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LINKED TWIST MAPPINGS : ERGODICITY

by

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Contents

§ 0. Introduction

In the paper we study ergodic properties of a simple class of conservative dynamical systems (piecewise smooth homeomorphisms of surfaces) called linked twist mappings.

Using elementary geometry we prove that if the twists are strong enough, then together with the fact that Lyapunov exponents are nonzero, this implies the l.t.m. and all its powers are ergodic, so Bernoulli. We define and study a large family of l.t.m.'s. But let us start with the following example : Let T^2 be the standard torus $\mathbb{R}^2/\mathbb{Z} \times \mathbb{Z}$ and let P, Q be closed annuli in T^2 defined by

$$P = \{(x,y) \in T^2 : y_o \leq y \leq y_1 \ , \quad x \ \text{arbitrary}\}$$
$$Q = \{(x,y) \in T^2 : x_o \leq x \leq x_1 \ , \quad y \ \text{arbitrary}\} \ .$$

Take any nonzero integer k and a linear function $f : \langle y_o, y_1 \rangle \to \langle 0,k \rangle$ (or $\langle k,0 \rangle$ if $k < 0$) which satisfies the properties $f(y_o) = 0$, $f(y_1) = k$. Call the number $\alpha = \frac{df}{dy}$ the slope. A (k,α)-twist map (or k-twist map) F on P is defined by

$$F(x,y) = (x+f(y),y) \quad ,$$

so F is the identity on both components of the boundary of P and rotates each circle y = constant by an angle $f(y)$. Similarly, by interchanging the roles of x and y, we define an (ℓ,β)-twist map (or ℓ-twist map) G on Q

$$G(x,y) = (x,y+g(x)) \ , \text{ with the slope } \beta = \frac{dg}{dx} \ .$$

Extend F and G to $P \cup Q$ by the identity to $\hat{F}$ and $\hat{G}$. The toral linked twist mapping is the composition

$$H = H_{f,g} = \hat{G} \circ \hat{F} \quad \text{on} \quad P \cup Q \quad .$$

Observe that H preserves Lebesgue measure on T^2 . We shall consider H together with this measure. In § 1 of the presented paper we prove

Theorem A : If a toral linked twist mapping H is composed from (k,α) - and (ℓ,β)-twists where k and ℓ have opposite signs, $|k|, |\ell| \geq 2$ and $|\alpha \cdot \beta| >$ constant $C_o \simeq 17.24445$, then H and all its powers are ergodic. In fact H is a Bernoulli mapping. □

At the end of § 3 we show how the assumptions $|k| \geq 2$, $|\ell| \geq 2$ can be weakened. Toral linked twist mappings were introduced by Easton [5] (However it seems that the basic phenomena were observed earlier by Oseledec, see [9, ch. 3.8]) From Wojtkowski's paper [13] it follows that assumed $|\alpha\beta| > 4$ H is almost hyperbolic. So $P \cup Q$ decomposes into a countable family of K-components. In view of that, the ergodicity of all powers of H in Theorem A implies that H is a K-system and even a Bernoulli system. (We add to the paper an Appendix, in which we explain what we mean by some of the properties mentioned above and state some facts from Pesin Theory for mappings with singularities [15], [10] , useful in this paper).

Burton and Easton in [2] and Wojtkowski in [13] proved almost hyperbolicity and ergodicity for the case k and ℓ have the same signs. In their case, global stable and unstable manifolds intersect each other since they are very long and go, roughly speaking, in different directions.

Here the global stable (unstable) manifolds, although internally very long could have a very small diameter in $P \cup Q$. On Figure 0.1 we show what could happen with subsequent images of a local unstable manifold γ under iterations of H .

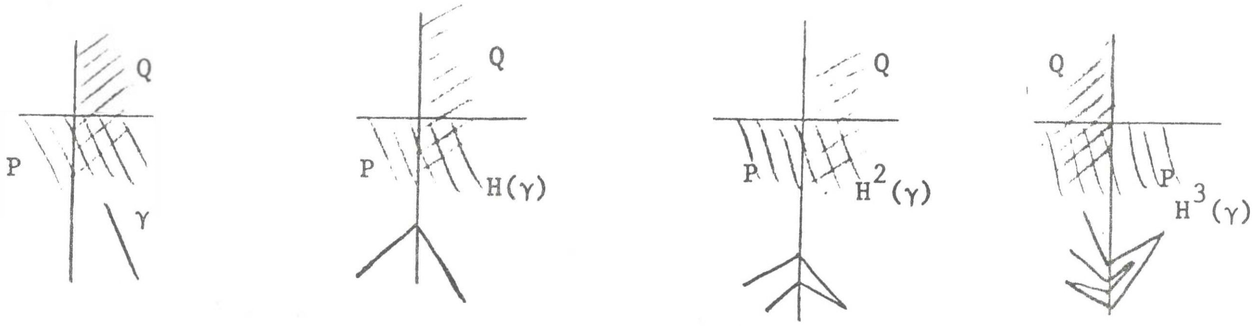


Figure 0.1

We prove however that this is not so, that piecewise linear global stable (unstable) manifolds contain segments winding along the whole annulus P (Q) . A similar phonomenon appears in an example studied by Wojtkowski in [14] .

We can replace P by a finite family of pairwise disjoint annuli $\{P_i\}$ in a surface M (more exactly it is enough if we have smooth embeddings of the interiors of the annuli into M). Similarly we can replace Q by $\{Q_j\}$ and assume that P_i and Q_j intersect transversally (strict definitions will be given in § 2). Let us assume also that $\bigcup_i P_i \cup \bigcup_j Q_j$ is connected. See Figure 0.2.

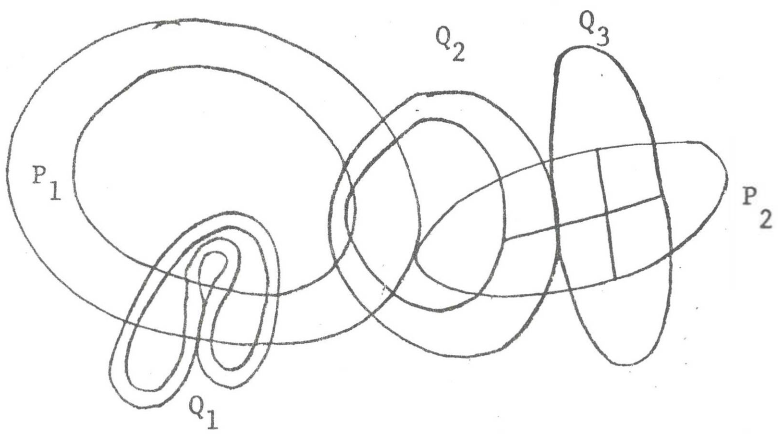


Figure 0.2

We call a composition of two mappings : k_i-twists on $\bigcup_i P_i$ with ℓ_j-twists on $\bigcup_j Q_j$ a linked twist mapping, l.t.m. (we may omit the assumption about linearity of twists and assume only they are C^2-functions.

We assume the existence of a measure on $\cup P_i \cup \cup Q_j$, equivalent to the Lebesgue measures on the annuli, with upper bounded density, invariant under our l.t.m. We consider l.t.m. together with such a measure. In § 2, we prove (and state exactly) the following

<u>Theorem B</u> : A l.t.m. which is built with twists sufficiently strong (i.e. the slopes are sufficiently large), for which $|k_i|, |\ell_j| \geq 2$, is a Bernoulli system.

Sufficient strength of the twists depends only on geometry of the intersection $P_i \cap Q_j$. Our l.t.m.'s generalize both : toral linked twist mappings ([2], [4], [5]) and an example of Bowen [1] , presented on Figure 0.3. (The invariant measure is the Lebesgue measure on the plane).

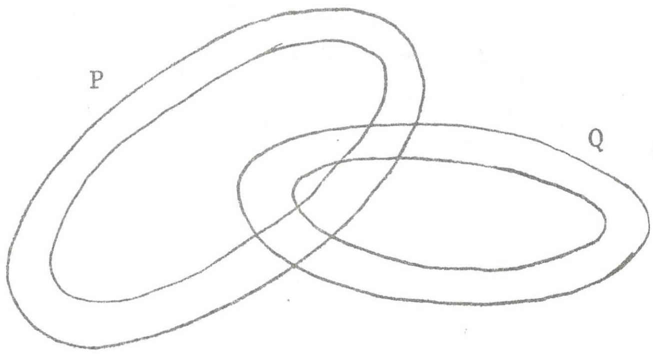


Figure 0.3

Wojtkowski proved that in the case of the Bowen l.t.m. if the twists on P and Q are strong enough, then the Bowen l.t.m. is almost hyperbolic. Theorem B implies that if additionally $|k|$, $|\ell| \geq 2$ then it is Bernoulli. (At the end of § 3 we show how the latter assumption can be weakened).

§ 3 is devoted to looking for assumptions about geometry of intersections $P_i \cap Q_j$ in P_i and Q_j (not only interior geometry of the intersections $P_i \cap Q_j$, as in Theorem B) or about the topology of $\bigcup_i P_i \cup \bigcup_j Q_j$ which could replace the assumptions $|k_i|$, $|\ell_j| \geq 2$ in Theorem B. We connect with the pair $(\{P_i\},\{Q_j\})$ some graphs and define and study their "transitivity". From considerations in this paragraph it immediately follows

Theorem C : Let $\{A_i\}_{i=1,\ldots,p}$ be a family of circles embedded into a surface M, pairwise disjoint. Let $\{B_j\}_{j=1,\ldots,q}$ be another such family. Assume that the pair $(\{A_i\},\{B_j\})$ is in generic position. More exactly assume that the circles A_i and B_j intersect transversally and for at least one circle A_i or B_j if A_i (respect. B_j) intersects $\bigcup_{j=1}^{q} B_j$ (respect. $\bigcup_{i=1}^{p} A_i$) in exactly two points, then these two points are not antipodal in A_i (respect. B_j) . Thicken the circles to annuli. If the thickness of the annuli is small enough then every (reasonable) l.t.m. on their union is Bernoulli . □

Beautiful examples of l.t.m.'s on compact surfaces are provided by some Thurston pseudo-Anosov diffeomorphisms, constructed with the use of "good" pairs of transversal families of circles, see [17, § 6] . Thurston thickens each family of circles to fill up the surface, rather than to make narrow strips, as in Theorem C .

In § 4 we describe briefly some facts concerning linked twist mappings, which we hope to study in more detail in the future :

a) We prove that for almost hyperbolic l.t.m. (as considered in Theorems B and C for example) hyperbolic periodic points and homoclinic points are dense in the whole domain :

b) We can further generalize l.t.m.'s . We can consider a composition of a finite number of families of twists alternately on the annuli $\{P_i\}$ and $\{Q_j\}$ (rather than to compose two families only), so that every annulus is twisted at least once and all twists on it go in the same direction. We prove that under analogous assumptions as in Theorem B, the mapping is Bernoulli .

This allows us to construct Bernoulli, piecewise linear homeomorphisms in every isotopy class of orientation preserving homeomorphisms on every compact orientable surface. This is due to the fact that our twists are exactly Dehn twists and Dehn twists generate all classes of isotopy of orientation preserving homeomorphisms (see [3] or [11]). (We can use at least double twists $|k_i|, |\ell_j| \geq 2$ since every Dehn twist D_α is isotopic with $D_\alpha^{n+1} \circ D_{\alpha'}^{-n}$ where α , α' is a pair of homotopic, embedded circles). We must of course blow up the annuli together with the invariant measure to obtain a set of measure 0 in the complement. (In case of Thurston's examples mentioned before, these homeomorphisms can be done pseudo-Anosov).

We compute an upper estimation for measure entropy. This implies that if the annuli are thickened circles but thickness tends to 0 (situation like in Theorem C) then the measure entropy tends to 0 .

c) If two annuli P_i, P_j (or Q_i, Q_j) have a common boundary circle S , then in the definition of l.t.m. H we do not need to assume that $H|_S = id$. We can prove almost hyperbolicity of H under the same assumptions about slopes of twists as in Theorem B. The proof is the same.

Consider an example on T^2 . Take $F(x,y) = (x+y,y)$. Define $a : \mathbb{R} \longrightarrow \mathbb{R}$, $a(x) = C\cdot(\min(x-[x],[x]+1-x) - \frac{1}{4})$ ($[x]$ means the integral part of x , $C > 0$ is a constant). For any integer $n \geq 0$ let us define

$a_n : \mathbb{R} \longrightarrow \mathbb{R}$, $a_n(x) = 2^{-n} \cdot a(2^n \cdot x)$. Let us define the "n-saw" $A_n : T^2 \longrightarrow T^2$, $A_n(x,y) = (x, y+a_n(x))$ and consider the linked twist mapping $H_n = A_n \circ F$.

The study of the above example has been suggested to me by Wojtkowski, see [13] . For $C \geq 4$, H_n is almost hyperbolic. Wojtkowski proved also in [14] , that assumed $C > 4{,}0329\ldots$ H_n , for $n = 0$, is Bernoulli . This implies immediately, by finite covering that for H_n , $n \geq 0$, T^2 decomposes into at most a finite number of ergodic components. In § 4. c) we fill a gap and prove that H_n , for every $n \geq 0$, is Bernoulli This gives an explicit C^o-arbitrarily small perturbation of the twist F , which is a Bernoulli system.

d) Using examples of Burton-Easton type one immediately obtains Bernoulli diffeomorphisms on T^2 (preserving Lebesgue measure) which are not Anosov but belong to the boundary of the space of Anosov diffeomorphisms (in the C^∞-topology). This simplifies the Katok construction [7] . Continuing by the Katok method one obtains a rich family of Bernoulli diffeomorphisms on the disc D^2 .

I started to do the present paper at IMPA, Rio de Janeiro, and I would like to thank IMPA for the hospitality there. I also want to thank Maciej Wojtkowski for stimulating discussions.

§ 1. Ergodicity of toral linked twist mappings. Proof of Theorem A .

We may assume $|\alpha| = |\beta|$. Otherwise we may change the coordinates on T^2 , taking the coordinates $(x, \sqrt{|\frac{\alpha}{\beta}|}\cdot y)$. Then we consider the torus $\mathbb{R}^2/\mathbb{Z} \times \sqrt{|\frac{\alpha}{\beta}|}\cdot\mathbb{Z}$. We may assume that $\alpha > 0$. Let us repeat according to [2] and [14] the proof of almost hyperbolicity of H . The matrix $DG\circ DF = \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ -\alpha & -\alpha^2+1 \end{pmatrix}$, for $\alpha > 2$, is hyperbolic. Its eigenvalues are

$$\lambda_{\pm} = \frac{-\alpha^2+2\pm\sqrt{\alpha^4-4\alpha^2}}{2}$$

and the expanding eigenvector (ξ_1, ξ_2) satisfies $\xi_1/\xi_2 = -(\frac{\alpha}{2})+\sqrt{(\frac{\alpha}{2})^2-1}$. Let us denote this number by L .

In $\mathbb{R}^2$ let us take the cone $C = \{(x,y) : L \leq \frac{x}{y} \leq 0\}$. Of course for any positive integers s,r, $DG^s\circ DF^r(C) \subset C$. There exists $\lambda > 1$ such that for any sequence of positive integers s_n, r_n , $n = ..-2,-1,1,2..$ there exists a vector $v \in C$ for which

$$||DG^{s_n}\circ DF^{r_n}\circ\ldots\circ DG^{s_1}\circ DF^{r_1}(v)|| > \lambda^n\, ||v|| \text{ for } n > 0$$

and

$$||(DG^{s_{-1}}\circ DF^{r_{-1}}\circ\ldots\circ DG^{s_n}\circ DF^{r_n})^{-1}(v)|| < \lambda^n ||v|| \text{ for } n < 0 .$$

(We consider the norm, maximum of coordinates. One can take $\lambda = |\lambda_-|$).

Denote $P \cap Q = S$. For almost every $x \in S$ we define the sequences (r_n) , (s_n) , $n > 0$ as follows : $r_1 > 0$ is the first time x hits S under F , $s_1 > 0$ is the first time $F^{r_1}(x)$ hits S under G , $r_2 > 0$ is the first time $G^{s_1}\circ F^{r_1}(x)$ hits S under F , and so on. Similarly, for G^{-1} , F^{-1} , r_n, s_n , for $n < 0$ are defined.

Then, operators in the formulas (1) correspond to the operators Dh^n on vectors tangent to S at x, where h denotes the induced map $H_S : S \longrightarrow S$ (i.e. the first return map).

For almost every $x \in S$ its H-orbit hits S with positive frequency (this is a corollary from Birkhoff Ergodic Theorem (see [2, Lemma 4.4]).

This fact, (1) and the analogous facts for H^{-1} and also the fact that almost every point $z \in P \cup Q$ hits S under H and H^{-1}, imply the existence of two H-invariant, measurable, tangent vector fields V^u, V^s and a measurable function Λ on $P \cup Q$ with the following properties : for almost every $x \in P \cup Q$, $\Lambda(x) > 1$

$$\|DH^n(V^u(x))\| > (\Lambda(x))^n \cdot \|V^u(x)\| \qquad \text{for } n > 0$$
$$\|DH^n(V^u(x))\| < (\Lambda(x))^n \cdot \|V^u(x)\| \qquad \text{for } n < 0$$
$$\|DH^n(V^s(x))\| > (\Lambda(x))^n \cdot \|V^s(x)\| \qquad \text{for } n < 0$$
$$\|DH^n(V^s(x))\| < (\Lambda(x))^n \cdot \|V^s(x)\| \qquad \text{for } n > 0 \quad .$$

In particular this proves that Lyapunov exponents are nonzero almost everywhere. Now we can refere to Pesin Theory in Katok-Strelcyn version. This gives existence of local stable and unstable manifolds $\gamma^s(x)$, $\gamma^u(x)$ for almost every $x \in X$ and absolute continuity.

[See Appendix. In fact our case is simpler than what it can happen in general and we could proceed directly. Using the Borel-Cantelli Lemma we could show the existence of $\gamma^{s(u)}(x)$. Absolute continuity of the families $\gamma^{s(u)}(x)$ (even almost everywhere, not only on each of an increasing sequence of sets almost exhausting X) follows easily (see [13]) from the obvious fact that $\gamma^{s(u)}(x)$ are linear segments] .

Hence $P \cup Q$ decomposes into a countable family of K-components.

We shall often consider the induced map $h = H_S = G_S \circ F_S : S \longrightarrow S$. h is uniformly hyperbolic (i.e. with a constant hyperbolicity coefficient $\lambda > 1$) on its domain of continuity and differentiability. The local stable and unstable manifolds for h are of course the same segments $\gamma^{s(u)}(x)$ as for H . (One could consider h directly from the beginning. (K-S) conditions for h hold, but checking (K-S,1) is not so trivial as for H, since Sing h is complicated).

According to the Appendix, to prove ergodicity of h and its powers, it is enough to show that for almost every $x,y \in S$ $h^m(\gamma^u(x))$ intersects $h^{-n}(\gamma^s(y))$ for integers m,n large enough. The same concerns H .

For any segment γ we denote by $\ell_h(\gamma)$ and $\ell_v(\gamma)$ the lengths of the orthogonal projections of γ to the horizontal, respect. vertical axes. We shall prove that for any linear segment $\gamma \subset h^m(\gamma^u(x))$, the image $F_S(\gamma)$, which is a union of linear segments, contains a segment γ' with $\ell_h(\gamma') > \delta \cdot \ell_v(\gamma)$ (for a constant $\delta > 1$ independent of γ) or γ' joins the left and right sides of S . In the latter case, due to $|\ell| \geq 2$, $G(\gamma')$ contains a segment joining the upper and lower side of S (Figure 1.1) (We shall call any segment in S joining the upper and lower sides of S a v-segment, and joining the left and right sides of S an h-segment.)

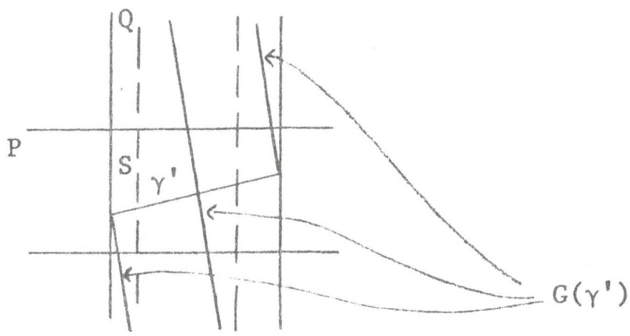


Figure 1.1.

Otherwise we act on γ' with G_S and so on. We get a sequence of segments of exponentially growing length. So it must finish with a v-segment or an h-segment. If we continue iterating with F and G alternately we find an h-segment or v-segment alternately at each step (since $|k|, |\ell| \geq 2$). The same happens for all sufficiently high iterations of H^{-1} on $\gamma^s(y)$. Concluding : $H^m(\gamma^u(x))$ contain v-segments and H^{-n} contain h-segments for all m,n sufficiently large. But the h-segments intersect v-segments.

So let us fix a segment $\gamma \subset H^m(\gamma^u(x))$. Let $m_1 > 0$ be the first time when $F^{m_1}(\gamma)$ intersects S . Then we have four possibilities :

1) $F^{m_1}(\gamma)$ contains an h-segment. This case has just been discussed.

2) The right side of $F^{m_1}(\gamma)$ intersects S (Figure 1.2).

3) The left side of $F^{m_1}(\gamma)$ intersects S (this case if fully analogous to the case 2)).

4) Both sides of $F^{m_1}(\gamma)$ intersect S (Figure 1.3.)

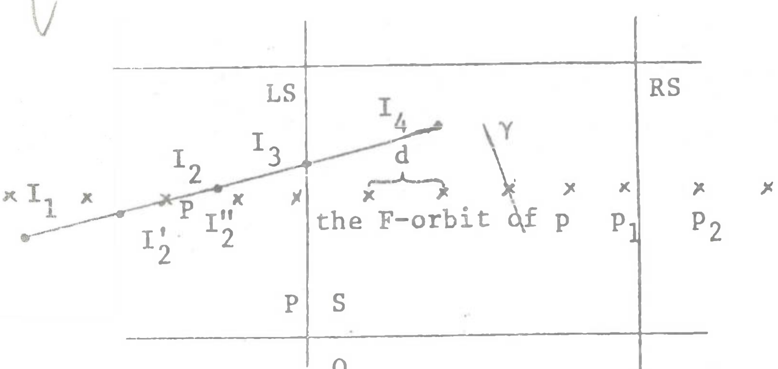


Figure 1.2.

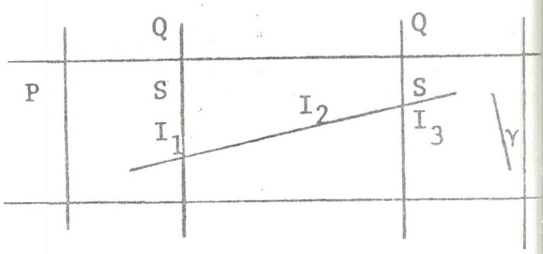


Figure 1.3.

We study the case 2) and make the assumption (*) that $F_S(\gamma)$ does not contain any h-segment. We divide $F^{m_1}(\gamma) \setminus S$ into three intervals I_1, I_2, I_3 . Let us denote $F^{m_I}(\gamma) \cap S = I_4$.

Along I_2 the rotation number $f(y)$ changes by $\alpha \ell_v(I_2)$. So, for any integer $n > 0$ such that $\frac{1}{n} < \alpha \cdot \ell_v(I_2)$ there exists a horizontal circle in P, intersecting I_2 on which the rotation number is $\frac{m}{n}$ for an integer m. So there exists an F-periodic point $p \in I_2$ with the period $[\frac{1}{\alpha \cdot \ell_v(I_2)} + 1]$. It divides I_2 into I_2' and I_2'' (Figure 1.2). The distance d between the different points of the F-orbit of p, $\mathrm{Orb}_F(p)$, is at least

$$\frac{1}{[\frac{1}{\alpha \cdot \ell_v(I_2)} + 1]}$$

Let us denote the last (to the right) point of $\mathrm{Orb}_F(p)$ in S by p_1 and the next one (to the right) by p_2 (Figure 1.2). Let $m_2 > 0$ be the first time when $F^{m_2}(p)$ is between p and RS - the right side of S (Including p and p_1. Observe however that $F^{m_2}(p) \neq p$. Otherwise p would have its F-orbit disjoint with S. But $F^{-m_1}(p) \in \gamma \subset S$, a contradiction).

We denote $J_o = I_2'' \cup I_3$ and $J_m = F(J_{m-1} \setminus S)$ for $m = 1,2,\ldots,m_2$. Then

$$\ell_h(J_m) \geq \min(d+\ell_h(I_2'' \cup I_3)\ ,\ \ell_h(I_2'' \cup I_3) + \alpha \cdot \ell_v(I_2'' \cup I_3))$$

for $m = 1,2,\ldots,m_2$.

If $F^{m_2}(p)$ is between p and LS (the left side of S), then $\ell_h(J_{m_2} \cap S) \geq d$. If $F^{m_2}(p) \in S \setminus \{p_1\}$, then

(2) $$\ell_h(J_{m_2} \cap S) \geq \min(d, \ell_h(I_2'' \cup I_3) + \alpha \cdot \ell_v(I_2'' \cup I_3)).$$

Assume $F^{m_2}(p) = p_1$. Let $\mathrm{dist}(p_1, RS) = \tau \cdot d$.

Then $\operatorname{dist}(RS,p_2) = (1-\tau)\cdot d$. We have either (2) satisfied or $\ell_h(J_{m_2} \cap S) = \tau\cdot d$ and $J_{m_2} \cap S$ touches RS with its right end.

Define $\tilde{J}_o = I_1 \cup I_2'$ and $\tilde{J}_m = F(\tilde{J}_{m-1} \setminus S)$ for $m = 1,..,m_2$: We have either

$$\ell_h(\tilde{J}_{m_2} \cap S) \geq \min((1-\tau)d\ ,\ \alpha\cdot\ell_v(I_1 \cup I_2')+\ell_h(I_1 \cup I_2'))$$

or $\tilde{J}_{m_2} \cap S$ touches LS with its left end (the number $(1-\tau)d$ appears because there can exist $m : 0 < m < m_2$ for which $F^m(p) = p_2$). So due to assumption (*) ,

$$\ell_h((J_{m_2} \cup \tilde{J}_{m_2}) \cap S) \geq \min(d,\alpha\cdot\ell_v(I_3)+\ell_h(I_3)),$$

$$\alpha\cdot\ell_v(I_1) + \ell_h(I_1)) \quad .$$

Thus in order to have $\ell_h((J_{m_2} \cup \tilde{J}_{m_2}) \cap S) \geq \delta\cdot\ell_v(\gamma)$ it is enough that the following inequalities hold :

(3) $\qquad d \geq \delta\cdot\ell_v(\gamma)$

(4) $\qquad \alpha\cdot\ell_v(I_3) + \ell_h(I_3) \geq \delta\cdot\ell_v(\gamma)$

(5) $\qquad \alpha\cdot\ell_v(I_1) + \ell_h(I_1) \geq \delta\cdot\ell_v(\gamma)$.

For (3) it is enough that $\dfrac{\alpha\cdot\ell_v(I_2)}{1+\alpha\cdot\ell_v(I_2)} \geq \delta\cdot\ell_v(\gamma)$ or

(6) $\qquad \ell_v(I_2) \geq \dfrac{\delta\cdot\ell_v(\gamma)}{\alpha(1-\delta\cdot\ell_v(\gamma))}$.

We can assume here $1-\delta\cdot\ell_v(\gamma) > 0$ since assumption (*) implies

(7) $\qquad \ell_v(\gamma)\cdot(L+\alpha) < 2$

and $L+\alpha \geq -1+\sqrt{17} > 3$. We can replace (4),(5) by

(8) (and (9)) $$\ell_v(I_{3(1)}) \geq \frac{\delta \cdot \ell_v(\gamma)}{L+2\alpha} .$$

(due to the fact that $\gamma \in C$ - the cone $L \leq \frac{x}{y} \leq 0$) The work would be also done if

$$\ell_h(I_4) \geq \delta \cdot \ell_v(\gamma)$$

which follows from

(10) $$\ell_v(I_4) \geq \frac{\delta \cdot \ell_v(\gamma)}{L+\alpha} .$$

There exists $\delta > 1$ and a partition of $F^{m_1}(\gamma) \setminus S$ into I_1, I_2, I_3 satisfying (6), (8), (9) or the inequality (10) is satisfied if

(11) $$\ell_v(\gamma) = \sum_{i=1}^{4} \ell_v(I_i) > \ell_v(\gamma)\left(\frac{1}{\alpha(1-\ell_v(\gamma))} + \frac{2}{2\alpha+L} + \frac{1}{\alpha+L}\right) .$$

We divide both sides by $\ell_v(\gamma)$ and due to (7) we obtain the condition

(12) $$1 > \frac{1}{\alpha(1-\frac{2}{L+\alpha})} + \frac{2}{2\alpha+L} + \frac{1}{\alpha+L}$$

(recall that $L = -(\frac{\alpha}{2}) + \sqrt{(\frac{\alpha}{2})^2-1}$.

In the case 4) the situation is simpler. $F^{m_1}(\gamma)$ divides into I_1, I_2, I_3 as on Figure 1.3. We need either $\ell_h(I_1) \geq \delta \cdot \ell_v(\gamma)$, or $\ell_h(I_3) \geq \delta \cdot \ell_v(\gamma)$, or $\ell_h F(I_2) - \ell(I_2) \geq \delta \cdot \ell_v(\gamma)$. (The sufficiency of the last inequality follows from the following: Lift everything to $\mathbb{R}^2$, denote two consecutive components of the lift of Q by Q_1 and Q_2. Then (assumed *) $F(I_2)$ has a components $\tilde{I}$ of its lift between left sides of Q_1 and Q_2 or between right sides of Q_1 and Q_2. Otherwise $\tilde{I}$ would intersect a component of the lift of I_2, which would imply the existence of an F-fixed point $q \notin S$. But $F^{-m_1}(q) \in \gamma \subset S$ - a contradiction).

For this it suffices that

$$(L+\alpha)\cdot \ell_v(I_1) \geq \delta\cdot \ell_v(\gamma) \quad ,$$

or

$$(L+\alpha)\cdot \ell_v(I_3) \geq \delta\cdot \ell_v(\gamma) \quad ,$$

or

$$\alpha\cdot \ell_v(I_2) \geq \delta\cdot \ell_v(\gamma) \quad .$$

For that it is enough if

$$\ell_v(\gamma) = \sum_{i=1}^{3} \ell_v(I_i) > \ell_v(\gamma)\left(\frac{2}{L+\alpha} + \frac{1}{\alpha}\right)$$

i.e.

(13) $$1 > \frac{2}{L+\alpha} + \frac{1}{\alpha} \quad .$$

(12) is satisfied for $\alpha > \alpha_o \simeq 4.152643$;

(13) is satisfied for $\alpha > \alpha_1 \simeq 3.239$.

This gives the constant $C_o = \alpha_o^2 \simeq 17.24445$ in the statement of Theorem A. □

Remark : If $\ell_v(\gamma)$ is small, then in (13) we can write $(L+m_1\alpha)$ instead of $L+\alpha$, where m_1 is large. So (13) can be replaced by

$$1 > \frac{1}{\alpha} \quad .$$

Also (12) can be replaced by

(14) $$1 > \frac{1}{\alpha} + \frac{2}{2\alpha+L} + \frac{1}{\alpha+L} \quad ,$$

since we can omit $\ell_v(\gamma)$ in the denominateur of the ratio $\frac{1}{1-\ell_v(\gamma)}$ of (11).

(14) holds for $\alpha > \alpha_2 \simeq 3.183590$.

In this case the f^m-images of any unstable segment $\gamma^u(z)$, for m sufficiently large, contain segments larger than a constant. Does it imply that there exists a decomposition into a finite number of K-components ?

§ 2. Ergodicity of linked twist mappings. Proof of Theorem B

We denote

$$P(y',y'';a) = \{(x,y) \in \mathbb{R}^2/a\mathbb{Z} \times \{0\} : y' \le y \le y''\}$$

$$Q(x',x'';b) = \{(x,y) \in \mathbb{R}^2/\{0\} \times b\mathbb{Z} : x' \le x \le x''\}$$

Take any sequences of numbers (y_i') , (y_i'') , (a_i) such that $y_i' < y_i''$, $a_i > 0$, $i = 1,\dots,p$ and (x_j') , (x_j'') , (b_j) such that $x_j' < x_j''$, $b_j > 0$, $j = 1,\dots,q$.

Denote

$$P_i = P(y_i',y_i'';a_i) \quad , \quad Q_j = Q(x_j',x_j'';b_j) \quad .$$

Take any smooth surface M and smooth embeddings

$$e_i : \text{int } P_i \longrightarrow M \ , \quad E_j : \text{int } Q_j \longrightarrow M$$

such that

$$e_i(\text{int } P_i) \cap e_j(\text{int } P_j) = \emptyset \qquad \text{for } i \ne j \quad ,$$

$$E_i(\text{int } Q_i) \cap E_j(\text{int } Q_j) = \emptyset \qquad \text{for } i \ne j \quad ,$$

and all the circles $e_i(\{y = \text{const.}\})$ for $y_i' < y < y_i''$ and $E_j(\{x = \text{const.}\})$ for $x_j' < x < x_j''$ intersect transversally. In the future, to simplify notations we shall omit the symbol int before P_i , Q_j when we act with e_i , E_j respectively.

For each C_{ijs} - a connected component of $e_i(P_i) \cap E_j(Q_j)$ we define the coordinates :

$$\Phi_{ijs}(z) = (E_j^{-1}(z)_x \ , \ e_i^{-1}(z)_y)$$

(subscripts x,y denote here x-th and y-th coordinates respectively).

Denote the set of all pairs (j,s) (respectively (i,s)) for which C_{ijs} exists by $\mathcal{J}_i$ (respect. $\mathcal{J}^j$). Denote Card $\mathcal{J}_i = q(i)$, Card $\mathcal{J}^i = p(j)$.

Our subsequent assumption is that for $(j,s) \in \mathcal{J}_i$, $(i,s) \in \mathcal{J}^j$ the mappings $\Phi_{ijs} \circ e_i$, $\Phi_{ijs} \circ E_j$ and the inverse mappings have upper bounded first derivatives and the mappings $e_i^{-1} \circ E_j$, $E_j^{-1} \circ e_i$ have upper bounded second derivatives.

Finally we assume that $\bigcup_{i=1}^{p} e_i(P_i) \cup \bigcup_{j=1}^{q} E_j(Q_j)$ is connected. We call $(\{e_i\}_{i=1,\dots,p}, \{E_j\}_{j=1,\dots,q})$, a <u>pair of transversal families of annuli</u>.

We introduce more notation : Denote $\bigcup_{i,j,s} C_{ijs} = C$, $\bigcup_{(j,s)\in\mathcal{J}_i} C_{js} = C_i$, $\bigcup_{(i,s)\in\mathcal{J}^j} C_{ijs} = C^j$ and $R_{ijs} = \Phi_{ijs}(C_{ijs})$. Define R as the disjoint union $R = \bigcup_{i,j,s} R_{ijs}$. Let $\Phi : C \longrightarrow R$ be equal to Φ_{ijs} on each C_{ijs} .Denote $\Phi(C_i)=R_i$, $\Phi(C^j) = R^j$, $e_i^{-1}(C_i) = P_i$; $E_j^{-1}(C^j)= Q_j$.

Define functions φ_i, ψ_j on the sets P_i, Q_j respect. by the formulas :

$$\Phi\circ e_i(x,y) = (\varphi_i(x,y),y)$$

$$\Phi\circ E_j(x,y) = (x,\psi_j(x,y)) \quad .$$

Define functions φ_i', ψ_j' on the sets R_i, R^j respectively by

$$(\Phi\circ e_i)^{-1}(x,y) = (\varphi_i'(xy),y)$$

$$(\Phi\circ E_j)^{-1}(x,y) = (x,\psi_j'(x,y)) \quad .$$

We denote by $|\zeta|$ the supremum over its domain for any function ζ .

Now we shall define the twists. On each P_i take a (k_i,α_i)-twist F_i (k_i is a nonzero integer, α_i is a real number) defined as follows

$$F_i(x,y) = (x+f_i(y),y)$$

for f_i a C^2-function defined on $<y_i', y_i''>$, such that $f_i(y_i') = 0$, $f_i(y_i'') = k_i a_i$.

Assume the function $\frac{df_i}{dy}$ is nowhere zero. If it is positive (i.e. $k_i > 0$) define the slope $\alpha_i = \inf\limits_{<y_i',y_i''>} \frac{df_i}{dy}$. If $k_i < 0$, $\alpha_i = \sup\limits_{<y_i',y_i''>} \frac{df}{dy}$.

Take on each Q_j a (ℓ_j,β_j)-twist G_j defined analogously.

Define $\hat{F}$, $\hat{G}$: $\bigcup\limits_i e_i(P_i) \cup \bigcup\limits_j E_j(Q_j) \circlearrowleft$ by

$$\hat{F}(z) = \begin{cases} e_i F_i e_i^{-1}(z) & \text{for } z \in e_i(P_i) \\ z & \text{for } z \notin \bigcup\limits_{i=1}^{p} e_i(P_i) \end{cases}$$

$$\hat{G}(z) = \begin{cases} E_j G_j E_j^{-1}(z) & \text{for } z \in E_j(Q_j) \\ z & \text{for } z \notin \bigcup\limits_{j=1}^{q} E_j(Q_j) \end{cases} .$$

Define a linked twist mapping (l.t.m.) as $H = \hat{G}\circ\hat{F}$. Consider H together with an H-invariant probability measure ν on $\bigcup\limits_i e_i(P_i) \cup \bigcup\limits_j E_j(Q_j)$ such that on each P_i the measure $e_i^*(\nu)$ is equivalent to the Lebesgue measure ν_i , with bounded density with respect to ν_i and such that each $E_j^*(\nu)$ has the analogous property. (Assume of course that such a measure ν exists.)

Now Theorem B takes the form

<u>Theorem B</u> : Fix a pair of transversal families of annuli $(\{e_i\},\{E_j\})$

If an l.t.m. H on $\bigcup_i e_i(P_i) \cup \bigcup_j E_j(Q_j)$ is built from (k_i,α_i)-twists, $i = 1,\dots,p$ and (ℓ_j,β_j)-twists, $j = 1,\dots,q$ where $|\alpha_i|$, $|\beta_j|$ are large enough to satisfy Condition $\mathcal{H}$ below, then H is almost hyperbolic.

If α_i, β_j satisfy the stronger Condition $\mathcal{E}$ and $|k_i|$, $|\ell_j| \geq 2$ then H and all its powers are ergodic. So H is a Bernoulli system.

<u>Condition $\mathcal{H}$</u> :

($\mathcal{H}$1) $\operatorname{sgn} \tilde{\alpha}_i = \operatorname{sgn} \alpha_i$ for every $i = 1,\dots,p$

($\mathcal{H}$2) $\operatorname{sgn} \tilde{\beta}_j = \operatorname{sgn} \beta_j$ for every $j = 1,\dots,q$

($\mathcal{H}$3) $|\tilde{\alpha}_i \cdot \tilde{\beta}_j| > (1+\mu_i)(1+\mu^j)$ where

$$\mu_i = \left|\frac{d\varphi_i}{dx}\right| \cdot \left|\frac{d\varphi_i'}{dx}\right| \quad , \quad \mu^j = \left|\frac{d\psi_j}{dy}\right| \cdot \left|\frac{d\psi_j}{dy}\right| \quad ,$$

for every pair (i,j) such that $e_i(P_i) \cap E_j(Q_j) \neq \emptyset$. Here we denote

$$\tilde{\alpha}_i = (\operatorname{sgn} \alpha_i) \cdot ((|\alpha_i| - \left|\frac{d\varphi_i'}{dy}\right|) \cdot \left|\frac{d\varphi_i'}{dx}\right|^{-1} - \left|\frac{d\varphi_i}{dy}\right|)$$

$$\tilde{\beta}_j = (\operatorname{sgn} \beta_j) \cdot ((|\beta_j| - \left|\frac{d\psi_j'}{dx}\right|) \cdot \left|\frac{d\psi_j'}{dy}\right|^{-1} - \left|\frac{d\psi_j}{dx}\right|)$$

[$\tilde{\alpha}_i, \tilde{\beta}_j$ bound the slopes of the induced mappings $(F_i)_{P_i}, (G_j)_{Q_j}$ respectively in the coordinates $\Phi \circ e_i$, $\Phi \circ E_j$. We could replace $\tilde{\alpha}_i$ (similarly $\tilde{\beta}_j$) by smaller numbers

$$(\operatorname{sgn} \alpha_i) \cdot (|\alpha_i| - 2 \cdot \left|\frac{d\varphi_i'}{dy}\right|) \cdot \left|\frac{d\varphi_i'}{dx}\right|^{-1} \quad .$$

They have clearer geometric meaning, since $\left|\frac{d\varphi_i'}{dy}\right|$ denotes in fact, the supremum of cotangents of angles between the horizontal circles $\{y = \text{const.}\} \subset P_i$ and images of the vertical circles $\{x = \text{const.}\} \subset Q_j$ in P_i , i.e. $e_i^{-1} \circ E_j(\{x = \text{const.}\}) \subset P_i$ for $j \in \mathcal{J}_i$.]

Condition E

(E1) sgn $\overset{\simeq}{\alpha}_i$ = sgn $\tilde{\alpha}_i$ = sgn α_i for every $i = 1,\dots,p$

(E2) sgn $\overset{\simeq}{\beta}_j$ = sgn $\tilde{\beta}_j$ = sgn β_j for every $j = 1,\dots,q$

(E3) $|\overset{\simeq}{\alpha}_i \cdot \overset{\simeq}{\beta}_j| > (\max(X(i),1+\mu_i))\cdot\max(Y(j),1+\mu^j)$ for every pair (i,j) such that $e_i(P_i) \cap E_j(Q_j) \neq \emptyset$. Here we denote

$$\overset{\simeq}{\alpha}_i = (\operatorname{sgn}\alpha_i)\cdot\left(|\alpha_i| - (2\cdot q(i)+3\cdot X(i))\left|\frac{d\varphi_i'}{dy}\right|\right)\cdot\left|\frac{d\varphi_i'}{dx}\right|^{-1}$$

$$\overset{\simeq}{\beta}_j = (\operatorname{sgn}\beta_j)\cdot\left(|\beta_j| - (2\cdot p(j)+3\cdot Y(j))\left|\frac{d\psi_j'}{dx}\right|\right)\cdot\left|\frac{d\psi_j'}{dy}\right|^{-1}$$

$X(i)$, respect. $Y(j)$, is the largest solution of the equation

$$1 = \frac{2q(i)}{X} + \frac{q(i)}{X-3q(i)} + \frac{2}{X-\mu_i} \quad , \quad \text{respectively}$$

$$1 = \frac{2p(j)}{Y} + \frac{p(j)}{Y-3p(j)} + \frac{2}{Y-\mu^j} \quad .$$

(We treat $\overset{\simeq}{\alpha}_i$, $\overset{\simeq}{\beta}_j$ as artificial "subslopes". For toral linked twist mappings, Devaney generalized toral l.t.m.'s, see [4] and Thurston examples [17 § 6] $\Phi \circ e_i = \Phi \circ E_j$ = identity. So $\overset{\simeq}{\alpha}_i = \tilde{\alpha}_i = \alpha_i$, $\overset{\simeq}{\beta}_j = \tilde{\beta}_j = \beta_j$).

Remark : Conditions H and E have a local character. If we treat the embeddings e_i , E_j as charts on the manifold $\cup\, e_i(P_i) \cup \cup\, E_j(Q_j)$, Conditions H , E about each individual α_i depend only on the geometry and topology (i.e. number of components) of the intersections $e_i(P_i) \cap E_j(Q_j)$ for all j for which this intersection is nonempty but do not depend on Q_j's which are far away. The same concerns the β_j's.

Proof of Theorem B :

The idea of the Proof is similar to that of Theorem A. However one should modify it a little since for each horizontal annulus P_i the number of components of intersections with the vertical annuli Q_j, number denoted by $q(i)$, can be greater than 1 (similarly it can happen that $p(j) > 1$). The fact that the images of the vertical circles, $e_i^{-1}E_j$ ({x = const.}) in P_i, need not be orthogonal to the horizontal circles {y = const.} and the fact that the maps $e_i^{-1}E_j$ need not be isometries leads only to new constants in the estimations.

Almost hyperbolicity

It is enough to prove that Condition $\mathcal{H}$ implies that the induced mapping $h = H_C$ has nonzero Lyapunov exponents and to check the (K-S) conditions for H (see Appendix). We shall consider $\tilde{h} = \Phi h \Phi^{-1}$ on $\mathcal{R}$. Denote also $\tilde{F} = \Phi \hat{F}_C \Phi^{-1}$, $\tilde{G} = \Phi \hat{G}_C \Phi^{-1}$. Of course $\tilde{h} = \tilde{G} \circ \tilde{F}$. Denote by $\ell_h(w)$ (respect. $\ell_v(w)$) the horizontal (respect. vertical) coordinate of a vector w in euclidean coordinates, denote the basic vectors at z, by $\frac{\partial}{\partial x}(z)$, $\frac{\partial}{\partial y}(z)$.

If α_i is positive define $\tilde{\alpha}_i' = \inf_{z \in \mathcal{R}_i} \ell_h D\tilde{F} \frac{\partial}{\partial y}(z)$, if negative $\tilde{\alpha}_i' = \sup_{z \in \mathcal{R}_i} \ell_h D\tilde{F} \frac{\partial}{\partial y}(z)$, for $i = 1,\dots,p$. $\tilde{\beta}_j'$ for $j=1,\dots,q$ is defined analogously. (Remember that $\tilde{F}$ and $\tilde{G}$ are defined and differentiable only out of a closed, nowhere dense, subset of $\mathcal{R}$ of zero measure. For simplicity of notation we will not make distinction between this domain and $\mathcal{R}$).

We shall now estimate the slopes $\tilde{\alpha}_i'$, $\tilde{\beta}_j'$ by passing through the original coordinates on P_i and Q_j.

Assume for example $\alpha_i > 0$. Take any point $z \in \mathcal{R}_i$ and assume

$m > 0$ is the first time when $F^m((\Phi\circ e_i)^{-1}(z)) \in P_i$. Then

$$\ell_h \, D\widetilde{F} \frac{\partial}{\partial y}(z) = \ell_h \, D(\Phi\circ e_i \circ F^m \circ e_i^{-1} \circ \Phi^{-1})(\frac{\partial}{\partial y}(z)) =$$

$$= \ell_h \, D(\Phi\circ e_i \circ F^m)(\frac{d\varphi_i'}{dy}(z) \cdot \frac{\partial}{\partial x}(\Phi\circ e_i)^{-1}(z) + \frac{\partial}{\partial y}(\Phi\circ e_i)^{-1}(z)) \geq$$

$$\geq \ell_h \, D(\Phi\circ e_i)((\frac{d\varphi_i'}{dy}(z)+\alpha_i)\cdot\frac{\partial}{\partial x}(F^m\circ(\Phi\circ e_i)^{-1}(z)) + \frac{\partial}{\partial y}F^m\circ(\Phi\circ e_i)^{-1}(z)) \geq$$

$$\geq (-|\frac{d\varphi_i'}{dy}| + \alpha_i) \cdot |\frac{d\varphi_i'}{dx}|^{-1} - |\frac{d\varphi_i}{dy}| = \widetilde{\alpha}_i \quad .$$

If we assume the last term is greater than 0 , which is just Condition $H1$, then $\operatorname{sgn} \widetilde{\alpha}_i' = \operatorname{sgn} \alpha_i$ (besides, we have used the assumption $\widetilde{\alpha}_i > 0$ in the last inequality above, which can be false without that). $\alpha_i < 0$ can be treated similarly. Analogously we show that Condition $H2$ implies $\operatorname{sgn} \widetilde{\beta}_j' = \operatorname{sgn} \beta_j$.

Take for every $z \in R_i$, for $i = 1,\dots,p$, the cone $C_z = \{(\xi_1,\xi_2) \in T_z R_i : |\xi_1/\xi_2| \leq \varepsilon_i\}$, for a positive number ε_i . Take for every $z \in R^j$, for $j = 1,\dots,q$, the cone

$$C^z = \{(\xi_1,\xi_2) \in T_z R^j : |\xi_1/\xi_2| \leq \varepsilon^j\} \quad ,$$

for a positive number ε^j . Assume that for any pair (i,j) such that $e_i(P_i) \cap E_j(Q_j) \neq \emptyset$

(1) $$\varepsilon_i \cdot \varepsilon^j > 1 \quad .$$

Then for every $z \in R$, $C_z \cup C^z = T_z R$. So, in order to obtain $D\widetilde{F}(\bigcup_{z\in R} C_z) \subset \bigcup_{z\in R} C^z$ and $D\widetilde{G}(\bigcup_{z\in R} C^z) \subset \bigcup_{z\in R} C_z$ it is enough if

$$D\widetilde{F}(\bigcup_{z\in R} C_z) \cup \bigcup_{z\in R} C_z = \emptyset$$

and

$$D\widetilde{G}(\bigcup_{z\in R} C^z) \cap \bigcup_{z\in R} C^z = \emptyset$$

One can easily compute that for this, it is sufficient that

$$|\tilde{\alpha}_i'| \geq (1 + |\frac{d\varphi_i}{dx}||\frac{d\varphi_i'}{dx}|)\varepsilon_i \qquad \text{and}$$

(2)

$$|\tilde{\beta}_j'| \geq (1+ |\frac{d\psi_i}{dy}||\frac{d\psi_j'}{dy}|)\varepsilon^j$$

(on the horizontal circles of P_i, the F_i's are rotations, so restricted to these circles, they have derivatives 1 . In R_i the module of the derivatives of $\tilde{F}_i$ and $\tilde{F}_i^{-1}$ restricted to the horizontal intervals are bounded above by

$$\mu_i = |\frac{d\varphi_i}{dx}||\frac{d\varphi_i'}{dx}| \quad .$$

The numbers $\mu^j = |\frac{d\psi_j}{y}|\ |\frac{d\psi_j'}{y}|$ play the analogous role for the maps $\tilde{G}_j$).
So, if for all pairs (i,j) such that $e_i(P_i) \cap E_j(Q_j) \neq \emptyset$

(3) $$(|\tilde{\alpha}_i'|\cdot|\tilde{\beta}_j'|)/(1+\mu_i)(1+\mu^j) > 1$$

then there exists a system of positive numbers $\{\varepsilon_i, \varepsilon^j\}$, $i = 1,\dots,p$, $j = 1,\dots,q$ satisfying (1), (2) and the family $\{C_z\}$ of the cones, such that

$$D\tilde{h}(\bigcup_{z\in R} C_z) \subset \bigcup_{z\in R} C_z \quad .$$

(One can take $\varepsilon_i = |\tilde{\alpha}_i'| / (1+\mu_i)$, $\varepsilon^j = |\tilde{\beta}_j'|/(1+\mu^j))$.
But (3) follows from (H3).

For any $w \in C_z$, $\ell_h(D\tilde{F}(w)) \geq \varepsilon_i \ell_v(w)$ and $\ell_v(D\tilde{G}\circ D\tilde{F}(w)) = \ell_v(D\tilde{h}(w)) \geq \varepsilon^j\cdot\varepsilon_i\cdot\ell_v(w) > \ell_v(w)$. Here $z \in R_i$, $\tilde{F}(z) \in R^j$.
This and the analogous consideration for $\tilde{h}^{-1}$ imply that Lyapunov exponents of $\tilde{h}$, hence h, are nonzero. Since almost every $z \in \bigcup e_i(P_i) \cup \bigcup E_j(Q_j)$

hits C with positive frequency, then Lyapunov exponents of H are positive. The (K-S) conditions (see Appendix) for H are trivially satisfied. The assumptions that the second derivatives of $e_i^{-1}E_j$, $E_j^{-1}e_i$, φ_i, ψ_j and density of the invariant measure with respect to Lebesgue measures on P_i, Q_j are bounded above, have been fixed especially for this aim.

Ergodicity :

Assume Condition H is satisfied and choose a system of positive numbers $\{\varepsilon_i, \varepsilon^j\}$, $i = 1,\dots,p$, $j = 1,\dots,q$ such that $|\tilde{\alpha}_i'| \geq (1+\mu_i)\varepsilon_i$, $|\tilde{\beta}_j'| \geq (1+\mu^j)\cdot\varepsilon^j$ and $\varepsilon_i\cdot\varepsilon^j > 1$ for all i, j for which $e_i(P_i) \cap E_j(Q_j) \neq 0$. Denote $\min\{\varepsilon_i\cdot\varepsilon^j : e_i(P_i) \cap E_j(Q_j) \neq 0\} = \Delta > 1$.

We shall compute an additional condition for α_i so that for any local unstable manifolds for H, $\gamma = \gamma^u(z)$ where $z \in C_{ijs}$ ($\gamma \subset C_{ijs}$, by definition) either $\hat{F}_C(\gamma)$ contains a curve γ inside a set $C_{ij's'}$ which joins left and right sides of $C_{ij's'}$ or

$$\ell_h(\Phi_{ij's'}(\gamma')) \geq \varepsilon_i \cdot \ell_v(\Phi_{ijs}(\gamma)) \quad . \tag{4}$$

(If we assume $\alpha_i > 0$ then ℓ_h, respect. ℓ_v, denote here the horizontal, respect. vertical lengths of upper oriented curves in R_i. More exactly we consider inside P_i, respect. R_i, only curves which transversally intersect the horizontal circles, respect. intervals. Then ℓ_h means the x-th coordinate of the upper end minus the x-th coordinate of the lower end of the curve so the horizontal length, ℓ_h, can be as well positive as negative. ℓ_v, the difference between y-th coordinates is here positive. If $\alpha_i < 0$ we change the sign of ℓ_h.)

We shall compute analogously a condition for β_j so that for any curve $\gamma \subset C_{ijs}$, $\gamma \subset \hat{F}_C(\gamma^u(z))$, $\hat{G}_C(\gamma)$ contains a curve $\gamma' \subset C_{i'js'}$

which joins the upper and lower sides of $C_{i'js'}$ or

$$\ell_v(\Phi_{i'js'}(\gamma')) \geq \varepsilon^j \cdot \ell_h(\Phi_{ijs}(\gamma)) \qquad (4')$$

(For β_j we consider the right side orientation on curves transversally intersecting vertical circles or intervals).

So, beginning with $\gamma^u(z)$ and taking successive images under $\hat{F}_C$, $\hat{G}_C$, ... we obtain at each second step a curve Δ-times longer. So we will finish with a curve $\gamma' \subset \hat{G}_C \circ \hat{F}_C \ldots \circ \hat{G}_C \circ \hat{F}_C(\gamma^u(z))$ joining upper and lower sides of a $C_{i'j's'}$ or $\gamma'' \subset \hat{F}_C \circ \hat{G}_C \circ \ldots \circ \hat{G}_C \circ \hat{F}_C$ joining left and right sides. Since we assume $|k_i|$, $|\ell_j| \geq 2$ for all i,j then $H^{\min(p,q)}(\gamma')$, or $H^{\min(p,q)}\hat{G}_C(\gamma'')$, contains curves winding around all annuli $e_i(P_i)$, which will finish the proof (see Figure 1.1. in § 1). This is the unique place we use the assumption that $|k_i|$, $|\ell_j| \geq 2$.

Fix i , assume $\alpha_i > 0$, fix $\gamma = \gamma^u(z) \subset C_{ij_os_o}$. Until the end of the proof we shall usually omit the subscript i .

Make the <u>assumption</u> * that $\hat{F}_C(\gamma)$ does not contain any curve joining left and right sides of any $C_{js} = C_{ijs}$ for $(j,s) \in \mathcal{J}_i$.

Let m_1 be the smallest $m > 0$ for which $\hat{F}^m(\gamma)$ hits C (i.e. $\hat{F}^{m_1}(\gamma) \cap C_{j_1s_1} \neq \emptyset$ for some $(j_1,s_1) \in \mathcal{J}_i$). Either $F^{m_1}(\gamma) \subset C$ (denote this case by (i)) , or it hits $C_{j_1s_1}$ with its upper end (or with the lower end, which is an analogous case) as on Figure 2.1.

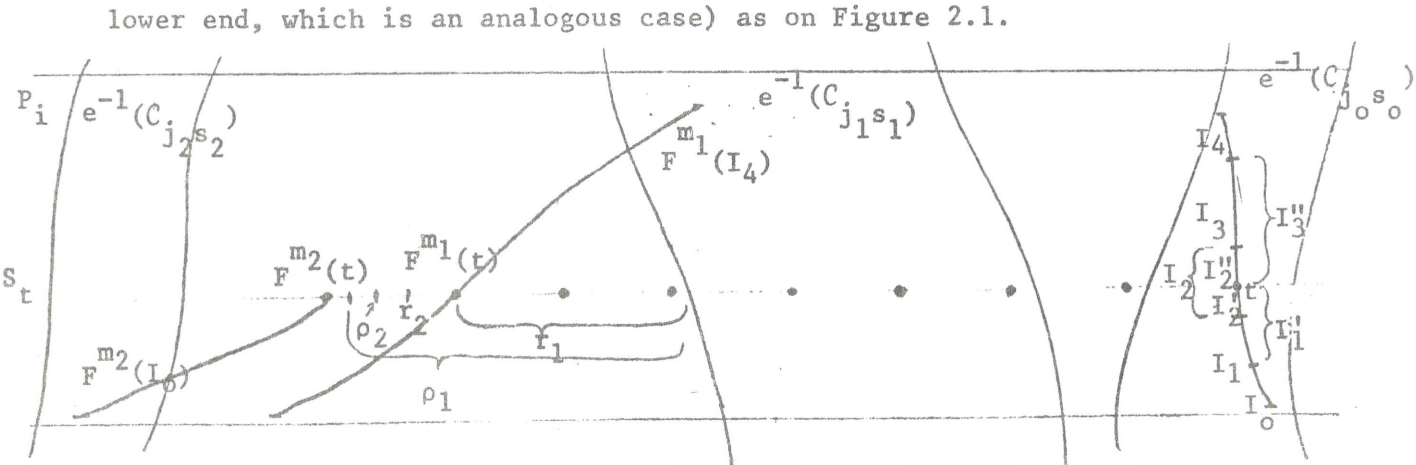


Figure 2.1

Let us consider the second case. Denote $e^{-1}(\gamma) = I$. Denote $I_4 = F^{-m_1}(F^{m_1}(I) \cap e^{-1}(C_{j_1 s_1}))$. Take the sequence of curves

$$J_{m_1} = \hat{F}^{m_1}(\gamma)$$

$$J_m = \hat{F}(J_{m-1} \setminus C) \quad \text{for} \quad m > m_1 \ .$$

There may not exist any $m \geq m_1$ for which J_m hits C with its lower end. Denote this case by (ii).

However let us consider the case when such an m exists. Denote the first such m by m_2 and the appropriate C_{js} by $C_{j_2 s_2}$. Denote $I_o = e^{-1}\hat{F}^{-m_2}(J_{m_2} \cap C) \subset I$. It may happen that $I \setminus (I_o \cup I_4)$ is empty or is one point. Denote that case by (iii) . Consider now the case when there is a nondegenerate curve in I between I_o and I_4 .

Divide it into three curves I_1, I_2, I_3 as on Figure 2.1. There exists an F-periodic point $t \in I_2$ with the minimal distance "d" between the points of Orb(t) (the F-orbit of t) satisfying

(5) $$d \geq \frac{\alpha.\ell_v(I_2)}{1+\frac{\alpha\cdot\ell_v(I_2)}{a}}$$

(we recall that $a = a_i$ is length of the annulus P_i).

By assumption (*)

(6) $$\alpha\cdot\ell_v(I) < 3\cdot a \quad .$$

Otherwise a continuous function (F(z)-z)x-th coordinate would have growth on I at least $3a$. So $F(I)$ would intersect I in at least three points. So between the first and the third intersection it would fully intersect every set $e^{-1}(C_{js})$ for $(j,s) \in \mathcal{J}_i$.

(One could now deduce from (5) and (6), replacing I (in (6)) by I_2, that $d \geq \frac{\alpha \cdot \ell_v(I_2)}{4}$. We shall however use (5) and (6) later in a better way).

Denote left and right components of the boundary of $e^{-1}(C_{js})$ in $\operatorname{int} P_i$ by $LS_{j,s}$, $RS_{j,s}$ respectively. Observe that the Lipschitz constants of $LS_{j,s}$, $RS_{j,s}$ treated as graphs of functions of the y-th variable to the x-th variable are bounded above by $N = |\frac{d\varphi'}{dy}|$ (these functions are even differentiable since the mappings $e_i^{-1} \circ E_j$ have upper bounded second derivatives, but that is not important here).

Denote by S_t the horizontal circle in P_i containing our F-periodic point t.

We denote

$$t_{j,s} = LS_{j,s} \cap S_t \quad , \quad t'_{j,s} = RS_{j,s} \cap S_t$$

for $(j,s) \in \mathcal{J}_i$.

The point t divides I_2 into I'_2 and I''_2 (see Fig. 2.1). Denote $I'_1 = I_1 \cup I'_2$ and $I'_3 = I_3 \cup I''_2$.

Denote $r_1 = t_{j_1 s_1} - F^{m_1}(t)$ and

$$\rho_1 = \min((\ell_h(F^{m_1}(I'_3)) + \alpha \cdot \ell_v(I'_3) - N \cdot \ell_v(I'_3) - \eta \cdot \ell_v(I) - r_1) \cdot \frac{q(i)}{q(i)-1} + r_1, d + r_1) .$$

In the above formula we denote

$$\eta = |\frac{d\varphi'}{dy}| + |\frac{d\varphi'}{dx}| \cdot \varepsilon .$$

This coefficient is motivated by the fact that if for any m and $\tilde{I} \subset I$ such that $F^m(\tilde{I}) \subset e^{-1}(C)$,

$$\ell_h(F^m(\tilde{I})) \geq \eta \cdot \ell_v(I) \quad , \quad \text{then} \tag{7}$$

(7')
$$\ell_h(\Phi\circ e(F^m(\tilde{I}))) \geq \varepsilon\cdot \ell_v(I) \quad .$$

(Proof : Join the ends of $\Phi\circ e(F^m(\tilde{I}))$ by an interval $\hat{I} \subset R$. Then

$$\ell_h(\Phi\circ e)^{-1}(\hat{I}) \leq \left|\frac{d\varphi'}{dy}\right|\cdot \ell_v(\hat{I}) + \left|\frac{d\varphi'}{dx}\right| \ell_h(\hat{I}) = A \quad .$$

If (7') were false, then

$$A < \left(\left|\frac{d\varphi'}{dy}\right| + \left|\frac{d\varphi'}{dx}\right| \varepsilon\right) \ell_v(\hat{I}) = \eta \ell_v(\hat{I}) \quad ,$$

which would contradict (7).)

In definition of ρ_1 , if $q(i) = 1$, we mean $\frac{q(i)}{q(i)-1} = +\infty$.
We assume

(8)
$$\rho_1 > r_1 \quad .$$

In fact we shall need more.

Due to the term r_1+d in definition of ρ_1 , for each $(j,s) \in \mathcal{J}_i$ the arc$(t_{j,s}-\rho_1, t_{j,s}-r_1) \subset S_t$ contains at most one point from the set $\mathrm{Orb}(t)$. For the pair (j_1,s_1) such an arc contains no points from $\mathrm{Orb}(t)$, since its right end belongs to $\mathrm{Orb}(t)$.

There exist numbers ρ_2, r_2 such that $\rho_1 \geq \rho_2 > r_2 \geq r_1$, for each $(j,s) \in \mathcal{J}_i$

$$(t_{j,s}-\rho_2, t_{j,s}-r_2) \cap \mathrm{Orb}(t) = \emptyset$$

and

$$\rho_2 - r_2 = \frac{1}{q(i)}(\rho_1 - r_1) \quad .$$

Denote by m_3 the first $m > m_1$ such that $F^m(t) \in \bigcup_{(j,s)\in\mathcal{J}_i} \langle t_{j,s}-r_2, t'_{j,s}\rangle$.

Denote the case $m_2 \geq m_3$ by (iv). Now let us consider the case

$m_2 < m_3$. In this case, we repeat the above construction on the right sides of the sets C_{js} .

Denote $r_1' = F^{m_2}(t) - t'_{j_2 s_2}$

$$\rho_1' = \min((\ell_h(F^{m_2}(I_1'))+\alpha\cdot\ell_v(I_1')-N\cdot\ell_v(I_1')-\eta\cdot\ell_v(I)-r_1')\cdot \cdot\frac{q(i)}{q(i)-1} + r_1' \ , \ d+r_1') \quad .$$

Assume

$$(8') \qquad \rho_1' > r_1'$$

and as before find ρ_2', r_2' such that

$$\rho_1' \geq \rho_2' > r_2' \geq r_1' \quad ,$$

for each $(j,s) \in \mathcal{J}_i$

$$(t'_{j,s}+r_2', t'_{j,s}+\rho_2') \cap \mathrm{Orb}(t) = \emptyset$$

and

$$\rho_2'-r_2' = \frac{1}{q(i)}(\rho_1'-r_1') \quad .$$

Let m_4 be the first $m > \max(m_1,m_2)$ such that

$$F^{m_4}(t) \in \langle t_{j_4,s_4}-r_2, t'_{j_4,s_3}+r_2' \rangle$$

for a pair $(j_4,s_4) \in \mathcal{J}_i$.

Define

$J(m_1) = F^{m_1}(I_3')$ and

$J(m) = F(J(m-1) \setminus e^{-1}(C))$ for $m_1 < m \leq m_4$.

$J'(m_2) = F^{m_2}(I_1')$ and

$$J'(m) = F(J'(m-1) \setminus e^{-1}(C)) \quad \text{for} \quad m_2 < m \le m_4 .$$

We have

$$\ell_h(J(m)) \ge \min(\ell_h(F^{m_1}(I_3')) + \alpha \cdot \ell_v(I_3') \ , \ \rho_2 - N \cdot \ell_v(I_3')) = A$$

$$\ell_h(J'(m)) \ge \min(\ell_h(F^{m_2}(I_1')) + \alpha \cdot \ell_v(I_1') \ , \ \rho_2' - N \cdot \ell_v(I_3')) = A' .$$

If $F^{m_4}(t) \in \langle t_{j_4,s_4} - r_2, t_{j_4,s_4} \rangle$ then

$$\ell_h(J(m_4) \cap e^{-1}(C_{j_4,s_4})) \ge A - r_2 - N \cdot \ell_v(I_3') =$$

$$= \min(\ell_h(F^{m_1}(I_3')) + \alpha \cdot \ell_v(I_3') - N \cdot \ell_v(I_3') - r_2 \ , \ \rho_2 - 2N \cdot \ell_v(I_3') - r_2).$$

For (4) it suffices if this is greater than or equal to $\eta \cdot \ell_v(I)$.

For the first term in the minimum bracket, this follows from (8). (The complicated formula defining ρ_1 has been adjusted especially to this aim).

Rewrite the inequality for the second term

(9) $$\rho_2 - r_2 - 2N \cdot \ell_v(I_3') \ge \eta \cdot \ell_v(I) \quad .$$

Similarly, in the case $F^{m_4}(t) \in \langle t'_{j_4 s_4}, t'_{j_4 s_4} + r_2' \rangle$ for (4) it suffices that :

(9') $$\rho_2' - r_2' - 2N \cdot \ell_v(I_1') \ge \eta \cdot \ell_v(I) \quad .$$

If $F^{m_4}(t) \in (t_{j_4 s_4}, t'_{j_4 s_4})$ then either

$$J(m_4) \subset e^{-1}(C_{j_4 s_4}) \quad \text{or} \quad J'(m_4) \subset e^{-1}(C_{j_4 s_4})$$

by assumption *. This leads to the inequalities $A \ge \eta \cdot \ell_v(I)$ and $A' \ge \eta \cdot \ell_v(I)$, which follow from (9),(9'). (8),(8'),(9),(9') follow from the inequalities

$$(10)\quad \begin{cases} \dfrac{\alpha\cdot \ell_v(I_3')-2N\cdot \ell_v(I_3')-\eta\cdot \ell_v(I)}{q(i)-1} -2N\cdot \ell_v(I_3') \geq \eta\cdot \ell_v(I) \\ \text{if } q(i) > 1 \text{ , or} \\ \alpha\cdot \ell_v(I_3')-2N\cdot \ell_v(I_3') > \eta\cdot \ell_v(I) \quad \text{if } q(i) = 1 \text{ ,} \end{cases}$$

$$(11)\qquad \frac{d}{q(i)} - 2N\ell_v(I_3') \geq \eta\cdot \ell_v(I)$$

and analogous inequalities (10'), (11') with I_1' instead of I_3'.

We can replace (10) and (10') by

$$(12,12')\qquad \ell_v(I_{3(1)}) \geq (\frac{\alpha}{q(i)} - 2N)^{-1}\cdot \eta\cdot \ell_v(I) \quad .$$

Inequality (11), due to (5), follows from

$$\frac{1}{q(i)}\cdot \frac{\alpha\cdot \ell_v(I_2)}{1+\dfrac{\alpha\cdot \ell_v(I_2)}{a}} \geq (\eta+2N)\cdot \ell_v(I) \quad .$$

This is equivalent to

$$\ell_v(I_2) \geq \frac{q(i)\cdot(\eta+2N)\ell_v(I)}{\alpha(1-\dfrac{q(i)(\eta+2N)\ell_v(I)}{a})}$$

(assuming the denominator of the right side ratio is positive).

Now we use (6) for $\ell_v(I)$ in the denominator and obtain a sufficient condition :

$$(13)\qquad \ell_v(I_2) \geq (\frac{\alpha}{q(i)} - 3(\eta+2N))^{-1}(\eta+2N)\ell_v(I)$$

together with the assumption that the right side of (13) is positive. For (4) it also suffices that

(14) $$\ell_h(\Phi \circ e(F^{m_1}(I_4)) \geq \varepsilon \cdot \ell_v(I)$$

or that the analogous inequality (14') hold for I_o and m_2 instead of I_4 , m_1 respectively.

The vectors tangent to the curve $\Phi \circ e(I)$ belong to the cones $|\xi_1/\xi_2| < \varepsilon$. This allows us to replace (14), (14') by

(15,15') $$\ell_v(I_{4(0)}) \geq (\tilde{\alpha} - \varepsilon \cdot \mu)^{-1} \cdot \varepsilon \cdot \ell_v(I) \quad .$$

We add the inequalities (12),(12'),(13),(15),(15'), divide by $\ell_v(I)$ and obtain

(16) $$1 \geq 2q(i) \cdot (\alpha - 2Nq(i))^{-1} \cdot \eta + q(i) \cdot (\alpha - 3 \cdot q(i) \cdot (\eta + 2N))^{-1}(\eta + 2N)$$
$$+ 2(\tilde{\alpha} - \varepsilon \cdot \mu)^{-1} \cdot \varepsilon \quad .$$

The conclusion is, that if this inequality is satisfied and the terms $\alpha - 2N \cdot q(i)$ and $\alpha - 3q(i) \cdot (\eta + 2N)$ are positive, then either (15) or (15') is satisfied, or there exists a partition of $I \setminus (I_o \cup I_4)$ into I_1, I_2, I_3 such that (12),(12') and (13) are satisfied. This implies the inequality (4).

The inequality (16) implies (4) also in the omitted cases (i)-(iv). Indeed in the case (i) $I = I_4$ and we need only (15). In the case (ii) $I = I_2 \cup I_3 \cup I_4$ and we need (12), (13) and (15). In the case (iii) $I = I_o \cup I_4$, we need (15) and (15'). In the case (iv) also the same inequalities as in the main case suffice (we even do not need (12')).

We shall replace (16) by stronger, but simpler inequalities. Observe that

$$\eta + 2N \leq 3\left|\frac{d\varphi'}{dy}\right| + \left|\frac{d\varphi'}{dx}\right| \cdot \varepsilon \leq \left(3\left|\frac{d\varphi}{dy}\right| + \varepsilon\right) \cdot \left|\frac{d\varphi'}{dx}\right| \quad .$$

Denote $3\left|\frac{d\varphi}{dy}\right| + \varepsilon = \underline{\varepsilon}$ and $(\alpha - 2\left|\frac{d\varphi'}{dy}\right| \cdot q(i)) \cdot \left|\frac{d\varphi'}{dx}\right|^{-1} = \underline{\alpha}$ (observe that $\tilde{\alpha} \geq \underline{\alpha}$) , we replace (16) by

(17) $$1 \geq (2q(i) . \underline{\alpha}^{-1} + q(i)(\underline{\alpha} - 3q(i)\underline{\varepsilon})^{-1} + 2(\underline{\alpha} - \mu \cdot \underline{\varepsilon})^{-1})\underline{\varepsilon} \ .$$

Denote

$$\underline{\alpha} = X\underline{\varepsilon} \ .$$

(17) holds if $X \geq X(i)$, the largest solution of the equation

$$1 = \frac{2q(i)}{X} + \frac{q(i)}{X - 3q(i)} + \frac{2}{X - \mu} \ .$$

If we denote $\tilde{\tilde{\alpha}} = \underline{\alpha} - X(i) \cdot 3 \cdot \left|\frac{d\varphi}{dy}\right|$, then using (18) we conclude finally that if

(19) $$\tilde{\tilde{\alpha}}_i \geq X(i) \cdot \varepsilon_i \ ,$$

then (4) holds.

By an analogous consideration for G_j one obtains for (4') the condition

$$|\tilde{\tilde{\beta}}_j| \geq Y(j) \cdot \varepsilon^j \ .$$

The proof is finished. By assumption (E3) we can find a right system $\{\varepsilon_i, \varepsilon^j\}$, $i = 1,\dots,p$, $j = 1,\dots,q$, defining for example

$$\varepsilon_i = |\tilde{\tilde{\alpha}}_i| \ / \ \max(X(i), \mu_i + 1)$$

$$\varepsilon^j = |\tilde{\tilde{\beta}}_j| \ / \ \max(Y(j), \mu^j + 1)$$ □

§ 3. Graphs of linkage of l.t.m.'s

Definition 1 : For any pair $P = (\{P_i\}_{i=1,\dots,p}, \{Q_j\}_{j=1,\dots,q})$ of transversal families of annuli define a (nondirected) graph $\Gamma(P)$ as follows : The vertices of $\Gamma(P)$ are the sets P_i and Q_j . For any pair (P_i,Q_j) we take as many edges joining P_i with Q_j as the number of components of the intersection $P_i \cap Q_j$. (There are no edges joining P_{i_1} with P_{i_2} or Q_{j_1} with Q_{j_2}).

We shall use the notation v_i , $i = 1,..,p+q$ for the vertices of $\Gamma(P)$, $v_i = P_i$ for $i = 1,\dots,p$, $v_i = Q_{i-p}$ for $i = p+1,\dots,p+q$. By u_{ijs} we shall denote the edge joining v_i and v_j , corresponding to the component $C_{i,j-p,s}$ if $i \leq p < j$, $C_{i-p,j,s}$ if $j \leq p < i$, together with the chosen direction from v_i to v_j (so each edge of $\Gamma(P)$ gives two directed edges). The set of all directed edges u_{ijs} will be denoted by $U(P)$.

Notation 2 : For any l.t.m. H on P we call a curve $\gamma \subset C_{ijs}$ joining lower and upper sides of C_{ijs} and such that for some $z \in C$, $m \geq 0$, $\gamma \subset h^m(\gamma^u(z))$, a v-curve. Analogously we call $\gamma \subset C_{ijs}$ joining left and right sides of C_{ijs} , $\gamma \subset \hat{F}_C \circ h^m(\gamma^u(z))$, an h-curve (Recall that $h = H_C$ is the induced map). We shall use the same terminology for $e_i^{-1}(\gamma)$ and $E_j^{-1}(\gamma)$.

Definition 3 : For any l.t.m. H on P define a graph $\Gamma(H)$ by adding to the graph $\Gamma(P)$ a set $U_o(H)$ of new edges as follows : join any vertex P_i with itself by an edge denoted u_{ii} (or u_{ii1}) if for every $(j,s) \in \mathfrak{J}_i$, for every v-curve $\gamma \subset C_{ijs}$, $\hat{F}_C(\gamma)$ contains an h-curve in the same C_{ijs} (and in all other $C_{ij's'}$, but this immediately follows from the definition of twists). We define a set $W(u_{ii})$ of "admissible weights" of u_{ii} as follows : a nonnegative integer n belongs to $W(u_{ii})$

if for all (j,s) , $(j',s') \in \mathcal{J}_i$ and v-curve $\gamma \subset C_{ijs}$ there exists an h-curve $\gamma' \subset \hat{F}_C(\gamma) \cap C_{ij's'}$ such that $\gamma' = \hat{F}^{n+1}(\hat{F}_C^{-1}(\gamma'))$. Similarly we join Q_j with itself by $u_{p+j,p+j} \in U_o(H)$ if for any $(i,s) \in \mathcal{J}^i$ and any h-curve $\gamma \subset C_{ijs}$, $\hat{G}_C(\gamma)$ contains a v-curve in C_{ijs} . The set $W(u_{j+p,j+p})$ is defined analogously as $W(u_{ii})$ for $i \leq p$. (There is no a priori obstruction to u_{ii} existing but with $W(u_{ii}) = \emptyset$). Denote $U(H) = U(P) \cup U_o(H)$.

Definition 4 : A sequence (r_n) of elements of $U(H)$ (or $U(P)$) is called a walk on $\Gamma(H)$ (or $\Gamma(P)$) if for any two consecutive elements $r_k = u_{ijs}$, $r_{k+1} = u_{i'j's'}$ we have $j = i'$ and $(i,s) \neq (j',s')$. We call a walk with weight a walk on $\Gamma(H)$ such that for each element of the form $u_{ii} \in U_o(H)$, $W(u_{ii}) \neq \emptyset$ and an admissible weight $w \in W(u_{ii})$ is chosen. Then we write $u_{ii}(w)$. By length of a walk (r_n) we call the number of the indices n for which $r_n \in U(P)$, minus 1 . (So, we do not compute edges u_{ii} , they are introduced artificially to allow us, after walking u_{jis} , to turn back and walk u_{ijs}). By length of a walk with weight we mean the length of the underlying walk plus double the sum of all weights of its elements of the form u_{ii} .

Definition 5 : We call $\Gamma(H)$ (or $\Gamma(P)$) transitive if for every two elements u_{ijs} , $u_{i'j's'} \in U(P)$ there exists a walk on $\Gamma(H)$ (or $\Gamma(P)$) which begins with u_{ijs} and finishes with $u_{i'j's'}$.

We call $\Gamma(H)$ (or $\Gamma(P)$) strongly transitive if there exists an integer N_o such that for any $N \geq N_o$ and u_{ijs} , $u_{i'j's'} \in U(P)$ where $i > p \geq j$, $i' > p \geq j'$ there exists a walk with weight, on $\Gamma(H)$ (or $\Gamma(P)$) which begins with u_{ijs} , finishes with $u_{i'j's'}$, with length $2N$.

Notation 6 : The degree of a vertex v_i in the graph $\Gamma(H)$ (or $\Gamma(P)$) is the number of edges incident with v_i (the edges u_{ii} are computed doubly!)

We use the notation $\deg_H v_i$ for $\Gamma(H)$ and $\deg_P v_i$ for $\Gamma(P)$.

Lemma 7 : 1. If for each vertex v_i of $\Gamma(H)$ $\deg_H(v_i) \geq 2$ (i.e. $\Gamma(H)$ has no "ends") and for at least one vertex v_{i_o} , $\deg_H(v_{i_o}) \geq 3$ (i.e. $\Gamma(H)$ is not a cycle), then $\Gamma(H)$ is transitive.

2. If additionally one of the following conditions holds :

(i) For each i , $\deg_P(v_i) \geq 3$,

(ii) For each i if $\deg_P(v_i) = 1$, then there exists $u_{ii} \in U_o(H)$ with $W(u_{ii}) \neq 0$. There exists i_o such that $W(u_{i_o i_o}) \supset \{m, m+1\}$ for an integer $m \geq 0$.

(iii) In (ii) replace condition about i_o by $\deg_P(v_{i_o}) \geq 2$ and $W(u_{i_o i_o}) \ni 1$. Then $\Gamma(H)$ is strongly transitive.

The same is true for $\Gamma(P)$ (with H replaced by P , the conditions (ii) and (iii) omitted). □

Proof 1. Transitivity : Since $\Gamma(H)$ is connected there exists a walk from u_{ijs} or u_{jis} (i.e. the nondirected edges) to $u_{i'j's'}$ or $u_{j'i's'}$. So we need to find a walk from u_{ijs} to u_{jis} (and from $u_{i'j's'}$ to $u_{j'i's'}$). We start walking at $u_{ijs} = u_{i_o i_1 s_o}$. Since for every v_i , $\deg_H(v_i) \geq 2$ we can always continue a walk from $u_{i_k i_{k+1} s_k}$ to $u_{i_{k+1} i_{k+2} s_{k+1}}$, $k = 0, 1, \ldots$. Let n be the first integer $n > 0$ such that there exists m , $0 \leq m < n$, for which $v_{i_n} = v_{i_m}$. If $m > 0$, then from v_{i_n} we can continue walking to $v_{i_{m-1}}$, $v_{i_{m-2}}, \ldots$ back to $u_{i_1 i_o s_o}$ (see Figure 3.1).

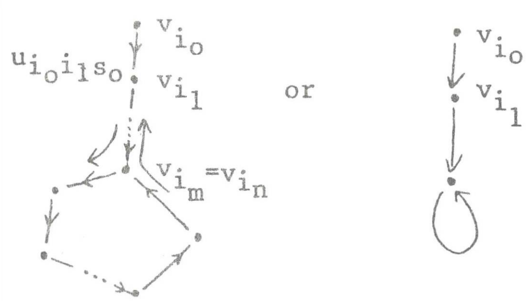


Figure 3.1

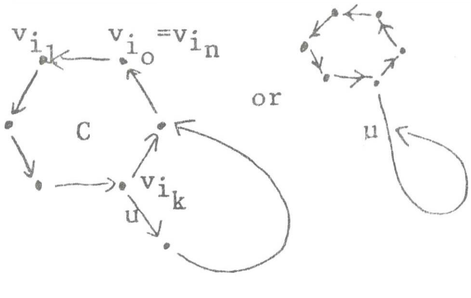


Figure 3.2

If $m = 0$ then we have a cycle C (Figure 3.2), but there exists $k : 0 \leq k < n$ such that v_{i_k} is incident with a third edge u. We walk along this u. Next, we always manage to continue a walk so that after some time we are back at C. Then we walk along C backward to $u_{i_1 i_o s_o}$.

Strong transitivity : To prove this, it is enough to find a finite family of cycles, i.e. periodic walks with weights, with lengths $2N_1, 2N_2, \ldots, 2N_r$, where the highest common factor of $N_1, \ldots, N_r$ $(N_1, \ldots, N_r) = 1$. Consider the three cases (i)-(iii) :

(i) Let for each $i = 1, \ldots, p+q$, $\deg_P(v_i) \geq 3$. Choose any $u_{i_o i_1 s_o} \in U(P)$. At v_o at least two different directed edges $u_{i_{-1} i_o s_{-1}}$, $u_{i'_{-1} i_o s'_{-1}}$ different from $u_{i_1 i_o s_o}$ finish. At v_1 at least two different directed $u_{i_1 i_2 s_1}$, $u_{i_1 i'_2 s'_1}$, different from $u_{i_1 i_o s_o}$, start. Extend the walks $u_{i_{-1} i_o s_{-1}} u_{i_o i_1 s_o} u_{i_1 i_2 s_1}$ and $u_{i'_{-1} i_o s'_{-1}} u_{i_o i_1 s_o} u_{i_1 i'_2 s'_1}$ to cycles C_1 and C_2 by walks w_1 and w_2 respectively (see Figure 3.3.)

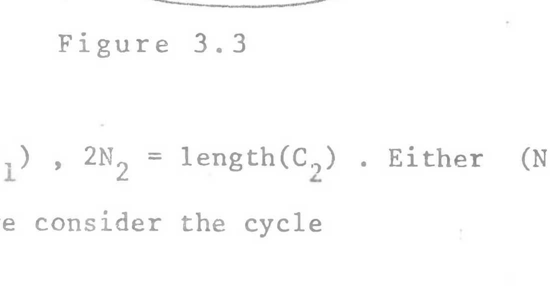


Figure 3.3

Denote $2N_1 = \text{length}(C_1)$, $2N_2 = \text{length}(C_2)$. Either $(N_1, N_2) = 1$ which proves the Lemma, or we consider the cycle

$C_3 = u_{i_1 i_2 s_1} w_1 u_{i_{-1} i_o s_{-1}} u_{i_o i'_{-1} s'_{-1}} - w_2 \, u_{i'_2 i_1 s'_1}$ with the length $2N_3 = 2(N_1 + N_2 - 1)$. Then $(N_1, N_2, N_3) = 1$.

(ii) Assume $\deg_P(v_{i_o}) = 1$, $\{m, m+1\} \subset W(u_{i_o i_o})$. Then $u_{i_o i_o}(m)$ can be extended to a periodic walk with weight, with some length $2N$. If we replace, in this walk with weight, $u_{ii}(m)$ by $u_{ii}(m+1)$, then we obtain length $2(N+1)$. But $(N,N+1) = 1$.

iii) Assume $\deg_{P}(v_{i_o}) \geq 2$, $1 \in W(u_{i_o i_o})$. Then there exists a periodic walk with weight going through v_{i_o} : $\ldots u_{ji_o s}$, $u_{i_o j's'}\ldots$ $(j \neq i_o \neq j')$. But we can enlarge its length by 2 taking $\ldots$ u_{jis} , $u_{ii}(1)$, $u_{ij's'}\ldots$ □

Proposition 8 : If the l.t.m. H satisfies all the assumptions of Theorem B except the assumption $|k_i|, |\ell_j| \geq 2$ and if $\Gamma(H)$ and $\Gamma(H^{-1})$ are transitive, then H is ergodic. If additionally H is strongly transitive, then all powers of H are ergodic, so H is a Bernoulli system.

Proof : Take any $z \in C$ for which there exists a local unstable manifold $\gamma^u(z)$. By the proof of Theorem B there exists an h-curve, (or v-curve) $\gamma_o \subset C_{i_o j_o s_o}$ such that $\gamma_o \subset \hat{F}H^{n_o}(\gamma^u(z))$ (or $\gamma_o \subset H^{n_o}(\gamma^u(z))$) for an integer $n_o \geq 0$. For every $u_{ijs} \in U(P)$ there exists a walk $W = (u_{i_o,j_o+p,s_o}$ (or u_{i_o+p,j_o,s_o}), $\ldots, u_{ijs})$ on the graph $\Gamma(H)$.

Assume γ_o is an h-curve, for example. Denote by $r_o,\ldots,r_n = u_{ijs}$ the consecutive edges of this walk with omitted elements of $U_o(H)$. We say that r_k has property (γ) if there exists an h-curve

$$\gamma_k \subset C_{i(r_k),j(r_k),s(r_k)} \cap \hat{F}_C \circ h^{(k-2)/2} \circ \hat{G}_C(\gamma_o) \quad ,$$

when $r_k = u_{i(r_k),j(r_k)+p,s(r_k)}$ or a v-curve $\gamma_k \subset C_{i(r_k),j(r_k),s(r_k)} \cap h^{(k-1)/2} \circ \hat{G}_C(\gamma_o)$, when $r_k = u_{j(r_k)+p,i(r_k),s(r_k)}$. By the definition of walk if r_k has property (γ) then r_{k+1} has property (γ) . So, by transitivity, in every C_{ijs} there exists a v-curve $\gamma \subset h^m(\gamma^u(z))$ for some $m = m(ijs)$. By Theorem B for H^{-1} and transitivity of $\Gamma(H^{-1})$, if for $z' \in C$ a local stable manifold $\gamma^s(z')$ exists, then there exists $C_{i_1 j_1 s_1}$, an integer $m \geq 0$ and a curve $\gamma \subset h^{-m}(\gamma^s(z'))$ joining left and right sides of $C_{i_1 j_1 s_1}$. This implies that $h^{m(i_1 j_1 s_1)}(\gamma^u(z))$ intersects

$h^{-m}(\gamma^s(z'))$. So h and H are ergodic.

Assume now that $\mathcal{W}$ is a walk with weight, $\gamma_o \subset C_{i_o j_o s_o}$ is a v-curve and $r_n = u_{ijs}$ satisfies : $i > p \geq j$. Then there exists a v-curve $\gamma_n \subset C_{i(r_n),j(r_n),s(r_n)} \cap H^{(1/2)\mathrm{length}(\mathcal{W})}(\gamma_o)$ (this key fact follows immediately from the definitions of a walk with weight and its length, these definitions were adjusted especially to this aim).

Concluding, by strong transitivity of $\Gamma(H)$, for any C_{ijs} and N - an integer sufficiently large, there exists a v-curve $\gamma \subset C_{ijs} \cap H^N(\gamma_o)$. This yieds ergodicity of the mappings H^k for every integer k . □

[Instead of graph $\Gamma(P)$ one can consider its derived directed graph $\Gamma^d(P)$ defined as follows : the vertices of $\Gamma^d(P)$ are directed edges u_{ijs} of $\Gamma(P)$. There exists a directed edge in $\Gamma^d(P)$ which starts at u_{ijs} and ends at $u_{i'j's'}$ if $j = i'$ and $(i,s) \neq (j',s')$. The graph $\Gamma^d(H)$ is defined by adding new edges to $\Gamma^d(P)$ as follows : We add a directed edge which starts at u_{ijs} and ends at u_{jis} if for P_j and $(i-p,s) \in \mathcal{J}_j$ when $i > p \geq j$ or Q_{j-p} and $(i,s) \in \mathcal{J}^{j-p}$ when $j > p \geq i$, the property described in Definition 3 is satisfied. For $\Gamma^d(P)$ and $\Gamma^d(H)$ a walk is defined in a standard way .]

Lemma 7 and Proposition 8 give the topological condition about P (i.e. $\deg_P(v_i) \geq 3$ for $i = 1,\dots,p+q$) which implies that every l.t.m. on P satisfying the assumptions of Theorem B , even except $|k_i|$, $|\ell_j| \geq 2$ is Bernoulli. If P does not satisfy this condition, then the question of whether H is Bernoulli reduces to studying the existence of u_{ii} , and studying the set $W(u_{ii})$. Proposition 9 will be devoted to this question. But first, we introduce more notation.

Number the $q(i)$ components of $e_i^{-1}(C)$ in P_i and denote them $Q_j^o,\dots,Q^{q(i)-1}$, starting from any component and going to the right. Denote the left and right sides of Q^j by LS^j, RS^j respectively. Denote by $S(y_o)$ the horizontal circle $S(y_o) = \{(x,y) \in P_i : y = y_o\}$. Since for any two points $z_1, z_2 \in S(y)$ there exist two arcs in $S(y)$ joining them, we fix that (z_1,z_2) denotes the arc oriented to the right, with the begin at z_1, the end at z_2. $\ell(z_1,z_2)$ denotes its length.

Denote

$$\mathcal{D}(P_i) = \max_{j=0,\dots,q(i)-1} \ \inf_{y\in(y_i',y_i'')} \ell(RS^j \cap S(y), LS^{j+1 (\mathrm{mod}\ q(i))} \cap S(y)) .$$

Proposition 9 : For H a l.t.m. satisfying assumptions of Theorem B except $|k_i|,|\ell_j| \geq 2$, for $1 \leq i \leq p$ the existence of u_{ii} follows from each one of the following conditions :

1. $|k_i| \geq 2$
2. $q(i) = 1$ and $\tilde{\alpha}_i \geq X(i) \cdot \frac{x_j''-x_j'}{y_i''-y_i'}$, where $P_i \cap Q_j \neq \emptyset$;
3. $\mathcal{D}(P_i) \geq \frac{1}{2} a_i$.

In the case 1, $W(u_{ii}) \ni 0$. In the case 3., if $\mathcal{D}(P_i) \geq \frac{n}{n+1} a_i$ for $n \geq 1$, then $W(u_{ii}) \ni n$. (We leave writing the analogous conditions for $p < i \leq p+q$ to the reader).

Proof : The case 1. is explained on Figure 1.2. In the case 2. we need the inequality (4) in § 2 for $\varepsilon_i' = \max(\varepsilon_i, \frac{x_j''-x_j'}{y_i''-y_i'})$ instead of ε_i. This follows from the inequality (19. §2) with ε_i' instead of ε_i.
Consider the case 3. Assume for example $k_i > 0$ (the case $k_i < 0$ is analogous). Assume $\mathcal{D}(P_i) \geq \frac{n}{n+1} a_i$ for $n \geq 1$.

Lift $F_i : P_i = \mathbb{R}\times\langle y_i',y_i''\rangle / a_i\mathbb{Z} \times \{0\} \longrightarrow P_i$ to $\tilde{F} : \mathbb{R}\times\langle y_i',y_i''\rangle \circlearrowleft$ so that $\tilde{F}|_{\{(x,y):y=y_i'\}}$ = identity.

Take any v-curve $\gamma \subset Q^j$, $0 \leq j < q(i)$, and choose a covering curve $\tilde{\gamma}$ in a component $\tilde{Q}^j$ of the set covering Q^j. Denote the consecutive components covering $Q^{j\pm1 \pmod{q(i)}}$, $Q^{j\pm2 \pmod{q(i)}}$, ... by $\tilde{Q}^{j\pm1}$, $\tilde{Q}^{j\pm2}$, ..., the left and right sides of $\tilde{Q}^j$ by $L\tilde{S}^j$, $R\tilde{S}^j$ and the lines $\mathbb{R} \times \{y\}$ by $\tilde{S}(y)$. For any two points lying on the same line we use the standard relations $\leq$ and $<$. Denote by j_o such j at which the maximum in the definition of $\mathcal{D}(P_i)$ is attained. There exists $(x_1,y_1) \in \tilde{\gamma}$ such that $\tilde{F}(x_1,y_1) \in R\tilde{S}^{j_o}$ (provided $j < j_o$, otherwise we replace j_o by $j_o+q(i)$).

Since $|\tilde{F}^m(x_1,y_1)-\tilde{F}^{m-1}(x_1,y_1)| = |\tilde{F}(x_1,y_1)-(x_1,y_1)|$ for any integer m, we have

$$|\tilde{F}^{n+1}(x_1,y_1)-L\tilde{S}^{j_o+1-q(i)} \cap \tilde{S}(y_1)|$$

$$\leq (n+1)\cdot|R\tilde{S}^{j_o} \cap \tilde{S}(y_1)-L\tilde{S}^{j_o+1-q(i)} \cap \tilde{S}(y_1)| \leq a_i ,$$

hence $\tilde{F}^{n+1}(x_1,y_1) \leq L\tilde{S}^{j_o+1} \cap \tilde{S}(y_1)$ and $\tilde{F}^n(x_1,y_1) < L\tilde{S}^{j_o+1} \cap \tilde{S}(y_1)$. So there exists $(x_2,y_2) \in \tilde{\gamma}$ such that $y_2 > y_1$ and $\tilde{F}^n(x_2,y_2) \in L\tilde{S}^{j_o+1}$.

$$|\tilde{F}^{n+1}(x_2,y_2)- R\tilde{S}^{j_o} \cap \tilde{S}(y_2)| \geq a_i \quad .$$

hence $\tilde{F}^{n+1}(x_2,y_2) \geq R\tilde{S}^{j_o+q(i)} \cap \tilde{S}(y_1)$.

We conclude that the curve $\gamma_1 \subset \gamma$ with the ends at (x_1, y_1) and (x_2,y_2) has the property that $F^m(\gamma_1) \cap Q^j = \emptyset$ for every $m =1,..,n$ and $j = 0,...,q(i)-1$ and $F^{n+1}(\gamma_1)$ contains an h-curve in each Q^j. But this means that $n \in W(u_{ii})$. □

Theorem C follows immediately from Theorem B, the case 3 of Proposition 9. (if $\deg_p(v_i) = 1$ we need $\mathcal{D}(P_i) \geq \frac{2}{3}a_i$ for $i \leq p$ and $\mathcal{D}(Q_{i-p}) \geq \frac{2}{3} b_{i-p}$ for $i > p$, if $\deg_p(v_i) = 2$ it is enough to replace

$\frac{2}{3}$ by $\frac{1}{2}$), Lemma 8. and Proposition 9.

Remark : It is also true under the assumptions of Theorem C (but with the property "if A_i intersects $\bigcup_{j=1}^{q} B_j$ in exactly two points then these two points are not antipodal in A_i" assumed for every A_i and respective property for every B_j) that $h = H_C$, the induced mapping is Bernoulli. To prove it, one needs consider walks without weight, i.e. not consider weight in the definition of length. This gives information about the curves $h^n(\gamma^u(z))$ instead of $H^n(\gamma^u(z))$. Strong transitivity of $\Gamma(H)$ follows in this case from the fact : $\deg_H(v_i) \geq 3$ for each v_i (we leave a proof to the reader). □

To show how to apply the above results we shall study the examples from the Introduction.

Example 1 : Consider H , a toral linked twist mapping, with twists as strong as in Theorem A but not necessarily satisfying $|k|, |\ell| \geq 2$.
We shall discuss the following properties which H may additionally satisfy:

(a_1) $|\alpha| \geq \sqrt{C_o} \cdot \frac{x_1 - x_o}{y_1 - y_o}$ (b_1) $|\beta| \geq \sqrt{C_o} \cdot \frac{y_1 - y_o}{x_1 - x_o}$

(a_2) $x_1 - x_o \leq \frac{1}{2}$ (b_2) $y_1 - y_o \leq \frac{1}{2}$

(a_3) $x_1 - x_o \leq \frac{1}{3}$ (b_3) $y_1 - y_o \leq \frac{1}{3}$

(a_4) $|k| \geq 2$ (b_4) $|\ell| \geq 2$.

The graph $\Gamma(P)$ looks as in Figure 3.4.

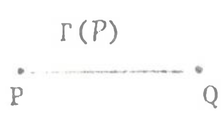


Figure 3.4

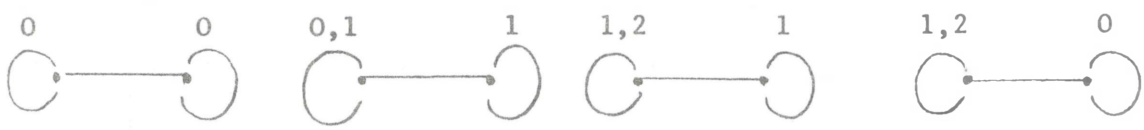
Figure 3.5

If one of the conditions (a_i) , $i = 1,\dots,4$ and one of the conditions (b_i) , $i = 1,\dots,4$ is satisfied, then $\Gamma(H)$ and $\Gamma(H^{-1})$ are transitive so H is ergodic (the induced map $h = H_{P\cap Q}$ is even Bernoulli). See Figure 3.5.

If the conditions of one of the following sets of conditions are satisfied :

(i) $\{(a_4),(b_4)\}$, (ii) $\{(a_2),(b_2),(a_4)\}$,
(iii) $\{(a_2),(b_2),(b_4)\}$, (iv) $\{(a_3),(b_2)\}$,
(v) $\{(a_3),(b_4)\}$, (vi) $\{(b_3),(a_2)\}$,
(vii) $\{(b_3),(a_4)\}$,

then $\Gamma(H)$ is strongly transitive, so all powers of H are ergodic, hence H is Bernoulli. The graphs $\Gamma(H)$ in some of the above cases are presented in Figure 3.6. (numbers on the edges joining P (respect. Q) with itself denote admissible weights)



case (i) case (ii) case (iv) case (v)

Figure 3.6.

Example 2 : Consider H , the Bowen example (see Introduction) with twists as strong as in Theorem B, but not necessarily satisfying $|k|$, $|\ell| \geq 2$. Discuss the following properties :

(a) $\mathcal{D}(P) \geq \frac{1}{2}a$ (c) $\mathcal{D}(Q) \geq \frac{1}{2}b$

(b) $|k| \geq 2$ (d) $|\ell| \geq 2$.

(We recall that (a) (analogously (b)) means that no circle $\{y = \text{const.}\} \subset P$ contains a pair of antipodal points contained in $e^{-1} \circ E(Q)$.)

The graph $\Gamma(\mathcal{P})$ is presented in Figure 3.7.

Figure 3.7.

If one of the conditions (a)-(d) is satisfied, the $\Gamma(H)$ is strongly transitive so all powers of H are ergodic, hence H is Bernoulli. (Also $h = H_{P \cap Q}$ is Bernoulli). The graphs $\Gamma(H)$ in these cases are presented at Figure 3.8.

case (a) case (b) case (c) case (d)

Figure 3.8.

§ 4. a) Density of periodic and homoclinic points

Assume H be an l.t.m. (but omit the assumption $|k_i|, |\ell_j| \geq 2$). Assume also that Condition H (from Theorem B, § 2) is satisfied (which gives almost hyperbolicity). Then hyperbolic periodic points of H and homoclinic points are dense in $\bigcup_{i=1}^{p} e_i(P_i) \cup \bigcup_{j=1}^{q} E_j(Q_j)$ (we keep the notation from § 2). This follows easily from Pesin Theory. The technique which gives it as a by-product has been worked out by Katok, see [8] . However, we shall give a sketch of proof :

Denote by ν_H the H-invariant measure under consideration. Denote $G^{u(s)}(\varepsilon) = \{z \in C$: there exists a local unstable (stable) differentiable manifold $\gamma^{u(s)}(z)$ with z in its middle, length $(\gamma^{u(s)}) \geq \varepsilon\}$.

Then for each point $x \in C$ and small $\delta > 0$ $(\delta \ll \varepsilon)$ there exists a ball $B(y,\delta) \subset C$ close to x , such that

$$\nu_H(B(y,\delta) \cap G^u(\varepsilon) \cap G^s(\varepsilon) \neq \emptyset \; .$$

Denote $B(y,\delta) \cap G^u(\varepsilon) \cap G^s(\varepsilon) = B$. There exists $z \in H^{n_1}(B) \cap H^{-n_2}(B) \cap B$ for n_1, n_2 large enough and $n_1/n_2 \simeq 1$.

Take a small rectangle S built using the cross $H^{n_1}(\gamma^s(H^{-n_1}(z))) \cup H^{-n_2}(\gamma^u(H^{n_2}(z))$ and curves with tangent vectors in the horizontal (respectively vertical) H-invariant cones C_h (resp. C_v) (these cones are complements of the cones C_z (resp. C^z) from § 2.). See Fig 4.1.

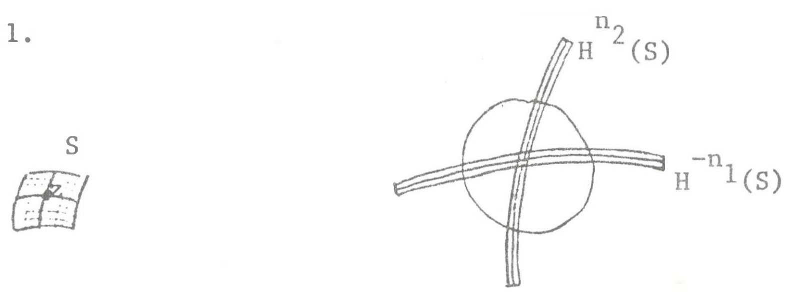


Figure 4.1

An argument like use of the λ-lemma, and standard geometric reasons, yield the existence of a periodic point in $H^{n_2}(S) \cap H^{-n_1}(S)$. To prove density of homoclinic points use two rectangles S_1, S_2 in two neighbouring discs $B(y_1,\varepsilon)$, $B(y_2,\varepsilon)$, Figure 4.2.

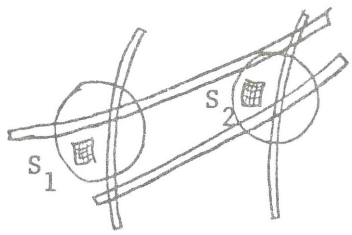


Figure 4.2.

Our proof of the density of periodic (and homoclinic) points in fact is right for any measure preserving dynamical system H with singularities, satisfying Katok-Strelcyn (K-S) conditions (see [10]) with Lyapunov exponents almost nowhere 0 (density in support of the measure, of course).

It seems that even Katok's [8] estimate for entropy $h_{\nu_H}(H) \leq \limsup_{n\to\infty} n^{-1} \log(\mathrm{Card}\ \mathrm{Per}_n H)$ holds. (Katok has confirmed my opinion).

It is an intriguing question whether existence of at least one periodic point in C implies density of Per H, even without assuming existence of a good invariant measure ν_H. For the case when all the k_i, ℓ_j have the same sign (on the torus T^2) this has been proved by Devaney [4]. He has used the fact that if the global stable and unstable manifolds of a periodic point p, $\gamma^s_{glob}(p)$, $\gamma^u_{glob}(p)$ pass close to each other, then they have a nonempty intersection. Here I do not know how to exclude the following possibility (see Figure 4.3)

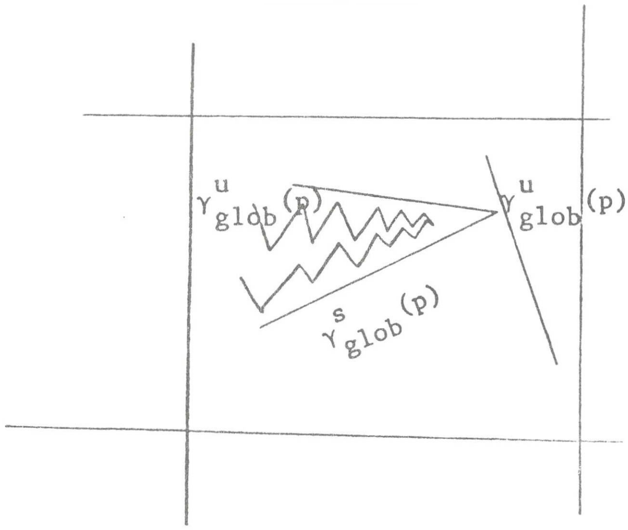


Figure 4.3.

b) Further generalization of linked twist mappings

Take a pair of transversal families of annuli $\mathcal{P} = (\{P_i\},\{Q_j\})$ as in Theorem B (more exactly, take a pair of embeddings $(\{e_i\}_{i=1,...,p}, \{E_j\}_{j=1,...,q})$, see § 2.) . For a positive integer N, for every $n = 0,1,...,N-1$ choose $\mathcal{J}(P,n) \subset \{1,...,p\}$ and $\mathcal{J}(Q,n) \subset \{1,...,q\}$. Assume that

$$\bigcup_{n=0}^{N-1} \mathcal{J}(P,n) = \{1,...,p\}$$

$$\bigcup_{n=0}^{N-1} \mathcal{J}(Q,n) = \{1,...,q\} \quad .$$

For each $i \in \mathcal{J}(P,n)$ take a $k_{i,n}$-twist $F_{i,n}$ on P_i, for each $j \in \mathcal{J}(Q,n)$ take an $\ell_{j,n}$-twist $G_{j,n}$ on Q_j . Assume that for each i (respectively j) the signs of all $k_{i,n}$ (resp. $\ell_{j,n}$), $n = 0,...,N-1$ are the same. Define F_n, G_n, $n = 0,1,...,N-1$ on $\bigcup_{i=1}^{p} e_i(P_i) \cup \bigcup_{j=1}^{q} E_j(Q_j)$ by

$$F_n(x) = \begin{cases} e_i F_{i,n} e_i^{-1}(x) & \text{if } x \in e_i(P_i) \text{ and } i \in \mathcal{J}(P,n) \\ x & \text{otherwise.} \end{cases}$$

$$G_n(x) = \begin{cases} E_j\, G_{j,n} E_j^{-1}(x) & \text{if } x \in E_j(Q_j) \text{ and } j \in J(Q,n) \\ x & \text{otherwise.} \end{cases}$$

Define $H = G_{N-1} \circ F_{N-1} \circ \dots \circ G_o \circ F_o$.

Assume finally that H preserves a measure $\nu = \nu_H$ equivalent to Lebesgue measures on P_i , Q_j with upper bounded densities.

We prove that assuming the twists are sufficiently strong (more exactly assuming Condition H from § 2 for all pairs of twists $F_{i,m}$, $G_{j,m}$, such that $e_i(P_i) \cap E_j(Q_j) \neq \emptyset$) H has almost everywhere nonzero Lyapunov exponents.

For every $n = 0,\dots,N-1$ define

$H_n = G_n \circ F_n \circ \dots \circ G_o \circ F_o \circ G_{N-1} \circ F_{N-1} \circ \dots \circ G_{n+1} \circ F_{n+1}$. Consider H_n together with the H_n-invariant measure $(G_n \circ \dots \circ F_o)_*(\nu_H)$. For a fixed n denote $m_k = n+k+1 \pmod N$. We have $H_n = G_{m_{N-1}} \circ F_{m_{N-1}} \circ \dots \circ G_{m_o} \circ F_{m_o}$. For every $j \in J(Q,n)$ denote

$$A(j,n) = \bigcup_{i=0}^{N-1} G_{m_o}^{-1} \circ \dots \circ G_{m_{i-1}}^{-1} (E_j(Q_j) \cap (\bigcup_{t \in J(P,m_i)} e_t(P_t))) .$$

Due to this definition every point $z \in A(j,n)$, under the induced mapping $(H_n)_{A(j,n)}$ at least once undergoes a horizontal twist, by F_{t,m_i} , and at the end undergoes the vertical twist $G_{j,n}$. This gives positive Lyapunov exponents for $(H_n)_{A(j,n)}$. Since almost every H_n-trajectory starting in $A(j,n)$ hits $A(j,n)$ with positive frequency, we can deduce that almost every $z \in A(j,n)$ has a positive Lyapunov exponent.

For almost every $z \in \bigcup e_i(P_i) \cup \bigcup E_j(Q_j)$ there exist s,n,j such that $G_n \circ F_n \circ \dots \circ G_o \circ F_o \circ H^s(z) \in E_j(Q_j)$ and $j \in J(Q,n)$ (only points in circles where rotations $F_{N-1} \circ \dots \circ F_o$ are rational can behave differently).

Next for almost every $z \in E_j(Q_j)$, if $j \in \mathcal{J}(Q,n)$, there exists $s \geq 0$ such that $(H_n)^s(z) \in A(j,n)$. This proves that almost every $z \in \cup e_i(P_i) \cup \cup E_j(Q_j)$ has a positive Lyapunov exponent. Analogously we prove that the second Lyapunov exponent is almost everywhere negative.

In fact we can prove also a generalization of the ergodic part of Theorem B.

Theorem : Assume that Condition E is satisfied (replace $q(i)$ by $q(i)\cdot N$ and $p(j)$ by $p(j)\cdot N$) and $|k_{i,n}|$, $|\ell_{j,n}| \geq 2$. Then H is Bernoulli.

Sketch of proof : We proceed as in the proof of Theorem B. The difference is that we consider an unstable arc $\gamma \subset C_{ijs}$ and its images under $F_o,\dots,F_{N-1},F_o,\dots,F_{N-1},F_o,\dots$ and so on, instead of under F all the time. Next we consider only intersections of the images $F_n\circ\dots\circ F_o\circ H^k(\gamma) \subset e_i(P_i)$ with such C_{ijs} that $j \in \mathcal{J}(Q,n)$. (We find an H-periodic point $t \in \gamma$ and consider its orbit under $F_o,\dots,F_{N-1},F_o,\dots$)

We reach the situation when there exist P_i (or Q_j) , k , n and C_{ijs} such that $F_n\circ\dots\circ G_o\circ F_o\circ H^k(\gamma) \cap C_{ijs}$ ($\gamma = \gamma^u(z)$, a local unstable manifold) contains an arc joining the left and right sides of the C_{ijs} (we call it an h-curve) and $j \in \mathcal{J}(Q,n)$ (or $G_n\circ F_n\circ\dots\circ F_o\circ H^k(\gamma)$ contains an arc inside a C_{ijs} joining its lower and upper sides (a v-curve) and $i \in \mathcal{J}(P,n+1(\mathrm{mod}\ N))$) . Then $G_n\circ F_n\circ\dots\dots\circ F_o\circ H^k(\gamma)$ contains an arc joining left and right sides of the C_{ijs} , winding at least twice around $E_j(Q_j)$ (or... ; we start to omit the second case). We shall call such an arc a spiral. Denote the above indices by k_o,n_o,j_o .

We call a subset $A \subset \{1,\dots,p+q\}$ an $\mathcal{F}_{k,n}$-set if :

1) For every $i \in A$, $i \leq p$ one of the following conditions is satisfied :

a) $e_i(P_i)$ contains a spiral ;

b) Every C_{ijs} such that $(j,s) \in \mathcal{J}_i$ contains either an h-curve, or a v-curve and then $j+p \in A$. The set of v-curves is nonempty.

2) For every $i \in A$, $i > p$ one of the following conditions is satisfied

a) $E_{i-p}(Q_{i-p})$ contains a spiral ;

b) Every $C_{j,i-p,s}$ such that $(j,s) \in \mathcal{J}^{i-p}$ contains either a v-curve or an h-curve and then $j \in A$. The set of h-curves is nonempty.

All spirals and v,h - curves in this definition are contained in the set $F_n \circ \dots \circ G_o \circ F_o \circ H^k(\gamma)$.

We call $A \subset \{1,\dots,p+q\}$ a $\mathcal{G}_{k,n}$-set if it satisfies the above conditions but with $G_n \circ F_n \circ \dots \circ F_o \circ H^k(\gamma)$ rather than $F_n \circ \dots \circ G_o \circ F_o \circ H^k(\gamma)$.

We find a sequence of $\mathcal{F}_{k,n}$- and alternately $\mathcal{G}_{k,n}$-sets. We start with the $\mathcal{G}_{k_o,n_o}$-set

$$A_o = \{p+j_o\}$$

When a $\mathcal{G}_{k,n}$-set A_m is defined one can easily prove that

$$\begin{aligned} A_{m+1} = A_m \cup \{i : \text{there exists } j \text{ such that } e_i(P_i) \cap E_j(Q_j) \neq \emptyset \\ p+j \in A_m,\ e_i(P_i) \cap E_j(Q_j) \text{ contains a v-curve in} \\ G_n \circ F_n \circ \dots \circ F_o \circ H^k(\gamma) \text{ and } i \in J(P, n+1 (\mathrm{mod}.N))\} \end{aligned}$$

is an $\mathcal{F}_{k,n+1}$-set if $n < N-1$ or an $\mathcal{F}_{K+1,0}$-set if $n = N-1$.

Analogously we define A_{m+1} if A_m is an $\mathcal{F}_{k,n}$-set. One can easily prove that for all m sufficiently large $A_m = \{1,\dots,p+q\}$.

So for every k sufficiently large $\{1,\dots,p+q\}$ is a $\mathcal{G}_{k,N-1}$-set. For $j \in J(Q,N-1)$, $E_j(Q_j)$ contains a spiral $S_k \subset H^{k+1}(\gamma)$.

Now we repeat the whole consideration for a local stable manifold $\gamma^s(z')$ and obtain for an ℓ (large), in $H^{-\ell}(\gamma^s(z')) \cap E_j(Q_j)$ an h-curve or a spiral, but twisted in a different direction than S_k.
(It may happen that $\mathcal{J}(Q,N-1) = \emptyset$. Then we replace N-1 by any n such that $\mathcal{J}(Q,n) \neq \emptyset$ and consider H_n instead of H. But H_n and H are conjugate).

We shall estimate the measure entropy $h_\nu(H)$ for the almost hyperbolic H.

Proposition : Assume that $\nu((F_n \circ \ldots \circ G_o \circ F_o)^{-1}(C)) \leq \varepsilon$ and $\nu((G_n \circ F_n \circ \ldots \circ G_o \circ F_o)^{-1}(C)) \leq \varepsilon$ for every $n = 0,\ldots,N-1$ (in particular $\nu(C) \leq \varepsilon$). Assume that the slopes of the twists are bounded above by $C_1 \cdot \varepsilon^{-1}$. Then

$$h_\nu(H) \leq 2N\sqrt{\varepsilon}\log(2N\, C_1^2 C_2^2 \varepsilon^{-5/2})$$

(here C_2 is an upper bound for $\|D(E_j^{-1} \circ e_i)\|$, $\|D(e_i^{-1} \circ E_j)\|$).

Proof : Denote by $\lambda^+(x)$, the positive Lyapunov exponent. Denote $\bigcup_{i=1}^{p} e_i(P_i) \cup \bigcup_{j=1}^{q} E_j(Q_j) = X$. We know that

(1) $$h_\nu(H) \leq \int_X \lambda^+(x)\, d\nu(x)$$, see [15], [10] or Appendix

Denote by χ_C the characteristic function of C. For $n=0,\ldots,N-1$ denote

$$\hat{\chi}_{n,P}(x) = \lim_{k\to\infty} \frac{1}{k} \sum_{\ell=0}^{k-1} \chi_C \circ F_n \circ \ldots \circ G_o \circ F_o(H^\ell(x)) \quad \text{and}$$

$$\hat{\chi}_{n,Q}(x) = \lim_{k\to\infty} \frac{1}{k} \sum_{\ell=0}^{k-1} \chi_C \circ G_n \circ \ldots \circ G_o \circ F_o(H^\ell(x))$$

(for almost every x, lim exists by the Birkhoff Ergodic Theorem).
Denote $X_1 = \{x \in X$: there exists n such that $\hat{\chi}_{n,P}(x) \geq \sqrt{\varepsilon}$ or $\hat{\chi}_{n,Q}(x) \geq \sqrt{\varepsilon}\}$. Denote $X_2 = X \setminus X_1$. Since $\int_X \hat{\chi}_{n,P(Q)}(x)\,d\nu(x) \leq \varepsilon$, we have $\nu(X_1) \leq 2N\sqrt{\varepsilon}$. $\lambda^+(x) \leq \log 2NC_1C_2\varepsilon^{-1}$ for almost every $x \in X$ so

(2) $$\int_{X_1} \lambda^+(x)d\nu(x) \leq 2N\sqrt{\varepsilon} \log(2NC_1C_2\varepsilon^{-1})$$

Take an arbitrarily small $\delta > 0$. For almost every $x \in X_2$, $0 \leq n \leq N-1$, $a_{n,P(Q)}(k) = \text{Card}\,\{\ell : 0 \leq \ell < k\,,\ (G_n\circ)F_n\circ\ldots G_o\circ F_o\circ H^\ell(x) \in C\} < k\cdot(\sqrt{\varepsilon}+\delta)$ for every $k \geq k(x)$ sufficiently large.

So the $2Nk$ consecutive iterates of x under $F_o, G_o, \ldots, G_{N-1}, F_o, \ldots$ divide into at most $b_k = 2N\cdot k(\sqrt{\varepsilon}+\delta)$ blocks and on each block we twist in the same annulus. So

$$||DH^k(x)|| \leq \left(\left(\frac{2Nk}{b_k}\right)\cdot C_1\cdot\varepsilon^{-1}\cdot C_2\right)^{b_k} ,$$

($||\cdot||$ has not here a strict sense since a Riemannian metric on X has not been defined. So we need to use the constant C_2).

$$\lambda^+(x) = \lim_{k\to\infty} \frac{1}{k} \log||DH^k(x)|| \leq \lim_{k\to\infty} \frac{1}{k}\, 2Nk(\sqrt{\varepsilon}+\delta)\log(C_1C_2(\sqrt{\varepsilon}+\delta)^{-1}\varepsilon^{-1}) .$$

Since δ has been chosen arbitrarily we can neglect it. So

$$\lambda^+(x) \leq 2N\sqrt{\varepsilon}\log(C_1C_2\cdot\varepsilon^{-3/2}) .$$

Thus using this, (1) and (2) we have

$$h_\nu(H) \leq \int_{X_1} \lambda^+(x)d\nu(x) + \int_{X_2} \lambda^+(x)d\nu(x)$$

$$< 2N\sqrt{\varepsilon}\log(2NC_1^2C_2^2\varepsilon^{-5/2}) .$$

c) <u>Perturbation of a twist by "a saw"</u>

We keep the notation from the Introduction. We prove that $H = H_n = A_n\circ F$ and all its powers are ergodic.

We introduce more notation. Choose lifts $\widetilde{A}$, $\widetilde{F}$ of A_n , F to

$\mathbb{R}^2$. Denote $\tilde{H} = \tilde{A} \circ \tilde{F}$. Denote $L_s = \{(x,y) \in \mathbb{R}^2 : x = s \cdot 2^{-n}\}$, $R_s = \{(x,y) \in \mathbb{R}^2 : x = s \cdot 2^{-n} + 2^{-n-1}\}$, Q_s denotes the strip between L_s and R_s and Q'_s the strip between R_s and L_{s+1} , for every integer s . We show schematically on Figure 4.4 how $\tilde{A}$ acts. $\tilde{A}$ is identity on the dotted lines and pushes L_s , R_s according to arrows

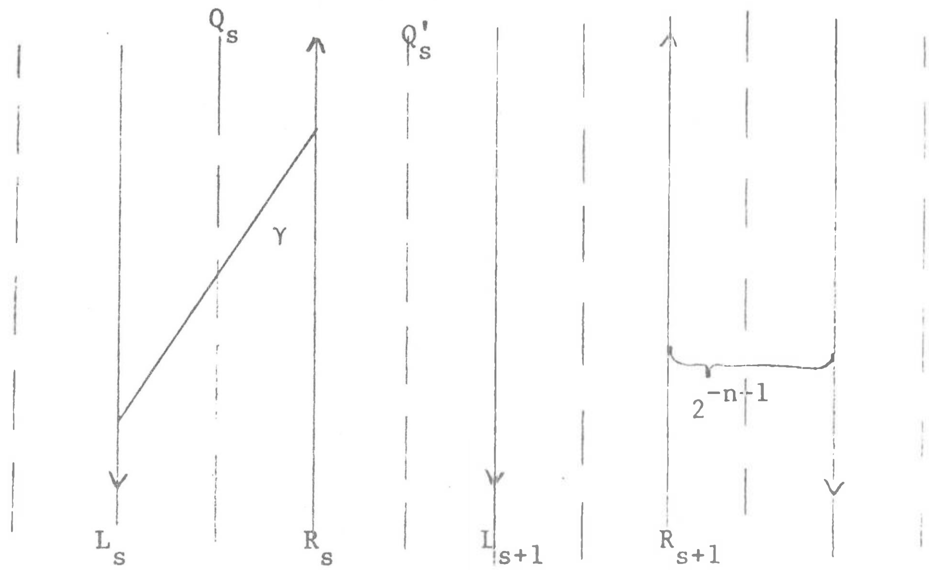


Figure 4.4.

When we consider an oriented curve δ embedded into $\mathbb{R}^2$ we denote by $b(\delta)$, $e(\delta)$ the points of the beginning respectively the end of the δ . By Pr_x , resp. Pr_y we denote the orthogonal projections on x-th , resp. y-th, axes. Let v be the expanding eigenvector of the matrix $DA(z) \circ DF$ for $z \in Q'_s$ and $(\xi_1, \xi_2) = (DA(z) \circ DF)(v)$ for $z \in Q_s$. Of course $\xi_1 / \xi_2 > 0$. Denote

$$S^+(z) = \{(\eta_1, \eta_2) \in T_z\mathbb{R}^2 : \xi_1/\xi_2 \leq \eta_1/\eta_2\} .$$

Observe that for each lift to $\mathbb{R}^2$ of a local unstable manifold, $\tilde{\gamma}^u(z)$, for every integer n , and every point z' of differentiability of $\hat{H}^n(\tilde{\gamma}^u(z))$ such that $z' \in \bigcup_{s \in \mathbb{Z}} Q_s$, the tangent space $T_{z'}\hat{H}^n(\tilde{\gamma}^u(z)) \subset S^+(z')$.

Take a point $z \in T^2$ for which $\gamma^u(z)$ exists. From [14] it follows that if $C > 4{,}0329\ldots$ then there exists an integer $m_o \geq 0$ and a segment $\gamma \subset Q_s$ which joins L_s with R_s for an integer s and covers a segment in $H^{m_o}(\gamma^u(z))$; see Figure 1. Orient γ from L_s to R_s and denote this oriented segment by g_o . We shall construct by induction a sequence of piecewise linear, oriented curves g_k , $k = 0,1,\ldots$ such that $g_k \subset \tilde{H}^k(\gamma)$.

Denote the first (last) segment of g_k by $g_{k,o}(g_{k,1})$ (with orientations inherited from g_k) . Assume that there exist $s(k,0)$, $s(k,1)$ such that $b(g_{k,0}) \in L_{s(k,0)}$, $e(g_{k,0}) \in R_{s(k,0)}$, $b(g_{k,1}) \in L_{s(k,1)}$ and $e(g_{k,1}) \in R_{s(k,1)}$.

Since $\text{length}\ (Pr_x\ \tilde{F}(g_{k,0})) = \text{length}(Pr_y(g_{k,0})) + 2^{-n-1} \geq 5\cdot 2^{-n-1}$, there exists s such that $\tilde{F}(g_{k,0})$ intersects L_s and R_s . (Here and until the end we use only the inequality $C > 4$). Denote by $s(k+1,0)$ the smallest such s . Similarly define $s(k+1,1)$ as the largest s such that $\tilde{F}(g_{k,1})$ intersects R_s .

Finally define

$$g_{k+1} = \tilde{A}\circ\tilde{F}(g_k) \setminus (\{(x,y) \in \tilde{A}\circ\tilde{F}(g_{k,0}) : x < s(k+1,0)\cdot 2^{-n}\} \cup$$

$$\cup \{(x,y) \in \tilde{A}\circ\tilde{F}(g_{k,1}) : x > s(k+1,1)\cdot 2^{-n} + 2^{-n-1}\}$$

with orientation transported by $\tilde{A}\circ\tilde{F}$ from g_k .

Observe (!) that for $k = 0,1,\ldots$

$$Pr_y(e(g_{k+1,1})) - Pr_y(e(g_{k,1})) > C-4 \quad \text{and}$$

$$Pr_y(b(g_{k+1,0}) - Pr_y(b(g_{k,0})) < 4-C \quad .$$

Observe also that the sequence of acute angles between $g_{k,0}$ (or $g_{k,1}$)

and the x-axis is decreasing with growing k .

This and (1) imply for every $k \geq 1$

(2) $$Pr_y b(g_{k,1}) - Pr_y b(g_{k,0}) = Pr_y e(g_{k,1}) - Pr_y e(g_{k,0}) > 0 .$$

From this, or directly, we deduce that for $k = 1$, $g_{k,0} \neq g_{k,1}$, i.e. $s(k,0) < s(k,1)$. This and inductive reasoning with use of (2) give $s(k,0) < s(k,1)$ for every $k \geq 1$. So, for every $k \geq 0$

(3) $$Pr_x b(g_{k,1}) - Pr_x b(g_{k,0}) \geq 0$$

(1) and (3) imply

$$\mathrm{dist}(Pr_x(g_{k,1}), Pr_x(g_{k,0})), \mathrm{dist}(Pr_y(g_{k,1}), Pr_y(g_{k,0})) \xrightarrow[k\to\infty]{} \infty$$

If we do the same construction with F^{-1} , A_n^{-1} and $\gamma^s(z')$ we obtain a similar sequence of curves, but in (2) the inequality must have different direction than in (3). This shows for example that there is no H-invariant circle embedded into T^2 . But we still do not know whether sufficiently far H-images of $\gamma^u(z)$ intersect sufficiently for counter-images of $\gamma^s(z')$ (see Figure 4.5, fortunately $\tilde{H}^{-\ell}(\hat{\gamma}^s(z'))$ cannot go around $\tilde{H}^k(\gamma)$ as on that Figure, see consideration below). We shall prove it now.

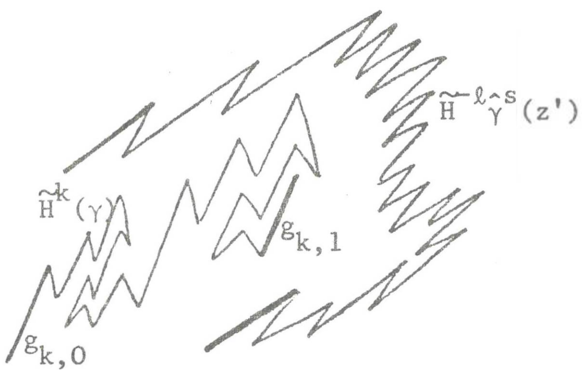


Figure 4.5.

Denote by W_k the rectangle between $L_{s(k,0)}$, $R_{s(k,1)}$ and the two horizontal lines containing $b(g_{k,0})$ and $e(g_{k,1})$ respectively.

Our aim is to show that for every $k \geq 1$ there exists a piecewise linear curve $\hat{g}_k \subset g_k$ which satisfies the following properties :

1. $b(\hat{g}_k)$ belongs to the lower side od W_k, $e(\hat{g}_k)$ belongs to the upper side of W_k ;

2. the line $\ell_{k,0} = b(\hat{g}_k) + \tau\cdot(\xi_1,\xi_2)$ lies left of the line $\ell_{k,1} = e(\hat{g}_k) + \tau\cdot(\xi_1,\xi_2)$;

3. $\hat{g}_k$ is contained in the parallelepiped W_k^1 between lower and upper sides of W_k, $\ell_{k,0}$ and $\ell_{k,1}$. See Figure 4.6. (on that Figure g_k has been

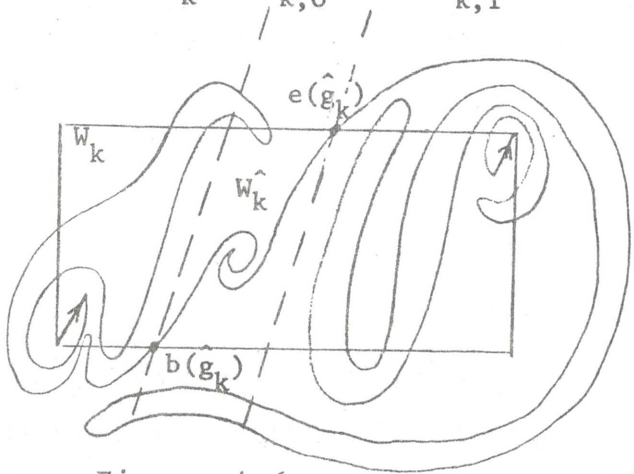


Figure 4.6

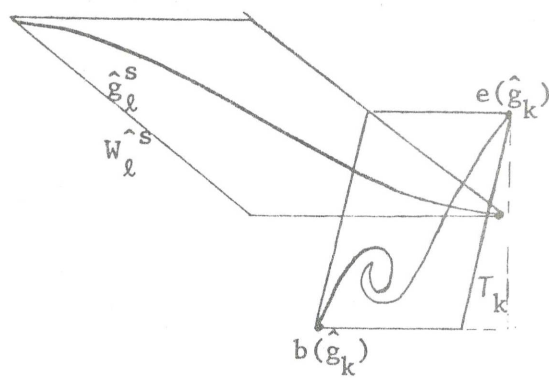


Figure 4.7

drawn only schematically, the fact that it is piecewise linear, with the segments in the prescribed sectors has been neglected).

Existence of such a $\hat{g}_k$ would allow to finish our proof. Indeed, the length and height of W_k^1 tend to ∞ when $k\to\infty$, but the angles are fixed. For a local stable manifold $\gamma^s(z')$ we could similarly find the curves $\hat{g}_\ell^s$ contained in lifts of $H^{-\ell}(\gamma^s(z'))$, for large ℓ, to $\mathbb{R}^2$ lying inside corresponding left twisted (rather than right twisted) parallelepipeds $W^{1,s}$, joining the acute angles of $W_\ell^{1,s}$. The sides of $W_\ell^{1,s}$

would get large with ℓ large and we could choose lifts so that lower ends of $\hat{g}^s_\ell$ were inside the triangle T_k ; see Figure 4.7. This would give the desired intersection of $H^{k+m_o}(\gamma^u(z))$ with $H^{-\ell}(\gamma^s(z'))$. (This part of proof goes more or less as for the Burton-Easton example).

Proof of existence of $\hat{g}_k$:

We start with definitions. Let g be any oriented piecewise linear curve embedded into $\mathbb{R}^2$, $g: \langle t',t'' \rangle \to \mathbb{R}^2$, $t' < t''$, oriented from $g(t')$ to $g(t'')$.

Define a mapping $\Phi(g): \langle t',t'' \rangle \to S^1$ ($S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle) as follows. Let $t' = t_o < t_1 < \ldots < t_m < t_{m+1} = t''$ denote the consecutive points at which g is not differentiable. Choose $\varepsilon : 0 < \varepsilon \ll \min_{i=0,\ldots,m} (t_{i+1} - t_i)$. Define

$$\Phi(g)(t) = Dg \frac{\partial}{\partial t}(t) / \| Dg \frac{\partial}{\partial t}(t) \|$$

for $t \in \bigcup_{i=0}^{m} \langle t_i+\varepsilon, t_{i+1}-\varepsilon \rangle$. For $t = \tau(t_i-\varepsilon)+(1-\tau)(t_i+\varepsilon)$, $0 \le \tau \le 1$, $i=1,\ldots,m$:

$$\Phi(g)(t) = \frac{\tau \cdot \Phi(g)(t_i-\varepsilon)+(1-\tau)\cdot\Phi(g)(t_i+\varepsilon)}{\| \tau\Phi(g)(t_i-\varepsilon)+(1-\tau)\Phi(g)(t_i+\varepsilon) \|} ,$$

for $t \in \langle t', t'+\varepsilon \rangle$:

$$\Phi(g)(t) = \Phi(g)(t'+\varepsilon)$$

and finally for $t \in (t''-\varepsilon, t'')$:

$$\Phi(g)(t) = \Phi(g)(t''-\varepsilon) .$$

Define the index $\mathrm{ind}(g) = \frac{1}{2\pi i} (\log \Phi(g)(t'') - \log \Phi(g)(t'))$ where log means its branch continuous along the curve $\Phi(g)(\cdot)$. Clearly it does not depend on ε .

Define a mapping $\Psi = \Psi(g,b(g)) : \langle t',t''\rangle \to S^1$ as follows : For $t \in \langle t',t'+\varepsilon\rangle$, $\Psi(t) = \Phi(g)(t)$, For $t \geq t'+\varepsilon$, $\Psi(t) = (g(t)-b(g))/\|g(t)-b(g)\|$.

Define an index with respect to $b(g)$:

$$\mathrm{ind}_{b(g)}(g) = \frac{1}{2\pi i}(\log \Psi(g)(t'')-\log\Psi(g)(t'')) \quad .$$

Denote by $\bar{g}$, g with reversed orientation. Define an index of g with respect to $e(g)$ $\mathrm{ind}_{e(g)}(g) = -\mathrm{ind}_{b(\bar{g})}(\bar{g})$

Lemma 1 : $\mathrm{ind}(g) = \mathrm{ind}_{b(g)}(g) + \mathrm{ind}_{e(g)}(g)$.

We go now back to our curves g_k , taken together with parametrizations $g_k : \langle 0,1\rangle \to \mathbb{R}^2$.

Lemma 2 : a) For every $k \geq 0$, $t \in (0,1\rangle$, $\mathrm{ind}(g_k) = 0$, $\mathrm{ind}(g_k|_{\langle 0,t\rangle}) \leq 0$.

b) $|\mathrm{ind}_{b(g_k)}(g_k)| < \frac{1}{4}$, $|\mathrm{ind}_{e(g_k)}(g_k)| < \frac{1}{4}$.

Proof : Goes by induction with respect to k . To prove part b) use also inequalities (2) and (3).

Now, having Lemma 2, we can forget about our dynamics and deduce existence of $\hat{g}_k$ only from properties of curves in $\mathbb{R}^2$.

Fix $k \geq 1$. Let us blow up $b(g_k)$, $e(g_k)$ to small discs D_0 , D_1 . Consider a universal covering X of $\mathbb{R}^2 \setminus (\mathrm{int}\, D_0 \cup \mathrm{int}\, D_1)$ and a lift $\widehat{W}^0 \subset X$ of $\mathrm{int}(W_k)$. Denote $\widehat{W} = \mathrm{cl}(\widehat{W}^0)$. Denote by Π the projection onto $\mathbb{R}^2$.

By Lemma 2b) there exists an isotopy $\Gamma : \langle 0,1\rangle \times \langle 0,1\rangle \to \mathbb{R}^2$ such that $\Gamma_1 = \Gamma(1,.) = g_k$, $\Gamma_0 = \Gamma(0,.) = \Delta$, $\Gamma(s,t) = \Gamma_s(t) = \Gamma_0(t)$ for

$t \in \langle 0,\varepsilon \rangle \cup \langle 1-\varepsilon,1\rangle$ for a small ε and

$$\Gamma(\langle 0,1\rangle \times (0,1)) \subset \mathbb{R}^2 \setminus \{b(g_k),e(g_k)\}$$ (Δ on $\langle \varepsilon,1-\varepsilon\rangle$ is defined as linear, see Figure 4.8).

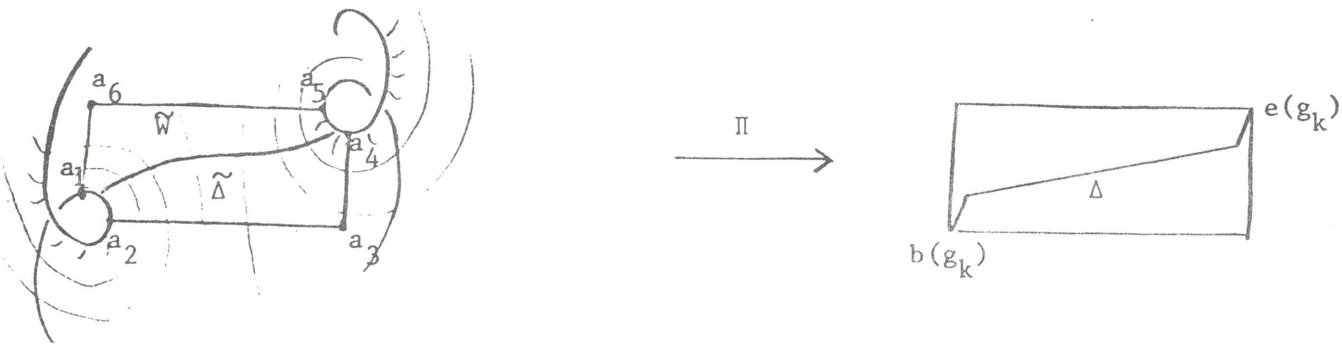


Figure 4.8.

Lift Γ to $\widetilde{\Gamma}$ on X so that $\widetilde{\Gamma}_o = \widetilde{\Delta}$, where $\Pi\circ\widetilde{\Delta} = \Delta$ and $\widetilde{\Delta}(\langle 0,1\rangle) \subset \widetilde{W}$ ($\widetilde{\Gamma}$ keeps the ends $\widetilde{\Gamma}_s(0)$, $\widetilde{\Gamma}_s(1)$ fixed). Denote by C_1 (respect. C_2) the curve embedded into X , inside $Fr(\widetilde{W})$, joining the points a_1 and a_4 through a_2 , a_3 (respect. through a_6 , a_5) , Figure .

Let $t = t_2$ be the first parameter t for which $\widetilde{\Gamma}_1(t) \in C_2$. Let $t = t_1$ be the last parameter $t < t_2$ for which $\widetilde{\Gamma}_1(t) \in C_1$. We define $\hat{g}_k = \Pi\circ\widetilde{\Gamma}_1|_{\langle t_1,t_2\rangle} = g_k|_{\langle t_1,t_2\rangle}$ and prove that it has the required properties.

(4) $$\widetilde{\Gamma}_1(\langle t_1,t_2\rangle) \subset \widetilde{W} .$$

This follows from the following : C_2 disconnects X into X_1 containing int $\widetilde{W}$ and X_2 . $\widetilde{\Gamma}_1(0) \in X_1$, so t_2 is the first time when $\widetilde{\Gamma}_1(g)$ goes out of X_1 . Now we use similarly the fact that C_1 disconnects X_1 .

(4) implies immediately that for every t , $t_1 \leq t \leq t_2$,

(5) $$-\frac{1}{4} < \operatorname{ind}_{b(g_k)}(g_k|_{\langle 0,t\rangle}) < \frac{1}{4} .$$

(since $\tilde{\Gamma}_1|_{<0,t>}$ is homotopic with fixed ends, to a curve γ fully contained in $\tilde{W}$ and $\pi\circ\gamma$ obviously has property (5)).

Since $\tilde{\Gamma}_1|_{<0,t_2>} \subset X_1$, which is simply-connected, then for a small $\varepsilon > 0$, there exists a homotopy $h : <0,1> \times <0,t_2> \to X_1$ such that $h_1 = h(1,.) = \tilde{\Gamma}_1|_{<0,t_2>}$, $h_s(t) = h_o(t)$ for $t \in <0,\varepsilon> \cup <t_2-\varepsilon, t_2>$ and $s \in <0,1>$, $\pi\circ h_o|_{<\varepsilon,t_2-\varepsilon>}$ is linear and $h(<0,1> \times <\varepsilon, t_2-\varepsilon>) \not\ni \tilde{\Gamma}_1(t_2)$.

Denote by B the group of all covering transformations on X (i.e. $b \in B$ if $\pi\circ b = id$). Let e be the neutral element of B. Then if $b \in B \setminus \{e\}$ we have

$$b\circ h_1(<0,t_2>) \cap h_1(<0,t_2>) = \emptyset$$

since $\pi\circ h_1 = g_k|_{<0,t_2>}$ has no self-intersections. But each $b\circ h_1$ begins at $Fr(X)$ and the rest is contained in $int(X)$. So we can improve the homotopy h to h^1 so that $h_o = h^1_o$, $h_1 = h^1_1$ and additionally to all properties of h described above, h^1 satisfies :

$h^1(<0,1> \times <0,t_2>) \cap \bigcup_{b\in B\setminus\{e\}} b\circ h_1(<0,t_2>) = \emptyset$. This implies that

$\pi\, h^1(<0,1> \times <0,t_2>) \not\ni g_k(t_2)$, so

$$ind_{g_k(t_2)}(g_k|_{<0,t_2>}) > -\frac{1}{4} \quad .$$

By Lemma 1, this and (5) imply

$$(6) \qquad ind(g_k|_{<0,t_2>}) > -\frac{1}{2} \quad ,$$

hence by Lemma 2 a) for $t = t_2$, $T_{g_k(t_2)}g_k \subset int\, S^t_{g_k(t_2)}$. (if g_k is not differentiable at t_2 , then by $T_{g_k(t_2)}g_k$ we mean the left side tangent space).

Observe that $g_k(t_2)$ belongs to the upper side of W_k, (otherwise if it belonged to the left side we might estimate in (5) from below by 0, hence in (6) by $-\frac{1}{4}$, a contradiction). This implies that in (6) we can replace $-\frac{1}{2}$ by $-\frac{1}{4}$.

(In fact one can easily prove even that $\mathrm{ind}(g_k|_{<0,t_2>}) = 0$ but it needs going back to the dynamics of H).

We prove that $\hat{g}_k$ lies left of $\ell_{k,1}$. Assume that there exists t_3 such that $t_1 \le t_3 < t_2$ and $g_k(t_3) \in \ell_{k,1} \cap W_k$ (see Figure 4.9). Choose the largest such t_3.

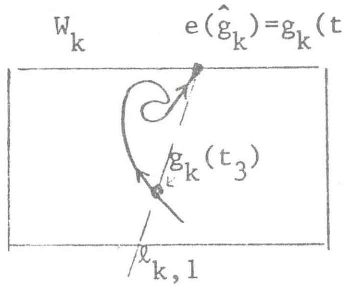


Figure 4.9

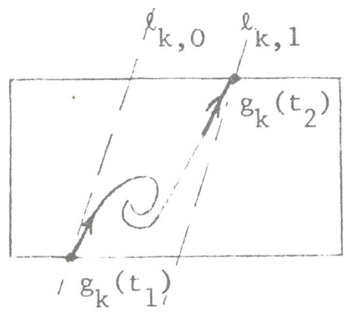


Figure 4.10

Then $\mathrm{ind}_{g(t_3)}(g_k|_{<t_3,t_2>}) \le 0$ and $\mathrm{ind}_{g(t_2)}(g_k|_{<t_3,t_2>}) < 0$. So, using Lemma 1, $\mathrm{ind}(g_k|_{<t_3,t_2>}) < 0$. We can assume that g_k is differentiable at t_3, otherwise we could slightly rotate $\ell_{k,1}$ around $g_k(t_2)$ in the negative direction. So

$\mathrm{ind}(g_k|_{<0,t_3>}) + \mathrm{ind}(g_k|_{<t_3,t_2>}) = \mathrm{ind}(g_k|_{<0,t_2>})$ which using (6) with $-\frac{1}{4}$ gives

$$\mathrm{ind}(g_k|_{<0,t_3>}) > -\frac{1}{4} \quad .$$

But for this index the values between $-\frac{1}{4}$ and 0 are forbidden, since the vector $D_{g_k}(\frac{\partial}{\partial t}(t_3))$ is directed left of $\ell_{k,1}$. So $\mathrm{ind}(g_k|_{<0,t_3>}) > 0$, which contradicts Lemma 2a.

Now observe that $|\mathrm{ind}(g_k|_{<t_1,t_2>})| < \frac{1}{2}$ (by Lemma 1, see Figure 4.10), hence by Lemma 2a) $T_{g_k(t_1)}(g_k) \in \mathrm{int}\, S^t_{g_k(t_1)}$. In fact even $\mathrm{ind}\, g_k|_{<0,t_1>} = 0$. To prove that $\hat{g}_k$ lies right of $\ell_{k,0}$ we proceed similarly to the proof that it lies left of $\ell_{k,1}$.

d) Bifurcations of the toral automorphism $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

Take a mapping $H_{2f,2g} : T^2 \longrightarrow T^2$ as described at the beginning of the Introduction, with the annuli $P = Q = T^2$ and C^∞-functions $f(y)$, $g(x)$ such that $f = g$ (after change of x to y), $\frac{d^k f}{dy^k}(y) = 0$, $f(y) = y$ for $y = 0, \frac{1}{2}, 1$ and $\frac{df}{dy}(y) > 0$ for $y \neq 0, \frac{1}{2}, 1$ (see Figure 4.11).

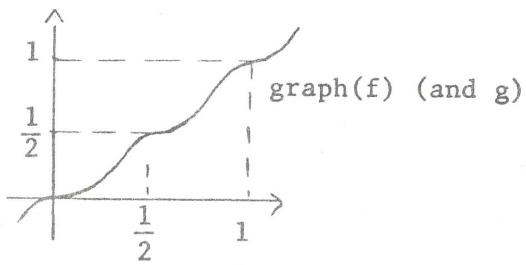


Figure 4.11

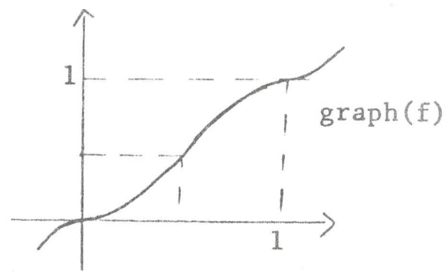


Figure 4.12

From [2] it follows that $H_{f,g}$ is Bernoulli. Repeat briefly the rest of Katok construction, [7]. Act on T^2 with the involution $\mathrm{inv}(z) = -z$. It commutes with $H_{f,g}$. T^2/inv is a sphere S^2 (we introduce on T^2/inv a smooth structure around four singularities of action by inv : $(0,0)$, $(0,\frac{1}{2})$, $(\frac{1}{2},0)$, $(\frac{1}{2},\frac{1}{2})$). Next remove from S^2 a pole, for example the image of $(0,0)$ from T^2.

On T^2 we can study an even simpler example $H_{f,g}$, where we take f as above, but assume $\frac{df}{dy}(\frac{1}{2}) > 0$ (see Figure 4.12) and $g = \mathrm{id}$. Denote by X a vector field on $S^1 = \mathbb{R}/\mathbb{Z}$ which pushes from the point $\frac{1}{2}$ to 0 and 1 (i.e. $X(t) = (-\sin 2\pi t)\cdot\partial/\partial t$) and X_t its flow.

Consider the two-parameter family of functions

$f_{t,s} = s\cdot id + (1-s)\cdot(f\circ X_t)$ for $s,t \in \mathbb{R}$, $s \leq 1$.

We obtain an intriguing two-parameter family of diffeomorphisms on T^2

$$H(t,s) = H_{f_{t,s},g} .$$

We describe below some properties of $H(t,s)$:

1. For $0 < s \leq 1$, $t \in \mathbb{R}$, $H(t,s)$'s are clearly Anosov diffeomorphisms.

2. For $s = 0$ they are Bernoulli (by [2]) but not Anosov, since $DH(t,0)(0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

3. For $0 \leq s \leq 1$, $t \in \mathbb{R}$ all $H(t,s)$ are topologically conjugated with the algebraic automorphism $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

This follows from the fact that $H(t,s)$ is homotopic to A which allows to prove semiconjugacy i.e. the existence of a continuous function $h : T^2 \longrightarrow T^2$ such that $h\circ H(t,s) = A\circ h$. In fact h is a homeomorphism since a lift $\widetilde{H}(t,s)$ of $H(t,s)$ to $\mathbb{R}^2$ is expansive with constant of expansiveness arbitrarily large. See [16] .

4. Measure entropy with respect to Lebesgue measure $h_\ell(H(t,s))$ is a continuous function of (t,s) for $0 \leq s \leq 1$, $t \in \mathbb{R}$.

This follows from the fact that stable and unstable subbundles depend continuously on (t,s) for $s > 0$. For $(t_n,s_n) \to (t_o,0)$ we can prove pointwise convergence (almost everywhere) of stable (unstable) subbundles $E^{s(u)}$ of $H(t_n,s_n)$ to those of $H(t_o,0)$. Next use the formula

$$h_\ell(H(t,s)) = \int_{T^2} \log\| DH(t,s)|_{E^u(x)} \| \, d\ell(x) .$$

It is easy to prove also that subbundles $E^{s(u)}$ are continuous on a set of full measure. This holds, in fact, for all linked twist mappings considered in this paper.

5. Fixed $s : 0 \leq s \leq 1$, $h_\ell(H(t,s)) \xrightarrow[t\to+\infty]{} 0$. (The proof is similar to item b) of this paragraph). This together with 3. and 4. shows that for any number α , $0 < \alpha \leq h(A) = \log(\frac{3+\sqrt{5}}{2})$ there exists an Anosov diffeomorphism A_α isotopic and conjugated to A , preserving Lebesgue measure such that $h_\ell(A_\alpha) = \alpha$.

6. If we take the pointwise limit $A_o = \lim_{t\to\infty} H(s,t)$ we have $h_\ell(A_o) = 0$, but A_o is not defined on the set $\{(x,y) \in T^2 : y = \frac{1}{2}\}$ of measure 0 . However one can find a lot of entropy zero homeomorphisms of the form $H_{f,g}$ in the boundary (in C^o-topology) of the space of smooth Anosov diffeomorphisms conjugated to A .

For example take $H_{f,id}$ where f is the standard Cantor function, i.e. the monotone function of $<0,1>$ onto $<0,1>$ defined on the Cantor set $C = \{x \in <0,1> : x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, a_n = 0,2\}$ by $f(\sum_{n=1}^{\infty} \frac{a_n}{3^n}) = \sum_{n=1}^{\infty} \frac{\frac{1}{2}a_n}{2^n}$. Then $H_{f,id}$ satisfies (K-S)-conditions [see Appendix] with $\mathrm{Sing}(H_{f,id}) = <0,1> \times C$ and its Lyapunov exponents are zero.

(This is not strange since Lind and Thouvenot proved in [12] that any ergodic automorphism of the Lebesgue space is equivalent to a homeomorphism on T^2 preserving Lebesgue measure, topologically conjugate to A with conjugacy isotopic to identity. Since a conjugating homeomorphism can be C^o-approximated by a smooth diffeomorphism, see [6, Appendix] , then Lind, Thouvenot homeomorphisms belong to the C^o-boundary of the smooth Anosov diffeomorphisms isotopic to A).

7. The above constructions can be done for any orientation preserving

hyperbolic automorphism $A : T^2 \longrightarrow T^2$. This follows from the fact, see [18] , that we can decompose the matrix A as follows: $A = \pm a^{s_1} b^{t_1} \dots a^{s_n} b^{t_n}$ (up to conjugacy) where $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $t_j, s_j > 0$ for $j = 1,\dots,n$. (I due this remark to discussion with J.H. Przytycki).

<u>Question</u> : What happens to the Lyapunov exponents and measure entropy of $H(t,s)$ when $s < 0$? (Observe that after s passes the bifurcation parameter 0 and gets negative, an elliptic motion around $0 \in T^2$ occurs).

Appendix

We prove in this paper ergodic properties of mappings using the so-called Pesin Theory, but in a more general situation than considered by Pesin in [15] , since our mappings have singularities. Pesin Theory for maps with singularities has recently been considered by Katok and Strelcyn [10] , but the results are still in the form of a preprint, so for the comfort of the reader we list them below.

Katok-Strelcyn, (K-S) conditions

Let X be a complete metric space with a metric ρ . Let $N \subset X$ be an open subset which is a Riemannian manifold with a Riemannian metric inducing $\rho|_N$. Assume that there exists a number $r > 0$ such that for each $x \in N$, $\exp_x$ restricted to the ball $B(x) = B(x, \min(r, \mathrm{dist}_\rho(x, X \setminus N)))$ is injective.

Let μ be a probability measure on X and Φ be a μ - preserving, C^2 - , 1-1 mapping defined on an open set $V \subset N$, into N. Denote $\mathrm{sing}\,\Phi = X \setminus V$.

(K-S,1) There exist positive constants a , C_1 such that for every $\varepsilon > 0$

$$\mu(B(\mathrm{sing}\Phi, \varepsilon)) \leq C_1 \varepsilon^a ,$$

$B(\mathrm{sing}\Phi, \varepsilon)$ means the neighbourhood of $\mathrm{sing}\,\Phi$ with radius ε .

(K-S,2)
$$\int \log^+ \|D\Phi(x)\| \, d\mu(x) < \infty$$
$$\int \log^+ \|D\Phi^{-1}(x)\| \, d\mu(x) < \infty .$$

(K-S,3) There exist positive constants b, C_2 such that for every $x \in X \setminus \mathrm{sing}\,\Phi$

$$\|D^2\,\Phi(x)\| \leq C_2(\mathrm{dist}(x,\mathrm{sing}\,\Phi))^{-b} .$$

(By $\|D^2\Phi(x)\|$ we mean $\sup\{\|D^2(\exp_z^{-1}\circ\Phi\circ\exp_y)\| : x \in B(y),\ \Phi(x) \in B(z)\}$).

18 **Remark** : If $\Phi = \Phi_n\circ\ldots\circ\Phi_1$ and Φ preserves μ we can replace the (K-S) conditions for Φ by the analogous conditions for each Φ_i separately, with Sing Φ_i and $\mu_i = (\Phi_{i-1}\circ\ldots\circ\Phi_1)_*(\mu)$ respectively. ~~(with μ Φ-invariant, not necessarily Φ_i-invariant).~~

Theorem : a) If Φ satisfies the (K-S)-conditions then for almost every $x \in X$, for all vectors tangent at x, there exist Lyapunov exponents and there exist local stable and unstable manifolds $\gamma^s(x)$, $\gamma^u(x)$, corresponding to negative and positive exponents respectively. Denote by $\Lambda^{s(u)}(k)$ the set of points where the number of negative (positive) Lyapunov exponents computed with multiplicities is equal to k (i.e. $\dim\gamma^{s(u)}(x) = k$). Consider $\Lambda^{s(u)}(k)$ if $\mu(\Lambda^{s(u)}(k)) > 0$. Then for a sequence of sets $\Lambda^{s(u)}(k,m)$ increasing with m which exhaust almost all of $\Lambda^{s(u)}(k)$, the families $\{\gamma^{s(u)}(x) : x \in \Lambda^{s(u)}(k,m)\}$ are absolutely continuous.

[I cannot refrain from explaining how to use in the proof the key condition (K-S,1). Take for an arbitrarily small δ the sequence $B_n = B(\mathrm{Sing}\,\Phi, (1-\delta)^n)$. By (K-S,1), $\sum_{n=0}^{\infty}\mu(\Phi^{-n}(B_n)) < \infty$. Hence by the Borel-Cantelli Lemma for almost every $x \in X$, $\mathrm{dist}(\Phi^n(x),\mathrm{sing}\,\Phi) \geq (1-\delta)^n$ for all $n \geq n(x)$ sufficienlty large. When we pass with Φ, using exp, to small balls in the tangent spaces, we extend $\exp^{-1}_{\Phi^{n+1}(x)}\circ\Phi\circ\exp_{\Phi^n(x)}$ to $\hat{\Phi}_n$ and prove the existence of stable manifolds γ_n^s for $\hat{\Phi}_n$, $n = 0,1,\ldots$, which shrink quicker than $(1-\delta)^n$ and then the

$\exp_{\Phi^n(x)}(\gamma_n^s)$ are stable manifolds for Φ in $X \setminus \mathrm{Sing}\,\Phi$].

b) If we assume additionally that the measure μ is equivalent to the Riemannian measure on N and all Lyapunov exponents are almost everywhere different from 0, then X decomposes into a countable family of positive measure μ, Φ-invariant, pairwise disjoint sets $X = \bigcup_{i=1}^{\infty (\text{or } N)} \Lambda_i$ such that for every i, $\Phi|_{\Lambda_i}$ is ergodic and $\Lambda_i = \bigcup_{j=1}^{j(i)} \Lambda_i^j$ where $\Lambda_i^j \cap \Lambda_i^{j'} = \emptyset$ for $j \neq j'$, $\Phi|_{\Lambda_i}$ permutes Λ_i^j and for each j, $\Phi^{j(i)}|_{\Lambda_i^j}$ is ~~a K-system.~~ Bernoulli (In the above situation we sometimes call the system almost hyperbolic and say that it decomposes into a countable family of ~~K-~~ Bernoulli components).

19 c) If additionally for almost every $z, z' \in X$ there exist integers m, n such that $\Phi^m(\gamma^u(z)) \cap \Phi^{-n}(\gamma^s(z')) \neq \emptyset$ then in the decomposition of X we have only one set $\Lambda_i = \Lambda_1$ (i.e. $N = 1$). In particular Φ is ergodic.

d) If additionally for almost all points $z, z' \in X$ and every pair of integers m, n large enough $\Phi^m(\gamma^u(z)) \cap \Phi^{-n}(\gamma^s(z')) \neq \emptyset$ then all powers of Φ are ergodic. This implies $j(1) = 1$ so Φ is a ~~K-~~ Bernoulli system.

koniec (4)

<u>Remark</u> : [oral communication of F. Ledrappier]. In fact such K-systems are Bernoulli systems. This follows from the fact that every finite regular partition of X is weak Bernoulli. This follows from the adaptation of the demonstration of the analogous fact for Anosov diffeomorphisms given in part 1 of M. Ratner "Anosov flows with Gibbs measure are also Bernoulli ". Isr. J. Math. 17(1974) pp. 380-391.

<u>Theorem</u> : If Φ satisfies the (K-S)-conditions then

$$h_\mu(\Phi) \leq \int_X \lambda^+(x)\, d\mu(x)$$

where $\lambda^+(x)$ denotes sum of the positive Lyapunov (with multiplicities) at x .

If we assume additionally that μ is equivalent to the Riemann measure on N then the Pesin formula for entropy holds :

$$h_\mu(\Phi) = \int_X \lambda^+(x)\, d\mu(x) \quad .$$

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