alternative use of (7.1) or (7.2), respectively.

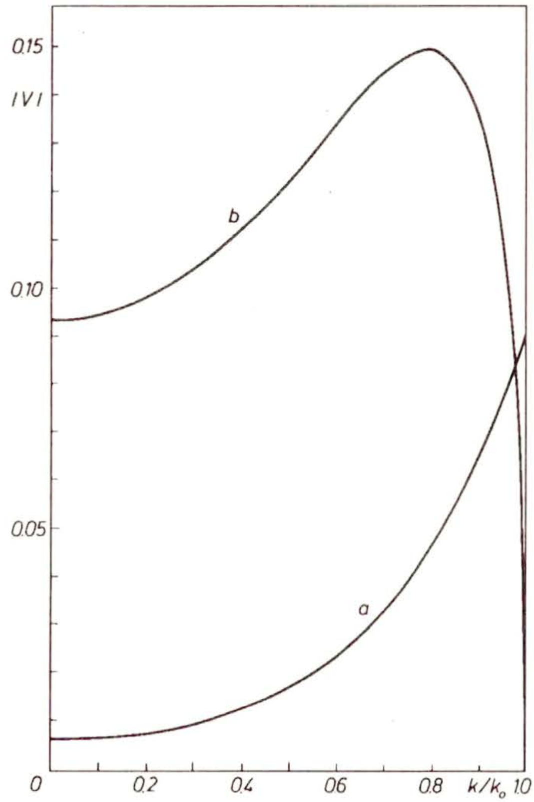

FIG. 1. Reflection coefficient $|V| = |R(0)|$ as a function of k/k_0 for a layer described by Eq. (7.1) (curve a) or by Eq. (7.2) (curve b).

Equation (6.15) has been numerically integrated along with the boundary condition $R(Z = 1) = 0$, adopting Eqs. (7.1), (7.2) and accounting for (7.4). The reflection coefficient $|V| = |R(0)|$ has been derived for all possible values of k $\left(0 \le k < k_0, \text{ with } k_0 = \left[\frac{\varrho\omega^2}{\Gamma^0_{66}} - c\frac{\Gamma^0_{55}}{\Gamma^0_{66}}\frac{\pi^2 Q^2}{1+2Q}\right]^{1/2}\right)$. The values of V have been substituted into the boundary conditions $(6.4)_{1,2}$ in order to integrate Eq. (6.2). Then, both solutions for u_1 and R have been exploited to obtain the wave splitting within the layer, according to (6.10). The results are shown in Figs. 1–5 for

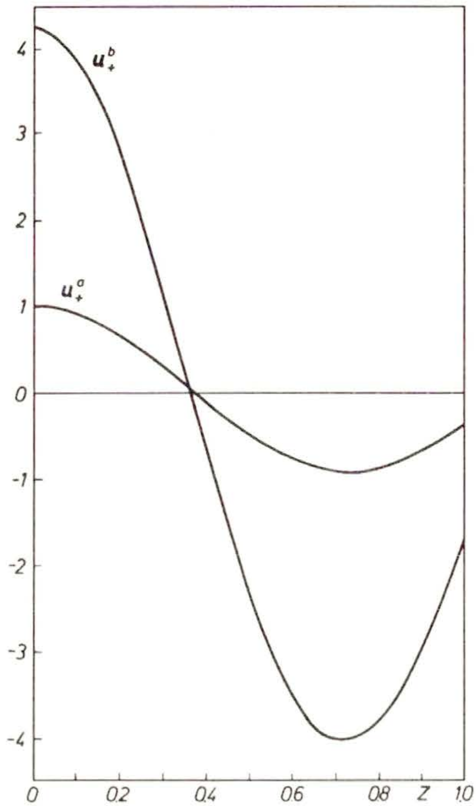

FIG. 2. Real and imaginary parts of the forward wave component within the "monotone" layer (see Eq. (7.1)).

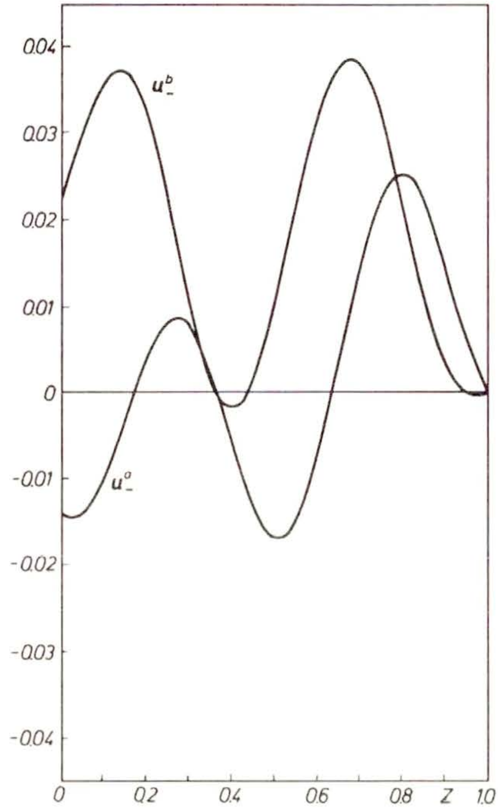

FIG. 3. Real and imaginary parts of the backward wave component within the "monotone" layer (see Eq. (7.1)).

a layer of zinc ($\varrho_0 = 7135\,\mathrm{kg/m^3}$, $\Gamma_{55}^0 = 39\cdot10^9\,\mathrm{Pa}$, $\Gamma_{66}^0 = 63\cdot10^9\,\mathrm{Pa}$) with $Q = 0.1$ and $\omega = 10^4\,\mathrm{Hz}$. In particular, Fig. 1 shows $|V|$ versus k for the "monotone" layer described by Eq. (7.1) (curve a), and for the symmetric layer described by Eq. (7.2) (curve b). Figures 2 and 3 show the real and the imaginary parts $u_\pm^a$ and $u_\pm^b$ of the opposite modes in the split wave-field for normal incidence ($k = 0$) in the "monotone" layer (see Eq. (7.1)). Analogous results are shown in Figs. 4, 5 for the symmetric layer (see Eq. (7.2)). From Figs. 3 and 5 is evident the phase shift between u_-^a and u_-^b which shows the mixing effect of the reflectivity R on the real and imaginary parts of the field inside the layer. We also observe that the reflection coefficient $|V|$ for normal incidence in the symmetric layer is by one order of magnitude greater than that of the "monotone" layer (Fig. 1). This fact, which is also evident from the results of the reflected amplitudes $u_-^{a,b}$ (Figs. 3, 5), is due to the steeper profile of the constitutive properties in the symmetric layer. We note, however, that this behaviour is reversed when incidences are considered which are close to the limiting value k_0.

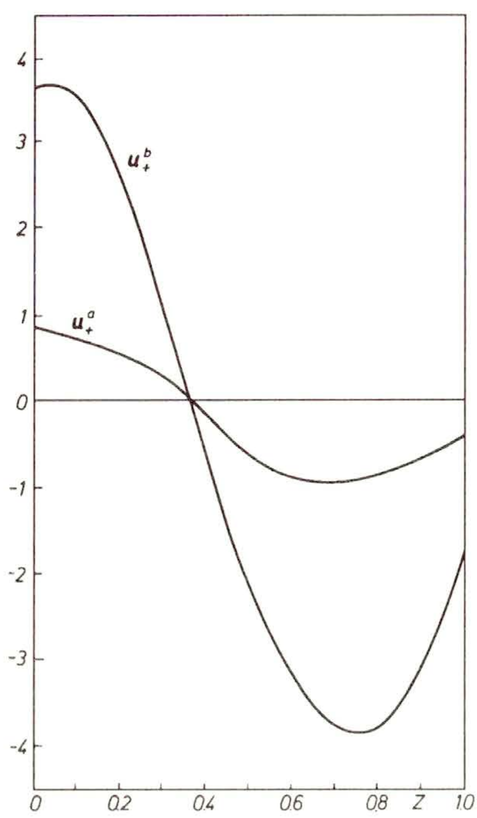

FIG. 4. Real and imaginary parts of the forward wave component within the symmetric layer (see Eq. (7.2)).

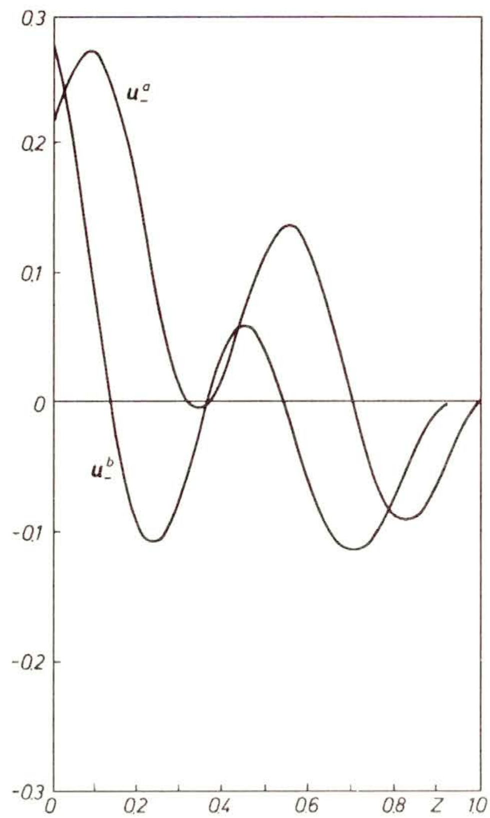

FIG. 5. Real and imaginary parts of the backward wave component within the symmetric layer (see Eq. (7.2)).

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DIPARTIMENTO DI INGEGNERIA BIOFISICA ED ELETTRONICA
UNIVERSITÀ, GENOVA, ITALY.

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The Wigner potential method in the investigation of thermal properties of regular composites

S. MAY, S. TOKARZEWSKI and A. ZACHARA (WARSZAWA)

FOR PERIODIC, two-dimensional potentials satisfying the Laplace equation, a new functional basis, different from that used by RAYLEIGH [1], has been derived. This basis allowed us to construct a simple recurrence formulae for evaluation of an effective transport coefficients for regular two-dimensional composites. As an example, the power expansion of an overall conductivity for square array of circular cylinders has been evaluated.

1. Governing equations

THE TEMPERATURE distribution and the effective conductivity of composites of regular structure were first investigated by RAYLEIGH [1]. He performed calculations for rectangular arrays of circular as well as spherical inclusions. The approach of Rayleigh was next developed by many other authors [2–4]. In this paper, we present a method of solving the two-dimensional periodic problems by using a new functional basis different from that used by Rayleigh. This basis appears to be very convenient for seeking the solutions of Laplace equation and leads to very effective algorithms.

Let us consider a material composed of circular cylinders of conductivity λ_d, embedded in a matrix of conductivity λ_c. The composite is subjected to an external linear temperature field. The elementary cell is presented in Fig. 1. Let a be the cylinder radius, l – the distance between the cylinder axes, T^d and T^c – the temperature of inclusions and matrix, respectively. The temperature field in a unit cell fulfills the conductivity equations

$$(1.1)\qquad \begin{aligned} \nabla^2 T^c &= 0 \qquad \text{for} \quad r > a,\\ \nabla^2 T^d &= 0 \qquad \text{for} \quad r < a, \end{aligned}$$

and the boundary conditions for $r = a$

$$(1.2)\qquad \begin{aligned} T^c &= T^d,\\ \lambda_c \frac{\partial T^c}{\partial r} &= \lambda_d \frac{\partial T^d}{\partial r}, \end{aligned}$$

where r, θ are polar coordinates with the origin located on the cylinder axis.

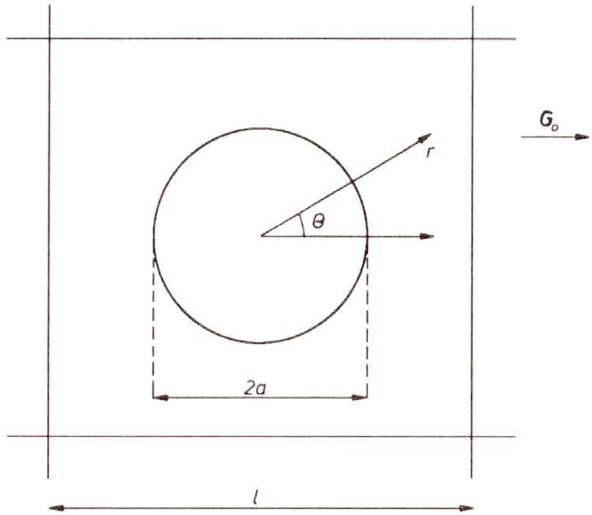

FIG. 1.

Rayleigh obtained the solution of Eqs. (1.1) in the form:

$$\begin{aligned} T^c(r,\theta) &= \sum_{k=1}^{\infty}\left(a_k r^k + \frac{b_k}{r^k}\right)\cos k\theta\,, \\ T^d(r,\theta) &= \sum_{k=1}^{\infty} c_k r^k \cos k\theta\,. \end{aligned} \tag{1.3}$$

The solution may be interpreted as generated by an infinite system of multipoles located at the cylinder axes. We have here three infinite sets of coefficients a_k, b_k, c_k, ($k = 1, 3, \ldots$) since, due to the symmetry conditions, only odd values of k are allowed [1]. With the aid of the boundary conditions (1.2), the coefficients a_k and c_k may be expressed as linear functions of b_k. To determine b_k, Rayleigh made an assumption that the part of the potential in the unit cell corresponding to the term of Eq. $(1.3)_1$ which is non-singular in the unit cell center $r = 0$ resulted from two sources. The first of them is the external gradient of temperature. The second one is a joint influence of the multipoles from the other cells corresponding to the terms of Eq. $(1.3)_1$ which are singular in the centers of these cells [1]. This assumption leads to the following infinite system of equations for the coefficients b_k,

$$\delta_{k,1} + k!\,\frac{(u+2)}{ua^{2k}}b_k = \sum_{j=1}^{\infty}\frac{(k+j-1)!}{(j-1)!}S_{k+j}b_j, \qquad k = 1,3,5\ldots, \tag{1.4}$$

where

$$u = \frac{\lambda_d}{\lambda_c} - 1, \tag{1.5}$$

$$S_m = \sum_{\{n\}} \frac{1}{(x_n + iy_n)^m}. \tag{1.6}$$

Symbols S_m denote the Rayleigh sums, (x_n, y_n) are Cartesian coordinates of the centers of the cells, i is an imaginary unit and $\{n\}$ denotes summation to infinity in the directions of x and y over all cylinder centers lying outside the unit cell. The sums S_m depend on the geometrical properties of the array. Numerical values of S_m for square and hexagonal arrays are given in [2].

The approximate values of b_k can be calculated from Eq. (1.4) subjected to truncation. The effective conductivity of a composite depends on the coefficient b_1 according to the formula derived by RAYLEIGH [1],

$$\mu = \frac{\lambda_{\text{cf}}}{\lambda_c} = 1 - 2\pi b_1. \tag{1.7}$$

For a square array of cylinders, the temperature distribution $T^i(r, \theta; \varphi, u)$ and the effective conductivity $\mu(\varphi, u)$ depend on two dimensionless quantities: the cylinder volume fraction φ and physical properties of the components represented by u (1.5). Coefficients a_k, b_k and c_k appearing in (1.3) are functions of φ and u.

It is well known that the Rayleigh method provides the non-unique solutions for λ_{ef}, since the second Rayleigh's sum S_2 over the infinite array of cylinders is only conditionally convergent, i.e., it depends on the shape of the exterior boundary of the composite. This was the reason why, for a long time, many authors were questioning the correctness of the Rayleigh approach [5]. In 1979 MCPHEDRAN *et al.* [2] pointed out that an infinite, flat layer of a composite subjected to the external temperature gradient is the only correct sample shape for calculation of λ_{ef} by the Rayleigh method.

An interesting approach has been proposed by ZUZOVSKI, BRENNER [6] and SANGANI, ACRIVOS [7]. Their methods avoid all the difficulties of the Rayleigh method mentioned above. They decomposed the temperature field into two components. The first one is a macroscopic shape-dependent component $T^{i,m}$, and the second one is periodic, depending on the geometry and physical properties of the composite $T^{i,p}$,

$$T^i = T^{i,m} + T^{i,p}, \tag{1.8}$$

where $i = c, d$. In view of the periodicity of the temperature field and the square symmetry of the array, the normal derivative of the periodic component of temperature is equal to 0 on the cell boundary,

$$\mathbf{n} \cdot \nabla T^{c,p} = 0, \tag{1.9}$$

where $\mathbf{n}$ is a unit vector normal to the boundary of the cell. Condition (1.9) may be considered as equivalent to the equations of Rayleigh (1.4).

It was shown in [6, 7] that the periodic component may be expressed by an infinite set of derivatives of a certain function T_0 called the Wigner potential [8]. By using these derivatives ZUZOVSKI, BRENNER [6] and SANGANI, ACRIVOS [7] investigated the effective conductivity of regular arrays of spheres. They noticed that successive derivatives of T_0 formed a functional basis convenient for representation of solutions of three-dimensional Laplace equations.

The main aim of this paper is to construct a new functional basis for periodic, two-dimensional potentials generated by the Laplace equation. As an example, we will derive a simple recurrence formula for evaluation of the effective conductivity, in the form of a power series in u, for a square array of circular cylinders.

2. The functional basis

The Rayleigh functional basis consists of multipoles located in the centers of single cells. These basic functions do not fulfill the periodic boundary condition on the boundary of the cell. Our aim is to find a basis, the elements of which fulfill identically the periodicity conditions. Such a basis can be built up with the aid of the Wigner potential. In this section we shall limit our investigation merely to the periodic term $T^{i,p}$ of the temperature field. For the sake of convenience, the upper index p in $T^{i,p}$ will be omitted, i.e. $T^{i,p} \equiv T^i$.

Let us consider an infinite system of point heat sources of intensity q, located in the nodes of a square array of period l, accompanied by neutralizing fuzzy sources of uniform density $\tau = -q/l$ of the opposite sign. In such a grid, the global intensity of sources is equal to 0. The temperature field generated by such a system of sources fulfills the Poisson equation (2.1)

$$\nabla^2 T = -2\pi q \cdot \left(\delta(\mathbf{r}) - \frac{1}{l^2}\right), \tag{2.1}$$

and the boundary condition (1.9), where $\delta(\mathbf{r})$ is the Dirac function. The solution of equations (2.1) and (1.9) was given by CICHOCKI and FELDERHOF [8] in the form:

$$T_0(\mathbf{r}) = q \cdot \left(-\ln r + \frac{1}{2}\pi r^2 + \sum_{m=4}^{\infty} A_m r^m \cos m\theta\right). \tag{2.2}$$

Coefficients A_m were found in the process of summation over an infinite grid of cells, with the exception of the cell located in the center of the coordinate system. Coefficients A_m are related to the Rayleigh sums S_m (1.6) as follows,

$$A_m = \frac{S_m}{m}.$$

The index m in (2.2) is a multiple of 4, because $T_0(\mathbf{r})$ is independent of rotation of the frame of reference by the angle $\pi/2$. The first term in the parentheses of (2.2) represents the influence of a single source located in the center of the cell, the second one is generated by the fuzzy sources, while the third term given by the infinite sum is due to the sources located in the external cells. Function (2.2), called the Wigner potential [8], can be used as a starting point for the construction of the functional basis for periodic two-dimensional potentials.

A multipole of order k includes 2^k point sources, and it is defined by the scalar intensity q_k and k unit vectors $\mathbf{n}_s$, $s = 1, 2, \ldots, k$, representing the directional properties of the multipole (see for example [9]). The multipole potential is proportional to the k-th directional derivative of the point source potential in directions $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_k$, respectively.

$$T_k(\mathbf{r}) = q_k \cdot \left[\prod_{s=1}^{k} (-1)^k (\mathbf{n}_s \cdot \nabla) \right] T_0(\mathbf{r}), \qquad k = 1, 2, \ldots . \tag{2.3}$$

For a classical multipole, $T_0(\mathbf{r})$ is the potential of a point source in an infinite region: in the 2-dimensional case

$$T_0(\mathbf{r}) = -\ln r. \tag{2.4}$$

Instead of a single multipole, one may consider an infinite system of identical multipoles of order k located in the nodes of a square array. To determine the potential of such a grid, one should apply the operator of the right-hand side of (2.3) to the function $T_0(\mathbf{r})$ defined by (2.2). Positive and negative fuzzy sources balance each other and they have no influence on the global potential.

In the operator $\mathbf{n}_s \cdot \nabla$ of directional derivative, one may distinguish two components of different types of symmetry,

$$\mathbf{n}_s \cdot \nabla = \alpha_s \mathcal{U} + \beta_s \cdot \mathcal{V}, \tag{2.5}$$

where

$$\mathcal{U} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta},$$
$$\mathcal{V} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}.$$

Action of the operator $\mathcal{U}$ on the symmetric or antisymmetric functions produces the results of the same type of symmetry: symmetric or antisymmetric, respectively. On the other hand, operator $\mathcal{V}$ changes the type of symmetry to the opposite one. Let us introduce the notations:

$$T_k^{c,s} = (-1)^k (k-1)! \frac{\cos k\theta}{r^k} + \pi(\delta_{1k} r \cos\theta + \delta_{2k}) + \sum_{m=1}^{\infty} \frac{(m+k)!}{m!} r^m A_{k+m} \cos m\theta, \tag{2.6}$$

$$T_k^{c,a} = (-1)^k (k-1)! \frac{\sin k\theta}{r^k} + \pi \delta_{1k} r \sin\theta + \sum_{m=1}^{\infty} \frac{(m+k)!}{m!} r^m A_{k+m} \sin m\theta . \tag{2.7}$$

The differential operators $\mathcal{U}$ and $\mathcal{V}$ acting on functions $T_k^{c,s}(\mathbf{r})$ and $T_k^{c,a}(\mathbf{r})$ have the following properties:

$$\begin{aligned} \mathcal{U}T_k^{c,s} &= T_{k+1}^{c,s}, & \mathcal{U}T_k^{c,a} &= T_{k+1}^{c,a}, \\ \mathcal{V}T_k^{c,s} &= T_{k+1}^{c,a}, & \mathcal{V}T_k^{c,a} &= T_{k+1}^{c,s} 1. \end{aligned} \tag{2.8}$$

Applying the operator $(\mathbf{n}_s \cdot \nabla)$, $s = 1, \ldots, k$, to the function $T_0(\mathbf{r})$ k-times, one obtains a sum of the following terms:

$$\mathcal{U}^j \mathcal{V}^{k-j} T_0(\mathbf{r}), \qquad j = 0, 1, \ldots, k;$$

hence

$$T_k(\mathbf{r}) = C_k T_k^{c,s}(\mathbf{r}) + D_k T_k^{c,a}(\mathbf{r}), \tag{2.9}$$

where C_k and D_k are certain known numerical coefficients.

The functions $T_k^{c,s}$ and $T_k^{c,a}$ defined by (2.6) and (2.7) constitute a basis for the solution in the elementary cell outside the cylinder. According to (2.9), the potential of the grid of k-th order multipoles is a linear combination of symmetric and antisymmetric functions of k-th order.

The functions $T_k^{c,s}$ and $T_k^{c,a}$ have singularities on the axis of the cylinder, and they can not be used for representing the solution inside the cylinder. In this region, we assume the basis (2.10), (2.11) without singularity

$$T_k^{d,s} = (-1)^k (k-1)! \frac{r^k}{a^{2k}} \cos k\theta + \pi(\delta_{1k} r \cos\theta + \delta_{2k}) + \sum_{m=1}^{\infty} \frac{(m+k)!}{m!} r^m A_{k+m} \cos m\theta , \tag{2.10}$$

$$T_k^{d,a} = (-1)^k (k-1)! \frac{r^k}{a^{2k}} \sin k\theta + \pi \delta_{1k} r \sin\theta + \sum_{m=1}^{\infty} \frac{(m+k)!}{m!} r^m A_{k+m} \sin m\theta . \tag{2.11}$$

It is easily seen that for $r = a$, the corresponding functions in both the bases are equal,

$$\mathbf{T}_k^{c,s}(a,\theta) = \mathbf{T}_k^{d,s}(a,\theta), \qquad \mathbf{T}_k^{c,a}(a,\theta) = \mathbf{T}_k^{d,a}(a,\theta). \tag{2.12}$$

Representing the solution in the bases (2.6)–(2.7) and (2.10)–(2.11), we fulfill identically the condition (1.9) of periodicity on the cell wall, and continuity of temperature on the cylinder surface. The condition of equality of normal components of a heat flux on the cylinder boundary determine uniquely the coefficients of expansion of λ_{ef} in our basis.

Let us introduce the symbols for the basic functions in both components of the composite:

$$T_k^s = \begin{cases} T_k^{d,s} \\ T_k^{c,s} \end{cases} \qquad T_k^a = \begin{cases} T_k^{d,a} & r \leq a, \\ T_k^{c,a} & r \geq a. \end{cases} \tag{2.13}$$

Derivatives of the functions T_k^s and T_k^a are discontinuous for $r = a$. Both functions (2.13) fulfill the Poisson equations (2.14)–(2.15) given by

$$\nabla^2 T_k^s = -\delta(r-a)\cos k\theta, \tag{2.14}$$

$$\nabla^2 T_k^a = -\delta(r-a)\sin k\theta. \tag{2.15}$$

The relations (2.14)–(2.15) enable another interpretation of the basic functions, as a potential generated by the sources located at the cylinder boundary. The cosine and sine heat sources generate the symmetric functions (2.6) and (2.10), and the antysymmetric functions (2.7) and (2.11), respectively. In this interpretation, the intrinsic ties between the singular functions for the region outside the cylinders and non-singular functions inside the cylinders, are easily seen. For the case of circular cylinders arranged in a square array, the solution of (1.1)–(1.3) is a symmetric function. Hence we shall not consider in the sequel the basic functions T_k^a.

3. Recurrence algorithm

Using the functional basis given by the symmetric functions (2.6) and (2.10), we shall express the temperature field of the matrix ($i = c$) and inclusions ($i = d$), determined by Eqs. (1.1), (1.2) and (1.9), in the form of a power series expansion in u,

$$T^i(r,\theta;\varphi,u) = \tau^{(0)} + \sum_{m=1}^{\infty} \tau^{(i,m)}(r,\theta;\varphi)u^m. \tag{3.1}$$

Here, according to the previous definition (1.8), the function T^i in (3.1) is the sum of both the macroscopic $\tau^{(0)}$ and periodic (the sum for $m \geq 1$) parts of the temperature field. In this respect, the notations of Eqs. (3.1) and (1.8) are different. Following BERGMAN [10], we rewrite Eqs. (1.1) in a form valid for both the matrix and the inclusion in a unit cell,

$$\nabla \cdot (1 + u\theta_d)\nabla T^i = 0, \tag{3.2}$$

where θ_d is the characteristic function of inclusions. By inserting the series (3.1) into Eq. (3.2) and collecting terms with the same power of u, we obtain the general recurence formula for the coefficients $\tau^{(i,m)}$:

$$\begin{aligned} \nabla^2\tau^{(0)} &= 0, \\ \nabla^2\tau^{(i,m)} &= -\nabla\theta_d \cdot \nabla\tau^{(i,m-1)}, \qquad m = 1, 2, \dots . \end{aligned} \tag{3.3}$$

The composite is subjected to an external temperature gradient equal to unity. Hence the solution of $(3.3)_1$ is given by $\tau^{(0)} = r\cos\theta$.

Now let us turn our attention to the periodic part of the solution determined by Eq. $(3.3)_2$. Taking into account properties of the scalar product and of the characteristic function θ_d, we rearrange its right-hand side and Eq. $(3.3)_2$ takes the form [11]

$$\nabla^2\tau^{(i,m)} = \delta(r-a)\frac{\partial\tau^{d,m-1)}}{\partial r}, \tag{3.4}$$

where the functions $\tau^{(d,m-1)}$ $(m = 1, 2, \dots)$ are defined inside the cylinder, and a denotes the radius of the cylinder. Note that the functions $\tau^{(i,m)}$ determined by $(3.3)_2$ are periodic and can be represented by the series

$$\tau^{(i,m)} = \sum_{k=1}^{\infty} c_k^{(m)} T_k^i, \qquad \text{for} \quad m = 1, 2, \dots, \tag{3.5}$$

where T_k^i are the basic functions given by (2.6) and (2.10), while $c_k^{(m)}$ are real coefficients.

Now let us present the basic functions in a renormalized form which will be more convenient for further considerations. Superscript s will be here disregarded since only symmetric basic functions are the subject of our interest,

$$T_j^c = \frac{a^{j+1}}{2j!}\left[(j-1)!\,\frac{\cos j\theta}{r^j} + \sum_{k=1}^{\infty} p_{jk} r^k \cos k\theta\right], \tag{3.6}$$

$$T_j^d = \frac{a^{j+1}}{2j!}\left[(j-1)!\,\frac{r^j}{a^{2j}}\cos j\theta + \sum_{k=1}^{\infty} p_{jk} r^k \cos k\theta\right], \tag{3.7}$$

where

$$p_{jk} = -\frac{(j+k)!}{k!}\left(A_{j+k} + \frac{1}{2}\pi\delta_{j+k,2}\right). \tag{3.8}$$

Inserting (3.5) into (3.4) and making use of (2.14), we obtain the recursion formula for the coefficients $c_k^{(m)}$

$$\sum_{k=1}^{\infty} c_k^{(m+1)} \cos k\theta = -\sum_{j=1}^{\infty} c_j^{(m)} \frac{\partial}{\partial r} T_j^d. \tag{3.9}$$

Next, introducing T_k^d given by (3.7) into (3.9) and collecting the terms with $\cos k\theta$, we finally arrive at:

$$c_k^{(m+1)} = -\sum_{j=1}^{\infty} c_j^{(m)} \left(\frac{1}{2}\delta_{kj} + p_{kj} \frac{ka^{k+j}}{2j!} \right), \qquad k = 1, 2, \ldots, \tag{3.10}$$

where the term p_{jk} is given by (3.8).

The input data for algorithm (3.7) are

$$c_k^{(1)} = \delta_{1k}, \qquad k = 1, 2, \ldots, \tag{3.11}$$

since the gradient of external temperature field is equal to unity. The recurrence formula (3.10) allows us to compute the coefficients $c_k^{(m)}$ $(m = 1, 2, \ldots)$ in (3.5), and hence to determine by means of (3.1) the temperature field T^i inside the unit cell.

It is worth to note that the solution of Eqs. (1.1) presented by (3.1) with (3.5)–(3.8) and (3.10) satisfies the boundary conditions (1.2), in spite of the fact that they were not introduced here explicitly. In fact, the boundary condition $(1.2)_1$ is fulfilled owing to the form of the basic functions assumed, what can be seen from Eqs. (2.12). The condition $(1.2)_2$ can be rewritten, with the aid of (1.5), to the following form:

$$\left.\frac{\partial T^c}{\partial r}\right|_{r=a+0} - \left.\frac{\partial T^d}{\partial r}\right|_{r=a-0} = u \left.\frac{\partial T^d}{\partial r}\right|_{r=a-0}. \tag{3.12}$$

Inserting (3.1), (3.5), (3.6) and (3.7) to (3.12) and collecting terms with equal powers of u we get first $c_1^{(1)} = 1$ (see (3.11)), and then the recurrence expression (3.10) for the successive coefficients $c_k^{(m+1)}$. Thus we can see that the procedure presented here satisfies both boundary conditions (1.2).

4. Calculation of effective conductivity

Now we shall use the recurrence algorithm to calculate the effective conductivity of the composite. To this end let us consider the temperature field in the matrix which can be expressed by Eq. (3.1) with the aid of (3.5) and the basic functions (3.6). This expression may be transformed to the Rayleigh form $(1.3)_1$ which allows us to calculate the coefficient b_1 of the term $\cos\theta/r$ of the power series of u. Inserting b_1 into (1.7) we obtain the formula for effective conductivity of the composite

$$\mu(u) = 1 + \sum_{n=1}^{\infty} C_n u^n, \tag{4.1}$$

where

$$C_n = \varphi c_1^{(n)}. \tag{4.2}$$

The coefficients $c_1^{(n)}$ can be obtained from the recurrence formula (3.10) which for $k = 1$ takes the following form

$$c_1^{(m+1)} = -c_1^{(m)}\frac{1}{2}(1-\varphi) + 2\sum_{n=1}^{\infty} c_{4n-1}^{(m)} n A_{4n} (\varphi/\pi)^{4n}, \tag{4.3}$$

while coefficients $c_{4n-1}^{(m)}$ are calculated directly from (3.10).

We start our calculations with (3.11), and then from Eq. (4.3) we obtain the successive coefficients. The first four of them are listed below:

$$c_1^{(1)} = 1, \tag{4.4}$$

$$c_1^{(2)} = -\frac{1}{2}(1-\varphi), \tag{4.5}$$

$$c_1^{(3)} = -c_1^{(2)}\frac{1}{2}(1-\varphi) + 4\sum_{n=1}^{\infty}(4n-1)(nA_{4n})^2(\varphi/\pi)^{4n}, \tag{4.6}$$

$$c_1^{(4)} = -\left[c_1^{(3)}\frac{1}{2}(1-\varphi) + 2(2-\varphi)\sum_{n=1}^{\infty}(4n-1)(nA_{4n})^2(\varphi/\pi)^{4n}\right]. \tag{4.7}$$

The process could be continued, however the expressions for coefficients of higher order are more complex and they will not be presented here. The coefficients of higher order were calculated numerically from the formula (4.3). The first nonvanishing Wigner coefficients A_m which appear in Eqs. (4.3) and (4.6)–(4.7) are given below:

$$A_4 = 0.7878030005, \qquad A_8 = 0.5319716294,$$
$$A_{12} = 0.3282374177, \qquad A_{16} = 0.2509809396.$$

The values of coefficients $C_n(\varphi)$ were obtained from the formula (4.2). Several low order coefficients (up to C_6) are gathered in Table 1.

Now we compare Eq. (4.1) with the Maxwell-Garnett formula (see [11, 12]) which is the first approximation of the effective conductivity coefficient. The Maxwell-Garnett formula may be presented as a function of u and φ in the following form:

$$\mu = 1 + \frac{\varphi u}{1 + u(1-\varphi)/2}. \tag{4.8}$$

Table 1. Coefficients of the power series expansion of effective conductivity μ for a square array of cylinders.

φ	C_1	C_2	C_3	C_4	C_5	C_6
0.10	0.10	−0.04500	0.020250	−0.009113	0.004101	−0.001846
0.20	0.20	−0.08000	0.032024	−0.012830	0.005148	−0.002068
0.30	0.30	−0.10500	0.036936	−0.013086	0.004682	−0.001698
0.40	0.40	−0.12000	0.036784	−0.011662	0.003884	−0.001381
0.50	0.50	−0.12500	0.033646	−0.010208	0.003615	−0.001488
0.60	0.60	−0.12000	0.029979	−0.010181	0.004465	−0.002255
0.70	0.70	−0.10500	0.028735	−0.012751	0.006975	−0.004169
0.75	0.75	−0.09375	0.030114	−0.015261	0.009200	−0.006077

If we expand (3.20) into a power series of u, we obtain

$$\mu = 1 + \varphi\left[u - \frac{1}{2}(1-\varphi)u^2 + \frac{1}{4}(1-\varphi)^2u^3 - \frac{1}{8}(1-\varphi)^3u^4 + \ldots\right]. \tag{4.9}$$

Although this expression (4.9) is only a rough approximation of μ, certain resemblance to the formula (4.1) and (4.2)–(4.7) can easily be seen. In fact, the coefficients at the first and the second power of u which appear in (4.9), and those calculated from (4.4) and (4.5), are identical. The other coefficients of (4.9) are identical merely with the leading terms of the expressions (4.6) and (4.7).

5. Continued fraction expansion

The power series expression (4.1) is not an effective form for representing μ because of the small convergence radius and very slow rate of convergence. It is much better to express $\mu(\varphi, u)$ in the form of a continued fraction (see [11, 12]). Comparison of the two forms (4.8) and (4.9) illustrates how convenient and effective may be the rational representation, as compared with infinite series.

If we substitute $s = 1/u$ into Eq. (4.1), we can present the series in the form of a J-fraction [13]

$$\mu(\varphi, s) = 1 + \frac{k_1}{l_1 + s -}\,\frac{k_2}{l_2 + s -}\,\frac{k_3}{l_3 + s -}\,\frac{k_4}{l_4 + s -}\,\ldots, \tag{5.1}$$

where coefficients $k_n(\varphi)$ and $l_n(\varphi)$ can be determined using the coefficients C_n (Table 1), on the basis of another recurrence algorithm given in the Appendix.

The coefficients of the first level of the J-fraction calculated in the Appendix (A.4) are

$$k_1 = \varphi, \qquad l_1 = (1-\varphi)/2. \tag{5.2}$$

Inserting (5.2) into (5.1) and assuming the other coefficients to be equal to zero, we get

$$\mu = 1 + \frac{\varphi}{(1-\varphi)/2 + 1/u} = 1 + \frac{\varphi u}{u(1-\varphi)/2 + 1}\,. \tag{5.3}$$

We can see that Eq.(5.3) is identical with the Maxwell-Garnett formula (4.8), the accuracy of which is limited to small values of u and φ. However, it is an advantageous feature of the continuous fraction expansion that successive approximants of the fraction rapidly increase its accuracy. The results presented in [11] indicate that for $u \to \infty$ and $\varphi = 0.7$, which is a rather high value, only three or four levels of the fraction are sufficient to preserve a good accuracy. Nevertheless in the asymptotic case, if $\varphi \to \varphi_{\max} = \pi/4$, the method presented here fails and an analysis of a different kind is needed [14].

In the present paper the algorithm has been applied to a composite which consists of a square array of cylinders embedded in a matrix. The algorithm was also applied to the composites of hexagonal geometry [15].

6. Conclusion

A new functional basis derived in this paper allowed us to obtain a simple recurrence algorithm for calculating the effective transport coefficient of regular two-dimensional composites (3.10), (3.11). The algorithm is simply recursive and does not involve the solution of a large number of coupled equations. The results are used as input data to express the effective transport coefficient in the form of a rapidly convergent continuous fraction expansion.

Appendix

The algorithm presented below enables a recurrence calculation of the J-fraction coefficients k_n and l_n, on the basis of the given coefficients C_n of the power series (4.1). The coefficients are calculated from the following formulae [13]:

$$k_{n+1} = \frac{\sigma_n}{\sigma_{n-1}}\,, \qquad l_{n+1} = \tau_{n-1} - \tau_n\,, \tag{A.1}$$

where

$$\sigma_n = C_{2n+1} + \sum_{j=1}^{n} b_{nj} C_{2n+1-j}\,, \tag{A.2}$$

$$\tau_n = \frac{1}{\sigma_n}\left[C_{2n+2} + \sum_{j=1}^{n} b_{nj} C_{2n+2-j}\right]. \tag{A.3}$$

We start with $n = 0$. The required initial values of parameters are

$$\sigma_{-1} = 1, \qquad \tau_{-1} = 0.$$

Hence we have from (A.1)–(A.3)

$$k_1 = C_1 = \varphi, \qquad l_1 = -\frac{C_2}{C_1} = \frac{1}{2}(1-\varphi). \tag{A.4}$$

The successive values of k_n, l_n are then calculated from (A.1). Several auxiliary parameters b_{nj} in (A.2) and (A.3) have the following values:

$$b_{n-1,-1} = 0, \qquad b_{n,n+1} = 0, \qquad b_{n+1,0} = 1, \qquad b_{0,0} = 1,$$

the other ones must be determined from the relation

$$b_{n,j} = b_{n-1,j} + l_n b_{n-1,j-1} - k_n b_{n-2,j-2}\,. \tag{A.5}$$

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