2.2.3. – teoria związków konstytutywnych
5.3.1. – teoria ogólna

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SOME ASPECTS OF INVARIANT THEORY IN PLASTICITY

Part II. Constitutive relations for perfectly locking materials. Comments on perfectly plastic solids

Summary

The aim of this contribution is twofold. Firstly, by using the representation theory of isotropic tensor functions, general form of the constitutive relationship for perfectly locking materials is derived. The homogeneity condition of degree zero imposed on the locking behaviour permits to obtain the general form of the locking locus. Some particular cases are also studied.

Secondly, to account for the dissipation density dependent on hydrostatic pressure, the theory proposed by Sawczuk and Stutz [1] is generalized. Consequently, incompressible behaviour of isotropic perfectly plastic materials obeying pressure sensitive yield condition is described.

1. Introduction

The constitutive relations for perfectly locking materials seem to have been first proposed by Prager [2]. Counterparts of the limit analysis theorems were formulated by Čyras [3]. Next, more rigorous mathematical study of not necessarily perfectly locking solids have been undertaken in Refs [4-7].

Our aim here is to derive the general form of the constitutive equation (2.1) provided that (2.2) is satisfied. The condition (2.2) expresses the fact that Eq. (2.1) should not depend upon the time scale. Locking behaviour can strongly depend upon the density ρ of the material, c.f. Refs [8-10]. Therefore, we assume that the tensor function $\hat{\mathbf{E}}$ depends not only on $\dot{\mathbf{T}}$, but also on ρ.

The representation theory of tensor functions is a convenient tool for the study of Eq. (2.1) satisfying the condition (2.2). Having derived the general form of the tensorially nonlinear constitutive relationship describing perfectly locking behaviour, two-dimensional cases are discussed. Several particular locking loci are proposed.

The locking law derived is, in general, not associated with the corresponding locking condition. The associated locking law is obtained under an additional condition.

The second problem discussed in the paper concerns a generalization of the model of isotropic perfectly plastic solids, proposed by Sawczuk and Stutz [1]. That model yields the dissipation density d as a function of the rate of deformation tensor $\mathbf{D}$ only. Thus incompressible materials obeying pressure dependent yield conditions are precluded by the model presented in [1]. For the model proposed in Section 10, d depends not only on $\mathbf{D}$ but also on a scalar parameter ξ. Particularly one may take $\xi = tr\mathbf{T} = T_{ii}$, where $\mathbf{T}$ is the stress tensor.

The comprehensive paper by Spencer [11], the book [12] and review paper [13] provide an exhaustive source of informations related to both the theory of scalar invariants and tensor functions as well as to their applications to the formulation of various constitutive relationships, c.f. also Refs [14-24].

An influence of a fabric tensor on perfectly locking and perfectly plastic behaviour will be studied in a separate paper; for earlier results the reader should refer to Refs [25-29].

2. General form of isotropic constitutive relationship for perfectly locking materials

The general constitutive relationship for isotropic, density dependent locking materials has the following form

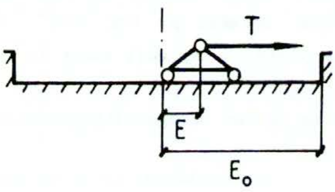
$$\mathbf{E} = \hat{\mathbf{E}}\,(\rho,\ \dot{\mathbf{T}}) \tag{2.1}$$

subject to

$$\frac{\partial \mathbf{E}}{\partial \dot{\mathbf{T}}}\,.\dot{\mathbf{T}} = 0 \quad \left(\text{or} \quad \frac{\partial E_{ij}}{\partial \dot{T}_{kl}}\,\dot{T}_{kl} = 0\right) \quad \text{if} \quad \frac{\partial \mathbf{E}}{\partial \dot{\mathbf{T}}} \neq \underline{\mathbf{0}}\,. \tag{2.2}$$

Here $\mathbf{E} = (E_{ij}) \in \mathbb{E}^3_S$ is the strain tensor, $\dot{\mathbf{T}} = (\dot{T}_{ij})$ denotes the stress rate tensor, $\mathbb{E}^3_S$ is the space of symmetric 3×3 metrices and $\underline{\mathbf{0}}$ ($\mathbf{0}$) is the zero tensor of the fourth (second) order. The homogeneity condition (2.2) expresses time-scale independence of strains.

One-dimensional case is illustrated below, see Fig.1 and Fig.2, c.f. [7].



$|E| \le E_o$

and

$|E| < E_o \Rightarrow T = 0$

$|E| = E_o \Rightarrow T = \lambda\,\dfrac{E}{|E|}$

Fig.1

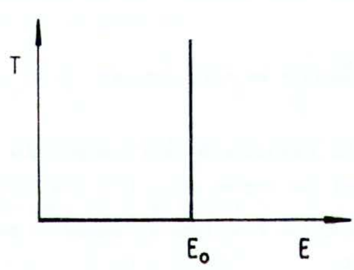


Fig.2

To study the tensorially nonlinear constitutive equation (2.1) under the condition (2.2) a procedure similar to that used by Sawczuk and Stutz [1] for isotropic perfectly plastic materials will consequently be used, c.f. also [15-19]. According to the theory of representation of isotropic tensor functions the relation (2.1) is expressed in the following way

$$\mathbf{E} = \alpha_0 \mathbf{I} + \alpha_1 \dot{\mathbf{T}} + \alpha_2 \dot{\mathbf{T}}^2 , \tag{2.3}$$

where

$$\alpha_h = f_h(\rho,\ tr\dot{\mathbf{T}},\ tr\dot{\mathbf{T}}^2,\ tr\dot{\mathbf{T}}^3) , \tag{2.4}$$

and $tr\dot{\mathrm{T}} = \dot{T}_{ii}$, $tr\dot{\mathrm{T}}^2 = \dot{T}_{ij}\dot{T}_{ji}$, $tr\dot{\mathrm{T}}^3 = \dot{T}_{ij}\dot{T}_{jk}\dot{T}_{ki}$. The summation convention is used throughout the paper. Moreover f_h are arbitrary functions of the basic invariants $tr\dot{\mathrm{T}}$, $tr\dot{\mathrm{T}}^2$, $tr\dot{\mathrm{T}}^3$ and the density ρ, c.f. the formula (A.8) given in Appendix.

On account of the homogeneity condition (2.2), the relation (2.3) cannot be uniquely inverted. We recall that (2.2) implies $det(\partial\hat{\mathbf{E}}/\partial\dot{\mathrm{T}}) = 0$. Moreover, as will be demonstrated in the sequel, (2.2) implies the existence of a scalar relation, called locking locus, between ρ, $tr\mathbf{E}$, $tr\mathbf{E}^2$, $tr\mathbf{E}^3$. The locking locus is a counterpart of the yield locus, well known in the theory of plasticity.

Let us investigate the consequences of the homogeneity condition (2.2). When applied to Eq. (2.3) it yields

$$\left(\frac{\partial\alpha_0}{\partial\dot{\mathrm{T}}} . \dot{\mathrm{T}}\right)\mathrm{I} + \left(\frac{\partial\alpha_1}{\partial\dot{\mathrm{T}}} . \dot{\mathrm{T}} + \alpha_1\right)\dot{\mathrm{T}} + \left(\frac{\partial\alpha_2}{\partial\dot{\mathrm{T}}} . \dot{\mathrm{T}} + 2\alpha_2\right)\dot{\mathrm{T}}^2 = 0 , \tag{2.5}$$

where $\mathbf{a}.\mathbf{b} = a_{ij}b_{ji}$. The last relation is satisfied for an arbitrary $\dot{\mathrm{T}}$ provided that

$$\frac{\partial\alpha_k}{\partial\dot{\mathrm{T}}} . \dot{\mathrm{T}} + k\alpha_k = 0 \ ; \ k=0,1,2. \tag{2.6}$$

Eqs (2.6) imply that the functions α_k are homogeneous with respect to the stress rate tensor and of the degree zero, (-1) and (-2), respectively; c.f. (A.7). As we already know, the functions α_k depend on basic invariants of $\dot{\mathrm{T}}$, hence Eqs (2.6) reduce to

$$\frac{\partial\alpha_k}{\partial tr\dot{T}} tr\dot{T} + 2\frac{\partial\alpha_k}{\partial tr\dot{T}^2} tr\dot{T}^2 + 3\frac{\partial\alpha_k}{\partial tr\dot{T}^3} tr\dot{T}^3 + k\alpha_k = 0 , \quad k=0,1,2. \tag{2.7}$$

By introducing new variables

$$t_1 = tr\dot{T}, \quad t_2 = tr^{1/2}\dot{T}^2, \quad t_3 = tr^{1/3}\dot{T}^3, \tag{2.8}$$

the system of equations (2.7) is transformed to

$$\frac{\partial\alpha_k}{\partial t_i} t_i + k\alpha_k = 0 ; \quad k=0,1,2. \tag{2.9}$$

Now the functions α_k depend on t_1, t_2 and t_3, yet the same notation is preserved. To solve the system (2.9) new variables are introduced, c.f. [30,31]

$$x = \ln t_2, \quad p = t_1/t_2, \quad q = t_3/t_2, \quad t_2 > 0 \quad . \tag{2.10}$$

We set

$$\tilde{\alpha}_k(x,p,q) = \alpha_k[t_1(x,p),\ t_2(x),\ t_3(x,q)] , \tag{2.11}$$

where

$$t_1(x,p) = t_2 p = e^x p , \quad t_2 = e^x , \quad t_3(x,q) = t_2 q = e^x q . \tag{2.12}$$

Hence

$$\frac{\partial\tilde{\alpha}_k}{\partial x} = \frac{\partial\alpha_k}{\partial t_i}\frac{\partial t_i}{\partial x} = \frac{\partial\alpha_k}{\partial t_1} e^x p + \frac{\partial\alpha_k}{\partial t_2} e^x + \frac{\partial\alpha_k}{\partial t_3} e^x q = \frac{\partial\alpha_k}{\partial t_i} t_i . \tag{2.13}$$

Thus the system (2.9) transforms to

$$\frac{\partial\tilde{\alpha}_k}{\partial x} + k\tilde{\alpha}_k = 0, \quad k=0,1,2, \tag{2.14}$$

and vice versa, obviously.

The solution of (2.14) is given by

$$\alpha_0 = A(\rho,p,q) , \quad \alpha_1 = \frac{1}{t_2} B(\rho,p,q) , \quad \alpha_2 = \frac{1}{t_2^2} C(\rho,p,q) , \tag{2.15}$$

where A, B and C are arbitrary functions of their arguments and the tilda has been dropped.

Substituting (2.15) into (2.3) we obtain

$$E = AI + \frac{1}{t_2} B\dot{T} + \frac{1}{t_2^2} C\dot{T}^2 . \tag{2.16}$$

Taking account of (2.16), (A.9) and (A.11)-(A.13) we calculate

$$E^2 = \alpha I + \frac{\beta}{t_2}\dot{T} + \frac{\gamma}{t_2^2}\dot{T}^2 , \tag{2.17}$$

where

$$\alpha = A^2 + 2BC(\tfrac{1}{6}p^3 - \tfrac{1}{2}p + \tfrac{1}{3}q^3) + C^2(\tfrac{1}{6}p^4 - \tfrac{1}{2}p^2 + \tfrac{1}{3}pq^3),$$

$$\beta = 2AB + BC(1 - p^2) + \tfrac{1}{3}C^2(q^3 - p^3), \tag{2.18}$$

$$\gamma = 2AC + B^2 + 2BCp + \tfrac{1}{2}C^2(p^2 + 1).$$

By using (2.16)-(2.18) we find

$$\frac{\dot{T}}{t_2} = \lambda\ (\beta_0 I + \beta_1 E + \beta_2 E^2), \tag{2.19}$$

where

$$\lambda = (B - \frac{\beta C}{\gamma})^{-1},\ \beta_0 = \frac{C\alpha}{\gamma} - A ,\quad \beta_1 = 1 ,\ \beta_2 = -\frac{C}{\gamma} . \tag{2.20}$$

The constitutive relation (2.19) is an inverse one with respect to Eq. (2.16).

We now pass to the derivation of the general form of the locking locus. By using the relations (2.16), (2.17) and the identities (A.10)-(A.13) we have

$$trE = 3A + Bp + C ,$$

$$trE^2 = 3\alpha + \beta p + \gamma , \tag{2.21}$$

$$trE^3 = 3A\alpha + (\alpha B + \beta A)p + (\alpha C + \beta B + \gamma A) + (\beta C + \gamma\beta)q^3 + $$
$$+ \frac{\gamma C}{6}(p^4 + 8pq^3 - 6p^2 + 3) .$$

Eliminating the parameters p and q in (2.21), provided that such an elimination can be performed, the general form of the locking condition is obtained:

$$f(\rho,\ trE,\ tr^{1/2}E^2,\ tr^{1/3}E^3) = 0 . \tag{2.22}$$

In general, the locking law (2.19) is not associated with the locking locus (2.22). Thus without additional assumptions one has, in general,

$$\dot{T} \neq \Lambda\frac{\partial f}{\partial E} ,\quad \Lambda \geq 0 . \tag{2.23}$$

The associated locking law is obtained provided that, c.f. Section 3

$$\frac{\partial \hat{E}_{ij}}{\partial \dot{T}_{kl}} = \frac{\partial \hat{E}_{kl}}{\partial \dot{T}_{ij}} . \tag{2.24}$$

It should also be noted that the set

$$K(\rho) = \{\mathbf{E} \in \mathbb{E}^3_s \mid f(\rho, tr\mathbf{E}, tr^{1/2}\mathbf{E}^2, tr^{1/3}\mathbf{E}^3) \le 0\} , \tag{2.25}$$

is not necessarily convex. Obviously, convexity of the set $K(\rho)$ imposes a restriction on the function f, and consequently on A, B and C, c.f. [32].

If the set $K(\rho)$ is convex and closed, then the general subdifferential form of the associated locking law is given by

$$\dot{\mathbf{T}} \in \partial I_{K(\rho)}(\mathbf{E}) , \tag{2.26}$$

where

$$I_{K(\rho)}(\mathbf{E}) = \begin{cases} 0, \text{ if } \mathbf{E} \in K(\rho) , \\ + \infty, \text{ otherwise} . \end{cases} \tag{2.27}$$

Denoting by E_i (i=1,2,3) the principal strains, the locking locus may equivalently be written in the following way, c.f. Appendix

$$g(\rho, E_1, E_2, E_3) = 0 . \tag{2.28}$$

For $C \equiv 0$ we obtain the so called tensorially linear relation and (2.19) simplifies to

$$\frac{\dot{\mathbf{T}}}{t_2} = \frac{1}{B} (-A\mathbf{I} + \mathbf{E}), \tag{2.29}$$

The parametric form of the yield locus (for $C \equiv 0$) is then given by

$$\begin{aligned} trE &= 3A + Bp , \\ trE^2 &= 3A^2 + B^2 + 2ABp , \\ trE^3 &= 3(A^3 + A^2Bp + AB^2) + B^3q^3 . \end{aligned} \tag{2.30}$$

In the sequel, the above relations will be used for a specification of particular locking loci and locking laws, c.f. Sections 6, 7 and 8.

3. The associated locking law

The associated locking law is obviously a particular case of the general nonassociated law given by (2.19). Firstly, however, the direct law is investigated. Eq. (2.24) implies

$$\frac{\partial \hat{E}_{ij}}{\partial \dot{T}_{kl}} \dot{T}_{kl} = \frac{\partial \hat{E}_{kl}}{\partial \dot{T}_{ij}} \dot{T}_{kl} . \tag{3.1}$$

We set

$$\mathfrak{L} = \frac{\partial \hat{E}_{ij}}{\partial \dot{T}_{kl}} \dot{T}_{kl} , \qquad \mathfrak{P} = \frac{\partial \hat{E}_{kl}}{\partial \dot{T}_{ij}} \dot{T}_{kl} , \tag{3.2}$$

$$K_1 = tr\dot{\mathbf{T}}, \quad K_2 = tr\dot{\mathbf{T}}^2, \quad K_3 = tr\dot{\mathbf{T}}^3 . \tag{3.3}$$

By using (2.3) and (3.2) we readily obtain

$$\begin{aligned} \mathfrak{L} = & \left(\frac{\partial \alpha_0}{\partial K_1} K_1 + 2\frac{\partial \alpha_0}{\partial K_2} K_2 + 3\frac{\partial \alpha_0}{\partial K_3} K_3 \right)\mathbf{I} + \\ & + \left(\frac{\partial \alpha_1}{\partial K_1} K_1 + 2\frac{\partial \alpha_1}{\partial K_2} K_2 + 3\frac{\partial \alpha_1}{\partial K_3} K_3 + \alpha_1 \right)\dot{\mathbf{T}} + \\ & + \left(\frac{\partial \alpha_2}{\partial K_1} K_1 + 2\frac{\partial \alpha_2}{\partial K_2} K_2 + 3\frac{\partial \alpha_2}{\partial K_3} K_3 + 2\alpha_2 \right)\dot{\mathbf{T}}^2, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \mathfrak{P} = & \left(\frac{\partial \alpha_0}{\partial K_1} K_1 + \frac{\partial \alpha_1}{\partial K_1} K_2 + \frac{\partial \alpha_2}{\partial K_1} K_3 \right)\mathbf{I} + \\ & + \left(2\frac{\partial \alpha_0}{\partial K_2} K_1 + 2\frac{\partial \alpha_1}{\partial K_2} K_2 + 2\frac{\partial \alpha_2}{\partial K_2} K_3 + \alpha_1 \right)\dot{\mathbf{T}} + \\ & + \left(3\frac{\partial \alpha_0}{\partial K_3} K_1 + 3\frac{\partial \alpha_1}{\partial K_3} K_2 + 3\frac{\partial \alpha_2}{\partial K_3} K_3 + 2\alpha_2 \right)\dot{\mathbf{T}}^2. \end{aligned} \tag{3.5}$$

Because $\mathfrak{L} = \mathfrak{P}$, hence (3.4) and (3.5) imply

$$2\frac{\partial \alpha_0}{\partial K_2} = \frac{\partial \alpha_1}{\partial K_1} , \qquad 3\frac{\partial \alpha_0}{\partial K_3} = \frac{\partial \alpha_2}{\partial K_1} , \qquad 3\frac{\partial \alpha_1}{\partial K_3} = 2\frac{\partial \alpha_2}{\partial K_2} . \tag{3.6}$$

If (3.6) is satisfied then a potential $G(\rho, \dot{\mathbf{T}}) = G_1(\rho, K_1, K_2, K_3)$ exists such that

$$\mathbf{E} = \partial G/\partial \dot{\mathbf{T}} . \tag{3.7}$$

The potential G may then be identified with the locking work rate, see Section 5.

By noting that the functions α_0, α_1 and α_2 are given by (2.15), the condition (3.6) assumes the following form

$$p\frac{\partial A}{\partial p} + q\frac{\partial A}{\partial q} = -\frac{\partial B}{\partial p},$$

$$\frac{\partial A}{\partial q} = q^2\frac{\partial C}{\partial p}, \tag{3.8}$$

$$q^2\left(2C + p\frac{\partial C}{\partial p} + q\frac{\partial C}{\partial q}\right) = -\frac{\partial B}{\partial q}.$$

Let us now investigate the inverse locking law (2.19), which is written as follows

$$\dot{\mathbf{T}} = \lambda(\gamma_0\mathbf{I} + \gamma_1\mathbf{E} + \gamma_2\mathbf{E}^2), \tag{3.9}$$

where $\gamma_h = \gamma_h(\rho, L_1, L_2, L_3)$, (h=0,1,2), and $L_1 = tr\mathbf{E}$, $L_2 = tr\mathbf{E}^2$, $L_3 = tr\mathbf{E}^3$.

If the symmetry condition

$$\frac{\partial \dot{T}_{ij}}{\partial \hat{E}_{kl}} = \frac{\partial \dot{T}_{kl}}{\partial \hat{E}_{ij}}. \tag{3.10}$$

is satisfied then a potential $F(\rho, \mathbf{E})$ exists such

$$\dot{\mathbf{T}} = \lambda\partial F/\partial\mathbf{E}. \tag{3.11}$$

The condition (3.10) is fulifilled provided that, c.f. (3.6)

$$2\frac{\partial\gamma_0}{\partial L_2} = \frac{\partial\gamma_1}{\partial L_1}, \qquad 3\frac{\partial\gamma_0}{\partial L_3} = \frac{\partial\gamma_2}{\partial L_1}, \qquad 3\frac{\partial\gamma_1}{\partial L_3} = 2\frac{\partial\gamma_2}{\partial L_2}. \tag{3.12}$$

Under the conditions specified by (3.12), the potential F may be identified with the locking locus.

4. Spherical and deviatoric parts of the locking law

Let $\mathbf{E}^d$ and $\dot{\mathbf{S}}$ denote the deviatoric parts of the strain tensor $\mathbf{E}$ and the stress rate tensor $\dot{\mathbf{T}}$, respectively, c.f. (A.14). For physical reasons, it is convenient to split up Eq. (2.3) into the spherical

$$tr\mathbf{E} = 3\phi_0, \tag{4.1}$$

and deviatoric

$$\mathbf{E}^d = \phi_1\dot{\mathbf{S}} + \phi_2[\dot{\mathbf{S}}^2 - \tfrac{1}{3}(tr\dot{\mathbf{S}}^2)\mathbf{I}], \tag{4.2}$$

parts, respectively, where

$$\phi_0 = \alpha_0 + \tfrac{1}{3}\alpha_1 tr\dot{\mathbf{T}} + \tfrac{1}{3}\alpha_2 tr\dot{\mathbf{S}}^2 + \tfrac{1}{9}\alpha_2 tr^2\dot{\mathbf{T}}^2, \tag{4.3}$$

$$\phi_1 = \alpha_1 + \frac{2}{3}\alpha_2 tr\dot{T} , \qquad \phi_2 = \alpha_2 . \tag{4.4}$$

Taking account of the identities

$$\begin{aligned} \dot{T}^2 &= \dot{S}^2 + \frac{2}{3}(tr\dot{T})\dot{S} + \frac{1}{9}(tr^2\dot{T})I , \\ \dot{T}^3 &= \dot{S}^3 + (tr\dot{T})\dot{S}^2 + \frac{1}{3}(tr^2\dot{T})\dot{S} + \frac{1}{27}(tr^3\dot{T})I , \\ tr\dot{T}^2 &= tr\dot{S}^2 + \frac{1}{3}tr^2\dot{T} , \\ tr\dot{T}^3 &= tr\dot{S}^3 + tr\dot{T}tr\dot{S}^2 + \frac{1}{9}tr^3\dot{T} , \end{aligned} \tag{4.5}$$

and applying the homogeneity condition (2.2) to (4.1) and (4.2) we obtain

$$\frac{\partial\phi_k}{\partial s_i} s_i + k\phi_k = 0 ; \; k=0,1,2. \tag{4.6}$$

where

$$s_1 = t_1 = tr\dot{T}, \quad s_2 = tr^{1/2}\dot{S}^2, \quad s_3 = tr^{1/3}\dot{S}^3 . \tag{4.7}$$

Functions

$$\phi_0 = a(\rho,r,s), \quad \phi_1 = \frac{1}{s_2} b(\rho,r,s) , \; \phi_2 = \frac{1}{s_2^2} c(\rho,r,s) \tag{4.8}$$

solve the system (4.6), where

$$r = s_1/s_2 , \qquad s = s_3/s_2 . \tag{4.9}$$

Obviously, the functions ϕ_k, $(k=0,1,2)$, are homogeneous functions of the degree zero, minus 1 and minus 2, respectively.

By using (4.2) and the identities (A.15)-(A.17) we find

$$\frac{1}{s_2}\dot{S} = \lambda\{E^d - \frac{6c}{6b^2-c^2}[(E^d)^2 - \frac{1}{3}(tr(E^d)^2)I]\} , \tag{4.10}$$

where

$$\lambda = [b - \frac{2c^2}{6b^2-c^2}(b + cs^3)]^{-1}. \tag{4.11}$$

Now, similarly as in the previous section we derive the locking locus in the parametric form

$$trE = 3a ,$$

$$tr(E^d)^2 = b^2 + 2bcs^3 + \frac{1}{6}c^2 = (b + cs^3)^2 + c^2(\frac{1}{6} - s^6), \qquad (4.12)$$

$$tr(E^d)^3 = \frac{1}{2}bc(b + cs^3) + \frac{1}{3}c^3(s^6 - \frac{1}{12}) + b^3s^3 =$$

$$= s^3(b + cs^3)^3 + \frac{1}{2}c(1 - 6s^6)[b(b + cs^3) - \frac{1}{18}c^2(1 - 6s^6)] .$$

Eliminating the parameters r and s from (4.12) we obtain the general form of the locking condition

$$f(\rho, \ tr\mathbf{E}, \ tr^{1/2}(\mathbf{E}^d)^2, \ tr^{1/3}(\mathbf{E}^d)^3) = 0 . \qquad (4.13)$$

For c=0, the relation (4.10) takes a simple form

$$\frac{\dot{\mathbf{S}}}{s_2} = \frac{1}{b}\mathbf{E}^d . \qquad (4.14)$$

Hence

$$s = \frac{tr^{1/3}\dot{\mathbf{S}}^3}{tr^{1/2}\dot{\mathbf{S}}^2} = \frac{tr^{1/3}(\mathbf{E}^d)^3}{tr^{1/2}(\mathbf{E}^d)^2} . \qquad (4.15)$$

Thus the parametric form of the locking condition is given by

$$trE = a(\rho, \ \frac{tr^{1/3}(\mathbf{E}^d)^3}{tr^{1/2}(\mathbf{E}^d)^2}) ,$$

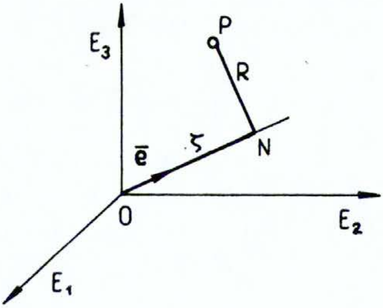
$$(4.16)$$

$$tr^{1/2}(\mathbf{E}^d)^2 = b(\rho, \ \frac{tr^{1/3}(\mathbf{E}^d)^3}{tr^{1/2}(\mathbf{E}^d)^2}) .$$

By an elimination of the parameter r we obtain the general form of the locking condition in the tensorially linear case:

$$f(\rho, \ trE, \ tr^{1/2}(\mathbf{E}^d)^2, \ \frac{tr^{1/3}(\mathbf{E}^d)^3}{tr^{1/2}(\mathbf{E}^d)^2}) = 0 . \qquad (4.17)$$

The invariants of the deformation tensor **E** occuring in the last formula have a simple geometric interpretation. Let the strain state be described by the principal strains E_k (k=1,2,3) and characterized by the vector $\overline{OP}$, where $P(E_k)$, $O(0,0,0)$. We set $\overline{OP} = \overline{ON} + \overline{NP}$, c.f. Fig.3.



$$\bar{e} = \frac{1}{\sqrt{3}} [1,\ 1,\ 1]$$

Fig.3

The straight line determined by $\bar{e}$ makes equal angels with the axes. On the other hand, the vector $\overline{NP}$ belongs to the plane orthogonal to this line. This plane is called the deviatoric plane and is denoted by Π. Its equations is $E_1 + E_2 + E_3 = 0$. The length of the vector $\overline{ON}$ is

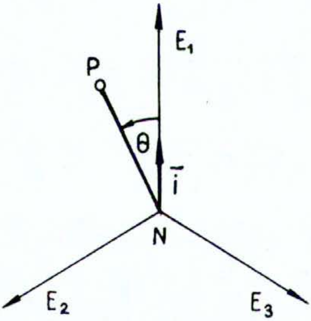
$$|\overline{ON}| = \zeta = tr\mathbf{E}/\sqrt{3}\ , \tag{4.18}$$

whereas $|\overline{NP}|$ characterizes the deviatoric part of $\mathbf{E}$

$$|\overline{NP}| = R = tr^{1/2}(\mathbf{E}^d)^2\ . \tag{4.19}$$

The angle Θ (Fig.4) between $\overline{NP}$ and $\bar{i}$ is

$$\Theta = arc\ cos[\sqrt{6}(\ \frac{tr^{1/3}(\mathbf{E}^d)^3}{tr^{1/2}(\mathbf{E}^d)^2}\ )^3]/3\ . \tag{4.20}$$



$$\bar{i} = \frac{1}{\sqrt{6}} [2,\ -1,\ -1]$$

Fig.4

5. Locking work rate

In our case the density W of the locking work rate is given by

$$
\begin{aligned}
W(\rho,\dot{\mathrm{T}}) &= \dot{\mathrm{T}}_{ij}\mathrm{E}_{ij} = \alpha_0(\rho,\mathrm{p},\mathrm{q})\,tr\dot{\mathbf{T}} + \alpha_1(\rho,\mathrm{p},\mathrm{q})\,tr\dot{\mathbf{T}}^2 + \alpha_2(\rho,\mathrm{p},\mathrm{q})\,tr\dot{\mathbf{T}}^3 = \\
&= \mathrm{A}(\rho,\mathrm{p},\mathrm{q})\mathrm{t}_1 + \mathrm{B}(\rho,\mathrm{p},\mathrm{q})\mathrm{t}_2 + \mathrm{C}(\rho,\mathrm{p},\mathrm{q})\mathrm{q}^2\mathrm{t}_3 = \\
&= \dot{\mathrm{S}}_{ij}\mathrm{E}_{ij} + \tfrac{1}{3}\,tr\dot{\mathbf{T}}\,tr\mathbf{E} = \phi_0(\rho,\mathrm{r},\mathrm{s})\,tr\dot{\mathbf{T}} + \phi_1(\rho,\mathrm{r},\mathrm{s})\,tr\dot{\mathbf{S}}^2 + \phi_2(\rho,\mathrm{r},\mathrm{s})\,tr\dot{\mathbf{S}}^3 = \\
&= \mathrm{a}(\rho,\mathrm{r},\mathrm{s})\mathrm{s}_1 + \mathrm{b}(\rho,\mathrm{r},\mathrm{s})\mathrm{s}_2 + \mathrm{c}(\rho,\mathrm{r},\mathrm{s})\mathrm{s}^2\mathrm{s}_3,
\end{aligned}
\tag{5.1}
$$

where the functions α_h, A, B, C, ϕ_h, a, b and c are defined by (2.15) and (4.8), respectively. The density ρ may be assumed to be a function of the strain, e.g.:, c.f. [8]

$$\rho = \hat{\rho}(\mathbf{E}) \ . \tag{5.2}$$

Then one has

$$\mathfrak{W}(\mathbf{E},\dot{\mathbf{T}}) := W(\hat{\rho}(\mathbf{E}),\dot{\mathbf{T}}) \ . \tag{5.3}$$

Particularly, $\rho = \hat{\rho}(tr\mathbf{E})$ and then we have

$$\mathfrak{W}_1(tr\mathbf{E},\dot{\mathbf{T}}) = W(\hat{\rho}(tr\mathbf{E}),\dot{\mathbf{T}}) \ . \tag{5.4}$$

In the constitutive relation (2.3) and (2.4) ρ may be treated as a scalar parameter. Above, we have interpreted it as a density. A dependence of ρ on strains (E_{ij}) implies that the density of the locking work rate depends not only on stress rate $(\dot{\mathrm{T}}_{ij})$ but also, explicitly, on **E**. Such a dependence significantly enlarges the class of non-assciated isotropic locking laws.

If (3.6) is satisfied then we may set G = W because

$$\mathbf{E} = \frac{\partial}{\partial\dot{\mathrm{T}}}(\dot{\mathrm{T}}_{ij}\mathrm{E}_{ij}) = \mathbf{E} + \frac{\partial \mathrm{E}_{ij}}{\partial\dot{\mathrm{T}}}\dot{\mathrm{T}}_{ij} = \mathbf{E} = \frac{\partial W}{\partial\dot{\mathrm{T}}} = \frac{\partial \mathrm{G}}{\partial\dot{\mathrm{T}}} \ , \tag{5.5}$$

provided that (2.2) and (3.1) are satisfied.

6. Plane stress state

The present Section is concerned with practically important case when the stress tensor has the following form

$$T = \begin{bmatrix} \underline{\mathfrak{T}} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad \underline{\mathfrak{T}} = \begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix} \tag{6.1}$$

In this particular case the constitutive relation (2.16) and the identities (A.19), (A.20) yield

$$\mathbf{E} = \begin{bmatrix} \underline{\mathfrak{E}} & & 0 \\ & & 0 \\ 0 & 0 & E_{33} \end{bmatrix} \quad , \quad \underline{\mathfrak{E}} = [\mathfrak{E}_{\alpha\beta}]; \ \alpha,\beta = 1,2. \tag{6.2}$$

where

$$\begin{aligned} \underline{\mathfrak{E}} &= \mathfrak{B}(\rho,p)\underline{\mathfrak{I}} + \frac{\mathfrak{C}(\rho,p)}{t_2} \dot{\underline{\mathfrak{T}}} \ , \\ E_{33} &= \mathfrak{A}(\rho,p) \end{aligned} \tag{6.3}$$

and

$$\mathfrak{A} = A \ , \quad \mathfrak{B} = A + \frac{C}{2}(1 - p^2) \ , \ \mathfrak{C} = B + Cp \ . \tag{6.4}$$

Here $t_2 = tr^{1/2}\dot{\underline{\mathfrak{T}}}^2$, $p = tr\dot{\underline{\mathfrak{T}}}/tr^{1/2}\dot{\underline{\mathfrak{T}}}^2$ and $\underline{\mathfrak{I}} = [\delta_{\alpha\beta}]$. Obviously $E_{13} = E_{23} = 0$. The material functions $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ depend on ρ and p solely because, c.f. (A.20)

$$q^3 = \frac{tr\dot{T}^3}{tr^{3/2}\dot{T}^2} = \frac{tr\dot{\underline{\mathfrak{T}}}^3}{tr^{3/2}\dot{\underline{\mathfrak{T}}}^2} = \frac{1}{2}(3p - p^3) \ . \tag{6.5}$$

The relation inverse to $(6.3)_1$ takes the form

$$\frac{1}{tr^{3/2}\dot{\underline{\mathfrak{T}}}^2} \dot{\underline{\mathfrak{T}}} = - \frac{\mathfrak{B}}{\mathfrak{C}} \underline{\mathfrak{I}} + \frac{1}{\mathfrak{C}} \underline{\mathfrak{E}} \ , \tag{6.6}$$

whereas the parametric locking condition is expressed by

$$\begin{aligned} tr\mathbf{E} &= tr\underline{\mathfrak{E}} + E_{33} = \mathfrak{A} + 2\mathfrak{B} + \mathfrak{C}p \ , \\ tr\mathbf{E}^2 &= tr\underline{\mathfrak{E}}^2 + E_{33}^2 = \mathfrak{A}^2 + 2\mathfrak{B}^2 + 2\mathfrak{B}\mathfrak{C}p + \mathfrak{C}^2. \end{aligned} \tag{6.7}$$

As previously, eliminating the parameter p from the system (6.7) we obtain the general form of locking condition for the plane stress state:

$$f(\rho, \ tr\mathbf{E}, \ tr^{1/2}\mathbf{E}^2) = 0 \ , \tag{6.8}$$

or equivalently

$$f(\rho, \ tr\underline{\mathfrak{E}} + E_{33}, \ (tr\underline{\mathfrak{E}}^2 + E_{33}^2)^{1/2}] = 0 \ . \tag{6.9}$$

We note that the results given above can also be obtained in a somewhat diffrent manner. The second approach consists in substitution of (6.1), (A.19) and (A.20) into (2.3); next the homogeneity condition (2.2) is exploited.

The locking locus and the locking law for the plane stress state are derived similarly as in Section 4. There is an analogy with Telega's [19] study of the plane flow for perfectly plastic materials.

The deviatoric part of the stress rate tensor takes the form

$$\dot{\underline{\mathfrak{S}}} = \dot{\underline{\mathfrak{T}}} - \frac{1}{3}(tr\dot{\underline{\mathfrak{T}}})\underline{\mathfrak{I}} ,$$
$$\dot{S}_{33} = -\frac{1}{3}\, tr\dot{\underline{\mathfrak{T}}} , \quad \dot{S}_{13} = \dot{S}_{23} = 0 , \tag{6.10}$$

where $\dot{\underline{\mathfrak{S}}} = (\dot{\mathfrak{S}}_{\alpha\beta})$ $(\alpha,\beta = 1,2)$; we recall that $tr\dot{\mathfrak{T}} = tr\dot{\mathbf{T}}$.

Taking account of (A.18) and (A.20) we have

$$tr\dot{\mathbf{S}}^2 = tr\dot{\underline{\mathfrak{S}}}^2 + \frac{1}{9}\, tr^2\dot{\mathbf{T}} ,$$
$$tr\dot{\mathbf{S}}^3 = \frac{1}{2}\, tr\dot{\mathbf{T}} tr\dot{\mathbf{S}}^2 - \frac{1}{9}\, tr^3\dot{\mathbf{T}} . \tag{6.11}$$

The relation (6.11) imply that now the functions a, b and c occurring in (4.8) depend on ρ and r only, because

$$s = \frac{tr^{1/3}\dot{\mathbf{S}}^3}{tr^{1/2}\dot{\mathbf{S}}^2} = \left(\frac{1}{2}\, r - \frac{1}{9}\, r^3\right)^{1/3} . \tag{6.12}$$

By applying (6.12) and the Cayley-Hamilton theorem in the two-dimensional case (c.f. (A.18)) we calculate

$$\dot{\underline{\mathfrak{S}}}^2 = \frac{1}{3}\,(tr\dot{\mathbf{T}})\dot{\underline{\mathfrak{S}}} - \frac{1}{2}\left(\frac{1}{9}\, tr^2\dot{\mathbf{T}} - tr\dot{\underline{\mathfrak{S}}}^2\right)\underline{\mathfrak{I}} . \tag{6.13}$$

Substituting (6.10) and (6.12) into (4.2) we obtain

$$\underline{\mathfrak{D}} = \underline{\mathfrak{E}} - \frac{1}{3}\,(tr\mathbf{E})\underline{\mathfrak{I}} = \mathfrak{b}(\rho,r)\underline{\mathfrak{I}} + \mathfrak{c}(\rho,r)\dot{\underline{\mathfrak{S}}}/s_2 ,$$
$$E^d_{33} = \mathfrak{a}(\rho,r), \quad E^d_{13} = E^d_{23} = 0 , \tag{6.14}$$

where

$\mathfrak{a}(\rho,r) = \frac{1}{3}\left(\frac{1}{3}\, cr^2 - br - c\right)$, $\mathfrak{b}(\rho,r) = \frac{1}{3}\, c\left(\frac{1}{2} - \frac{1}{6}\, r^2\right)$, $\mathfrak{c}(\rho,r) = b + \frac{1}{3}\, cr$,

and

$$\mathbf{E}^d = \begin{bmatrix} \underline{\mathfrak{Y}} & & 0 \\ & & 0 \\ 0 & 0 & E^d_{33} \end{bmatrix} \tag{6.15}$$

The equation "inverse" to Eq. $(6.14)_1$ has the following form

$$\frac{\dot{\underline{\mathfrak{S}}}}{s_2} = \lambda\ [\underline{\mathfrak{Y}} - b\underline{\mathfrak{I}}]\ , \tag{6.16}$$

where

$$\lambda = c^{-1} = 3/(3b + cr)\ . \tag{6.17}$$

Taking account of $(4.12)_{1,2}$ and (6.14) one obtains a parametric form of the locking condition

$$\begin{aligned} &tr\mathbf{E} = 3a\ , \\ &tr(\mathbf{E}^d)^2 = tr\underline{\mathfrak{Y}}^2 + (E^d_{33})^2 = \tfrac{1}{9}\ [b^2(9 + r^2) + \\ &\qquad + 5c^2(\tfrac{1}{4} + \tfrac{1}{6}\ r^2 + \tfrac{1}{36}\ r^4) + 8bcr - \tfrac{2}{3}\ cr^2(br + c^2)]\ . \end{aligned} \tag{6.18}$$

Eliminating the parameter r from (6.18), a general form of the locking condition is derived:

$$f(\rho,\ tr\mathbf{E},\ tr^{1/2}(\mathbf{E}^d)^2) = 0\ , \tag{6.19}$$

Here $\mathbf{E}$ and $\mathbf{E}^d$ are given by (6.2) and (6.15) respectively.

For the plane stress state the density of the locking work rate is given by

$$\begin{aligned} W &= \dot{\mathfrak{T}}_{\alpha\beta}\mathfrak{E}_{\alpha\beta} = \mathfrak{B}(\rho,p)t_1 + \mathfrak{C}(\rho,p)t_2 = [A + C(1 - p^2)/2]t_1 + (B + Cp)t_2 = \\ &= \dot{\mathfrak{S}}_{\alpha\beta}\mathfrak{Y}_{\alpha\beta} + \tfrac{1}{3}\ tr\dot{\mathbf{T}}tr\mathbf{E} = a(\rho,r)s_1 + c(\rho,r)tr\dot{\underline{\mathfrak{S}}}^2/s_2 = \\ &= [a - r^2(b + cr/3)/9s_2)]s_1 + (b + cr/3)tr\dot{\underline{\mathfrak{S}}}^2/s_2 . \end{aligned} \tag{6.20}$$

where the functions $\mathfrak{B}$, $\mathfrak{C}$ and c are defined by (6.4) and (6.14), respectively.

7. Plane deformation

By definition, the deformation tensor has now the following form

$$\mathbf{E} = \begin{bmatrix} \underline{\mathfrak{E}} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{bmatrix}\ , \qquad \underline{\mathfrak{E}} = [\mathfrak{E}_{\alpha\beta}];\ \alpha,\beta = 1,2. \tag{7.1}$$

Taking account of (7.1) and (A.18) in (2.16) leads to

$$\underline{\mathfrak{E}} = d_0\dot{\underline{\mathfrak{I}}} + d_1\dot{\underline{\mathfrak{T}}}\ , \tag{7.2}$$

$$0 = A + \frac{1}{t_2} B\dot{T}_{33} + \frac{1}{t_2^2} C\dot{T}^2_{33} , \tag{7.3}$$

where

$$d_0 = A + C(tr\dot{\underline{\mathfrak{T}}}^2 - tr^2\dot{\underline{\mathfrak{T}}})/2t_2^2, \quad d_1 = (B/t_2) + Ctr\dot{\underline{\mathfrak{T}}}/t_2^2 , \tag{7.4}$$

and

$$\dot{\mathbf{T}} = \begin{bmatrix} \dot{\underline{\mathfrak{T}}} & & 0 \\ & & 0 \\ 0 & 0 & \dot{T}_{33} \end{bmatrix} . \tag{7.5}$$

Now the basic invariants of the stress rate tensor are given by

$$\begin{aligned} tr\dot{\mathbf{T}} &= tr\dot{\underline{\mathfrak{T}}} + \dot{T}_{33} , \\ tr\dot{\mathbf{T}}^2 &= tr\dot{\underline{\mathfrak{T}}}^2 + \dot{T}^2_{33} , \\ tr\dot{\mathbf{T}}^3 &= tr\dot{\underline{\mathfrak{T}}}^3 + \dot{T}^3_{33} = \tfrac{3}{2}\, tr\dot{\underline{\mathfrak{T}}}\,tr\dot{\underline{\mathfrak{T}}}^2 - \tfrac{1}{2}\, tr^3\dot{\underline{\mathfrak{T}}} + \dot{T}^3_{33}. \end{aligned} \tag{7.6}$$

Taking account of $(2.21)_{1,2}$ and (7.2) we find a parametric form of the locking condition

$$\begin{aligned} tr\mathbf{E} &= tr\underline{\mathfrak{E}} = 2A + \frac{1}{t_2} Btr\dot{\underline{\mathfrak{T}}} + \frac{1}{t_2^2} Ctr\dot{\underline{\mathfrak{T}}}^2 , \\ tr\mathbf{E}^2 &= tr\underline{\mathfrak{E}}^2 = 2d_0^2 + 2d_0 d_1 tr\dot{\underline{\mathfrak{T}}} + d_1^2 tr\dot{\underline{\mathfrak{T}}}^2 . \end{aligned} \tag{7.7}$$

By virtue of (A.20) we write

$$tr\underline{\mathfrak{E}}^3 = \frac{3}{2}\, tr\underline{\mathfrak{E}}\,tr\underline{\mathfrak{E}}^2 - \frac{1}{2}\, tr^3\underline{\mathfrak{E}} . \tag{7.8}$$

Hence the locking condition (c.f. (2.20)) depends on ρ, $tr\underline{\mathfrak{E}}$ and $tr\underline{\mathfrak{E}}^2$ only. Thus we may write

$$f(\rho,\ tr\underline{\mathfrak{E}},\ tr^{1/2}\underline{\mathfrak{E}}^2) = 0 . \tag{7.9}$$

The constitutive relation (2.19) reduces to

$$\begin{aligned} \frac{\dot{\underline{\mathfrak{T}}}}{tr^{1/2}\dot{\mathbf{T}}^2} &= \lambda\ \{[\beta_0 + \frac{\beta_2}{2}(tr\underline{\mathfrak{E}}^2 - tr^2\underline{\mathfrak{E}})]\underline{\mathfrak{I}} + (\beta_1 + \beta_2 tr\underline{\mathfrak{E}})\underline{\mathfrak{E}}\} , \\ \frac{\dot{T}_{33}}{tr^{1/2}\dot{\mathbf{T}}^2} &= \lambda\ \beta_0 , \end{aligned} \tag{7.10}$$

where λ and β_h (h=0,1,2) are given by (2.18). Now the material functions λ

and β_h depend on ρ and p solely, c.f. (7.8).

For the plane deformation the density of the locking work rate is given by

$$W = \dot{\Sigma}_{\alpha\beta}\mathcal{E}_{\alpha\beta} = d_0 tr\dot{\underline{\Sigma}} + d_1 tr\dot{\underline{\Sigma}}^2 \ . \tag{7.11}$$

8. Particular locking laws and locking conditions

In this Section we shall consider simple locking laws and locking conditions resulting from (4.14) and (4.16), respectively.

Example 8.1.

Assume that $3a = a_0(\rho)$, $b = 0$. The locking condition has the following form

$$tr\mathbf{E} = a_0(\rho) \ , \tag{8.1}$$

and $tr(\mathbf{E}^d)^2 = 0$. Thus the locking behaviour influences the spherical part only.

Example 8.2.

Let $a = 0$ and $b = b_0(\rho)$. Then (4.16) reduces to

$$tr^{1/2}(\mathbf{E}^d)^2 = b_0(\rho) \ , \tag{8.2}$$

while the locking law is given by

$$\frac{\dot{S}}{tr^{1/2}\dot{S}^2} = \frac{1}{b_0(\rho)}\mathbf{E}^d \ . \tag{8.3}$$

The condition (8.2) is a counterpart of the Huber-Mises yield condition.

Example 8.3.

Let us assume that $3a = a_1(\rho)r$ and $b = b_0(\rho) + b_1(\rho)r$. The locking condition takes the form

$$a_1 \, tr^{1/2}(\mathbf{E}^d)^2 - b_1 tr\mathbf{E} - b_0 a_1 = 0 \ , \tag{8.4}$$

Now the locking law is

$$\frac{\dot{S}}{tr^{1/2}\dot{S}^2} = \frac{1}{b_0 + \frac{b_1}{a_1} tr\mathbf{E}}\mathbf{E}^d \ . \tag{8.5}$$

The locking condition (8.4) is a counterpart of the Drucker-Prager yield condition. We recall that the last is given by

$$c_1 trT + \frac{1}{\sqrt{2}} tr^{1/2}S^2 = k , \tag{8.6}$$

where c_1 is a material constant and k the shear yield limit.

Example 8.4.

Let $3a = a_0 + a_1 r$, $b = b_0 + b_1 r$; a_0, a_1, b_0 and b_1 may depend on ρ. An easy calculation yields the locking condition

$$\tilde{b}_2(trE - a_0) - tr^{1/2}(E^d)^2 = b_0 , \tag{8.7}$$

and the locking law

$$\frac{\dot{S}}{tr^{1/2}\dot{S}^2} = [\tilde{b}_2(trE - a_0) + b_0]^{-1}E^d , \tag{8.8}$$

where $\tilde{b}_2 = b_1/a_1$. Thus only three material constants (or functions) are needed: a_0, b_0 and $\tilde{b}_2$.

Example 8.5.

We take $3a = a_0 + a_1 r$, $b = b_0 + b_2 r^2$; as previeusly a_0, a_1, b_0 and b_2 may depend on ρ. Now the locking condition has the following form

$$(trE - a_0)^2 - \frac{a_1^2}{b_2} tr^{1/2}(E^d)^2 + \frac{b_0 a_1^2}{b_2} = 0 . \tag{8.9}$$

The locking law is

$$\frac{\dot{S}}{tr^{1/2}\dot{S}^2} = [b_0 + \frac{b_2}{a_1^2} (trE - a_0)^2]^{-1}E^d . \tag{8.10}$$

Similarly as in the previous example only three material functions intervene: a_0, b_0 and b_2/a_1^2. The locking condition (8.9) is a counterpart of Mróz-Buyukozturk yield condition [33].

Taking more general form of b_0:

$$b_0 = \tilde{b}_0(\rho, \frac{tr^{1/3}(E^d)^3}{tr^{1/2}(E^d)^2}) , \tag{8.11}$$

we obtain a counterpart of Nilsson-Glemberg criterion [33].

9. Decomposition of the tensor function (2.3) in orthogonal bases

Blinowski investigated the decomposition of the isotropic tensor function in orthogonal bases [34]. His approach is now adopted to the tensor function (2.3).

We will now show, after the paper [34], that (2.3) can be transformed into a form which makes it possible to determine all scalar functions ϕ_h (h=0,1,2) or a, b and c in a simple way.

We can rewrite (2.3) or (4.1) and (4.2) as follows

$$E = \mu_0 I + \mu_1 \dot{S} + \mu_2 \dot{S}^{\bullet} , \tag{9.1}$$

where

$$\dot{S}^{\bullet} = (\dot{S}^2)^d - s^2 s_3 \dot{S} , \quad (\dot{S}^2)^d = \dot{S}^2 - \frac{1}{3} s_2^2 I , \tag{9.2}$$

and

$$\begin{aligned} \mu_0 &= \phi_0 - \frac{1}{3} \phi_2 s_2^2 = a - \frac{1}{3} c , \\ \mu_1 &= \phi_1 + \phi_2 s^2 s_3 = (b + cs^3)/s_2 , \\ \mu_2 &= \phi_2 = c/s_2^2 . \end{aligned} \tag{9.3}$$

Obviously, the functions μ_α (α=1,2), are homogeneous functions of the degree (-1) and (-2), respectively. Performing the contraction of (9.1) with $\dot{S}$ and $\dot{S}^{\bullet}$, respectively, we obtain

$$\mu_1 = E.\dot{S}/s_2^2 , \qquad \mu_2 = E.\dot{S}^{\bullet}/tr\dot{S}^{\bullet 2}. \tag{9.4}$$

Making use of the Cayley-Hamilton theorem (see Appendix) and (4.8) we find

$$\begin{aligned} b &= [E.\dot{S}s_2^3 - E.(\dot{S}^2)^d ss_3^2]/m , \\ c &= 6[E.(\dot{S}^2)^d - 6E.\dot{S}s^2 s_3]/m \end{aligned} \tag{9.5}$$

provided that $\dot{S} \neq 0$; here

$$m = s_2^6 - 9s_3^6 . \tag{9.6}$$

The diagonal representation of (9.1) takes the form (see [34])

$$diag\ (E_1, E_2, E_3) = \mu_0 I + \kappa_1 diag\ (\dot{S}_1, \dot{S}_2, \dot{S}_3) \pm \kappa_2 diag\ (\dot{S}_3 - \dot{S}_2, \dot{S}_1 - \dot{S}_3, \dot{S}_2 - \dot{S}_1), \tag{9.7}$$

where

$$\kappa_1 = \mu_1 + \mu_2 s^2 s_3, \qquad \kappa_2 = \mu_2 (m/6s_2^2)^{1/2} . \tag{9.8}$$

Here $\dot{S}_i$, (i=1,2,3), are eigenvalues of $\dot{S}$ (note that for equal eigenvalues κ_2 = 0). We observe that now κ_α (α=1,2) are not homogeneous functions.

Among other possible orthogonal representations (in the sense of scalar product) of the tensor function (9.1) of interest is the following one

$$E = \eta_0 I + \eta_1 K_1 + \eta_2 K_2 , \qquad (9.9)$$

where

$$K_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad K_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad (9.10)$$

where

$$\eta_0 = \mu_0, \quad \eta_1 = \phi_1 sin\gamma - \frac{1}{\sqrt{6}} s_2^2\phi_2 cos2\gamma, \quad \eta_2 = \phi_1 cos\gamma - \frac{1}{\sqrt{6}} s_2^2\phi_2 sin2\gamma, \quad (9.11)$$

and

$$\gamma = arcsin\ (-\sqrt{6}s^3)/3 . \qquad (9.12)$$

We recall that the parameter s_i $(i=1,2,3)$ and s are defined by (4.7) and (4.9).

10. Comments on plastic dissipation density and yield condition for isotropic perfectly plastic solids

The constitutive relation proposed by Sawczuk and Stutz [1] precludes a dependence of the dissipation density on hydrostatic pressure. Particularly, in the case of a plane flow the flow rule reduces to the classical von Mises theory [35] and a pressure dependent yield criterion is ruled out.

A simple generalization of the model developed in [1] allows to remove the drawbacks mentioned above.

Let $D = (D_{ij})$ be the rate of deformation tensor and ξ a scalar parameter (or a set of scalar parameters). The tensorially non-linear constitutive relations for perfectly plastic isotropic materials is assumed in the following form

$$T = \hat{T}(\xi, D) \qquad \text{subject to} \qquad \frac{\partial \hat{T}}{\partial D} . D = 0 \quad \text{and} \quad \frac{\partial \hat{T}}{\partial D} \neq 0 . \qquad (10.1)$$

If the tensor function $\hat{T}$ is independent of ξ then the model of Sawczuk and Stutz [1] is recovered. In the sequel the parameter ξ will mainly denote the hydrostatic pressure, though such an interpretation is not the only possible. As previously, it may stand for the density of a material.

The representation of the isotropic tensor function $\hat{T}$ has the form

$$T = \alpha_0 I + \alpha_1 D + \alpha_2 D^2 . \qquad (10.2)$$

Here α_h $(h=0,1,2)$ are isotropic functions of the basic invariants of D nad ξ; moreover they are of degree 0, -1, -2 in the rates of deformation, respectively, c.f.[1]. The third term on the r.h.s. of (10.2) is responsible for second order effects.

We set

$$\mathbf{D}^{d} = \mathbf{D} - \frac{1}{3}\,(tr\mathbf{D})\mathbf{I}\;, \qquad \mathbf{S} = \mathbf{T} - \frac{1}{3}\,(tr\mathbf{T})\mathbf{I}\;. \tag{10.3}$$

and choose the following set of basic invariants

$$\begin{aligned} &tr\mathbf{D}\,, \quad tr(\mathbf{D}^{d})^{2}\,, \quad tr(\mathbf{D}^{d})^{3}\,, \\ &tr\mathbf{T}, \quad tr\mathbf{S}^{2}\,, \quad tr\mathbf{S}^{3}. \end{aligned} \tag{10.4}$$

The relation (10.2) is decomposed into the spherical and deviatoric parts

$$tr\mathbf{T} = 3\Phi_{0}\,, \quad \mathbf{S} = \Phi_{1}\mathbf{D}^{d} + \Phi_{2}[(\mathbf{D}^{d})^{2} - \frac{1}{3}\,(tr(\mathbf{D}^{d})^{2})\mathbf{I}], \tag{10.5}$$

where Φ_0, Φ_1 and Φ_2 are functions of ξ and the kinematical invariants $(10.4)_1$. The state of plastic motion is conveniently described in terms of the dimensionless parameters

$$t = \frac{tr^{1/3}(\mathbf{D}^{d})^{3}}{tr^{1/2}(\mathbf{D}^{d})^{2}}\;, \quad w = \frac{tr\mathbf{D}}{tr^{1/2}(\mathbf{D}^{d})^{2}}\;, \text{ where } \quad tr^{1/2}(\mathbf{D}^{d})^{2} > 0. \tag{10.6}$$

The condition of homogeneity yields

$$\Phi_{0} = A(\xi, t, w)\;, \quad \Phi_{1} = \frac{B(\xi, t, w)}{tr^{1/2}(\mathbf{D}^{d})^{2}}\;, \quad \Phi_{2} = \frac{C(\xi, t, w)}{tr(\mathbf{D}^{d})^{2}}\;. \tag{10.7}$$

Obviously, the functions A, B and C are not to be confused with the functions entering into the formula (2.15).

The law of plastic distorsion takes eventually the form, c.f. [1] or Section 4 in this paper,

$$\mathbf{D}^{d} = \lambda\,\{\mathbf{S} - \frac{6C}{6B^{2} - C^{2}}\,[\mathbf{S}^{2} - \frac{1}{3}(tr\mathbf{S}^{2})\mathbf{I}]\}\;, \tag{10.8}$$

$$\lambda = (tr^{1/2}(\mathbf{D}^{d})^{2})[B - \frac{2C^{2}}{6B^{2} - C^{2}}\,(B + Ct^{3})]^{-1}\;. \tag{10.9}$$

We note that the parameter ξ, for instance the hydrostatic pressure, influences also the plastic multiplier λ.

The flow law (10.8), if substituted into (10.5), results in the following expression for the stress invariants

$$trT = 3A\ ,$$
$$tr\mathbf{S}^2 = (B + Ct^3)^2 + C^2(1 - 6t^6)/6, \qquad (10.10)$$
$$tr\mathbf{S}^3 = t^3(B + Ct^3)^3 + C(1 - 6t^6)[B(B + Ct^3) - C^2(1 - 6t^6)/18]/2.$$

A yield locus is obtained when t and w are eliminated from (10.10)

$$\mathrm{f}(\xi,\ tr\mathbf{T},\ tr\mathbf{S}^2,\ tr\mathbf{S}^3) = 0\ . \qquad (10.11)$$

Assuming that $(10.10)_1$ can be resolved in ξ one obtains

$$\xi = \psi(tr\mathbf{T},\ t,\ w)\ . \qquad (10.12)$$

If ψ does not dependent on t and w then $\xi = \psi_1(tr\mathbf{T})$; particularly $\xi = tr\mathbf{T}$.

In general, the flow law (10.8) is not associated with the yield condition (10.11). Under an additional condition

$$\frac{\partial \hat{T}_{ij}}{\partial D_{kl}} = \frac{\partial \hat{T}_{kl}}{\partial D_{ij}}\ , \qquad (10.13)$$

the associated flow rule follows, c.f. Section 3 of the present paper and [18].

By using (10.5) the general form of the dissipation density d is readily calculated

$$d = d(\xi,\mathbf{D}) = T_{ij}D_{ij} = \Phi_0 tr\mathbf{D} + \Phi_1 tr(\mathbf{D}^d)^2 + \Phi_2 tr(\mathbf{D}^d)^3\ . \qquad (10.14)$$

Incompressible flow. Now we have $tr\mathbf{D} = 0$ and consequently $w = 0$. The governing equations take the form

$$\mathbf{D} = \lambda\ \{\mathbf{S} - \frac{6C_1}{6B_1^2 - C_1^2}\ [\mathbf{S}^2 - \tfrac{1}{3}(tr\mathbf{S}^2)\mathbf{I}]\}\ , \qquad (10.15)$$

where $B_1 = B(\xi, t, 0)$, $C_1 = C(\xi, t, 0)$, whereas

$$\lambda = (tr^{1/2}\mathbf{D}^2)[B_1 - \frac{2C_1^2}{6B_1^2 - C_1^2}\ (B_1 + C_1 t^3)]^{-1}\ . \qquad (10.16)$$

Eliminating the parameter t from the relations $(10.10)_2$ and $(10.10)_3$, the yield locus is obtained in the form

$$\mathrm{f}(\xi,\ tr\mathbf{S}^2,\ tr\mathbf{S}^3) = 0\ . \qquad (10.17)$$

Hence we conclude that for $\xi = tr\mathbf{T}$ the yield condition can still depend on the hydrostatic pressure, though the constitutive relation describes the initial, incompressible flow. Moreover, the dissipation density is given by

$$d = d(\xi,\mathbf{D}) = \tilde{\Phi}_1 tr(\mathbf{D}^d)^2 + \tilde{\Phi}_2 tr(\mathbf{D}^d)^3 \ , \tag{10.18}$$

where

$$\tilde{\Phi}_1 = \frac{B_1(\xi,t)}{tr^{1/2}(\mathbf{D}^d)^2} \ , \quad \tilde{\Phi}_2 = \frac{C_1(\xi,t)}{tr(\mathbf{D}^d)^2} \ . \tag{10.19}$$

Plane incompressible flow. In this case $D_{13} = D_{23} = D_{33} = 0$ and $tr\mathbf{D} = 0$, $tr(\mathbf{D}^d)^3$ 0; hence $t = w = 0$. Consequently the material functions B_1 and C_1 depend on ξ solely. We put

$$\mathbf{D} = \begin{bmatrix} \underline{\mathfrak{D}} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad b_1(\xi) = B_1(\xi,0), \quad c_1(\xi) = C_1(\xi,0) \ . \tag{10.20}$$

The stress tensor $\mathbf{T}$ has the form

$$\mathbf{T} = \begin{bmatrix} \underline{\mathfrak{T}} & & 0 \\ & & 0 \\ 0 & 0 & T_{33} \end{bmatrix} , \tag{10.21}$$

while the constitutive relationship is given by

$$\underline{\mathfrak{D}} = \lambda \ [\underline{\mathfrak{T}} - \tfrac{1}{2} (tr\underline{\mathfrak{T}})\underline{\mathfrak{I}}] \ , \quad T_{33} = \tfrac{1}{2} (tr\underline{\mathfrak{T}} - c_1) \ , \tag{10.22}$$

where $\lambda = tr^{1/2}\underline{\mathfrak{D}}^2/b_1$, $tr\underline{\mathfrak{T}} = T_{11} + T_{22}$.

Consider now the dissipation density and the yield condition. If $\xi = tr\underline{\mathfrak{T}}$, then $b_1 = b_1(tr\underline{\mathfrak{T}})$ and $c_1 = c_1(tr\underline{\mathfrak{T}})$. For $\xi = tr\mathbf{T} = tr\underline{\mathfrak{T}} + T_{33}$ we have

$$T_{33} = \tfrac{1}{2} \ [tr\underline{\mathfrak{T}} - c_1(tr\underline{\mathfrak{T}} + T_{33})] \ . \tag{10.23}$$

Assuming that Eq. (10.23) can be resolved for T_{33} we may write $T_{33} = \tilde{T}_{33}(tr\underline{\mathfrak{T}})$. Thus in both cases the functions b_1 and c_1 may be assumed to depend on $tr\underline{\mathfrak{T}}$. Therefore the dissipation density becomes

$$D = D(tr\underline{\mathfrak{T}},\underline{\mathfrak{D}}) = tr(\mathbf{S}\mathbf{D}^d) = tr^{1/2}\underline{\mathfrak{D}}^2 b_1(tr\underline{\mathfrak{T}}) \ . \tag{10.24}$$

On the other hand

$$D = tr\underline{\mathfrak{D}}\underline{\mathfrak{T}} = \frac{tr^{1/2}\underline{\mathfrak{D}}^2}{b_1} \ (tr\underline{\mathfrak{T}}^2 - \tfrac{1}{2} \ tr^2\underline{\mathfrak{T}}) \ . \tag{10.25}$$

Hence by using (10.24) we finally obtain the yield condition in the following form

$$2tr\underline{\mathfrak{T}}^2 - tr^2\underline{\mathfrak{T}} = 2b_1^2 \ . \tag{10.26}$$

In the case of plane incompressible flow when b_1 and c_1 do not depend on ξ we have $b_1 = b_0$ and $c_1 = c_0$, where b_0 and c_0 are material constants. The flow rule reduces then to the classical von Mises theory and a pressure dependent yield criterion is precluded [35].

Appendix

For second order Cartesian tensors $\mathbf{A} = (A_{ij})$, $\mathbf{B} = (B_{kl})$, $(i,j,k,l=1,2,3)$ the following notation is used

$$tr\mathbf{A} = A_{ii} = A_{11} + A_{22} + A_{33}\,, \quad tr^p\mathbf{A} = (tr\mathbf{A})^p\,, \tag{A.1}$$

$$\mathbf{A}.\mathbf{B} = tr(\mathbf{AB}) = tr\mathbf{AB} = A_{ij}B_{ij}\,, \tag{A.2}$$

$$(\mathbf{AB})_{ik} = A_{ij}B_{jk}\,, \tag{A.3}$$

$$(\mathbf{A}\otimes\mathbf{B})_{ijkl} = A_{ij}B_{kl}\,, \tag{A.4}$$

$$(\mathbf{A}^n)_{ik} = (\mathbf{A}^{n-1}\mathbf{A})_{ik} = \underbrace{A_{ij}A_{jm}\ldots A_{pk}}_{n\ \text{times}}\,, \tag{A.5}$$

$\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$.

The notion of a homogeneous functions of degree m is essential for our considerations.

Definition A.1. [30-32]. A function $f = F(\mathbf{A})$ is called homogeneous of degree m if

$$F(t\mathbf{A}) = t^m F(\mathbf{A}) \tag{A.6}$$

for each $t > 0$.

We note that F is not necessarily a scalar function.

Euler's theorem for homogeneous functions yields

$$\frac{\partial F}{\partial \mathbf{A}}\,.\mathbf{A} = mF \qquad \text{or} \qquad \frac{\partial F}{\partial A_{ij}}\,A_{ij} = mF\,. \tag{A.7}$$

As basic scalar invariants of $\mathbf{A}$ one may assume

$$tr\mathbf{A}\,, \quad tr\mathbf{A}^2\,, \quad tr\mathbf{A}^3\,. \tag{A.8}$$

Cayley-Hamilton's theorem yields

$$\mathbf{A}^3 - I_1\mathbf{A}^2 + I_2\mathbf{A} - I_3\mathbf{I} = \mathbf{0}\,, \tag{A.9}$$

where $\mathbf{I}$ denotes the unit tensor; $I_i(\mathbf{A})$ $(i=1,2,3)$ are scalar invariants, being the coefficients of the following polynomial in λ:

$$det(\mathbf{A} + \lambda \mathrm{I}) = \lambda^3 + \mathrm{I}_1\lambda^2 + \mathrm{I}_2\lambda + \mathrm{I}_3 \tag{A.10}$$

The invariants (A.8) and I_i are interrelated by

$$\mathrm{I}_1 = tr\mathbf{A}\ , \quad \mathrm{I}_2 = \frac{1}{2}\,(tr^2\mathbf{A} - tr\mathbf{A}^2)\ ,$$
$$\mathrm{I}_3 = det\mathbf{A} = \frac{1}{6}\,tr^3\mathbf{A} - \frac{1}{2}\,tr\mathbf{A}tr\mathbf{A}^2 + \frac{1}{3}\,tr\mathbf{A}^3 \tag{A.11}$$

By using the Cayley-Hamilton's theorem we readily calculate

$$\mathbf{A}^4 = \frac{1}{2}\,(tr^2\mathbf{A} + tr\mathbf{A}^2)\mathbf{A}^2 + \frac{1}{3}\,(tr\mathbf{A}^3 - tr^3\mathbf{A})\mathbf{A} +$$
$$+ (\frac{1}{6}\,tr^4\mathbf{A} - \frac{1}{2}\,tr^2\mathbf{A}tr\mathbf{A}^2 + \frac{1}{3}\,tr\mathbf{A}tr\mathbf{A}^3)\mathbf{I}\ , \tag{A.12}$$

and

$$tr\mathbf{A}^4 = \frac{1}{6}\,(tr^4\mathbf{A} + 8tr\mathbf{A}tr\mathbf{A}^3 + 3tr^2\mathbf{A}^2 - 6tr\mathbf{A}^2tr^2\mathbf{A})\ . \tag{A.13}$$

Let us recall that if $\mathbf{A} = (\mathrm{A}_{ij})$ is a tensor, then its deviator is defined by

$$\mathbf{A}_d = \mathbf{A} - \frac{1}{3}\,(tr\mathbf{A})\mathbf{I}\ . \tag{A.14}$$

The following relations hold true

$$\mathbf{A}_d^3 = \frac{1}{2}\,(tr\mathbf{A}_d^2)\mathbf{A}_d + \frac{1}{3}\,(tr\mathbf{A}_d^3)\mathbf{I}\ , \tag{A.15}$$

$$\mathbf{A}_d^4 = \frac{1}{2}\,(tr\mathbf{A}_d^2)\mathbf{A}_d^2 + \frac{1}{3}\,(tr\mathbf{A}_d^3)\mathbf{A}_d, \tag{A.16}$$

$$tr\mathbf{A}_d^4 = \frac{1}{2}\,tr^2\mathbf{A}_d^2\ . \tag{A.17}$$

For a two-dimensional tensor $\underline{\mathfrak{U}} = (\mathfrak{U}_{\alpha\beta})$ $(\alpha,\beta = 1,2)$ the Cayley-Hamilton's theorem has the form

$$\underline{\mathfrak{U}}^2 - (tr\underline{\mathfrak{U}})\underline{\mathfrak{U}} + \frac{1}{2}\,(tr^2\underline{\mathfrak{U}} - tr\underline{\mathfrak{U}}^2)\underline{\mathfrak{I}} = \underline{\mathfrak{O}}\ . \tag{A.18}$$

Hence

$$\underline{\mathfrak{U}}^3 = \frac{1}{2}\,(tr\underline{\mathfrak{U}}^2 + tr^2\underline{\mathfrak{U}})\underline{\mathfrak{U}} + \frac{1}{2}(tr\underline{\mathfrak{U}}tr\underline{\mathfrak{U}}^2 - tr^3\underline{\mathfrak{U}})\underline{\mathfrak{I}}\ , \tag{A.19}$$

$$tr\underline{\mathfrak{U}}^3 = \frac{3}{2}\,tr\underline{\mathfrak{U}}tr\underline{\mathfrak{U}}^2 - \frac{1}{2}\,tr^3\underline{\mathfrak{U}}\ . \tag{A.20}$$

For a matrix $\mathbf{A} = (\mathrm{A}_{ij})$ its principal values A_i $(i=1,2,3)$ can be calculated by using the following formula

$$\mathrm{A}_i = \frac{1}{3}\,\mathrm{I}_1 + \frac{2}{3}\,\mathrm{A}_H cos\,[\frac{2}{3}\,\pi(i-1) - \psi]\ , \tag{A.21}$$

where

$$A_H = (I_1^2 - 3I_2)^{1/2} , \tag{A.22}$$

$$\cos 3\psi = (2I_1^3 - 9I_1I_2 + 27I_3)/2A_H^2 . \tag{A.23}$$

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