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The Development of the Ancient Theories of Proportions*

ABSTRACT: This paper is devoted to the reconstruction of the development of ancient theories of proportion. I argue that the development of ancient theories of proportion was motivated by the quest for one general mathematical theory and by inquiries of the mutual relations between the highest principles of Plato's *protology*: the One (i.e. that which is arithmetical) and the Dyad (i.e. that which is geometrical). Six main types of the theory of proportion are analyzed, with a historical, mathematical and philosophical background.
KEYWORDS: proportions • ancient mathematics • ancient philosophy • history of mathematics • Euclid • Eudoxus • Theaetetus

1. The significance of proportions in ancient mathematics

The ancient theories of proportion were studied for ages. It is impossible to understand the mathematics of ancient Greece without understanding the theories of proportion. Even more, without this, it is also impossible to understand quite a large part of ancient and medieval philosophy. The references to the proportion, harmony, and fundamental problems of mathematics are numerous in Pythagorean philosophy, in Plato and Aristotle. Moreover, many philosophical texts are based on some mathematical theories and results.

The reconstruction of the ancient theories of proportion reveals in full flagrancy the differences between the ancient and modern way of thinking in mathematics.

The main aim of the present paper is not to present a detailed exposition of the ancient theories of proportion, but only to give a general introduction and description of the main lines of their development, and of the internal logic of this development.

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In the times of Pythagoras, in his school, Greek mathematics had one and only one principle: the number. In the background of this theory were some facts that are astonishing even today, such as the emergence of the harmony and the concordance of vibrating strings, of the volumes of air in the cavities of some instruments or of water contained in bowls. The mutual relations between their diameters were described by small (natural) numbers[1]. Irregular divisions of the above objects did not produce a consonance. The Pythagoreans supposed at first that, for every single thing, it was possible to give the numerical description of its essential qualities. In particular, it was possible to describe "all mathematics" using numbers, for instance: to give a description of geometrical lines, surfaces and their correlations "at once" in terms of numbers.

A number for the Greeks was solely a natural number bigger than the number 1. A number was a multitude of monads. The absolutely indivisible "1" was not a number, but the principle of numbers.

A discovery of the incommensurability of the side and the diagonal of a square was that of the mutual irreducibility of that which is arithmetical and that which is spatial or geometrical. For the Greeks, the proof of the incommensurability was not indirect, contrary to the modern (as well as to Aristotle's) reconstructions using the law of excluded middle[2]. Their proof was positive: it was to show that the diagonal is not a number at all and that there are two different types of mathematical entities, i.e. numbers and magnitudes. If a side is a number, i.e. is measured by the line which corresponds to "1", then the diagonal is an odd and even number at the same time[3]. The mathematical reality was broken into two mutually irreducible

[1] Cf. Theon of Smyrna, *Mathematics useful for understanding Plato by Theon of Smyrna, Platonic Philosopher*, Ch. Toulis (Ed.), translated by R. Lawlor and D. Lawlor with an appendix of notes by J. Depuis, Secret Doctrine Reference Series, Wizzard Bookshelf, San Diego, 1979, Book II, especially section XIIa *The discovery of the numerical laws of consonances*. (Cf. also Theon of Smyrna, *Theonis Smyrnaei philosophi platonici expositio rerum mathematicarum ad legendum Platonem utilium*, E. Hiller (Ed.), Teubner, Lipsiae 1878.)

[2] Besides the law of excluded middle they had the principle that every number must be either odd or even. If something is odd and even simultaneously, it means that this "something" is not a number. A very interesting fact is that the ancients have had not a single series of numbers but many of them; cf. for instance the explanations given by Theon of Smyrna. In particular, the basic series were those of odd and even numbers. Every series starts with the unity ("1"), and every one has its own mechanism of generation. Therefore, for the ancients, numbers do not create one domain. For Plato, there is no one idea of "all numbers"; cf. *Nicomachean Ethics* 1096a 17–19, *Eudemian Ethics* 1218a 1–10, *Politics* 1275a 34–65, *etc.*

[3] Cf. Aristotle's *Analytica Priora* 41a: "For all who effect an argument *per impossibile* infer syllogistically what is false, and prove the original conclusion hypothetically when some-

realms: the geometrical and the numerical. Every number is measured by "1", thus all numbers are commensurable.

We know now that the discovery of the incommensurability of some geometrical magnitudes resulted in the mutual irreducibility of geometry and arithmetic. The proof of the incommensurability of the diagonal and the side of a square demonstrated that not "everything is a number". Therefore, it was impossible in antiquity to think about geometrical lines and figures in terms of some metrical concepts, such as the (numerical) lengths of lines or numerically measured areas. For us, the moderns, it is self-evident that the lines can be ordered and compared. However, when we try to grasp the situation of the ancient mathematical problem without our real numbers, or even fractions, or square roots and powers, then it seems obvious how difficult a problem it was to compare lines, figures and natural numbers. The ancients did not know about real or even rational numbers.

The continuum, i.e. the Dyad, was inexhaustible for them. Moreover, the Dyad was really indefinite and inexpressible. Therefore, they had to grasp only the chosen objects from the continuum. It was unclear how many other possible objects there are "in" the Dyad. Even the names of "irrational" lines support these ancient convictions about the Dyad. It was not obvious that the

thing impossible results from the assumption of its contradictory; e.g. that the diagonal of the square is incommensurate with the side, because odd numbers are equal to evens if it is supposed to be commensurate. One infers syllogistically that odd numbers come out equal to evens, and one proves hypothetically the incommensurability of the diagonal, since a falsehood results through contradicting this. For this we found to be reasoning *per impossibile*, viz. proving something impossible by means of an hypothesis conceded at the beginning". (tr. A. J. Jenkinson; cf. A. J. Jenkinson, *Analytica Priora*, in A. J. Jenkinson (Ed.), *Aristotle* A. J. Jenkinson 12 v., Clarendon Press, Oxford, 1928. Retrieved 2001, from http://etext.virginia.edu/etcbin/toccer-new2?id=AriPrio.xml\&images=images/modeng\&data=/texts/english/modeng/parsed\&tag=public\&part=all). Cf. also reconstructions of this proof: theorem X. 117 (an unoriginal interpolation but a genuine reconstruction) in the *Elements*, or T. L. Heath, *The thirteen books of Euclid's 'Elements' translated from the text of Heiberg with introduction and commentary*, vols. 1–3, University Press, Cambridge, 1908; retrieved 2008, from http://www.wilbourhall.org/pdfs/, vol. III, pp. 2–3. The same kind of reasoning was used in the proof fundamental for ancient mathematics that a line (or any other geometrical construct) is not composed of indivisible parts. Proclus says: "For a finite line would consist of either an odd or an even number of parts", and, therefore "a magnitude consists of parts infinitely divisible"; cf. Proclus, *A Commentary On the First Book of Euclid's Elements*, G. R. Morrow (Ed.), Princeton University Press, Princeton, New Jersey, 1992, 277,25 – 279,4 (in Greek: idem, *Procli Diadochi in Primum Elementorum Librum Commentarii*, G. Friedlein (Ed.), Leipzig 1873, repr. G. Olms, Hildesheim 1967; retrieved 2009, from http://www.wilbourhall.org/). Cf. also T. L. Heath, *The thirteen books …*, vol. I, p. 268.

continuum had a "flat" well-ordered structure. The modern proof of this fact is based on the very specific concept of the arithmetized continuum, usually determined by many hidden assumptions. Other concepts of continuum are possible. In my opinion, the ancient concept of continuum is irreducible to the modern set-theoretic concepts. Therefore the "hunting for geometrical objects" that was possible in the Dyad was seen as the imposition of the strict limits forced by the One. One more such limit was imposed by proportions: a method that is alternative to the constructions from the assigned basic line. There was also the third method based on Plato's theory of participation and his structure a one over many; I do not explain this point, however, in this paper[4].

If we are going to compare the ancient continuum with a modern theory, then non-well-founded sets[5] are better than Zermelo-Fraenkel set theory. Non-well-founded sets correspond at least to the ancient fundamental property of continuum: continuum is not composed of indivisible parts. On the other hand, Robinson's non-standard analysis with

[4] This point is the crux of my reconstruction of the foundations of ancient arithmetic as based on the highest principles: the One and the Dyad.; cf. my book Z. Król, *Platon i podstawy matematyki współczesnej. Pojęcie liczby u Platona*, (*Plato and the Foundations of Modern Mathematics. The concept of Number by Plato*), Wydawnictwo Rolewski, Nowa Wieś 2005. About *protology*, the One and the Dyad, and Plato's philosophy, cf. also S. Blandzi, *Platoński project filozofii pierwszej*, Wydawnictwo IFiS PAN, Warszawa 2002; J. N. Findlay, *Plato: The Written and Unwritten Doctrines*, Humanities Press, New York 1974; D. H. Fowler, *The Mathematics of Plato's Academy. A New Reconstruction*, Clarendon Press, Oxford University Press, Oxford 1987; K. Gaiser, *Platons Ungeschriebene Lehre. Studien zur systematischen und geschichtlichen Begründung der Wissenschaften in der Platonischen Schule. Appendix: Testimonia Platonica. Quellentexte zur Schule und mündlischen Lehre Platons*, Stuttgart 1963, (IInd ed. 1968); V. Hösle, *Zu Platons Philosophie der Zahlen und deren mathematischer und philosophischer Bedeutung*, „Theologie und Philosophie", 1984, 59, 321–355; G. Junge, *Platos Ideen-Zahlen*, „Classica et Mediaevalia. Revue Danoise de Philologie et D'Histoire", Librairie Gyldental, pp. 18–38, Copenhague 1949; Z. Markowic, *Platons Theorie über das Eine und die unbestimmte Zweiheit und ihre Spuren in der griechischen Mathematik*, in: O. Becker (Ed.), *Zur Geschichte der griechischen Mathematik*, pp. 308–318, Darmstadt 1965; G. B. Matthews, S. Marc Cohen, *The One and the Many*, "Review of Metaphysics", 1968, 21, 630–665; G. Martin, *Platons Lehre von der Zahl und ihre Darstellung durch Aristoteles*, „Zeitschrift für philosophische Forschung", Bd. VII, 1953, 191–203; R. D. Mohr, *The Number Theory in Plato's Rep. VII and Philebus*, "Isis", 1981, 72, 620–627; L. Robin, *La Théorie Platonicienne des Idées et des Nombres d'après Aristote. Étude Historique et Critique*, Georg Olms Verlag, Paris 1908; J. Stenzel, *Zahl und Gestalt bei Platon und Aristoteles*, B. G. Teubner, Leipzig, Berlin 1924; A. Wedberg, *Plato's philosophy of mathematics*, Almquist and Wicksell, Stockholm 1965.

[5] Cf. P. Aczel, *Non-Well-Founded Sets*, Center for the Study of Language and Information, Stanford, 1988.

some infinitely small magnitudes has nothing in common with the ancient continuum[6].

Putting oneself in the real ancient situation is an effect of what I call the reconstruction of the hermeneutical horizon. Therefore, the first condition of such a reconstruction is to suspend all modern interpretations of ancient theories of proportion in terms of modern algebra. Here, I do not present the details of this method. I would only like to present some results of this method, which are understandable without an analysis of the way in which they were obtained.

We also know that the ancient classifications of the incommensurable lines are a testimony to some unifying tendencies in mathematics[7]. The ancients set their minds to the o n e mathematics, i.e. to the only one and universal mathematical theory in which it would be possible to compare numbers and geometrical magnitudes. The problem is similar to the modern quest for the physical "theory of everything" or quantum gravity.

Therefore, in this paper I would like to give some evidence that the ancient theories of proportion form the second stream – apart from the classifications of incommensurable magnitudes – of ancient efforts to unify arithmetic and geometry.

As a starting point, it is necessary to remind the modern reader one important fact. Fractions were absent from theoretical arithmetic and mathematics. The numerical proportion $2 : 4 = 4 : 8$ is not a statement about the equality of numbers or fractions, and it is not reducible to the equality $1/2 = 1/2$. The proportion $2 : 4 = 4 : 8$ affirms the equality of two r a t i o s , and these ratios are not numbers at all, though all the terms are numbers.

The discovery of incommensurability exiled the Greeks from the earlier u n i q u e paradise of numbers taking all of reality under its control. Therefore, after a previous naive metrical approach to geometrical magnitudes, the Greeks had to create separate theories of proportion for numbers and for geometrical magnitudes. As I argue below, the first attempt to unify arithmetic and geometry, after the discovery of incommensurability, was Theaetetus' theory of partially mixed proportions. In this theory, it was possible to compare a numerical ratio with some geometrical ratios, how-

[6] Cf. for instance A. Robinson, *Non-Standard Analysis*, Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam 1966. Leśniewski's mereology is closer to the ancient view; cf. St. Leśniewski, *On Various Ways of Understanding the Expressions 'Class' and 'Collection'*, in: S. J. Surma, D. I. Barnet & V. F. Rickey (Eds.), *St. Leśniewski 'Collected Works'*, Nijhoff International Philosophy Series vol. 44/1, PWN, Kluwer Academic Publishers, Dordrecht-Boston-London 1927.

[7] Cf. Z. Król, *Platon i podstawy matematyki* …, op.cit.

ever, it was impossible to compare the mixed ratios of the form "number to geometrical magnitude". The first theory of fully mixed ratios was Eudoxus' famous theory of proportions.

From the technical point of view, the theories of proportion create the central part of an ancient mathematical method. The development of these theories was the *sine qua non* of the famous works by Archimedes, Apollonius and others.

On the other hand, from the philosophical point of view, these theories are understood mainly in a metaphoric way, usually without even a basic mathematical knowledge about them. Speaking about a "dignity of numbers and proportion" explains nothing. Proportions are manifestations of the highest principle: the One, and they are not some unclear metaphors.

2. Early theories of proportion[8]: T_1

T_1 is just a mark because, in reality, it does not correspond to one theory, but rather to many problems gathering early views and methods concerning proportions before the discovery of incommensurability (i.e. around 460–430 B.C.). At this stage only a naive metrical approach is present.

To avoid any confusion, I would like to explain that what I mean by "naive metrical approach" is different to that of W. R. Knorr[9]. Knorr distinguishes two approaches in pre-Euclidean mathematics. Firstly, the topological approach is present in the books I, III and IV of the *Elements*[10]. The

[8] Cf. D. H. Fowler, *Ratio in Early Greek Mathematics*, "Bulletin of the American Mathematical Society (New Series)", 1979, 1, 807–848; idem, *Book II of Euclid's Elements and a pre-Eudoxan theory of ratio*, "Archive for History of Exact Sciences", 1980, 22, 5–36; idem, *Book II of Euclid's Elements and a pre-Eudoxan theory of ratio, Part 2: Sides and diameters*, "Archive for History of Exact Sciences", 1982, 26, 193–209.

[9] Cf. W. R. Knorr, *The Evolution of the Euclidean Elements*, D. Reidel Publishing Co., Dordrecht 1975, pp. 6–8.

[10] Cf. the following editions and translations of Euclid's works: J. L. Heiberg, *Euclidis Elementa*, Teubner, Leipzig 1883–1888. Retrieved 2007, from: http://www.perseus.tufts.edu/cgi-bin/ptext?lookup=Euc.+toc or from http://www.wilbourhall.org/; J. L. Heiberg, *Euclidis opera omnia*, J. L. Heiberg & H. Menge (Eds.), Bibliotheca Scriptorum Graecorum et Romanorum Teubneriana, vols. I–VIII, in: Aedibus B. G. Teubneri, Lipsiae 1883–9, (Vol. I: *Euclidis Elementa, Libros I–IV Continens*, Edidit Et Latinae Interpretatus Est I. L. Heiberg, Lipsiae 1883; Vol. II: *Euclidis Elementa, Libros V–IX Continens*, Edidit Et Latinae Interpretatus Est I. L. Heiberg, Lipsiae 1884; Vol. III: *Euclidis Elementa, Librum X Continens*, Edidit Et Latinae Interpretatus Est I. L. Heiberg, Lipsiae 1886; Vol. IV: *Euclidis Elementa, Libros XI–XIII Continens*, Edidit Et Latinae Interpretatus Est I. L. Heiberg, Lipsiae 1885; Vol. V: *Continens Elementorum Qui Peruntur Libros XIV–XV Et Scholia In Elementa Cum Prolegmenis Criticis Et Appendicibus*, Edidit Et Latinae Interpretatus Est I. L. Heiberg, Lipsiae 1888; Vol. VI: *Euclidis Data Cum Commentario Marini Et Scholiis*

topological approach is dominant in Ionic mathematics, the climax of which are the works of Hippocrates of Chios. The material of these works is extant in (some parts of) Books I, III, VI and, in a part concerning the measure of a circle, in Book XII, and also in some other sources. The main interest in this approach (or tradition) is "the examination of the topological relations of point, line, plane figure; the angle is of particular interest; the triangle is the principal plane studied".

Secondly, the metrical approach, present in Books II, IV, X, XIII and VI, more recent than the topological approach, was formed in the times of Theodorus. "The principal problem is the measurement of area; the Euclidean treatment has the appearance of formalizing the naive metrical tradition typified by the still extant Heronian corpus". Knorr explains the mutual relations between these two traditions[11].

My naive metrical approach means that mathematicians, being unaware of the incommensurability of some magnitudes, could create some part of geometry and arithmetic using only an intuitive concept of proportion. They considered numbers implicitly as measures of every line and figure.

The traces of such an approach are extant even in the form of the proof of incommensurability, in which it is demonstrated that not everything is a number; cf. above. The proof is possible only based on the assumption of such an approach. It is known also that the discovery of incommensurability was shocking for the ancients and this shock is understandable in connection with the naive metrical approach. Therefore, my naive metrical approach is at least not later than Knorr's topological tradition.

There were, of course, different schools in mathematics, and it was mainly a Pythagorean opinion that "everything is a number". It is known also that numbers for Pythagoreans were spatially extended and

Antiquis, Edidit Henrikus Menge, Lipsiae 1896; Vol. VII: *Euclidis Optica, Opticorum Recensio Theonis, Catoptrica Cum Scholiis Antiquis*, Edidit I. L. Heiberg, Lipsiae 1895; Vol. VIII: *Suplementum: Anaritii In Decem Libros Elementorum Euclidis Commentarii*, Edidit Maximilianus Curtze, Lipsiae 1899, (retrieved 2009, from http://www.wilbourhall.org/)); idem, *Les trois livres de Porismes d'Euclide*, M. Chasles (Ed.), Mallet-Bachelier, Imprimeur Libraire, Paris 1860; idem, *Die Data von Euklid, nach Menges Text aus dem griechischen übersetztz und herausgegeben von Clemens Thaer*, C. Thaer (Ed.), Springer-Verlag, Berlin, Göttingen, Heidelberg 1962; idem, *Euclid Book 'On divisions of figures' with a restoration based on Woepcke's text and on 'Practica geometrie' of Leonardo Pisano*, R. C. Archibald (Ed.), Cambridge at the University Press 1972; idem, *Euclid's Elements of Geometry.*, R. Fitzpatrick (Ed.), Richard Fitzpatrick (2007), (retrieved 2008, from http://farside.ph.utexas.edu/euclid.html); D. E. Joyce, *The Euclid's 'Elements'*, Clark University (retrived 1997, from: http://alepho.clarku.edu/~djoyce/ java/elements/elements.html).

11 Cf. W. R. Knorr, *The Evolution …*, op.cit., pp. 7–8.

corporal, which is one evidence more for the naive metrical attitude at that time.

To the early beginnings of Greek mathematics belong many mathematical results based on the naive metrical concept of proportion and on its basic properties. For example, the Delian problem of the duplication of a cube or the considerations on the lunes (of Hippocrates). Hippocrates' investigations confirm the existence of the early theories of geometrical proportion. In **T_1**, also three kinds of proportion or means are known, i.e. the arithmetic, geometric and harmonic, the golden section, *etc*. Nicomachus of Gerasa[12] and Iamblichus[13] inform us that the three proportions were discovered by Pythagoras[14]. Filolaus speaks about proportion in the fragment 6 (Diels[15]). Nicomachus[16] also affirms that the concepts of the harmonic and geometric mean were used by Filolaus. Iamblichus even says that Pythagoras was acquainted with the golden section by the Babylonians[17].

To **T_1** belong also partial investigations of the multiple and epimoric (*epimorion diasthema*, *superparticularis*) ratios. Later, in the scope of the theory designated **T_2**, it was possible to prove f o r n u m b e r s (Archytas) that every numerical epimoric ratio is of the form $n : (n + 1)$; cf. Book VIII of the *Elements*. In the same way, the specification of the theory of prime numbers is done in **T_3**.

[12] Cf. Nicomachus of Gerasa, *Introductio arithmetica*, R. Hoche (Ed.), B. G. Teubner, Leipzig 1866., II. 22.1. Cf. also idem, *Introduction to arithmetic, translated into English by Martin Luther D'Ooge; with studies in Greek arithmetic, by Frank Egleston Robbins and Louis Charles Karpinski*, The Macmillan company, New York; Macmillan and company, ltd., London 1926.

[13] Cf. Iamblichus, *In Nichomachi arithmeticam introductionem liber*, H. Pistelli (Ed.), B. G. Teubner, Leipzig 1894, (reprint B. G. Teubner, Stuttgart 1975), p. 118, 23. Cf. also, idem, *Theologoumena arithmeticae*, V. De Falco (Ed.), B. G. Teubner, Leipzig 1922, (reprint B. G. Teubner, Stuttgart 1975).

[14] Cf. also Theon, *Mathematics useful …*, op.cit., pp. 47, 116. Theon mentions even some other kinds of proportion.

[15] Cf. H. Diels, *Die Fragmente der Vorsokratiker*, Bd. I–II, Weidmannsche Buchhandlung, Berlin 1922, and H. Diels, & W. Kranz, D*ie Fragmente der Vorsokratiker*, VIth edition, Weidmannsche Buchhandlung, Berlin 1951. Cf. also C. A. Huffman, *Philolaus of Croton Pythagorean and Presocratic. A Commentary on the Fragments and Testimonia with Interpretive Essays*, Cambridge University Press, Cambridge 1993, and idem, *Archytas of Tarentum: Pythagorean, Philosopher and Mathematician King*, Cambridge University Press, Cambridge 2005. Cf. also A. C. Bowen, *The foundations of early Pythagorean harmonic science: Archytas, fragment 1*, "Ancient Philosophy", 1982, 2, 79–104, and M. Timpanaro Cardini, *Pitagorici, Testimonianze e frammenti*, 3 vols., La Nuova Italia, Firenze 1958–64; G. S. Kirk, J. E. Raven, & M. Schofield, *The Presocratic Philosophers: A Critical History with a Selection of Texts*, Cambridge Univ., Cambridge 1983.

[16] Cf. Nicomachus of Gerasa, *Introductio arithmetica*, op.cit., II. 26.2.

[17] Cf. Iamlichus, *In Nichomachi arithmeticam …*, op.cit.

T_1 is divided into two stages in a rather natural way:

[Stage I.] At this stage, **T_1** is developed as o n e theory of proportions, i.e. ignoring the fact of incommensurability. It was also motivated by some problems in musical harmony. In early Pythagorean harmonics, the natural attitude was to consider jointly numerical and geometrical proportions; cf. the division of a canon.

[Stage II.] After the discovery of incommensurability, **T_1** entered two other lines of development, **T_1a** and **T_1b**, however, every line was based on the previous work done at Stage I. **T_1a** contains the early arithmetic and theories of numerical proportions[18]. **T_1b**, on the other hand, contains the early theories, or sometimes only observations, concerning geometrical proportions, and their scope was determined by the current mathematical needs of problem solving, but mathematicians became aware about the problems connected with the numerical description of geometry.

The existence of **T_1a** and **T_1b** is indicated, among other things by parallel studies on geometrical algebra and arithmetic, the existence of two versions of many theorems, i.e. arithmetic and geometric[19] (including the theorem of Pythagoras), the concept of similar numbers and the analysis of related problems and theorems from the early Pythagorean arithmetic, e.g. theorem IX. 30[20]. The theorems about the gnomic divisions of numbers in the so called *psephoi*-arithmetic[21] together with Pythagorean number triples[22].

[18] The basic sources for the reconstruction of the content of **T_1a** are the works of Nicomachus, Theon, Proclus and the Pythagoreans. In the present paper I have to limit the exposition to the most general remarks and observations.

[19] Knorr writes: "The division of the proofs on incommensurability into separate arithmetic and geometric parts is standard in the historical accounts of these studies"; cf. for instance, B. L. Van der Waerden *Arithmetik der Pythagorerer*, p. 682ff. Cf. also our Chapters VII and VIII." (cf. W. R. Knorr, *The Evolution …*, op.cit., p. 107, footnote 106). Of course, this confirms also the existence of **T_1b** as well. Cf. also B. L. Van der Waerden, *Science Awakening*, English tr. A. Dresden, P. Noordhoff, Holland 1954, and B. L. Van der Waerden, *Die Harmonielehre der Pythagoreer*, „Hermes", 1943, 78, 163–199.

[20] Note the conceptual similarity of Hippocrates' definition of proportion quoted below and theorem IX. 30.

[21] Numbers in early Pythagorean arithmetic were represented by discrete objects: small stones, sticks, dots, *etc*. Only the theories of proportion **T_5** and **T_6** made it possible to represent numbers by continuous lines. *Psephoi*-arithmetics was reconstructed by O. Becker. For more information, cf. also Knorr, *The Evolution …*, op.cit., Chapter V. The problem of number representations is described by W. R. Knorr (*The Evolution …*, op.cit.), especially in Chapters V, VII and VIII.

[22] Cf. for instance theorem 11, p. 155 in W.R. Knorr, *The Evolution …*, op.cit.

This theorem is connected with early Pythagoreanism by Proclus[23] and Heron[24], i.e. numbers satisfying Pythagoras' theorem, and some theorems on the divisibility of numbers[25], also belong to **T_1a**. Ancient harmonics had to be based on the intuitive foundation which was exactly described by Theaetetus' theory **T_5**. The last fact indicates also the Greek names of the musical intervals and the concept of *diastema*[26].

The main representative of **T_1b** was Hippocrates of Chios. From the historical sources (especially from Simplicius' information about Hippocrates' squaring of lunes[27]), it seems that theorems I. 47, II. 12, II. 13, and a version of XII. 2 were known to him. Therefore a part of mathematics from the books I, II and VI of the *Elements* was known in the times of Hippocrates.

On the other hand, Proclus[28] informs us that Hippocrates reduced the problem of the duplication of a cube to the construction of two geometric means between two given lines.

Simplicius (and Eudemus) quotes Hippocrates' definition of proportional magnitudes:

> Similar segments are the same parts of the circles respectively, as for instance a semicircle is similar to a semicircle and a third part of a circle to a third.

Knorr adds the following important comment which fully supports my naive metrical approach:

> That is, he is employing a conception of 'part' which is valid only for commensurable magnitudes, although many of the magnitudes in his constructions are, in fact, incommensurable[29].

23 Proclus, *Procli Diadochi in Primum …*, op.cit., p. 428.

24 Cf. Heron of Alexandria, *Heronis Alexandrini opera quae supersunt omnia*. Volumen IV, *Heronis Definitiones cum variis collectionibus, Heronis quae feruntur Geometrica*, I. L. Heiberg (Ed.), B. G. Teubner, Leipzig 1912, (reprint B. G. Teubner, Stuttgart 1976), IV, 218–220.

25 Heron uses some such theorems concerning the divisibility by 3 and 4, which are formulated in terms of Pythagorean triples.

26 Cf. Á. Szabó, *The Beginnings of Greek Mathematics*, Akademiai Kiado, Budapest; D. Reidel, Dordrecht 1978, Part II. Cf. Also, idem, *Die frühgriechische Proportionenlehre im Spiegel ihrer Terminologie*, "Archive for History of Exact Sciences", 1965, 2, 197–270.

27 Cf. Simplicius' *Commentary to Physics of Aristotle*; *Comm.*, 60.22 – 68.32. Cf. also Proclus, *Procli Diadochi in Primum …*, *op.cit*, p. 66, and W. R. Knorr, *The Evolution …*, op.cit., pp. 40–41 (and footnotes 60–62), B. L. Van Der Waerden *Science Awakening*, op.cit., pp. 131–136, and T. L. Heath, *The Thirteen Books of Euclid's Elements*, vols. I–III, Cambridge University Press, Cambridge 1926, vol. I, pp. 182–209.

28 Cf. Proclus, *Procli Diadochi in Primum …*, op.cit., p. 213.3–11.

29 Cf. W. R. Knorr, *The Evolution …*, op.cit., p. 41.

This fact indicated by Knorr corroborates the general character of **T_1**, as based not on the mature theory of proportion but rather on some non-explicit assumptions and metrical intuitions[30]. In any case, it was possible to obtain a part of the material from Book II of the *Elements* with the use of some naive metrical intuitions, i.e. a part of ancient geometrical algebra, as well as a part of the theory of the similarity of figures. The theory of the similarity of figures in Book VI is also based on some metrical intuitions only. A more exact account of these intuitions was possible in the frames of **T_5** and **T_6**.

Proclus[31], using the *History* of Eudemus of Rhodes, ascribes the discovery of the method of the application of areas (which is the main method in geometrical algebra) to the early Pythagoreans. However, only some of the material from the books II and VI of the *Elements* was known to the Pythagoreans, i.e. before the times of Theodorus and Archytas. In the times of Hippocrates, a large part of the theory of the similarity of triangles was known[32].

The naive metrical approach assumes the possibility of the comparison of geometrical magnitudes. The main difficulty in this approach is the lack of a unit of common measure for the magnitudes to be compared, because in many cases, they are incommensurable. Therefore, the fact which is obvious for almost every modern student as well as scholar, that the lines can be ordered based on their metrical properties (or even well-ordered) was, in fact, problematic and unclear following the discovery of incommensurable magnitudes. This problem caused the emergence of the ancient classifications of incommensurable lines, and is testified by the mutual relations between the ancient theories of proportion and the classifications. For instance, Knorr explains the connection between the geometrical algebra from Book II with Theodorus' investigations of incommensurabilities[33].

Presenting more historical sources concerning **T_1** is beyond the scope of this paper.

3. Theories of numerical proportions motivated by the inquiries of incommensurabilities: **T_2** and **T_3**

After **T_1** had entered into Stage II, theories **T_2** and **T_3** are the next stages of **T_1a**. They are theories of numerical proportions in arithmetic. **T_2** is

[30] Cf. also ibidem, p. 41, and footnote 62.

[31] Cf. Proclus, *Procli Diadochi in Primum …*, op.cit., pp. 176, 186.

[32] Cf. W.R. Knorr, *The Evolution …*, op.cit., pp. 204–205, footnote 18.

[33] Cf. ibidem, p. 96.

Archytas' (according to Van der Waerden) theory of proportions, extant mainly in Book VIII of the *Elements*. **T_3**, known mainly from Book VII of the *Elements*, was most probably created by Theaetetus of Athens, who is also the author of the material from Books X, XIII and, partly, from Book VI. Van der Waerden argued that Book VII contains an older theory of proportion than Book VIII[34], because the book VII is based on the definition of proportion very similar to the definition quoted by Hippocrates. However, in my opinion, these two definitions are different because the first (i.e. that of Hippocrates) concerns geometrical magnitudes, and the second, i.e. Definition VII. 20[35], based on Definitions VII. 3[36] and VII. 4[37], is concerned only with numbers. The definitions are only apparently similar.

Theaetetus' authorship is settled by strict connections between Book VII and the classification from Book X of the *Elements*.

Some procedures in the ancient theories of proportions for numbers and for magnitudes are usually identified by scholars, which is an error. For instance, there was no single *anthyphairetic* procedure, but rather many different procedures: those for numbers, for magnitudes, and one more in **T_5**, because they were used by the different theories of proportion.

In **T_2** and **T_3**, all terms in the proportion, and every ratio, are only numbers. In **T_3**, one can exchange the "places" of some terms: if $a : b = c : d$, where "a", "b", "c", "d" are numbers, then $a : c = b : d$, by theorem VII. 13.

Theories **T_2** and **T_3** were created in order to solve specific mathematical problems. For example, Theaetetus' theory of proportion **T_3** creates the basis for the further, i.e. in **T_5**, division of geometrical magnitudes on parts corresponding to square and cube numbers, and contains the arithmetical preliminaries for **T_5**. The connection of both **T_2** and **T_3** with the classification of commensurable magnitudes is excellently explained by W. R. Knorr[38].

On the other hand, the mutual relations between these theories are relatively well examined. Still, it is necessary to explain the connections of **T_3** with the new theory **T_5**. One more problem is the dependence of the mathematics of **T_2** on the historically younger **T_3**.

In general, **T_2** is based on an intuition of Hippocrates' definition of proportional magnitudes applied to numbers. This concept was precisely

[34] Cf. B. L. Van Der Waerden, *Science Awakening*, op.cit., pp. 49, 107–116.

[35] Def. VII. 20: "Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.".

[36] Def. VII. 3: "A number is a part of a number, the less of the greater, when it measures the greater".

[37] Def. VII. 4: "But parts when it does not measure it".

[38] Cf. W.R. Knorr, *The Evolution ...*, op.cit., mainly in Chapters VII and VIII.

defined by Theaetetus in Definition VII. 20 in arithmetic, i.e. for commensurable magnitudes.

The additional motivation for the creation of Archytas' **T_2** was some problems in ancient harmonics. According to Á. Szabó[39], the discovery of incommensurable magnitudes is connected with the problem of the division of a canon in harmonics. At first, these enquiries were confined to the scopes of **T_1**. However, the biggest part of the *Sectio Canonis* and the *De Institutione Musica* of Boetius is connected with **T_2**[40]**.** The fragments from Archytas' treatise on harmonics[41] contains similar definitions of the mathematical means as Plato's *Timaeus*[42].

T_4, the next theory of proportion, was motivated also by some problems in ancient harmonics.

4. Theory of purely geometric proportions: T_4

This theory is extant mainly in Book VI of the *Elements*. In **T_4**, all terms are exclusively some geometrical magnitudes, such as lines, circles and plane figures. This theory was elaborated by Theaetetus of Athens in a systematic way.

T_4 is independent from Eudoxus' theory of Book V (here designated **T_6**). The theory was reconstructed by Töplitz[43], Becker[44] and Van der Waerden in the quoted works.

[39] Cf. Á. Szabó, *The beginnings …*, op.cit., *passim*.

[40] It concerns, for example, Archytas' famous theorem that there are no numerical means between numbers *n* and *n+1*. The problem, of course, has a geometrical solution, although such a geometric division of a canon is not endowed with beauty and harmony. All basic musical intervals are of the form *n: (n +1)*; e.g.: *2:1*, *4:3*, *3:2*, *9:8*. The empirical observation of the emergence of consonants, if the division of a canon is based on some simple numerical proportions, linked ancient harmonics and arithmetic, and, in addition, created one more evidence for the role of the One and the Dyad. Cf. also Boetius, *Boetius und die griechische Harmonik. Des Anicius Manlius Severinus Boetius 'Fünf Bücher Über die Musik'*, O. Paul (Ed.), Verlag von F. E. C. Leuckart (Constanin Sander), Leipzig 1872.

[41] The fragment is extant, for example, in Porphyry's *In Ptolemai Harmonica*; cf. Diels, *On Harmonics*.

[42] Cf. *Tim*. 31c – 32a, 36a–b.

[43] O. Töplitz, *Das Verhältnis von Mathematik und Ideenlehre bei Platon*, „Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik", Abt. B.1, 1931, 3–33, and O. Töplitz, *Die mathematische Epinomisstelle*, „Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik", Abt. B.1, 1931, 334–346.

[44] Cf. O. Becker, *Mathematische Existenz. Untersuchungen zur Logik und Ontologie mathematischer Phänomene*, „Jahrbuch für Philosophie und phänomenologische Forschung", 1927, 8, 539–809; idem, *Die dihairetische Erzeugung der platonischen Idealzahlen*, „Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik", Abt. B., 1931, 1,

Becker showed how to obtain a geometric analogue of the theorems VII. 13 and V. 16 using only geometric reasoning[45]. From proportion $\mathcal{A} : \mathcal{B} = \mathcal{C} : \mathcal{D}$ it follows that:

rec. *AD* = **rec.** *BC* = **rec.** *DA* = **rec.** *CB*[46].

From the above we have that $\mathcal{A} : \mathcal{C} = \mathcal{B} : \mathcal{D}$[47]. Here we use the fact trivial but extremely interesting in antiquity that $\mathcal{A} : \mathcal{C}$ = **rec.** *AD* : **rec.** *CD*, which connects lines with surfaces[48].

Another assumption in the proof is that:

rec. *AD* : **rec.** *BD* = $\mathcal{A} : \mathcal{B} = \mathcal{C} : \mathcal{D}$ = **rec.** *BC* : **rec.** *BD*.

The only thing remaining is to prove that, if in a geometric proportion the second terms are equal, the first terms are equal as well, i.e. **rec.** *AD* = **rec.** *BC*. This requires Archimedes' lemma, which follows – as is demonstrated also by Becker – from theorem X. 1. All the theorems used were known before the times of Eudoxus.

Now, it is possible to establish which theorems from Book VI of the *Elements* were know in **T_1b**, and which are some new theorems belonging to **T_4**. Knorr demonstrates that, for the proofs of the theorems from Book X, the following theorems are necessary: VI. 1, 14, 16, 17 and 22[49]. These theorems form a separate group in the book VI.

5. First attempt to metricize geometry: Theaetetus' theory of mixed proportions T_5

This new theory of proportions is an intermediate stage between **T_4** and Eudoxus' theory **T_6**. **T_5** is also the first theory after the discovery of incommensurability which successfully, though in part only, unified numerical and geometric theories of proportions in one theory of mixed proportions.

464–501; idem, *Eudoxos-Studien I. Eine voreudoxische Proportionslehre und ihre Spuren bei Aristoteles und Euklid*, „Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik", Abt. B. 1, 1933, 311–333; idem, *Lehre vom Geradem und Ungeradem im Neunten Buch der euklidischen Elemente*, „Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik", Abt. B. 3, 1936, 533–553; idem, *Die Aktualität des Pythagoreischen Gedankens*, in: D. Henrich, W. Schulz & K.–H. Volkmann-Schluck (Eds.), Die Gegenwart der Griechen im neueren Denken. Festschrift für Hans Georg Gadamer zum 60. Geburtstag, pp. 7–30, J. C. B. Mohr, Tübingen 1960.

45 Cf. O. Becker, *Eudoxos-Studien I*, op.cit., and B. L. Van der Waerden, *Science Awakening*, op.cit., pp. 175–179.

46 „**rec.** AB" means „rectangle AB", and „**sq.** AB" means "square AB"

47 Cf. O. Becker, *Eudoxos-Studien I*, op.cit., p. 311.

48 Cf. Aristotle's *Topica* 158b.

49 Cf. W. R. Knorr, *The Evolution* ..., op.cit., p. 305.

In my opinion, the possibility to equal a numerical ratio with a geometric ratio required a particular theory of proportions, different from **T_4**. Therefore **T_5** is a theory the terms of which are numbers and some geometric magnitudes, i.e. the same as in **T_4**. **T_5** makes it possible to compare numerical ratios and geometric ratios. Their terms are numbers: *a*, *b*, *c*, …, and magnitudes: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, … . For example, $\mathcal{A} : \mathcal{B}$ can be a ratio composed of two lines, or of two squares, or of a line and a polygonal plane figure. The only limitation is the impossibility to compare the ratio of a number to a magnitude with the other ratios, i.e. it is impossible to prove an equivalent of theorems V. 16 and VII. 13 from the *Elements*. Thus, **T_5** contains proportions of the form:

$a : b = \mathcal{A} : \mathcal{B}$, but the proportions of the kind $a : \mathcal{B} = \mathcal{A} : b$ are excluded.

The initial theorems from Book X, which seem to be superfluous from the point of view of Eudoxus' theory, create the base for **T_5**. In contradiction to what is usually assumed, it is **T_5** that the author of *Epinomis* indicates, instead of **T_6**[50]. The fragment 819d – 820e (cf. especially 820c 4) of the *Epinomis* is concerned with **T_5**, and not **T_6**, because in **T_6**, magnitudes and numbers are comparable (cf. theorem V. 16). Therefore, E. Sachs is right: there is a connection between Book X of the *Elements* and this fragment[51].

Moreover, **T_5** and **T_4** are connected with the book X, together with Becker's reconstruction presented in the previous section, make it possible to eliminate **T_6** from every proof in Book X altogether. The use of the theory from Book V becomes needless.

Also, after Euclid's edition of the material from Book X, there are still traces of **T_5**. For example, Knorr writes:

> A second problem is that the Euclidean proof [of the theorem X. 9 – Z.K.] employs two conceptions of proportion (V, Def. 5 for magnitudes and VII, Def. 20 for integers) without having proved their equivalence in the case of commensurable magnitudes.
> At some point the arithmetic and geometric parts of the theory [i.e. the theory from Book X – Z.K.] were separated, for only the latter is contained in the Elements; we have seen how this separation gave rise to a logical flaw in the Euclidean theory, whereby a strictly geometric theorem, X. 9, came to be applied as an arithmetic condition of commensurability.
> Euclid proves this as X. 5 and X. 6. While a modern theory of rational magnitudes would treat this as a definition, Euclid is correct to provide

50 Cf. O. Töplitz *Das Verhältnis von Mathematik und Ideenlehre bei Platon*, op.cit., pp. 13–15.

51 Cf. E. Sachs, *Die fünf Platonischen Körper*, Weidmann, Berlin 1917.

> it as a theorem, since his own definition of commensurable magnitudes (X, Def. 1) is based on the existence of a common measuring magnitude. But in his proof of X. 5 he appears to err in the same way as we mentioned in connection with X. 9. That is, he applies VII, Def. 20, the definition of proportion for integers, to the case of a proportion in which two of the terms are not integers, but rather commensurable magnitudes. What is needed, therefore, is a proof that a proportion of magnitudes (in the sense of Book V), where the magnitudes are commensurable, satisfies the properties of a proportion in the sense of Book VII. The absence of this step indicates that the original form of X. 5 did not resort to the Eudoxean definition, but that Euclid failed to perceive the necessity of revision[52].

The above facts indicate that:

1. Euclid replaced **T_5** by Eudoxus' **T_6**.
2. The basic form of the initial part of Book X is taken directly from a work of Theaetetus, and this initial part is almost unchanged.
3. Lack of the realization of **T_5** is responsible for the problems raised by Knorr (and others).

The main method in **T_5** is the construction with the ruler and compasses of two lines or areas in the ratio as a number has to a number (and vice versa). I made the reconstruction of this method[53]. The method is based on antyphairetic reasoning.

The second possible operation in **T_5** is the geometric finding the greatest common divisor of two numbers by finding of the corresponding greatest common measure of two lines (or of two areas), and vice versa. The latter operation indicates the so-called *antyphairetic* theory of proportions as a method of **T_5**[54]. It is also possible that the antyphairetic theory of proportions was developed earlier than **T_5**. If we are aware of the existence of different (i.e. numerical and geometrical) antyphairetic methods, *antyphairesis* is likely a part of **T_5**.

T_5 together with **T_3** and **T_4** make it possible to prove every theorem from Book X without Eudoxus' theory from Book V. In particular,

[52] Cf. W. R. Knorr, *The Evolution …*, op.cit., p. 253, 238), and pp. 253–254.

[53] Cf. the proof of Lemma 6 in Z. Król, *Platon i podstawy …*, op.cit. It is known also that Hippocrates of Chios was able to construct two lines such that the squares on them were in proportion as *2 : 3* and *6 : 1*; cf. B. L. Van der Waerden *Science Awakening*, op.cit., p. 136.

[54] This theory, based on Euclid's algorithm, is reconstructed by Knorr, who uses some of Becker's findings. Cf. W. R. Knorr, *The Evolution …*, op.cit., Chapter VIII and Appendix B. However, as I mentioned before, Knorr does not discern antyphairetic methods in geometry and in arithmetic.

theorem V. 16 is replaced by its purely geometric analogue reconstructed by Becker.

Let us notice that **T_5** is independent from the choice of the assigned basic line, although such a connection can be made.

6. Eudoxus' theory of proportions: T_6

A very natural generalization of Theaetetus' theory **T_5** is Eudoxus' famous theory of proportions, extant in Book V of the *Elements*. Nevertheless, even **T_6** does not unify arithmetic and geometry into a single theory, because there are still many geometrical objects which are not accessible with the methods of this theory. For instance, it is impossible to apply **T_6** to infinite and indefinite objects. This theory concerns only the so-called Archimedean magnitudes. Two magnitudes are "Archimedean" if there is a number *n* such that the lesser magnitude, when increased *n*-times, will become equal to or bigger than the second magnitude. Obviously, the famous Archimedes' lemma assumes only finite magnitudes[55]. And there are still many non-Archimedean magnitudes, such as horned angles. Therefore, even with the use of **T_6**, the Dyad was still inexhaustible, and not reducible to the limiting One.

In Antiquity establishing which magnitudes are Archimedean was a separate problem, and it was necessary to prove this fact for each given magnitude. For example, we know that there was a dispute in Antiquity about the nature of an angle, if it is a quality (Eudemus) or a quantity (Plutarch of Athens, Apollonius, Carpus of Antioch) or a relation (Euclid)[56].

The emergence of **T_6** creates a revolution in ancient mathematics because it was the first n o n - c o n s t r u c t i v e theory. O. Becker explains it in the *Mathematische Existenz*. The fundamental Eudoxean definition V. 5 is non-constructive:

> Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever are taken

[55] "If there be (two) unequal lines or (two) unequal areas, the excess by which the greater exceeds the less can, by being [continually] added to itself, be made to exceed any given magnitude among those which are comparable with [it and with] one another", cf. *On spirals*, (cf. T. L. Heath, *The Works of Archimedes Edited in Modern Notation with Introductory Chapters by T. L. Heath with a Suplement 'The Method of Archimedes' Recently Discovered by Heiberg*, Dover Publications, Inc., New York 1912; retrieved 2009, from http://www.wilbourhall.org/, p. 155). Cf. also *The quadrature of the parabola*, idem, op.cit. p. 234.

[56] Cf. Proclus, *The Commentary …*, op.cit., pp. 100–102.

of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

The above definition is formulated for every possible combination of natural numbers and therefore, it is impossible to check this condition explicitly. Also, definition V. 5 uses also concepts of actually infinite scopes.

T_6 makes it possible to compare mixed ratios, such as "a number to a line" or "a figure to a number". Moreover, it is possible to transform every proportion of mixed ratios onto Theaetetus' equality of purely numerical and geometrical ratios, and vice versa. Therefore, for example, from $a : b = \mathcal{A} : \mathcal{B}$, by theorem V. 16, we get: $a : \mathcal{A} = b : \mathcal{B}$.

We can imagine now, how it was possible for Eudoxus to find his famous definitions of proportional magnitudes V. 1 – V. 7 with the use of **T_5**. Eudoxus was aware of the existence of the groups ("families") of lines that are commensurable and incommensurable. He was aware also that that there is an indefinite number of families of lines commensurable in length, and incommensurable in length with the lines from the other families. It is possible to choose a local assigned basic line for every family. If so, to every Theaetetus' proportion in one family, there is a corresponding Theaetetus' proportion in the other, i.e. the corresponding proportions are determined by the ratio of the same numbers. For example, we can consider in every family the lines which are in the same ratio *4 : 2* in relation to the basic line in the given group.

We can find such proportions by the construction of the greatest common measure of any two lines that are commensurable in length. Lines are commensurable in length if they are in ratio as a number has to a number, but in relation to the basic line (in a group) which corresponds to 1. The greatest common measure of any two lines that are commensurable in length in the given group is equal to the line obtained from the division of the basic line in the ratio equal to the numerical ratio between these lines.

Regarding the comparison of the lines from the different groups, it is sufficient to know which measure, i.e. which basic line, is lesser, equal or greater. The relation of being lesser, greater or equal for the basic lines is transitive, for all the other lines, and conservative. For instance, if in one group of lines we have two lines in the least ratio *4 : 2*, and if the common measure in the first group is greater than the measure in the second group, then the corresponding lines in the second group are lesser than their counterparts in the first pair of lines. Using these intuitions it is possible to see how V. 16

"works" and that the *sine qua non* condition of the proportionality of lines is Archimedes' lemma. Therefore, to create **T_6**, it is necessary to operate with the groups of infinite scopes. It is also visible now that the ancient extensions of the classification of incommensurable lines from Book X were subsequent steps leading to **T_6**.

I recommend the reader interested in a historical and philological analysis of Greek terminology concerning the ancient theories of proportion to consult the works of Á. Szabó[57].

Here the theory of proportions by Ommar Khayyam, a mystical poet and a mathematician (XIth century A.D.), is worth noting. It is a peak of the development of the ancient theories of proportions. In the *Discussion of Euclid's difficulties*, Khayyam gives the definition of proportionality of four magnitudes, irrespective of whether they are discrete or continuous[58].

Four magnitudes are proportional ($\mathcal{A} : \mathcal{B} = \mathcal{C} : \mathcal{D}$), if some numbers, obtained in the following way, are equal. Assuming that $\mathcal{B}$ is greater than $\mathcal{A}$, and $\mathcal{D}$, than $\mathcal{C}$, we can subtract from $\mathcal{B}$ the multiple of $\mathcal{A}$, given by the number constructed as in the initial theorems of Book X of the *Elements* or in my Lemma 6[59]. We obtain a remainder smaller than $\mathcal{A}$ (and $\mathcal{B}$). Next, we repeat the above procedure using the remainder instead of $\mathcal{A}$. As a result we get a second number. There are two possibilities: 1) the process will terminate in a finite number of steps (if $\mathcal{A}$ is equal to $\mathcal{B}$ – in the first step) in the case when $\mathcal{A}$ and $\mathcal{B}$ are commensurable, or, 2) we can repeat the procedure to infinity, if the magnitudes are incommensurable. We can do the same with the magnitudes $\mathcal{C}$ and $\mathcal{D}$. Khayyam defined that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are proportional if the numbers obtained for $\mathcal{A}$ and $\mathcal{B}$, at every stage, are the same as the numbers obtained for $\mathcal{C}$ and $\mathcal{D}$. He next demonstrated that this *antyphairetic* definition of proportionality is equivalent to Eudoxus' definition from Book V. He established also that the ratios are equal to, greater than or smaller than the other ratios iff the corresponding numbers are respectively equal, greater or smaller. Khayyam determined also the product of ratios. The latter was undefined in ancient mathematics.

Since than, the proportion between any f i n i t e magnitudes could be considered as determined by some s e t of numbers. Nasir ad-Din at-Tusi treats proportions in this way. Geometry becomes gradually an arithmetized theory.

[57] Cf. also J-L. Gardies, L'héritage épistémologique d'Eudoxe de Cnide. Un essai de reconstitution, Libraire Philosophique J. Vrin, Paris 1988.

[58] Cf. B. L. Van der Waerden, *A History of Algebra. From al-Khwārizmī to Emmy Noether*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo 1985., pp. 29–31.

[59] Cf. Z. Król, *Platon i podstawy …*, op.cit.

The crux of Khayyam's theory is the ancient theory of the infinite divisibility of continuum. For, he establishes the possibility to find in every case the fourth proportional magnitude to any given three magnitudes on this ground.

The next steps of the development of the theories of proportion are connected with the emergence of the modern infinite model for Euclidean geometry. A very important stage leading to the absolute space was Nicolas Oresme's theory of proportion.

7. Concluding remarks

The proposed reconstruction of the main lines of the development of the ancient theories of proportion[60] is supported by many sources and results, cf. for instance O. Becker's works.

[60] Cf. also I. Grattan-Guiness, *Numbers, Magnitudes, Ratios and Proportions in Euclid's Elements: How Did He Handle Them?*, "Historia Mathematica", 1966, 23, 355–75; M. Hand, *Mathematical structuralism and the Third Man*, "Canadian Journal of Philosophy", 1993, 23, 179–192; H. Hasse, & H. Scholz, *Die Grundlagenkrisis der griechischen Mathematik*, unknown binding 1928; T. L. Heath, *A history of Greek mathematics*, Oxford University Press, Oxford 1921; J. Itard, *Les livres arithmétiques d'Euclide*, "Histoire de la pensée 10", Hermann, Paris 1961; J. Klein, *Die griechische Logistik und die Entstehung der Algebra*, I. Tl., „Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik", Abt. B.3, 1936, 18–105; idem, *Greek Mathematics and the Origin of Algebra*, Cambridge Mass. & London 1968, (republished Dover 1992); I. Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*, MIT Press, Cambridge, Mass./London 1981; A. Jones, (Ed.), *Pappus of Alexandria. Book 7 of the Collection. Part 1: Introduction, Text, and Translation. Part 2: Commentary, Index, and Figures*, Sources in the History of Mathematics and Physical Sciences vol. 8, Springer-Verlag, New York 1986; P. Palmieri, *The Obscurity of the Equimultiples: Clavius' and Galileo's Foundational Studies of Euclid's Theory of Proportions*, "Archive for History of Exact Sciences", 2001, 55, 535–597; Pappus d'Alexandrie, *La Collection Mathématique. OEuvre traduite pour la première fois du Grec en Français, avec une introduction et des notes*, 2 vols., P. Ver Eecke, Desclée De Brouwer, Paris and Bruges 1933; K. Saito, *Duplicate Ratio in Book VI of Euclid's Elements*, "Historia Scientiarum", 2nd Ser., 1993, 3–2, 115–135; idem, *Phantom Theories of Pre-Euclidean Proportions*, "Science in Context", 2003, 16(3), 331–347; H. Scholz, *Warum haben die Griechen die Irrationalzahlen nicht aufgebaut?*, „Kant-Studien", 1928, 33: 35–72; J. Vogt, *Zur Entdeckungsgeschichte des Irrationalen*, „Bibliotheca Mathematica", (3d.s.), 1913–4 14, 9–29; K. Von Fritz, K., *The discovery of incommensurability by Hippasus of Metapontum*, "Annals of Mathematics", 1945, 46, 242–64; H. G. Zeuthen, *Geschichte der Mathematik im Alterturn und im Mittelalter*, Kopenhagen, Leipzig 1902; idem, *Sur l'origine historique de la connaissance des quantités irrationnelles,* "Bulletin of the Royal Academy of Sciences of Denmark", 1910, 395-435; Aristarchus of Samos, *Aristarchus of Samos, the Ancient Copernicus. A History of Greek Astronomy to Aristarchus Together With Aristarcus's Treatise 'On the sizes and distances of the Sun and Moon'*, T. Heath (Ed.), Clarendon Press, Oxford 1913; Apollonius of Perga, *Apollonii Pergaei Que Graece Extant Cum Commentariis Antiquis.* Edidit Et Latine Interpretatus Est I. L. Heiberg, vols. I, II, I. L. Heiberg (Ed.), B. G. Teubner, Lipsiae 1891 (retrieved 2008, from http://www.wilbourhall.org/).

Aristotle in the *Nichomachean Ethics* explicitly discerns numerical (1131b 12) and geometric (1132a 1) proportions. The schema of the development of the theories of proportions, presented in this paper, is corroborated also by fragments 74b and 99a from the *Posterior Analytics*[61].

Pappus in his commentary on the tenth book of the *Elements* explains, in paragraphs 8 and 6 (Part I), that the term "proportion" is used for numbers and spatial magnitudes in different meanings:

> Not every ratio, therefore, is to be found with the numbers; nor do all things that have a ratio to one-another, have that of a number to a number, because in that case all of them would be commensurable with one-another, and naturally so, since every number is homogeneous with finitude (or the finite), number not being plurality, the correspondence notwithstanding, but a defined (or limited) plurality[62].

The next part of this paragraph indicates that, in ancient mathematics, the differences between two types of wholes were essential: "one over determined plurality" and "one over undefined plurality".[63]

[61] Cf. H. Tredenick, *Aristotle. Aristotle in 23 Volumes*, vols. 17, 18, translated by H. Tredennick, Harvard University Press, Cambridge, MA 1933; William Heinemann Ltd., London 1989, (retrieved 1997, from: http://www.perseus.tufts.edu/).

[62] Cf. W. Thomson, & G. Junge, *The Commentary of Pappus on Book X of Euclid's Elements. Arabic Text and Translation by William Thomson with Introductory Remarks, Notes, and a Glossary of Technical Terms by Gustav Junge and William Thomson*, Harvard Semitic Series, vol. VIII, Cambridge, London 1930, p. 71.

[63] Cf. Z. Król, op.cit., and idem, *Intuition and History: Change and the Growth of Mathematical Knowledge*, "International Journal for Knowledge and Systems Science", Japan Advanced Institute of Science and Technology (JAIST), Japan 2006 2(3), 22-32. Cf. also idem, *O platonizmie w teorii mnogości*, (*On Platonism in set theory*), „Roczniki Filozoficzne KUL", 2003, 3(51), 225-252; idem, *Platon i podstawy matematyki współczesnej*, in A. Motycka & S. Blandzi (Eds.), *Spotkania platońskie. W dobie rozumu rozproszonego wracamy do korzeni.*, pp. 56-63, Wydawnictwo IFiS PAN, Warszawa 2004; idem, *Platonizm matematyczny i hermeneutyka*, (*Hermeneutics and Mathematical Platonism*), Wydawnictwo IFiS PAN, Warszawa 2006; idem, *Geometria starożytna i filozofia Platona na podstawie Komentarza Pappusa do X księgi Elementów Euklidesa*, (*Ancient geometry and Plato's philosophy. Remarks concerning the Commentary of Pappus on Book 10 of Euclid's 'Elements'*), „Kwartalnik Historii Nauki i Techniki", 2006, 3-4, 1-35; idem, *Wprowadzenie do starożytnych teorii proporcji*, (*Introduction to the ancient theories of proportion*), „Kwartalnik Historii Nauki i Techniki", (in Polish), 2007 52(1), 73-91; idem, *The Emergence of New Concepts in Science,* in: *Creative Environments: Issues for Creativity Support for the Knowledge Civilization Age.* A.P. Wierzbicki, Y. Nakamori eds. Chapter XVII, pp. 415-442, Springer Verlag 2007.

On the other hand, paragraph 6[64] indicates other types of proportions:

> It should be pointed out, however, that the term, proportion, is used in one sense in the case of the w h o l e , i.e. the finite and homogeneous continuous quantities, in another sense in the case of the commensurable continuous quantities, and in still another sense in the case of the continuous quantities that are named rational. For with reference to continuous quantities the term, ratio, is understood in some cases only in the sense that it is the relation of finite continuous quantities to one another with respect to greatness and smallness [cf. **T_4** – Z.K.]; whereas in other cases it is understood in the sense that it denotes some such relation as exists between the numbers [cf. **T_5** – Z.K.], all commensurable continuous quantities, for example, bearing, as is evident, a ratio to one-another like that of a number to a number; and finally, in still other cases, if we express the ratio in terms of a definite, assumed measure, we become acquainted with the distinction between rationals and irrationals.

Another impulse to develop the ancient theories of proportions was due to the divisions of figures and the classifications of incommensurable magnitudes.

It is possible to see that the main stream of efforts in mathematics was directed to the creation of one mathematical theory, unifying – as much as it was possible – arithmetic and geometry. The fundamental role of the highest principles: the One and the Dyad, is indicated[65].

Hence, the existence of some hidden assumptions and implicit determinants (cf. for instance n a i v e m e t r i c a l a p p r o a c h) for the understanding and creation of non-formal ancient mathematics is demonstrated. They, and their reconstruction, are essential for understanding mathematical theories in any time. The reconstruction results in the detection of the evolution of some Platonic methods in mathematics, such as the ability to operate with the wholes of actually infinite scopes. The reconstruction of such global implicit determinants for the modern and strictly formalized mathematics is also possible[66]. Therefore, the reconstruction determines the "style in force" of mathematical investigations in the given historical

[64] W. Thomson, & G. Junge, *The Commentary of Pappus …*, op.cit.

[65] Theon demonstrates that every kind of proportion is generated from the One and the Dyad.

[66] Cf. Z. Król, *Uwagi o stylu historycznym matematyki i rozwoju matematyki* (*Remarks concerning the historical style of mathematics and the development of mathematics*), in: *Światy matematyki. Tworzenie czy odkrywanie?*, I. Bondecka-Krzykowska, J. Pogonowski (eds.), Wydawnictwo UAM, Poznań 2010.

period. The scientific revolutions, such as the emergence of Newtonian physics and differential calculus are closely connected with the evolution of these implicit determinants which are a part of the hermeneutical horizon for mathematics. Such matters are the subject of the hermeneutics of mathematics[67].

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