YOUNG'S

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OF THE HIGHER

EQUATION

SECOND EDITION

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THEORY AND SOLUTION

OF

ALGEBRAICAL EQUATIONS

OF

THE HIGHER ORDERS.

BY J. R. YOUNG,

PROFESSOR OF MATHEMATICS IN BELFAST COLLEGE.

SECOND EDITION, ENLARGED.

LONDON:
SOUTER AND LAW, 131, FLEET STREET.
1843.

Printed by C. Adlard, Bartholomew Close.

PREFACE.

The present work is an endeavour to exhibit a comprehensive view of the Theory and Solution of Algebraical Equations; especially in reference to that more important department of the subject, which relates to the practical analysis and computation of the roots when the coefficients of the equation are given numbers.

I am not aware of the existence of any work in which so ample a discussion of the general problem of the solution of numerical equations is furnished as that which is attempted, however imperfectly, in the following pages.

The main topics of this discussion are indeed but very recent contributions to science, made by different persons, at comparatively short intervals, within the last twenty or thirty years; and are as yet too imperfectly known and estimated to have become generally incorporated into our systems of analysis.

But the importance of these recent additions to our knowledge, in a branch of inquiry of so much practical consequence as the solution of numerical equations, in which all existing methods were felt to be so limited and imperfect, seemed sufficient to justify even reiterated endeavours to introduce them into general notice, and to urge their claims to general adoption.

Under this conviction, the first edition of the present work, and the subsequent introductory volume on the Analysis and Solution of Cubic and Biquadratic Equations, were prepared :—I believe they have both been as instrumental as I had any right to expect, in accomplishing the objects of their publication.

In the introductory work just alluded to, it was my more immediate aim to invite the attention of the young analyst to STURM'S infallible method of analysing numerical equations; and to show the efficiency and general practicability of that method, in conjunction with the subsequent development of HORNER, to accomplish, with more ease and certainty than had hitherto been done, the solution of equations of the third and fourth degrees.

It is a remarkable circumstance connected with the recent improvements adverted to above, that they have presented, in rapid succession, three independent methods for analysing a numerical equation; thus affording a choice of means by which the most formidable obstacle that had hitherto stood in the way of a complete solution of the problem might be overcome. These methods are respectively due to BUDAN, FOURIER, and STURM: each is characterized by distinctive peculiarities, and each has its own advocates, who have perhaps sometimes allowed their partialities, in favour of one method, to influence their judgment in estimating the merits of the others.

What more especially distinguishes the method of STURM from each of the other methods is its unfailing certainty, and its entire freedom from tentative operations :—two qualifications of the first importance in practical science. It was on these grounds chiefly that I had always regarded STURM'S method as the best; and that I had ventured, more than once, to affirm its superiority even as regards equations of a higher degree than the fourth. Up to this point my conviction of the superior eligibility of this method, when conducted conformably to the plan which I have recommended, remains unchanged. Nor should I now see any reason to modify my former views, as to the more extended application of STURM's theorem, but for the improvements which I think I have in this volume effected in other methods of analysis; more especially in that proposed by FOURIER, which, however easy and concise in certain particular applications of it, had but very slender claims, in the state in which it was left by its distinguished author, to rank with the theorem of STURM, whether we regard the certainty of its conclusions, or the general practicability of reaching them.

It is probable that the modifications here adverted to may contribute to bring the method of FOURIER into more general use in the analysis of the higher equations, whenever, from the magnitude of the coefficients, STURM's process might be expected to involve numbers inconveniently large. It is not likely however that any future discovery will ever lead to the entire abandonment of STURM's method. Within the limits above stated, it is still upon the whole the best that can be given. And the simple character of its operations, and the undeviating certainty of its results, will always command for it a prominent position in every exposition of the doctrine of numerical equations.

By far the greater part of the following work is devoted to the analysis and development of these theories. I have endeavoured to place each of the new methods before the reader in the best form I could; and, by copiously illustrating them all, have afforded him the means of forming a correct estimate of their comparative merits, and of drawing his own conclusions, as to the preference to be given to any one in particular, in cases of more than ordinary difficulty.

The critical examinations and discussions in which I have indulged will, I trust, be found to have been conducted with impartiality. In matters of science and demonstration it is absurd to entertain preferences and predilections. By unduly extolling a favorite author or a favorite method, a writer may, for a time, easily supplant the just claims of others; and, as far as his influence extends, may actually impede the science he would wish to advance.

In expounding the recent improvements in the doctrine of equations, I have attributed the first step in those improvements to BUDAN. This step consists in an extension of the theorem of DESCARTES; an extension which, although very obvious, and easily made, was nevertheless first publicly announced in the *Nouvelle Méthode*, &c. of BUDAN in 1807.

The same theorem was afterwards given by FOURIER, in his *Analyse des Equations Déterminées;* and published, after the death of the author, in 1831; since which time it has been common, with English writers, to call the extension in question the "Theorem of FOURIER," without any regard to the prior claim of BUDAN. It would be difficult to discover upon what grounds this misappropriation is persisted in. It is certain that the leading men in France—LAGRANGE and LEGENDRE, the colleagues and associates of FOURIER—regarded the theorem as due to BUDAN;* and there is no evidence to show that FOURIER himself ever disputed the claim, although there are strong reasons for concluding that FOURIER was led to the same thing by his own

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* The academicians appointed to examine the Mémoire of BUDAN were LAGRANGE and LEGENDRE, who make no mention in their report of any similar theorem by FOURIER. The report of the commissioners on BUDAN's paper closes as follows:—"Nous croyons que le théorème trouvé par M. BUDAN mérite l'attention de la classe, comme étant une extension de la règle de DESCARTES, et que son Mémoire peut etre imprimé dans le Recueil des Mémoires présentes, accompagné du présent rapport. *Signé*—LAGRANGE; LEGENDRE, *rapporteur*." See the *Nouvelle Méthode* of Budan, 2d edit. 1822.

independent investigations into the theory of equations. There is a passage in the *Histoire de Mathématiques* of MONTUCLA which justifies this conclusion. I have extracted it at page 151. It is very remarkable that NAVIER, the editor of FOURIER'S posthumous work, and all the other advocates of FOURIER'S priority, should have overlooked testimony so strongly confirmatory of the position they have taken such pains to establish. Perhaps the fairest way would be to consider the theorem in question as the common property of FOURIER and BUDAN.

But this theorem only partially accomplishes the object to which it is applied—the analysis of a numerical equation. Additional principles were required to complete the decomposition thus partially effected. These were accordingly supplied both by BUDAN and FOURIER: and at this point of the process the two methods become perfectly distinct and independent.

This more advanced theorem of FOURIER has, however, met with comparatively but little notice in this country. By the "Theorem of FOURIER" is generally meant merely the preliminary theorem, noticed above, as common to BUDAN and FOURIER; the additional principle, by which this theorem is perfected, and which in a peculiar manner displays the genius and resources of the author, being altogether overlooked.

I have thought it necessary therefore to give a very full exposition of this second theorem of FOURIER, which I have endeavoured to free from the principal imperfections which precluded its successful application beyond very narrow limits. These modifications are proposed and explained in the Ninth Chapter, and the importance of them practically illustrated in Chapters Eleven and Twelve.

I have also consigned to this twelfth chapter some new views and developments, which, as far as they extend, I believe to be accessions to our knowledge; and to increase our practical facilities in a difficult and delicate department of the subject.

The methods proposed in this chapter for distinguishing real from imaginary roots, in cases of doubt, will be found to be simple in their theory and easy in their practical application. It is desirable that the doctrine of algebraical equations should be rendered independent of the more advanced principles of analysis. To establish FOURIER's criterion for the testing of these doubtful cases, we must borrow assistance from the analytical theory of curves, or from the theorem of LAGRANGE on the limits of TAYLOR's series: both of which subjects involve the principles of the differential calculus. The methods here proposed effect the objects of FOURIER's criterion by aid of only the common algebraic theories.

Some apology may be necessary for a new term which I have ventured to introduce into these discussions:—the term *imperfect roots;* a name by which I have designated certain real values, by which a particular class of imaginary roots may be replaced, and which are shown to furnish real approximate solutions to the equation. FOURIER had noticed the fact that imaginary roots divide themselves into two classes, distinguished by very marked peculiarities; but I think he did not develop the theory of this interesting principle with sufficient fulness. I have considered it with more detail in the eighth chapter One of these two classes of imaginary roots suggests real values actually available in calculation, and having full claim to the character of approximate numerical solutions: these are the values that I have called *imperfect* roots, to mark their defect from that strict accuracy which belongs to the other real roots of the equation.

The formulas given in this twelfth chapter for determining two roots of a numerical equation, after the others have been computed, will be found useful on many occasions. One of these formulas has suggested a very simple investigation of a general expression for the solution of equations of the fourth degree: it will be found in the Eighteenth Chapter. This investigation might no doubt be derived from the general theory of symmetrical functions; but the steps supplied by that theory would be much more complicated and difficult.

In several other portions of the book I have also introduced particulars which appear to me to have claim to originality. But I am anxious to express myself with all becoming caution and reserve in reference to these matters; more especially as I have found, since the completion of the work, an anticipation of a principle which I had thought had hitherto escaped notice: I have made the proper acknowledgment in a NOTE at the end.

A cursory glance at the more advanced sheets of the present volume will show that the work abounds in calculations, several of which involve a good deal of numerical labour. It was necessary to enter upon these, in order to furnish a satisfactory view of the actual capabilities of the new methods. In these practical illustrations, as well as in the theoretical developments on which they are founded, I have tried to secure rigid accuracy: and although it is likely, that in such a large amount of calculation, one or two numerical errors may have escaped detection, I am persuaded that these, if any occur, are very few in number.

It may be mentioned in conclusion, that in animadverting upon the methods proposed by FOURIER and BUDAN, I have sought to establish the validity of my objections by practical illustrations, taken, for the most part, from the favourers of the methods themselves. These, however, I have considered it necessary carefully to recompute; for I have found, from experience, that such calculations are not to be taken upon trust. Examples, incorrectly worked by the original proposers, or erroneously copied by their immediate successors, are often transmitted from book to book, without any correction. The well known cubic equation, by which **Newton** illustrated his method of approximation, is in this latter predicament: it is, in general, inaccurately calculated. I have given the correct figures at page 333.

With this brief account of its general scope and intention, I submit the work to the impartial examination of those interested in the progress of the important department of analysis on which it treats; I ought perhaps to say, to their *indulgent* examination; since from the extent of ground gone over, much of which has been as yet but little trodden, and the innovations I have ventured to introduce into various parts of the subject, imperfections may have escaped me, which may render such indulgence necessary to secure to the book the favorable reception that was so largely given to the former edition.

J. R. YOUNG.

Belfast College;
Oct. 1, 1842.

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ERRATA.

Page 31, line 19, *for* a and b *read* b and a.
60, to line 1 *add* " if the roots are all real."
168, lines 8 and 11, *for* right *read* left, and *for* left, right.
170, line 3, *for* $f_4(x)$ *read* $f_4(x)=0$.
177, head line, *for* STURM *read* FOURIER.
256, in second column *dele* **4** and *for* **5** *read* **4**.
266, line 15, after root-figure *add* " or that figure plus 1."
300, line 25 *for* 300 *read* 298.
444, line 14, *supply* . . . . [5]

THEORY AND SOLUTION

OF

ALGEBRAICAL EQUATIONS.

INTRODUCTION.

(Art. 1.) The great object of all computation is the determination of numerical values for unknown quantities by help of the given relations which they bear to other quantities already known; and it is the office of algebra to express these relations in a symbolical form, by means of the marks and signs which constitute the notation of that science.

Every such algebraical expression of the conditions which connect the known and unknown quantities together, in any mathematical enquiry, when reduced to its most convenient form, furnishes us with an *equation;* and it is thus that the *solution of equations,* as the evolution of the unknown quantities involved in them is called, becomes the chief business of algebra, and the grand problem to the full discussion of which all its rules and processes are merely subsidiary.

But such is the difficulty of the subject that all the resources of algebra have been hitherto found inadequate to effect the solution of it in general and finite terms—that is, without the aid of infinite series—except in the simple cases where the unknown quantity rises to no higher a degree than the second. The rigorous solution of an equation of the second degree, by means

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of a general formula for the values of the unknown quantity, is readily accomplished by common algebra. But the discovery of a like formula for equations of the third degree has hitherto resisted every effort. The same may be said with respect to equations of the fourth degree, while for those of the fifth and more advanced orders it has been demonstrated that the existence of any such general formula is impossible.

(2.) It is true that the early Italian algebraists, Ferrari, Tartaglia, and Cardan, investigated general methods for equations of the third and fourth degrees, and that these methods, expressed in the notation of modern algebra, furnish formulas which do really represent the sought values in terms of the coefficients of the equation and under a finite form. But, for certain relations among the coefficients, these formulas, and all others that have since been proposed for a like purpose, involve *imaginary expressions*, which, except in certain particular cases, render the actual computations impracticable till the formulas are developed in an infinite series; and the imaginary terms, by mutually opposing one another, become exterminated.

(3.) It is observable that the well-known formula for equations of the third degree—usually called the formula of Cardan, and to which every other known method is reducible—is generally in the predicament here described whenever the values of the unknown quantity are all *real*. The formula which represents these values, and from which they are to be evolved, turns out in nearly every such case to be merely a compact symbolical expression for an *infinite series;* and it is only from this series, and not immediately from the undeveloped formula itself, that the real values concealed under the imaginary expressions which enter the formula can be computed or approximated to.

(4.) This failure of Cardan's formula to solve the cubic in a general form, when the values of the unknown quantity are all real, has always been regarded as a very remarkable and often indeed as an anomalous circumstance in the doctrine of equations. But it will be shown hereafter, when we come to that department of our subject to which the discussion of the general formulas proposed for the solution of equations belongs, that this circumstance might have been anticipated, as a necessary consequence of the laws of algebra; and that the actual exhibition of the real values, in a finite computable form, by CARDAN's method, would be a contravention of those laws, and therefore a really anomalous occurrence. Thus the method becomes essentially inapplicable as a general method whenever the values of the unknown quantity are all real.

(5.) Every general form for equations of the fourth degree, involves in it the above-mentioned subsidiary expression for equations of the third; and thus, without extending our enquiries to equations of the higher orders, we are compelled to acknowledge that algebra has not as yet rewarded the labours of analysts with effective formulas for the solution of equations of the third and fourth degrees, by means of which the real values may always be computed without the aid of infinite series. As far as the discovery of such formulas is concerned, no advance of consequence has been made beyond the point reached by CARDAN and DESCARTES; and the search after general symbolical forms for the solution of the higher equations, as also the attempt to improve and rationalize those already alluded to, are occupations which have always proved so barren of success that analysts have at length ceased to prosecute their enquiries in reference to these objects.

(6.) But although the solution of equations above the second degree by general algebraical formulas competent to supply the unknown values under all relations of the coefficients has thus been found to be impracticable, yet the actual evolution of these values by common arithmetical processes may always be effected whenever such values really exist; so that, however the powers of algebra may in general transcend those of arithmetic, we have here, in the problem of the general solution of equations, a very remarkable example of the fact that we may arrive at numerical results by certain general and uniform arithmetical operations which, although very simple in their character, it is not within the power of algebra, with its present symbols of operation, to represent in any finite formula. In fact the operation by which the numerical solution of equations of all orders is now effected, is an inverse operation of so fundamental and purely elementary a character that it does not seem to admit of analysis into any simpler and subordinate processes, except in the single case of the quadratic; so that to represent the general operation symbolically it would seem that we must devise a new notation for that purpose, and not attempt the representation by combining together other supposed component forms.

(7.) The numerical evolution of the values of the unknown quantity in a quadratic equation is an operation of which the arithmetical extraction of the square root is but a particular case; so that this latter process may be regarded, theoretically, as one of a simpler and more elementary character; and to such a process we really can reduce the operation for solving a quadratic equation, as the common algebraical formula for quadratics shows. Our success in this particular class of equations is owing entirely to the power we possess of making the unknown member of every such equation a complete square, by the introduction of a certain *known* quantity. The case is very different with equations of the next higher and still more advanced degrees: CARDAN's formula for cubics—admitting its general irreducibility when the values sought are real—so far from furnishing a finite algebraic expression for those values in terms of the coefficients, actually proclaims that the existence of such an expression is an impossibility.

(8.) CARDAN's formula, if we except a few particular cases, effects no more for cubics whose roots are all real, than a certain well-known imaginary expression for a circular arc effects for the rectification of the circle.* Each formula is but a compact symbolical expression for certain infinite series.

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* The expression here alluded to is the following, due to JOHN BERNOULLI, viz. $\pi = \frac{2 \log \sqrt{-1}}{\sqrt{-1}}$

(9.) Such forms, however, although utterly useless for the purpose of actual computation, are nevertheless valuable on other grounds; and although formulas, like the irreducible case of CARDAN'S, if they could be discovered for the higher equations, would be of no value in the numerical solution of such equations, yet they would be received as important acquisitions into other departments of analytical enquiry; since they would enable us to exhibit, at least symbolically, the elements of which every rational and integral algebraic polynomial is composed, however difficult it might be to determine when these symbolical forms stand for real quantities, and when they are purely imaginary.

(10.) The great labour which analysts have from time to time bestowed upon this research after general algebraic forms for the solution of the higher equations, although wholly unsuccessful, or very nearly so, as respects the ostensible object of enquiry, has been rewarded by the discovery of most of those interesting and important truths which constitute the general theory of equations, many of which have directly contributed to advance towards its present perfection the *method of numerical solution*, which the recent efforts of BUDAN, FOURIER, HORNER, and STURM, have rendered entirely effective and general.

(11.) This method, as already noticed, is a purely arithmetical process, performed upon the *numerical coefficients* of the proposed equation, universally applicable without regard to the degree of the equation, and altogether independent of any such general algebraic model, or formula, as that which analysts have so long sought in vain to discover.

(12.) It is our intention in the following pages to present a connected, and, as far as we are able, a perspicuous view of the researches just adverted to, with such modifications and additions as appear to us to be real improvements, calculated to increase our facilities in the analysis and solution of the higher equations. The more elementary details, as far as equations of the first four degrees are concerned, have already been discussed with sufficient copiousness in an introductory volume.* But in order to render the present work complete, as a comprehensive exposition of the modern theory and solution of algebraical equations, it will be necessary to resume and generalize some of the theoretical investigations given in that Introduction; the previous study of which, however, as an epitome of the leading topics in the present volume, is earnestly recommended to the student.†

(13.) Before terminating these preliminary remarks, it will be necessary to say a word or two as to the notation to be generally employed in this treatise; and to explain the sense in which certain terms of frequent occurrence in the higher departments of analysis will be used in the following pages.

The notation for the different classes of equations involving one unknown quantity will usually be as follows, although we shall occasionally depart from it in the advanced parts of the work.

A simple equation will in general be expressed thus:

$$\mathrm{A}x + \mathrm{N} = 0.$$

A quadratic equation,

$$\mathrm{A}_2 x^2 + \mathrm{A}x + \mathrm{N} = 0.$$

A cubic equation,

$$\mathrm{A}_3 x^3 + \mathrm{A}_2 x^2 + \mathrm{A}x + \mathrm{N} = 0.$$

A biquadratic equation,

$$\mathrm{A}_4 x^4 + \mathrm{A}_3 x^3 + \mathrm{A}_2 x^2 + \mathrm{A}x + \mathrm{N} = 0.$$

And, in general, an equation of the nth degree will be written,

$$\mathrm{A}_n x^n \; . \; . \; . \; . + \mathrm{A}_3 x^3 + \mathrm{A}_2 x^2 + \mathrm{A}x + \mathrm{N} = 0;$$

in which the absolute term N, and the coefficients A, A_2, A_3, &c.

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* The Analysis and Solution of Cubic and Biquadratic Equations, 1842.

† In the course of the following work we shall presume the reader to be in some degree acquainted with the introductory treatise alluded to, and shall occasionally refer to it for some practical details respecting equations of the third and fourth degrees, which need not be repeated here.

usually represent real numbers, either positive or negative, integral or fractional. The polynomial on the left of the sign of equality we shall frequently call the *first side* or the *first member* of the equation.

(14.) It is common in algebraical enquiries, involving frequent reference to complicated expressions, to designate those expressions by some more brief and commodious form; and to facilitate this abridgment, a new word, the word *function*, has been introduced into algebra, and represented symbolically by the initial letter f, or F or ϕ, or f', or f_1, &c.

Thus any expression involving x, as, for instance, the left-hand member of either of the foregoing equations, is called, in brief, a *function of* x, and represented by one or other of the forms

$$f(x),\quad F(x),\quad \phi(x),\quad \psi(x),\quad f'(x),\quad F'(x),\quad f_1(x),\ \&c.$$

when, however, one of these forms is fixed upon to represent any algebraical expression, it is plain that, in order to avoid confusion, we must adhere to that form of representation throughout the enquiry; and must not employ the same form to characterize other expressions, or other functions.

If, for example, we agree to represent the foregoing general equation of the nth degree by $f(x) = 0$, we are not afterwards at liberty to represent any other different function, occurring in the same enquiry, by the characteristic f, any more than we are at liberty to denote two different magnitudes by one and the same algebraical character. We see, therefore, that while the term *function* has the most extended signification, comprehending all algebraical combinations possible, yet, by varying the form of the initial letter, or characteristic, which stands for the word, the various forms of functions may all be represented in the proposed notation by distinctive symbols.

(15.) The expression $f(x) = 0$, which we have just employed to denote, in short, the general equation of the nth degree, includes in it, of course, all the particular equations written above, as n may be any positive and integral exponent whatever. The symbol $f'(x)$ or $f_1(x)$, &c. denotes, as already remarked, a function of the same quantity, x, although different from the function $f(x)$; yet, as the preceding forms are derived from this last, by simply supplying an accent, or subscribed numeral, they are the forms usually employed to express functions derived from, or dependent on, a primitive function $f(x)$. For example, if the function $ax^6 + bx^5$ be represented by $f(x)$, and we have occasion to exhibit the successive quotients which arise from dividing this primitive function by x repeatedly, it would be convenient to use the following notation:

$$f(x) = ax^6 + bx^5$$

$$f_1(x) = ax^5 + bx^4$$

$$f_2(x) = ax^4 + bx^3$$

$$f_3(x) = ax^3 + bx^2$$

&c. &c.

where $f(x)$ is the primitive, and the others the derived functions, each being derived from the preceding, by a repetition of a known process, viz. the process of division by x. Again, suppose we had to deduce from the function, $3x^4 + 5x^3 - 2x^2 + 7x - 12$, a series of others in succession, by the following uniform process, viz. each term in the derived function is to be deduced from the corresponding term in the preceding function by multiplying that term by the exponent of x in it, and then diminishing the exponent by unity; the several functions would be as follows:

primitive function, $f(x) = 3x^4 + 5x^3 - 2x^2 + 7x - 12$

1st derived function, $f_1(x) = 12x^3 + 15x^2 - 4x + 7$

2d derived function, $f_2(x) = 36x^2 + 30x - 4$

3d derived function, $f_3(x) = 72x + 30$

4th derived function, $f_4(x) = 72$

This last expression, 72, not containing x, cannot in strictness be regarded as a function of that quantity; its symbolical representation, however, $f_4(x)$, carrying the subscribed numeral 4, informs us that it has arisen from four repetitions of some uniform process to a primitive function, $f(x)$.

If in any function we change the quantity of which it is a function for any other, preserving however the form of the function unaltered, then we must introduce a like change in the abridged representation, merely altering the letter inclosed in the parenthesis, without changing the characteristic outside: thus, if $f(x)$ denote as in the last example, then $f(y)$, $f(a)$, &c. will be the respective representatives of

$$3y^4 + 5y^3 - 2y^2 + 7y - 12, \; 3a^4 + 5a^3 - 2a^2 + 7a - 12, \&c.$$

(16.) By a *rational function* of any quantity is to be understood an algebraical expression into which that quantity enters only in a rational form, that is, without the encumbrance of fractional exponents or radical signs. If the quantity enter the expression with any such appendage, that expression is an *irrational function* of the quantity.

And by an *integral function* of a quantity, is meant an expression into which the quantity enters only in an integral form; that is, it never occurs in the denominator of a fraction: wherever it does so occur, the expression involving it is a *fractional* function of that quantity.

(17.) In thus classifying algebraical functions it is plain that we have regard only to the quantity, or quantities, in reference to which the function is considered; no attention being paid to the forms under which other quantities may enter the expression. The left-hand member of the foregoing equation of the nth degree is a rational and integral function of x, whatever be the constitution of the coefficients with which x is connected.

(18.) The expression *root of an equation* is applied to every quantity, whether real or imaginary, which, when substituted for the unknown, actually reduces the first member of the equation to zero, thus satisfying the condition implied in the equation: so that if there exist p quantities, which when substituted for x in the polynomial $f(x)$, reduce that polynomial to zero, then the equation $f(x) = 0$ has those p quantities for its roots.

(19.) The determination of all the roots is called the *solution* of the equation: and the problem which has for its object the determination of all the roots of an equation by a *general formula*, applicable to all particular values of the coefficients without restriction, aims at the discovery of such a function of those coefficients as will in itself embody all the values of x, both real and imaginary. This is called the problem of the general solution of algebraical equations.

We have already observed that the search after such a function has been attended with complete success only as respects equations of the first and second degrees.

(20.) The problem which furnishes directions for evolving all the *real* roots, one after another, in *numbers*, by aid of the given *numerical coefficients* which any particular equation may offer, is called the problem of the general solution of numerical equations. It is a problem which, like the former, has exercised the talents of the ablest analysts of all countries for the last two hundred years: but the satisfactory completion of it is an achievement of very recent date.

(21.) In the present treatise we propose to ourselves the accomplishment, in moderate space, of the four following primary objects: We shall endeavour first to develop the theory upon which both the general problems just noticed equally depend: secondly, to explain the principles of the numerical solution, in connexion with the recent researches and improvements by which that solution has been perfected; showing the practical efficiency of these principles by their successful application to advanced equations of very considerable difficulty: thirdly, to discuss, with sufficient detail, the other and more general problem, in so far at least as any real approach has been made towards a successful solution of it: and lastly, to blend with these leading and paramount topics, certain collateral and subsidiary enquiries usually expected to have a place in every treatise on the theory of equations.

CHAPTER I.

FUNDAMENTAL PROPERTIES, PREPARATORY TO THE GENERAL THEORY OF EQUATIONS.

(22.) The simple expedient, first adopted by Harriot, of arranging all the significant terms of an equation upon one side of the sign of equality, and leaving merely zero on the other side, has proved a preparatory step of considerable importance in the theory and analysis of equations. In the actual determination of the roots of an equation such a preliminary arrangement of its terms is not always necessary: but in the antecedent examination as to whether the things called roots necessarily exist for every equation; in the search after the number and nature of these roots; their connexion with the coefficients; and, in short, in all enquiries into the structure of equations, the preparation of Harriot must always form the initial step in the investigation.

That every equation has a root, either real or imaginary, is a principle which Harriot and succeeding algebraists have, till lately, assumed. But, as this is the fundamental principle upon which nearly the whole theory of equations is based, it is of importance that it should be firmly established by a rigorous demonstration. Several attempts have accordingly been made, of late, to supply such a demonstration. Of these the most recent, probably the most satisfactory, and unquestionably the most simple and elementary, is by Cauchy; it is that which we shall adopt, in substance, in the present exposition.

We have observed that this principle is the foundation of *nearly* the whole of the present theory of equations. It is proper to make this slight qualification, because two or three interesting propositions belonging to this theory may be readily established without its aid. These it will be convenient to dispose of before entering upon a demonstration of the principle referred to: and it will be farther necessary to establish some preliminary theorems respecting polynomials in general, which theorems are in frequent request in analysis, and are indispensable here as lemmas to the principal proposition.

They are as follow:

PROPOSITION I.

(23.) In any polynomial

$$f(x) = A_1x + A_2x^2 + A_3x^3 + \ldots\ldots A_nx^n$$

in which all the coefficients are real and finite quantities, and which does not contain any term independent of x, it will always be possible to assign a value to x, such as to render the entire expression less in value than any proposed quantity L.

Let A_k be the greatest coefficient without regard to sign; then for every positive value of x, the proposed polynomial will necessarily be less than

$$A_k\,(x + x^2 + x^3 + \ldots\ldots x^n)$$

But if x be less than unity, the series within the parenthesis will only be equal to $\frac{x}{1-x}$, even should it go on to infinity (*Algebra*, art. 76). Hence, when x is less than unity, we must have

$$f(x) < A_k\frac{x}{1-x}$$

Consequently, the proposed condition $f(x) < \mathrm{L}$ will be fulfilled, provided either

$$A_k\frac{x}{1-x} = \mathrm{L}, \text{ or } A_k\frac{x}{1-x} < \mathrm{L}.$$

From the first of these conditions we have

$$x = \frac{\mathrm{L}}{A_k + \mathrm{L}} \ldots\ldots [1]$$

from the second

$$x < \frac{L}{A_k + L} \dots . [2]$$

Hence the condition $f(x) < L$ may always be satisfied.

The student must be careful not to infer from this proposition more than the reasoning warrants. The only object has been to show that *a value* of x [1] exists, for which, and for every smaller value, [2], the condition $f(x) < L$ is necessarily satisfied. But whether or not other values exist, too large to come within the conditions [1], [2], which nevertheless satisfy the inequality $f(x) < L$, we are not authorized to say from anything that is proved above. Generally speaking such other values do exist; but at present we are only interested in the fact that *a value* for x sufficiently small may be assigned such that it and all values below it, down to zero itself, when severally substituted for x, cause $f(x)$ to become smaller than any proposed quantity.

(24.) It is obvious that the preceding demonstration applies to the case in which $f(x)$ is an *infinite series* of the proposed form, n being indefinitely great; provided, as above, that the coefficients are all finite.

Consequently, whether the series be finite or infinite, we may always give to x a value sufficiently small to render any proposed term in it numerically greater than the sum of all the terms which follow: that is, $A_p x^p$ being any proposed term, we may always satisfy the condition

$$A_p x^p > (A_{p+1} x^{p+1} + A_{p+2} x^{p+2} + \dots .)$$

For the series within the parenthesis, which may be written

$$x^p (A_{p+1} x + A_{p+2} x^2 + \dots .)$$

may be rendered less than $x^p L$, L being any finite quantity.

Let L be equal to A_p; then we can fulfil the condition

$$(A_{p+1} x^{p+1} + A_{p+2} x^{p+2} + \dots .) < A_p x^p$$

which is the condition proposed.

PROPOSITION II.

(25.) In any polynomial

$$f(x) = \mathrm{A}_n x^n \ldots\ldots\ldots \mathrm{A}_3 x^3 + \mathrm{A}_2 x^2 + \mathrm{A}x + \mathrm{N}$$

in which the coefficients are all real and finite, it will always be possible to assign to x a value that will render the first term numerically greater than the sum of all the terms which follow.

Let A_k be the greatest coefficient without regard to sign, then for every positive value of x we shall have

$$f(x) - \mathrm{A}_n x^n < \mathrm{A}_k (x^{n-1} \ldots\ldots x^3 + x^2 + x + 1)$$

The first member of this inequality expresses the sum of all the terms after the first: so that in order that the first may exceed this sum, it will be fully sufficient that it exceed the second member: that is, that we have the condition

$$\mathrm{A}_n x^n > \mathrm{A}_k (x^{n-1} \ldots\ldots x^3 + x^2 + x + 1)$$

or, summing the geometric series,

$$\mathrm{A}_n x^n > \mathrm{A}_k \frac{x^n - 1}{x - 1}$$

or,

$$x^n > \frac{\mathrm{A}_k}{\mathrm{A}_n} \frac{x^n - 1}{x - 1}$$

And this is evidently satisfied provided that x be such as to render $x - 1$ either equal to, or greater than, $\frac{\mathrm{A}_k}{\mathrm{A}_n}$. Hence to fulfil the proposed condition we have only to assume x so that

$$x = \text{or} > \frac{\mathrm{A}_k}{\mathrm{A}_n} + 1$$

which we may of course always do.

It thus appears that, the leading term of the polynomial $f(x)$ being positive, we can always give to x a positive value a such that $f(a)$ shall necessarily be positive, whatever be the values or signs of the subsequent coefficients; or however we alter the signs in any proposed case.

Hence if the same value a be taken negatively instead of positively, $f(-a)$ will still be positive, provided n be *even;* because $f(-a)$ will differ from $f(a)$ only as respects the *signs* of the terms after the first. But if n be *odd,* then when $-a$ is put for x the leading term will be negative: and since, as just shown, this leading term will be numerically greater than the sum of all that follow, we infer that in this case $f(-a)$ must be negative.

(26.) If the polynomial considered in the present proposition were the first member of an equation, that is, if we had $f(x)=0$, we might remove the coefficient A_n by division, without disturbing the condition implied in the equation: this is usually done in discussing the properties of equations, for the purpose of avoiding all unnecessary complication in expressing their general forms. Considering A_n to be unity, conformably to this practice, A_k representing that coefficient which is numerically the greatest, as before, we may conclude, from what is shown above—

1. That the first member of the equation $f(x)=0$ will always be positive if for x we put the positive quantity A_k+1 or any greater value.

2. That the first member will in like manner always be positive if for x we put the negative value $-(A_k+1)$, or any negative value still greater, provided the equation be of an *even* degree.

3. And that the first member will be negative for the substitutions in last case provided the equation be of an odd degree.

These conclusions lead to important truths. They show that in every equation of an odd degree two values can always be found, which, when separately substituted for the unknown quantity, will furnish two results with opposite signs; and that in every equation of an even degree, two such values can also be assigned whenever the final term, or absolute number, is *negative.* For in this case the substitution of zero for x will give a *negative* result, viz., the absolute number itself, and the substitution of either $+(A_k+1)$ or $-(A_k+1)$ will give a *positive* result.

From these inferences it may be proved without difficulty that every equation of an odd degree without exception, and every equation of an even degree, provided its final term be negative, must necessarily have a root. This conclusion we might indeed deduce immediately from what has just been established, provided it be conceded that every polynomial $f(x)$, which gives results of opposite signs when two values a, b, are successively given to x, passes from $f(a)$ to $f(b)$ continuously, through all intermediate values, as x passes continuously from a to b: since, if this be admitted, $f(x)$ cannot pass from *plus* to *minus*, or from *minus* to *plus*, without first becoming *zero* for one or more of the values of x intermediate between a and b. But this is a principle that requires demonstration. It is the object of the next proposition to establish it with the necessary rigour.

PROPOSITION III.

(27.) If, in the polynomial

$$f(x) = x^{n} + A_{n-1}\, x^{n-1} \ldots . + A_2\, x^2 + Ax + N$$

x be supposed to vary continuously from $x = a$, to $x = b$, then the function $f(x)$ will vary continuously from $f(a)$ to $f(b)$.

Let a' be any value intermediate between a and b. Substitute $a' + h$ for x in the polynomial, and it will become

$$f(a' + h) = (a' + h)^{n} + A_{n-1}\,(a' + h)^{n-1} \ldots . + A_2\,(a' + h)^2$$
$$+ A\,(a' + h) + N$$

that is, actually developing by the binomial theorem, and arranging the results according to the powers of h,

$$\begin{array}{lll|l|l}
f(a'+h) = & a'^{n} & + n\,a'^{n-1} & h + n(n-1)\,a'^{n-2} & \dfrac{h^2}{2} \ldots . + h^{n} \\
& + A_{n-1}\,a'^{n-1} & + (n-1)\,A_{n-1}\,a'^{n-2} & + (n-1)(n-2)\,A_{n-1}\,a'^{n-3} & \\
& \vdots & \vdots & \vdots & \\
& + A_2\;\, a'^2 & + 2\,A_2\,a' & + 2\,A_2 & \\
& + A a' & + A & & \\
& + N & & &
\end{array}$$

Whence it appears that the development of $f(a' + h)$ will be a series of the form

$$P + Qh + R\frac{h^2}{2} + S\frac{h^3}{2 . 3} + T\frac{h^4}{2 . 3 . 4} + \ldots . h^n,$$

where P, Q, R, &c. stand for the compound coefficients above.

These coefficients, it will be observed, are all *finite* quantities, because the coefficients of the original polynomial, as well as the quantities a and n, are finite. Moreover, the first, P, is evidently no other than the function $f(a')$, the second, Q, is derived from P by multiplying each term of P by the exponent of a' in that term, and then diminishing the exponent by unity; R is derived from Q in a similar manner, and so on in succession, the law of derivation being that already adverted to and illustrated at article 15. Employing then the notation recommended in that article, replacing P by $f(a')$, Q by $f_1(a')$, R by $f_2(a')$, &c. we shall have

$$f(a'+h)=f(a')+f_1(a')h+f_2(a')\frac{h^2}{2}+f_3(a')\frac{h^3}{2.3}+f_4(a')\frac{h^4}{2.3.4}+..h^n$$

the functions of a' which form the coefficients being all finite.

Now by (23) a value so small may be given to h that the sum of the terms after $f(a')$ shall be less than any assignable quantity, however small. Hence, whatever intermediate value a' between a and b be fixed upon for x in $f(x)$, in proceeding to a neighbouring value, by the addition to a' of a quantity h ever so minute, we obtain for $f(a' + h)$ a like minute increase of the preceding value $f(a')$. In other words, in proceeding continuously from a to b, in our substitutions for x, the results of those substitutions must be in like manner continuous, or all connected together without any unoccupied interval; for we have just seen that no such unoccupied interval adjacent to any result $f(a')$ can possibly exist, however small the interval is supposed to be.

PROPOSITION IV.

(28.) If two real quantities be separately substituted for the unknown quantity in any arranged equation, and furnish results having different signs, that is, one plus and the other minus, then that equation must have at least one root of a value intermediate between the values substituted.

What this proposition affirms is this, viz., that if a quantity a be found which, when substituted for x in any rational and integral polynomial $f(x)$, gives a positive result; and another b be found which, when substituted, gives a negative result, then of necessity there exists some *one* value, at least, between the values a and b which, if substituted for x, will render $f(x)$ *zero*, and thus be a root of the equation $f(x) = 0$.

The truth of this immediately follows from the last proposition; since it is there shown that in proceeding continuously from a to b in our substitutions for x, the results—which never become infinite—proceed continuously from $f(a)$ to $f(b)$, leaving no unoccupied interval, but passing through every value between $f(a)$ and $f(b)$. But *zero* is one of these intermediate values, inasmuch as the results *change sign* somewhere in the interval, passing from positive to negative, or from negative to positive, which it is obvious a continuous series of finite quantities can never do without first becoming zero. Hence, there necessarily exists some value between a and b for which $f(x)$ becomes zero; that is, the equation $f(x) = 0$ has a root between a and b.

PROPOSITION V.

(29.) 1. Every equation of an odd degree has at least one real root of a contrary sign to that of its last term.

2. Every equation of an even degree, whose last term is negative, has at least two real roots, one positive and the other negative.

Let the equation be

$$f(x) = x^n + \quad . \ . \ . \ . \quad A_3x^3 + A_2x^2 + Ax + N = 0;$$

and let A_k be the coefficient which has the greatest numerical value, disregarding signs.

Suppose first that n is odd, and that N is *negative*. Then for $x = 0$, $f(x)$ is reduced to the last term, and is therefore *negative*. But for $x = A_k + 1$, $f(x)$ is *positive* (26). Consequently (28) the equation $f(x) = 0$ has at least one real *positive* root between 0 and $A_k + 1$.

Suppose now, n being still odd, that N is *positive*. Then $f(0)$, being as before reduced to the last term, is positive. But (26) $f(-A_k - 1)$ is *negative;* hence, in this case also, the equation has a root (28) comprised between 0 and $-(A_k + 1)$, and therefore *negative*.

Again, let n be even, and N negative. Then, as in the first case, $f(0)$ is *negative*, while both $f(A_k + 1)$ and $f(-A_k - 1)$ are *positive* (26). Consequently the equation $f(x) = 0$ has at least two real roots (28): one a positive root between 0 and $A_k + 1$, and the other a negative root between 0 and $-(A_k + 1)$.

(30.) If this second part of the proposition could be readily generalized like the first part, that is, if we could now prove that an equation of an even degree must have a root, though the final term be *positive*, we might here complete the basis upon which the whole of the subsequent theory of equations is constructed.

But to establish rigorously this particular case of the general proposition is by no means an easy task, although one which must necessarily be accomplished, unless we exclude from the general theory every equation of an even degree whose final term is positive.

If by means of any algebraic transformation, or of any arithmetical operations performed upon the first member of an equation, we could always convert it into another whose final term should be negative, the difficulty would be removed; but although a great variety of changes may in this way be effected upon an equation, yet no transformation can *generally* convert an equation of an even degree, whose last term is positive, into another whose final term shall have a contrary sign. It will be hereafter shown that such a change is impossible. And as the principles hitherto established are inadequate to meet the exigencies of this particular case of the problem, we must have recourse to other considerations and other arguments. These, as remarked at the outset (22), have been supplied by several modern analysts; and among others by CAUCHY,[*] whose investigation, a little improved by STURM,[†] we propose to give in the third chapter. This investigation accomplishes somewhat more than the case actually demands; but it is well for the student to notice the exact amount of difficulty which stands in the way of a complete solution to the fundamental problem by aid of only the ordinary elementary principles.

But, before entering upon the proposed enquiry, we shall offer a few remarks upon the nature and signification of certain imaginary forms which the roots of equations sometimes assume. This seems to be the more necessary, since the answer usually made to the objections brought against expressions of this kind, viz., that the results reached through their aid have always proved valid when submitted to other tests, is far from satisfactory, as it can apply, at farthest, only within the limits of actual experience, and can afford no ground of confidence in any future extension of science which the employment of these expressions may effect. Among the earlier algebraists it was common to reject all but the *positive* roots of an equation; those affected with the negative sign being called *false* roots, and those involving the symbol $\sqrt{-1}$, *imaginary*. The scruples about negative roots have long been removed;[‡] and the few observations which follow, on the other class of expressions, may tend to confirm their claim also to a place among the legitimate instruments of analysis.

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[*] Cours d'Analyse.

[†] Traité d'Algèbre par MAYER et CHOQUET.

[‡] MASERES and FREND were the last writers who stood out against the admission of negative roots. The objections of the former are of frequent recurrence throughout his voluminous productions: those of the latter will be found in his ably written work on *The Principles of Algebra*, 1796, and in his *True Theory of Equations*, 1799.

CHAPTER II.

ON CERTAIN IMAGINARY EXPRESSIONS.

(31.) It has been proved in the preceding chapter, that every rational equation has at least one real root, provided the equation be not of an even degree with its last term positive. This is, in effect, the same as proving that the first member of every equation, with the exception just mentioned, has at least one real binomial factor of the form $x - a$. We have already adverted to the fruitlessness of every attempt that might be made to bring this case of exception under the same general conclusion by resorting to algebraical artifice for the purpose of changing the sign of the final term. No such artifice could succeed except in particular cases, since it is not generally true that an equation of an even degree with the final term positive *has* a real root, or is capable of division by a real binomial factor, without leaving a remainder. Our knowledge of the constitution of equations of the second degree—common quadratics—the theory of which is fully established by elementary algebra, is sufficient to authorize this assertion, since equations of this kind, when the final term is positive, often have only *imaginary roots*. Instead therefore of searching after real *binomial* factors in equations of an even degree with the last term positive, analysts have addressed themselves to the enquiry whether or not every such equation admits of a *real trinomial divisor* of the form $x^2 + px + q$, and thence of at least *two* roots either real or imaginary.

Demonstrations of the necessary existence of such a divisor under all circumstances, have been given by several algebraists. The most satisfactory and conclusive of these are by GAUSS, LAGRANGE, LEGENDRE, and IVORY. But none of them have a character sufficiently elementary to answer the purpose of instruction, as they presuppose a command over the artifices and refinements of analysis but seldom acquired by the student till he has arrived at a stage of his progress far in advance of that at which a proof of the present proposition becomes necessary. The demonstration by CAUCHY, that every equation must have a root of the form $a + b\sqrt{-1}$, where a and b are real, or one of them zero, and from which the necessary existence of a real trinomial factor in every equation belonging to the class referred to easily follows, is much more nearly commensurate with a student's capabilities and previous attainments, and therefore better entitled to a place in any elementary exposition of the theory of which it forms so important a part. It is this which we shall introduce, with some modifications, in the next chapter.

(32.) It will be observed, that the proposition which thus affirms that every equation without exception has a root, of which $a + b\sqrt{-1}$ is the general type, not only announces that every equation has a root either real or imaginary, but expressly declares the unvarying form of the latter. If b be zero, the symbol of impossibility will vanish, and the form will then express a real root, in all other cases the expression continues imaginary and unchanged in form. Such imaginary expressions, however, have often led to controversy, and have by some been altogether rejected from the subject we are now discussing, as involving impossibilities and contradictions irreconcilable with every rational system of algebra. But it should be remembered that these expressions are not the invention of the analyst, arbitrarily and artificially contrived to effect a purpose: he is involuntarily, and unavoidably led to them, by the recognized operations of the science, performed upon, or at least applied to, real quantities; they naturally and necessarily arise out of these operations; and therefore cannot be otherwise than consistent with them, however inexplicable they may seem to be. For example, the operations requisite for the solution of a quadratic equation $ax^2 - bx + c = 0$

are all applied to the real coefficients a, b, c, and are all indicated symbolically by the general expression

$$x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

which spontaneously assumes the reprobated form whenever the relations among the real quantities entering it are such that $b^2 < 4ac$, where a and c are both of the same sign. Under these circumstances the operation implied in the symbol $\surd$ cannot be performed, and to affirm the possibility of its performance would be to contradict established principles. But such an affirmation is never made. Whatever may have been the preparatory process whence the foregoing expression has issued, in proceeding with that expression through the reverse process, we shall undoubtedly arrive at the original equation; in other words, this expression, when substituted for x, renders the first member zero, and is thus entitled to be called a *root*, or rather the *roots*, of that equation. The expression in question, under the circumstances supposed, is never regarded as referring to a final and complete result, in which all the operations implied in it terminate, but as necessarily coming short of such a result by the interposition of an insuperable barrier—the demand of an impracticable process; and this demand, be it remembered, must remain in full force, whatever new operations the expression be submitted to, till it becomes counteracted and neutralized by another of directly opposite import. And it is only thus that imaginary quantities can be rendered available as instruments of investigation in the doctrine of algebraic magnitude in general. Operations which it is not possible to execute are *indicated:* these must be accounted for; and, whenever they disappear in the result of any investigation into which they have once entered, they do so solely by the similar entrance of counteracting operations directly the reverse of the former.

(33.) It is by the repeated application of this same principle of reverse operations that we clear our way, step by step, to the solution of a simple equation, and to many other algebraic results.

Transposition, removing fractions—radical signs—coefficients, &c., are only so many reverse operations suggested at once by the step in which the direct operations occur. The operation of squaring necessarily removes $\sqrt{}$ from $\sqrt{a}$; or, without actually attempting to perform *either* process, we safely write down a, as the necessary result of *both*, and that whether a be positive or negative. And it is in this way that imaginaries become converted into real effective quantities.

(34.) In the higher analysis, imaginary quantities are frequently introduced into exponential, logarithmic, and other transcendental expressions, for the purpose of denoting, in a finite form, certain analytical developments. This employment of them is in some degree conventional, and requires our assent to certain extensions of notation naturally enough suggested by the individual cases before us. These extensions and generalizations of the ordinary notation do not come under consideration here; our present remarks having reference only to the common imaginary form $a + b\sqrt{-1}$, and to its competency, when other operations are combined with the impracticable one implied in it, to yield real results.

There can be no more objection to the employment of the terms addition, subtraction, multiplication, &c., in reference to these expressions, than there is to the use of the same terms in reference to algebraical forms in general. In neither case is the actual execution of the operations necessarily implied, nor mere arithmetical results intended. Nothing more need be meant by these terms, than the simply linking together, by the signs $+$, $-$, $\times$, &c., the expressions to be combined; remembering, however, the offices they are competent to perform when called into activity, but especially observing their neutralizing influences upon one another, with a view to the reduction of the combination to its utmost simplicity of form. The redundancies being thus removed, the simplified result may then be turned over to the processes of common arithmetic, as actually pointed out by the signs of operation. Should any of these imply an impossibility, we then infer at once the impossibility of the arithmetical result sought; a conclusion, however, which must not be come to, till we are sure that account has been taken of all the neutralizing operations.

This is, in fact, as observed above, all that is really done in the greater part of algebraical investigations. Signs of operation are abundantly employed throughout such investigations, which operations are thus *implied*, not *performed*. The final result embodies, and is accountable for, the aggregate of all these. Those that may have neutralized one another on the way leave no trace of their existence; and whether, if separately called into activity, they could have performed their offices or not, is matter of no moment; we are not interested in the enquiry whether an obstacle could have been overcome, which extraneous assistance has removed out of the way.

(35.) The vague denominations *imaginary* and *impossible*, as applied to the peculiar expressions here considered, and which convey no idea of their proper character, nor of their connexion with real quantity, have, no doubt, operated upon some minds in excluding them, as mere creatures of the imagination, from among the instruments of analysis. Yet these expressions are connected with each other, and with the real values furnished by any general result, by the same universally recognized principle that unites all the other particular cases of that result—the principle of *continuity*. The continuous series of real values—values arising from the actual performance of all the operations—may terminate, and be succeeded by a continuous series of expressions still involving an unsatisfied stipulation; these again may arrive at an extreme limit, and there originate another continuous series of real values; but throughout all these changes the law, impressed upon the general formula by the signs of operation which enter it, is uninterruptedly preserved; and is impressed with equal distinctness upon each individual case.

The *law* is continuous throughout, the effects of it are presented to us in different continuous forms claiming a distinct classification, though having, in virtue of the common law, a common bond of connexion.

These circumstances are well exemplified by the application of analysis to curves: at present it will be sufficient to refer for such illustration to the hyperbola.

The specific form which every imaginary result thus takes—every symbol in it, whether of operation or of quantity, being controlled by the same specific conditions as those that govern the real values, gives a character and definiteness to it, of which not the slightest notion is conveyed by the terms *imaginary quantity* and *impossible quantity*. We might in many ways alter the imaginary roots of a quadratic equation without making them in any respect more or less imaginary or impossible; yet any such alteration, however minute, would violate specific conditions, and introduce error. The name *imaginary* or *impossible* is adequate to characterize *fully* only such objects of analytical research as cannot admit of expression, or algebraic representation, by any combination whatever of the symbols of algebra.

For example, the problem which requires the determination of a value or algebraical expression for x that shall fulfil the condition

$$(2x - 5) + \sqrt{x^2 - 7} = 0$$

where the *plus* before the radical sign implies the positive root of $\sqrt{x^2 - 7}$, is strictly impossible. No expression, either real or imaginary, can satisfy the condition, or represent a root of the proposed irrational equation. We shall have occasion to advert again to impossible relations of this kind in the next chapter; but for a full explanation of the circumstances to which such impossibilities are traceable, the student is referred to the treatise on *Algebra,* page 128.

(36.) Our object in this digression has been to convey definite notions respecting a class of analytical expressions of frequent and unavoidable occurrence in the theory of equations. The preceding observations upon the meaning of the imaginary form—its claim to the character of a root, and the wide difference between every imaginary expression and such an algebraic impossibility as that adduced above, may help to place these expressions in a clearer light before the mind of the student; and to give him the same confidence in the real results derived from them that is so readily yielded to all the other deductions of analysis.

(37.) It is a remarkable fact that, however we combine expressions of this kind together, by the operations of addition, subtraction, multiplication, and division, the results are always of the same form as the original expressions.

1. Thus by addition and subtraction,

$$(a \pm b\sqrt{-1}) \pm (a' \pm b'\sqrt{-1}) =$$
$$(a \pm a') \pm (b + b')\sqrt{-1}$$

which is of the form

$$\mathrm{A} + \mathrm{B}\sqrt{-1}$$

like the original expressions: and the same form would of course be preserved if more terms were added or subtracted.

2. By multiplication,

$$(a \pm b\sqrt{-1})(a' \pm b'\sqrt{-1}) =$$
$$(aa' - bb') \pm (a'b + ab')\sqrt{-1}$$

a result, as before, of the form

$$\mathrm{A} + \mathrm{B}\sqrt{-1}$$

3. By division, or rationalizing the denominator,

$$\frac{a \pm b\sqrt{-1}}{a' \pm b'\sqrt{-1}} = \frac{(aa' + bb') \pm (a'b - ab')\sqrt{-1}}{a'^2 + b'^2}$$

which is still of the same form.

From the second of these conclusions we infer that any integral positive power of $a + b\sqrt{-1}$ is of the same form; so that, taking account of the first conclusion also, it follows that if

$a + b\sqrt{-1}$ be substituted for x in any rational and integral function of x, as for instance in the first member of an algebraic equation, the result will always be of the form $P + Q\sqrt{-1}$, whether the coefficients of x in the proposed polynomial be real, or imaginary quantities of the form $A + B\sqrt{-1}$.

This is a conclusion of importance. It might, if necessary, be rendered more comprehensive. In fact the third inference above justifies our affirming that the form would remain the same though *negative* integral powers entered the function; since these negative powers might each be replaced by unity divided by positive powers. And it will be an easy inference from the chief proposition in next chapter, that the same form is reproduced when the power is fractional;* so that whatever ordinary algebraic operations be performed upon the quantity $a + b\sqrt{-1}$ we are invariably conducted to the same form. The conclusion has indeed been extended even further than this, and operations not within the limits of ordinary algebra have been shown still to terminate in the same form. These general views were first propounded by D'ALEMBERT: a brief account of them will be found in the *Traité de la Résolution*, &c. of LAGRANGE, Note ix.

(38.) We know from the theory of quadratic equations that if one root of such an equation be of the form $a + b\sqrt{-1}$, there must necessarily be another, differing from it only in the sign which connects the imaginary part with the real, that is, there must be another root of the form $a - b\sqrt{-1}$. We shall hereafter find that this peculiarity has place in all equations. Such roots or expressions are called *conjugate* roots, or *conjugate* expressions: and we thus say that the roots of a quadratic equation, when imaginary, are *conjugate*.

Another term has been introduced by CAUCHY into the arithmetic of imaginary quantities, the term *modulus* which it will be convenient to define here.

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* This inference also follows from the application of the binomial theorem to the proposed expression; but the form thence deduced is not finite. (*Algebra*, page 197.)

The *modulus* of an imaginary quantity, $a + b\sqrt{-1}$, is the expression, $\sqrt{a^2 + b^2}$, formed by taking the square root of the sum of the squares of the real quantities which enter it. For example $\sqrt{9 + 16}$ or 5 is the modulus of $3 - 4\sqrt{-1}$; the same is also the modulus of $3 + 4\sqrt{-1}$. Thus two conjugate expressions have the same modulus.

(39.) The following properties respecting these moduli will be found useful in next chapter.

1. The sum of two quantities has a modulus comprised between the sum and difference of the moduli of the quantities themselves.

Let the two quantities be

$$a + b\sqrt{-1},\ a' + b'\sqrt{-1}$$

and let r, r' represent their moduli; that is, let

$$r^2 = a^2 + b^2,\ r'^2 = a'^2 + b'^2$$

Let also R be the modulus of the sum of the proposed quantities; then we shall evidently have

$$\begin{aligned} R^2 &= (a + a')^2 + (b + b')^2 \\ &= a^2 + a'^2 + b^2 + b'^2 + 2(aa' + bb') \\ &= r^2 + r'^2 + 2(aa' + bb') \end{aligned}$$

Now multiplying r^2, r'^2 together, we have

$$\begin{aligned} r^2 r'^2 &= a^2 a'^2 + b^2 b'^2 + a^2 b'^2 + a'^2 b^2 \\ &= (aa' + bb')^2 + (ab' - ba')^2 \end{aligned}$$

Hence the numerical value of $aa' + bb'$ must be less than, or at most equal to rr'; and consequently R^2 must be comprised between

$$r^2 + r'^2 + 2rr' \text{ and } r^2 + r'^2 - 2rr'$$

or, which is the same thing, between

$$(r + r')^2 \text{ and } (r - r')^2$$

Therefore the modulus R is comprised between the sum and difference of the moduli r, r'; and can never be *less* than the difference.

2. The product of two quantities has for modulus the product of their moduli.

For by multiplication

$$(a + b\sqrt{-1})\,(a' + b'\sqrt{-1}) =$$

$$aa' - bb' + (ab' + ba')\sqrt{-1}$$

and taking the modulus of this result, we have

$$\sqrt{\{\,(aa' - bb')^2 + (ab' + ba')^2\,\}} =$$

$$\sqrt{\{\,a^2a'^2 + b^2b'^2 + a^2b'^2 + b^2a'^2\,\}} =$$

$$\sqrt{\{\,(a^2 + b^2)\,(a'^2 + b'^2)\,\}}$$

and the moduli of the original expressions are

$$\sqrt{a^2 + b^2} \text{ and } \sqrt{a'^2 + b'^2}$$

Hence the product of any number of factors must have for modulus the product of all the moduli of those factors; so that when the factors are all equal, and in number n, we may express this conclusion by saying that the nth power of an imaginary quantity has for modulus the nth power of the modulus of that quantity itself.

3. The quotient of two quantities has for modulus the quotient of the modulus of the dividend by the modulus of the divisor.

For by division,

$$\frac{a + b\sqrt{-1}}{a' + b'\sqrt{-1}} = \frac{(aa' + bb') + (a'b - ab')\sqrt{-1}}{a'^2 + b'^2}$$

The square of the modulus of this last expression is

$$\frac{(a^2a'^2 + b^2b'^2) + (a'^2b^2 + a^2b'^2)}{(a'^2 + b'^2)^2} = \frac{(a'^2 + b'^2)\,(a^2 + b^2)}{(a'^2 + b'^2)^2} =$$

$$\frac{a^2 + b^2}{a'^2 + b'^2}$$

which is the square of the modulus of the dividend divided by the square of the modulus of the divisor.

It may be proper to add to what is here said respecting moduli, that the *absolute values* only of the expressions so called are recognized, signs being disregarded: that the modulus of a *real* quantity is the absolute value of that quantity itself; and that, in order for an imaginary expression to become zero, it is necessary and sufficient that its modulus be zero. For $a + b\sqrt{-1}$ cannot be zero unless both $a = 0$ and $b = 0$; and in these circumstances, and in these only, can $\sqrt{a^2 + b^2}$ or $a^2 + b^2$ be zero.

It is interesting further to notice, that the modulus of $a + b\sqrt{-1}$ is no other than the expression for the radius of the circle, in reference to which b and a are the respective sine and cosine of the same arc θ. For putting $a = \mathrm{R}\cos\theta$, and $b = \mathrm{R}\sin\theta$, we have

$$a + b\sqrt{-1} = \mathrm{R}\,(\cos\theta + \sin\theta\sqrt{-1}) \text{ and } \sqrt{a^2 + b^2} = \mathrm{R}.$$

(40.) We shall conclude these remarks with the following theorem, also from Cauchy, which shows, in a remarkable manner, the efficiency of imaginary expressions as instruments in the investigation of the properties of real quantities.

If two numbers, of which each is the sum of two squares, be multiplied together, the product must be also the sum of two squares.

Let the two numbers be

$$a^2 + b^2 \text{ and } a'^2 + b'^2.$$

The first of these may be considered as the product of the factors

$$a + b\sqrt{-1}, \text{ and } a - b\sqrt{-1},$$

and the second as the product of the factors

$$a' + b' \sqrt{-1}, \text{ and } a' - b' \sqrt{-1},$$

so that the product of the proposed numbers will be the product of the four factors

$$a + b \sqrt{-1},\ a - b \sqrt{-1},\ a' + b' \sqrt{-1},\ a' - b' \sqrt{-1}.$$

Actually multiplying the first and third, and then the second and fourth, we have the following pair of *conjugate* expressions, viz.

$$(aa' - bb') + (ab' + ba') \sqrt{-1},\ (aa' - bb') - (ab' + ba') \sqrt{-1},$$

of which the product is

$$(aa' - bb')^2 + (ab' + ba')^2,$$

which is therefore equal to the product of the original numbers; and proves that that product must, like each of the proposed factors, be the sum of two squares.

If we interchange the numbers a and b, or the numbers a', b', the terms of the product just deduced will be different: thus putting a' for b', and b' for a', which produces no essential change in the proposed numbers, we have

$$(a^2+b^2)(a'^2+b'^2)=(aa'-bb')^2+(ab'+ba')^2=(ab'-ba')^2+(aa'+bb')^2$$

Consequently there are two ways of expressing, by the sum of two squares, the product of two numbers, each of which is itself the sum of two squares, thus:

$$(5^2 + 2^2)\ (3^2 + 2^2) = 11^2 + 16^2 = 4^2 + 19^2$$

$$(2^2 + 1^2)\ (3^2 + 2^2) = 4^2 + 7^2 = 1^2 + 8^2$$

&c. &c.

CHAPTER III.

ON THE PROPERTY THAT EVERY EQUATION HAS A ROOT.

(41.) In order to demonstrate the principal proposition of the present chapter in the most general manner, it will be convenient first to consider a particular and very simple case of it; the case namely in which the equation is of the form $x^m \pm N = 0$, where N may be limited to the values 1 and $\sqrt{-1}$.* We shall give to this preparatory step the form of a lemma.

PROPOSITION I. LEMMA.

Each of the equations

$$x^m = \pm 1, \qquad x^m = \pm \sqrt{-1}$$

has a root comprehended in the general form $a + b\sqrt{-1}$.

This is evidently the case with respect to the equation $x^m = +1$, whether the number m be even or odd; since $x = 1$ always satisfies it. It is also as plainly true of the equation $x^m = -1$ when m is odd, because then $x = -1$ satisfies it.

When m is even, it must either be some power of 2, or else some power of 2 multiplied by an odd number; if it be a power of 2, then the value of x will be obtained after the extraction of the square root repeated as many times in succession as there are units in the said power. Now the square root of the form

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* The previous consideration of this latter value is not absolutely necessary, but it may be included without adding much to the length or difficulty of the argument.

$a+b\sqrt{-1}$ is always of the same form.* Hence when m is a power of 2, each of the equations $x^{m} = -1$, $x^{m} = \pm\sqrt{-1}$ has a root of the form announced. When m is a power of 2 multiplied by an odd number, then, if we extract the root of this odd degree first, there will remain to be extracted only a succession of square roots.

We have therefore merely to show, that when m is odd, a root of $\pm\sqrt{-1}$ is of the predicted form.

Now the odd *powers*, 1, 3, 5, &c. of $+\sqrt{-1}$, are

$$+\sqrt{-1},\ -\sqrt{-1},\ +\sqrt{-1},\ \ldots.$$

and the same powers of $-\sqrt{-1}$ are

$$-\sqrt{-1},\ +\sqrt{-1},\ -\sqrt{-1},\ \ldots.$$

Consequently, when m is odd, a *root* of $\pm\sqrt{-1}$ is either $+\sqrt{-1}$ or $-\sqrt{-1}$. Hence the predicted form occurs whether m be odd or even.

It follows from this proposition that whatever positive whole number m may be, $(-1)^{\frac{1}{m}}$ and $(\sqrt{-1})^{\frac{1}{m}}$ will always be of the form $a+b\sqrt{-1}$; or more generally $(-1)^{\frac{n}{m}}$ and $(\sqrt{-1})^{\frac{n}{m}}$ will always be of this form, n and m being any integers positive or negative (37).

PROPOSITION II.

(42.) Every algebraical equation, of whatever degree, has a root of the form $a+b\sqrt{-1}$, whether the coefficients of the equation be all real, or any of them imaginary and of the same form.

Let

$$f(x) = x^{n} + A_{n-1}x^{n-1} + \ldots A_{3}x^{3} + A_{2}x^{2} + A_{1}x + N = 0 \ \ldots \ [1]$$

represent any equation the coefficients of which are either real or imaginary.

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* Algebra, p. 113.

If in this equation we substitute $p + q\sqrt{-1}$ for x, p and q being real, the first member will furnish a result of the form $P + Q\sqrt{-1}$, P and Q being real (37). Should $p + q\sqrt{-1}$ be a root of the equation, this result must be zero; or, which is the same thing, the modulus of $P + Q\sqrt{-1}$, viz. $\sqrt{P^2 + Q^2}$, must be zero (39). And we have now to prove that values of p and q always exist that will fulfil this latter condition.

In order to this it will be sufficient to show that whatever value of $\sqrt{P^2 + Q^2}$, greater than zero, arises from any proposed values of p and q, other values of p and q necessarily exist, for which $\sqrt{P^2 + Q^2}$ becomes still smaller; so that the *smallest* value of which $\sqrt{P^2 + Q^2}$ is capable must be zero; and the particular expression $p + q\sqrt{-1}$, whence this value has arisen, must be a root of the equation.

For the purpose of examining the effect upon any function $f(x)$ of changes introduced into the value of x, the development exhibited at (27) is very convenient. By changing x into $x + h$ the altered value of the function is thus expressed by

$$f(x+h) = f(x) + f_1(x)h + f_2(x)\frac{h^2}{1\,.\,2} + f_3(x)\frac{h^3}{1\,.\,2\,.\,3} \ldots h^n \quad [2]$$

where $f(x)$ is the original polynomial, and $f_1(x)$, $f_2(x)$ &c. contain none but integral and positive powers of x (27).

The first of these functions $f(x)$ becomes $P + Q\sqrt{-1}$ when $p + q\sqrt{-1}$ is substituted for x; the other functions may some of them vanish for the same substitution, for aught we know to the contrary; but *all* the terms after $f(x)$ cannot vanish; the last h^n, which does not contain x, must necessarily remain.

Without assuming any hypothesis as to what terms of $f(x+h)$ vanish for the value $x = p + q\sqrt{-1}$ which causes the first of those terms, $f(x)$, to become $P + Q\sqrt{-1}$, let us represent by h^m the *least* power of h for which the coefficient does not vanish when $p + q\sqrt{-1}$ is put for x. This coefficient will be of the form $R + S\sqrt{-1}$, in which R and S cannot *both* be zero.

When $p + q\sqrt{-1}$ is put for x, we have represented $f(x)$ by

$P+Q\sqrt{-1}$. In like manner, when $p+q\sqrt{-1}+h$ is put for x we may represent the function by $P'+Q\sqrt{-1}$. The development [2] will then be

$$P'+Q'\sqrt{-1}=(P+Q\sqrt{-1})+(R+S\sqrt{-1})h^{m}+\text{terms in } h^{m+1},\ h^{m+2},\ \ldots\ h^{n}.$$

Now h is quite arbitrary:—we may give to it any sign and any value we please, provided only it come under the general form $a+b\sqrt{-1}$. Leaving the *absolute value* still arbitrary, we may therefore replace it by either $+k$ or $-k$, or $\pm(-1)^{\frac{1}{m}}k$; and thus render h^{m} either positive or negative, whichever we please, whatever be the value of m; and we have seen that $(-1)^{\frac{1}{m}}$ comes within the stipulated form (41). Hence we may write the foregoing development thus, the sign of k^{m} being under our own control:

$$P'+Q'\sqrt{-1}=(P+Q\sqrt{-1})+(R+S\sqrt{-1})k^{m}+\text{terms in } k^{m+1},\ k^{m+2},\ \ldots\ k^{n}.$$

But in any equation of this kind the real terms in one member are together equal to those in the other; and the imaginary terms in one to the imaginary terms in the other. Consequently,

$$P'=P+Rk^{m}+\text{the } \textit{real} \text{ terms in } k^{m+1},\ k^{m+2},\ \ldots\ k^{n}$$

$$Q'=Q+Sk^{m}+\textit{real} \text{ terms involving powers above } k^{m}.$$

Hence the square of the modulus of $P'+Q'\sqrt{-1}$ is

$$P'^{2}+Q'^{2}=P^{2}+Q^{2}+2(PR+QS)k^{m}+\text{real terms in } k^{m+1},\ k^{m+2}..k^{2n}$$

Now k may be taken so small that the sum of all the terms after $P^{2}+Q^{2}$ may take the same sign as $2(PR+QS)k^{m}$ by (25), which sign we can always render negative whatever $PR+QS$ may be, because, as observed above, k^{m} may be made either positive or negative, as we please.

Hence we can always render

$$P'^2 + Q'^2 < P^2 + Q^2, \text{ or } \sqrt{P'^2 + Q'^2} < \sqrt{P^2 + Q^2}.$$

In other words, whatever values of p and q, in the expression $p + q\sqrt{-1}$, cause the modulus $\sqrt{P^2 + Q^2}$ to exceed zero, other values exist for which the modulus will become smaller; and consequently one case at least must exist, for which the modulus, and consequently the expression $P + Q\sqrt{-1}$, must become zero.

This conclusion presumes however that $PR + QS$ is not zero. If such should be the case, then our having chosen the form of h as to secure a command over the *sign* of $2(PR + QS)$ will have been unnecessary. The form must then be so chosen that a command may be secured over the sign of the first term *after* $2(PR + QS)k^m$, in the above series for $P'^2 + Q'^2$, which does not vanish; when the preceding conclusion will follow.

PROPOSITION III.

(43.) The values of a and b in the expression $a + b\sqrt{-1}$, which when put for x in $f(x)$ cause that polynomial to vanish, can never be infinite.

We may write $f(x)$ as follows, viz.,

$$f(x) = x^n \left(1 + \frac{A_{n-1}}{x} + \frac{A_{n-2}}{x^2} + \dots . \frac{N}{x^n}\right)$$

or putting $P + Q\sqrt{-1}$ for what $f(x)$ becomes, when $p + q\sqrt{-1}$ is substituted for x, we have

$$P + Q\sqrt{-1} =$$

$$(p+q\sqrt{-1})^n\left(1+\frac{A_{n-1}}{p+q\sqrt{-1}}+\frac{A_{n-2}}{(p+q\sqrt{-1})^2}+\dots.\frac{N}{(p+q\sqrt{-1})^n}\right)$$

Now the modulus of a quotient is the quotient of the modulus of the dividend by the modulus of the divisor (39). In each of the dividends A_{n-1}, A_{n-2}, &c. above, the modulus is finite by hypothesis. Hence if either p or q be infinite, and consequently the modulus of every denominator or divisor, also infinite, the modulus of each quotient must be *zero*. Hence in this case each of the above fractions must itself be zero (39), and therefore the modulus of the entire quantity within the parenthesis simply 1; and the modulus of a product is the product of the moduli of the factors, so that the modulus of the preceding product, viz., $\sqrt{P^2+Q^2}$, is the modulus of $(p+q\sqrt{-1})^n$. But the nth power of $p+q\sqrt{-1}$ has for modulus the nth power of the modulus of $p+q\sqrt{-1}$, that is, the nth power of $\sqrt{p^2+q^2}$ (39) which is infinite: consequently $\sqrt{P^2+Q^2}$ must be infinite. But when $p+q\sqrt{-1}$ is a root of the equation $f(x)=0$, $\sqrt{P^2+Q^2}$ is zero. Hence in this case neither p nor q can be infinite.

(44.) An objection may be brought against the preceding reasoning that ought not to be concealed. It may be denied that the modulus of the product above referred to is simply the modulus of $(p+q\sqrt{-1})^n$ in the case of p or q infinite; for it may be maintained, that although in this case all the quantities within the parenthesis after the 1 become *zero*, yet the combination of these with $(p+q\sqrt{-1})^n$, which involves *infinite* quantities, *may* produce quantities also infinite; and thus the modulus of the product may differ from the modulus of $(p+q\sqrt{-1})^n$ by a quantity infinitely great. It is not to be denied that there is weight in this objection. But it is not difficult to see that although the true modulus may thus differ from the modulus of $(p+q\sqrt{-1})^n$ by an infinite quantity, yet the modulus of $(p+q\sqrt{-1})^n$, involving higher powers than enter into the part neglected, is infinitely greater than that part. This part therefore is justly regarded as nothing in comparison to the part preserved, the former standing in relation to the latter as a finite quantity to infinity.

But the proposition may be established somewhat differently, as follows:

Substituting $(p+q\sqrt{-1})$ for x in $f(x)$, we have

$$P+Q\sqrt{-1}=$$

$$(p+q\sqrt{-1})^n+A_{n-1}(p+q\sqrt{-1})^{n-1}+\ldots A_1(p+q\sqrt{-1})+N.$$

Call the agregate of all these terms after the first, $P'+Q'\sqrt{-1}$; then it is plain that the modulus of the first term, that is, $(\sqrt{p^2+q^2})^n$, must infinitely exceed the modulus $\sqrt{P'^2+Q'^2}$, of the remaining terms whenever p or q is infinite; because in this latter modulus so high a power of the infinite quantity p or q cannot enter as enters into the former. Now the modulus of the whole expression, that is, of the sum of $(p+q\sqrt{-1})^n$ and $P'+Q'\sqrt{-1}$, is not less than the difference of the moduli of these quantities themselves (39), which difference is infinite. Hence, as before, $\sqrt{P^2+Q^2}$ must be infinite when p or q is infinite.

(45.) Several different versions are given by the continental algebraists of Cauchy's argument to prove the existence of a root for every equation. Cauchy himself has given two distinct forms to his demonstration: one in the 18th number of the *Journal de l'Ecole Polytechnique,* and the other in the 10th chapter of his *Cours d'Analyse.* Of the other modifications that have been proposed, that contained in the *Algèbre* of Lefebure de Fourcy, and the one in the treatise of Mayer et Choquet, for which the authors state themselves to be indebted to Sturm, are to be preferred on the ground of involving fewer perplexities than the other demonstrations.*

The method of investigation here employed has been mainly modelled upon this last form of proof; but it differs from it in many respects; and from the departure thus made we think that the reasoning has not only been comprised in a smaller compass, but that it has also received additional simplicity.

(46). It will have been noticed that this reasoning proceeds entirely upon the hypothesis that the equation is rational as respects the unknown quantity x; the coefficients may be rational or not; the only condition with respect to them being that they

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* The student may also consult with advantage, Peacock's View of the Present State of Analysis, in the "Report of the Third Meeting of the British Association." pp. 297-305.

must come under the form $a + b\sqrt{-1}$, where a and b are real and finite, but either rational or irrational.

We have already adduced an example of an irrational equation which has *no root* (35); and it would be very easy to multiply such examples. Mr. **Horner**, in a paper in the *Philosophical Magazine*,[*] has commented at some length on equations of this kind. But the peculiarities there dwelt upon have been often remarked by preceding writers; though very erroneous doctrines have sometimes been held respecting them. **Garnier**[†] has instanced several examples of this kind, and has properly accounted for the *foreign roots* which are found to be involved in the equation when rendered rational. Thus the equation

$$-\sqrt{x-1} = 1 - \sqrt{x-4}$$

when rendered rational, gives $x = 5$. But this value, when put for x, fails to satisfy the equation; the equation really satisfied by it is

$$+\sqrt{x-1} = 1 + \sqrt{x-4}$$

and no value or expression whatever can possibly satisfy the proposed equation: in other words it has no root.

The mistake, however, of confounding the rational equation with the irrational from which it has been deduced, is often committed; and the roots of the former erroneously attributed to the latter equation. **Rolle**, in his attacks upon the Differential Calculus, fell into this error more than once, from disregarding the restrictions which he himself had imposed upon the signs of his irrational terms.[‡]

When the signs thus prefixed to the radicals which enter an equation are regarded merely as links to connect the several terms together, and as exercising no control over the symbol of irrationality, then it will involve no more restrictions than the rational equation deduced from it; and the roots of the one will be

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[*] Phil. Mag., 1836.

[†] Analyse Algébrique, 1814, p. 335.

[‡] See **Montucla**, Histoire des Mathematiques, tom. iii, p. 111.

also the roots of the other. Some writers have contended for this view of the office of the signs in all such circumstances. But it is at variance with the general principle, conformably to which, analytical conditions are expressed; and when adopted, should be done so by special agreement in reference to the enquiry in hand. MONTUCLA, in animadverting upon the errors of ROLLE, fully recognizes the propriety of his original restrictions as to the signs of the irrational terms; and in accordance with the common practice of analysts always considers the symbol of irrationality, when forming part of the data of a problem, as controlled by the plus or minus sign prefixed to it. Yet when this symbol is *introduced* in the course of an algebraic investigation, and is thus unrestricted by the original conditions of the problem, the ambiguous sign is always to be regarded as involved in it.

We must direct the student for a more complete discussion of the present topic to the source already pointed out.* It was proper thus briefly to invite his attention to it here, in order that he might clearly see the restrictions under which the general principle established in the present chapter is to be received; and be furnished with the reasons which render it necessary, in discussing the theory of equations—of which theory the principle adverted to is the foundation—that *irrational* equations be excluded, or at least that they be rendered rational by a preparatory process.

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* Algebra, Third Edition, p. 128.

CHAPTER IV.

ON THE GENERAL PROPERTIES OF EQUATIONS.

(47.) THE properties to be developed in the present chapter are those which belong to all rational equations containing but one unknown quantity, without regard to the degree of the equation, and generally without any stipulation as to whether the coefficients be real or imaginary. Whenever the property announced requires that the coefficients be real, we shall introduce the necessary restriction into the enunciation of it; otherwise we shall consider them to be indifferently either real or imaginary. In our occasional numerical illustration of particular truths we shall however always choose examples with *real* coefficients; because in actual practice they generally present themselves in a real form. Moreover, we shall, for simplicity sake, usually suppress the coefficient of the highest power of the unknown quantity, or rather shall regard this coefficient as unity, retaining it in a general form only when any practical operation to be deduced from the investigation would seem to be limited in generality by suppressing it. It is plain that the coefficient of the highest power may in all cases be reduced to unity by division; and that the new coefficients, thence resulting, will still preserve the prescribed form (39).

PROPOSITION I.

(48.) If a be a root of any equation

$$f(x) = \mathrm{A}_n x^n + \;.\;.\;.\;.\; + \mathrm{A}_3 x^3 + \mathrm{A}_2 x^2 + \mathrm{A}x + \mathrm{N} = 0,$$

then the first member of it, $f(x)$, will necessarily be divisible by $x - a$. And conversely, if the polynomial $f(x)$ be divisible by the binomial $x - a$, then a will be a root of the equation $f(x) = 0$.

1. Let the division of $f(x)$ by $x - a$ be performed. Then, as the divisor contains only the first power of x, the operation will evidently proceed till we arrive at a remainder independent of x; and the quotient, like the dividend, can contain only integral and positive powers of x. Call this quotient Q and the remainder R; then we have

$$f(x) = (x - a)\,Q + R.$$

Now if a be a root of the equation, the first member of this identity must become zero when a is put for x; and consequently the second member also. But when a is put for x, the second member is reduced to R; therefore R must be zero; that is, if a be a root, $x - a$ will divide the polynomial without leaving a remainder.

2. Again, let $x - a$ divide $f(x)$ without leaving a remainder; then will a be a root of the equation $f(x) = 0$.

For, calling the quotient Q, which can contain none but positive integral powers of x, we have the identity

$$f(x) = (x - a)\,Q.$$

Put a for x in the second member, and it becomes reduced to zero; consequently the same substitution for x reduces the first member to zero; that is, a is a root of the equation $f(x) = 0$.

(49.) The actual operation of dividing the first member of an equation by a binomial of the form $x - a$, whether a be a root of that equation or not, is one of very considerable importance in the process of numerical solution. It is plainly an operation of the simplest character, belonging to the first rudiments of algebra, and requiring little or no ingenuity or address in the performance of it. Nevertheless, it is to an improved and more compact form of arranging the elements of this simple operation that the perfection at present attained in the solution of numerical equations of the higher degrees, is in a great measure attributable.

Many of the accessions which, from time to time, abstract science has received, are clearly traceable to previous changes and improvements in the mere symbols of operation. A compact and significant notation has been a fertile source of modern analytical discovery; and the recent advances in the solution of numerical equations show, in a striking manner, how much mere arithmetical arrangement may effect in facilitating the practical development of a complicated theory. But for a few innovations upon the common methods of performing certain simple numerical operations; it is probable that the solution of equations—as far as practicability is concerned—would be now in the very state in which Newton and his immediate disciples left the problem.

(50.) In the process for dividing $f(x)$ by $x - a$, viewed in reference to its connexion with this important problem of the general solution of equations, it is the *remainder* that is more the object of search than the quotient; the determination of the quotient, however, always forms part of the operation for finding the remainder in the arrangement here to be recommended, although the remainder *may* be formed very readily without reference to this quotient. Thus calling the quotient Q, and the remainder R, as before, we have

$$A_n x^n + \ . \ . \ . \ . + A_3 x^3 + A_2 x^2 + Ax + N = (x - a)\,Q + R,$$

and if in this identity we put a for x, we get the remainder at once, viz.

$$A_n a^n + \ . \ . \ . \ . + A_3 a^3 + A_2 a^2 + Aa + N = R \ . \ . \ . \ . \ [1],$$

showing, what is worthy of observation, that the remainder arising from dividing the polynomial by $x - a$ will always be exhibited by simply changing, in that polynomial, x for a. As to the quotient Q, it will, as remarked above, contain only positive integral powers of x; and it will moreover be in degree a unit lower than the proposed polynomial. In other words, if $f(x)$ be of the fifth degree, Q will be of the fourth; and if $f(x)$ be of the fourth, Q will be of the third; and so on. For simplicity we shall assume the polynomial to be of the fifth degree. It will be readily seen that the generality of the reasoning is not in the least impaired by this assumption. Putting then

$$f(x) = A_5x^5 + A_4x^4 + A_3x^3 + A_2x^2 + Ax + N,$$

the quotient Q will be a polynomial of the form

$$Q = A'_4x^4 + A'_3x^3 + A'_2x^2 + A'x + N',$$

from which we may return to the dividend, or original function $f(x)$, by multiplying it by $x - a$, and then adding R to the product; that is to say, upon actually executing this multiplication, we shall have the identity

$$\begin{array}{l} A'_4x^5+(A'_3-aA'_4)x^4+(A'_2-aA'_3)x^3+(A'-aA'_2)x^2+(N'-aA')x-aN'+R \\ =A_5x^5 \quad + \quad A_4x^4 \quad + \quad A_3x^3 \quad + \quad A_2x^2 \quad + \quad Ax + N \end{array}$$

so that by equating the coefficients of the like powers of x, we shall have the relations which connect together the coefficients of the dividend and those of the quotient; and from which the latter may all be determined from the former without going through the usual process of actual division. The relations are as follow:

$$\begin{array}{llll} & & & A'_4 = A_5 \\ & A'_3 - aA'_4 = A_4; & \text{consequently,} & A'_3 = aA'_4 + A_4 \\ & A'_2 - aA'_3 = A_3 & ,, & A'_2 = aA'_3 + A_3 \\ & A' - aA'_2 = A_2 & ,, & A' = aA'_2 + A_2 \\ & N' - aA' = A & ,, & N' = aA' + A \\ \text{also} & R - aN' = N & ,, & R = aN' + N. \end{array}$$

From the equations on the right, it appears that the first coefficient A'_4, in Q, will be the same as the first in $f(x)$; that the second will be found by multiplying A'_4 by a, and adding the second in $f(x)$; and that, generally, every coefficient in Q will be derived by the same uniform process of multiplying the preceding coefficient of Q by a, and adding to the product the corresponding coefficient in $f(x)$; which process, extended up to N′, the coefficient of x^0 in Q, furnishes also the remainder R.

(51.) Thus is the operation of division, which sometimes by the common method extends itself to a tedious length, reduced to a form of remarkable compactness and simplicity. We shall have merely to arrange the coefficients of the dividend in a horizontal row, to multiply the first by a, and add in the second; then to multiply this result by a, adding in the third; then again this new result by a, adding in the fourth; and so on to the end. The last result will be the remainder; the preceding results will be the several coefficients of the quotient in order, with their proper signs, commencing with the second coefficient; the first being the same as the first number in the horizontal row with which we commence. We shall give an example or two for illustration; premising that when terms are absent from the proposed polynomial, that is, when the expression is *incomplete*, the absent terms must be supplied by *zeros*, and must occupy their proper places in the row of coefficients.

1. Required the quotient and remainder arising from the division of the polynomial,

$$x^5 + 7x^4 + 3x^3 + 17x^2 + 10x - 14;$$

by the binomial $x - 4$.

$$\begin{array}{rrrrrr} 1 & +\,7 & +\,3 & +\,17 & +\,10 & -\,14 \\ & 4 & +\,44 & +\,188 & +\,820 & +\,3320 \\ \hline & 11 & +\,47 & +\,205 & +\,830 & +\,3306. \end{array}$$

Hence the quotient is

$$x^4 + 11x^3 + 47x^2 + 205x + 830,$$

and the remainder $+ 3306$.

2. Required the quotient and remainder arising from the division of

$$3x^6 - 6x^4 + 4x^3 - x - 45624,$$

by the binomial $x + 5$.

Supplying the vacant terms in this *incomplete* expression, we have

$$\begin{array}{r}
3 + \ \ 0 - \ \ 6 \ \ \ + 4 \ \ \ \ \ + 0 \ \ \ \ \ - 1 \ - 45624 \\
- 15 + 75 - 345 + 1705 - 8525 + 42630 \\
\hline
- 15 + 69 - 341 + 1705 - 8526 - \ \ 2994
\end{array}$$

Hence the quotient is

$$3x^5 - 15x^4 + 69x^3 - 341x^2 + 1705x - 8526,$$

and the remainder — 2994.

(52.) It is this simple process for obtaining the coefficients of the quotient, and thence the remainder, due to the division of a polynomial $f(x)$ by a binomial $x - a$, that gives to Horner's method of approximating to the roots of numerical equations much of that practical facility which distinguishes it from all other operations for the same purpose. The theoretical principle upon which it depends is too obvious not to have been long known before the discovery of the method referred to, and it is accordingly adverted to by several preceding writers. It is distinctly enough stated by Garnier, in his *Elemens d'Algèbre*, 1811, p. 399, and still more so by Francœur, in his *Mathématiques*, 1819, tom. II. p. 37; but its practical bearing upon the solution of equations was not observed till the publication of Horner's researches.* Had Newton adopted, in his own method of approximation, the numerical arrangement suggested by the foregoing proposition, Horner's more perfect mode of proceeding would have been obviously so small an advance upon it, that it could scarcely have escaped being suggested to the minds of some

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* Mr. Horner's original investigation was published in the *Philosophical Transactions* for 1819. But it involved reasonings so intricate, and made so large a demand upon the more recondite departments of analysis, that, notwithstanding the subsequent modification of certain parts of it in *Leybourn's Repository*, it attracted but little notice from mathematicians, and seemed likely to fall into unmerited neglect. We believe that the investigation owes the simple and elementary form in which it now appears, principally to the author of the present Treatise.

of the many persons who aimed at improving the Newtonian operation. This will be seen when we come to compare the two methods.

(53.) The expression [1], deduced at (50) for the remainder arising from the division of a rational and integral polynomial $f(x)$ by $x - a$, shows that the computation of this remainder is the same thing as the computation of the value of the polynomial itself for the particular value a of x: so that the operation described above, for finding the *remainder*, is that which furnishes the value of the polynomial for a given value of x.

Thus, in the first of the preceding examples, 3306 is the numerical value of the polynomial when 4 is put for x. In the second example the value of the polynomial for $x = -5$ is -2994.

It may be remarked, finally, that the remainder resulting from the division of *any* function of x by $x - a$, and the value of that function for $x = a$, will always be identical, provided only that the quotient, furnished by the division, do not become infinite for the proposed value of x. This is evidently the only condition necessary in order that the product of quotient and divisor may become *zero* when the divisor does, that is, when $x = a$; the corresponding value of the dividend being then expressed by the remainder.

PROPOSITION II.

(54.) Every equation has exactly as many roots as there are units in the exponent of the highest power of the unknown quantity in it; that is, an equation of the nth degree, has n roots and no more.

Let

$$f(x) = x^n + \ldots . A_3x^3 + A_2x^2 + Ax + N = 0$$

be any algebraical equation whatever. It necessarily admits of *one* root a_1, either real or imaginary (42); and therefore (48), the first member, $f(x)$, is divisible by at least one binomial divisor

$x - a_1$. Hence, calling the quotient arising from this division, $f'(x)$, we have

$$f(x) = (x - a_1) f'(x),$$

where $f'(x)$ is a rational and integral polynomial of a degree inferior to $f(x)$ by *unity*.

If we equate this latter polynomial to zero, or assume the equation $f'(x) = 0$, a value for x will exist that will satisfy that equation (42); so that calling this value a_2, it follows that there is at least one binomial divisor, $x - a_2$, of $f'(x)$; that is, putting $f''(x)$ for the quotient of the division, we shall have

$$f'(x) = (x - a_2) f''(x),$$

where $f''(x)$ is a polynomial of a degree *two units* below the original one.

In like manner the equation $f''(x) = 0$ has also a root a_3, by the same general principle (42); so that $f''(x)$ has also a binomial divisor, $x - a_3$; and the corresponding quotient will be in degree *three units* below the original polynomial. The continuation of this process would thus furnish a series of polynomials $f'(x)$, $f''(x)$, &c., descending continually in degree by unity, and thus at length reaching a simple binomial $x - a_n$, there being of course as many such results as there have been divisions. Consequently

$$\begin{aligned} f(x) &= (x - a_1) f'(x) \\ &= (x - a_1)(x - a_2) f''(x) \\ &= (x - a_1)(x - a_2)(x - a_3) f'''(x) \\ &\quad \cdots\cdots\cdots\cdots \\ &\quad \cdots\cdots\cdots\cdots \\ &= (x - a_1)(x - a_2)(x - a_3) \dots (x - a_n). \end{aligned}$$

And thus the polynomial of the nth degree $f(x)$ is decomposable into n binomial factors of the first degree.

In order that $f(x)$ may become zero, it is sufficient therefore that any one of these simple factors become zero; that is to say, it is sufficient that x take any one of the n values,

$$a_1,\ a_2,\ a_3\ .\ .\ .\ .\ .\ a_n.$$

Consequently, every equation of the nth degree must have n roots.

It is plain that a polynomial of the nth degree cannot have more than n factors of the form $x - a$; because if more than n factors be multiplied together, the product will be a polynomial of a higher degree than the nth.

It does not follow from the foregoing reasoning that the factors, and consequently the roots, are all necessarily *unequal*: any two or more may be equal to one another. All that the reasoning shows is, that the *number* of simple factors in the first member of an equation in x, of the nth degree, is n, and that the number of roots is therefore n. There is no restriction as to the relative magnitudes either of the factors or of the roots.

(55.) From this important proposition it follows, that we may always represent the first member of an equation of the nth degree under the form

$$(x - a_1)\ (x - a_2)\ (x - a_3)\ .\ .\ .\ .\ (x - a_n)$$

or may consider such an equation as compounded (by multiplication) of as many simple equations,

$$x - a_1 = 0,\ x - a_2 = 0,\ x - a_3 = 0,\ \&c.$$

as there are units in the number denoting its degree, and of no more.

We may infer also, when one of the roots (a_1) of an equation has been discovered, that we shall arrive at an equation containing all the remaining roots by dividing the first member of the proposed equation by $x - a_1$, and then equating the quotient to zero. The new equation, thus derived from the proposed, is called the *depressed* equation. It is very easily obtained by employing the process at (51) instead of the common operation for division. In like manner if two roots, a_1, a_2, be known, we may deduce the depressed equation containing the remaining roots by dividing the first member of it by the simple factors $(x - a_1)$, $(x - a_2)$, in succession, or by the quadratic factor $(x - a_1) \times (x - a_2) = x^2 - (a_1 + a_2)\,x + a_1\,a_2$ at once, and then equating the quotient to zero. Hence, if we were in possession of a general method for determining a single root of every equation, we should have means sufficient for the discovery of *all* the roots of every equation; and should thus be enabled to decompose every rational and integral polynomial, with a single unknown quantity, into its constituent simple factors. Thus the component factors of the polynomial $f(x)$ would be obtained by equating $f(x)$ to zero; then determining one after another, as above described, all the roots of $f(x) = 0$; and finally writing down the binomials furnished by connecting each of these roots in succession, by a changed sign, with x. The actual determination of the roots of an equation constitutes the problem of the *general solution of equations;* and it is one, as already remarked in the Introduction, for which algebra has effected comparatively but little, except in the case of *numerical equations;* that is, equations whose coefficients are known numbers, and not general symbolical expressions or letters. In this latter case little or no advance has been made beyond equations of the fourth degree.* When the equation is numerical all the real roots may be obtained by the combined methods of STURM and HORNER, as will be hereafter explained, so that the roots which these methods leave undetermined will all be imaginary. LAGRANGE has explained a method by help of which even these may be actually exhibited:† but the calculation of the real roots, and the discovery of the

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* And as observed in the Introduction, the symbolical expressions for the roots of equations of the third and fourth degrees are useless for the purposes of actual numerical computation except in particular cases.

† Traité de la Résolution des Equations Numériques, 1826, p. 19 and p. 177. The principles of this method will be explained hereafter.

number merely of those that are imaginary, comprise all that is usually sought to be accomplished in the numerical solution of equations.*

But whatever difficulties may attend the decomposition of a given polynomial into its simple factors, the reverse operation, that of compounding the expression from given simple elements, or of constructing an equation that shall have given roots, is very easily accomplished. The roots being given, the simple factors, formed by connecting each, by a changed sign, with x, are all given; and the product of these is the first member of the required equation, the other member being zero.

(56.) It has already been proved (42) that every equation $f(x) = 0$ has a root of the form $a + b\sqrt{-1}$; therefore the depressed equation $f'(x) = 0$ has also a root of the same form; in like manner the next depressed equation $f''(x) = 0$ has a root of this form, and so on, through the entire series. Hence all the n roots of an equation of the nth degree are of the same form, $a + b\sqrt{-1}$. It will be further shown in next proposition, that when the coefficients of the equation are all real, and any of the roots are actually imaginary, b being different from zero, they must occur in *conjugate pairs* (38); that is, every imaginary root $a + b\sqrt{-1}$, must be accompanied by another, $a - b\sqrt{-1}$.

The conclusion just obtained, respecting the general form of all the roots of an equation, enables us to affirm with confidence that every fractional power of $a \pm b\sqrt{-1}$ must be of the same

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* Still the determination of the imaginary roots of numerical equations is not without its use in certain practical enquiries, more especially in the Integral Calculus. But it is a branch of the general doctrine of numerical equations which has not been cultivated with the success that has attended the researches of algebraists in reference to the *real* roots. The method of LAGRANGE, adverted to in the preceding note, is too laborious to be of much practical value. The same may be said of the method of BERNOULLI, by *recurring series;* so that a practicable process for calculating the imaginary roots is still a desideratum in the doctrine of numerical equations.

form. For, representing this power by x, we have

$$x = (a \pm b\sqrt{-1})^{\frac{m}{n}}$$

$$\therefore x^n = (a \pm b\sqrt{-1})^m$$

The integral power m has already been shown to be of the form $P + Q\sqrt{-1}$, whether m be positive or negative (37); so that the last equation may be written thus,

$$x^n = P + Q\sqrt{-1} \text{ or } x^n - (P + Q\sqrt{-1}) = 0$$

And it has been shown above that the values of x, in this equation, are all of the form $\alpha + \beta\sqrt{-1}$. Hence the principle announced is fully established.

PROPOSITION III.

(57.) Every equation whose coefficients are real, and which has one imaginary root, has necessarily another, conjugate to it: that is, imaginary roots enter into equations in conjugate pairs.

Let the equation contain the imaginary root $a + b\sqrt{-1}$; then if this root be substituted for x in $f(x)$, that is in

$$N + Ax + A_2x^2 + \cdots\cdots + x^n = 0$$

we shall have

$$N + A(a+b\sqrt{-1}) + A_2(a+b\sqrt{-1})^2 + \cdots\cdots + (a+b\sqrt{-1})^n = 0$$

Now it is obvious that if the several terms in the first member of this equation be developed by the binomial theorem, or by actual multiplication, we shall have a series of monomials, of which all those that involve *odd* powers of $b\sqrt{-1}$ will be imaginary; and all the others real: that is, calling the sum of the real terms P, and the sum of the imaginary terms $Q\sqrt{-1}$, the equation may be written

$$P + Q\sqrt{-1} = 0 \quad . \; . \; . \; [1]$$

which equation can be satisfied only by the conditions

$$P = 0, \ Q = 0\,;$$

for if these have not place an imaginary quantity would be equal to a real quantity.

Now we know from the binomial theorem that the developments of $(p + q)^m$ and of $(p - q)^m$ differ only in the *signs* of the terms involving the odd powers of q. Hence if $a - b\sqrt{-1}$ had been substituted for x in $f(x)$, instead of $a + b\sqrt{-1}$, we should have been led to a result

$$P - Q\sqrt{-1}$$

differing from the former [1] only in the sign of Q. And since $P = 0$ and $Q = 0$, this result must like the former be zero; that is

$$P - Q\sqrt{-1} = 0 \ . \ . \ . \ . \ [2]$$

so that both $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$ are equally roots of the proposed equation.

The first member of that equation is therefore divisible by both of the simple divisors

$$x - a - b\sqrt{-1}$$

$$x - a + b\sqrt{-1}$$

and consequently by their product, the *real* quadratic divisor

$$x^2 - 2ax + a^2 + b^2, \text{ or } (x - a)^2 + b^2 \ . \ . \ . \ . \ [3]$$

If the depressed equation which results from this division have an imaginary root, then from what has just been shown, there must exist another real quadratic divisor, composed of another pair of conjugate roots. And in general there must exist as many real quadratic factors in $f(x)$ as there are pairs of imaginary roots in the equation $f(x) = 0$, besides those formed from pairs of real roots.

(58.) It follows therefore that every equation of an even degree, $2n$, with real coefficients, may be decomposed into n *real* quadratic factors; whatever be the character of the roots entering it.

And further, in conjunction with what is proved at (29), every equation of an odd degree, $2n + 1$, with real coefficients, may be decomposed into $n + 1$ *real* factors; n of these being of the second degree and one of the first.

It is scarcely necessary to remark that, in both these cases, if more real roots than two enter the equation, they may be combined in pairs in different ways; so that there may be different sets of n quadratic factors, all equally composing the original polynomial. And when we say that an equation *may* be decomposed into its n quadratic factors, we merely imply the *existence* of these factors, in its first member. As remarked at (55) the actual determination of them requires the accomplishment of the general solution of equations.

(59.) When the roots of an equation with real coefficients are *all* imaginary, then each of the real quadratic factors [3], into which it is decomposable being the sum of two squares, it follows that whatever number be substituted in the equation for x, the result of that substitution must always be *positive;* so that if a *negative* result arise from any substitution for x, *all* the roots of the equation cannot be imaginary. That the final term of the equation, or that independent of x, must be positive is an inference from (29) as well as from what is here established.

By reasoning exactly similar to that employed above, may it be proved that if an equation with rational coefficients have a root of the form $a + \sqrt{b}$, it must have another of the form $a - \sqrt{b}$; so that quadratic surd roots, whether real or imaginary, always enter into equations in pairs.

PROPOSITION IV.

(60.) To determine the forms of the functions which the coefficients in the general equation $f(x) = 0$ are of the roots.

It has already been shown (55) that the first member of the general equation

$$x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + A_{n-3}x^{n-3} + \dots + A_3x^3 +$$
$$A_2x^2 + Ax + N = 0$$

is no other than the product of the n binomial factors

$$x - a_1,\ x - a_2,\ x - a_3,\ .\ .\ .\ .\ .\ x - a_n$$

in which $a_1, a_2, a_3, \ldots\ a_n$, are the n roots of the equation. By the actual multiplication of these we shall arrive therefore at the polynomial $f(x)$, and thus discover the manner in which the n roots enter into the formation of the coefficients. Two or three steps of this multiplication will be sufficient to make known the general law which connects the coefficients and roots together.

$$\begin{array}{l}
x - a_1 \\
x - a_2 \\
\hline
\end{array}$$

$$\begin{array}{ll|l}
x^2 & - a_1 & x + a_1 a_2 \\
 & - a_2 &
\end{array}$$

$$\begin{array}{l}
x - a_3 \\
\hline
\end{array}$$

$$\begin{array}{ll|l|l}
x^3 & - a_1 & x^2 + a_1 a_2 & x - a_1 a_2 a_3 \\
 & - a_2 & + a_1 a_3 & \\
 & - a_3 & + a_2 a_3 &
\end{array}$$

$$\begin{array}{l}
x - a_4 \\
\hline
\end{array}$$

$$\begin{array}{ll|l|l|l}
x^4 & - a_1 & x^3 + a_1 a_2 & x^2 - a_1 a_2 a_3 & x + a_1 a_2 a_3 a_4 \\
 & - a_2 & + a_1 a_3 & - a_1 a_2 a_4 & \\
 & - a_3 & + a_2 a_3 & - a_1 a_3 a_4 & \\
 & - a_4 & + a_1 a_4 & - a_2 a_3 a_4 & \\
 & & + a_2 a_4 & & \\
 & & + a_3 a_4 & &
\end{array}$$

Hence by continuing this process, we have, for the coefficients of the proposed equation, the values

$$A_{n-1} = -a_1 - a_2 - a_3 - \ldots\ldots - a_n$$

$$A_{n-2} = a_1 a_2 + a_1 a_3 + a_2 a_3 + \ldots\ldots + a_{n-1} a_n$$

$$A_{n-3} = -a_1 a_2 a_3 - a_1 a_2 a_4 - \ldots\ldots - a_{n-2} a_{n-1} a_n$$

. . . .

. . . .

$$N = a_1 a_2 a_3 a_4 \ldots\ldots a_n (-1)^n.$$

We infer, therefore, that in any equation in which the first term, or highest power of x, has the coefficient unity, the coefficient of the second term is equal to the sum of the roots with their signs changed; the coefficient of the third term is equal to the sum of the products of every two roots with their signs changed; the coefficient of the fourth term is equal to the sum of the products of every three roots with their signs changed; and so on: and the last term is equal to the product of all the roots with their signs changed. It is proper to observe however, that in the composition of the *third*, *fifth*, *seventh*, &c. coefficients, it is indifferent whether the signs of the roots be changed or not; since the products that form these consist each of an *even* number of factors. It follows from this:

1. That if the coefficient of the second term in any equation be 0, that is, if the term be absent, the sum of the positive roots must be equal to the sum of the negative roots.

2. When the coefficients are all whole numbers, and the first unity, every integral root of the equation will be found among the integral factors of the last term; for this last term, divided by a root, will express the product of the remaining roots. This product will therefore be the last term of the depressed equation involving those remaining roots; and it is plain, from the process for deducing this equation, adverted to at p. 51, that *all* the resulting coefficients will be integral if the original coefficients, and the root by which they have been depressed, be all integral. This last consideration indeed justifies the more general inference that, whether the leading coefficient be unity or any other integer, still every integral root will accurately divide the last term.

3. If the roots of an equation be all positive, the terms will be alternately positive and negative; and if the roots be all negative, the terms will be all positive.

It appears, moreover, that if one root only of an equation be changed, every one of the coefficients will be changed.

For a different and very simple method of deriving the law of the coefficients, the student is referred to the *Analysis and Solution of Cubic and Biquadratic Equations*, page 25.

PROPOSITION V.

(61.) If the signs of the alternate terms in an equation be changed, the signs of all the roots will be changed.*

Let

$$x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + A_{n-3} x^{n-3} + \&c. = 0 \quad . \quad . \quad . \quad . \quad [1]$$

be any equation, and a a root; then, if a be substituted for x in the first member, the result will be zero; and if we change the alternate signs, writing the equation thus,

$$x^n - A_{n-1} x^{n-1} + A_{n-2} x^{n-2} - A_{n-3} x^{n-3} + \&c. = 0 \quad . \quad . \quad . \quad . \quad [2],$$

and instead of a substitute $-a$ for x, the result, should n be even, will be the very same as before, and consequently zero; but if n be odd, then the result will differ from what it was before only in this, viz. that all the signs merely of the polynomial will be changed, so that as it was zero before, it must be zero still. Hence, for every root a in [1], there is an equal root, with contrary sign, $-a$, in [2].

Thus the positive roots of any equation may be converted into negative, and the negative into positive, by simply changing the alternate signs of the equation, commencing at the second, and taking care to allow for the absent terms in incomplete equations.

It is obvious that if the signs of *all* the terms are changed, the roots remain unaltered; because whatever values of x cause the polynomial to become zero in one case, make it zero in the other also.

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* The equation is understood to be *complete*. If any term be absent it must be replaced by a cipher.

PROPOSITION VI.

(62.) If all the coefficients of an equation be whole numbers, and that of the leading term unity, the equation cannot have a fractional root.

If possible, let the equation

$$x^n + A_{n-1}x^{n-1} + \ldots . + A_3x^3 + A_2x^2 + Ax + N = 0,$$

whose coefficients are all integral, have a fractional root; and let the fraction in its lowest terms be $\frac{a}{b}$. Then, putting this for x, we have

$$\left(\frac{a}{b}\right)^n + A_{n-1}\left(\frac{a}{b}\right)^{n-1} + \ldots + A_3\left(\frac{a}{b}\right)^3 + A_2\left(\frac{a}{b}\right)^2 + A\left(\frac{a}{b}\right) + N = 0.$$

Or, multiplying by b^{n-1},

$$\frac{a^n}{b} + A_{n-1}a^{n-1} + \ldots + A_3a^3b^{n-4} + A_2a^2b^{n-3} + Aab^{n-2} + Nb^{n-1} = 0.$$

Now in this polynomial every term after the first is integral. Hence, transposing these to the other side, the first member, which will then be $\frac{a^n}{b}$, must also be integral. But $\frac{a}{b}$ being in its lowest terms, a and b have no factor in common; and it is obvious that there can be no simple factors in a^n that are not also in a itself: consequently a^n and b have no factor in common; that is the fraction $\frac{a^n}{b}$ is in its lowest terms: and yet it is equal to an integer: which is absurd. Therefore the proposed equation cannot have a fractional root.

Hence when the coefficients of an equation are whole numbers, the first being unity, every real root of the equation must either be a whole number or an interminable decimal. The latter are called *incommensurable* roots. It follows from (60) that if there be if the roots are all real

one of these in the equation, there must be one more at least; whether the coefficients be integral or not, provided only that they be all rational; or that the second merely be rational.

Moreover if, under the former hypothesis as to the coefficients, none of the divisors of the last term, whether taken positively or negatively, can satisfy the conditions of the equation, when they are severally substituted for x, we may conclude that the equation has neither an integral nor yet a fractional root: so that those roots which are not imaginary must be incommensurable.

In next chapter this conclusion will be generalized; for it will be proved that, whatever integer the leading coefficient A_n may be, every fractional root must have for its numerator a divisor of N, and for its denominator a divisor of A_n.

(63.) It is an important step towards the solution of a numerical equation to discover by a previous examination how many of its roots are real, and thence how many are imaginary. A more minute analysis indeed than this must be effected before the numerical value of the real roots can be actually developed:—we must know how many of these are positive, and how many are negative; and lastly, if they be not whole numbers themselves, between what pair of whole numbers each is situated. Now if we happen to know that the roots of any proposed equation are all *real*, there is a rule—called the rule of DESCARTES—which will enable us to infer, from the mere inspection of the signs of the coefficients, how many of these roots are positive, and how many negative. The rule will do more; it will show us, without any stipulation as to the character of the roots, the greatest number of each kind of real roots the equation can possibly have, consistently with the signs which connect its terms together; and the knowledge of these limits, in certain cases to be hereafter noticed, will lead at once to the discovery of the number of imaginary roots entering the equation. The rule therefore is of considerable importance in the analysis of equations; and that importance has been increased of late by the extension and greater efficiency which have been given to it by BUDAN and FOURIER; an account of whose researches will be found in a subsequent chapter. The rule is enunciated as follows:

PROPOSITION VII.

(64.) A complete equation cannot have a greater number of positive roots than there are *changes* of sign from + to —, and from — to +, in the series of terms forming its first member. Nor can it have a greater number of negative roots than there are *permanencies,* or repetitions of the same sign in proceeding from term to term.

To demonstrate this remarkable proposition, it will be necessary merely to show that, if any polynomial, whatever be the signs of its terms, be multiplied by a factor $x - a$, corresponding to a *positive* root, the resulting polynomial will present at least one more *change of sign* than the original; and that if it be multiplied by $x + a$, corresponding to a *negative* root, the result will exhibit at least one more *permanence of sign* than the original. This is the form to which the proposition was first reduced by Segner,[*] whose demonstration is distinguished from all others by its simplicity. It is in substance as follows:

Suppose the signs of the proposed polynomial to succeed each other in any order, as

$$+ - - + - + + + - - +,$$

then the multiplication of the polynomial, by $x - a$, will give rise to two rows of terms, which, added vertically, furnish the product. The first row will, obviously, present the very same series of signs as the original; and the second, arising from the multiplication by the negative term $-a$, will present the same series of signs as we should get by changing every one of the signs of the first row. In fact, the two rows of signs would be

$$\begin{array}{cccccccccccc}
+ & - & - & + & - & + & + & + & - & - & + & \\
 & - & + & + & - & + & - & - & - & + & + & - \\
\hline
\end{array}$$

$$\text{and signs of prod. } + - \pm + - + \pm \pm - \pm + -.$$

---

[*] Mémoires de Berlin, 1756. Lagrange, Traité, &c. p. 156.

We have written the ambiguous sign ± in the product when the addition of unlike signs in the partial products occurs; and it is very plain that these ambiguities will, in this and in every other arrangement, be just as numerous as the permanencies in the proposed: thus, in the present arrangement, the proposed furnishes four permanencies, viz. $- \; -$, $+ \; +$, $+ \; +$, $- \; -$; and there are, accordingly, in the product four ambiguities, the other signs remaining the same as in the proposed, with the exception of the final sign, which is superadded, and which is always contrary to the final sign in the proposed.

It is an easy matter, therefore, when the signs of the terms of any polynomial are given, to write down immediately the signs in the product of that polynomial, by $x - a$, as far, at least, as these signs are determinable without knowing the values of the quantities employed; for we shall merely have to change every permanency in the proposed into a sign of ambiguity, and to superadd a final sign, unlike that with which the proposed row terminates. For instance, if the proposed arrangement were

$$+ \; - \; + \; + \; - \; - \; - \; + \; - \; + \; + \; +,$$

the signs of the product would be

$$+ \; - \; + \; \pm \; - \; \pm \; \pm \; + \; - \; + \; \pm \; \pm \; -.$$

Again, if the signs in the proposed were in the order

$$+ \; + \; + \; - \; + \; - \; + \; - \; - \; -$$

the signs in the product would be in the order

$$+ \; \pm \; \pm \; - \; + \; - \; + \; - \; \pm \; \pm \; +$$

As therefore in passing from the multiplicand to the product, it is the *permanencies* only of the former that can suffer any change, it is impossible that the *variations* can ever be diminished however they may be increased. Consequently the most unfavorable supposition for our purpose is, that the permanencies (omitting the superadded sign,) remain the same in number; and in this case, if the proposed terminate with a variation, the superadded sign in the product, being unlike the sign that immediately precedes it, will introduce another variation; but if it terminate with a permanency, then the corresponding *ambiguity* in the result will obviously, substitute for it what sign we will, form a variation, either with the preceding or with the superadded sign. It follows, therefore, that no equation can have a greater number of positive roots than variations of sign.

To prove the second part of the proposition it will suffice to remark that, if we change the alternate signs in an equation, we change the roots from positive to negative, and vice versâ. The equation thus changed would have its permanencies replaced by variations, and its variations by permanencies; and, since by the foregoing, the changed equation cannot have a greater number of positive roots than variations, the proposed cannot have a greater number of negative roots than permanencies.

(65.) This proposition constitutes the celebrated *rule of signs;* and serves to point out limits which the number of the positive and negative roots of an equation can never exceed. It does not, however, furnish us with the means of ascertaining how many real roots, of either kind, any proposed equation may involve; nor indeed does it enable us to affirm that even one positive or negative root actually exists in any equation; it merely shows that *if* real roots exist, those which are positive, or those which are negative, cannot exceed a certain number; they may, however, fall greatly short of this number, and, indeed, be all imaginary. But, as remarked at (63), the rule is not without its use, even in detecting imaginary roots; as it sometimes discovers discrepancies incompatible with the existence of real roots, in those equations which are *incomplete,* or have terms wanting. For example, suppose we wished to ascertain the nature of the roots of the cubic equation

$$x^3 + Ax + N = 0,$$

in which A and N are positive. Putting the equation in a complete form, it is

$$x^3 \pm 0x^2 + Ax + N = 0.$$

Now, when we take the second term, +, there are no variations, so there can be no positive root; but when we take the same term, —, there is only one permanence, so that there cannot be more than one negative root; these conclusions would be contradictory if the roots were real; we therefore infer that the proposed has a pair of imaginary roots.

If the equation had been

$$x^3 - Ax + N = 0,$$

we could not have pronounced anything respecting the nature of the roots from the application of the *rule of signs;* for, supplying the absent term, we have

$$x^3 \pm 0x^2 - Ax + N = 0;$$

which presents one permanence and two variations, whichever sign we give to the second term; so that all we can affirm is, that *if* the roots are real, two must be positive and one negative. Two roots, however, *may* be imaginary, in which case the third will be negative, because the sign of N is positive (29).

(66.) Unfailing criteria for the detection of imaginary roots will be given in a subsequent chapter; it only remains for us to deduce here one or two obvious particulars, the most important of which is, that when we know beforehand that an equation contains none but real roots, then the rule of DESCARTES will enable us to ascertain exactly the number of each kind, as may be readily proved as follows:

Let n be the degree of the equation, p the number of permanencies, and v the number of variations; then $n = p + v$. Let also p' be the number of negative roots, and v' the number of positive roots; then $n = p' + v'$: whence

$$p + v = p' + v'.$$

Now it is proved above that p' cannot be greater than p, nor can v' be greater than v; hence, necessarily,

$$p = p', \text{ and } v = v';$$

consequently, when the roots are all real, the number of positive roots will be equal to the number of variations, and the number of negative roots equal to the number of permanencies.

A second inference is, that when the signs of a complete equation are alternately positive and negative the equation cannot have a negative root; and when the signs are all alike it cannot have a positive root.

(67.) In the preceding investigation the equation is supposed to be rendered complete, when any terms are wanting, by the insertion of those terms with zero-coefficients. But, as these zeros must be accompanied by the double sign ±, their intervention merely adds to the number of *ambiguities;* so that if a variation exist between two consecutive signs in the incomplete equation the ambiguities which intervene, however they be interpreted, cannot possibly destroy this variation, or convert it into a permanency. Hence no variations can be *lost* by these insertions; nor, if we give to all the intervening zeros the same sign, can any be gained. We may therefore suppress the condition as to the equation being complete, as far as the first part of the proposition is concerned, and conclude that, whether the equation be complete or not, the number of positive roots can never exceed the number of changes of sign. And as the negative roots of an equation are always convertible into positive by simply substituting $-x$ for x, (61,) we may conclude farther, that the number of negative roots can never exceed the number of changes that would take place, if x were turned into $-x$. Thus the rule of signs may be freed from all restriction as to the complete form of the equation: and notice need be taken of the absent terms only with a view to detect the existence of imaginary roots.

(68.) This method of searching after indications of imaginary roots is of considerable importance in the analysis of equations; it is involved in the following general principles:

1. The absence of an even number of consecutive terms from an equation is an indication that there are at least that number of imaginary roots in the equation.

Let the $2m$ absent terms in the equation of the nth degree be replaced by $2m$ zeros; and suppose, first, that the two terms between which they occur have like signs. Give to the $2m$ zeros these same signs. Then, considering only the two terms and the intervening zeros, we shall evidently have $2m + 1$ permanencies; so that if the terms omitted furnish k permanencies more, it will follow that the number of *positive* roots cannot exceed $n - k - 2m - 1$.

Change now the sign of the first, and that of every alternate, zero; then, their number being even, the first and last must evidently have unlike signs. Hence the last zero and the term following it have a permanence; and this is the only permanence within the proposed limits: hence the number of negative roots cannot exceed $k + 1$. Consequently the entire number of real roots cannot exceed $(n - k - 2m - 1) + (k + 1) = n - 2m$. But there are n roots altogether; therefore there must be $2m$ imaginary roots at least.

If the vacancies occur between unlike signs, then, when the zeros are all plus, there will be only $2m$ permanencies; and when they are alternately plus and minus, no permanencies, in the interval. Hence, supposing, as before, that the terms not taken into account furnish k permanencies, it will follow from the first arrangement that the number of positive roots cannot exceed $n - k - 2m$, nor the number of negative roots, k; so that the entire number cannot exceed $n - k - 2m + k = n - 2m$: therefore in this case also there must be $2m$ imaginary roots at least.

2. The absence of an odd number of consecutive terms from an equation is an indication that the equation has that number of imaginary roots, plus or minus one; according as the vacancies occur between like or between unlike signs.

When the extreme signs are like and the intervening $2m + 1$ zeros are written with the same signs, we shall have $2m + 2$ permanencies. And when the signs of the zeros are made alternately positive and negative we shall have *no* permanencies.

From the first arrangement it follows, if k be the permanencies furnished by the omitted terms, that the number of positive roots cannot exceed $n - 2m - 2 - k$; and from the second arrangement that the number of negative roots cannot exceed k; so that the entire number of real roots cannot exceed $n - 2m - 2$. Hence $2m + 2$ of the roots at least must be imaginary.

When the extreme signs are unlike there will be but $2m + 1$ permanencies under the first arrangement, and *one* under the second; so that in this case the number of positive roots cannot exceed $n - 2m - 1 - k$, nor the number of negative roots $k + 1$. Hence, the number of real roots cannot exceed $(n - 2m - 1 - k) + (k + 1) = n - 2m$; so that $2m$ of the roots, at least, must be imaginary.

(69.) By the aid of these principles we may examine the several intervals that occur in an incomplete equation, and thus determine a limit to the total number of its imaginary roots. And in order to this it will be unnecessary to keep in remembrance the formal enunciations we have given to those principles above: for, from the character of the preceding investigation, and the conclusions deduced from it, it is plain that, when absent terms occur in an equation, we have nothing to do but to fill up each chasm by the requisite group of zeros, giving to the individual zeros of each group the sign of the immediately preceding term, so that the *greatest* number of permanencies may be secured. We have then to write the several signs of the entire series of terms anew, giving however a changed sign to every alternate zero in each group, so that the *least* number of permanencies may be obtained. The number of permanencies thus *lost* in proceeding from the first row of signs to the second, or, which amounts to the same thing, the number of variations lost in proceeding from the second to the first, will be equal to the number of imaginary roots which the equation must have, at least.

This gives the necessary completion to the rule of DESCARTES, and is included in the general theorem of FOURIER and BUDAN, which will be discussed hereafter. It is usually ascribed to DE GUA, who was the first to give a general demonstration of the rule of DESCARTES,* and who deduced several interesting consequences from it as respects the connexion between zero-coefficients and imaginary roots. The leading principle, however, established by DE GUA, in the paper referred to, is the important one that an equation can never have all its roots real unless the equations of inferior degree derived from it, as at p. 17, have all their roots real also; from which he inferred that if for any value of x one of these derived polynomials vanished, and at the same time caused the immediately preceding and succeeding polynomials to furnish results with like signs, imaginary roots necessarily existed in the proposed equation.† From the same general principle he further deduced an expression for the number of conditions necessary to be fulfilled in order that all the roots of an equation may be real. LAGRANGE has given a condensed and able view of these researches of DE GUA, in a note appended to his work on *Equations,* and has noticed the remarkable circumstance, that the conditions of reality to which DE GUA was led are the very same in number as those to which he himself was conducted by a very different route; although these conditions were found to be unnecessarily numerous in equations of the third and fourth degree, and were suspected by LAGRANGE to be capable of reduction in those of the higher orders, a suspicion which the discovery of STURM has fully confirmed.

To these observations we shall merely add, that the rule of DESCARTES was first published in his *Geometry* in 1637, but without demonstration. And as the work of HARRIOT, published in 1631—ten years after the author's death,‡ had already exhibited the mode of effecting the composition of the first member of an equation from the simple factors involving the several roots—illustrating and confirming the method by numerical examples, it has been generally maintained by English algebraists that the rule

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* *Mémoires de l'Académie des Sciences,* 1741. The general theorem of FOURIER includes that of DESCARTES, as well as the deductions of DE GUA noticed in the text.

† These propositions will be proved in the chapter on the limits to the real roots of equations.

‡ *Artis Analyticæ Praxis.*

of DESCARTES was plainly set forth in HARRIOT'S researches, and that it should bear HARRIOT'S name. The learned Dr. WALLIS strenuously supports HARRIOT'S claim: but the precision with which the principle is announced by DESCARTES, although very probably but a mere conjectural inference from HARRIOT'S composition of equations, will no doubt justify the common practice of giving his name to the rule.*

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* For further particulars respecting this subject reference may be made to WALLIS'S *Algebra,* to the before-mentioned paper of DE GUA, to HUTTON'S *History of Algebra,* and to a note at p. 220 of vol. I. of HUTTON'S *Course,* by DAVIES, 1841. It may be proper to add that DE GUA, though maintaining the claims of DESCARTES to the discovery of the theorem, very naturally concludes, in the absence of all demonstration, that DESCARTES arrived at it only by induction: and we may here add, that for such an induction HARRIOT'S examples were fully sufficient. FOURIER, however, combats this opinion of DE GUA, on the ground that a demonstration of the theorem was possible from the composition of equations, which composition however HARRIOT was the first to make known. See FOURIER, *Analyse des Equations,* p. v.

CHAPTER V.

ON THE TRANSFORMATION OF EQUATIONS.

(70.) ALGEBRAICAL EQUATIONS do not always present themselves in the most convenient forms for solution, and hence the expediency of being provided with the means of changing them from one form to another. Depriving the leading term of its coefficient, by division or otherwise, is the most simple change of this kind, and is a desirable preparative to the usual methods of solution, as it simplifies the form without affecting the roots of the equation. In most transformations, however, the roots themselves become also changed, but still bear such known and simple relations to those of the original equation, as to render the determination of these latter from them an easy operation. Generally indeed, to change the roots into others bearing given relations to them, is the direct object of the transformation; so that this part of the subject, in its full extent, involves the solution of the following comprehensive problem, viz. To transform an equation into another such that the roots of the latter shall be any given functions of those of the former. Under this form the subject will offer itself for discussion in a subsequent chapter; but we have no occasion to enter upon the investigation of so general a problem here, our attention at present being confined to those transformations which are useful or necessary in the actual solution of numerical equations, and which may be comprised in the four propositions following:

PROPOSITION I.

(71.) To transform an equation into another whose roots shall differ, either in excess or defect, from the roots of the original by any given quantity.

Let us suppose that the original equation is

$$A_4x^4 + A_3x^3 + A_2x^2 + Ax + N = 0 \quad . \; . \; . \; . \quad [1],$$

and that we wish to transform it into another whose roots shall be the same in number, but shall differ from them in magnitude each by the quantity r; then the relation between the x in the original equation and the x' in the transformed, will be

$$x = x' + r, \text{ or } x' = x - r$$

in which r will be plus or minus, according as the new roots, or values of x', are to differ from the original roots, or values of x, in defect or excess. If we actually substitute this value of x in the original, we shall obviously have the transformed equation, which will be of the form

$$A_4x'^4 + A'_3x'^3 + A'_2x'^2 + A'x' + N' = 0 \quad . \; . \; . \; . \quad [2];$$

and it is in this way that the result is generally obtained. But the method of actual substitution is unnecessarily laborious; and may be entirely superseded by a very simple process, which we shall now explain.

Instead of x' put its value $x - r$ in the equation [2]; we shall then have

$$A_4(x-r)^4 + A'_3(x-r)^3 + A'_2(x-r)^2 + A'(x-r) + N' = 0 \quad . \; . \quad [3];$$

an equation which, when reduced to a series of monomials by actually developing the terms, must be identical with the original; for, in fact, we have now returned from [2] to [1], by restoring to x' its value $x - r$. Hence we have the identity

$$A_4(x - r)^4 + A'_3(x - r)^3 + A'_2(x - r)^2 + A'(x - r) + N' =$$

$$A_4x^4 + A_3x^3 + A_2x^2 + Ax + N.$$

It is plain that if we divide the first member of this by $x - r$, the remainder must be N'; but, the two members being identical, the division of either by $x - r$ must give the same remainder, and the same quotient. The division, therefore, of the second member, that is of the original polynomial, by $x - r$, gives, for remainder N′, and for quotient,

$$A_4 (x - r)^3 + A'_3 (x - r)^2 + A'_2 (x - r) + A'.$$

Also, dividing this by $x - r$, we have for remainder A′, and for quotient,

$$A_4 (x - r)^2 + A'_3 (x - r) + A'_2.$$

Again, dividing this by $x - r$, the remainder becomes A'_2, and the quotient,

$$A_4 (x - r) + A'_3.$$

And lastly, dividing this by $x - r$, we have, for the final remainder, A'_3; and, for the final quotient, A_4; and in this manner may the coefficients in the transformed equation [2] be severally determined.

Although, to avoid unnecessary complication, we have assumed the equation [1] to be of only the fourth degree, yet it is plain that the process by which the unknown coefficients in [2] have been derived one after another, in reverse order, is perfectly general. The division of the original polynomial by $x - r$ furnishes for remainder N′; the division of the quotient by $x - r$ furnishes for remainder A′; the division of the second quotient gives for remainder A_2'; and thus, by noting the remainders supplied by these successive divisions, we discover the several coefficients of the transformed equation one after another, till we finally arrive at the coefficient A'_{n-1}; the one preceding this, that is the leading coefficient, being the same as that in the original equation.

Now we have exhibited at (51) a very easy way of performing the division of a polynomial, $f(x)$, by such a divisor as $x - r$; and, by employing that method in the present problem, the required transformation may always be rapidly effected, as the following examples will readily show.

1. Transform the equation

$$x^4 + 5x^3 + 4x^2 + 3x - 105 = 0,$$

into another, whose roots shall be less than those of the proposed, by 2. Here the constant divisor is $x-2$, and the process directed by the above investigation, and conducted according to the plan at (51), will be as follows:

$$\begin{array}{lllll}
A_4 & A_3 & A_2 & A & N \\
1\ + & 5\ + & 4\ + & 3\ - & 105\ (2=r \\
 & 2\ + & 14\ + & 36\ + & 78 \\ \cline{1-5}
1\ + & 7\ + & 18\ + & 39\ - & 27\ \therefore N'=-27 \\
 & 2\ + & 18\ + & 72 & \\ \cline{1-4}
1\ + & 9\ + & 36\ + & 111\ \therefore A'=111 & \\
 & 2\ + & 22 & & \\ \cline{1-3}
1\ + & 11\ + & 58\ \therefore A'_2=58 & & \\
 & 2 & & & \\ \cline{1-2}
1\ + & 13\ \therefore A'_3=13. & & &
\end{array}$$

Hence the transformed equation is

$$x'^4+13x'^3+58x'^2+111x'-27=0.$$

(72.) After what has been done in Proposition I. p. 42, it is presumed that the student will require no verbal explanation of the foregoing process. It will no doubt be sufficient to remark that, calling the numbers below the black lines *results*, each result is formed by adding r times the result immediately before it to the result immediately above it. We may observe, however, that the operation would be somewhat abbreviated by omitting the repetition of the first coefficient in the commencement of each row of results, by suppressing the plus signs, and by reserving the determinations of A'_3, A'_2, A', and N', till we come to the last result, thus:

$$\begin{array}{ccccl}
1 & 5 & 4 & 3 & -105\ (2 \\
 & \underline{2} & \underline{14} & \underline{36} & 78 \\
 & 7 & 18 & 39 & \\
 & \underline{2} & \underline{18} & 72 & \\
 & 9 & 36 & & \\
 & \underline{2} & 22 & & \\
 & 11 & & & \\
 & 2 & & & \\
\hline
\end{array}$$

$$x'^4 + 13x'^3 + 58x'^2 + 111x' - 27 = 0.$$

By suppressing also the several addends, and performing the addition operations mentally, we should, of course, abridge the space occupied by the process, very considerably. The whole would then, in fact, be reduced to this, viz.

$$\begin{array}{ccccl}
1 & 5 & 4 & 3 & -105\ (2 \\
 & 7 & 18 & 39 & -\ 27 \\
 & 9 & 36 & 111 & \\
 & 11 & 58 & & \\
 & 13 & & & \\
\end{array}$$

Other means might be easily contrived for shortening the apparent work; but we would recommend to the student the exhibition of the entire process rather than incur the risk of error by suppressing any of the steps. When r is 1, then indeed, as there is no effective multiplication, the process naturally takes the form here given, as in the following example.

2. It is required to transform the equation

$$2x^4 - 13x^2 + 10x - 19 = 0$$

into another, whose roots shall be less than the roots of this equation by 1.

$$\begin{array}{cccccl}
2 & 0 & -13 & 10 & -19\ (1 & \\
 & 2 & -11 & -1 & -20 & \therefore N' = -20 \\
 & 4 & -7 & -8 & & \therefore A' = -\ 8 \\
 & 6 & -1 & & & \therefore A'_2 = -\ 1 \\
 & 8 & & & & \therefore A'_3 = 8. \\
\end{array}$$

Hence, the transformed equation is

$$2x'^4 + 8x'^3 - x'^2 - 8x' - 20 = 0.$$

3. It is required to transform the preceding equation into another, whose roots are less by 3.

$$\begin{array}{ccccc}
2 & 0 & -13 & 10 & -19\ (3 \\
 & 6 & 18 & 15 & 75 \\
\cline{2-4}
 & 6 & 5 & 25 & \\
 & 6 & 36 & 123 & \\
\cline{2-3}
 & 12 & 41 & & \\
 & 6 & 54 & & \\
\cline{2-2}
 & 18 & & & \\
 & 6 & & & \\
\hline
\end{array}$$

trans. equa. $2x'^4 + 24x'^3 + 95x'^2 + 148x' + 56 = 0.$

4. It is required to transform the equation

$$6x^3 - 3x^2 + 4x - 1 = 0,$$

into another, whose roots shall exceed the roots of this by 3.

Here the multiplier will be -3.

$$\begin{array}{cccc}
6 & -3 & 4 & -1\ (-3 \\
 & -18 & 63 & -201 \\
\cline{2-3}
 & -21 & 67 & \\
 & -18 & 117 & \\
\cline{2-2}
 & -39 & & \\
 & -18 & & \\
\hline
\end{array}$$

trans. equation, $6x'^3 - 57x'^2 + 184x' - 202 = 0.$

The foregoing operation for diminishing or increasing the roots of an equation by any proposed number, deserves the student's special attention. The process is purely numerical; and arranges itself in a form that renders it of great value in the numerical solution of equations, which is conducted, step by step, by a series of operations of this kind.

5. Transform the equation

$$x^3 - 7x + 7 = 0$$

into one whose roots shall be less than the roots of this by 2.

The transformed equation is $x^3 + 6x^2 + 5x + 1 = 0$.

6. Transform the equation

$$19x^4 - 22x^3 - 35x^2 - 16x - 2 = 0$$

into another, in which the roots shall be diminished by 3.

The transformed equation is

$$19x^4 + 206x^3 + 793x^2 + 1232x + 580 = 0.$$

7. Transform the equation

$$3x^4 - 13x^3 + 7x^2 - 8x - 9 = 0$$

into another, whose roots shall each be smaller than those of the proposed by $\frac{1}{3}$.

The transformed equation is $3x^4 - 9x^3 - 4x^2 - \frac{65}{9}x - \frac{34}{3} = 0$.

PROPOSITION II.

(73.) To transform an equation into another whose roots shall be the reciprocals of those of the former.

In the proposed equation

$$N + Ax + A_2 x^2 + A_3 x^3 + \;.\;.\;.\;.\; A_n x^n = 0,$$

substitute $\frac{1}{y}$ for x, then the values of $\frac{1}{y}$ will be the same as those of x, and, consequently, the values of y will be the reciprocals of those of x; that is, the roots of the equation

$$N + \frac{A}{y} + \frac{A_2}{y^2} + \frac{A_3}{y^3} + \ldots \frac{A_n}{y^n} = 0,$$

or, rather of

$$Ny^n + Ay^{n-1} + A_2y^{n-2} + A_3y^{n-3} + \ldots . A_n = 0,$$

will be the reciprocals of the roots of the proposed equation. Hence the transformed equation is deduced from the original, simply by reversing the order of the coefficients; as many terms, therefore, as are absent in the original equation, so many and no more will be absent in the transformed.

Hence we may transform an equation into another, whose roots shall be less or greater than the reciprocals of those of the proposed, by applying the process employed in last proposition to the coefficients of the given equation, written in reverse order. For example, let it be required to transform the equation

$$x^4 - 12x^2 + 12x - 3 = 0,$$

into another, whose roots shall be equal to the reciprocals of those of the given equation, diminished by 1.

$$\begin{array}{rrrrrl}
-3 & 12 & -12 & 0 & 1 & (1 \\
 & 9 & -3 & -3 & -2 & \\
 & 6 & -3 & 0 & & \\
 & 3 & 6 & & & \\
 & 0 & & & &
\end{array}$$

Hence, the transformed equation is

$$-3y^4 \pm 0y^3 + 6y^2 \pm 0\,y - 2 = 0;$$

or rather

$$3y^4 - 6y^2 + 2 = 0.$$

(74.) If the coefficients of the proposed equation be the same when taken in reverse order, as when taken in direct order, both in signs and numerical values, it is obvious that the reciprocals of the roots will furnish the same series of quantities as the roots themselves, seeing that the equation which involves the reciprocal roots will be the same as the original equation; the roots of the original equation must, therefore, under such circumstances, be of the form

$$a, \frac{1}{a}; \ a_2, \frac{1}{a_2}; \ a_3, \frac{1}{a_3}; \ \&c.$$

$$\text{or } -1; \ a_1, \frac{1}{a_1}; \ a_2, \frac{1}{a_2}; \ a_3, \frac{1}{a_3}; \ \&c.$$

of which the reciprocals produce the same series of quantities.

The first row of roots, where each is accompanied by its reciprocal, will belong to equations of an even degree: the second to those of an odd degree, the signs of the coefficients observing the law just stated. It will appear presently that the single isolated root, in an equation of an odd degree, must be, as written above, — 1, the first term of the equation being, as usual, positive.

If the equation be of an odd degree, and the coefficients, taken in reverse order, be in magnitude the same as when taken in direct order, but in signs all different, then also will the roots of the transformed equation be identical with those of the original equation; for, by changing all the signs of the transformed equation, which of course produces no change in the roots, the equation will become the same as the original, and must, therefore, have the same roots. The same thing evidently has place in equations of an even degree, under like circumstances, provided only the middle term be absent. If the middle term be present, then the signs taken in reverse order cannot all be contrary to those taken in direct order: the middle term will interfere with this arrangement of the signs.

Equations whose coefficients exhibit either of these laws, and whose roots are, in consequence, of the above form, are called *recurring equations*, or *reciprocal equations*. They will be more fully treated of hereafter.

In a recurring equation of an odd degree, one root will always be + 1 or — 1, according as the sign of the last term is — or + ; for, as the roots of the transformed are always the same as those of the original in recurring equations, and yet at the same time the roots of the transformed are the reciprocals of those of the original, one of the odd number of roots must be + 1, or — 1 ; moreover, as the remaining roots consist of pairs, having the same sign, the last term of the equation, which is the product of all the roots with their signs changed, must take the opposite sign to the root unity, being — when that root is +, and + when the root is —.

PROPOSITION III.

(75.) To transform an equation into another, whose roots shall be given multiples or submultiples of those of the proposed equation.

Let the given equation be freed by division from the coefficient of the first term;* then, in the resulting equation, the coefficient of the second term will be the sum of the roots with contrary signs; the next coefficient, the sum of the products, two and two; the next, the sum of the products, three and three, signs being changed, and so on (prop. iv. p. 55) : hence, for the roots to be m times as great, we must multiply the second term by m, the third by m^2, the fourth by m^3, and so on. These multiplications being effected we may introduce any additional factor into all the transformed coefficients without disturbing the roots: and thus the coefficient of the leading term of the equation, temporarily removed by division, may now be restored by multiplication. In practice however we may evidently leave the original coefficient undisturbed, and proceed at once with the multiplications here directed.

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* It is necessary to say freed by *division*, in order that the roots may be preserved unaltered. The present proposition furnishes other means of removing the first coefficient, but not without changing the roots.

If, for example, it be required to transform the equation

$$2x^3 - 5x^2 + 7x - 12 = 0,$$

into another, whose roots are three times as great, we shall merely have to multiply the second term by 3, the third by 9, and the fourth by 27; the transformed will therefore be

$$2x^3 - 15x^2 + 63x - 324 = 0.$$

It is an obvious inference from the preceding rule, that if in an equation the coefficients of the second, third, fourth, &c. terms be divisible by m, m^2, m^3, &c., respectively, the roots will have the common measure m.

(76.) We may now easily prove the property mentioned at p. 60, viz., that when an equation, with integral coefficients, has a fractional root, the final coefficient, N, is divisible by the numerator of the fraction, and the leading coefficient, A_n, by the denominator. For let $\frac{a}{b}$ be the fractional root, a and b having no common factor: then if the equation be transformed into another whose roots are b times as great, the final term of the transformed will be b^nN. One root of this transformed equation will be the integer a, therefore (p. 57) a is a factor of b^nN. But as a has no factor in common with b, by hypothesis, a cannot be a factor of b^n: hence a must be a factor of N.

Again, let the proposed equation be converted into another whose roots are the reciprocals of the original roots (73). The leading coefficient of the transformed equation will be N, and the absolute term A_n: also $\frac{b}{a}$ will be a root of this equation. As before, let the last equation be transformed into another whose roots are a times as great: the absolute term will then be a^nA_n, which (p. 57) is divisible by b. But, as before, b cannot be a factor of a^n: hence it must be a factor of A_n.

It thus appears that when the first member of an equation whose coefficients are all integral, is divided by the first member of the simple equation involving one of the roots,—numerical factors common to both terms of the divisor being expunged,—the extreme terms of the quotient must be free from fractions. But this is only a particular case of a much more general property, viz., that if the first member of an equation be divided by the first member of any other equation whose roots all belong to the former—the coefficients of both dividend and divisor being free from fractions, and those of the latter free from common factors —then *all* the terms of the quotient will be free from fractions. This general proposition will be found of considerable use in the analysis of equations: it may be established as follows:

Let the first member of the proposed equation, when the coefficients are freed from common factors, be

$$ax^{n} + bx^{n-1} + cx^{n-2} + \dots . k \dots . [1],$$

and the first member of the equation involving m of its roots

$$a' x^{m} + b' x^{m-1} + \dots . k' \dots . [2],$$

common factors being expunged.

All the simple factors of [2] occur among those of [1], so that [1] is divisible by [2] without remainder. Suppose that, in order to preclude the entrance of fractions into the quotient arising from this division, it be necessary to multiply [1] by some factor P, and let the quotient whose coefficients are thus all rendered integral be

$$a'' x^{p} + b'' x^{p-1} + \dots . k'' \dots . [3].$$

This quotient has no factor common to all its terms, otherwise P would involve this factor unnecessarily, and would therefore be needlessly large.

Now since the operation terminates without a remainder, we must have

$$[2] \times [3] = \mathrm{P} \times [1].$$

The second member of this equation is divisible by the number P: hence the first member must be divisible by P. But neither [2] nor [3] is divisible by any number except unity. Consequently P = 1,* so that the division of [1] by [2] furnishes

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* We may prove that the first member of the above equation can have no numerical factor, as follows:—Suppose q to be any prime number, and let R represent the aggregate of those terms in [2], whose coefficients contain the

a quotient free from fractions, without any previous preparation.

It follows from this that, when the proposed equation is free from fractions, every depressed equation, arising from the elimination of any number of its roots—by dividing by the integral polynomial involving those roots—must also be free from fractions. Hence, when it is suspected that a polynomial, free from fractions and common factors, involves roots all of which belong to a proposed equation, we may proceed to divide the latter polynomial by the former; and, if a fractional coefficient occur in the quotient, may discontinue the operation, and conclude that the supposed connexion between the polynomials has not place; but if no fraction occur, and the operation terminate without remainder, then of course the suspicion is verified. The same conclusions follow although the suspected divisor be divided by any integer—as for instance by the leading coefficient in it—notwithstanding the fractional coefficients thence introduced; since the only effect upon the quotient will be to introduce this integer, as a factor, into all its terms.

(77.) By help of the transformation in (75), the coefficient of the first term of an equation may be removed without intro-

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factor q, R′ representing the remaining terms. In like manner let S be the sum of the terms in [3], whose coefficients all involve q, S′ being the sum of the remaining terms; then

$$R + R' = [2]$$
$$S + S' = [3]$$

Hence, multiplying these together,

$$RS + R'S + RS' + R'S' = [2] \times [3].$$

And since by hypothesis q divides R and S, it divides the first three terms of this equation: hence, that it may divide the product of [2] and [3], it must divide the fourth term R′S′: but this is impossible; for let hx^r, $h'x^{r'}$ be the two terms in R′ and S′ respectively, affected with the *highest* power of x; then since their product, $hh'x^{r+r'}$, must form a distinct term of the product R′S′, because all the other component parts of it involve inferior powers of x, it follows that hh' must be divisible by q; which is absurd, because there is no factor either in h or h' equal to q, seeing that all terms containing this factor have been excluded from R′ and S′.

ducing fractions; for, if m be the coefficient of the first term, and we transform the equation into another, whose roots are m times those of the former, the factor m will then enter into all the other terms; dividing by it will therefore free the first term, and introduce no fractions. The transformed equation will therefore be obtained by expunging the coefficient of the first term, preserving the second term, multiplying the third by m, the fourth by m^2, &c. and the roots of the transformed will be m times those of the original. Thus, taking the equation

$$3x^3 - 5x + 2 = 0,$$

which, completed, is

$$3x^3 + 0x^2 - 5x + 2 = 0;$$

we have for the transformed, whose roots are three times as great, the equation

$$x^3 + 0x^2 - 15x + 18 = 0,$$

or, rather

$$x^3 - 15x + 18 = 0.$$

Fractions may be removed from an equation by transforming the equation into another, whose roots are those of the former, multiplied by the product of the denominators of the fractions, or by a common multiple of the denominators. For example, the equation $x^3 + \frac{1}{2}x^2 - \frac{1}{3}x + 2 = 0$, will be transformed into $x^3 + 3x^2 - 12x + 432 = 0$, by multiplying the terms, commencing at the second, by the successive powers of 6; and, if the roots of the former equation be a_1, a_2, a_3, those of the latter will be $6a_1$, $6a_2$, $6a_3$.

This method will always effect the removal of the fractions, although not always without introducing a new inconvenience; namely, very large coefficients. If the attainment of simplicity therefore be the ultimate object, we shall often find it better to clear the fractions in the ordinary way; that is, by multiplying all the terms by the least common multiple of the coefficients, although we thus introduce a coefficient into the leading term. But which of these two methods is to be preferred in any particular case, must be determined from the character of the denominators themselves, which may happen to be so related to one another that they may all be removed according to the first method by employing only a small *submultiple* of their common multiple. For instance, the equation

$$x^4 + \frac{x^3}{2} - \frac{x^2}{3} - \frac{x}{36} + \frac{5}{648} = 0,$$

will, agreeably to the first method, be converted into

$$x^4 + 3x^3 - 12x^2 - 6x + 10 = 0,$$

an equation whose roots are six times those of the former, by employing the multipliers 6, 6^2, 6^3, 6^4, instead of the corresponding powers of the common multiple 648; 6 being a small submultiple of this.

(78.) In what has preceded, the roots of the proposed equation are considered to be *multiplied* by m in the transformed: but if they be regarded as divided by m, then the terms of the equation, commencing with the second, will have to be divided by m, m^2, m^3, &c. respectively; and as these divisors may all be removed by multiplying each of the terms by the highest power of m, it follows that the general equation of the nth degree

$$A_n x^n + A_{n-1} x^{n-1} + \ldots A_3 x^3 + A_2 x^2 + Ax + N = 0$$

will become

$$A_n m^n x^n + A_{n-1} m^{n-1} x^{n-1} + \ldots A_3 m^3 x^3 + A_2 m^2 x^2 + Amx + N = 0$$

when its roots are all divided by m. And it may be further remarked, that whatever powers of x be absent in the original equation, the same powers will be absent in the transformed, whether the roots be multiplied or divided.

PROPOSITION IV.

(79.) To transform an equation into another, in which any proposed term shall be absent.

If the transformed equation is to be deprived of its *second* term, which is the term generally required to be removed, the transformation may be effected by the process in Problem I. p. 70, as it will be merely required to diminish the roots by such a quantity, r, as will cause the second coefficient in the resulting equation to vanish. Now, in the process of diminishing the roots, it is seen, that when the leading coefficient is unity, r is added to the second term n times; so that for the result of these additions to be zero, r must be minus the nth part of the second coefficient in the proposed equation. To illustrate this, let it be required to remove the second term from the equation

$$x^4 - 12x^3 + 17x^2 - 9x + 7 = 0.$$

Here $r = \frac{12}{4} = 3$, and the operation is as follows:

$$\begin{array}{rrrrrl}
1 & -12 & 17 & -9 & 7 & (3 \\
 & 3 & -27 & -30 & -117 & \\
\cline{2-4}
 & -9 & -10 & -39 & & \\
 & 3 & -18 & -84 & & \\
\cline{2-3}
 & -6 & -28 & & & \\
 & 3 & -9 & & & \\
\cline{2-2}
 & -3 & & & & \\
 & 3 & & & & \\
\hline
x^4 & +\,0x^3 & -\,37x^2 & -\,123x & -\,110 = 0. &
\end{array}$$

hence the transformed equation is

$$x^4 - 37x^2 - 123x - 110 = 0,$$

the roots of which are those of the proposed diminished by 3.

(80.) But in order to determine the value of r, necessary to cause any other coefficient to vanish, let us actually substitute $x' + r$ for x, in the general equation

$$x^n + A_{n-1}\, x^{n-1} + \ldots . \, Ax + N = 0,$$

and then develop the several powers by the binomial theorem, arranging the result according to the decreasing powers of x'; we shall thus have

$$\begin{array}{r|ll|ll|l}
x'^n + nr & x'^{n-1} & + \frac{n(n-1)}{2} r^2 & x'^{n-2} + \ldots . & + r^n & =0 \\
+ A_{n-1} & & + (n-1)\, A_{n-1}\, r & & + A_{n-1}\, r^{n-1} & \\
& & + A_{n-2} & & + A_{n-2}\, r^{n-2} & \\
& & & & \ldots\ldots & \\
& & & & \ldots\ldots & \\
& & & & + Ar & \\
& & & & + N &
\end{array}$$

In order that the second term of this transformed equation may vanish, we must have the condition

$$nr + A_{n-1} = 0 \therefore r = -\frac{A_{n-1}}{n},$$

as before determined.

That the third term may vanish, we must have the condition

$$\frac{n(n-1)}{2} r^2 + (n-1)\, A_{n-1}\, r + A_{n-2} = 0;$$

which, being a quadratic equation, will furnish two values for r, each of which will cause the third term in the transformed equation to vanish.

The determination of values for r, that will cause the fourth term to vanish, will require the solution of an equation of the third degree; and, to remove the last term N, would require the solution of the following equation of the nth degree in r, viz. the equation

$$r^n + A_{n-1} r^{n-1} + \dots . Ar + N = 0;$$

which is no other than the proposed, x being replaced by r; so that the removal of the last term requires a preparatory process, equivalent to solving the original equation.

(81.) It may be remarked here, that methods have been investigated for removing from an equation as many intermediate terms as we please, with the view of reducing equations of the advanced degrees into soluble forms. This method was first proposed by TSCHIRNHAUSEN in 1683; and it has been recently revived and treated with much analytical skill and address by Mr. JERRARD, in his *Mathematical Researches.* But like all other attempts to extend the limits of the general problem of the solution of algebraical equations by finite formulæ, beyond equations of the fourth degree, these methods have proved unsuccessful: the imperfection common to all of them being that, when applied beyond these bounds, their application requires the solution of equations higher in degree than that which is proposed to be solved.

The only term which in the present state of algebra it is of any consequence to remove from an equation is the second; in most of the methods proposed for solving equations the absence of this term conduces to the simplicity of the operation, whether the solution be by an algebraical formula or by a process purely numerical. In the latter mode of treating the problem, however, the advantage of this absence is felt but in a trifling degree, and that chiefly in the preparatory analysis of the equation for the purpose of discovering the nature of the roots.

We may further observe with respect to the transformed equation to which the removal of this term leads, that although the roots of this equation will differ from those of the original, yet the *differences* of the roots of the original will be the same as the differences of the roots of the transformed; so that when these differences are to be determined, with a view to the discovery of the roots themselves—as in the method of LAGRANGE, to be hereafter explained—we may substitute the transformed equation for the original, and this effects considerable saving in the labour attendant upon the method adverted to.

By removing the second term from a quadratic equation, we shall be immediately conducted to the well-known formula for its solution. Thus, the equation being

$$x^2 + \mathrm{A}x + \mathrm{N} = 0,$$

the transformed in $x' + r$, will be

$$\begin{array}{r|l} x'^2 + 2r & x' + \left.\begin{array}{l} r^2 \\ + \mathrm{A}r \\ + \mathrm{N} \end{array}\right\} = 0; \\ + \mathrm{A} & \end{array}$$

and, that its second term may vanish, we must have

$$2r + \mathrm{A} = 0 \therefore r = - \tfrac{1}{2}\mathrm{A},$$

which condition reduces the transformed to

$$x'^2 - \tfrac{1}{4}\mathrm{A}^2 + \mathrm{N} = 0$$

$$\therefore x' = \pm \sqrt{\tfrac{1}{4}\mathrm{A}^2 - \mathrm{N}}$$

$$\therefore x = x' + r = - \tfrac{1}{2}\mathrm{A} \pm \sqrt{\tfrac{1}{4}\mathrm{A}^2 - \mathrm{N}};$$

which is the common formula for the solution of a quadratic equation.

(82.) All the problems discussed in the present chapter, as well as that just commented upon, equally deserve notice, as furnishing the necessary operations preparatory to the actual solution of an equation. To facilitate the process of solution, or to convert the equation from an inconvenient form into another better adapted to certain rules and formulæ for evolving the roots, is the only object proposed to be accomplished by the transformation of equations. The property inferred at (61), and which suggests the solution of the problem—to transform an equation into another whose roots shall be the same in value but opposite in sign—does in strictness belong to this department of our subject. It is indeed included in Proposition III, as the roots of the transformed equation are no other than those of the original multiplied by -1. This transformation is useful in the analysis of equations; as the examination need be applied only to the detection of positive roots, into which the negative are converted by simply changing the alternate signs.

The more general transformation, of which this is a particular case, finds its application as a preparatory step in the search after commensurable roots. We have already seen that if the leading coefficient be unity and the others integral, the commensurable roots must be integral also. But if the leading coefficient be other than unity, the commensurable roots may be fractional; and cannot therefore, like the former, be found among the integral divisors of the absolute term. The transformation referred to, by changing these fractional roots into integral, will sometimes be found to facilitate the search after the commensurable roots.

The same transformation may also be usefully applied even in connexion with the very effective mode of solution that will be more especially dwelt upon in the practical part of this work—HORNER's method: and which is independent of the preparatory operations that most other methods require. By means of this transformation however, each of the commensurable roots will always be presented by HORNER's process in a finite form; whether they be integral or fractional, instead of appearing, in the latter case, in the form of an incomplete or approximative decimal. An objection that has been hastily made against the method in question—viz. that *approximate values* only are furnished by it when the roots are fractional—is thus completely removed.

The transformation at (73), by which roots are converted into their reciprocals, is another preparation essential to the success of certain methods of solution. More especially is this the case as respects the method of Budan, to be discussed hereafter; and advantage may be occasionally derived from it even in the more perfect process of Horner, as will be sufficiently seen when we come to examine into the capabilities of that method.

It is further worthy of notice that, by aid of this transformation, the general proposition in (79) may be considerably simplified. In examining the different cases of that proposition, it was seen that the difficulty of applying it increased as the term to be removed approached nearer to the final term of the polynomial. But if instead of removing a term near the end, we were to remove the term as near to the beginning, and then transform the resulting equation into another whose roots are the reciprocals of it, an equation would be obtained in which the term at the proposed distance from the final term would be zero; since the coefficients in the direct equation, reckoning from the first to the last, are the same as those in the reciprocal equation, reckoning from the last to the first. Hence, if any method of solution were proposed that would be facilitated by the removal of the last term but one, as certain existing methods are facilitated by the removal of the second term, the requisite preparation would be easily made by removing the second term, and then reversing the order of the coefficients; observing that the roots of the equation thus deduced are the reciprocals of those of the original equation, after the removal of its second term.

The importance of the transformation proposed in Proposition I will be distinctly seen in connexion with Newton's rule for finding a superior limit to the roots of an equation; and, as before observed, in connexion with the numerical operation for obtaining the roots themselves.

CHAPTER VI.

ON DETERMINING THE LIMITS TO THE REAL ROOTS OF EQUATIONS.

(83.) In the concluding observations in last chapter we have briefly explained the objects which analysts have in view in attempting certain changes in the form merely of algebraical equations, preparatory to the application of methods of solution. When these transformations are effected, wherever they are found necessary, the first step in the actual solution may be commenced; and this, as far at least as numerical equations are concerned, consists in the determination of the places in the numerical scale which the several real roots occupy. The satisfactory settlement of this important point constitutes what is called the *analysis of the equation.* And we are fortunately at length in possession of methods—the methods of Sturm and Fourier—by which this analysis may always be completed; that is to say, when any two numbers are proposed, we can always discover whether or not roots lie between them, and how many roots are so situated; and then, by narrowing the limits between which they are thus found to lie, we can find exactly between what two consecutive numbers each root, or each group of nearly equal roots, must be posited. In other words, we can completely analyse the equation.

In commencing this analysis it would clearly be unwise to fix upon two numbers at random; for these might be very remote from the extreme limits of the entire series of roots; and it is for the purpose of guiding us to a judicious selection of superior and inferior limits to the roots—that is, numbers above the greatest root and below the least—that the propositions in the present chapter are of any value. It is scarcely necessary to remark upon the advantage of such limits, in diminishing the range of divisors of the last term, in seeking for the commensurable roots.

The remotest limits to every set of positive roots are obviously the extreme values 0 and $\frac{1}{0}$. The corresponding limits to the negative roots are 0 and $-\frac{1}{0}$. But the *imaginary* roots of an equation cannot be regarded with propriety as being comprehended within these; because, in passing continuously from zero to infinity, that is from $x=0$ to $x=\pm\frac{1}{0}$, we should never pass an *imaginary* value of x; so that imaginary roots have no *real* limits: and therefore, in the subsequent propositions respecting limits, it must be borne in mind that these limits are always spoken of in reference to the real roots only.

(84.) We may define a superior limit to the positive roots of an equation, $f(x)=0$, as any positive number which is greater than the greatest root of the equation. Its distinguishing character is therefore such, that when it, or any number greater than it, is substituted for x in the polynomial $f(x)$, the result will always be too great; that is, always positive.

An inferior limit to the negative roots is a number nearer to $-\frac{1}{0}$ than the numerically greatest of these. Its character is such that, whatever number still nearer to $-\frac{1}{0}$, be substituted for x in the polynomial $f(x)$, the sign of the result shall invariably be the same as the sign furnished by it itself. This sign may be either positive or negative; it will be *positive* if the degree of the equation be *even*, and *negative* if the degree be *odd*; since an even power of $-\frac{1}{0}$ is plus, and an odd power minus.

Hence, in searching for close superior and inferior limits, our object will be to find the pair of smallest numbers, positive and negative such that, commencing with the positive, and proceeding continuously onwards towards $+\frac{1}{0}$, the results of the substitutions can never either change sign or become zero; and commencing with the negative limit, and proceeding in like manner towards

$-\frac{1}{0}$, the results can neither change sign nor become zero. But as remarked at (82) it will in all cases be sufficient that we know how to find close limits to *positive* roots; since the negative roots become positive by taking the alternate terms of the equation with contrary signs.

It may be proper to add further, that in the following propositions we shall consider the coefficient of the leading term of the equation to be unity. This condition is essential in the first three propositions: those that succeed these however apply independently of any such restriction.

PROPOSITION I.

(85.) In any equation whose second term is negative and all the other terms positive, the coefficient of the second term, taken positively, is a superior limit to the positive roots.

Let the equation be

$$x^n - A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + \dots + Ax + N = 0$$

Then, since

$$x^n - A_{n-1}x^{n-1} = (x - A_{n-1})\, x^{n-1}$$

the equation may be written thus, viz.

$$(x - A_{n-1})x^{n-1} + A_{n-2}x^{n-2} + \dots + Ax + N = 0$$

If A_{n-1} be substituted for x in the left-hand member, the first term will vanish: and as the other terms are all positive, the entire result of this substitution must be positive. If a quantity greater than A_{n-1} be substituted, the terms after the first must still continue positive, while the first itself, instead of becoming zero, will be also positive. Hence the result is always positive for values of x not less than A_{n-1}. This quantity, therefore, is a superior limit to the positive roots.

PROPOSITION II.

(86.) In any equation the greatest negative coefficient taken positively and increased by unity is a superior limit to the positive roots.

This proposition has been demonstrated, virtually, at (26), where it is shown that if the greatest negative coefficient be increased by unity, then it, and every quantity greater than it, when substituted for x in the equation, will render the first member of it always positive. Hence the greatest negative coefficient so increased must be a superior limit to the positive roots.

This proposition is an obvious inference from the method proposed by NEWTON for finding a superior limit, and which will be given hereafter. It was thus inferred by MACLAURIN;* and is hence often called *Maclaurin's limit*.

PROPOSITION III.

(87.) In any equation of the nth degree, if x^{n-k} be the power involved in the first negative term, and $-$ P be the greatest negative coefficient, then will $P^{\frac{1}{k}} + 1$ be a superior limit to the positive roots.

Let us take the most unfavorable case, viz. that in which all the terms, from the term involving x^{n-k} inclusive, are negative, and affected with the coefficient P. Then it is plain that the proposed polynomial will necessarily be positive for every value of x which renders the first term greater than the sum of all these: that is, the polynomial will be positive provided we satisfy the condition

$$x^n > P\,(x^{n-k} + x^{n-k-1} + \ldots + x + 1)$$

or, which is the same thing, the condition

$$x^n > P\,\frac{x^{n-k+1} - 1}{x - 1}.$$

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* MACLAURIN'S Algebra, page 174.

Now, assuming x greater than unity, this latter condition is always satisfied when

$$x^n > P\frac{x^{n-k+1}}{x-1}$$

since the right-hand member of this inequality is greater than the right-hand member of the former.

Multiplying each member by $x - 1$, and then dividing each by x^{n-k+1}, we have

$$(x-1)\,x^{k-1} > P$$

which condition is satisfied whenever either of the following is, viz.

$$(x-1)^k = P, \text{ or } (x-1)^k > P,$$

because the first member of either of these is less than the first member of the former. We have then to determine x so that

$$x = P^{\frac{1}{k}} + 1, \text{ or } x > P^{\frac{1}{k}} + 1$$

Hence $P^{\frac{1}{k}} + 1$ exceeds the greatest root of the equation.

The following examples will serve to show the application of this proposition.

1. $x^4 - 5x^3 + 37x^2 - 3x - 4 = 0$

$\therefore P = 5,\ k = 1\ \therefore\ P^{\frac{1}{k}} + 1 = 6 =$ *superior limit.*

2. $x^5 + 7x^4 - 12x^3 - 49x^2 + 52x - 13 = 0$

$\therefore P = 49,\ k = 2\ \therefore\ P^{\frac{1}{k}} + 1 = 8 =$ *sup. limit.*

3. $x^4 + 11x^2 - 25x - 67 = 0$

$\therefore P = 67,\ k = 3\ \therefore\ P^{\frac{1}{k}} + 1 = 5 + \ldots \therefore 6 =$ *sup. limit.*

4. $3x^3 - 2x^2 - 11x + 4 = 0$

$\therefore P = \frac{11}{3},\ k = 1\ \therefore\ P^{\frac{1}{k}} + 1 = 4\frac{2}{3}\ \therefore 5 =$ *sup. limit.*

PROPOSITION IV.

(88.) If, in an equation, — P be the greatest negative coefficient, and if, among those positive terms which precede the first negative term, the greatest coefficient S be taken, then will $\frac{P}{S}+1$ be a superior limit to the positive roots of the equation.

The most unfavorable case will be that in which all the terms that follow the first negative term are also negative, and their coefficients equal to P. Under these circumstances, the equation may be written

$$-P(1+x+x^2+\dots x^m)+Sx^{m+1}+Tx^{m+2}+Ux^{m+3}+\dots=0\dots[1]$$

Now the negative portion of this polynomial will be

$$-P\frac{x^{m+1}-1}{x-1},$$

which, by substituting $\frac{P}{S}+1$ for x, becomes

$$-S\left\{\frac{P}{S}+1\right\}^{m+1}+S.$$

Also the positive portion, by a like substitution, becomes

$$S\left\{\frac{P}{S}+1\right\}^{m+1}+T\left\{\frac{P}{S}+1\right\}^{m+2}+U\left\{\frac{P}{S}+1\right\}^{m+3}+\dots$$

of which the first term alone exceeds the former portion; therefore the aggregate of both portions must be positive. If the coefficient S belonged to a term more advanced, it is obvious that the excess of the positive portion above the negative would be increased. It is easy to see that, when any value of x is found that will cause the positive part of [1] to exceed the negative, every higher value of x will have a similar effect; for, if we divide both portions by x^{m+1}, the first will consist of a series of fractions in x, and will consequently diminish as x increases; while the second part will continually increase with x. Hence $\frac{\text{P}}{\text{S}} + 1$ is a superior limit to the positive roots of the equation.

Applying this method of finding a limit to the examples in the preceding proposition, we have, for the limit in the first example,

$$\text{1st, } \frac{\text{P}}{\text{S}} + 1 = \frac{5}{1} + 1 \therefore 6 = \textit{superior limit}$$

$$\text{2d, } \frac{\text{P}}{\text{S}} + 1 = \frac{46}{7} + 1 \therefore 8 = \textit{superior limit}$$

$$\text{3d, } \frac{\text{P}}{\text{S}} + 1 = \frac{67}{11} + 1 \therefore 8 = \textit{superior limit}$$

$$\text{4th, } \frac{\text{P}}{\text{S}} + 1 = \frac{11}{3} + 1 \therefore 5 = \textit{superior limit.}$$

The limits given by this method are, in these examples, the same as those before determined, with the exception of that in the 3d example, to which the former method is applied with more success. In the following example, however, this latter method of finding a near superior limit has greatly the advantage:

$$x^4 + 16x^3 - 2x^2 - 12x - 6 = 0,$$

$$\therefore \frac{\text{P}}{\text{S}} + 1 = \frac{12}{16} + 1 \therefore 2 = \textit{superior limit.}$$

By the former method the limit would be

$$\text{P}^{\frac{1}{2}} + 1 = 12^{\frac{1}{2}} + 1 \therefore 5 = \textit{superior limit.}$$

The limit established in this proposition was first given by M. Vène in the *Mémoires de l'Académie de Bruxelles*, 1822.

PROPOSITION V.

(89.) If each negative coefficient be taken positively, and divided by the sum of all the positive coefficients which precede it, the greatest quotient thus obtained, when increased by unity, will be a superior limit to the positive roots.

The demonstration of this proposition depends upon the known expression for the sum of a geometrical series, from which we infer that

$$\frac{x^m - 1}{x - 1} = x^{m-1} + x^{m-2} + x^{m-3} + \; . \; . \; . \; . + x + 1$$

and consequently that every power x^m, of x, may itself be represented by a polynomial of the form

$$x^m = (x - 1) \left\{ x^{m-1} + x^{m-2} + x^{m-3} + \; . \; . \; . \; . + x + 1 \right\} + 1$$

where the series within the braces involves *all* the powers of x inferior to the power represented.

Let the *positive* terms of the equation

$$A_n x^n + \; . \; . \; . \; . \; A_2 x^2 + Ax + N = 0 \; . \; . \; . \; . \; [1]$$

be each replaced by its equivalent polynomial: we shall have for the first of these

$$A_n x^n = A_n(x-1)x^{n-1} + A_n(x-1)x^{n-2} + A_n(x-1)x^{n-3} + \, . \, . \, A_n(x-1) + A_n$$

and if the other polynomials be written under this, so that like powers of x may range vertically, we shall have for the sum of all these positive terms a polynomial of the form

$$A_n(x-1)x^{n-1} + A'_{n-1}(x-1)x^{n-2} + A'_{n-2}(x-1)x^{n-3} + \, . \, . \, A'(x-1) + N' \, . \, . [2]$$

in which all the powers of x, inferior to the leading power x^n, necessarily occur.

Let the different negative terms of [1], that have been omitted, be now introduced in their proper places under the like terms of [2]. It is plain that more than one negative term cannot occur under any positive term of [2], since in different negative terms the powers of x are different.

The negative terms of [1] being thus introduced into the sum of the positive terms of [1] as represented by [2], we shall of course obtain the original polynomial [1] under a new form; and it is from contemplating it under this change of form that the proposition announced discovers itself. For it is plain that the polynomial in question will always be positive provided that x be greater than 1, and that moreover no negative term exceeds the positive term in [2] under which it is placed; since by the first condition all the terms of [2] are positive, and by the second, however many of these may be balanced by the negative terms written under them, none can be overbalanced; so that the aggregate must continue positive. These then are the only conditions we have to fulfil.

Suppose the first negative term that occurs in [1] to be $-A_p x^p$; the proper place for this under [2] will be below the term $A'_{p+1}(x-1)x^p$; so that, with respect to these, the second condition will be fulfilled, that is, the sum of these two terms will be positive, provided

$$A'_{p+1}(x-1) > A_p, \text{ or } x > \frac{A_p}{A'_{p+1}} + 1$$

and the same sum will be zero if these two expressions be equal.

In like manner, if the next negative term that occurs in [1] be $-A_q x^q$, the sum of it and the like term above it in [2], will be positive, provided

$$A'_{q+1}(x-1) > A_q, \text{ or } x > \frac{A_q}{A'_{q+1}} + 1$$

and it will be zero if x be equal to, instead of greater than, this expression. And by thus taking every negative term of [1], and comparing it with the like term of [2], we shall obtain the several partial conditions which, if simultaneously satisfied for any value of x, will render that value a superior limit to the roots of [1]: for the other condition, $x > 1$, is, we see, necessarily comprehended in each of the foregoing.

Now it is obvious that each of the partial conditions referred to will be fulfilled if from among the fractions which enter them, viz.,

$$\frac{A_p}{A'_{p+1}}, \ \frac{A_q}{A'_{q+1}}, \text{ \&c.}$$

we select the *greatest*, and cause x to fulfil the condition involving it alone. But the coefficient, A'_{q+1}, of any term $A'_{q+1}(x-1)x^q$, in the development [2], is no other than the sum of all the *positive* coefficients in [1] which precede the term $A_q x^q$. Hence, if fractions be formed by taking each negative coefficient positively, and dividing it by the sum of all the positive coefficients that precede it, the greatest of these fractions, increased by unity, will be a superior limit to the positive roots of the equation.

(90.) The preceding proposition was first given by M. Bret, in the sixth volume of the *Annales des Mathématiques*, and as a general principle, may be regarded as the most effective that has yet been proposed for finding a close superior limit to the positive roots of an equation. The limit of Maclaurin, given in Proposition II, and which, from the readiness of its application, is that which is most frequently employed, is evidently included in the limit of Bret as a particular case.

The most unfavorable case for the application of the present proposition will be that in which the greatest negative coefficient is preceded by positive coefficients whose sum is comparatively small; when the method of Vène may give a limit equally close, and that in Proposition II, one still closer. It is plain that the limit determined by the former of these methods can never be closer than that found by the present proposition; and hence we may infer that when our object is to obtain the closest limit in any case, we need apply only the two propositions (87) and (89).

If we take the examples already given (87), the limits determined by the present proposition are as follow:

$$1.\quad \frac{5}{1}+1\ \therefore\ \text{limit}=6;\quad 2.\quad \frac{49}{8}+1\ \therefore\ \text{limit}=8;$$

$$3.\quad \frac{67}{12}+1\ \therefore\ \text{limit}=7;\quad 4.\quad \frac{11}{3}+1\ \therefore\ \text{limit}=5.$$

In the third of these examples the method of (87) gives a closer limit.

Suppose the following examples were proposed, viz.

1. $4x^5-8x^4+23x^3+105x^2-80x+11=0,$

then it is plain that the greatest fraction is $\frac{8}{4} = 2$; therefore 3 is a superior limit to the positive roots.

2. $2x^7 + 11x^6 - 10x^5 - 26x^4 + 31x^3 + 72x^2 - 230x - 348 = 0.$

The fractions are

$$\frac{10}{13}, \frac{26}{13}, \frac{230}{116}, \frac{348}{116},$$

the last of which, equal to 3, is the greatest; therefore 4 is a superior limit.

These propositions on limits refer exclusively to the *positive* roots of the equation; the only class of roots which, as remarked at (82), need be attended to in the enquiry, on account of the facility with which the negative roots may be converted into positive roots. When this conversion has been effected, and the superior limit determined by any of the preceding propositions, this limit taken negatively, will be numerically greater than the greatest negative root of the proposed equation. If the coefficients of the proposed be all positive, then it can have no real roots but negative roots; so that the foregoing propositions can apply only after the change adverted to, of the negative roots into positive, has been made.

With respect to the determination of *inferior* limits to the roots, it will be sufficient to remark that the propositions already established are applicable to the discovery of these after the equation has, by (73), been converted into another whose roots are the reciprocals of those of the proposed. For the greatest root in the transformed equation will be the reciprocal of the least in the original equation, and vice versâ; so that if a superior limit be found from the transformed equation, the reciprocal of this will be an inferior limit in reference to the original equation; and this whether we regard the positive roots or the negative roots.

It is sufficient therefore that the propositions for the determination of limits to the roots of an equation, comprehend all that concerns the *superior* limits to the *positive* roots; since everything else respecting the limits may be brought within the scope of these propositions by easy preparatory transformations. It should be observed, however, that in many cases these propositions may be altogether dispensed with, and a superior limit, closer even than they would give, inferred from roughly estimating the comparative importance of the positive and negative terms ; in which estimate we shall generally be assisted by bringing all the positive terms to the left of the sign of inequality $>$, and the negative terms to the right, and then seeking for the smallest value of x, for which, and for all higher values, the assumed inequality may be satisfied. The number sought will be most readily suggested if both members of the inequality be first divided by the highest power of x.

Let example 4 at (87) be taken, viz.

$$3x^3 - 2x^2 - 11x + 4 = 0\ ;$$

then we have to satisfy the condition

$$3x^3 + 4 > 2x^2 + 11x$$

or

$$3 + \frac{4}{x^3} > \frac{2}{x} + \frac{11}{x^2}.$$

For $x = 2$, the second member a little exceeds the first; but for $x = 3$, the inequality is fulfilled, and obviously for every higher number, as the first member can never decrease below 3. Hence the limit is 3.

Again, let the equation be

$$x^4 + x^3 - 15x^2 - 19x - 3 = 0.$$

Then we must have

$$x^4 + x^3 > 15x^2 + 19x + 3$$

or

$$1 + \frac{1}{x} > \frac{15}{x^2} + \frac{19}{x^3} + \frac{3}{x^4}$$

which is readily seen to be satisfied for $x = 5$, and for every higher number. Hence 5 is a superior limit. By (89) the limit would be 11 ; and by (87) it would be 6.

Besides the methods here explained there is another, that of NEWTON, by means of which a close limit to the positive roots may be obtained. It is however but seldom applied to this purpose, as it does not offer the facilities of the other methods. We shall postpone the consideration of it to next chapter; because it is intimately connected with the inquiries there to be discussed, and forms a suitable introduction to the methods of BUDAN and FOURIER for separating the roots of equations.

PROPOSITION VI.

(91.) If the real roots of an equation, ranged in the order of their magnitudes, be

$$a_1, a_2, a_3, a_4, \ldots$$

a_1 being the greatest, or that nearest to $+\infty$, a_2 the next in magnitude, and so on to the least, or that nearest to $-\infty$; and if a number b_1, greater than a_1, be substituted for x, the result will be positive; if a number b_2, in magnitude between a_1 and a_2, be substituted for x, the result will be negative; if a number b_3, between a_2 and a_3, be substituted, the result will be positive, and so on.

The first member of the proposed equation, after removing the leading coefficient, is the product of the simple factors

$$(x - a_1)\ (x - a_2)\ (x - a_3)\ (x - a_4)\ \ldots$$

multiplied by the quadratic factors involving the imaginary roots. Omitting these latter for the present, let us examine the effect of our proposed substitutions upon the product of the real factors. Putting then b_1 for x in these factors, we have

$$(b_1 - a_1)\ (b_1 - a_2)\ (b_1 - a_3)\ (b_1 - a_4) = \text{a } \textit{positive} \text{ number,}$$

because all the factors are positive.

Putting b_2 for x, we have

$$(b_2 - a_1)\ (b_2 - a_2)\ (b_2 - a_3)\ (b_2 - a_4) = \text{a } \textit{negative} \text{ number,}$$

because the first factor is negative, and all the others positive.

Putting b_3 for x, we have

$$(b_3 - a_1)\,(b_3 - a_2)\,(b_3 - a_3)\,(b_3 - a_4) = \text{a } positive \text{ number},$$

because the first two factors are positive, and the others negative. And by continuing these substitutions we should obviously thus obtain as many changes of sign in the results as there are real and unequal roots, and no more. Now the quadratic factors which we have omitted always give a positive result for every real value of x (59); consequently the introduction of these factors would cause no change in the foregoing results.

Hence, we may deduce the following inferences, viz.

1 :—If two numbers be successively substituted for x in any equation, and give results affected with *different* signs, then there lie between those numbers, one, three, five, or some *odd* number of roots :

And 2 :—If two numbers, when substituted successively for x, give results affected with the *same* sign, then there lie between those numbers, two, four, six, or some *even* number of roots, or else none at all.

(92.) From the preceding proposition it is plain that the places which the real positive unequal roots of an equation occupy would all be detected provided we knew a number either equal to, or less than, the smallest of the differences of every pair of these roots. For if $\triangle$ be such a number, we could never pass over more than a single root at a time by employing for our successive substitutions the arithmetical progression

$$0,\ \triangle,\ 2\triangle,\ 3\triangle,\ 4\triangle,\ \&c.$$

and thus the place for every positive and unequal root would be made known by a change of sign being furnished by the consecutive substitutions between which it lies. It is obvious too that instead of commencing our series of substitutions with 0, we may begin with the inferior limit to the positive roots; never extending them beyond the superior limit.

These considerations led WARING, and afterwards LAGRANGE, to seek for general methods of determining in all cases a suitable value of $\triangle$; these were readily seen to be dependent upon a transformed equation of which the roots should be the *differences* of the roots of the proposed equation: and hence the celebrity of the *equation of the differences of the roots,* upon which the method proposed by LAGRANGE for the analysis of numerical equations is founded; and of which an account will be given hereafter. It is plain that the actual *solution* of the equation of the differences is not necessary in order to furnish a proper value of Δ, since it will be sufficient that we know an inferior limit merely to the positive roots of the transformed equation, and that we put this limit for Δ. Theoretically, this method of separating the real unequal roots of an equation is perfect; but the practical difficulties attendant upon the calculation of the coefficients of the transformed equation are insuperable in equations of a high degree.

The method has however altogether yielded to those of STURM, FOURIER, and BUDAN, which when modified and improved, as hereafter proposed, will be found to leave but little to be desired in the practical solution of this important problem.

(93.) The proposition next following is one of considerable importance in the analysis of equations. Its object is to discover an equation of a lower degree than the equation proposed, whose real roots shall have the remarkable property of lying either singly, or in groups, between every two real roots of the original equation, thus effectively separating all the latter by interposing themselves.

Every equation whose roots thus separate those of another, by lying in the several intervals between them, is called a *limiting equation* to that other. If the roots of the original equation were known, limiting equations to it might be constructed in endless variety, for we should only have to assume values lying in the intervals which separate the given roots, and to construct an equation having these values for roots. But the roots of the original equation are supposed to be unknown; and our present object is to deduce at once a limiting equation whose roots, when determined, shall make known the situations of those of the primitive equation. Such an equation may be derived with great ease from the coefficients merely of the equation proposed, by a uniform operation applicable to all cases. The derived equation being thus always connected with the primitive by a constant law, is called *the* limiting equation, in order to distinguish it from all other equations whose roots may also separate those of the original. We shall now show how the limiting equation is to be deduced.

PROPOSITION VII.

(94.) An equation being given to determine another, a unit lower in degree, such that the real roots of the latter may separate all the real roots of the former.

Let the proposed equation, deprived of its leading coefficient, be

$$f(x) = x^n + A_{n-1} x^{n-1} + \dots + A_3 x^3 + A_2 x^2 + Ax + N = 0 \dots [1],$$

and let its real roots, taken in the order of their magnitude, commencing with the greatest and descending to the least, be

$$a_1, a_2, a_3, a_4, \dots$$

Let also the imaginary roots, if any, be represented by

$$k_1, k_2, k_3, k_4, \dots$$

then it is required to determine an equation of the $n - 1$*th* degree, whose real roots shall arrange themselves in the several intervals between the real roots of the series above.

It is evident that if the roots of the given equation be all diminished by r; that is, if $x' + r$ be put for x, the roots of the transformed will be

$$a_1 - r,\ a_2 - r,\ a_3 - r, \dots;\ k_1 - r,\ k_2 - r, \dots [2],$$

the equation itself being of the same degree as that proposed; that is, of the form

$$f(x' + r) = x'^n + A'_{n-1} x'^{n-1} + \dots A'_3 x'^3 + A'_2 x'^2 + A'x' + N' = 0.$$

Now it is only the last term but one in this equation, as we shall presently see, that need occupy our attention; and we know from (60), that the coefficient A′, of this term, is found from the roots exhibited in [2] by multiplying every $n-1$ of them together with changed signs, and adding the several results; that is, by taking the sum of all the partial products which the above factors, with changed signs, furnish, omitting each factor in succession.

As only one factor is thus omitted in each partial product, and as every one is omitted in its turn, it follows that wherever a *single* imaginary factor enters without its conjugate, *there* must *all* the real factors occur; for the omission of this conjugate necessitates the entrance into the product of all the other factors. It is of importance to observe this, because it authorizes us in concluding that, whenever we suppose any one of the real factors to become zero, thus causing all the partial products to vanish, except that from which this real factor has been omitted, the product preserved can contain only *real* simple factors, and the real quadratic factors formed from the different pairs of imaginary roots, the latter factors being always positive for every value of r (59.)

The last term but one in the above transformed equation being $A'x'$, the composition of A' will be as follows:

$$\begin{aligned}
A' = & (r-a_1)(r-a_2)(r-a_3)\dots(r-k_1)(r-k_2)\dots \\
& +(r-a_1)(r-a_2)(r-a_4)\dots(r-k_1)(r-k_2)\dots \\
& +(r-a_1)(r-a_3)(r-a_4)\dots(r-k_1)(r-k_2)\dots \\
& \vdots \\
& +(r-a_2)(r-a_3)(r-a_4)\dots(r-k_1)(r-k_2)\dots \\
& \qquad\qquad \&c. \qquad\qquad\qquad \&c.
\end{aligned}$$

Now this same coefficient is easily obtained from the coefficients of the original equation by the process described at (50); and the result so obtained will be an arranged polynomial descending according to the powers of r.* The same polynomial may however be obtained still more easily by a process which will be explained presently. But without seeking the arranged polynomial in r, to which the above expression for A' is equivalent, we may at once prove that the real roots of that polynomial equated

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* This method of obtaining A' is exhibited in full in the *Analysis and Solution of Cubic and Biquadratic Equations*, p. 63.

to zero, that is to say, the real values of r in the equation $A' = 0$, will separate all the real roots of the proposed equation $f(x) = 0$; in other words, the equation $A' = 0$ will be the *limiting equation* to the equation $f(x) = 0$.

For if in A' we put a_1 for r, each partial product must vanish, except one, since there is only one partial product from which $(r - a_1)$ is absent, and this will become

$$(a_1 - a_2)\ (a_1 - a_3)\ (a_1 - a_4)\ \ldots, \textit{ positive};$$

because, as a_1 is the greatest root, all the real simple factors will be positive: and the real quadratic factors, as observed above, will be positive also.

If we put a_2 for r, all will vanish except the product from which $(r - a_2)$ is absent, so that this will become

$$(a_2 - a_1)\ (a_2 - a_3)\ (a_2 - a_4)\ \ldots, \textit{ negative},$$

because a_2 is the greatest root that enters, except a_1, and the quadratic factors are positive.

In like manner, putting a_3 for r, we shall have

$$(a_3 - a_1)\ (a_3 - a_2)\ (a_3 - a_4)\ \ldots, \textit{ positive},$$

and so on. But when a series of quantities a_1, a_2, a_3, &c., substituted for the unknown in any equation, give results alternately positive and negative, every pair of these quantities must comprehend an odd number of the real roots of that equation. Consequently the real roots of $f(x) = 0$ are necessarily separated, the interval between every pair being occupied by some odd number, at least *one*, of the real roots of $A' = 0$. Hence $A' = 0$ is the limiting equation to $f(x) = 0$.

We have now to show how A' may be deduced from $f(x)$ in an arranged form. In order to this we remark, that the transformed equation, whose roots are [2], will be obtained by substituting $x' + r$ for x in [1], and developing the several terms by the binomial theorem. The coefficient A' in this transformation will be that which multiplies the simple power x'; and instead of being the last coefficient but one, it will become the *second* coefficient, provided we develop $f(r + x')$ instead of $f(x' + r)$, because we shall thus reverse the order of the terms.

Substituting, then, $r + x'$ for x in [1], and developing by the binomial theorem, we have

$$
\begin{array}{ll|l|l|l}
r^{n} & +nr^{n-1} & x' + \frac{n(n-1)}{2} r^{n-2} & x'^{2} + \frac{n(n-1)(n-2)}{2\,.\,3} r^{n-3} & x'^{3} + \dots x'^{n} = 0. \\
+A_{n-1}r^{n-1} & +(n-1)A_{n-1}r^{n-2} & +\frac{(n-1)(n-2)}{2}A_{n-1}r^{n-3} & +\frac{(n-1)(n-2)(n-3)}{2\,.\,3}A_{n-1}r^{n-4} & \\
+A_{n-2}r^{n-2} & +(n-2)A_{n-2}r^{n-3} & +\frac{(n-2)(n-3)}{2}A_{n-2}r^{n-4} & +\frac{(n-2)(n-3)(n-4)}{2\,.\,3}A_{n-2}r^{n-5} & \\
+A_{n-3}r^{n-3} & +(n-3)A_{n-3}r^{n-4} & +\frac{(n-3)(n-4)}{2}A_{n-3}r^{n-5} & +\frac{(n-3)(n-4)(n-5)}{2\,.\,3}A_{n-3}r^{n-6} & \\
\vdots & \vdots & \vdots & \vdots & \\
+A_{3}r^{3} & + \quad 3A_{3}r^{2} & + \quad 3A_{3}r & A_{3} & \\
+A_{2}r^{2} & + \quad 2A_{2}r & + \quad A_{2} & & \\
+Ar & + \quad A & & & \\
+N & & & & \\
\end{array}
$$

that is, adding up the coefficients in these vertical columns,

$$N' \quad + A'x'^1 \quad + A'_2x'^2 \quad + A'_3x'^3 \ldots\ldots + x'^n = 0,$$

so that

$$A' = nr^{n-1} + (n-1)A_{n-1}r^{n-2} + (n-2)A_{n-2}r^{n-3} + \ldots 3A_3r^2 + 2A_2r + A = 0 \ldots [3]$$

is the limiting equation required; and thus the polynomial A′ will be no other than the first *derived function* of $f(r)$, deduced as already explained at (15), and there represented by $f_1(r)$. In like manner the other coefficients A'_2, A'_3, &c., are the second, third, &c., functions derived according to the same law, and divided respectively by 2, by 2·3, &c.; that is, as before shown (27),

$$f(r+x') = f(r) + f_1(r)x' + \frac{f_2(r)}{2}x'^2 + \frac{f_3(r)}{2\cdot3}x'^3 + \frac{f_4(r)}{2\cdot3\cdot4}x'^4 + \ldots x'^n \ldots [4].$$

As it matters not by what letter we represent the unknown in any equation, we may change r in [3] into x, and write the limiting equation thus:

$$nx^{n-1} + (n-1)A_{n-1}x^{n-2} + (n-2)\,A_{n-2}x^{n-3} + \ldots 3A_3x^2 + 2A_2x + A = 0 \ldots [5],$$

the first member being the first derived function $f_1(x)$ of $f(x)$, so that the limiting equation may be written down at once from inspecting the original; for any term in the limiting equation is obtained from the corresponding term in the proposed by multiplying this latter by the exponent of x in that term, and diminishing the exponent by unity.

For example, if the original equation be

$$f(x) = 2x^4 - 7x^3 + 4x^2 + 2x - 12 = 0,$$

the limiting equation will be

$$f_1(x) = 8x^3 - 21x^2 + 8x + 2 = 0.$$

In like manner the limiting equation to this is

$$f_2(x) = 24x^2 - 42x + 8 = 0.$$

And, finally, the limiting equation to this last is the simple equation

$$f_3(x) = 48x - 42 = 0.$$

so that the descending series of limiting equations, or the successive derived functions $f_1(x)$, $f_2(x)$, &c., may be severally deduced from an original polynomial with great ease.

(95.) When the roots of the primitive equation are all real, the roots of the limiting equation must all be real too; otherwise the real roots of the latter would be too few to allow of one occupying an interval between every two of the former. In this case, therefore, the roots of the limiting equation must be situated relatively to those of the proposed equation as follows:

$$\begin{array}{ccccccccc} a_1 & & a_2 & & a_3 & & a_4 & & \cdots \\ & r_1 & & r_2 & & r_3 & & & \cdots \end{array}$$

But when imaginary roots enter the original, then, as the limiting equation *may* have more real roots than the former by one, or three, or five, &c. we cannot pronounce with certainty upon the exact distribution of the roots of the limiting equation among those of the original, as in the case above: all that we can affirm is that every interval between the latter roots will be occupied by an odd number of the former roots (91). There must, therefore, be at least m real roots in the limiting equation if there are $m + 1$ in the original; so that the entrance of imaginary roots into the limiting equation will be a sure indication that as many imaginary roots, at least, must enter the primitive equation.

The same conclusion may be extended to the subsequent derived or limiting equations; for imaginary roots cannot enter into any one of these without the same number, at least, entering the preceding; and so on up to the original equation.

(96.) It is, likewise, an inference from the form [4] above, that if any of the functions derived from the first member of an equation, $f(x) = 0$, vanish for a real value of r, such that the same value, when substituted in the function immediately preceding, and also in that immediately succeeding, furnishes results with like signs, the equation $f(x) = 0$ must have imaginary roots. For the equation $f(r + x) = 0$, where r is *real*, cannot of course have more or fewer imaginary roots than the equation $f(x) = 0$. But, in the case supposed, the value of r is such that the polynomial $f(r + x)$, as exhibited in the second member of [4], in its properly arranged form, has zero-coefficients between terms with like signs. Hence, (68) $f(r + x) = 0$, and consequently $f(x) = 0$ must have imaginary roots.

It is scarcely necessary to remark that the values of $f(r)$, $f_1(r)$, $f_2(r)$, &c. for any values of r, are the same as those of $f(x)$, $f_1(x)$, $f_2(x)$, &c. for equal values of x.

It follows from the preceding conclusion, that when all the roots of $f(x) = 0$ are real, then every value of x which causes either of the derived functions to vanish, must cause the immediately adjacent functions on each side to take opposite signs: the contrary taking place in any instance will be a sure indication of the existence of imaginary roots in the proposed equation.

These consequences were first deduced by De Gua, in his paper before referred to;* and they are included in the general theorem of Fourier to be discussed in next chapter. But the connexion between the roots of an equation and those of the several equations of inferior degrees derived from it as above, was, it seems, first noticed by Rolle,† who, by the help of the derived equations of the second degree, proposed to find limits to the roots of the preceding equation of the third degree; thence limits to those of the antecedent equation of the fourth degree; and so on till we should finally arrive at the limits of the real roots of the proposed equation. This was called the *method of Cascades;* but independently of the uncertainty as to the number of imaginary roots entering the proposed equation, of the existence of which, even when they are actually present, the several dependent equations may preserve no trace, the length of the calculations has caused this method of searching for the situations of the real roots of an equation to be entirely abandoned.

It is easy to see, however, when the real roots of the limiting equation are actually determined, how the number and places of

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* See Lagrange, Note VIII.

† Algebre, 1690.

the roots of the equation whence it has been derived may be accurately found. For as every interval between the roots of the proposed is occupied by one root, or some odd number of roots, of the limiting equation, it is plain that if we call these latter roots, arranged in descending order of magnitude, $r_1, r_2, r_3, \dots . r_k$, the several terms of the series

$$\infty, r_1, r_2, r_3, \dots . . r_k, -\infty$$

when substituted for x in succession in the proposed equation, will furnish exactly as many changes of sign as there are real roots in the latter; because never more than *one* of these roots can be passed over at a time, and all lie within the extreme limits $+\infty, -\infty$. In those cases, therefore, where the real roots of the limiting equation can be found, the number and situations of the real roots of the primitive equation can always be determined. Thus a cubic equation can always be analysed by this method, since the derived quadratic can always be solved: but for farther details on this subject we must refer the student to the introductory volume on the *Analysis, &c. of Cubic and Biquadratic Equations*, pp. 67-70.

Theory of Vanishing Fractions.

(97.) From the principles established in the foregoing proposition, we readily derive the following consequences, viz.:

Since

$$f(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4) \dots .$$

and

$$f_1(x) = (x - a_1)(x - a_2)(x - a_3) \dots + (x - a_1)(x - a_2)(x - a_4) \dots + \&c.$$

it follows that

$$\frac{f_1(x)}{f(x)} = \dots \frac{1}{x - a_4} + \frac{1}{x - a_3} + \frac{1}{x - a_2} + \frac{1}{x - a_1} \dots [1].$$

In like manner, for any other equation $F(x) = 0$, we have

$$\frac{F_1(x)}{F(x)} = \dots \frac{1}{x - b_4} + \frac{1}{x - b_3} + \frac{1}{x - b_2} + \frac{1}{x - b_1} \dots [2].$$

Suppose the two equations

$$f(x) = 0, \quad F(x) = 0,$$

have a root in common, viz. $a_1 = b_1$, then, dividing [1] by [2], we have

$$\frac{f_1(x)}{F_1(x)} \cdot \frac{F(x)}{f(x)} = \frac{\ldots \frac{1}{x-a_4} + \frac{1}{x-a_3} + \frac{1}{x-a_2} + \frac{1}{x-a_1}}{\ldots \frac{1}{x-b_4} + \frac{1}{x-b_3} + \frac{1}{x-b_2} + \frac{1}{x-b_1}}$$

Hence, multiplying numerator and denominator of the second member by $x - a_1$, and then substituting for x, its value $x = a_1$, we have

$$\frac{f_1(a_1)}{F_1(a_1)} \cdot \frac{F(a_1)}{f(a_1)} = 1$$

$$\therefore \frac{f_1(a_1)}{F_1(a_1)} = \frac{f(a_1)}{F(a_1)};$$

from which we learn, that if any two equations have a common root a, and their limiting equations be taken, the ratio of the original polynomials, when a is put for x, will be equal to the ratio of the limiting polynomials when a is put for x.

This property furnishes us with a ready method of determining the value of a fraction, such as $\frac{f(x)}{F(x)}$, when both numerator and denominator vanish for a particular value of x, as, for instance, for $x = a$. For we shall merely have to replace the polynomials in numerator and denominator by their limiting polynomials, and then make the substitution of a for x. If, however, the terms of the new fraction should also vanish for this value of x, we must treat it as we did the original, and so on, till we arrive at a fraction of which the terms do not vanish for the proposed value of x. The following examples will sufficiently illustrate this method:

1. Required the value of $\frac{x^2 - a^2}{x - a}$, when $x = a$.

Here $\frac{f_1(a)}{F_1(a)} = \frac{2a}{1} = 2a$, the required value.

2. Required the value of

$$\frac{nx^{n+1} - (n+1)\,x^n + 1}{(1-x)^2},$$[*]

when $x = 1$.

$$\frac{f_1(x)}{F_1(x)} = \frac{n\,(n+1)\,x^n - n(n+1)\,x^{n-1}}{-2(1-x)}.$$

This still becomes $\frac{0}{0}$ for $x = 1$,

$$\therefore \frac{f_2(x)}{F_2(x)} = \frac{n^2(n+1)x^{n-1} - n(n+1)\,(n-1)x^{n-2}}{2}.$$

$$\therefore \frac{f_2(1)}{F_2(1)} = \frac{n(n+1)}{2},$$

the value sought.

3. Required the value of

$$\frac{1-x^n}{1-x},$$

when $x = 1$.

$$\frac{f_1(1)}{F_1(1)} = \frac{-n}{-1} = n.$$

4. Required the value of

$$\frac{b(a-\sqrt{ax})}{a-x},$$

for $x = a$.

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[*] This is the expression for the sum of n terms of the series

$$1 + 2x + 3x^2 + 4x^3 + \&c.$$

We may here put $\sqrt{x} = y$, and thus change the fraction into

$$\frac{b(a - a^{\frac{1}{2}} y)}{a - y^2}$$

$$\frac{f_1(y)}{\mathrm{F}_1(y)} = \frac{-ba^{\frac{1}{2}}}{-2y} \therefore \frac{f_1(a^{\frac{1}{2}})}{\mathrm{F}_1(a^{\frac{1}{2}})} = \frac{b}{2} \text{ the value required.}$$

5. Required the value of

$$\frac{f(y)}{\mathrm{F}(y)} = \frac{(a+x)^{\frac{m}{n}} - (a+y)^{\frac{m}{n}}}{x - y},$$

when $x = y$. (See *Algebra*, third edition, page 200.)

Put $a + y = z^n$, then the fraction is changed into

$$\frac{(a + x)^{\frac{m}{n}} - z^m}{x - z^n + a}$$

$$\therefore \frac{f_1(z)}{\mathrm{F}_1(z)} = \frac{-mz^{m-1}}{-nz^{n-1}} = \frac{m}{n} \cdot \frac{z^m}{z^n} = \frac{m}{n} \cdot \frac{(a + y)^{\frac{m}{n}}}{a + y};$$

and therefore the value, when $x = y$, is

$$\frac{m}{n} \cdot \frac{(a + x)^{\frac{m}{n}}}{a + x}$$

Theory of Equal Roots.

(98.) The foregoing proposition also readily leads to a method of freeing an equation from all repetitions of the same root, whenever such occur; as also of ascertaining whether an equation has equal roots or not. For, as in the limiting equation $f_1(x) = 0$, the polynomial $f_1(x)$ consists of the sum of the products arising from multiplying together every $n - 1$ of the factors of $f(x)$, each group of factors in $f_1(x)$ will differ from $f(x)$ only by the absence of a single factor. Hence, if there be *two* equal factors in $f(x)$, that is, if $f(x) = 0$ have two equal roots, one of these factors must occur in each of the groups which compose $f_1(x)$, so that $f(x)$ and $f_1(x)$ have this factor for a common measure. If there be *three* equal roots in $f(x) = 0$, then will $f(x)$ and $f_1(x)$ have for a common measure the quadratic factor involving two of them, because more than one of the equal factors cannot be absent from any of the terms of $f_1(x)$. And generally if $f(x) = 0$ have p roots equal to a, then will $(x - a)^{p-1}$ be a common measure of $f(x)$ and $f_1(x)$; since in none of the component parts of $f_1(x)$ can more than one of the p equal factors be absent.

Again, if besides the p factors equal to $(x - a)$, there also enter q factors equal to $(x - b)$ in the composition of $f(x)$, then, besides the former common measure, the polynomials $f(x)$, $f_1(x)$, will also have the common measure $(x - b)^{q-1}$, for reasons similar to those which have already been assigned. And generally, if the equation $f(x) = 0$ have p roots equal to a, q roots equal to b, r roots equal to c, &c. then the greatest common measure of the polynomials $f(x)$, $f_1(x)$, will be

$$(x - a)^{p-1} (x - b)^{q-1} (x - c)^{r-1} \dots .$$

In order, therefore, to discover whether or not an equation $f(x) = 0$ has equal roots, we have only to ascertain whether or not $f(x)$ and $f_1(x)$ have a common measure $\phi(x)$; if they have, the division of $f(x)$ by $\phi(x)$ will give a polynomial involving the roots of the proposed equation without any repetition. It is indeed practicable to deduce a polynomial which shall involve only those roots which enter singly into the proposed, as we shall shortly show in general terms; at present, we shall apply the method to one or two particular examples.

1. It is required to determine whether the equation

$$f(x) = 2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0,$$

has equal roots.

$$f_1(x) = 8x^3 - 36x^2 + 38x - 6,$$

the greatest common measure $\phi(x)$ of the polynomials $f(x)$, $f_1(x)$, is $x-3$; hence the equation has *two* roots each equal to 3. Dividing, therefore, $f(x)$ by $(x-3)^2$, we have $2x^2+1$; hence the other roots are involved in the equation

$$2x^2+1=0 \therefore x=\pm\frac{1}{2}\sqrt{-2},$$

that is, the four roots of the proposed equation are

$$3,\ 3,\ \frac{1}{2}\sqrt{-2},\ -\frac{1}{2}\sqrt{-2}.$$

If we had divided $f(x)$, simply by the common measure $x-3$, the quotient would have been a polynomial of the third degree; involving, besides the two unequal roots just determined, one of the equal roots, as already explained. But by always increasing the exponent of every distinct binomial composing the common measure by unity, and then, performing the division, we obtain, as in this example, a quotient involving only those roots which occur without repetition.

2. It is required to determine whether the equation

$$f(x)=x^7+5x^6+6x^5-6x^4-15x^3-3x^2+8x+4=0,$$

has equal roots.

$$f_1(x)=7x^6+30x^5+30x^4-24x^3-45x^2-6x+8,$$

$$\phi(x)=x^4+3x^3+x^2-3x-2.$$

The equation has therefore equal roots involved in the equation $\phi(x)=0$. As in this last equation the roots all occur once less often than in the original, they would be all different if those of the original could enter only in pairs; but, as that equation cannot have *eight* roots, the roots of $\phi(x)=0$ cannot be all different: hence $\phi(x)=0$ will also contain equal roots. Let us therefore ascertain these.

The limiting polynomial derived from $\phi(x)$ is

$$\phi_1(x)=4x^3+9x^2+2x-3,$$

the common measure of $\phi(x)$, $\phi_1(x)$, is $x+1$; hence the equation $\phi(x)=0$ has *two* roots equal to -1; and, consequently, the equation $f(x)=0$ must have *three* roots equal to -1.

By division,

$$\frac{\phi(x)}{(x+1)^2} = x^2 + x - 2,$$

and from

$$x^2 + x - 2 = 0$$

we get

$$x = 1, \quad x = -2;$$

hence

$$\phi(x) = (x+1)^2 (x-1)(x+2),$$

and consequently,

$$f(x) = (x+1)^3 (x-1)^2 (x+2)^2,$$

that is, the roots of the proposed are

$$-1, -1, -1, 1, 1, -2, -2.$$

If the equal roots in the proposed had all entered in pairs, $\phi(x)$, $\phi_1(x)$ would have had no common measure; and the determination of the equal roots would have required the solution of the equation $\phi(x) = 0$, which would have contained each of those roots once; and the remaining roots—those that enter the original equation without repetition—would have been found by dividing $f(x)$ by the square of $\phi(x)$, and equating the quotient to zero. And in general the solution of the proposed equation, when equal roots enter, may always be reduced to the solution of a series of others of inferior degrees, of which the first contains only the unequal roots of the proposed, the second each one of the double roots, the third each of the triple roots, &c. This may be proved as follows:

Let X represent the product of the factors which enter *singly.*

X^2_2 . . the product of all the *pairs.*

X^3_3 . . the product of all the *threes.*

X^4_4 . . the product of all the *fours.*

&c.

so that

$$f(x) = X\, X^2_2\, X^3_3\, X^4_4 \;.\;.\;.\;.$$

then the greatest common divisor, $\phi(x)$, of $f(x)$ and $f_1(x)$, will be

$$\phi(x) = X_2\, X^2_3\, X^3_4\, X^4_5 \;.\;.\;.\;.$$

Again, calling the greatest common measure of $\phi(x)$, and its derived function $\phi_1(x)$, $\phi'(x)$, we have

$$\phi'(x) = X_3 X^2{}_4 X^3{}_5 \dots$$

In like manner, calling the greatest common measure of $\phi'(x)$, and its derived function $\phi'_1(x)$, $\phi''(x)$, and continuing the operation, we have

$$\phi''(x) = X_4 X^2{}_5 \dots$$

$$\phi'''(x) = X_5 \dots$$

&c. &c.

Hence, by division,

$$F(x) = \frac{f(x)}{\phi(x)} = X X_2 X_3 X_4 X_5 \dots$$

$$F'(x) = \frac{\phi(x)}{\phi'(x)} = X_2 X_3 X_4 X_5 \dots$$

$$F''(x) = \frac{\phi'(x)}{\phi''(x)} = X_3 X_4 X_5 \dots$$

$$F'''(x) = \frac{\phi''(x)}{\phi'''(x)} = X_4 X_5 \dots$$

&c. &c.

and, consequently, the determination of the roots of the proposed equation is reduced to the solution of the following series of equations, viz.

$$\frac{F(x)}{F'(x)} = X = 0,$$

$$\frac{F'(x)}{F''(x)} = X_2 = 0,$$

$$\frac{F''(x)}{F'''(x)} = X_3 = 0,$$

&c. &c.

The first of these equations involves the single roots only, the second each one of the double roots, the third each one of the triple roots, &c.

(99.) Since each multiple root in $f(x)=0$ enters once less often in the first derived equation $f_1(x)=0$, it follows that, if we continue the derivation, it will enter twice less often in the second derived equation $f_2(x)=0$; three times less often in the third derived equation $f_3(x)=0$; and so on till it disappears altogether. Thus the degree of multiplicity of every root will be equal to the number of derivations which are necessary in order to cause that root to disappear. If one of the equal roots be known, this method might therefore be adopted to discover its degree of multiplicity. For instance, if it be known that one root of the equation in example 2 be -1, we should find how often this root enters by substituting -1 in the successive derived functions, till we arrived at one which did not become zero for this substitution; the number of derivations thus employed would express the number of times the root -1 entered the equation: thus,

$$f(x)= x^7+5x^6+6x^5-6x^4-15x^3-3x^2+8x+4$$

$$f_1(x)= 7x^6+30x^5+30x^4-24x^3-45x^2-6x+8$$

$$f_2(x)= 42x^5+150x^4+120x^3-72x^2-90x-6$$

$$f_3(x)=210x^4+600x^3+360x^2-144x-90$$

The function $f_3(x)$ is the first that does not vanish for $x=-1$: hence the root -1 enters the equation three times.

This method of detecting the existence of equal roots, and of determining their degree of multiplicity, was first noticed by HUDDE, and published, together with some other researches of the same able algebraist, in 1659, by SCHOOTEN, in his *Commentary on the Geometry of* DESCARTES.

(100.) What has now been delivered contains the complete theory of equal roots; and furnishes all necessary directions for the elimination of these from any proposed equation, or for reducing such an equation to others, of inferior degree, which shall involve among them all the roots of it without any repetitions. It is obvious, therefore, that the general problem which has for its object the actual solution of numerical equations, that is, the calculations of all the real roots, may be regarded as completely solved when the difficulties connected with the solution of equations whose roots are all *unequal* are overcome. And accordingly, in the different methods of treating the general problem that have hitherto been proposed, the elimination of the equal roots has always been considered as a preliminary, essential to the successful application of the method proposed, or at least essential to the prosecution of every such method to its ultimate conclusion.

The practical difficulties however of this preparatory process in the analysis of an equation have been altogether overlooked; and thus very erroneously estimated by theoretical writers on this subject. It is common, even in the most recent publications on the theory of equations, to see the operation characterized as one of very easy performance; as if to clear an equation of equal roots, and to clear it of fractions, were preliminaries that might be disposed of with equal expedition. The truth is, however, that even in equations of but a moderately high degree—those for instance containing the fifth or sixth power of the unknown quantity—the operation for finding the common measure of the proposed polynomial and its first derived function, involves in it a considerable amount of numerical labour; so considerable indeed that any method of analysing an equation, which should imply as much calculation as is thus expended upon the preliminary preparation, would be practicable only within very narrow limits.

On these grounds we do not place much practical importance upon the theory just expounded, nor upon any method of analysing equations into which these operations for the common measure enter,—not as constituting the substance of the method itself, but as a mere preparative for its application. It is singular that LAGRANGE, who was so fully alive to, and conversant with, the practical difficulties connected with the analysis of numerical equations:—it is singular that he should have so overlooked this objection to his own and all existing methods, as he evidently did from his uniformly regarding the test of equal roots as one of very ready application. POINSOT, too, in the analysis he has given of the *Equations Numériques*, as prefixed to the last edition of that work (1826), considers the preparation for equal roots as of no moment in an estimate of the practical difficulties implied in the analysis and solution of a numerical equation.* And such indeed, as before remarked, has been the general doctrine hitherto held upon this subject.

The real amount of numerical work actually entering into the operation for finding the common measure of a polynomial and its derived function when, by an improved mode of arrangement the labour is economised to the utmost, will be exhibited when we come to discuss STURM's method of analysing an equation: a method which, unlike all that has preceded it, is wholly comprised in the operation adverted to; and which therefore attains the object in view with only the same labour that has usually been expended in order to prepare the equation for the efficient application of other modes of analysis; so that in fact STURM's method may be said to discover to us the desired results as soon as we reach the point from which other methods set out in search of them.

But these other methods have been hitherto unnecessarily encumbered with operations of this kind; since, as will be hereafter shown, the information which they supply may in general be obtained from far simpler considerations. It would be impossible to dispense with the process for the common measure in the theorem of STURM, as that process does itself constitute the method. But in the methods of FOURIER and BUDAN the common measure enters as a mere auxiliary, for the removal of the doubt as to whether or not equal roots exist within proposed limits; for any other purpose the operation for the common measure is useless, and may, therefore, be dispensed with whenever the doubt in question can be resolved by simpler means.

(101.) We have been led, by these considerations, to seek for a more readily applicable criterion of equal roots than that which the common measure supplies; and have in some degree succeeded in the search, by aid of the general principle established at (76), in conjunction with the following inferences from the preceding theory :—

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* "S'il y a des racines égales, il sera facile de les reconnaître, et de les dégager de l'équation."—POINSOT *Analyse du Traité*, &c. p. viii.

1. If an equation whose coefficients are commensurable have a pair of equal roots and no greater number, these roots must be commensurable: for the common measure of the first member of this equation, and the function derived from it, will be a binomial expression of the first degree with finite coefficients, and which when equated to zero will furnish one of the equal roots; these roots, therefore, must be commensurable; that is, either integers or fractions.

2. If the leading coefficient in the supposed equation be unity, and the others integral, the equal roots must be integral, because no fractional root can exist under these conditions (62).

3. If an equation with commensurable coefficients, have three equal roots, and no more, these also must be commensurable: for in this case the common measure will be of the second degree, and when equated to zero will give *two* of the equal roots: these roots, as just remarked, must be commensurable, hence all the three roots must be commensurable. And, as before, if the leading coefficient be unity, and the others integral, the equal roots will be integral.

4. By the same reasoning, if an equation with commensurable coefficients have m equal roots, and no other groups of equal roots, these m roots must be commensurable; and they will be integral if the leading coefficient be unity and the other coefficients integers.

5. When the leading coefficient is unity, and the other coefficients whole numbers, and m equal integral roots enter, we may infer, from the formation of the coefficients (60), that the absolute number, and the coefficient of the immediately preceding term, that is, the coefficient of x, will admit of a common measure involving $m - 1$ of these roots; that the coefficients of x and x^2 will have a common measure involving $m - 2$ of them; and so on till we come to the coefficients of x^{m-2} and x^{m-1}, which will have a common measure involving the multiple root once. For if the depressed equation containing only the unequal roots be considered, it will involve none but integral coefficients (76); so that if the equal roots be now introduced, as at (60), they can combine with none but integral factors. Hence, if the root occur twice, it will be found among the integral factors

of the common measure of the coefficients N and A; if it occur three times, it will be found among the factors of the common measure of N, A, and A_2: and so on. And, therefore, by trying several factors of the common measure in question, by actually substituting them for x in the proposed equation, when from any circumstance multiple roots are suspected to exist, we may remove all doubt on the subject. In analysing an equation the doubts that may arise as to the entrance of equal roots are confined to certain definite intervals, or within determinate numerical limits; so that of the factors adverted to above only those falling within these limits need be regarded.

And further, if the repeated root occur but twice, the square of it must be a factor of x^0 or N; if it occur three times, the cube of it must be a factor of N, and the square of it a factor of A; if it occur four times, the fourth power of it must be a factor of N, the cube of it a factor of A, and the square of it a factor of A_2, and so on. And thus, of the factors of N to be tested, those only need be used whose powers also are factors, entering, as here described, according to the multiplicity of the roots.

6. These inferences may be easily generalized: they apply, whatever be the integral value of the leading coefficient, and whether the repeated root be integral or fractional. Thus let the repeated root be $x = \frac{a}{b}$, a and b having no common factor; then if the root enter m times, the original polynomial will be divisible by $(bx - a)^m$, giving a quotient involving the remaining roots, and into which none but integral coefficients enter (76). Let us now return to the original polynomial by multiplying this quotient by $bx - a$, m times: the first multiplication by $bx - a$ will evidently give a product, into the first term of which b must enter as a factor, and into the last of which a must enter: the next multiplication must therefore give a product into the first term of which b^2 must enter, into the second b, into the last a^2, and into the last but one a: the third multiplication therefore must give a product whose first three terms involve b^3, b^2, b respectively; and last three, a^3, a^2, a, reckoning these last in reverse order, and so on. Hence the coefficients A_n, A_{n-1}, A_{n-2}, &c. will be divisible by b^m, b^{m-1}, b^{m-2}, &c. respectively, down to b;

and the coefficients N, A, A_2, &c., by a^m, a^{m-1}, a^{m-2}, &c., down to a. In other words the coefficients taken in order, reckoning from the beginning, will be divisible by the corresponding decreasing powers of the *denominator* of the repeated root; and the coefficients, reckoning from the end, will be divisible by the like powers of the *numerator*.

7. The inferences still have place, whatever be the degree of the multiple factor entering the proposed polynomial; so long as this factor, as well as the original polynomial, have none but integral coefficients. This is plain from the reasoning in the preceding case, which remains the same, as respects the entrance of the factors b, a, whether the repeated multiplier be $bx - a$, or $bx^m + \ldots . + a$.

These conclusions will greatly simplify the research after equal roots; and will either enable us wholly to dispense with the laborious process for the common measure, or will, at least, render the more tedious steps of it unnecessary. We shall more fully show this to be the case when we come to examine FOURIER'S method for analysing an equation, into which method the operation for the common measure has been supposed necessarily to enter. At present we shall merely refer to the two examples already considered.

The first of these, at page 117, can have no *fractional* multiple roots, because the leading coefficient 2, has no factor a perfect power: the equal roots, if any, must therefore be integral. Unity, which *always* has claim to be tried, does not succeed; and from the factors of 9 and 6, it is plain that + 3 and — 3 are the only other numbers to be tested; and, as no higher power of 3 than the square enters 9, we infer that more than *two* equal roots cannot have place in the equation. By testing 3 we find this to be one of a pair of equal roots. Equal *quadratic* factors could not possibly enter the equation; since, as the first coefficient shows, the polynomial is not a complete square.

In the example at page 118 no fractional equal roots can enter. Applying, therefore, + 1 and — 1 we discover the unit roots, as at page 119; and hence the remaining equal roots from the resulting quadratic.

CHAPTER VII.

ON THE METHOD OF NEWTON FOR FINDING A SUPERIOR LIMIT; AND ON THE SEPARATION OF THE ROOTS BY THE METHODS OF BUDAN AND FOURIER.

(102.) To find a number greater than the greatest root of an equation, Newton proposed to transform the equation into another whose roots should be less than those of the former by an undetermined quantity r, and then to determine r by trial, so as to cause all the coefficients in the transformed equation to become positive. Such a value of r would obviously exceed the greatest positive root of the proposed equation; for the real roots of the transformed, which are those of the original diminished by r, would all be negative (64), so that the greatest positive root of the original equation must have been diminished by a number greater than itself. As an example of Newton's method, let us take the equation

$$x^3 - 5x^2 + 7x - 1 = 0\,;$$

then, substituting $x' + r$ for x, the transformed is

$$\begin{array}{rl|rl|l}
0 = x'^3 + & 3r & x'^2 + & 3r^2 & x' + r^3 \\
 & -5 & & -10r & -5r^2 \\
 & & & +7 & +7r \\
 & & & & -1
\end{array}$$

Now, after a few trials, we find that 3 is the smallest value for r, which causes the several compound coefficients to become positive; therefore 3 exceeds the greatest positive root of the equation.

(103.) We should arrive at the same result, by diminishing the roots of the proposed successively by unity, according to the process in (71), and stopping as soon as the transformed coefficients become all positive; thus:

$$\begin{array}{rrrrl}
1 & -5 & 7 & -1 & (1 \\
 & -4 & 3 & 2 & \\
 & -3 & 0 & & \\
 & -2 & & & \\
\hline
1 & -2 & \pm 0 & +2 & (1 \\
 & -1 & -1 & 1 & \\
 & 0 & -1 & & \\
 & 1 & & & \\
\hline
1 & +1 & -1 & +1 & (1 \\
 & 2 & 1 & 2 & \\
 & 3 & 4 & & \\
 & 4 & & & \\
\hline
1 & +4 & +4 & +2 &
\end{array}$$

It may be observed of this method, that it not only furnishes a superior limit to the greatest positive root in every case, but when the roots are all real the limit thus determined is *immediately* above the true value of the greatest root; that is, the preceding number in the arithmetical scale is the first figure of that root. This is obvious, for the coefficients of the transformed become all positive as soon as all the roots become negative, and not before (66).

Even without knowing whether the roots are all real, we can pronounce the limit thus found to be the immediately superior limit, if the last coefficient in the immediately preceding set be negative; so that, in this case, we shall also know the first figure of the greatest root. This will appear plain, from considering that the last coefficient in any set (which is in fact the absolute number) is the result of the corresponding polynomial for $x = 0$; and that the last coefficient in the succeeding set is the result of the same polynomial for $x = 1$: and since these results, in the case supposed, have opposite signs, one root at least must have been passed over, and that the greatest, as the final coefficients are all positive.

The same process, as we go on, supplies a like indication of every passage we make over a single real root, or over any odd number of roots; every such indication being a change of sign in the last terms of two consecutive transformations. In the example above, the very first transformation presents a change of sign in the last term; we infer, therefore, that a root of the equation lies between 0 and 1.

If, however, the last term vanish in any transformed, the circumstance will prove that our last diminution has exhausted one of the roots; for one root of the transformed will then be zero, this being the value which it is obvious will always satisfy every equation whose final or absolute term is zero. Should not only the last, but also the last but one, vanish, we may, in like manner, conclude that two roots have been exhausted; and, if p last terms vanish, p roots will have been exhausted; so that the equation proposed will have p roots, each equal to the integer which expresses the number of transformations. In seeking, therefore, the superior limit by the foregoing process, we shall always detect in our progress every positive integral root of the equation.

Again, if any intermediate term vanish from one of the transformed equations, the circumstance may lead to the detection of imaginary roots of the equation; for, if on each side of the vanishing term the contiguous terms have *like* signs, the rule of DESCARTES will show that the roots cannot be all real; such an occurrence will, therefore, be a sure indication of the existence of at least one pair of imaginary roots in the transformed equation, and, consequently the same number in the original; because the increasing or diminishing the roots of an equation by any *real* number, can never either increase or diminish the number of imaginary roots. The following example will illustrate these remarks.

2. Let the equation be

$$x^3 - 3x^2 + 4x - 2 = 0,$$

and diminish the roots by unity:

$$\begin{array}{cccc} 1 & -3 & 4 & -2\,(1 \\ & -2 & 2 & 0 \\ & -1 & 1 & \\ & 0 & & \end{array}$$

At the close of the first step, we immediately infer that $x = 1$ is a root of the equation. The other two roots are involved in the equation

$$x^2 \pm 0x + 1 = 0;$$

and, as 0 occurs between the two like signs +, we infer that both roots are imaginary.

In seeking the superior limit, therefore, by the process recommended, we may *sometimes* detect the existence of imaginary roots, although they do not *always* furnish the above indication of their presence.*

3. Again, let us take the example

$$x^4 - 4x^3 + 10x^2 - 12x + 9 = 0,$$

which, as it has no permanencies, cannot have any negative roots. Diminishing the roots by 1, we get the transformed coefficients

$$1 \quad 0 \quad 4 \quad 0 \quad 4.$$

This transformation detects the existence of two pair of imaginary roots; we need not, therefore, proceed to another transformation, but conclude immediately that all the roots of the proposed are imaginary.

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* Unequivocal tests for detecting the existence of imaginary roots will be furnished hereafter.

The foregoing advantages, with some others which might be mentioned, are considerable; and are peculiar to this method of applying Newton's rule to the discovery of limits.

It may be enquired, however, here—Is it under all circumstances, possible to obtain, by successive diminutions of the roots, a transformed equation involving *only positive* coefficients? To this it may be replied, that, whenever we diminish the roots by a number exceeding the greatest positive root, the result of the real simple factors in the polynomial is necessarily positive in every term; and it continues so for every further diminution. Now, if there be any imaginary factors, the continual diminutions of which we speak must at length annihilate the real parts of these imaginaries, or render them positive, in which case every quadratic factor into which the several pairs of imaginaries enter, will have all its coefficients essentially positive, and therefore those of the transformed polynomial will be all positive.

(104.) But the same conclusion may be otherwise established as follows: It is evident, in diminishing the roots of an equation by 1, 2, 3, &c., that the second coefficient in any transformed is always equal to the second coefficient in the preceding equation, plus a certain number of times the first; so that, should there be a variation of sign between the first two terms of the proposed, we may, by continuing the transformations, change this variation into a permanency; whilst, on the contrary, if there be a permanency between the first two terms of the original, no transformation of the kind spoken of can change it into a variation. A permanency of sign may, therefore, in all cases be established between the first two terms of a transformed equation.

Again, since the third coefficient in any transformed is always equal to the third in the preceding transformation, plus a certain number of times the second, plus a certain number of times the first, it is plain that a variation between the second and third terms of a transformed, whose first and second terms have like signs, must be eventually converted into a permanency; whilst, on the contrary, if the first three terms had originally a permanency of sign, no subsequent transformation, arising from diminishing the roots, could introduce among them a variation.

By similar reasoning, we prove that, having obtained a permanency for the first three terms, we shall arrive, by continuing the transformations, at a permanency between the third and fourth, and so on, till we shall necessarily be led at length to a transformed equation exhibiting only permanencies of sign. Of course this *necessary* increase of permanencies in the leading terms of the successive transformed polynomials, will not prevent an *accidental* increase of them among other terms to the right, and these will facilitate the close of the process.

Let us take for a fourth example the equation given at p. 45:

$$x^4 + 3x^3 + 2x^2 + 6x - 148 = 0$$

$$\begin{array}{cccccl} 1 & 3 & 2 & 6 & -148 & (1 \\ & 4 & 6 & 12 & -136 & \end{array}$$

The -136 in this step is indication sufficient that 1 is not the limit. Diminishing then by 2, we find, for the final term, -88; hence 2 is not the limit: but, by diminishing by 3, the numbers in the first step are

$$6 \qquad 20 \qquad 66 \qquad 50,$$

which being all positive, the succeeding numbers must be positive; so that, without continuing the process, we infer that 2 is the first figure of the greatest positive root of the equation. We might, in like manner, have stopped the work at the second step of the third transformation, in the example at page 128, and have inferred the value of the limit.

(105.) Hitherto we have considered only the positive roots of the equation; but this might seem sufficient for our purpose, because, by changing the signs of the alternate terms of an equation, the negative roots become changed into positive, and, after this change, the superior limit to the positive roots would, when taken with the negative sign, be the inferior limit to the negative roots.

There is, however, no absolute necessity to effect this change in the signs of the terms of an equation. For it is plain, after the foregoing reasoning, that, if instead of diminishing we increase the roots of the proposed by 1, 2, 3, &c., we shall ultimately obtain a *variation* between the first and second term, then a variation between the second and third, then between the third and fourth, and so on ; so that we shall finally arrive at a set of transformed coefficients, presenting only *variations* of sign, and the number of transformations required to lead to this will express the number, which, taken negatively, is the inferior limit of the negative roots ; that is, a larger negative number than any of them. Whenever, in the progress of these transformations, we pass over a single, or indeed over any odd number of negative roots, a change of sign in the last coefficient will always give notice of the circumstance ; and, if we should entirely exhaust a negative root by these continual additions of unity, the reduction to zero of the same coefficient will apprize us of the fact.

(106.) From what has now been said of the progressive tendency of the successive transformations to terminate, when the roots are diminished, in a series of permanencies, and when they are increased, in a series of variations, we may conclude that,

1. If p, q, be any positive numbers, of which p is less than q, and if the roots of an equation be diminished first by p and then by q, the coefficients of the first transformed equation, that is, of the equation in $(x - p)$, cannot have fewer variations than the coefficients of the second transformed, that is, of the equation in $(x - q)$.

2. If the roots be *increased* first by p, and then by q, the coefficients of the *second* transformed equation, or that in $(x + q)$, cannot have fewer variations than the coefficients of the transformed in $(x + p)$.

Hence, under no circumstances can the number of variations, furnished by any transformed equation in $(x \pm r)$, be increased by further diminishing the roots, or diminished by further increasing the roots.

(107.) We are now prepared to demonstrate the following theorem, which may be regarded as an extension of the rule of DESCARTES :

Theorem of Budan.

Let p *and* q *be any two numbers, with signs like or unlike, but such that* q *is nearer to* $+\infty$ *than* p; *then, if an equation in* x *has* m *real roots comprised between* p *and* q, *the transformed equation in* (x — p) *has at least* m *variations more than the transformed in* (x — q).

Suppose first, that but one real root lies between p and q; then (91) the last terms of the transformations in $(x-p)$ and $(x-q)$ must have contrary signs, which requires that these transformations have not the same number of variations; for when the signs of the first and last terms of any equation are like, the number of variations must evidently be *even*, whatever be the number of intermediate terms; and when the extreme signs are unlike, the number of variations must be odd. But, by what is shown above, the first cannot have *fewer* variations than the second: it must necessarily, therefore, have at least one variation more.

Again: let there be m real roots comprised between p and q, and let us suppose them to be all unequal, and represented in the order of their increasing magnitude by

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad . \ . \ . \ . \ a_m.$$

Let, moreover, the numbers

$$b_1, \quad b_2, \quad b_3, \quad b_4, \quad . \ . \ . \ . \ b_{m-1}$$

be respectively comprised between a_1 and a_2; between a_2 and a_3; between a_3 and a_4, &c.; so that we may have the continued inequality

$$p < a_1 < b_1 < a_2 < b_2 < a_3 \ . \ . \ . \ . \ < a_{m-1} < b_{m-1} < a_m < q;$$

it will then follow, that if we form successively the equations in $(x-p)$, in $(x-b_1)$, in $(x-b_2)$, in $(x-b_3)$, . . . $(x-b_{m-1})$, up to that in $(x-q)$, each of these equations will have at least one variation more than the following one. Hence the equation in $(x-p)$ must have at least m variations more than the equation in $(x-q)$: which was to be proved. When the roots are all real, it is obvious that the number of variations which disappears in the successive transformations is precisely equal to the number of roots comprised between p and q.

It will have been remarked, that in the foregoing examination we have supposed that the real roots between p and q are unequal. We know, however, that previously to seeking the nature and situation of the roots, the first member of an equation may always be disencumbered of its multiple factors; though, as remarked at (100), the ordinary process by which this is effected is laborious. We shall show, however, presently that the theorem announced above equally holds, whether equal roots enter the equation or not.

(108.) The substance of what has now been proved amounts to this, viz.

1. If two transformed equations, the one in $(x - p)$, and the other in $(x - q)$, both exhibit the same number of variations, there is no root comprised between p and q.

2. If there be a variation between the last term in one, and the last term in the other, an odd number of roots must be comprehended between p and q, and there cannot be an odd number without this variation.

3. It is very obvious, that the loss of a *single* variation, in passing from one transformed to another, can never take place, except a change occur in the sign of the *final* term. Hence, when but a single variation is lost in passing from the transformation in $(x - p)$ to that in $(x - q)$, then one root, and only one, lies between p and q: for that *one* root, at least, is so situated follows from the preceding inference; and more than one there cannot be, otherwise there would be more roots than variations lost between the two transformations.

It further appears that when the sign of the final term remains the same, if any changes are lost, two, or some even number, must be lost.

4. If the number of variations lost be two, the equation *may*

have two real roots between p and q; but it may happen also that there are none in this interval. It is certain that the equation cannot have more than two roots in the interval p, q, otherwise the series would have lost more than two variations.

5. It is easy to see how the rule of DESCARTES follows from the theorem at (107). For if the equation have m positive roots between 0 and any number q, then by the theorem, in proceeding from the equation in x to that in $x - q$, m variations at least must be lost; and therefore the equation in x must have at least that number of variations to lose; so that there cannot be a greater number of positive roots than there are variations of sign in the proposed equation. If we change all the signs, commencing with the second, the negative roots will be converted into positive, and the permanencies into variations. Hence the equation cannot have a greater number of negative roots than there are permanencies of sign in the proposed equation.

6. The theorem at (107) may be expressed in a form somewhat different; and may be further amplified as follows. It is given in this form both by BUDAN and FOURIER*:—

If m variations be lost in passing from the transformed equation in $(x - p)$ to that in $(x - q)$, the equation in x *may* have m real roots between the limits p, q; but it cannot have a greater number.

If the number of real roots be not m, then the true number can differ from m only by an *even* number k; and the additional loss of variations will be attributable to k imaginary roots in the proposed equation. This may be proved as follows:—

(109.) If, between the two transformed equations which we are considering, we could interpose all the intermediate transformations which would arise from passing continuously from p to q, we should readily detect the cause of this additional loss of an even number of variations between the extreme transformations; for, as no quantity, whose range is confined within finite limits,

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[*] BUDAN, *Nouvelle Méthode pour la Résolution des Equations Numériques*, 1807. FOURIER, *Analyse des Equations Determinées*, 1831.

can proceed continuously from + to —, or from — to +, without first passing through zero (28), we should necessarily arrive, in the course of our intermediate transformations, at one or more containing vanishing terms. The corresponding terms, in the immediately preceding transformation, would make known the *signs* with which the consecutive ones vanished; and the corresponding terms, in the immediately subsequent transformation, would also make known the proper signs in which the same terms would vanish, in returning from the latter transformation to the former. Now should it happen that when the signs of the zeros, determined in the former way, or by means of the antecedent transformation, cause the terms among which these zeros occur to have more variations than when the signs are determined by the subsequent transformation, it is plain that this loss of variations will never be replaced in the following transformations, but will go to augment the loss arising from passing over roots between p and q. But a loss of variations, anywhere within the extreme terms of any transformed equation, implies the change of *two* variations into two permanencies (page 135); hence an *even number* of variations is thus lost, and yet the real roots of the transformed, involving the zeros, remain the same. It follows, therefore, by the rule of DESCARTES, which we may now assume from the inference 5, above, that this equation (and consequently the proposed,) has that even number of imaginary roots.

If the signs of the zeros in the transformation in question present no ambiguity, whether determined from the antecedent or from the subsequent equation, then the several transformed equations must all exhibit the same number of variations, till we arrive at a root, when the last term will vanish, and in the next transformed reappear with a changed sign. This will continue till all the real roots between the proposed limits are passed over, when there will have disappeared as many variations as roots between p and q. Hence the additional variations, which may have disappeared, can have done so from no other cause than that above stated; and these additional disappearances therefore mark the number of imaginary roots.

We have noticed before (page 130) the importance of attending to the signs of the terms contiguous to any simple vanishing term in a transformed equation; and have shown that when the contiguous terms have like signs we may infer the existence of a pair of imaginary roots; a conclusion which harmonizes with that just deduced, and which is, in fact, included in it, as the case referred to is contained in the more general one here considered. When, however, but one term vanishes, the signs are very readily supplied, the zero being always of one sign, +, or —, when the term is deduced from the antecedent contiguous equation, and of the opposite sign, —, or +, when it is deduced from the subsequent contiguous equation. But when several terms vanish, we must actually write down the two series of signs which the contiguous equations referred to exhibit, and which, as before remarked, may equally replace the intermediate series, in order to discover the indications of imaginary roots. This supposes, of course, that we know what the contiguous series of signs are; and that we may in all cases find them with great ease, will be seen from the following considerations.

(110.) Let us suppose that in the course of any transformations we have arrived at an equation or at a series of coefficients containing zeros, and that we want to determine the series of signs due to the immediately succeeding transformation. Represent the indefinitely small quantity by which the roots of the transformed at which we have arrived must be diminished, in order to furnish the next transformation, by δ; then, from what has been said about the influence of the signs in one transformation upon those of the next (104), it will be seen that, on account of the minuteness of δ, the sign of any term to be deduced must always be the same as that of the corresponding term above it; for, by making the multiplier δ smaller and smaller, we may render every product by it as small as we please; so that the final addend, which, added to any term in the proposed series, is to produce the desired term in the new one, may always be made smaller than the term to which it is added, when that term is of any magnitude at all, and therefore the new term will have the same sign as the corresponding term preceding; when, however, this corresponding term is zero, then the sign of the result will obviously be the same as that immediately before this zero. For example, if the series in which the zeros occur be

$$+\ 0\ \ 0\ \ 0\ \ 0 -\ 0\ \ 0\ \ 0 - +\ 0 +\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ -$$

the immediately subsequent series must be

$$+ + + + + - - - - - + + + + + + + + -$$

To form the immediately preceding series from the proposed, and thus to go back a step, requires that we regard our minute factor δ as negative; and as multiplying by $-\delta$ has the effect of changing the sign of every addend, which we must always remember is numerically less than the term to which it is to be added, on account of the minuteness of the multiplier which forms it, the antecedent series will be

$$+ - + - + - + - + - + - + - + - + - -$$

The order, therefore, and the signs, of the three consecutive transformations are as follow:

$$\begin{array}{c}
+ - + - + - + - + - + - + - + - + - - \\
+\ 0\ \ 0\ \ 0\ \ 0 -\ 0\ \ 0\ \ 0 - +\ 0 +\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ - \\
+ + + + + - - - - - + + + + + + + + -
\end{array}$$

in which the lower series has fourteen changes of sign fewer than the upper series, showing that, in the insensibly minute transit from the first to the third, fourteen variations have been lost, and yet no real root passed over: hence the equation from which such results have been deduced contains fourteen imaginary roots, besides whatever others may manifest themselves in transforming between other intervals; and it is obvious that every zero gives rise to a variation in the antecedent series, and to a permanence in the subsequent one; so that every passage through zero converts a variation, on the left, into a permanency.

(111.) The foregoing considerations lead to this *rule of the double sign*, viz.

To obtain the upper series, repeat the signs in the middle series, commencing at the left hand, till we come to zero, over which write the contrary sign to that last inserted, so that every sign exhibited in the middle series is to have the same sign above it in the upper series; and every zero is to have above it a sign contrary to that previously written in the upper series.

To obtain the lower series, put under every zero the same sign as that last inserted instead of the contrary sign; in other respects proceed as in the former precept.

It is plain that, although when but one zero occurs, the upper and lower series *may* preserve the same variations, yet, when two or more consecutive zeros occur, this will be impossible; so that when any transformation has two or more consecutive vanishing terms we may be sure of the existence of imaginary roots. The rule will make known how many are indicated.

In examining these cases of consecutive zero coefficients, we have all along supposed that the vanishing terms do not extend up to the last in the series, thus causing the series to terminate with consecutive zeros. Should however such be the case, it is plain that the proposed equation will thus be depressed as many units in degree as there are consecutive zeros at the extremity of the series; and will consequently have just so many roots all equal to the number from which the transformation in question has arisen.

The converse of this is equally plain, viz. that when equal roots exist in the proposed equation, the transformation which results from diminishing all the roots by one of these—thus reducing each of the latter to *zero*—will terminate with as many consecutive zeros as there are roots equal to the number employed in the transformation; because the evanescence of so many of the final terms is necessary in order that the equation may be divisible by x as often as there are zero roots.

From what is proved above it appears that the passage through these zeros is attended with the loss of just so many variations. Hence when equal roots are passed over, their number is exactly equal to the number of variations lost in the passage: and consequently the theorem at (107), as well as all the deductions from it, remains unaffected by the entrance of equal roots into the equation. It follows too that when a *single* root is passed over, causing a change in the final sign, the immediately preceding sign remains undisturbed by the passage.

(112.) From what has now been said we gather the following directions for determining the nature and situation of the roots of an equation.

1. From the given equation deduce a series of transformed equations, by means of the multipliers

$$\ldots\ldots -1000,\ -100,\ -10,\ -1,\ 0$$

$$1,\ 10,\ 100,\ 1000\ \ldots\ldots$$

taken in order, commencing sufficiently near to the limit $-\infty$ to cause the terms in the first transformed equation to have variations only. If our first transformed exhibit any permanencies we are not to reject the step, but to ascend from it, through the preceding transformations, till we arrive at a series of variations. This is to be regarded as the first series. The last series, or that which terminates the process in the other direction, is to present only permanencies. The interval between the first *transforming multiplier* and the last, will comprise all the real roots of the equation, and will also conceal the indications of the imaginary roots.*

2. When zeros occur in any of the transformations, the signs of the terms are to be ascertained by the *rule of the double sign.*

3. Those partial intervals, from step to step, during which no loss of variation occurs, are to be rejected, as no roots can lie in this region of the entire interval.

4. Those partial intervals, wherein but *one* variation is lost, embrace one real root of the equation, and only one.

5. Those partial intervals, in which any *odd* number of varia-

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* In practice it will be usually found more convenient to effect the transformations due to — 1, — 10, — 100, &c. by changing these from negative to positive, and using the original coefficients with their alternate signs changed. The results arising out of these modifications will be those sought when alternate signs are again changed, as in the first example following.

tions is lost, comprehend at least one real root; and *may* inclose as many real roots as there are variations lost. When the number of variations lost exceeds the number of real roots, this excess will mark the number of imaginary roots, indications of which occur in the interval.

6. Those partial intervals, in which any even number of variations disappear, *may* comprehend as many real roots. They either actually do this, or else they comprehend indications of as many imaginary roots as will make up that number.

(113.) The last two of these statements point to certain regions of doubt, occurring within the entire interval which limits the range of the system of roots. To remove this doubt, and to evolve the information respecting the roots, which really lies concealed in these regions, would agreeably to the foregoing theory, require us to pass continuously over the space, without allowing the minutest interval to escape examination. This tedious scrutiny may, however, be dispensed with in practice, and the desired information obtained by the application to the doubtful intervals of certain criteria, by means of which, the indications of the real and of the imaginary roots are much more readily detected. The investigation of these criteria will be given in the next chapter.

We shall now show the application of the foregoing principles to one or two examples.

1. Let there be proposed the equation

$$x^5 - 3x^4 - 24x^3 + 95x^2 - 46x - 101 = 0.$$

To determine the intervals, between which the roots are to be found.

In order to this we must deduce a series of transformed equations, or rather a series of transformed coefficients, by means of the multipliers . . . . — 10, — 1, 0, 1, 10, ... , which we shall call *factors of transformation*, or *transforming factors.*

These coefficients are obtained as follows:

$$\begin{array}{l}
1-3-24+95-46-101\ (1 \\
\quad -2-26+69+23-78 \\
\quad -1-27+42+65 \\
\qquad 0-27+15 \\
\qquad 1-26 \\
\qquad 2 \\
\hline
1+2-26+15+65-78
\end{array}$$

$$\begin{array}{l}
1-3-24+95-46-101\ (10 \\
\qquad 7+46+\ \&c.
\end{array}$$

This operation need not be continued, as we see that the resulting transformed coefficients must necessarily be all *plus*.

Changing now the alternate signs of the proposed equation, commencing with the second term, and proceeding as above, we have

$$\begin{array}{l}
1+3-24-\ 95-\ 46+101\ (1 \\
\quad 4-20-115-161-60 \\
\quad 5-15-130-191 \\
\quad 6-\ 9-139 \\
\quad 7-\ 2 \\
\quad 8
\end{array}$$

$$\begin{array}{l}
1+\ 3-24-95-46+101\ (10 \\
\qquad 13+\&c.;\ \textit{all plus.}
\end{array}$$

Consequently by returning to the proper signs, we have the following series, viz.

(— 10) . . . . + — + — + —

(— 1) . . . . + — — + — +

(0) . . . . + — — + — —

(1) . . . . + + — + + —

(10) . . . . + + + + + +

As the first factor of transformation gives only variations, — 10 is the inferior limit to all the negative roots ; and as the factor of transformation 10 gives only permanencies, 10 is the superior limit to the positive roots. Hence the roots all lie between — 10 and 10, and within these limits lie concealed the indications of the imaginary roots.

By comparing the two series given by the factors — 10 and — 1, we conclude, from the change of sign in the final term of the latter, and from the circumstance that only one variation is lost, that one root exists between — 10 and — 1 ; and only one.

The series given by the factors — 1 and 0, intimate the existence of one root between these limits, for the final signs are contrary, and only one variation is lost.

The series given by the factors 0 and 1, show that no root exists between these limits, nor yet any indications of imaginary roots, for no variations are lost.

The series given by the factors 1 and 10 show, by the change in the final sign, that one root at least exists between these limits ; there may be three, because three variations are lost ; at all events, the interval [1, 10], is the only interval in which indications of imaginary roots can occur : it would, however, be tedious to seek for these indications by trying intermediate factors of transformation, and we have already promised a more convenient method of proceeding, to be given hereafter.

2. Let the equation

$$x^4 - 4x^3 - 3x + 23 = 0,$$

be proposed.

The transforming factors 0, 1, 10, give

$$(0) \ldots . + - 0 - +$$

$$(1) \ldots . + 0 - - +$$

$$(10) \ldots . + + + + +$$

Hence, applying the rule of the double sign, we have

$$\begin{array}{ll} (0)\left\{\begin{array}{l}(<0) \;\ldots\ldots\; + - + - + \\ (>0) \;\ldots\ldots\; + - - - + \end{array}\right. \\ (1)\left\{\begin{array}{l}(<1) \;\ldots\ldots\; + - - - + \\ (>1) \;\ldots\ldots\; + + - - + \end{array}\right. \\ (10) \qquad\quad \ldots\ldots\; + + + + + \end{array}$$

The first of these series gives four variations, and the second two, this loss of two variations indicates the existence of one pair of imaginary roots.

The third and fourth series exhibit the same number of variations; hence the zero, produced by the transforming factor (1), does not arise from imaginary roots.

Let us now examine the series (0) and (1), for which purpose we must compare the signs of (> 0) and (< 1), and we thus find that no root is comprised in the interval [0, 1], because there is no loss of variation.

For the interval [1, 10], we must examine the series (> 1) and (10), which we find to indicate the existence of two roots, because two variations are lost, but whether they are real or not cannot as yet be ascertained; this, however, is the only doubtful interval.

3. Let the proposed equation be

$$x^5 + x^4 + x^2 - 25x - 36 = 0.$$

The transforming factors

$$-10, \quad -1, \quad 0, \quad 1, \quad 10,$$

give the following series of results:

$$\begin{array}{rl} (-10) \;\ldots\ldots & + - + - + - \\ (-1) \;\ldots\ldots & + - + - - - \\ (0) \;\ldots\ldots & + + 0 + - - \\ (1) \;\ldots\ldots & + + + + - - \\ (10) \;\ldots\ldots & + + + + + + \end{array}$$

Applying the rule of the double sign, we have

$$\begin{array}{ll} (-1) & \dots\dots\ + - + - - - \\ (0)\begin{cases} (<0) \\ (>0) \end{cases} & \begin{array}{l} \dots\dots\ + + - + - - \\ \dots\dots\ + + + + - - \end{array} \\ (1) & \dots\dots\ + + + + - - \end{array}$$

Comparing now these results, we find:—

That all the real roots exist in the interval between — 10 and + 10.

That two of these roots *may* lie between — 10, and — 1, because, in passing over this interval, two variations have disappeared; the interval may, however, contain indications of two imaginary roots.

That a pair of imaginary roots are indicated by (0), because the signs of (< 0) and (> 0) differ by two variations.

That no root exists between — 1 and 0, because the series (— 1) and (< 0) have the same number of variations.

That no root exists between 0 and 1, because the series (> 0) and (1) have the same number of variations.

That one real root exists between 1 and 10, because one variation has disappeared.

The only doubtful interval here is that between — 10 and — 1.

We shall give but one more example of the determination of the intervals of the roots.

4. Let the proposed equation be

$$x^7 - 2x^5 - 3x^3 + 4x^2 - 5x + 6 = 0.$$

The transforming factors

$$-10,\ -1,\ 0,\ 1,\ 10,$$

give the following results:

$$\begin{array}{rl} (-10) & \dots\dots\ + - + - + - + - \\ (-1) & \dots\dots\ + - + - + + - + \\ (0) & \dots\dots\ + \ 0 - 0 - + - + \\ (1) & \dots\dots\ + + + + + - - + \\ (10) & \dots\dots\ + + + + + + + + \end{array}$$

And, applying the rule of the double sign, we have

$$\begin{array}{lll} (-1) & & \ldots\ldots + - + - + + - + \\ (0)\left\{\begin{array}{l} \\ \\ \end{array}\right. & \begin{array}{l} (<0) \\ (>0) \end{array} & \begin{array}{l} \ldots\ldots + - - + - + - + \\ \ldots\ldots + + - - - + - + \end{array} \\ (1) & & \ldots\ldots + + + + + - - + \end{array}$$

We deduce, therefore, the following particulars:

There is one root between the limits — 10 and — 1, and only one.

The series (0) shows the existence of two imaginary roots in the equation, because the series (<0) and (>0) differ by two variations.

There is no real root between — 1 and 0.

There *may* be two real roots between 0 and 1, as two variations disappear between (>0) and (1); but if there are not two real roots in this doubtful interval, there exists within it an indication of two imaginary roots.

There *may* also be two more real roots between 1 and 10.

The only intervals, therefore, in which we ought to seek for roots are those between — 10 and — 1, between 0 and 1, and between 1 and 10; and we know also that the equation has two imaginary roots at least.

(114.) It may not be improper to remark here, that when the equation proposed for examination has any of its terms wanting, as in the last three examples, we may always, by applying to it the rule of the double sign at once, determine the least number of imaginary roots that the equation can possibly have. Thus, in the last example, the signs of the proposed are

$$+ 0 - 0 - + - +$$

instead of which, the rule of the double sign gives the two series,

$$+ - - + - + - +$$

$$+ + - - - + - +$$

which, because they differ by two variations, establish the existence of at least two imaginary roots in the equation.

(115.) In equations of the form

$$x^n + N = 0$$

this method makes known the exact number of imaginary roots. For example, suppose the equation is

$$x^6 - 1 = 0$$

which gives the series

$$+ \; 0 \; 0 \; 0 \; 0 \; 0 \; -$$

and, by the rule of the double sign,

$$+ - + - + - -$$

$$+ + + + + + -$$

in which the upper series has five variations, and the lower but one. Hence, there are four imaginary roots in the equation, which is obviously the entire number; the two real roots being $+1$, -1.

From a mere inspection of this upper and lower series, it is obvious that, in all cases, when $m + 1$ zeros intervene in an equation between *unlike* signs, there must exist at least m imaginary roots; and when $m + 1$ zeros intervene between *like* signs, there must exist at least $m + 1$ imaginary roots. These are the conclusions that have been otherwise deduced at (68).

(116.) From what has already been shown at p. 110, it is evident that the coefficients of the transformed equation $f(x + r) = 0$, to which we are conducted by diminishing the roots of a given equation $f(x) = 0$, as in the preceding examples, by any number r, are no other than the successive functions

$$f(x), f_1(x), \frac{1}{2} f_2(x), \frac{1}{2 \cdot 3} f_3(x), \frac{1}{2 \cdot 3 \cdot 4} f_4(x), \text{\&c.} \ldots \ldots [1]$$

when written in reverse order and r substituted for x. For the proposed equation being

$$f(x) = A_n x^n + A_{n-1} x^{n-1} + \ldots + A_3 x^3 + A_2 x^2 + Ax + N = 0 \ldots [2]$$

we have for the transformed function $f(x+r)$, when the terms of the development are written in reverse order, the expression

$$f(x+r) = \frac{f_n(r)}{1 \cdot 2 \cdot 3 \ . \ . \ . \ n} x^n + \frac{f_{n-1}(r)}{1 \cdot 2 \cdot 3 \ . \ . \ . \ (n-1)} x^{n-1} + \ . \ . \ .$$

$$\frac{f_3(r)}{1 \cdot 2 \cdot 3} x^3 + \frac{f_2(r)}{1 \cdot 2} x^2 + \frac{f_1(r)}{1} x + f(r)$$

of which the coefficients are what those in [1] become when r, the factor of transformation, is substituted for x. For example, taking the first member of the equation proposed at page 142, and the several functions derived from it, we have

$$f(x) = x^5 - 3x^4 - 24x^3 + 95x^2 - 46x - 101$$

$$f_1(x) = 5x^4 - 12x^3 - 72x^2 + 190x - 46$$

$$\frac{1}{2} f_2(x) = 10x^3 - 18x^2 - 72x + 95$$

$$\frac{1}{2 \cdot 3} f_3(x) = 10x^2 - 12x - 24$$

$$\frac{1}{2 \cdot 3 \cdot 4} f_4(x) = 5x - 3$$

$$\frac{1}{2 \cdot 3 \cdot 4 \cdot 5} f_5(x) = 1$$

Putting now 1 for x and writing the results in reverse order, we have

$$1 + 2 - 26 + 15 + 65 - 78$$

and putting 10 for x we have all the results *positive.*

Also putting 0 for x we have

$$1 - 3 - 24 + 95 - 46 - 101$$

the coefficients of the proposed polynomial. And thus, whatever general relations are shown to exist among the derived functions [1], x being any value whatever, the same must exist among the coefficients of the original equation: for the functions [1] return to these coefficients when $x = 0$: in other words the original equation is

$$f(x) = \frac{f_n(0)}{1 \cdot 2 \cdot 3\,.\,.\,n} x^n + \frac{f_{n-1}(0)}{1 \cdot 2 \cdot 3\,.\,.\,.\,(n-1)} x^{n-1} + \ldots + \frac{f_3(0)}{1 \cdot 2 \cdot 3} x^3$$

$$+ \frac{f_2(0)}{1 \cdot 2} x^2 + \frac{f_1(0)}{1} x + f(0) = 0$$

(117.) The examples by which we have illustrated the theorem at (107), or rather that at page 136, are from the *Analyse des Equations* of Fourier; but the processes by which the several transformations involved in these examples are here effected are very different from those employed in that work. Fourier actually exhibits the several derived functions in every case, as above; and as the *signs* of these only, for particular values of x, are required, he disregards the numerical divisors 2; 2·3; 2·3·4; &c. and thus encumbers the several expressions from which the series of signs are to be deduced, with coefficients unnecessarily large. The method of transformation uniformly employed in the present work is that exhibited at length in the analysis of the first example at page 143: it reduces the operation to the utmost simplicity.

In the preceding exposition of this method of *partially analysing* a numerical equation we have united the names of Budan and Fourier, each of whom announced, independently it would seem of each other, the theorem at page 136, on which the method is founded. It is common with English writers to ascribe this theorem exclusively to Fourier—a singular preference; since the publication of it by Budan preceded the work of Fourier by nearly a quarter of a century. In the advertisement prefixed to this work, the editor, Navier, adduces evidence in favour of Fourier's prior claim to the theorem. This evidence however consists of individual attestations to the fact that Fourier had developed his theory in manuscript so early as 1797—ten years before the publication of Budan; and that he had publicly expounded it in his lectures at the Polytechnic School in 1803. But testimony of this kind must always be deficient in that distinctness; as to the precise character and extent of the communications made, which so eminently belongs to the printed publication of them. There is however no room to doubt that **Fourier** was really engaged in researches upon numerical equations long before the appearance of **Budan**'s work; and that he had advanced in the enquiry beyond his predecessors. There is very conclusive evidence of this in a printed statement which seems to have escaped the notice of **Navier**, and the other advocates of **Fourier**'s claims. We allude to a passage in **Montucla**'s *History of Mathematics*, which we quote below.*

It is probable that **Fourier** was withheld so long from the publication of his researches—which after all were not printed till after his death—on account of the inefficiency of his theorem to make known the exact character of those *doubtful intervals* which, as we have already seen, frequently occur within the extreme limits of the real roots of an equation. Attempts were made both by **Fourier** and **Budan** to remove this defect, by help of certain supplementary operations applied expressly to the intervals in question. In this further analysis of the equation the two methods are perfectly distinct. We shall discuss them separately in the next chapter.

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* In the passage referred to, **Montucla**, or rather **Lalande**, adverting to the previous inquiries of **De Gua**, and the general demonstration given by him of the rule of **Descartes**, proceeds as follows: "Je ne puis passer sous silence un mémoire sur la résolution des équations par le cit. **Fourier**, ancien professeur de mathématiques au College de Tonnerre, qui s'est aussi spécialement occupé de cette démonstration; il en donne deux, l'une géometrique et fondée sur la considération des courbes ci-dessus, l'autre purement analytique, et fondée sur des principes différens de ceux de l' abbé **De Gua**. Ses recherches le conduisent à beaucoup d'autres vérités utiles, qu'il est juste qu'il publie lui-même le premier."—(**Montucla**: *Hist. des Mathématiques*, tom. iii. p. 39, 1802.)

Fourier died just as his work on *Equations* was put to press: his mss. were consigned to the care of **Navier**, who published the first part in 1831. But the death of **Navier** himself, shortly afterwards, put a stop to the progress of the publication.

CHAPTER VIII.

ON THE ANALYSIS OF EQUATIONS FROM GEOMETRICAL CONSIDERATIONS: METHOD OF FOURIER.

(118.) From the investigations in the preceding chapter, it appears that it is no difficult matter, when any numerical equation is proposed, to determine close inferior and superior limits, within the interval of which shall lie concealed not only all the real roots, but likewise all the indications of imaginary roots. In fact, we can never be sure that all the real roots are actually comprehended within any proposed boundaries, till we have ascertained that the indications of the imaginary roots all lie between the same limits; so that when the extreme limits of the real roots are clearly determined, the complete analysis of the equation consists merely in a sufficiently minute subdivision of the interval between them.

In the foregoing chapter such a searching scrutiny has not been attempted; and accordingly the character of some of the component intervals, which our partial analysis has furnished, often remained doubtful. In some cases it would be impossible to completely remove this doubt by simply narrowing the component intervals, or increasing the number of subdivisions; that is, by making our factors of transformation less and less. Extraneous information would still be requisite before we could pronounce with confidence upon the character of an interval, however minute, in passing over which two changes of signs were lost: we should experience the same uncertainty as before, as to whether these two changes indicated a pair of imaginary roots, or a pair of real roots differing from each other so minutely as to lie both within the small interval referred to. And we could resolve the doubt only by knowing, from some independent source, either the least of the differences furnished by every pair of real roots, or else a number less than the least difference, as already explained at (92). The determination of such a number, though theoretically possible, is an operation so laborious, in equations beyond the fourth degree, as to be practically useless in the analysis of equations. Indeed, roots sometimes differ by numbers so exceedingly small, that, even supposing a limit below this difference to be found, yet the labour of passing over an interval by such minute advances would be a very long and tedious process. It would be well if we could exhibit to the eye the *continuous* series of results which the first member of any equation would furnish by substituting continuous values for x from the inferior up to the superior limit of the roots: we should then perceive at a glance all the passages of the polynomial through zero, and thus become acquainted with the exact number of the real roots. Of course the practical difficulties in the way of this are insuperable; yet the idea obviously suggests the geometrical representation of an algebraic polynomial by means of a continous curve line, which shall unite all the isolated values of that polynomial resulting from individual substitutions.

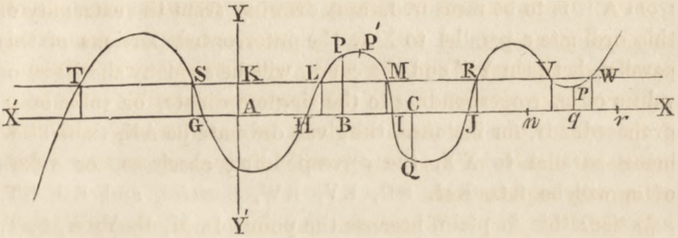
A contemplation of this curve would not only verify all the analytical results known to be implied in the equation, but, from purely geometrical considerations, new truths might discover themselves which had escaped observation in the abstract algebraical form. It was from examining in this manner the geometrical representation of an algebraic polynomial, that FOURIER was led to the method about to be explained for determining the character of the doubtful intervals occurring between the extreme limits of the roots of an equation, without having recourse to the problem for finding a number less than the least of the differences of the roots.

The idea of converting an algebraic polynomial into a continuous curve, embodying all the peculiarities of the symbolical expression in a geometrical form, first suggested itself to DESCARTES, and is a contrivance which has often been resorted to for clear illustration of certain particulars in the general theory of equations, for which illustration we shall shortly see it to be well adapted: but the geometrical property noticed by **Fourier** enabled him to advance a step nearer than his predecessors towards the complete analysis of a numerical equation.

(119.) If the student be already familiar with the principles of analytical geometry, he will readily perceive how this connexion between an algebraical equation and a geometrical curve subsists; and how the properties of the one become convertible into those of the other: but for those who may be unacquainted with this important branch of analysis, it will be necessary to offer a few preliminary explanations.

As usual let $f(x) = 0$ represent any algebraical equation in x: this is a *determinate equation,* because the unknown quantity x admits only of a determinate number of values to the exclusion of all other values. But if we remove the restriction which confines the first member to the single value zero, and imply that this value is arbitrary, by writing the equation in the more general form $f(x) = y$, we then render the equation *indeterminate,* since x admits of any value whatever; to each of which, however, there corresponds a certain *determinate* value of y, as implied in the sign of equality. We have then to exhibit the general law, which thus connects every value of x with the corresponding value of y, by means of a geometrical figure.

From any assumed point A draw an indefinite straight line AX towards the right, and extend it indefinitely in the opposite direction AX′. In like manner draw from A a perpendicular to X′X of indefinite length, AY, and prolong it indefinitely in the opposite direction AY′. These two lines are called *the axes*, and the point A, where they intersect, is called the *origin* of the axes. Now if we assume any series of positive numerical values for x, and measure each of these values from A towards X, according to any unit of length chosen at pleasure, we must measure a like series of negative values of x in the opposite direction from A, that is from A towards X′; for then the lengths, thus set off in each direction, will not only be correct representations of the absolute numerical values for which they stand, in reference to the linear unit previously agreed upon, but they will also imply, in the directions in which they are measured, the algebraic signs which those values take. In like manner, if positive numerical values of y are measured along the other axis, from A in the direction AY, then negative values must be measured in the opposite direction AY′.



This being premised, let AB represent any value of x; then the corresponding value of y will also have some linear representation, which may be set off upon the other axis, above X′X if y be positive, and below X′X if negative; or, which is better, upon a parallel to this axis drawn through B, the termination of the linear value of x: let BP be this length. Of the point P, thus determined, AB is called the *abscissa*, and BP the *ordinate:* together they are called the *coordinates* of the point P. Another assumed value of x will furnish another abscissa AC, and for the corresponding y another ordinate CQ, represented in the diagram as *negative*, being drawn *below* the axis X′X. These new coordinates introduce a second point Q. And thus if it were possible to construct the continuous series of values for x and y, setting out with $x = 0$, and proceeding towards $x = + \infty$ on the right, and towards $x = - \infty$ on the left, we should be furnished with a continuous series of points; that is, with an uninterrupted curve line. We may therefore consider this curve as traced out by the extremity P, of an ordinate BP $= y$, moving parallel to itself, along the axis X′X, and varying in length as its distance x from A varies, the law of variation being expressed by the relation $y = f(x)$. Conceiving the curve to be actually generated in this way, it is easy to see how, from any value of x being given, the corresponding value of y may be found, and vice versa.

Thus, if AC represent the given value of x, then the perpendicular CQ, extended till it meets the curve, will represent the corresponding value of y, which we shall know to be positive or negative according as Q is above or below XX′;—in the above diagram it is negative. In like manner, if the value of y be given, then setting off that value upon the axis YY′, attending to the algebraic sign of it, in order to ascertain in which direction from A it is to be measured, and drawing from the extremity of this ordinate a parallel to X′X, the intercepted portions of this parallel, between YY′ and the curve, will be so many abscissas, or values of x, corresponding to the single ordinate, or value of y, proposed. If, for instance, the given ordinate be AK, then TKW being parallel to X′X, the corresponding abscissas, or values of x, will be KL, KM, KR, KV, KW, *positive*, and KS, KT, *negative:* this is plain, because the points L, M, R, V, W, S, T, have these several abscissas, and one uniform length of ordinate, viz. the length AK.

(120.) As observed above, it is not possible actually to construct this curve; the utmost we could do would be to approximate to its form by means of a series of isolated points determined from a series of successive values of x; but, by making the intervals between these values very small, we could evidently form a tolerably accurate notion of the general character of the curve within any proposed limits taken for the values of x. Indeed, without any approximate construction at all, such a general notion may be formed from the nature of the polynomial whence it has been derived. Thus we may be quite sure that the undulating curve above exhibits the general character of such a polynomial; for the composition of the polynomial is such that to any value of x there corresponds but *one* value of y, as in the figure; while for particular values of y there may exist several values of x, as many indeed as there are units in the highest exponent of x. The curve, therefore, should be such that as many values as there are for x, corresponding to a given value AK for y, so many intersections must there be of the curve with the parallel to X′X through K. If y be zero, then the corresponding values of x must represent the real roots of the equation $f(x) = 0$: they will, therefore, be equal in number to the intersections of the undulating line with the axis X′X: those intersections to the right of A will indicate the number of positive roots; and those to the left the number of negative roots. In the diagram above three positive roots are indicated, and two negative roots. As, in order to produce an intersection with the axis of abscissas, the curve must pass from one side of that axis to the other, it follows that the ordinates, immediately before and immediately after the intersection, must have opposite signs; that is, the polynomial $f(x)$ changes sign while x passes through a single real root.

Following the progress of the curve, after this intersection, it is plain that no second change can take place in the sign of the ordinate till the curve again crosses the axis; that is, till another root is passed, when the ordinate emerges on the other side of the axis with a changed sign. Its length thence increases up to a certain limit, at which the curve again bends towards the axis, crosses it a third time, and gives rise to a new series of ordinates with signs opposite to those which vanished at the former point of intersection.

Thus we see that two values of x, which give for $f(x)$ or y results with *opposite* signs, must intercept either 1, or 3, or 5, &c. real roots; and two values which give results with like signs must intercept either 0, or 2, or 4, &c. real roots. When, in the polynomial $f(x)$, a number so great is substituted for x that the transformed equation, arising from diminishing the roots of the equation $f(x) = 0$ by this number, has all its terms positive, then we know (102) that the number in question exceeds the greatest positive root of the equation; and, moreover, that if a series of numbers, continually increasing beyond this, be successively substituted, that the results $f(x)$ or y, will also continually increase (103). In a similar manner will the results continually increase for substitutions for x continually tending towards $-\infty$, after a certain limit is reached, viz., that which furnishes a transformed equation with its terms alternately positive and negative. Hence, the curve, after having furnished as many intersections with the axis of abscissas as there are real roots, continues its course interminably on each side of the axis of ordinates; and, after a certain limit, all undulations must cease, and the ordinates become continually longer and longer without termination.

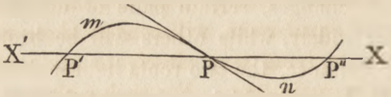
Whether the degree of the equation be odd or even, that part of the curve which is to the right of A will proceed on its unlimited course *above* the axis of abscissas; since the ordinates, after the limit referred to, must always be positive. But to the left of A the curve will extend above or below the axis of abscissas according as the degree of the equation is even or odd: this will appear from considering that when the terms are alternately positive and negative in an equation of an even degree, the final term,—which is that furnished by the polynomial $f(x)$ when the number by which the roots are diminished is substituted for x,—will be positive; and in an equation of an odd degree the same term will be negative. This shows that for an equation of an odd degree there must be at least *one* intersection of the curve with the axis of abscissas: as the curve proceeds without limit on *both* sides of that axis, which it cannot do without crossing the axis, should it cross a second time, it *must* cross a third, otherwise it could not proceed on opposite sides of X′X: for a like reason if it cross a fourth time, it must also cross a fifth time, and so on: the number of intersections being necessarily *odd;* that is, an equation of an odd degree must have an odd number of real roots. An equation of an even degree has not necessarily *any* real roots; as the curve need not of necessity cross the axis, because it proceeds without limit on one and the same side: but if there be *one* intersection, there must on this account necessarily be another; and if a third, then a fourth, and so on; so that when the degree is even the equation must have an even number of real roots, or else none at all.

(121.) If in the equation $f(x) = y$ a succession of values be given to y, from $y = 0$ to $y = a$, we shall have a succession of equations from $f(x) = 0$ to $f(x) = a$, or $f(x) - a = 0$, differing from one another only in the final or absolute term. If AK represent one of these values of y, the intersections L, M, R, V, W, S, T, will show the number of real roots in the corresponding equation; and by conceiving X′X to move parallel to itself, till it reach the distance a from its original position, the successive series of intersections will exhibit the number and character of the real roots of the several successive equations. It is easy to see that, by means of these changes, certain pairs of unequal roots will be separated more widely, while others will be brought nearer to equality: for instance, the two roots indicated originally by the intersections G, H, and which are separated by the interval GH, are changed into the roots indicated by the intersections S, L, which are wider apart, when the axis arrives at TW; whilst the two roots, differing originally by HI, now differ only by LM; and these are actually brought together, and rendered equal, when the axis has advanced so far as to touch at P′, rendering the separating interval LM zero. This then is the geometrical peculiarity of a pair of *equal* roots:—instead of *intersection*, as in the case of a single root, there is *contact* with the axis.

When the axis, by moving parallel to itself, advances still farther, the two roots that have been rendered equal no longer exist: as soon as the axis ceases to touch, a pair of intersections is lost; and thus a pair of roots becomes unaccounted for by the intersections that remain. This, therefore, is the geometrical peculiarity of a pair of *imaginary* roots:—the curve approaches towards the axis, bends before arriving at it, and completes an undulation without meeting it. There is such a peculiarity at p; X′X being the position of the axis.

It is worthy of notice that, by changing the value of the absolute term of an equation, without disturbing the other coefficients, we may always convert, as above, a pair of consecutive unequal roots into a pair of equal roots: but that that change will not generally suffice to render three unequal roots equal. In the curve above, no three intersections can be made to coalesce, and merge into a single point, by any change in the distance merely of the parallel TW from its original position.

In order that three points of intersection may merge into one there must be a change in the coefficients of $f(x)$, such that the geometrical equivalent is not a mere transference of the axis parallel to itself, but a change of direction in that axis. Thus the three points P′, P, P″ merge into the single point P when X′X, by turning round P, takes the position of a tangent to each of the portions PmP', PnP'', of the curve, at the point of inflexion P.



(122.) By thus giving a geometrical interpretation to the expressions of analysis, much clearness and distinctness may often be added to the ideas conveyed by our symbolical forms, and many interesting analytical truths at the same time suggested. One great advantage of presenting algebraical expressions under this form is, that instead of our attention being confined to isolated individual values merely, we are enabled to contemplate the law of continuity that unites them all. It is, indeed, solely from this law being presented to our view, in the continuous curve which replaces the analytical formula, that the geometrical representation can supply anything in addition to our analytical deductions. Of course the algebraical form is competent to furnish all the inferences deduced from the curve which represents it; but it often happens that what is so entirely concealed among the algebraic symbols, as to be evolved only by analytical artifice, may spontaneously offer itself to notice in the geometrical representation.

From what is shown above, much light is thrown upon the connexion between real and imaginary roots; and upon the fact of the necessary occurrence of the latter when the principle of continuity is carried fully out (35). The connexion here spoken of is not that between the roots of an individual equation; but that between the successive series of roots of a continuous series of equations. Instead of considering an isolated equation, the geometrical form enables us to trace the connecting circumstances of the entire series to which that one belongs; and thus to ascertain how its imaginary roots arise, and what real values have given place to them, or have merged into them. Considering in this way any proposed equation as one of a series of others, in which the right-hand members pass continuously over a series of values from $-a$ to $+b$, and, therefore, through zero, we clearly see how, by the operation of a uniform law, two unequal roots pass into equal ones, and thence into an imaginary pair; and also how a minute change in the absolute term of an equation having a pair of equal roots, will convert those roots either into two unequal roots, lying very closely to one another, or into a pair of imaginary conjugates. For such a change corresponds to a slight movement of the axis parallel to itself. If contiguous ordinates, or values of $f(x)$, one on each side of the ordinate 0 corresponding to the equal roots, be found to have increased by this displacement of the axis, it will show that the axis must have receded from the curve: the two roots will then have become imaginary. But if the same ordinates have diminished, we may then infer that the roots have continued real, and have become unequal. We shall find these circumstances of consequence in the analysis of equations.

It is of importance to observe, in reference to what has just been said respecting the geometrical indications of imaginary roots, that all the imaginary roots of any equation $f(x) = 0$ are not necessarily thus indicated in the curve line which completely represents the general equation $f(x) = y$. Only those are so distinguished of which each pair unites continuously, as above described, with a pair of real values of the equation $f(x) = a$, which real values approach towards equality as a approaches towards zero; or, referring to the geometrical representation, as TW approaches towards X′X. After this equality is reached the values pass from a real into an imaginary form, which passage is indicated in the diagram by the undulation—which first gave a pair of intersections, and then by the union of these a point of contact p—becoming altogether detached from the axis.

It is essential therefore to the existence of this undulation, that the imaginary roots indicated by it be those into which two equal roots of $f(x) - a = 0$ have merged by an alteration in the value of a. These equal roots are represented in the diagram by the line Aq, or rather by a line equal and parallel to Aq, touching at p. We know, from the theory of equal roots, that the repeated root enters also once into the derived equation $f_1(x) = 0$, which is the derived equation equally of $f(x) - a = 0$, and of

$f(x) = 0$, since a change in the absolute number of the primitive, causes no change in the derived equation. Aq therefore represents a root of $f_1(x) = 0$.

It thus appears that in passing over the interval nr, comprehending the indication of a pair of imaginary roots, that is, in substituting continuous values for x, from $x = \mathrm{A}n$, to $x = \mathrm{A}r$, in $f(x)$, we necessarily pass over a root of $f_1(x) = 0$; and the value of this (Aq) is such, as to render $f(x)$, or pq, a *minimum;* that is, less than the immediately preceding and succeeding values of $f(x)$. The two changes of sign, lost in the interval nr, thus arise from the passage of $f_1(x)$ through zero. In the signs of the derived functions, from the last up to $f_1(x)$ inclusive, only one change can be lost by this passage; and as two are lost when the next following function $f(x)$ is included, it follows that the value which makes $f_1(x)$ zero, causes $f(x)$ and $f_2(x)$ to take *like* signs.

Now not only has the equation $f(x) = 0$ a pair of imaginary roots when these circumstances have place in the last three functions $f(x)$, $f_1(x)$, $f_2(x)$, but also when similar circumstances have place in *any* three consecutive functions; for wherever an intermediate function vanishes for a value which renders the signs of the functions contiguous to it, on each side, *like*, a pair of imaginary roots in the primitive equation will be implied (page 129). These latter imaginary roots are not indicated then in the curve referred to: they have their indications in other curves, those which arise from constructing every equation $f_m(x) = y$, of which the first member $f_m(x)$ takes the same sign as $f_{m+2}(x)$ for a value which causes the intermediate function $f_{m+1}(x)$ to vanish.* This

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* These latter indications directly refer only to the imaginary roots of that derived function which immediately precedes the one that vanishes in the order of derivation, and the existence of imaginary roots also in the primitive equation is but an inference from this. These latter roots are thus merely *indicated*, and nothing respecting them beyond the simple indication of their existence is furnished to us. It is not so with respect to the other class of imaginary roots, whose presence is immediately made known, as above explained, by the first and second derived functions. Each pair of such roots is not only indicated, but to a certain extent the real parts are actually represented or expressed. In the diagram at page 155. Aq actually represents a portion of the root indicated, which portion becomes more and more important as pq diminishes. The numerical value of Aq is that which, when put for x in $f(x)$,

distinction of the imaginary roots into classes, suggested by the different curves in which their indications occur, is of considerable importance in reference to the researches of **Fourier**; who, as we shall now see, is careful to observe the principles implied in it in his analysis of those doubtful intervals which sometimes occur within the limits of the real roots of an equation.

(123.) An interval, anywhere within the extreme limits of the roots of an equation $f(x) = 0$, is doubtful, when the values of x which comprehend it produce no change of sign in $f(x)$, although an even number of changes are lost in the entire series of derived functions, in the passage of x from the smaller of those values to the greater. The indications lying in such an interval may imply real roots, either equal or unequal; or they may belong only to imaginary pairs: our object at present is to discover criteria by which the true character of the interval may be ascertained.

Suppose first that two roots only are indicated in the interval in question—the interval $[a, b]$; and let the geometrical representation of that interval be either that in figure 1 below, or that in

more nearly satisfies the condition $f(x) = 0$ than any neighbouring value: the defect of the result from zero is represented by pq: it can therefore be diminished, and finally annihilated, only by the preceding value taking an *imaginary increment*, or one of the form $\alpha + \beta\sqrt{-1}$, which, however, will become the more unimportant as the defect itself to be removed becomes smaller. Thus Aq, or the root of $f_1(x) = 0$, will be an approximation to the real part of the imaginary pair. When actually put for x in $f(x)$, the result will be nearer to zero than that given by any adjacent value: and if the defect from zero be so small as to warrant its disregard, in the inquiry in hand, the complementary imaginary part may unquestionably be rejected, and the real value taken for the root, or rather for one of two equal roots.

It is easy to see how all trace of the existence of the other class of imaginary roots becomes lost in the curve. We have considered our proposed equation as one of a continuous series of equations differing from one another only in the final term; and have taken note only of the intersections lost in passing over this series. But our equation may unite with an endless variety of varying equations, changing according to different laws. The curve at page 155 may have passed into that form through various preceding forms—forms which presented intersections that have gradually coalesced, and then disappeared. The manner of this disappearance, and of the passage from real values to imaginary, cannot of course be exhibited to the eye, although readily conceivable.

figure 2. The former, from what is shown above, will indicate a pair of real roots, because the curve cuts the axis; the latter, where no intersection takes place, will correspond to a pair of imaginary roots. And, from knowing the interval *ab*, and the ordinates *am*, *bn*, at its extremities, we have to determine which of these two representations belongs to the case under examination.

Fig. 1.

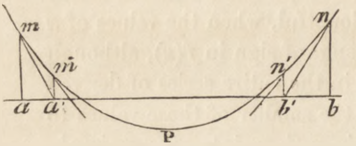
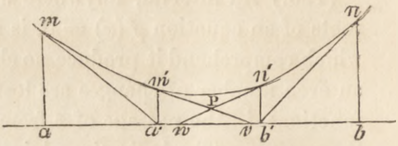


Fig. 2.



The following considerations suggested themselves to FOURIER for this purpose: If the second figure represent the true construction of the equation $f(x)=y$ in the doubtful interval, the following circumstances must have place, viz., if tangents be drawn from *m* and *n*, meeting the axis in *a'* and *b'*, and then again tangents from *m'* and *n'*, where the ordinates *a'm'* and *b'n'*, meet the curve, or where ordinates still closer together would meet it; and so on; it is plain that we shall at length arrive at a pair of tangents that must *cross one another* before they reach the axis. Let these be the tangents from *m'*, and *n'*, which, after crossing, meet the axis in *v*, *w*; then the sum of the two portions *a'v*, *b'w*, must necessarily *exceed* the interval *a'b'*. Each of these portions is, in geometrical language, called a *subtangent*, and is defined as the part of the axis between the ordinate and tangent; so that when the curve is that of figure 2, it is always possible, by bringing the ordinates *am*, *bn*, which bound the doubtful interval, closer together, that is, by narrowing the interval $[a, b]$, to arrive at a pair of subtangents whose sum shall exceed the interval thus contracted.

Now this can never be brought about in the first figure: nor can even the sum of the subtangents be rendered *equal* to the interval, as is evident; and in attempting to effect it by narrowing the interval $[a, b]$, we should be led *within* the limits of the real roots, and thus to a value of y or $f(x)$ opposite in sign to the values $f(a)$ and $f(b)$; so that the separation of the roots would be accomplished.* We are thus informed of the desired criterion for testing the character of the proposed interval; and it only remains to convert the geometrical operations, involved in its application, into the processes of analysis. The well-known theory of curves at once suggests these.

The subtangent corresponding to any abscissa a, is analytically expressed by $\frac{f(a)}{f_1(a)}$;† so that, disregarding the algebraic signs, and actually adding the subtangents at the limits a, b, the criterion of a pair of imaginary roots is

$$\left\{\frac{f(a)}{f_1(a)} + \frac{f(b)}{f_1(b)}\right\} = \text{or} > (b-a) \quad . \quad . \quad . \quad . \quad [\text{A}]$$

And in seeking to fulfil this condition, by making the interval $[a, b]$ narrower and narrower, we shall either actually succeed in doing so, or be led to a value of $y=f(a')$, or $y=f(b')$, of opposite sign to that of $f(a)$ and $f(b)$; and thus to a separation of the roots.

(124.) It must be observed, however, that it is all along presumed in the foregoing reasoning, that the curve has no sinuosities or points of inflexion, as in the figure at page 155, throughout the interval between m and n;—the existence of such a point would be fatal to the preceding conclusions.

This restriction requires that $f_2(x)$ preserves its sign unchanged throughout the interval $[a, b]$; for the analytical indication of

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* It should be observed in this latter case, that if the new ordinates, from whose extremities the new pair of tangents is drawn, always spring from the extremities of the last pair of tangents, the limits will contract at a continually diminishing rate; and we shall never be able to bring them within the points of intersection, and thus separate the roots: every new ordinate, therefore, should be distinct from that last taken by an interval which exceeds the length of the last subtangent; that is to say, in contracting the interval by assuming an intermediate value of x, this new value of x should differ from each of the former values by a quantity greater than either of the subtangents which those former values furnish. It is of importance to remember this.

† The proof of this, as well as of one or two other particulars in the next article, involves the elementary principles of the general theory of curve lines; for which the student may consult the second section of the author's *Differential Calculus*, chapters I and II.

a point of inflexion is $f_2(x)=0$. Moreover, the form assumed for the geometrical representation of the equation $f(x)=y$, within the limits $x=a$, $x=b$, is such as to imply the existence of a *point* P, at which the tangent is parallel to the axis: this implies the existence of a single value a' for x, between a and b, that will satisfy the analytical condition $f_1(a')=0$; so that while x passes over the interval $[a, b]$, $f_1(x)$ changes its sign.

The conditions therefore implied in the preceding constructions, and in the analytical inferences drawn from them, are that the two changes of signs lost in passing over the interval $[a, b]$, are lost entirely in the passage of the last *three* of the series of functions

$$f_n(x),\ f_{n-1}(x),\ \ldots\ f_3(x),\ f_2(x),\ f_1(x),\ f(x),$$

that is, in the three functions

$$f_2(x),\ f_1(x),\ f(x),$$

which give either the results

$$\begin{array}{lccccccc} x=a & + & - & + & & - & + & - \\ & & & & \text{or,} & & & \\ x=b & + & + & + & & - & - & - \end{array}$$

the preceding terms of the series losing no changes within these limits. Hence in the application of the criterion [A] we must proceed as follows:

(125.) Having substituted the two limits a and b in the series of functions above, and having compared the signs of the results, if we find that the second series of results has two changes of sign fewer than the first; but that omitting the last two signs of each series the second has just as many changes of sign as the first; then, in order to ascertain whether the two roots indicated are real or not, find the values of $\frac{f(a)}{f_1(a)}$, and $\frac{f(b)}{f_1(b)}$; and, disregarding their algebraic signs, see whether the sum of these fractions surpasses, or is at least equal to $b-a$: if such be the case, we may be assured that the two roots indicated are imaginary.

If the preceding condition have not place, the sum of the fractions being less that $b-a$, we must narrow the interval $[a, b]$

by taking some intermediate number c; but, to avoid the endless subdivisions of the interval which would attend the attempt to separate in this way two roots that might eventually prove to be *equal*, we ought to examine whether $f(x)$ and $f_1(x)$ have a common measure $\phi(x)$; and if so whether the equation $\phi(x)=0$ has a real root c comprised between a and b. If it have, the equation $f(x)=0$ has two real roots in the interval, each equal to c; and thus the character of the interval becomes determined.

But if the functions $f(x)$, $f_1(x)$ have no common divisor $\phi(x)$, or, having one, if the equation $\phi(x)=0$ have no root between a and b, which we may ascertain as above, then we must examine whether the two roots of $f(x)=0$, indicated between a and b, can be separated by the substitution of a number c intermediate between a and b. If upon the substitution of any such number the sign of $f(c)$ is different from that of $f(a)$ and $f(b)$, the two roots must be real; one lying between a and c, and the other between c and b. But if on the contrary, the sign of $f(c)$ is the same as that of $f(a)$ and $f(b)$, then we must conclude that the limits at first chosen were not sufficiently close to enable us to determine the character of the roots at the first operation.

For a second operation let us take for limits c, and that one of the former two of which the substitution in $f_1(x)$ gives a result of contrary sign to that of $f_1(c)$; and proceed with this interval as with that at first chosen; and so on till the condition [A] is fulfilled, or till the roots are separated.

(126.) As a first example, let the proposed equation be

$$f(x)=x^3+2x^2-3x+2=0.$$

Then, proceeding as in the last chapter, we find the following results for $x=0$, $x=1$.

	$f_3(x)$	$f_2(x)$	$f_1(x)$	$f(x)$
(0)	$+$	$+$	-3	$+2$
(1)	$+$	$+$	$+4$	$+2$

The actual values of $f_1(x)$, $f(x)$, for the proposed values of x, are written down; because these values are to be employed in the analysis of the interval [0, 1].

This example belongs to the class of equations to which the preceding rules apply: two changes of signs are lost in passing over the doubtful interval, $f_1(x) = 0$ has a single root in that interval, and $f_2(x) = 0$ has no indications of roots in the same interval. And we have now to ascertain whether the two roots indicated in $f(x) = 0$ are real or imaginary.

The limits in the present case are $a = 0$ and $b = 1$; therefore the ~~right~~left-hand member of the criterion [A], neglecting algebraic signs, is

$$\frac{f(0)}{f_1(0)} + \frac{f(1)}{f_1(1)} = \frac{2}{3} + \frac{2}{4}$$

and as this exceeds the ~~left~~right-hand member $b - a$ or 1, we conclude at once that the roots indicated are imaginary.

As a second example, let the equation

$$x^5 + x^4 + x^2 - 25x - 36 = 0$$

already partially analysed at page 145 be proposed. The analysis referred to shows the existence of a doubtful interval between the limits -10 and -1.

				$f_2(x)$	$f_1(x)$	$f(x)$
(-10)	$+$	$-$	$+$	$-$	$+\ 45955$	$-\ 89686$
$(-\ 1)$	$+$	$-$	$+$	$-$	$-\ 26$	$-\ 10$

In this case the first member of [A] is

$$\frac{89686}{45955} + \frac{10}{26}$$

which is evidently less than 9, the difference between the limits; so that the character of the interval still remains doubtful. It is possible that the roots indicated may be equal; so that before attempting to separate them, by subdividing the interval, it will be proper to examine whether or not such be the case; that is, whether or not the functions

$$x^5 + x^4 + x^2 - 25x - 36, \text{ and } 5x^4 + 4x^3 + 2x - 25$$

have a common measure. Upon trial we find that a common measure does not exist; and we may therefore proceed to narrow the interval, with the confident expectation that, sooner or later, the criterion [A] will be fulfilled, or else the roots separated.

Employing then an intermediate number, as — 2, we have

(− 10)	+	−	+	−	+	−
(− 2)	+	−	+	−	+	+
(− 1)	+	−	+	−	+	−

And this number separates the roots, as appears from the two changes which $f(x)$ presents. Hence one root lies between — 1 and — 2, and the other between — 2 and — 10.

These two examples will sufficiently illustrate the course to be adopted in the analysis of a doubtful interval whenever the series of signs, which the derived functions present in the passage of x over that interval, lose only two changes, and when, moreover, those losses are confined to the last three functions in the series.

It remains now to be shown that when these restrictions are removed, and the functions taken unconditionally, the same process may still be made effective in detecting the character of the doubtful interval.

(127.) In order to this, let the signs due to the series of functions, for the proposed limits, a, b, be written in two rows as before; and let there be inserted between these the numbers which express the changes lost in the first two terms, the first three, the first four, and so on to the end of the series: the last number inserted will of course express the greatest number of roots the proposed equation can have within the limits under examination. For instance, if the signs in the case of an equation of the fifth degree, for given values a, b, of x, be

	$f_5(x)$	$f_4(x)$	$f_3(x)$	$f_2(x)$	$f_1(x)$	$f(x)$
(a) . . .	+	+	−	+	+	−
		0	1	2	2	3
(b) . . .	+	+	+	+	+	+

the numbers expressing the changes lost in proceeding from term to term to the right-hand extremity of the series will be 0, 1, 2, 2, 3. These, **Fourier** calls the *indices* of the changes:—they show us that the equation $f_4(x)$ =0 has no root between a and b; that $f_3(x)=0$ has *one*, but not more; that $f_2(x)=0$ has indications of two roots, that $f_1(x)=0$ has like indications, and that $f(x)=0$ has indications of three roots in the interval.

It is necessary to notice that these indices can never succeed one another in an arbitrary manner: if δ represent any one, that immediately following must be either δ or $\delta-1$ or $\delta+1$. This is plain, because in passing on from one term to the next in the series, the number of changes is either left undisturbed or else increased or diminished by a single change only.

If the last index is 0, we shall infer that no root exists in the interval: if it is 1, that a single root exists but no more: the interval can be doubtful only when this index is greater than 1. Such then being the case, it is plain that *one* of the preceding indices, at least, must be 1: for the first index, if not itself 1, must be 0, in which case the second must be either 0 or 1, for no index can be *negative* (106), and as observed above, the difference between two consecutive indices can never exceed unity. It follows, therefore, however many zeros may succeed one another, the index 1 must at length occur, since by hypothesis the last index is 2, at least; so that wherever a zero occurs, a 1 must occur beyond it.

Let the index 1, nearest to the right-hand extremity of the series be taken, and let the corresponding function be $f_m(x)$. We may then infer, whatever be the composition of the functions beyond this, towards the right, that the equation $f_m(x)=0$ has only one root in the proposed interval.

The index next to this, on the right, must necessarily be 2: for it is not 1, by hypothesis; and it cannot be 0, since then a 1 would occur beyond it, as just remarked, and this also is contrary to the hypothesis. The index next to it on the left, however, may be either 1 or 0; and if it be 1, it may be made to become 0 by diminishing the interval $[a, b]$, as appears from the following considerations.

The equations $f_m(x)=0, f_{m+1}(x)=0$ cannot have equal roots within the limits a, b, since then two roots of the former would be equal (98), and thus the index corresponding to $f_m(x)$ could not be 1, implying only a single root. Whatever the actual value of this single root may be, we may consider it to be diminished down to the limit a', and increased up to the limit b', so that every root of $f_{m+1}(x) = 0$ may be excluded from the interval $[a', b']$: hence such an interval exists; or such values, a', b', may be given to x, that will reduce the index of $f_{m+1}(x)$ to zero, and the index of $f_m(x)$ to 1. By determining these values, the primitive interval $[a, b]$ becomes subdivided into the three partial intervals

$$[a, a']; \ [a', b']; \ [b', b].$$

In the first and third of these, it is plain that the equation $f_m(x) = 0$ cannot have any roots, since the only real root lying between the extreme limits a, b is comprised within the new limits a', b'. Consequently for each of the extreme intervals $[a, a']$ and $[b', b]$ the index corresponding to the function $f_m(x)$ will be 0; and thus, for each of these intervals the index 1, whose progress we are now tracing, is advanced further towards the right-hand extremity of the series; that is, nearer to $f(x)$, at which advanced point we may proceed anew as above.

It is possible, however, that this 1 may be postponed, by the occurrence of a succession of zeros, till we reach the last term $f(x)$ itself; in which case a single root will thus be detected; or the zeros may continue up to the end, in which case we shall know that roots are excluded from all but the middle interval $[a', b']$. As to this interval, we know that the index 1, corresponding to $f_m(x)$, is comprised within it; but it may happen that, with this contracted interval, $f_m(x)$ is no longer the last function to which the index 1 corresponds; there may, as in the extreme intervals, occur a 1 still more in advance; in which case the object we have in view, viz. to carry forward the final 1 either till it become the last of the series, or till it cause the last to be zero, will be promoted. But if this should not happen, $f_m(x)$ being still the last function whose index is 1, then in the partial interval $[a', b']$ we shall have for the functions

	$f_{m+1}(x)$	$f_m(x)$	$f_{m-1}(x)$
the indices	0	1	2

It thus appears that by successive subdivisions of the doubtful interval we can always advance the index 1, lying in any partial interval, so as to cause the last index to be either 1 or 0; or else we shall be led to the arrangement just exhibited; where the last index equal to 1 is preceded by 0, and followed by 2.

It is obvious that the intervals which lead to this result are the only ones involving any doubt; and we may, therefore, now confine ourselves to the examination of this single case.

(128.) And first we may remark that the equation $f_{m+1}(x)=0$ cannot have any root within the limits a', b', consistently with this arrangement; and, secondly, that the equation $f_m(x)=0$ has a single root between these limits, and no more. As to the equation $f_{m-1}(x)=0$, two roots are indicated, but whether they are real or imaginary remains to be determined.

Now this determination has already been effected by the rule at (125) by which the character of the two roots of $f_{m-1}(x)=0$ indicated by the limits a', b' may be ascertained. If they be real the criterion [A] will separate them, and the interval $[a', b']$ will be divided into two, for each of which the index corresponding to $f_{m-1}(x)$ will be 1; and thus the index 1, as in the other cases, will be advanced nearer to $f(x)$. But if the two roots of $f_{m-1}(x)=0$ prove to be imaginary, then we know (109) that two roots will be imaginary in every one of the subsequent equations:

$$f_{m-2}(x)=0,\ f_{m-3}(x)=0, \ldots f_2(x)=0,\ f_1(x)=0,\ f(x)=0$$

The two changes of sign lost in the series terminating in $f_{m-1}(x)$, in the passage of x from a' to b', in consequence of these two imaginary roots in $f_{m-1}(x)=0$, being confined to the three terms $f_{m+1}(x)$, $f_m(x)$, $f_{m-1}(x)$, must arise from the first and third of these taking the same sign for that value of x which renders the middle one zero; and, therefore (109) this loss is permanent throughout the entire series; that is, two roots of $f(x)=0$ are wanting in the interval $[a', b']$. Hence, in each of the indices corresponding to the functions from $f_m(x)$ onwards to the last $f(x)$ the 2, significant of the changes thus lost, necessarily enters. If this 2 be suppressed, account will then be taken, by the indices thus reduced, only of the roots the character of which still remains to be examined; and the index corresponding to $f_{m-1}(x)$ will become zero. Hence the index 1, which lies nearest to that furnished by $f(x)$, will occur in advance of $f_{m-1}(x)$, or else zeros only will make up the remainder of the series.

It follows, therefore, that in all cases, whether the two roots of $f_{m-1}(x) = 0$ be real or imaginary in the interval $[a', b']$, the preceding operations will give rise to new series of indices in which the index 1, nearest to the end, is still further advanced; so that we must at length obtain series of which the last term in each is itself 1, or else zero. The following examples will suffice to illustrate the preceding precepts.

(129.) The equation $x^5 - 3x^4 - 24x^3 + 95x^2 - 46x - 101 = 0$ partially analysed at (113) has indications between 1 and 10 of three roots, one of which is real and the other two doubtful. Let it be required to determine the true character of the latter.

	$f_5(x)$	$f_4(x)$	$f_3(x)$	$f_2(x)$	$f_1(x)$	$f(x)$
(1) . . .	$+ 120$	$+ 48$	$- 156$	$+ 30$	$+ 65$	$- 78$
	0	**0**	**1**	**2**	**2**	**3**
(10) . . .	$+ 120$	$+ 1128$	$+ 5136$	$+ 15150$	$+ 32654$	$+ 54939$

In proceeding along the series of indices from right to left we find the first 1 to correspond to the function $f_3(x)$: this index is followed by 2 and preceded by 0; and, therefore, the nature of the two roots of $f_3(x) = 0$, thus indicated, may be ascertained by the rule at (123). Applying then the criterion [A] we find for the sum of the fractions $\frac{30}{156} + \frac{15150}{5136}$ a value less than the difference 9, between the limits; so that the criterion is not satisfied: narrower limits, however, must satisfy it, or else separate the roots, provided that is, that the roots are not equal, and therefore inseparable. Agreeably to the rule we are to satisfy ourselves on this point first; that is, we are to ascertain whether the functions $f_2(x)$ and $f_3(x)$, which are

$$20x^3 - 36x^2 - 144x + 190, \text{ and } 60x^2 - 72x - 144$$

have a common measure. Upon trial, we find that no common measure exists; so that we may proceed to narrow the interval [1, 10] with the certainty of eventually detecting the character of the roots. The intervals [1, 2], [2, 3] give the following results:

$$(1) \;.\;.\;.\;.\; +120 + \;48 - 156 + 30 + 65 - 78$$

$$0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0$$

$$(2) \;.\;.\;.\;.\; +120 + 168 - \;48 - 82 + 30 - 21$$

$$0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 2$$

$$(3) \;.\;.\;.\;.\; +120 + 288 + 180 - 26 - 43 - 32$$

As in the first of these intervals [1, 2] the final index is 0, we infer that no real root exists in that interval.

As in the second interval [2, 3] the index 1, nearest to the termination of the series, is followed by 2, and preceded by 0, we have to apply to this interval the criterion [A], at page 165. The fractions $\frac{21}{30}$, $\frac{32}{43}$ give a sum greater than the difference 1 between the limits; so that the criterion is satisfied: and hence the roots indicated in the interval [2, 3] are imaginary. The third root, which is of course real, must therefore lie between 3 and 10.

2. As a second example let us take the equation

$$x^4 - 4x^3 - 3x + 23 = 0$$

which has been partially analysed at page 144.

The only doubtful interval here is the interval [1, 10], for which we have found the series

$$(1) \;.\;.\;.\;.\; +24 \mp 0 \quad -12 \quad -11 \quad +17$$

$$0 \quad 0 \quad 1 \quad 1 \quad 2$$

$$(10) \;.\;.\;.\;.\; +24 + 216 + 960 + 2797 + 5993$$

The index 1 nearest to the termination of the series of indices is followed by 2, but not preceded by 0. Hence the interval [1, 10] must be subdivided. Interposing the number 2 we have

$$(2) \;.\;.\;.\; +24 + 24 \;\mp 0 \quad -19 \quad +1$$

$$0 \quad 0 \quad 0 \quad 1 \quad 2$$

$$(10) \;.\;.\;.\; +24 + 216 + 960 + 2797 + 5993.$$

In passing over the interval [1, 2] no changes are lost, so that no root exists in this interval, but two roots are indicated in the interval [2, 10]. And as the index 1, within a place of the end, is followed by 2 and preceded by 0, the character of these roots is to be tested by the criterion [A]. Writing the fractions $\frac{1}{19}$, $\frac{5903}{2797}$, we see that their sum is less than 8, the distance between the limits 2 and 10; hence a number must be employed intermediate to these limits, unless the roots under examination prove to be equal; that is, unless the functions $f(x), f_1(x)$ which are

$$x^4 - 4x^3 - 3x + 23, \text{ and } 4x^3 - 12x^2 - 3$$

have a common measure. Upon trial we find that no common measure exists. Substituting then the intermediate number 3, we have the series of signs

$$(3) \ldots + \quad + \quad + \quad - \quad -$$

and comparing this with the series (2) and (10) above, we find that the roots are real, one lying between 2 and 3, and the other between 3 and 10.

We shall propose but one more example, of which we shall give the complete analysis; and, after the manner of **Fourier**, shall actually exhibit the several derived functions.

3. Let the equation be

$$\begin{aligned}
f(x) &= x^5 + x^4 + x^3 - 2x^2 + 2x - 1 = 0\\
\therefore f_1(x) &= 5x^4 + 4x^3 + 3x^2 - 4x + 2\\
f_2(x) &= 20x^3 + 12x^2 + 6x - 4\\
f_3(x) &= 60x^2 + 24x + 6\\
f_4(x) &= 120x + 24\\
f_5(x) &= 120.
\end{aligned}$$

(—1) . . . +		— 96	+ 42	— 18	+ 10	— 6
	0	1	2	2	2	2
(0) . . . +		+ 24	+ 6	— 4	+ 2	— 1
	0	0	0	1	2	3
(1) . . . +		+ 144	+ 90	+ 34	+ 10	+ 2

Comparing these results we find,

1 : That all the real roots are comprised within the limits -1, $+1$, as also the indications of imaginary roots:

2 : That indications of two roots occur between -1 and 0, and of three between 0 and 1.

In the interval $[-1, 0]$ there is only one index equal to 1; this is followed by 2, and preceded by 0: applying therefore the criterion [A], we find the sum of the fractions $\frac{42}{96}$, $\frac{6}{24}$ to be less than 1, the difference between the limits; so that the criterion is not satisfied, the limits not being sufficiently close to enable us to determine, by a single operation, whether the roots are real or imaginary. Before narrowing the interval we ought to ascertain whether the roots indicated are equal; that is, whether $f_3(x)$ and $f_4(x)$, which are

$$60x^2 + 24x + 6, \text{ and } 120x + 24$$

have a common measure. Upon trial we find that they have not. We must therefore contract the interval by interposing a number between -1 and 0. If we employ $-\frac{1}{2}$ we have the following results :

$$\begin{array}{lcccccc}
(-\frac{1}{2})\ldots & + & -36 & +9 & -\frac{13}{2} & +\frac{73}{16} & -\frac{83}{32} \\
 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} \\
(0)\ldots & + & +24 & +6 & -4 & +2 & -1.
\end{array}$$

The first partial interval, that between -1 and $-\frac{1}{2}$, cannot comprise any root, since no changes are lost in passing over it. Two roots are indicated in the second interval; and as the only index equal to 1 is followed by 2, and preceded by 0, we have to examine the fractions $\frac{9}{36}$ and $\frac{6}{24}$, in order to ascertain whether or not their sum surpasses, or is at least equal to $\frac{1}{2}$, the difference between the limits. We at once see that the sum is equal to $\frac{1}{2}$, and consequently that the roots indicated are imaginary.

The interval $[0, 1]$ still remains to be examined. In this interval the series of indices are 000123: hence, applying the criterion [A] to the functions to which the indices 1, 2 belong, we find that $\frac{2}{4} + \frac{10}{36}$ does not satisfy the condition [A]: the interval must therefore be contracted, unless the functions $f_1(x)$ and $f_2(x)$, which are

$$5x^4 + 4x^3 + 3x^2 - 4x + 2, \text{ and } 20x^3 + 12x^2 + 6x - 4$$

have a common measure. Upon trial we find that no common measure exists. Hence employing the intermediate number $\frac{1}{2}$, and thus subdividing the interval [0, 1] into the two [0, $\frac{1}{2}$] and [$\frac{1}{2}$, 1], we have the following results:

$$\begin{array}{lcccccc}
(0) \ldots + & + 24 & + 6 & - 4 & + 2 & - 1 \\
\quad\quad\;\; \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{2} \\
(\frac{1}{2}) \ldots + & + 84 & + 33 & + \frac{9}{2} & + \frac{25}{16} & - \frac{9}{32} \\
(1) \ldots + & + & + & + & + & +
\end{array}$$

The interval [$\frac{1}{2}$, 1] evidently comprehends a real root. The interval [0, $\frac{1}{2}$] indicates two roots: the indices require the application of the criterion [A] to the fractions $\frac{9}{4}$ and $\frac{25}{16} \div \frac{9}{2}$, the sum of which surpasses $\frac{1}{2}$: therefore the criterion is satisfied, and the two roots of $f_1(x) = 0$ are imaginary. Subtracting then the index 2, due to these imaginary roots, from the indices of $f_1(x)$ and $f(x)$, the last index becomes 0, so that no real root exists in the proposed interval. Hence the proposed equation has but one real root: this lies between ·5 and 1: the four remaining roots are imaginary.

(130.) Such is the method which Fourier has proposed for discovering the character of the roots of which indications exist between given limits; and which, when taken in combination with the theorem at (107), completes the analysis of the equation.

The theorem referred to shows that by substituting in $f(x)$, and in the several functions derived from it (112), a number increasing by insensible degrees from $-\frac{1}{0}$ to $+\frac{1}{0}$, the series of signs furnished by these substitutions will continue to present *variations* only, till the number substituted arrives at a certain limit a: after this the variations will gradually disappear, till we reach a second limit b, when they will become replaced by *permanencies*. We are thus made acquainted with the limits a, b, between

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which all the real roots, and the indications of the imaginary roots, must be sought.

It is evident that this loss of variations at any stage, can arise only from the number substituted causing one or more of the functions to become zero: and the evanescence of any one of the functions, not the last, $f(x)$, must be attended with one or other of these circumstances, viz., the immediately preceding function will have either the same sign as that immediately succeeding, or a contrary sign. In the latter case it is plain that no variation can be lost in the passage of the intermediate function through zero. In the former case two variations must be lost; and as, by hypothesis, the number substituted does not render $f(x)$ zero, this number is not a root of the primitive equation, but an *indicator of two imaginary roots;* a fact which, as before remarked (69), had been previously taken notice of by De Gua.

If a single real root exist in any of the partial intervals $[a', b']$, into which the whole interval $[a, b]$ is divided, as in the example at p. 128, the theorem (107) will enable us to detect it. If more than one *may* exist, the same theorem will indicate the possibility. To remove the doubt, the second theorem at (125) is to be resorted to; by aid of which, subdividing the interval as there directed, we shall arrive either at numbers which interpose themselves between the roots, and thus actually separate them, furnishing limits between which they severally lie, or else at numbers which are limits, not to real roots, but to *indicators* of conjugate imaginary roots.* And thus the character of any partial interval $[a', b']$ may be discovered, and the analysis of the equation completed. We shall offer some remarks upon this method of completing the analysis of an equation in the next chapter, and shall suggest means for reducing much of the labour attending its application.

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* The numbers thus called *indicators* are, as we have seen above, real roots of certain of the derived equations, each root being such that its substitution in the function immediately preceding, and in that immediately succeeding the vanishing function, gives like signs. Hence in narrowing an interval in which only indications of imaginary roots exist, we approximate to a root, not of the primitive, but of one of the derived equations, the passage over which root is attended with the loss of two variations. (See page 163.)

CHAPTER IX.

REMARKS ON THE PRECEDING METHOD, WITH SUGGESTIONS FOR ITS IMPROVEMENT.

(131.) The method which has just been explained is a very ingenious application of a simple property in the theory of curves to an important analytical purpose—the complete analysis of a numerical equation.

The examples which have been given to illustrate it, and which are all taken from the work of Fourier, show that the method may frequently be employed with success, and with the expenditure of but little trouble. But from these instances of its applicability it would be wrong to infer anything as to its merits as a general rule for the analysis of equations. The peculiar difficulties that have always opposed themselves to the success of every such general rule do not occur in the examples of Fourier, which have been so framed as not to involve the difficulties alluded to; and by a similar adaptation of the end to the means, the most limited and imperfect of methods might be exhibited to advantage. Theoretically, the process of Fourier is perfect; and is characterized by great ingenuity and analytical address: but it is proper to speak undisguisedly of its practical defects, beyond certain limits, and a certain kind of equations; because some recent writers, in dwelling upon its merits, have altogether overlooked the imperfections which render it, in its present form, altogether impracticable even in equations of so low a degree as the fourth, except certain conditions, beyond our control, happen to have place.

If, by a preliminary examination, we could always ascertain whether an equation proposed for analysis involved these favorable conditions or not, the value of FOURIER's method would be greatly increased; for then we should never, as at present, be exposed to the risk of entering upon a series of calculations which terminate too remotely for ordinary patience and perseverance to carry us through them.

The great blemish in the preceding method is, that it leaves us in doubt as to the character of contiguous roots, till by successive trials it either actually separates them, or else shows them to be imaginary. When they are real, and lie so closely together as to have four or five leading figures in common—a circumstance which at the outset we cannot foresee—it is plain, from the specimens already given, that the indications which determine their character would be delayed to so remote a step, that our patience would become exhausted long before reaching it; and the nature of the roots would still be left in doubt. The example at p. 221 of the introductory volume on the *Analysis and Solution of Cubic and Biquadratic Equations* offers an instance of this kind. By the method proposed by FOURIER, all the work of that example, as exhibited at pages 243, 244, 245, as well as the preliminary step at page 221, must be performed, in addition to the very tedious labour of narrowing the limits, by the successive steps above adverted to, till the roots, proceeding together to the extent of five places of decimals, are actually separated. It will be noticed that in pronouncing upon the impracticability of FOURIER's method, in cases such as this, we have altogether left the labour implied in the process for the common measure out of consideration; because we regard the necessity of carrying on the analysis by the tentative operations in last chapter, till the roots are actually separated, or till they are shown to be inseparable by being imaginary, as constituting the capital defect of FOURIER's method.

(132.) Every method which requires this separation to be accomplished is indeed essentially defective unless it also supply means by which the separation may be expeditiously effected. And it is solely because of this defect that all preceding rules for analysing an equation have been abandoned one after another. Whatever other imperfections may attach to any proposed plan for detecting the character of the roots of an equation this, at all events, must be removed, before such plan can be generally applicable. The presence of the defect here adverted to is the great drawback to the success of LAGRANGE'S mode of analysis; for it will be found upon examination that the impracticability of this mode is traceable solely to the circumstance that the separation of the real roots, or which is the same thing, the finding of a number less than the least of the differences of the roots, is indispensable to the discovery of the character of those roots by his method.

The method of LAGRANGE, however, is far less practicable than that of FOURIER; and it ought to be distinctly noticed, to the advantage of the latter method, that the labour of the separation is only in proportion to the proximity of the roots, whether real or imaginary; whilst the principal step in the method of LAGRANGE involves pretty nearly the same amount of difficulty whether the roots are nearly equal or not. On account of this peculiarity in the method of FOURIER, we should be encouraged to apply it, even in its present form, in cases where the method of LAGRANGE would appal us, from the known difficulties with which we should certainly have to contend.

When the roots of the original equation, as well as those of the derived equations, within the limits under examination, do not approach nearer to equality than by a coincidence in the leading figure of each, or at most in the first two figures, the analysis of it by the preceding method may perhaps be accomplishable even in an equation of the sixth or seventh degree, whilst no one would think of applying to any such equation the process of LAGRANGE. If roots do not approximate so closely even as here supposed, then FOURIER'S method will be comparatively easy; but when the labour of the several operations for the common measure is viewed in connexion with that due to the successive subdivisions of the interval—even for the amount of proximity above supposed—we feel justified in speaking of the eligibility of the method with some hesitation. We shall shortly see, however, that this labour may be altogether dispensed with, or greatly reduced.

(133.) Besides the advantage of FOURIER's method over others, when the real roots are moderately wide apart—that is, have no leading figures in common—there is another marked peculiarity in favour of it: it is, that it considerably reduces the embarrassment and uncertainty which the entrance of imaginary roots often occasion.* The considerations arising out of the ingenious contrivance of the subtangents lead to a rapid detection of the existence of such roots whenever they are not in the peculiar predicament of being rendered real by a very minute change in the absolute term of the equation involving them. In this latter case the imperfection before adverted to again discovers itself: for this case, and the case of nearly equal real roots, both involve the same peculiarities:—a minute change in the constitution of the equation being sufficient in either case to cause the doubtful roots to become equal.

These then are the kinds of equations that are excluded from the rule of FOURIER as it at present stands; namely, those whose roots are such that a very minute change in the final term, whether of the original or of any derived equation, suffices to render a pair of imaginary roots real or a pair of unequal roots equal. This exclusion, however, does not arise from any theoretical imperfection, but from the impracticability of performing the necessary calculations, without extraneous aid.

(134.) Any method that should with equal facility apply to all cases, the amount of labour being independent of the peculiar

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* We regard this peculiarity as a most important feature of FOURIER's method; and we shall show hereafter how it may be rendered available, in combination with the methods of STURM and HORNER, in completing the analysis of equations of more than ordinary difficulty. The assistance which the method of FOURIER is thus capable of affording to that of STURM will be seen to effect a great saving of numerical labour. The two methods have hitherto been regarded as distinct and independent:—we shall endeavour to show that considerable advantage may result from uniting them together, after the defects, which have hitherto attached to FOURIER's method—and which, as remarked in the text, preclude its application in extreme cases— are removed by means of the improvements suggested in the present chapter; these improvements will be brought into operation when we come to treat of the general solution of numerical equations.

relation of the roots just adverted to, must be expected, from this very comprehensiveness of character, to bring into operation, in the simpler examples to which more limited methods may be peculiarly adapted, a larger array of figures than those methods themselves; for it is essential to the perfection of so general a process that, whatever be the example to which it is applied, provision be always made by it for the demands of such critical and extreme cases as those we have been considering, whether the example in question present one of these or not; seeing that it is this which it devolves upon the method itself to make known: for we cannot determine, from the mere aspect of an equation, whether peculiar relations exist among its roots or not; nor, consequently, whether methods, which become inadequate under such relations, are applicable or not.

It would be very unfair therefore, for the purpose of instituting a comparison between a method but partially applicable and another of unfailing generality, to select an example within the powers of the former, place in juxta-position the processes of each, and thence draw our conclusions as to their relative merits: and the student requires to be cautioned against being misled, in this manner, when judging of the comparative claims of the theorems of FOURIER and STURM.

(135.) We are of course, throughout these remarks, speaking of the method of FOURIER as delivered by its author, and as expounded in the preceding chapter; and are not here considering whether, by blending other principles with its own peculiar process, the aid adverted to above, for rendering the separation of the roots always practicable, can be readily commanded or not. In seeking to separate the roots occupying a doubtful interval by continually contracting that interval, according to the method of last chapter, we are left too much to mere conjecture as to the situation in this interval which the real roots, or the indicators of imaginary roots, occupy; and hence, in subdividing an interval, for the purpose of inclosing the roots within narrower limits, several trial operations, fruitless in their results, must always be calculated upon. It is to reduce this useless labour—a labour which needlessly multiplies the work several fold—that extra assistance is so much required.

The method of HORNER, modified by a certain improvement to be hereafter explained, will be seen to furnish such assistance; so that by introducing this method in combination with that of FOURIER, in those cases where the character of the roots under examination unfolds itself only at a remote step of the analysis, the labour at present involved in FOURIER's process may be very considerably diminished. The advantage of this combination will be seen when we come to treat of the calculation of the roots.* It may be observed, however, that the analysis of an equation is thus made to depend upon its partial solution, instead of being entirely preparatory to that solution. This remark equally applies to the methods of LAGRANGE, FOURIER, and BUDAN; and must necessarily apply to every method that leaves the character of the roots undecided till those that are real are actually separated. The method of STURM is the only one which treats the analysis of an equation as a perfectly independent problem; always making known the exact number of real roots in any proposed interval without trenching upon the subsequent problem of seeking the actual development of those roots.

(136.) In the preceding observations upon the mode of analysis expounded in last chapter, and upon the inadequacy of it in its present form as a general method, we have, as before remarked, disregarded that part of the operation which involves the finding of common measures. FOURIER, and all who advocate this method, seem to entirely overlook the labour involved in these processes—two or three of which are sometimes required in the analysis of a single interval, (see example 3, page 175).

No attempt we believe has hitherto been made to remove this serious drawback to the application of the method, even in those cases where no remarkable proximity of roots occurs to preclude the hope of its success. We propose to show, however, that considerable improvement and reduction of labour may be introduced into this part of the operation, and that thus the application of FOURIER's method may in all cases of difficulty be greatly facilitated.

The operations for the common measure so frequently occurring in FOURIER's method of analysis, are always introduced for

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* See also the foot-note at page 162.

the purpose of ascertaining whether or not a proposed equation has a single pair of equal roots in a given interval. It has been already proved that when an equation has a pair of equal roots, and only a pair, that these roots must be commensurable (101); and, moreover, that the repeated root must be such that the square of its numerator must be a divisor of the final term N, the square of its denominator a divisor of the leading coefficient A_n, the numerator itself a divisor of A, and the denominator a divisor of A_{n-1}.

(A.) Hence, in order that the suspected pair of equal roots may exist, A and N must have a common factor (the numerator of the repeated root); and N must be divisible by the square of this factor. Also A_n and A_{n-1} must have a common factor (the denominator of the repeated root), and A_n must be divisible by the square of this factor.

(B.) These tests will generally be found more than sufficient to determine the point in question. But if in any case they are all fulfilled by any fraction $\frac{a}{b}$, then we must proceed to divide the proposed polynomial by $x - \frac{a}{b}$, according to the rapid process at (51). When b is unity the operation must be continued up to N, as at the article referred to; and the number under trial will be a root, or not, according as the final result is zero or not.*

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* In the former case, a root $\frac{a}{b}$ will have been determined, and the coefficients belonging to the depressed equation obtained; if the operation, repeated with these depressed coefficients, terminate in zero, the root $\frac{a}{b}$ will enter the equation twice: if not, one root only will be $\frac{a}{b}$; and thus the separation of the two will have been effected. It is worthy of observation that, in testing a suspected root in this manner, the doubt will often be more speedily removed when the number under trial is a fraction than when it is an integer; because in the latter case the operation must always be carried on up to the final result, or remainder; whereas, in the former case, the occurrence of a fraction, which may take place anywhere between the first and last result, at once puts a stop to the work. However, in the case of a suspected integral root we may proceed somewhat differently: we may reverse the order of the operations by *dividing* the last coefficient by the supposed root, adding the quotient to the preceding coefficient; dividing the result by the same number and adding the quotient to the next coefficient, and so on: a fractional quotient will, of course, stop the operation.

When b is not unity the same course must be adopted, and a corresponding conclusion drawn, unless we arrive at a fractional result; in which case the process need not be continued; for under these circumstances the number under trial cannot be a root (76).

(137.) We shall now apply these precepts to Fourier's examples, given in last chapter.

In example 2 at page 168, it is required to determine whether the equation

$$x^5 + x^4 + x^2 - 25x - 36 = 0$$

has a single pair of equal roots.

Applying the foregoing tests, we have first to see whether any integral factor of 25, except 1, has its square for a factor of 36. A glance is sufficient to show that it has not: hence equal roots do not enter the equation.

Again in example 1, page 173, we have to determine whether a pair of equal roots enters the equation

$$20x^3-36x^2-144x+190=0, \text{ or rather } 10x^3-18x^2-72x+95=0$$

Here neither of the numbers 95, 10 has a square factor except 1, so that equal roots cannot exist.

In like manner the example at page 174 requires it to be ascertained whether the equation

$$x^4 - 4x^3 - 3x + 23 = 0$$

has a pair of equal roots, which it evidently has not since 23 has no factor but unity; which is not a root.

Lastly, the example at page 175, requires it to be determined whether either of the equations

$$10x^2 + 4x + 1 = 0, \quad 5x^4 + 4x^3 + 3x^2 - 4x + 2 = 0$$

have a pair of equal roots; and a simple inspection shows that they have not, since neither of the numbers 10, 1, 5, 2 has a square factor different from 1; which is not a root.

In another of Fourier's examples, which we have not transcribed, the analysis of a doubtful interval requires it to be ascertained whether the equation

$$360x^2 - 1440x + 1440 = 0 \text{ or } x^2 - 4x + 4 = 0$$

has equal roots; which we at once see to be the case, each root being 2.

(138.) The analysis of the equation

$$x^4 + 312x^3 + 23337x^2 - 14874x + 2360 = 0$$

would be interrupted by a doubtful interval between 0 and 1; but we need not follow FOURIER'S directions, and seek the common measure of the first member of this and the first derived function—which is a work of very considerable labour— in order to ascertain whether the roots in this interval are equal or not; since we can confidently affirm at once that the roots cannot be equal; because the equation, having its first coefficient unity, cannot have a pair of equal roots which are not integral.

When, as in this last example, and in example 2, page 174, a common measure is to be sought, according to the directions of FOURIER, between $f(x)$ and $f_1(x)$, no other information is deduced from the process than is sufficient to remove all doubt as to the equality of the roots in the interval under examination.

FOURIER, indeed, was not aware that anything worth notice beyond this could be yielded by that process; and, consequently, as soon as the non-existence of equal roots was ascertained, he applied himself to the discovery of the real character of the roots by the method of successive subdivisions already explained.

But the more recent theorem of STURM, to be given in Chap. XI, shows that all this additional labour is superfluous; and that the process for the common measure does itself supply information, not only as to whether the roots in the proposed interval are equal or not, but also as to whether they are real or not. It is not remarkable that FOURIER should not have recognized this, as the theorem of STURM was a subsequent discovery: but it is singular that those who have since commented upon FOURIER'S method should not have adverted to the circumstance. It is certainly worthy of being adverted to, since it not only shows the superiority of STURM's method over that proposed by FOURIER, but discovers also the essential imperfection of the latter, in involving, as a mere auxiliary, an operation which is found to contain within itself the whole of the desired information. This operation, therefore, supplies much more than FOURIER's method needs; which is simply the resolution of the doubt as to whether or not equal roots exist in a given interval: and it would evidently be an improvement to exchange that operation for another of a less comprehensive, and therefore of a less laborious, character; and adequate to meet no more than the demands of the case.

(139.) The precepts (A) and (B), p. 185, by which this improvement is to some extent effected, suppose that the doubtful interval to be examined is the only one, for the same function, between the extreme limits of the roots. If another doubtful interval occur for the same functions *two* pairs of equal roots may exist between the extreme limits; and these need not necessarily be commensurable; because they may arise from the repetition of a quadratic factor which, when equated to zero, gives incommensurable roots. Still the conditions (A), omitting the parenthesis, must be fulfilled, even in this case (101); so that in all the examples hitherto discussed it is matter of indifference whether the function furnishes other doubtful intervals or not.

But when the conditions (A) are fulfilled, and yet from (B) we find that no equal commensurable roots exist, then if other doubtful intervals occur, for the same function, it is possible that there may be pairs of equal incommensurable roots entering as above stated. Generally speaking it will be best to put this to the actual test, by seeking the common measure between the function in question and the next derived function; carrying on the operation till a quadratic remainder is obtained. If the supposed equal roots exist, this quadratic, factors common to all its coefficients being suppressed, will accurately divide the original function; and fractions will be excluded from the quotient (76). We may either try if this be the case, or if the preceding cubic remainder be divisible by it without fractions: in either case the operation will be easy, because the squares of the extreme coefficients of the divisor must be smaller than, or at most equal to, the extreme coefficients of the original function, being factors of them (76). Thus the actual operation for the common measure need be entered upon only in those cases where the latter, and consequently the more laborious, steps of it become superfluous. It is scarcely necessary to remark that, when more than two equal quadratic factors enter into the composition of the proposed function, the operation for the common measure between it and its derived function will terminate at a still earlier stage.

(140.) It thus appears that even in those rare instances, in which the process for the common measure must be actually entered upon, the work terminates before that part of the operation is reached where by far the greater portion of the labour is accumulated.* But the calculation of even the earlier steps of the operation, by the common method of proceeding, is encumbered with much unnecessary work; and is apt to spread over an inconvenient extent of space. It would seem, therefore, in order to give to the method of Fourier all the practical facility which can render it available as a general rule of analysis, applicable to all cases, that, in addition to a more perfect method of narrowing the doubtful intervals, an easier and more compact form must be given to the numerical operation for the common measure. Such an improved form we have already proposed elsewhere,† and shall have occasion to recur to in the Chapter on Sturm's theorem.

To these remarks we may add that if an equation of the fifth degree have two pairs of equal roots the remaining root must be commensurable; and, generally, if an equation of the $2m + 1$ degree have m pairs of equal roots, the remaining root must be commensurable. This may be useful in ascertaining whether so many pairs of equal roots are possible in any proposed case: when they exist, the depressed polynomial, resulting from elimination of the aforesaid commensurable root, must be a complete square.

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* For an example of the great increase of labour involved in the computation of the last two steps of the common measure, when large coefficients enter, see the *Analysis of Cubic and Biquadratic Equations*, page 243.

† *Mathematical Dissertations*, pages 145 and 209.

(141.) It has been already observed (117) that **Fourier** always employs the auxiliary derived functions, which enter into his process of analysis, encumbered with numerical factors, from which they may be freed. (See page 175.) In the method of transformation employed in the present work, these factors never make their appearance; and thus our several results involve smaller numbers than the corresponding ones deduced by **Fourier**. In Chapter VII. the signs only of these results were the objects of examination: but in the further analysis of the equation, by the method taught in last chapter, the actual results themselves are brought into requisition. The criterion [A] at page 165 involves numerical values only, signs being disregarded. By continuing, therefore, to employ the same mode of derivation as that adopted in the first stage of the analysis, additional simplicity will be introduced into **Fourier**'s examination of the doubtful intervals. We shall only have to remember that in applying the formula [A] to two consecutive results $f_{m-1}(r)$, $f_m(r)$, we must multiply the denominator $f_m(r)$ by m, the number marking the place of the function in the series of derivations; as is plain from the expressions at (116.) As multiplying the second member of the criterion [A] by m is equivalent to suppressing this factor in the denominator of each fraction in the first member, we may always adopt this latter course, which will in general be the easier, and employ the numerical results in the formation of the fractions, without any modification. Thus in the example at page 173 the series of results deduced as here proposed would have been as follow: (see page 143.)

$$(1) \; . \; . \; . \; . \; +1+2 \quad -26 \quad +15 \quad +65 \quad -78$$

$$(10) \; . \; . \; . \; . \; +1+188 \quad +856 \quad +7575 \quad +32654 \quad +54939$$

The fractions submitted to the test would consequently have been $\frac{15}{26}$ and $\frac{7575}{856}$; and the number to which their sum is to be compared, 27.

(142.) The method of computing the derived functions exhibited at page 143 is always the most expeditious, even though these functions be written down before us, as at page 175. There are cases, of frequent occurrence, in which we would recommend these functions, when freed from superfluous factors, to be thus written down; not for the purpose of computing them, which is FOURIER's object, but with a view to extract from them certain information which they readily offer, and which will sometimes enable us to dispense with a tedious analysis. Thus, whenever, as in example 3, at page 175, the index 1, which is followed by 2 and preceded by 0, falls under the function of the first degree, the 2 being under the quadratic function, we would always recommend the derived functions to be actually written, before proceeding to analyse the interval; because the quadratic function would, in general, inform us, at a glance, whether the 2 under it referred to a pair of real roots, or to a pair of imaginary roots. In the example referred to, for instance, the quadratic function $f_3(x)$ is seen at once to be such as to give imaginary roots when equated to zero; since four times the product of the extreme coefficients exceeds the square of the middle one. And thus all the trouble of examining the interval [0, — 1], at page 176, might have been spared. Had this relation among the coefficients of $f_3(x)$ not subsisted, then we should have inferred that the index 2 indicated a pair of real roots; and should have sought, in the usual way, to separate them, without taking at each step the unnecessary trouble of applying the criterion. FOURIER evidently overlooked this means of simplifying his process.*

(143.) In addition to these suggestions for simplifying the operations of the preceding chapter, we have only to remark, in conclusion, that when, as is usually the case, the analysis of an equation is merely preparative to the actual computation of its real roots, we may allow a single pair of roots to remain doubtful, provided all the others are real, and may proceed at once to the calculation of these latter; because, as will be shown hereafter, when all the roots but two are determined, these two, whether real or imaginary, may be derived from the former with compara-

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* There are two other examples in FOURIER's work, which we have not transcribed, in which attention to this principle would have saved the trouble of analysing a doubtful interval. See pp. 148, 149, *Analyse des Equations Déterminée.*

tive ease. This is a truth of considerable importance in connexion with FOURIER's method of analysis; because it still further adds to the facility of its application to a class of equations that would otherwise be scarcely manageable without aid from HORNER's process of approximation to the roots. The equation at page 187 is one of this class: the analysis of the doubtful interval, agreeably to the steps proposed by FOURIER, is very tedious, on account of the great labour attendant upon the separation of the two roots in the interval [0, 1] by the slow and uncertain approach to them which those steps make. The remaining roots, however, are easily separated, and may therefore be expeditiously calculated; and thence, by the principle just stated, may the roots in the interval [0, 1] be readily and accurately determined without reference to any method for their separation. We shall exhibit the entire process in a future part of the work, when the value of the principle referred to will be more clearly seen.*

(144.) It may not be amiss in conclusion to briefly recapitulate the more important particulars dwelt upon in the present and two preceding chapters. And first we may notice that the method of

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* In the treatise on the *Analysis and Solution of Cubic and Biquadratic Equations* the two roots of the equation adverted to, which lie in the interval [0, 1], are separated, by HORNER's method of approximation, with comparative ease: the process supplying us with a series of transformed equations, such that if each, as it presents itself, be considered in conjunction with that which would arise from increasing the transforming factor (or root figure) by unity, we should have a pair of consecutive series of results at every step, to which FOURIER's tests would be immediately applicable. Hence, every step of the work, in the volume referred to, is to be accompanied by another, considered as a bye-operation, arising from transforming by an additional unit. When we arrive at the sixth step the unit-transformation connected with it will not require completion: the change of sign in the final term will show that the roots separate at this step. In fact this unit-transformation may be dispensed with throughout, provided we test the roots, at each step, by the first fraction only of the criterion [A]. But a still better test will be given in Chapter XII.

By thus blending HORNER's method of narrowing the interval, with these contrivances for testing the character of it, and combining with this two-fold process the facilitating principles delivered in the present chapter, we shall always proceed with the perfect certainty that every step we take is a real advance towards the removal of the doubt; and that no part of our labour will have been needlessly expended.

FOURIER divides itself into two principal parts—the theorem of BUDAN, and the subsequent process of Chapter VIII.

The first effects, in general, but a partial analysis of the equation: it makes known the extreme limits within which all the the real roots as well as all the indications of imaginary roots must be sought; and it actually separates those of the former which are sufficiently wide apart to have no leading figures in common. But when either imaginary roots or nearly equal roots enter the equation, the indications of their entrance being the same in both cases, we have no means of distinguishing the one class of roots from the other in any given example; and thus their character is left doubtful. Nevertheless, the theorem effectively prepares the way for the removal of this doubt by supplying the exact interval in which the necessary information must be sought.

The second process, which is exclusively due to FOURIER, addresses itself solely to this latter object; and aims at completing the analysis of the equation by extracting the desired information from the doubtful intervals. This involves a two-fold operation, or course of operations, viz. the continual narrowing of the interval, step by step, and the application of a certain test [A] at each contraction. If the concealed roots happen to be imaginary this test will sooner or later be satisfied; if they happen to be real and unequal, the continual diminution of the interval must at length separate them.

The application of the test is always easy: the separation of the roots often difficult and tedious. For as we are not always furnished with adequate means of knowing whereabouts in the doubtful interval the nearly equal roots may lie, supposing them to exist, or the indicators of the imaginary roots when real roots are wanting, we may become involved in tentative operations of a very fatiguing extent and labour. In order to be certain that these will ever terminate we must be sure that the roots under examination are not equal. It behoves us therefore to put this matter beyond doubt by a preliminary investigation; and thus has been combined with the trial-operations, adverted to above, the extra labour attendant upon finding common measures.

The method of STURM is wholly comprised in a single operation for the common measure; yet that method has been pronounced impracticable beyond very narrow limits. With much greater propriety might this be said of FOURIER's process, involving sometimes two or three operations of this kind; in addition to the great labour of separating the roots by trial-transformations.

To render then the method of FOURIER generally practicable, the following improvements must be effected:

1. When roots are indicated in an interval, means must be furnished for guiding us to that particular subdivision of the interval which the indicated roots occupy; and we must not be left to find our way to this subdivision through trial-operations only. This improvement we propose to accomplish by extending HORNER's method of approximating to the roots.

2. The necessity for the common measure must either be superseded, by some more readily applicable test for equal roots, or at least methods must be contrived for materially abridging the ordinary labour and extent of the operation. The former object is attained, in the generality of cases, by the simple criteria marked (A) and (B); and the latter object—the curtailment of the process for the common measure, when instances of rare occurrence render that process necessary—is effected by the considerations at page 186, which enable us to dispense with the more laborious steps of the work.

3. Lastly: when, as in cases like those just adverted to, the leading steps for the common measure must be performed, a compact method of working, purely numerical, like all the other parts of the process, and free from the needless encumbrances of the common method, is required to give additional facilities to the analysis in the more difficult cases. Such an improved mode of conducting the operation will be found in Chapter XI.

Thus improved, the method of FOURIER may with propriety be brought into advantageous comparison with that of STURM; and will probably henceforth take precedence of it in those cases where, from the magnitude of the coefficients, or of the leading exponent, very large numbers may be expected to enter the operation for the common measure.

CHAPTER X.

METHOD OF BUDAN FOR DETERMINING THE CHARACTER OF ROOTS IN DOUBTFUL INTERVALS.

(144.) THE method proposed by BUDAN for the analysis of a doubtful interval within the limits of the roots is theoretically much simpler than that of FOURIER. It depends upon the obvious principle that, whatever be the number of roots of a proposed equation lying between the limits 0, r, the very same number of roots of the equation whose roots are the reciprocals of the proposed, must lie between $\frac{1}{r}$ and $\frac{1}{0}$; for it is plain that to whatever roots 0 and r are the limits, in the proposed equation, to the reciprocals of those roots—and those only—will $\frac{1}{r}$ and $\frac{1}{0}$ be limits in the transformed equation.

The inference to be drawn from this is, that as many real roots as there are in the proposed equation, lying between 0 and r, so many *changes of sign*, at least, must there be in the reciprocal equation, after applying to it the transformation $\left(\frac{1}{r}\right)$; since between this transformation and the final one by $\left(\frac{1}{0}\right)$, this number of changes, at least, must be lost, otherwise there could not be so many roots of the reciprocal equation, in the interval $\left[\frac{1}{r}, \frac{1}{0}\right]$, as in the proposed in the interval $[0, r]$.

Should it happen therefore that, after applying the transformation $\frac{1}{r}$ to the reciprocal equation, fewer changes appear in the result than have disappeared by the tranformation (r) applied to the direct equation, we may conclude that these losses cannot *all* indicate real roots: as many imaginary roots, at least, must be indicated in the interval $[0, r]$ as the changes *lost* in the direct transformation exceed in number the changes *left* in the reciprocal transformation. But so long as the changes left in the latter transformation are the same in number as the changes lost in the former, or in fact so long as any changes at all are left, more than one, the interval $[0, r]$ will remain doubtful.

We may seek to remove this doubt by narrowing the interval $[0, r]$ in the direct equation; but the better plan will be to diminish the unlimited interval $\left[\frac{1}{r}, \frac{1}{0}\right]$ in the reciprocal equation, transforming successively by $\frac{1}{r} + 1$, $\frac{1}{r} + 2$, &c., till all the changes are lost. The *single* roots passed over in the course of these transformations will be indicated by so many changes of sign in the final term, and that number of real roots will thus be detected in the original interval $[0, r]$. If there are not as many single roots passed over as there are changes lost, then there will be doubtful intervals. To each of these the same treatment is to be applied as was applied to the interval $[0, r]$; that is, each of the new doubtful intervals is to be considered as belonging to a new direct equation, and the interval, as before, exchanged for another indefinitely wide belonging to a new reciprocal equation, and so on. Whenever, in passing from a direct transformation to its collateral reciprocal transformation, more signs are *lost* by the former than are *left* by the latter, a number of *imaginary roots* equal to the excess will be accounted for; and when, in narrowing the indefinite interval in a reciprocal equation, single roots are passed over, so many real roots will be accounted for. The operations are to be continued till as many roots are thus detected as there are changes lost in the doubtful interval $[0, r]$, when the analysis of that interval will be completed. The course of proceeding may therefore be described by the following precepts:

1. Let $[a', b']$ be any doubtful interval occurring in the partial

analysis of an equation by the method of (108). The equation due to the transformation (a') is the first direct equation; the interval between this and the transformation (b')—which is the doubtful interval—is $b' - a' = r$; that is, for the first direct equation, the interval to be examined is $[0, r]$.

2. Take the reciprocal of the equation (a'); that is, simply reverse its coefficients, and transform by $\left(\frac{1}{r}\right)$. If no changes are left after this transformation, all the roots indicated in the doubtful interval are imaginary.

3. If as many changes are left as are lost in the direct transformation (b'), the original doubt remains undiminished.

4. If fewer changes are left than are lost in (b'), the difference will imply that number of imaginary roots, so that the doubt will be partially removed.

5. Whether as many changes are left or fewer, we must cause them all to disappear—or at least all but one—by continuing to transform onwards towards $\frac{1}{0}$: and every time a *single* sign disappears, or an odd number of signs, a single real root will be indicated, and thus the doubt will be still further reduced.

6. If the doubt be not entirely removed, on account of the signs not all vanishing singly in the preceding series of transformations, new intervals of doubt will occur in this series. Let $[a'', b'']$ represent any such interval; and consider the transformation (a'') as a new direct equation, and the passage from it to the transformation (b''), as the corresponding direct transformation, the doubtful interval for (a'') being $b'' - a'' = r'$; that is, $[0, r']$, and proceed as with the former direct equation.

And this process is to be continued till there are as many roots detected—real and imaginary—as there are changes of signs lost in the first direct transformation; that is, in the passage over the interval under examination.

The following examples will sufficiently illustrate these precepts.*

1. Let it be required to analyse the equation

$$x^3 - 7x + 7 = 0,$$

the ordinary superior limit to whose positive roots is 8.

(0) . . . . $1 + 0 - 7 + 7$
(1) . . . . $1 + 3 - 4 + 1$
(2) . . . . $1 + 6 + 5 + 1$ *two variations lost.*

Here two variations are lost in the interval [1, 2]; or the transformation (1) loses two variations in the interval [0, 1]. This therefore is a doubtful interval. Hence taking the reciprocal of (1), and observing that 5 is a superior limit to the positive roots, we proceed as follows:

(0) . . . . $1 - 4 + 3 + 1$
(1) . . . . $1 - 1 - 2 + 1$ *two variations left.*
(2) . . . . $1 + 2 - 1 - 1$
(3) . . . . $1 + 5 + 6 + 1$

The transformation (3) furnishes no variations, so that this step is terminated, and the doubtful roots are found to be real, both lying in the interval [1, 2]. The third root is found to lie in the interval [-3, -4].

2. Let the proposed equation be

$$x^6 - 12x^4 - 2x^3 + 37x^2 + 10x - 10 = 0.$$

Here 13 is the ordinary superior limit to the positive roots: but if we diminish the roots by 10, all the resulting coefficients are

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* These examples are from a paper by VINCENT in the *Journal de Mathématiques* for October, 1836. But the transformations have been differently, and somewhat more compactly arranged, and some errors corrected. The work of the fourth example, in the paper referred to, is altogether erroneous.

found to be positive: hence 10 is a superior limit. Therefore subdividing the interval [0, 10] we have the following results:

(0) . . . . 1 + 0 − 12 − 2 + 37 + 10 − 10
(1) . . . . 1 + 6 + 3 − 30 − 26 + 36 + 24 *one var. lost.*
(2) . . . . 1 + 12 + 48 + 62 − 23 − 58 + 14
(3) . . . . 1 + + + + + + *two var. lost.*

The interval [2, 3] is doubtful: hence taking the reciprocal of (2), and observing that 6 is a superior limit to the roots of the reciprocal equation, by (86), we proceed as follows:

(0) . . . 14 − 58 − 23 + 62 + 48 + 12 + 1
(1) . . . 14 + 26 − 103 − 330 − 274 − 4 + 56 *two var. left.*
(2) . . . 14 + 110 + 237 − 202 − 1412 − 1740 − 615 *one var. left.*

This result is sufficient to show that the two roots in the proposed interval are both real; since the variation still left must disappear between (2) and (∞).

To determine the places of the negative roots, let the alternate signs of the proposed equation be changed: then we have

(0) . . . . 1 + 0 − 12 + 2 + 37 − 10 − 10
(1) . . . . 1 + 6 + 3 − 26 − 14 + 28 + 8 *one var. lost.*
(2) . . . . 1 + 12 + 48 + 66 + 1 − 30 + 6
(3) . . . . 1 + + + + + + *two var. lost.*

We have, therefore, to discover the character of the interval [2, 3]. Taking the reciprocal of (2), and proceeding as before, we have

(0) . . . . 6 − 30 + 1 + 66 + 48 + 12 + 1
(1) . . . . 6 + 6 − 59 − 110 + 42 + 196 + 104 *two var. left.*
(2) . . . . 6 + 42 + 61 − 166 − 492 − 220 + 185
(3) . . . . 6 + 78 + 361 + 618 − 114 − 1212 − 584 *one var. left.*

Hence, as there is only one more variation to lose, the two roots in the interval [2, 3] are real; so that the situations of the real roots are as follows: one root in each of the intervals [0, 1], [0, — 1], and two roots in each of the intervals [2, 3], [— 2, — 3].

3. Let the proposed equation be

$$x^6 - 6x^5 + 40x^3 + 60x^2 - x - 1 = 0$$

of which all the positive roots are included between 0 and 7.

(0) . . . 1 — 6 + 0 + 40 + 60 — 1 — 1

(1) . . . 1 + 0 — 15 + 0 + 135 + 215 + 93 *one var. lost*

(2) . . . 1 + 6 + 0 — 40 + 60 + 431 + 429

(3) . . . 1 + + + + 15 + 467 + 887 *two var. lost*

Recip. of

(2) . . . 429 + 431 + 60 — 40 + 0 + 6 + 1

(1) . . . 429 + + + + + + *no var. left.*

Hence the two roots indicated in the interval [2, 3] are imaginary.

Changing now the alternate signs of the equation and proceeding as before, we have

(0) . . . . 1 + 6 + 0 — 40 + 60 + 1 — 1

(1) . . . . 1 + 12 + 45 + 40 + + + *three var. lost*

Recip. of

(0) . . . . 1 — 1 — 60 + 40 — 0 — 6 — 1

(1) . . . . 1 + 5 — 50 — 90 — — — *one var. left.*

Hence, of the three roots indicated in the interval [0, — 1], two are imaginary, and one real.

4. Let the equation proposed for analysis be

$$4x^7 - 6x^6 - 7x^5 + 8x^4 + 7x^3 - 23x^2 - 22x - 5 = 0$$

Here 3 is a superior limit to the positive roots (89); so that the highest transforming factor will, at most, be 3.

(0) . . . 4 − 6 − 7 + 8 + 7 − 23 − 22 − 5

(1) . . . 4 + 22 + 41 + 23 − 11 − 30 − 58 − 44 *two var. lost*

(2) . . . 4 + + + + + + −

(3)* . . + *one var. lost*

Recip. of

(0) . . . 5 + 22 + 23 − 7 − 8 + 7 + 6 − 4

(1) . . . 5 + + + + + + + *no var. left*

Hence, in the positive region there is but one real root: it lies between 2 and 3.

Changing the alternate signs of the proposed, we have

(0) . . 4 + 6 − 7 − 8 + 7 + 23 − 22 + 5

(1) . . 4 + + + + + + + *four lost* (A)

Recip. of

(0) . . 5 − 22 + 23 + 7 − 8 − 7 + 6 + 4

(1) . . 5 + 13 − 4 − 33 − 15 + 16 + 14 + 8 *two left* (B)

(2) . . 5 + 48 + 179 + 317 + 248 + 33 − 38 + 4

(3) . . 5 + + + + + + + *two lost* (A′)

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* This transformation need not be computed, since we already know 3 to be the superior limit.

Recip. of

(2) .. 4− 38+ 33+ 248+ 317+ 179+ 48+5

(1)′.. 4− 10− 111− 17+ 1019+ 2462+ 2314+796

two left (B′)

(2)′.. 4+ 18− 87− 582− 219+ 4241+ 9640+6457

(3)′.. 4+ 46+ 105− 607− 2917− 424+14838+19472

(4)′.. 4+ 74+ 465+ 748− 3235−10993+ 3640+30517

(5)′.. 4+ 102+ 993+ 4323+ 6027−10366−22262+21220

(6)′.. 4+ 130+ 1689+10958+35429+45197− 2016+41

(6·05)′ 4+ + + + + + +

two lost (A″)

Taking now the reciprocal of (6)′, and transforming by $\frac{1}{\cdot 05} = 20$

(0) .. 41−2016+45197+35429+ 10958+1689+ 130+4

(20)..41+ + + + + + +

none left (B″)

The results (A), (B) show that two of the four roots indicated by the interval [0, 1], are necessarily imaginary. The results (A′), (B′) leave the remaining two roots still doubtful. The results (A″), (B″) show that these are also imaginary. Consequently the equation has no negative roots.

(145.) These examples are amply sufficient to illustrate the method of **Budan** when exhibited in its most improved and convenient form. The principle upon which it depends is very different from that upon which the operation of **Fourier** is founded; its characteristic peculiarity being that, instead of pursuing the doubtful interval through its successive contractions, till it can no longer conceal the character of the roots within its narrowed limits, the process of narrowing the interval is stopped as soon as fractional numbers would become necessary for a further subdivision of it. The small interval is then exchanged for another indefinitely wide, but embracing indications, of precisely the same import in reference to the doubtful roots, as those in the interval which it has replaced. This wide interval is in like manner gradually contracted, till the character of these indications discovers itself; or—if the information sought be delayed—till a further contraction would again introduce fractions, when the interval is as before replaced by another indefinitely wide; and so on, till the indications, thus transmitted through these successive intervals, unfold their character, and the operation terminates.

The proposition which justifies this transference of our operations, from an interval inconveniently narrow, to another indefinitely wide, is simply this: viz., that as many real roots of any equation as lie in the interval $[0, r]$, so many, and no more, belonging to the reciprocal of that equation, must lie in the interval $[\frac{1}{r}, \infty]$; that is, if r be taken equal to unity, in the interval $[1, \infty]$.

By substituting, therefore, this latter interval for the former, fractional transformations are evaded; and the proposed interval virtually narrowed without their aid. As in FOURIER's method so here, the contraction of the original interval till the roots, if real, actually separate, must be accomplished before the character of those roots can be decided upon. FOURIER effects this contraction by the direct method of minutely subdividing the interval itself. BUDAN accomplishes the same thing by the indirect method above described, thus avoiding FOURIER's minute subdivisions: but the two methods proceed to their ultimate object *pari passu*; and the same imperfection is common to both, each involving a vast amount of useless trial-operations in all cases where roots occur which a slight change in the coefficients of the equation would render equal.

With the exception of such cases as these the method, like that of FOURIER under the same restrictions, and independently of other aids, is readily practicable, as the foregoing examples sufficiently show; since the superfluous trial transformations—unavoidable where in our search after close limits so much is left to conjecture—are comparatively few; because the roots entering those examples, having no remarkable proximity to one another, disclose their character after a few steps of the work. But if the method of **Budan** be applied to the example at page 187, we shall find, like as in the unmodified process of **Fourier**, that the number and extent of the operations will exceed all reasonable bounds.

Another difficulty attendant upon the method, and the one which those who have cultivated it most have regarded as the most formidable, is that the case of equal roots is left unprovided for: and that, therefore, in order to ensure the success of the method it should previously be ascertained whether such roots enter the equation or not. This has been hitherto thought to involve the necessity of executing the operation for the common measure; by which the numerical labour is often very seriously increased: as, for instance, in such an example as that just referred to; and certain tests, all of which involve difficult and tedious operations, have been proposed, to supersede the necessity for the common measure.* But we have shown in the preceding

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* One of the most recent of these tests is given by **Cauchy**. It furnishes a limit to the number of steps towards the separation of the roots—whether those steps consist of successive transformations as above or not—within which, if the separation be not effected, we may infer the equality of the roots under trial. But this limit is in general so remote as to be practically useless. Another limit for the same purpose has been since proposed by **Vincent**: but this is still more unsuited to any practical purpose. Both these limits may be seen in the *Journal de Mathématiques*, (tom. ii. p. 235). Had the simple properties of equal roots, which we have established at (101), been known to these writers, the laborious investigations which conducted them to the useless forms adverted to would undoubtedly have been spared; and a very formidable obstacle to the success of the methods of **Budan** and **Fourier** would have been seen to yield to far more simple and practicable means. In reference to these means we may here observe that there is a peculiarity in the tests (A), (B) at page 185, which ought not to be left unnoticed, as it distinguishes them in a remarkable manner from all other tests for discriminating between equal and nearly equal roots. In these others the difficulty of applying the test, and the ambiguity attending it, if not pushed to its extreme limit, always increase with the proximity of the roots: thus the more nearly two roots approach to equality the more nearly will the final remainder, in the process for the common measure, tend to zero; and the greater will be the risk of confounding it with zero—and thus inferring equality of roots—if any abbreviations have been introduced into the work. And in like manner when other tests are applied any minute departure from strict accuracy in the numerical results would expose them to the same risk. But in applying the tests here

chapter that these laborious contrivances may be dispensed with; and that a few simple considerations, suggested by the general theorem which we have given at (76), and the inferences at (101) will either at once remove the doubt—which is their usual effect—or will so reduce the work for the common measure as to render it comparatively trifling. An important objection to BUDAN's method is thus provided against.

(146.) Considering each of the methods now discussed independently, and apart from the modifications which we have proposed to introduce into that of FOURIER, we should, upon the whole, be inclined to give the preference to BUDAN's mode of examining intervals of doubt, on the score of superior convenience to that of FOURIER; as it is in general easier to employ integral than fractional limits. But advantage will often accrue from combining both methods together: or rather the process of BUDAN will often admit of simplification from the introduction of FOURIER's criterion immediately before passing from the direct to its corresponding reciprocal equation. The necessity for this passage will thus sometimes be spared, from a very slight inspection of the coefficients in the last pair of transformations. Viewing, however, the actual determination of the real roots of an equation as the ultimate object which the previous analysis of the equation is to subserve, we shall find the method of FOURIER to unite, so much more readily than that of BUDAN, with HORNER's process for developing the roots, as to claim, in general, a decided preference over the latter method, in connexion with the rapid mode of solution just mentioned. If LAGRANGE's method of approximating to the roots of equations, by means of continued

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referred to we are free from apprehension of this kind. However closely the roots may lie together, if they are not accurately equal, the tests of equality will in general be just as wide of fulfilment as if no such proximity existed. This is more especially seen to be the case when the leading coefficient of the equation is unity; for then the equal pair of roots, if such exist, must be integral; and it is plain that, however many decimals in a pair of nearly equal roots may coincide, each of these roots may comparatively be widely different from any whole number. The example at page 187, is an instance of this, the nearly equal roots being ·316664 . . . . and ·316665 . . . .

fractions, and which Horner's method has superseded, were that which it were still proposed to employ in actual solution, then the analysis of Budan, combining more readily with that method, would have superior claims to adoption to that of Fourier, though, as remarked above, Fourier's test should be kept in view during the operation.* But the student will more fully understand the bearing of these remarks, the propriety of this distinction, and the grounds of our preference, when we come to examine the methods of solution as proposed by Lagrange and Horner.

We shall only observe finally that when, as at page 202, it is readily seen that the last of a series of direct transformations will equally lose all its variations whether the transforming factor be unity, as usual, or some convenient decimal, as ·5, ·05, &c., it will often, as in the example adverted to, save much subsequent calculation to employ the decimal instead of the unit. It is plain, without actually effecting the transformation of $(6)'$ by $(\cdot05)'$, that the results must be all *plus*: a glance at the last three terms of $(6)'$ is sufficient to convince us of this; so that all actual operation by decimals is still avoided.

For the analysis of the preceding equation by the method of Sturm reference may be made to the author's *Mathematical Dissertations*, page 195.

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* The remark, made at the close of last chapter, equally applies here, in reference to Budan's method of analysis: viz., that when the roots are to be computed, and a partial analysis of the equation has left but two roots in doubt —the others being real, the analysis of the doubtful interval need not be pursued: the values of the two roots indicated, whether they be real or imaginary, may be determined from the values of the other roots previously found. The formulas for two roots of an equation, in terms of the remaining roots, will be given in a future chapter.

CHAPTER XI.

ON THE METHOD OF STURM: WITH A COMPARISON OF IT WITH THE METHODS OF BUDAN AND FOURIER.

(147.) We now come to the theorem which Sturm has proposed for the analysis of an equation. It is distinguished from all other methods in three important particulars:—the simple character of its processes—the only operation involved in it being that for the common measure: the unfailing certainty of its results — it being entirely free from all tentative steps: and the basis furnished by it for the subsequent development of the roots—it being altogether independent of every method for their previous separation. This last is a striking peculiarity in the method of Sturm; and gives to it a character to which no other method can lay the remotest claim, since whatever other mode of proceeding we adopt we shall always be kept in suspense, as to the character of the roots indicated in any proposed interval, till those roots are actually separated, or till their separation is shown to be impossible.

In the introductory volume on the *Analysis of Cubic and Biquadratic Equations* we have given a very simple and elementary demonstration of this interesting theorem: the investigation which follows is more closely allied to that furnished by Sturm himself in his original memoir.*

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* Memoires présentés par des Savans Etrangers, &c. 1835. This *mémoire* was rewarded with the prize of the Academie Royale des Sciences, in 1834; and with a gold medal from the Royal Society of London, in 1840.

Let $X = 0$ be any equation whose coefficients are real, and whose roots are *unequal*;* and let X_1 be the polynomial, derived from X, agreeably to the process in (15.) Let the operation of finding the greatest common measure of X and X_1 be performed; and, in the several remainders which successively arise in the course of the process, change all the signs from + to −, and from − to +, and call the remainders thus modified, X_2, X_3, X_4 . . . . Put also the several quotients equal to Q_1, Q_2, Q_3 . . . ; then we shall obviously have these equations, viz.

$$\begin{aligned} X &= X_1\ Q_1 - X_2 \\ X_1 &= X_2\ Q_2 - X_3 \\ X_2 &= X_3\ Q_3 - X_4 \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ X_{m-2} &= X_{m-1}\ Q_{m-1} - X_m. \end{aligned}$$

The final remainder, X_m, is necessarily independent of x, and different from zero, since, by hypothesis, the equation has no equal roots (98). Suppose now, that in the several functions,

$$X,\ X_1,\ X_2,\ X_3,\ .\ .\ .\ .\ X_m,$$

two numbers, p, q, such that $p < q$ be successively substituted for x; these substitutions will furnish two series of signs; and it is the object of STURM's theorem to prove that

The difference between the number of variations of the first series, and that of the second, expresses exactly the number of real roots of the proposed equation, which are comprised between p *and* q.

Whence results the following rules for determining the entire number of real roots of an equation.

1. Apply to the two polynomials, X, X_1, the process for finding the greatest common measure, modifying every remainder, or new divisor, by changing the signs; we shall thus have the series of functions,

$$X,\ X_1,\ X_2,\ X_3,\ .\ .\ .\ .\ X_m;$$

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* The application of the theorem to the case of equal roots will be considered hereafter.

which are of continually decreasing dimensions in x, X_m being independent of x.

2. Substitute in this series, $-\infty$, and $+\infty$, successively, for x, noting the signs of the results.

3. Count the number of variations in each row of signs; the difference of these numbers expresses the total number of real roots in the equation.

Having now stated the nature and object of STURM's theorem, we shall proceed to establish the principles from which it is deduced. These are three in number, and are as follow:

1. The first principle is that, if r be a root of the equation

$$X = f(x) = 0$$

and $r + \delta$ be substituted for x both in $f(x)$, and in the first derived function $f_1(x)$, a value so small may be given to δ that $f(r + \delta)$ and $f_1(r)\,\delta$ shall have like signs.

For making the proposed substitution we have

$$f(r + \delta) = f(r) + f_1(r)\delta + \frac{f_2(r)}{2}\delta^2 + \frac{f_3(r)}{2\,.\,3}\delta^3 + \;.\;.\;.\;.\;\delta^n$$

in which $f_1(r)$, $f_2(r)$, &c. are derived from $f(r)$ as already explained.

Now since, by hypothesis, r is a root of $f(x) = 0$, we must have $f(r) = 0$; so that the foregoing development is

$$f(r + \delta) = f_1(r)\,\delta + \frac{f_2(r)}{2}\delta^2 + \frac{f_3(r)}{2\,.\,3}\delta^3 + \;.\;.\;.\;.\;\delta^n,$$

and it has been shown (25) that a value so small may be given to δ such that for it, and for all smaller values, the sign of the aggregate of the entire series on the right of the sign of equality will be the same as the sign of the leading term $f_1(r)\,\delta$; which term is always necessarily different from zero, because, by hypothesis, there are no equal roots in the proposed equation (98.) Hence a value for δ exists such that for it, and for all values still smaller, $f(r + \delta)$ and $f_1(r)\,\delta$ have like signs.

2. The next principle to be proved is, that if in the functions

$$X, \ X_1, \ X_2 \ . \ . \ . \ .$$

we put any number a for x, it can never happen that two consecutive functions vanish at once.

Let

$$X_{p-1}, \ X_p, \ X_{p+1},$$

be any three consecutive functions; then (p. 208,)

$$X_{p-1} = X_p \ Q_p - X_{p+1},$$

and if it were possible that there could exist together the conditions

$$X_{p-1} = 0, \ X_p = 0,$$

it would necessarily follow that

$$X_{p+1} = 0;$$

and, as moreover

$$X_p = X_{p+1} \ Q_{p+1} - X_{p+2},$$

it would further follow that

$$X_{p+2} = 0,$$

and so on. We should thus have finally the condition

$$X_m = 0,$$

that is to say, the last remainder would be zero; which is impossible, because, as there are no equal roots, X and X_1 cannot have a common measure.

This immediately leads to the third principles, viz.

3. If one of the functions, as X_p, become zero for any particular value of x, the two functions X_{p-1}, X_{p+1}, between which it is placed, have necessarily contrary signs for the same value of x. This is evident from the relation

$$X_{p-1} = X_p \ Q_p - X_{p+1}.$$

(148.) These principles being admitted, let us now represent by k any quantity, positive or negative, which may be nearer to $-\frac{1}{0}$ than any of the real roots of the equations

$$X = 0, \ X_1 = 0, \ X_2 = 0, \ . \ . \ . \ . \ X_{m-1} = 0;$$

and let k be conceived to increase continuously towards $+ \frac{1}{0}$, and that all the successive values are substituted for x in the functions

$$X, \ X_1, \ X_2, \ . \ . \ . \ . \ X_m,$$

the last of which, X_m, being independent of x, will of course remain unaffected by the substitutions; and, with respect to the others, we know that the *signs* of the results which they give will be continually reproduced in the same order, so long as k does not reach a value sufficiently great to render one of the functions zero.

Suppose, however, that such a value is attained, and let it be a: then the substitution of this value for x will either cause one or more of the functions

$$X_1, \ X_2, \ X_3, \ . \ . \ . \ . \ X_{m-1},$$

to become zero without rendering X zero, or else the substitution will render X zero, and may besides cause one or more of the other functions to vanish. Here are then two cases: and we shall now prove that in *the first case* no variation can be lost in the passage of x through the three consecutive states $a-\delta$, a, $a+\delta$; and that in *the second case* one variation will disappear, and only one, in passing from the state $x = a - \delta$ through $x = a$ to the immediately succeeding state $x = a + \delta$.

Let us examine the first case, viz. that in which one of the intermediate functions, as X_p, becomes zero for $x = a$, for which value x does not vanish.

As for the same value $x = a$, X_{p-1} and X_{p+1} give results with contrary signs (p. 210), it follows that the consecutive functions

$$X_{p-1}, \ X_p, \ X_{p+1},$$

must furnish one or other of these combinations of signs, viz.

$$\begin{array}{ccc} + & 0 & - \\ - & 0 & + \end{array}$$

so that, whether 0 be regarded as + or —, there is always one variation and one permanence; but whatever be the signs given by X_{p-1} and X_{p+1}, they have been preserved unaltered through all the passages of x, from $x = k$ up to $x = a$, as by hypothesis, no root of $X_{p-1} = 0$, or of $X_{p+1} = 0$, has been passed over in this interval; nor will these signs change in passing to the immediately succeeding state $x = a + \delta$, because, however near a may be to a root of one of these equations, yet δ may be made so small as to render it impossible that a root can be comprised between a and $a + \delta$.

We may, therefore, conclude that the three functions above, which for $x = a$ furnish one variation and one permanence, give equally a variation and a permanence for all values of x comprised between $x = k$ and $x = a + \delta$. No variation, therefore, is either lost or gained in the series X, X_1, X_2, ... in passing through the state $x = a$, however many of these functions may vanish in the passage.

Let us now consider the second case, or that in which X or $f(x)$ becomes zero for $x = a$.

Substitute in X and X_1, that is, in $f(x)$ and $f_1(x)$, the value $a + \delta$ for x, and we shall have (27)

$$f(a + \delta) = f(a) + f_1(a)\,\delta + \frac{f_2(a)}{2}\,\delta^2 + \frac{f_3(a)}{2\,.\,3}\,\delta^3 + \&c.$$

$$f_1(a + \delta) = f_1(a) + f_2(a)\,\delta + \frac{f_3(a)}{2}\,\delta^2 + \frac{f_4(a)}{2\,.\,3}\,\delta^3 + \&c.$$

But, by hypothesis,

$$f(a) = 0,$$

$$\therefore f(a + \delta) = f_1(a)\,\delta + \frac{f_2(a)}{2}\,\delta^2 + \frac{f_3(a)}{2\,.\,3}\,\delta^3 + \&c.$$

Hence, taking δ sufficiently small, $f(a + \delta)$ and $f_1\,(a + \delta)$ have respectively the same signs as $f_1(a)\,\delta$ and $f_1(a)$; and these have *like* signs when δ is positive, and *unlike* signs when δ is negative. Consequently, when δ is negative, $f(a + \delta)$ and $f_1(a + \delta)$ have contrary signs, and when δ is positive they have the same signs; so that in the passage from $x = a - \delta$ to $x = a + \delta$, a variation is changed into a permanence. No other loss of variation is due to this passage, because although other functions should vanish in the transition, yet, as we have seen above, their vanishing does not affect the number of variations.

It hence appears, that whatever be the previous state of the series

$$X, \quad X_1, \quad X_2, \ldots$$

with respect to signs, immediately before the passage of a root, one variation, and only one, will be lost in consequence of that passage; and that the variation thus lost, is that which always exists between X, and X_1, immediately before the passage of the root.

Now it is plain that this loss cannot be recovered in the interval between the passage of one root and of that next following; because, as in this interval X does not vanish, the variations throughout remain in number the same, as we have already proved. Yet, from the foregoing deductions, it clearly follows that immediately before the passage of the second root there must be a variation between the signs of the first two functions; we must conclude, therefore, that this change of a permanency into a variation cannot add to the total number of changes; hence the variations immediately before the passage of the second root, are precisely the same in number as immediately after the passage of the first. When the second root passes, a variation is necessarily lost, but only one; so that, immediately after the passage, the variations are in number fewer by two than at first; and thus the passage of every successive root is attended with the loss of one additional variation, and one only.

We may, therefore, now conclude, that the number of variations lost during the increase of x from $x = p$, to $x = q$, is exactly equal to the number of real roots which are comprised between p and q; and thus the theorem at (147) is fully established.

From the foregoing investigation we gather the following useful particulars, viz.

(149.) 1. In order to ascertain the total number of real roots in any equation, we shall not be required by this theorem first to determine close limits, $- L$ and $+ L'$: it will obviously be sufficient to substitute in the series of functions X, X_1, X_2, &c., the extreme values $- \infty$ and $+ \infty$, between which all the real roots are necessarily comprehended; and the difference between the variations furnished by these substitutions, will be equal in number to the number of real roots in the equation. Having thus ascertained how many real roots there are in the equation, we may determine their nearest extreme limits by substituting the successive numbers of the series

$$0, \quad -1, \quad -2, \quad -3, \text{ \&c.} \;.\;.\;.\;.\; (1),$$

till we have as many variations as were given by the substitution of $-\infty$; after which we may substitute, in like manner, the numbers of the series

$$0, \quad 1, \quad 2, \quad 3, \text{ \&c.} \;.\;.\;.\;.\; (2),$$

till we arrive at as many variations as were before given by $+\infty$: the numbers at which we stop will be the extreme limits, and, moreover, the intermediate numbers will mark out the situations of the roots themselves; as the difference between the variations given by one number, and those given by any other, will always express the number of real roots which lie between the numbers substituted. The extreme limits thus obtained will obviously be the nearest integral limits possible.

2. It must here be observed that $-\infty$ and $+\infty$ need be substituted only in the terms containing the highest power of x in each function; because this term must, for $x = \pm \infty$, be numerically greater than all the other terms in the function together, so that the sign of this first term will determine the sign of the whole.

It is, moreover, obvious that when all the roots are real, the functions must be $n + 1$ in number; more numerous than this they cannot be, because they are of continually descending dimensions, and, from x^n to x^0 inclusively, comprehends but $n + 1$ grades at most; nor can the number of functions be fewer than $n + 1$, in the case supposed, for else there would not be n variations to lose, and, therefore, not n real roots. These same functions, too, must have the leading terms all of one sign, in order that the substitutions in them, of $-\infty$ and $+\infty$ for x, may in the one case give all variations, and, in the other, all permanencies. When, therefore, the functions X, X_1, X_2, &c. are $n+1$ in number, and have the first term in each, uniformly +, or uniformly —, we may conclude that the roots are all real; when, however, such conditions have not place, then imaginary roots exist; of which, the exact number may be determined, as above directed.

3. But in all cases where there are so many as $n+1$ functions, however their leading signs may vary, the determination of the number of real and of imaginary roots, may still be effected by a rule easily deducible from, but more simple than, the general one just established; and it is of consequence to notice this simplification of the general theorem, because the functions of which we speak usually amount in number to $n+1$, inasmuch as in seeking the greatest common measure of X and X_1, each divisor is usually of a degree immediately below that of the preceding divisor. Now in every such case, the number of imaginary roots in the equation $X = 0$ may be readily discovered, by the simple inspection of the signs of the leading terms of the $n+1$ functions: in fact

The equation $X = 0$ *has as many pairs of imaginary roots as there are variations in the series of signs of the leading terms of the functions*

$$X_1, \quad X_2, \quad X_3, \quad . \quad . \quad . \quad . \quad X_n,$$

these being supposed to diminish in degree regularly by unity.

This is proved by Sturm thus:

It follows from the hypothesis which has just been admitted, that every two consecutive functions X_{p-1}, X_p, are the one of an even degree, and the other of an odd degree. Hence, if these two functions have the same sign for $x = +\infty$, they must have contrary signs for $x = -\infty$; and *vice versâ*, if they have contrary signs for $x = +\infty$, they must have the same sign for $x = -\infty$; so that if we write one below the other, the two series of signs of the functions

$$X, \quad X_1, \quad X_2, \quad . \quad . \quad . \quad . \quad X_n,$$

for $x = -\infty$, and for $x = +\infty$, each variation in either of these two series will correspond to a permanence in the other series;

therefore the number of permanencies for $x = -\infty$ is equal to the number of variations for $x = +\infty$.

Let i be the number of variations for $x = +\infty$, and which may be zero. These variations are entirely due to the signs of the leading terms in the n functions

$$X_1, \quad X_2, \quad X_3, \; . \; . \; . \; . \; X_n,$$

because the leading term of X and the leading term of X_1 are necessarily positive.

Now we have just seen that the series of signs for $x = -\infty$ must furnish i permanencies; it must contain then $n - i$ variations, since the functions X, X_1, ... X_n are $n + 1$ in number; and that in a series of $n + 1$ signs, the number of variations and permanencies combined amount to the sum n.

But, by the general theorem, the number of real roots of the equation $X = 0$, all comprised between $-\infty$ and $+\infty$, must equal the excess of the number, $n - i$, of variations due to $x = -\infty$, above the number, i, of variations due to $x = +\infty$. The equation $X = 0$ has, therefore, $n - 2i$ real roots, and consequently $2i$ imaginary roots: these we know enter in pairs of the form $a \pm b\sqrt{-1}$; hence the number of these pairs is i.

4. To this we may add that, whatever be the number of the functions, the substitution of 0 for x will furnish the same series of signs as the final terms of those functions; and the substitution of ∞ for x will furnish the same series as the row of leading terms: consequently, since all the positive roots are comprised within the limits 0, ∞, it follows that the number of positive roots will always be expressed by the excess of variations furnished by the final signs above those furnished by the leading signs. And further, since the negative roots all lie within the interval $[0, -\infty]$, the number of negative roots will always be expressed by the excess of variations furnished by the leading signs, after those attached to negative powers are changed, above those furnished by the final signs: and thus the number of each kind of roots, positive, negative, and imaginary, may always be ascertained by simply inspecting the leading and final rows of signs presented by the series of functions, without any actual substitution whatever.

5. If, in substituting two numbers, p and q, in the functions, in order to ascertain how many roots lie between them, we find that any intermediate function vanishes, we may pass over the zero in estimating the number of variations ; for, as it was shown that in such a case the contiguous functions are always of contrary signs, the intervening one, whether taken + or —, will cause the three to furnish but one variation, so that the number of variations will not be affected by its omission.

When the first function, X, vanishes, we may also omit the zero in estimating the variations : for the vanishing of X shows that the number substituted is a root, and that a variation has just been lost by the change of sign of X ; the remaining variations, therefore, are all that are concerned with the other roots.

6. If, after having obtained the series of functions, we find that one of them, as X_r, is of such a nature as always to preserve the same sign, whatever number between p and q be substituted for x in it, then, in order to ascertain the number of roots between p and q, we may reject all the functions beyond X_r, and confine our substitutions to the series

$$X, \quad X_1, \quad X_2, \; . \; . \; . \; . \; X_r;$$

for, so long as X_r preserves the same sign, and, consequently, does not pass through zero, no alteration can take place in the number of variations furnished by it, and the following functions ; which is proved precisely as for X in (148). Hence, whatever changes take place in the interval $[p, q]$, occur in the functions, as far as X_r only. From this the following consequences result, viz.

7. If in the course of the operations, by which X_1, X_2, X_3, &c. are determined, we ascertain that a certain function, X_r, can have only imaginary roots, then, as the result of every substitution in it must be positive, we need not extend the process to the other functions, X_{r+1}, X_{r+2}, &c.

8. As, therefore, in the case just supposed, the number of real

roots in the equation is determinable from an examination of the $r+1$, first functions only, viz. the functions

$$X, \quad X_1, \quad X_2, \ldots . X_r;$$

we may, obviously, apply to these all the remarks which have hitherto been made in reference to the entire series: we may affirm, for instance, that when these functions regularly diminish in degree, by unity, and have all the same leading sign, that the equation has r real roots, and no more; and further, that when the leading signs are not all the same, but present i variations, the number of real roots will be only $r - 2i$. Hence we may extend STURM's second rule, as follows:

When the series of functions

$$X, \quad X_1, \quad X_2, \quad X_3, \ldots . X_r,$$

in which X_r *is either the final quotient, or else such that the roots of* $X_r = 0$ *are imaginary, regularly descend in degree, by unity, and present* i *variations in their leading signs, there are exactly* $r - 2i$ *real roots in the equation* $X = 0$. *If* $i = 0$, *that is, if there are no variations, the equation has* r *real roots, but no more.*

9. Another useful deduction from the inference 6 above is, that let us stop at whatever function we may, we can always ascertain what roots of the original equation lie *without* the limits which inclose the roots of our final function as soon as these limits are determined. Thus, if by any of the rules in Chap. VI. we find p, q for the inferior and superior limits of $X_r = 0$, then the substitutions of $-\infty$ and p for x, in the incomplete series of functions above, will make known the number of roots of $X = 0$ lying in the interval $[-\infty, p]$. In like manner the substitutions of q and ∞, in the same series, will make known the roots in the interval $[q, \infty]$. Still further information respecting the roots of $X = 0$ may be obtained from the series terminating in X_r, provided we determine not merely the extreme limits of the real roots of $X_r = 0$, but the intervals in which they severally lie; for we shall then become acquainted with those partial intervals *within* the preceding limits p, q, from which roots of $X_r = 0$ are as much excluded as from the regions *without* those limits; so

that whatever roots of $X = 0$ may lie in these partial intervals, their existence will be detected, as in the former case, without any aid from the series of functions beyond X_r.

10. Lastly, from what has now been established, it follows that in order that all the roots of an equation of the nth degree may be real, it is necessary and sufficient that the series of functions be $n + 1$ in number, and that their leading coefficients present no variation of sign. If either of these conditions fail, we may conclude with certainty that imaginary roots enter the equation. Of these $n + 1$ coefficients the first two, viz., those which enter the leading terms of X and X_1, spontaneously fulfil the requisite condition. The leading coefficients of the remaining $n - 1$ functions are determined one after another by the successive steps of the process for the common measure. Hence, in order that the roots may all be real, certain determinate functions of the coefficients, $n - 1$ in number, must fulfil $n - 1$ conditions; the conditions being, that they all have the same sign as the leading coefficient in the proposed equation. Fewer conditions than these would be insufficient: a greater number would be superfluous. And thus does the theorem of STURM furnish us at the outset with a satisfactory solution of a problem to which all other modes of investigation have been applied in vain: the problem, namely, to determine the exact number of conditions which certain determinate functions of the coefficients of an equation must fulfil, in order that all the roots of that equation may be real.

We have incidentally noticed, at page 68, the researches of DE GUA and LAGRANGE in reference to this subject: both were led, though by very different routes, to the same expression for the number of conditions, viz., to the expression $\frac{n(n-1)}{2}$, which exceeds $n-1$ by $\frac{(n-2)(n-1)}{2}$; so that these determinations involve this latter number of superfluous conditions.*

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* It must be observed that in particular examples the necessary conditions may be fewer in number than the above formula indicates, as some may happen to be implied in others. The number can never exceed that expressed by the formula.

(150). It now merely remains for us to show the application of the theorem at (147) to those cases in which equal roots enter the equation proposed for analysis.

Let the equation $X = 0$ have equal roots: then the function X, and all the functions that follow, will have the last of them, as X_p for a common measure; and the equation $X_p = 0$ will contain the repeating roots once less often than the equation $X = 0$. Hence the equation $\frac{X}{X_p} = 0$ will involve all the *different* roots of $X = 0$ without any repetitions, and therefore all the preceding deductions will apply to it; and thus the number and situations of all the *different* roots of the proposed equation may be determined as before from the function $\frac{X}{X_p}$, and the subordinate functions deduced from it, as already explained. But the series of functions deduced in like manner from X differ from the former series only by the common factor X_p entering all of them; and whatever sign this factor may take, for any particular value of x in it, the entire number of variations, furnished for that value by the complete series, will be the very same as would be furnished if the common factor were suppressed: for when the sign of this factor is *plus*, it can have no effect on the signs of the quantities multiplied by it; and when it is minus, the effect is to change *all* the signs of the quantities multiplied. In both cases the number of variations remains undisturbed. Hence, as it is merely the number about which we are concerned in STURM's theorem, it follows that, in applying that theorem to the discovery of the character of the roots of $X = 0$, in any proposed interval, we need not first eliminate the equal roots: the theorem equally applies whether the roots are all different or not: the only thing to be observed, when equal roots enter, that is, when one of the subordinate functions X_{p+1} vanishes, is, that the series of functions, terminating in X_p, will make known the number and situations of all the real roots taken *singly*. The equation $X_p = 0$ will contain those remaining roots which, combined with the former, cause the repetitions, and it may be analysed by applying to it the same process. Or if, after the places of the roots of $X = 0$ taken singly are discovered, as above, we actually determine the values of those of them which are multiple, by aid of the principles at (101), or the precepts at page 185, and then diminish the roots of the equation by each in succession we shall ascertain their degree of multiplicity; for as many times as a root enters, so many of the latter terms of the transformed equation arising from diminishing by that root, must vanish. This method of finding how often the same root enters an equation is obviously the same, virtually, as the more cumbersome mode of proceeding adverted to at (99). If after the equal commensurable roots are eliminated in this way, equal incommensurable roots should still remain, the polynomial factors, involving the several sets of these, may then be determined by the process for the common measure, as already described at page 186.

(151.) Thus the complete analysis of a numerical equation according to the method of STURM is wholly comprised in the ordinary process for seeking a common measure between a given polynomial and its first derived function. We cannot dispense with this process as in the search after equal roots, because it is not merely the final conclusion of the operation that we wish to consult: the result of each intermediate step is equally in request; since these several results supply the series of functions necessary to the determination of the character of the roots.

It is easy to see, therefore, to what end our efforts should be principally directed in attempting to facilitate the application of STURM's rule: our aim should evidently be to reduce as much as possible the mere numerical work by which the successive steps for the common measure are wrought, and the sought functions deduced, as far at least as these are absolutely necessary to make known the situations of the roots. But it is not always that the entire series *is* necessary; whenever, for instance, upon arriving at the *quadratic* function, we find that four times the product of the extreme coefficients exceed the square of the middle one, we may discontinue the work; since the quadratic function, under these circumstances, will never undergo any change of sign whatever number be substituted in it for x (59), so that the two functions that follow this, in the order of derivation, viz. the function of the first degree, and the final constant, would, if introduced, neither increase nor diminish the number of variations furnished by the incomplete series of any value of x, p. 217; and therefore the computation of these two functions would be superfluous. It is therefore always worth while to pause at the quadratic function, and inquire whether the coefficients it presents have the above-mentioned relation or not: as it is an important matter to save the last two steps of the calculation. The final step—that which determines the constant—need never be fully worked out; since it is not the numerical value, but only the sign of this constant that we have any occasion for: and this sign we may often arrive at, very readily, by aid of the property at p. 210, which shows that if a value of x, which makes one of the functions become zero, be substituted in the immediately preceding function, the resulting sign will always be *opposite* to that furnished by the immediately succeeding function, for the same substitution. Hence, when we have arrived at the last function but one, that is, at the function $ax + b$, we need merely substitute the value of x which renders this zero, viz. $= -\frac{b}{a}$, in the preceding quadratic, and change the sign of the result in order to get the proper sign of the final constant. It is seldom necessary to make this substitution with any view to strict numerical accuracy: a glance will often suffice to inform us whether the result would be positive or negative.

In a somewhat similar manner to that by which the actual computation of the final function may be thus avoided, may the function immediately preceding it be also dispensed with; and the information afforded by it obtained in another way. The departure thus made from the direct method of STURM will generally be attended with decided advantage on the score of facility. We have very fully dwelt upon this mode of proceeding in the introductory volume; and shall again advert to it in the next chapter: our chief object at present is to exhibit what appears to us to be the shortest and most convenient method of conducting the operation for the common measure, that is, of determining, in general, the remainder, of degree $n - 2$, which arises from dividing a rational polynomial in x, of degree n, by another of degree $n - 1$, fractions being excluded from the process.

(152.) Let the dividend and divisor be represented respectively by

$$ax^{\mathrm{n}} \quad + bx^{\mathrm{n}-1} + cx^{\mathrm{n}-2} + dx^{\mathrm{n}-3} + \ldots . \; [1]$$

$$a'x^{\mathrm{n}-1} + b'x^{\mathrm{n}-2} + c'x^{\mathrm{n}-3} + d'x^{\mathrm{n}-4} + \ldots . \; [2]$$

the remainder sought will present itself after two terms of the quotient have been obtained: the first of these, when combined with the divisor, will destroy the first term of the dividend, furnishing a remainder of degree $n - 1$, the leading term of which is in like manner destroyed by the second term of the quotient: the resulting remainder being that sought. It is desirable that this, and every succeeding remainder, be free from fractions; and to preclude their entrance it is requisite that both terms of the quotient be free from fractions. That the first of these two may involve no fraction it is obviously sufficient to multiply the dividend by a'; but then the first remainder, having for its leading term $(a'b - ab')x^{\mathrm{n}-1}$, would in general necessitate the entrance of a fraction into the second term of the quotient; viz. the fraction $\frac{a'b - ab'}{a'}$; with which fraction, therefore, the next remainder would be affected.

In order, therefore, to provide against the entrance of fractions in all cases, it is generally necessary to multiply the dividend not only by a', but by a'^2; after which preparation it is easy to see that the quotient will be $a'ax + (a'b - ab')$. Let us then multiply each of these terms by the divisor, arranging the several partial products, under the like terms of the dividend, with changed signs, in order to convert the operation of *subtracting* them from the dividend into that of *adding*. The entire work of the step, the quotient being suppressed as we have no occasion for it, will then take the following arrangement:

$$\begin{array}{rrrl}
a'^2.ax^{\mathrm{n}} + & a'^2.bx^{\mathrm{n}-1} + & a'^2.cx^{\mathrm{n}-2} + & a'^2.dx^{\mathrm{n}-3} + \ldots. \\
- a'a.a'x^{\mathrm{n}} - & a'a.b'x^{\mathrm{n}-1} - & a'a.c'x^{\mathrm{n}-2} - & a'a.d'x^{\mathrm{n}-3} - \ldots. \\
 & -(a'b - ab')a'x^{\mathrm{n}-1} - & (a'b - ab')b'x^{\mathrm{n}-2} - & (a'b - ab')c'x^{\mathrm{n}-3} - \ldots. \\
\hline
0 + & 0 \quad + & a''x^{\mathrm{n}-2} + & b''x^{\mathrm{n}-3} + \ldots. [3]
\end{array}$$

From this mode of arrangement it is easy to discover what functions the coefficients a'', b'', &c. in the sought remainder, are of the coefficients in the original polynomials. But in order that these functions may be computed in the most expeditious manner the work should be conducted conformably to the following type of the operations, where the multipliers inserted in the margin are the factors which combine with the given coefficients in the operation exhibited above, but written in reverse order.

$$\begin{array}{r|l}
 & a' - b' \quad - c' \quad - d' \quad - \; . \; . \; . \; . \\
 & \;\times \\
\text{MULTIPLIERS.} & a + b \quad + c \quad + d \quad + \; . \; . \; . \; . \\
\cline{2-2}
A = a'b - ab' & \quad - Ab' - Ac' - Ad' - \; . \; . \; . \; . \\
B = aa' & \quad - Bc' - Bd' - Be' - \; . \; . \; . \; . \\
C = a'^2 & \qquad Cc + Cd + Ce + \; . \; . \; . \; . \\
\cline{2-2}
\multicolumn{1}{r}{} & \qquad a'' \; + b'' \; + c'' \; + \; . \; . \; . \; .
\end{array}$$

The complete remainder to which these coefficients belong is

$$a''x^{n-2} + b''x^{n-3} + c''x^{n-4} + \; . \; . \; . \; .$$

But the signs of all its terms are to be changed, agreeably to the theorem of STURM, before it is employed, as above, in combination with [2], in deducing the next following remainder. Hence the dividend and divisor, with which the next step of the work is performed, are respectively

$$a'x^{n-1} + b'x^{n-2} + c'x^{n-3} + d'x^{n-4} + \; . \; . \; . \; .$$
$$- a''x^{n-2} - b''x^{n-3} - c''x^{n-4} - d''x^{n-5} - \; . \; . \; . \; .$$

Conformably to the foregoing general type of the work involved in each step, this last polynomial is to have all its signs after the first changed; the coefficients only are to be written down, those of the preceding polynomial, with their proper signs, are to be placed under them, and the operations in the model performed. The formation of the multipliers which occupy the side column of the work is sufficiently indicated by the general symbols; but to assist the memory, in particular examples, it is recommended always to insert the cross between the first two terms of the dividend and divisor at the head of each step, as above: as by this contrivance the mode of forming the first multiplier is very intelligibly pointed out: the formation of the other multipliers is too simple for us to stand in need of any such guidance to it. By uniting the several steps of the operation the following will be a general type or working model of the entire calculation, where it is to be observed that the several *remainders* are written with changed signs.

$$\begin{array}{l} a'-b'-c'-d'-\ldots \\ \quad\times \\ a+b+c+d+\ldots \end{array}$$

$$\begin{array}{r|l} \cline{2-2} A=a'b-ab' & -Ab'-Ac'-Ad'-\ldots \\ B=\quad aa' & -Bc'-Bd'-Be'-\ldots \\ C=\quad a'^2 & \quad Cc+Cd+Ce+\ldots \\ \cline{2-2} \multicolumn{2}{r}{-a''-b''-c''-\ldots = \text{1st rem. with signs changed.}} \end{array}$$

$$\begin{array}{l} -a''+b''+c''+\ldots \\ \quad\times \\ \quad a'+b'+c'+\ldots \end{array}$$

$$\begin{array}{r|l} \cline{2-2} A'=-a''b'+a'b'' & A'b''+A'c''+\ldots \\ B'=-\quad a'a'' & B'c''+B'd''+\ldots \\ C'=\quad a''^2 & C'c'+C'd'+\ldots \\ \cline{2-2} \multicolumn{2}{r}{-a'''-b'''-\ldots = \text{2d rem. with signs changed.}} \end{array}$$

$$\begin{array}{l} -a'''+b'''+\ldots \\ \quad\times \\ -a''-b''-\ldots \end{array}$$

$$\begin{array}{r|l} \cline{2-2} A''=a'''b''-a''b''' & A''b'''+\ldots \\ B''=\quad a''a''' & B''c'''+\ldots \\ C''=\quad a'''^2 & -C''c''-\ldots \\ \cline{2-2} \multicolumn{2}{r}{-a''''-\ldots = \text{3d rem. with signs changed.}} \end{array}$$

&c. &c.

$$\begin{aligned}
\therefore\ X &= ax^{n} + bx^{n-1} + cx^{n-2} + dx^{n-3} + ex^{n-4} + \dots \\
X_1 &= a'x^{n-1} + b'x^{n-2} + c'x^{n-3} + d'x^{n-4} + \dots \\
X_2 &= - a''x^{n-2} - b''x^{n-3} - c''x^{n-4} - \dots \\
X_3 &= - a'''x^{n-3} - b'''x^{n-4} - \dots \\
X_4 &= - a''''x^{n-4} - \dots \\
&\&c. \quad \&c.
\end{aligned}$$

The first step of the preceding work may be abridged in consequence of the manner in which X and X_1 are always connected together.

Thus, since

$$X_1 = nax^{n-1} + (n-1)\,bx^{n-2} + (n-2)\,cx^{n-3} + \dots$$

the first step will be as follows, the factor *a* common to all the multipliers being suppressed :

$$\begin{array}{c}
na - (n-1)b \quad - \quad (n-2)c \quad - \quad (n-3)d \quad - \dots \\
\times \\
a + b \quad + \quad c \quad + \quad d \quad + \dots
\end{array}$$

$$\begin{array}{r|l}
\cline{2-2}
b & -(n-1)b^2 - (n-2)bc - (n-3)bd - \dots \\
na & -n(n-2)ac - n(n-3)ad - n(n-4)ae - \dots \\
n^2a & n^2ac + n^2ad + n^2ae + \dots \\
\cline{2-2}
\end{array}$$

$$\therefore X_2 = \begin{array}{r|r|r|l}
(n-1)b^2 & x^{n-2} + (n-2)bc & x^{n-3} + (n-3)bd & x^{n-4} + \dots \\
-2nac & -\ 3nad & -\ 4nae &
\end{array}$$

The coefficients of X_2 may, therefore, be obtained thus : Write the coefficients of X_1 in a row, with their proper signs. Underneath these write the coefficients of X, commencing with the *second*, after having multiplied them in order by the numbers 1, 2, 3, 4, &c. taking care, however, to put down all the results of these multiplications except the first, with changed signs. We shall thus have two rows of figures of equal extent. Cut off the leading term in each row for multipliers ; then if the lower of these be multiplied by the upper row and the upper by the lower row, and the several results added, the step will be completed : thus

$$\begin{array}{r|l}
na & +\ (n-1)b\ +\ (n-2)c\ +\ (n-3)d\ +\ \ldots \\
b & -\quad 2c\quad -\quad 3d\quad -\quad 4e\quad -\ \ldots \\
\hline
 & (n-1)b^2 + (n-2)bc + (n-3)bd + \ldots \\
 & -\ 2nac\ -\ 3nad\ -\ 4nae\ -\ \&c. \\
\hline
 & -\ a''\ -\ b''\ -\ c''\ -\ \&c.
\end{array}$$

We shall now give a few examples of the application of these forms, and for the purpose of comparison shall subjoin to Sturm's analysis, the necessary operations by the methods of Fourier and Budan, improved as suggested in Chapter IX.*

(153.) As a first example let us take the equation

$$\mathrm{X} = x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2 = 0$$

$$\therefore \mathrm{X}_1 = 5x^4 + 12x^3 + 6x^2 - 6x - 2$$

$$\begin{array}{r|l}
5 & +\ 12\ +\ 6\ -\ 6\ -\ 2 \\
3 & -\ 4\ +\ 9\ +\ 8\ +\ 10 \\
\hline
 & 36 + 18 - 18 - 6 \\
 & -20 + 45 + 40 + 50 \\
\hline
 & 16 + 63 + 22 + 44 \\
\hline
 & 16 - 63 - 22 - 44 \\
 & \times \\
 & 5 + 12 + 6 - 6 - 2
\end{array}$$

$$\begin{array}{r|l}
-123 & 7749 + 2706 + 5412 \\
80 & -1760 - 3520 \\
256 & 1536 - 1536 - 512 \\
\hline
 & -7525 + 2350 - 4900 \\
 & \text{or } \div\ 5^2 \\
 & -301 + 94 - 196
\end{array}$$

---

* In the present chapter we do not propose to enter into very minute detail respecting the practical operation of Sturm's theorem. A great variety of examples, and an exhibition of the best expedients for contracting the work, will be found in the *Analysis and Solution of Cubic and Biquadratic Equations*, and in the *Mathematical Dissertations*.

Having now arrived at the coefficients of the quadratic function we apply the test adverted to at page 221, and find it to be satisfied; that is, $4(301 \times 196) > 94^2$. Consequently the operation is terminated; and the functions as far as necessary are

$$\begin{aligned}
X &= x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2\\
X_1 &= 5x^4 + 12x^3 + 6x^2 - 6x - 2\\
X_2 &= 16x^3 + 63x^2 + 22x + 44\\
X_3 &= -\,301x^2 + 94x - 196
\end{aligned}$$

These give, for

$$\begin{array}{rcccccl}
x = -\infty \ldots & - & + & - & - & & \textit{two variations.}\\
+\infty \ldots & + & + & + & - & & \textit{one variation.}
\end{array}$$

Hence there is only one real root, which must be positive, since the final sign of X is negative. A superior limit to this root (89) is 2; so that no number beyond 2 need be substituted in X in order to determine its situation. Putting 0 for x in X the result is *minus*: putting 1 for x the result is still *minus*: therefore the root must lie between 1 and 2.

(154.) The analysis of the preceding equation is very easily accomplished either by the method of **Fourier** or by that of **Budan**: thus, transforming by (-1), (0), (1), (10), we have

$$\begin{array}{rcccccc}
(-1) \ldots & + & - & \overset{+}{\underset{-}{0}} & - & + & - \quad \textit{two imaginary roots.}\\
(0) \ldots & + & + & + & - & - & -\\
(1) \ldots & + & + & + & + & + & -\\
(10) \ldots & + & + & + & + & + & +
\end{array}\left.\begin{array}{l}\\ \\ \\ \\ \end{array}\right\}\textit{one real root.}$$

From this partial analysis of the equation, we see that the only doubtful interval is [0, − 1]. The transformations corresponding to the limits of this interval are

(− 1) . . . + − − − + −
1 1 0 1 2
(0) . . . + 1 + 3 + 2 − 3 − 2 − 2

There is no necessity to contract the interval in order to determine the character of the two roots indicated: for we may infer at once that they are imaginary, inasmuch as

$$\frac{f(0)}{f_1(0)} = \frac{2}{2} = 1$$

which, alone, being equal to the entire interval [0, − 1] we know from the criterion [A] at page 165, that the roots are imaginary. And as two other imaginary roots were indicated by the preliminary process above, we conclude that the equation has but the one real root lying, as determined by that process, between 1 and 10.

By Budan's method we proceed as follows, for the same doubtful interval [0, — 1], or rather, changing alternate signs, for the interval [0, 1]:

(0) . . . . 1 − 3 + 2 + 3 − 2 + 2

(1) . . . . 1 + 2 + 0 + 1 + 3 + 3 *four var. lost.*

Reciprocal (0) . . . . 2 − 2 + 3 + 2 − 3 + 1

(1) . . . . 2 + 8 + + + + *no var. left.*

Hence there are four imaginary roots.

As a second example let us take the equation

$$X = x^5 - 10x^3 + 6x + 1 = 0$$

$$\therefore X_1 = 5x^4 - 30x^2 + 6$$

and the remainder of Sturm's functions are found as follows:

$$\begin{array}{r|l} 5 & +\,0-30+0\;+6 \\ {}^{*}0 & +\,20-0\;-24-5 \\ \hline \end{array}$$

$$20-0\;-24-5$$

$$20+0\;+24+5$$

$$\times$$

$$5+0-30+0+6$$

$$\begin{array}{r|l} \hline 0 & \\ 1 & 24+5 \\ 4 & -120+0+24 \\ \hline \end{array}$$

$$96-5-24$$

$$96+5+24$$

$$\times$$

$$20-0-24-5$$

$$\begin{array}{r|l} \hline 100 & 500+2400 \\ 1920 & 46080 \\ 96^2 & -221184-46080 \\ \hline \end{array}$$

$$174604+43680$$

or ÷ 4

$$43651+10920$$

Hence the series of functions is

$$\begin{aligned} X &= x^5-10x^3+6x+1 \\ X_1 &= 5x^4-30x^2+6 \\ X_2 &= 20x^3-24x-5 \\ X_3 &= 96x^2-5x-24 \\ X_4 &= 43651x+10920 \\ \therefore X_5 &= + \end{aligned}$$

---

[*] The advantage of having zero for the second coefficient is readily seen from the first two steps of this example. This advantage was alluded to at page 87.

From the signs of the leading terms we infer that all the roots are real (149): and from the signs of the final terms that two roots are positive, p. 216, and consequently that the remaining three are negative. From MACLAURIN's limit we see that the numerical value of each of these must be below 11. To find their exact places we proceed as follows:

	X	X_1	X_2	X_3	X_4	X_5		X	X_1	X_2	X_3	X_4	X_5
$x=0$...	+	+	−	−	+	+	$x=0$...	+	+	−	−	+	+
1...	−	−	−	+	+	+	−1...	+	−	−	+	−	+
2...	−	−	+	+	+	+	−2...	+	−	−	+	−	+
3...	−	+	+	+	+	+	−3...	+	+	−	+	−	+
4...	+	+	+	+	+	+	−4...	−	+	−	+	−	+

Whence the places of the five roots are as follow:

$$[0,\ 1];\quad [3,\ 4];\quad [0,\ -1];\quad [0,\ -1];\quad [-3,\ -4].$$

(155.) The partial analysis of this equation by the theorem of BUDAN and FOURIER (107) leaves the interval [0, − 1] doubtful, the corresponding transformations being

(−1)	+1	−5	−0	+20	−19	+4
				0	1	2
(0)	+1	−0	−10	+0	+6	+1

As $\frac{4}{19} + \frac{1}{6}$ is less than the distance 1, between the limits we must contract the interval before we can determine the character of the roots. Previously to this there will be no occasion to seek the common measure of $f(x)$ and $f_1(x)$, which would imply all the work in the foregoing process by the method of STURM, for we know that the proposed equation cannot have equal fractional roots (101). Nor in narrowing the interval need we proceed entirely at random; for we know, that if the roots in this interval should happen to be real, they cannot both lie so near to the limit 0 as $-\frac{1}{6}$, nor so near to the other limit -1, as $-\frac{4}{19}$, else the criterion [A] would be fulfilled, and the roots would be imaginary. Consequently, $-\cdot 2$ is a suitable intermediate number, being somewhat greater than $-\frac{1}{6}$. Transforming therefore by this number we have

$$\begin{array}{llllllr}
1 & -0 & -10 & +0 & +6 & +1 & (-\cdot 2 \\
 & -\cdot 2 & \cdot 04 & 1\cdot 992 & -\cdot 3984 & -1\cdot 12032 & \\
\cline{2-6}
 & -\cdot 2 & -9\cdot 96 & 1\cdot 992 & 5\cdot 6016 & -\cdot 12032 &
\end{array}$$

We need not complete the transformation: the change of sign in the final term shows that $-\cdot 2$ separates the roots. Hence one root lies between $-\cdot 2$ and -1, and the other between 0 and $-\cdot 2$. Let us now apply the method of Budan to the same doubtful interval:

$$\begin{array}{rl}
(0) \ldots\ldots & +1 \quad -0 \quad -10 \quad +0 \quad +6 \quad +1 \\
(-1) \ldots\ldots & +1 \quad -5 \quad -0 \quad +20 \quad -19 \quad +4 \\
\text{Reciprocal} \quad (0) \ldots\ldots & +1 \quad +6 \quad +0 \quad -10 \quad -0 \quad +1 \\
(-1) \ldots\ldots & \qquad\qquad\qquad\qquad\qquad\qquad\qquad -4
\end{array}$$

As a change of sign is produced in the final term, we need not complete this last transformation, but may conclude at once that the roots are real.

(156.) We may remark here, that when Sturm's method is applied to an equation of a high degree, or when very large coefficients enter an equation of even a low degree, the operation for the common measure will involve multiplications by large numbers, which numbers may be expected to increase very rapidly in the final steps of the work. But, as in order to discover the character of the roots, the accurate computation of a large array of figures in the several results is not necessary, it will in general be sufficient to secure the three or four leading figures merely of the final numerical result: those that follow may be rejected as redundant. In order to save the labour of computing these redundant figures, we recommend their progressive increase to be checked, as soon as they threaten to become uselessly large, by an easy reduction of the subsequent sets of multipliers. Thus, referring to the general model at p. 225, if the numbers at the close of the second step, for example, are so large as to threaten unnecessarily long multiplications in the computation of the third step: then in preparing for that step we should employ the multipliers $A_{\prime\prime}$, $B_{\prime\prime}$, $C_{\prime\prime}$, instead of A'', B'', C'', these being a''' times the former.

$$\begin{array}{ll} A'' = a'''b'' - a''b''' & A_{\prime\prime} = b'' - \dfrac{a''b'''}{a'''} \\ B'' = \quad a''a''' & B_{\prime\prime} = \quad a'' \\ C'' = \quad a'''^2 & C_{\prime\prime} = \quad a''' \end{array}$$

And in calculating $\frac{a''b'''}{a'''}$, only so many decimals are to be preserved as will be sufficient to secure accuracy, upon the principle of contracted multiplication, in the proposed number of leading figures to be retained in the several products which enter the step. In the following example the last step of the work is curtailed in this way:

$$X \ = x^6 + x^5 - x^4 - x^3 + x^2 - x + 1 = 0$$

$$X_1 = 6x^5 + 5x^4 - 4x^3 - 3x^2 + 2x - 1$$

$$\begin{array}{r|rrrrr}
6 & +5 & -4 & -3 & +2 & -1 \\
1 & +2 & +3 & -4 & +5 & -6 \\
\hline
 & 5 & -4 & -3 & +2 & -1 \\
 & 12 & +18 & -24 & +30 & -36 \\
\hline
 & 17 & +14 & -27 & +32 & -37 \\
\hline\hline
 & 17 & -14 & +27 & -32 & +37 \\
 & \times & & & & \\
 & 6 & +5 & -4 & -3 & +2 \quad -1
\end{array}$$

$$\begin{array}{r|rrrr}
1 & -14 & +27 & -32 & +37 \\
102 & 2754 & -3264 & +3774 & \\
17^2 & -1156 & -867 & +578 & -289 \\
\hline
 & -1584 & +4104 & -4320 & +252
\end{array}$$

$$\text{or, } \div 4 \times 9$$

$$\begin{array}{rrrr}
-44 & +114 & -120 & +7 \\
\hline\hline
-44 & -114 & +120 & -7 \\
\times & & & \\
17 + 14 & -27 & +32 & -37
\end{array}$$

$$\begin{array}{r|rrr}
-1277 & 145578 & -153240 & +8939 \\
-22 \times 17 & -44880 & +2618 & \\
44 \times 22 & -26136 & +30976 & -35816 \\
\hline
 & -74562 & +119646 & +26877 \\
\hline\hline
 & -74562 & -119646 & -26877 \\
 & \times & & \\
 & -44 + 114 & & -120 + 7
\end{array}$$

$$\begin{array}{r|rr}
-21{\cdot}69 & 2595 & +583 \\
22 & -591 & \\
37281 & -4474 & +261 \\
\hline
 & 2470 & -844
\end{array}$$

$$\text{or, } \div 2$$

$$\begin{array}{rr}
1235 & -422
\end{array}$$

Consequently, the functions are as follow:

$$\begin{aligned}
X\ &= x^6 + x^5 - x^4 - x^3 + x^2 - x + 1 \\
X_1 &= 6x^5 + 5x^4 - 4x^3 - 3x^2 + 2x - 1 \\
X_2 &= 17x^4 + 14x^3 - 27x^2 + 32x - 37 \\
X_3 &= -44x^3 + 114x^2 - 120x + 7 \\
X_4 &= -74562x^2 + 119646x + 26877 \\
X_5 &= \quad 1235x \quad - \quad 422 \\
\therefore X_6 &= -
\end{aligned}$$

From the variations in the leading signs we infer that all the roots are imaginary (149).

It may be proper to observe here, in reference to the foregoing method of abbreviating the work, that should roots approaching very nearly to equality enter the equation — thus causing a tendency in the final remainder towards zero — our curtailments, if too freely made, may deprive that remainder of all its significant figures; and present us only with a row of zeros. The same would happen if the roots were imaginary, provided a minute change in the final term of the equation would render them real and equal. In such a case, the final sign would be doubtful; and could not be safely inferred from the preceding function of the first degree. We might, under such circumstances, disregard this function of the first degree, and ascend to that of the second; consider it the last of the series, and proceed with the analysis as explained in the introductory treatise, Chapter IV, and fully illustrated at pp. 214-226. But the peculiar case here noticed will come under special consideration in next chapter.

(157.) Let us apply the method of **Fourier** to the preceding example:—

$$\begin{array}{rrrrrrrl}
1 & +1 & -1 & -1 & +1 & -1 & +1 & (1 \\
 & 2 & +1 & +0 & +1 & +0 & +1 & \\
 & 3 & +4 & +4 & +5 & +5 & & \\
 & 4 & +8 & +12 & +17 & & & \\
 & 5 & +13 & +25 & & & & \\
 & 6 & +19 & & & & & \\
 & 7 & & & & & &
\end{array}$$

$$\begin{array}{lccccccccl}
(0) \ldots & +1 & +1 & -\ 1 & -\ 1 & +\ 1 & -1 & +1 & \textit{four variations} \\
 & & & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \\
(1) \ldots & +1 & +7 & +19 & +25 & +17 & +5 & +1 & \textit{no variation.}
\end{array}$$

The index **2** corresponds to $f_2(x)$: we shall proceed to examine whether $f_2(x) = 0$ can have a pair of equal roots in the interval $[0, 1]$.

$$f(x) = x^6 + x^5 - x^4 - x^3 + x^2 - x + 1$$

$$f_1(x) = 6x^5 + 5x^4 - 4x^3 - 3x^2 + 2x - 1$$

$$\frac{1}{2} f_2(x) = 15x^4 + 10x^3 - 6x^2 - 3x + 1$$

$$\frac{1}{2 \cdot 3} f_3(x) = 20x^3 + 10x^2 - 4x - 1$$

$$\frac{1}{2 \cdot 3 \cdot 4} f_4(x) = 15x^2 + 5x - 1$$

$$\frac{1}{2 \cdot 3 \cdot 4 \cdot 5} f_5(x) = 6x + 1$$

$$\frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} f_6(x) = 1$$

The equation $f_2(x) = 0$ cannot have a pair of equal fractional roots, because its leading coefficient has no square factor (101). We may therefore proceed to subdivide the interval $[0, 1]$; and as our object is to interpose between the foregoing transformations, one that will change the first index **1** into **0**, we shall be guided to a suitable number by inspecting the fourth derived function above, viz., $15x^2 + 5x - 1$, which we see becomes *plus* for $x = \cdot 3$: hence the transformation ($\cdot 3$) must produce the desired change in the leading index.

Transforming therefore by (·3) we have

1	+1	−1	−1	+1	−1	+1	(·3
	·3	·39	−·183	−·3549	·19353	−·241941	
	1·3	−·61	−1·183	·6451	−·80647	·758059	
	·3	·48	− 39	−·3666	8355		
	1·6	−·13	−1·222	·2785	−·72292		
	·3	·57	·132	−·3270			
	1·9	·44	−1·090	−·0485			
	·3	·66	·330				
	2·2	1·10	−·760				
	·3	·75					
	2·5	1·85					
	·3						
	2·8						

(0) . . .	+ 1	+1	−1	−1	+1	−1	+1
				0	**1**	**2**	**2**
(·3) . . .	+ 1	+2·8	+1·85	− ·760	− ·0485	−·72292	+ ·758059
				1	**1**	**1**	**2**
(1) . . .	+ 1	+ 7	+ 19	+ 25	+ 17	+ 5	+ 1

Two imaginary roots are indicated in the interval [0, ·3], for $\frac{f_1(0)}{f_2(0)} = \frac{1}{2}$ is greater than ·3. To determine the character of the remaining two roots, indicated in the interval [·3, 1], we observe, first, that $f(x) = 0$ can have no equal roots in that interval (101); we shall therefore seek to interpose a transformation that will change the second of the indices **1**, **1**, **1**, **2**, into **0**. An inspection of the third derived function, viz., $20x^3 + 10x^2 - 4x - 1$, suggests ·5 as a suitable number; as the function changes from − to + in the interval [·3, ·5]. Transforming therefore by (·5) we have

1	+ 1	−1	− 1	+ 1	− 1	+ 1	(·5
	·5	·75	−·125	−·5625	·21875	−·390625	
	1·5	−·25	−1·125	+·4375	−·78125	·609375	
	·5	1	·375	−·3750	3125		
	2·0	·75	−·750	·0625	−·75		
	·5	1·25	1	·1250			
	2·5	2	·250	·1875			
	·5	1·5	1·75				
	3·0	3·5	2				
	·5	1·75					
	3·5	5·25					
	·5						
	4						

(·5) . . . . + 1 + 4 + 5·25 + 2 + ·1875 − ·75 + ·609375

0 **1** **2**

(1) . . . . + 1 + 7 + 19 + 25 + 17 + 5 + 1

Since $\frac{\cdot609375}{\cdot75} = \cdot8\ldots$ is greater than the distance ·5 between the limits we infer that these two roots are also imaginary.

It remains for us to examine the negative intervals; for this purpose we have, by changing alternate signs, and then changing those of the result,

1	− 1	− 1	+ 1	+ 1	+ 1	+ 1	(1
	0	− 1	0	1	2	3	
	1	0	0	1	3		
	2	2	2	3			
	3	5	7				
	4	9					
	5						

$$(-1)\ .\ .\ .\ .\ +1-5+9-7+3-3+3$$

1 1 2 2 2 2

$$(0)\ .\ .\ .\ .\ +1+1-1-1+1-1+1$$

In order to reduce the first index to **0** the factor (— ·2) will be suitable

1 +1	— 1	— 1	+ 1	— 1	+ 1	(— ·2
—·2	—·16	·232	·1536	—·23072	·246144	
·8	—1·16	—·768	1·1536	—1·23072	1·246144	
—·2	—·12	+·256	·1024	—·25120		
·6	—1·28	—·512	1·2560	—1·48192		
—·2	—·08	+·272	480			
·4	—1·36	—·240	1·3040			
—·2	—·04	·280				
·2	—1·40	·040				
—·2	— 0					
0	—1·40					
—·2						
—·2						

$$(-1)\ .\ .\ .\ .\ +\ -\ +\ -\ +\ -\ +$$

0 1 1 2 2 2

$$(-\cdot 2)\ .\ .\ .\ .\ +\ -\ -\ +\ +\ -\ +$$

The character of the roots is still doubtful: the interval however is reduced to [—·2, —1]. Trying the intermediate number —·6 we have

1	+1	−1	−1	+1	−1	+1	(−·6
	−·6	−·24	·744	.1536	−·69216	1·015296	
	·4	−1·24	−·256	1·1536	−1·69216	2·015296	
	−·6	·12	·672	−·2496	−·54240		
	−·2	−1·12	·416	·9040	−2·23456		
	−·6	·48	·384	−·4800			
	−·8	−·64	·800	·4240			
	−·6	·84	−·120				
	−1·4	·20	·680				
	−·6	1·20					
	−2·0	1·40					
	−·6						
	−2·6						

(− 1)... +1 −5 +9 −7 +3 −3 +3

0 1 2 2 2

(−·6)... +1 −2·6 +1·4 +·68 +·424 −2·23456 +2·015296

$$\frac{f_2(-1)}{f_3(-1)}+\frac{f_2(-\cdot 6)}{f_3(-\cdot 6)}=\frac{3}{21}+\frac{\cdot 424}{2\cdot 04}=\cdot 35\ \ldots .$$

As this result is less than ·4, the distance between the limits, the interval [− ·6, −1] must be rendered still narrower. Transforming, therefore, by an additional unit, we have finally

(−·7).. +1 − 3·2 +2·85 −·16 +·3315 −2·30592 +2·242479

0 1 2 2 2

(−·6).. +1 − 2·6 + 1·4 +·68 + ·424 −2·23456 +2·015296

$$\frac{f_2(-\cdot 7)}{f_3(-\cdot 7)}=\frac{\cdot 3315}{\cdot 48}=\cdot 69\ \ldots .$$

This result being greater than ·1, the distance between the limits, we conclude that the roots indicated are imaginary.

Let us now apply the method of **Budan** to the intervals [0, 1], [0, −1].

(0) . . . . + + − − + − +

(1) . . . . + + + + + + + *four variations lost.*

Recip. (0) . . . . + − + − − + +

$$\left|\begin{array}{cccccccc} 1 & -1 & +1 & -1 & -1 & +1 & +1 & (1 \\ & 0 & 1 & 0 & -1 & 0 & 1 & \\ & 1 & 2 & 2 & 1 & 1 & & \\ & & & \&c. & & \&c. & & \end{array}\right.$$

(1) . . . . + + + + + + + *no var. left.*

Hence, the four roots indicated in the interval [0, 1] are all imaginary.

Again: changing the alternate signs, we have

(0) . . . . + − − + + + +

(1) . . . . + + + + + + + *two var. lost.*

Recip. (0) . . . . + + + + − − +

(1) . . . . + + + + + + + *no var. left.*

Therefore the two remaining roots are also imaginary.

(158.) In each of the preceding examples the method of **Budan** has appeared very much to advantage. Its superiority in the analysis just completed is very conspicuous: the operations of **Sturm** and **Fourier** both involve a good deal of calculation; but we think preference should be given to the former, on account of the confidence we can place in every step of the work, as directly contributing to the object in view, without the expenditure of a single useless or tentative operation. This important peculiarity of **Sturm**'s method should never be overlooked in comparing it with other processes, the steps of which, though apparently requiring much less work, are seldom to be made without cautious deliberation and that sort of tact which experience alone can impart to the analyst. The method of **Sturm** is quite independent of every aid of this kind: it leads us so unerringly and so directly to the object sought—although sometimes by a lengthy path—that whatever be the inherent difficulties of the question, we proceed unapprized of them along the same uniform track, with full confidence of arriving at the solution without any demand being made upon our ingenuity or address to provide for particular exigencies. The simple model at (152) is an unerring guide in every possible case. But, even on the score of mere numerical work, an unfavorable decision must not be pronounced against STURM's method from the evidence of a few particular examples. Without actual trial, it is sometimes impossible to select, from the three preceding methods, that which shall necessarily effect the required analysis, in any proposed case, with the least expenditure of calculation. In many instances the processes of FOURIER and BUDAN will terminate very speedily, and indeed unexpectedly, in cases, which by STURM's method, would involve us in long and laborious multiplications. But it is some disadvantage to those methods that, so far from their enabling us to foresee this, they furnish us with no means of even fixing any moderate limit to the amount of work that *may* be required by them. We can always form a pretty correct estimate of the extent of calculation by STURM's method from an inspection of the coefficients, and the degree of the equation; since these always regulate the length of the operation. And thus we shall in every case be forewarned, at the outset, when the method of STURM should be abandoned for that of FOURIER or of BUDAN, or for the methods proposed in next chapter: the ease with which a step or two of BUDAN's method may be executed, renders it especially deserving of notice as a preliminary test where imaginary roots are suspected.

(159.) We shall now give an example that will in some measure illustrate the preceding observations.

Let the equation proposed for analysis be

$$12x^3 - 120x^2 + 326x - 127 = 0.$$

From the low degree of the equation, we may be quite sure of accomplishing the analysis, by the method of STURM, with but a very trifling amount of numerical labour. In order to exhibit the whole of it, we shall actually compute the final function X_3; although, as in the last example, the sign of this, which is all we want, may be more easily deduced from the principle in (151).

In applying the rule at p. 226 to the original functions

$$X = 12x^3 - 120x^2 + 326x - 127$$

$$X_1 = 18x^2 - 120x + 163$$

we shall suppress the common factor 6, which enters the multipliers, and shall employ the reduced numbers 3 and -20 instead.

$$\begin{array}{r|r}
3 & -120 + 163 \\
-20 & -652 + 381 \\ \hline
\multicolumn{2}{r}{2400 - 3260} \\
\multicolumn{2}{r}{-1956 + 1143} \\ \hline
\multicolumn{2}{r}{444 - 2117} \\ \hline
\multicolumn{2}{r}{444 + 2117} \\
\multicolumn{2}{r}{\times } \\
\multicolumn{2}{r}{18 - 120 + 163} \\ \hline
-15174 & -32123358 \\
444^2 & 32133168 \\ \hline
\multicolumn{2}{r}{-9810} \\ \hline
\end{array}$$

As the final result is *minus* we infer at once that the equation has a pair of imaginary roots; and consequently but a single real root, which from the final sign of X, must be positive. By substituting 0, 1, &c. in this polynomial we find the root to lie between 0 and 1.

(160.) Let us now apply Fourier's method to this example:

$$\left.\begin{array}{lccccc}
(0) & \ldots\ldots & + & - & + & - \\
(1) & \ldots\ldots & + & - & + & +
\end{array}\right\} \text{ one root.}$$

$$\begin{array}{lccccc}
(2) & \ldots\ldots & + & - & - & + \\
(3) & \ldots\ldots & + & - & - & +
\end{array}$$

$$\left.\begin{array}{lccccc}
(4) & \ldots\ldots & +12 & +24 & -58 & +25 \\
 & & & \mathbf{0} & \mathbf{1} & \mathbf{2} \\
(5) & \ldots\ldots & +12 & +60 & +26 & +3
\end{array}\right\} \text{ two roots doubtful.}$$

Now before proceeding to subdivide the interval [4, 5] it is necessary to ascertain whether equal roots can lie in this interval. Referring, therefore, to the proposed equation, we find that the absolute number 127 contains no square factor. Hence the roots indicated are not equal (101).

In order to avoid useless transformations, let us observe, like as in the example at page 231, that as [illegible] $=\cdot4\ldots$, and $\frac{3}{26}=\cdot1\ldots$ the interval to be examined is contracted to [4·4, 4·9]. Interposing, therefore, the number 4·5 we have

$$\begin{array}{rrrrl}
12 & +24 & -58 & +25 & (\cdot5 \\
 & 6 & 15 & -21\cdot5 & \\ \cline{2-4}
 & 30 & -43 & 3\cdot5 & \\
 & 6 & 18 & & \\ \cline{2-3}
 & 36 & -25 & & \\
 & 6 & & & \\ \cline{2-2}
 & 42 & & &
\end{array}$$

$$(4\cdot5)\ \ldots\ldots + 12 + 42 - 25 + 3\cdot5.$$

The interval between this and the preceding limit (5) is ·5; and as $\frac{3}{26}+\frac{3\cdot5}{25}$ is not so great as this, the interval is not yet sufficiently narrow: it is contracted, however, to [4·5 $\frac{3\cdot5}{25}$, 4·9] the number to be interposed is therefore 4·7; so that we have to transform the last result by (·2)

$$\begin{array}{rrrrl}
12 & +42 & -25 & +3\cdot5 & (\cdot2 \\
 & 2\cdot4 & 8\cdot88 & -3\cdot224 & \\ \cline{2-4}
 & 44\cdot4 & -16\cdot12 & \cdot276 & \\
 & 2\cdot4 & 9\cdot36 & & \\ \cline{2-3}
 & 46\cdot8 & -6\cdot76 & & \\
 & 2\cdot4 & & & \\ \cline{2-2}
 & 49\cdot2 & & &
\end{array}$$

$$(4\cdot7)\ \ldots\ldots + 12 + 49\cdot2 - 6\cdot76 + \cdot276.$$

The interval between 4·7 and 5 is ·3; and, as the fraction $\frac{\cdot276}{6\cdot76}$

when increased by $\frac{3}{26}$ is less than ·3, we infer that the interval is still too wide. Advancing therefore by another unit, we find

$$(4·8) \ . \ . \ . \ 12 + 52·8 + 3·44 + ·104 \,;$$

so that 4·8 oversteps the roots, supposing them to be real. We have then to contract the interval (4·7, 4·8); and we shall find that either (4·75) or (4·76) will terminate the process: thus, employing (4·75), we have

$$(4·75) \ . \ . \ . \ 12 + 51 - 1·75 + ·0625.$$

The interval between this and the preceding transformation is ·05; and since

$$\frac{·0625}{1·75} + \frac{·104}{3·44} = ·06 \ . \ . \ .$$

we conclude that the two roots are imaginary.

It is obvious that the foregoing analysis, conducted, as it is, without any sure principle to guide us to a suitable transformation, is much inferior to the process by STURM's rule, both as respects simplicity and expedition.

We shall not attempt the analysis of the equation by the method of BUDAN: Mr. LOCKHART,* after determining the character of the roots by a peculiar process, adds, "If M. BUDAN's algorithm had been used, about thirty-eight transformations must have been employed."

These transformations, however, would no doubt be greatly reduced by introducing FOURIER's test as recommended at (147); or by narrowing BUDAN's intervals as there suggested.

(161.) It will be unnecessary to add to the foregoing examples in this place: further applications of the theorem of STURM will occur in next chapter, with additional illustrations of that of FOURIER, in connexion with the actual solution of equations. Sufficient has been already done, however, to unfold the peculiarities of the different methods of analysing an equation to which the last three chapters have been devoted. Of these it is obvious that that of STURM is the only one of which the practical difficulty does not increase with the proximity of the roots to one another; as it is

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* LOCKHART's Resolution of Equations, Oxford, 1837, page 28.

the only one which enables us to pronounce at once upon the character of an interval, however wide, without first inquiring whether, by rendering it narrower and narrower, the roots indicated in it can be actually separated or not. This gives a theoretical perfection to Sturm's theorem that all other methods want; but it is certainly counterbalanced by a practical disadvantage of no small moment; which is that, in general, it involves the same amount of numerical labour whether the equation submitted to it be of the easiest or of the most difficult character, and which labour increases solely with the magnitude of the coefficients and the degree of the equation. In the method of Fourier the only thing that very materially increases the tediousness of the operation is that of which Sturm's method is altogether independent—the proximity of the roots; and thus the labour is more nearly proportionate to the inherent difficulty of the case. It is on this account that the method of Fourier, modified as we have proposed in Chapter ix, will always be the more eligible when the degree of the equation is high, and the coefficients of the terms large numbers. Still it is an interesting and an important truth, that we *have* a method for analysing an equation of universal application, every step of which is characterized by unerring certainty; so simple in its principles that nothing beyond the theory of the common measure is requisite to comprehend it; and of which the only difficulties in the practice are merely those attendant upon common multiplication. The method, however, may be so modified as that, like that of Fourier, the length of its operations shall become proportionate to the proximity of the roots. The contrivance by which this is effected changes to a certain extent the character of the method; but by dispensing with the computation of the latter steps of the work, it often effects a considerable saving of numerical labour. We have fully explained and illustrated the nature of this modification in the introductory treatise on *Cubic and Biquadratic Equations;* where it is shown that, as far as equations of the fourth degree inclusive, all except the leading step of the operation for the common measure—which leading step may be executed as at p. 226 with great ease and rapidity—may be dispensed with. Up to equations of this order, at least, we consider Sturm's method, modified as there recommended, to be that which should always be employed —the modification being resorted to only when it is foreseen that large numbers will enter the closing steps of the ordinary process. In equations of the more advanced degrees it will be necessary, when large numbers occur, to check their increase by means of the abbreviations suggested at (156). The effect of these abbreviations will be seen in the next chapter, in the analysis of some equations of considerable difficulty.

By fairly exhibiting the actual labour attendant upon STURM's method, when applied to equations involving large numbers, and when the best expedients are adopted to economise the work, we shall place the student in a position to judge of its general practicability beyond certain limits: and to form his own conclusions as to the relative claims of the methods of STURM and FOURIER, beyond equations of the fourth degree, when large numbers are involved. In such cases we anticipate a decision in favour of the latter method, when improved and modified as already suggested. The labour of executing the operation for the common measure in the cases adverted to will, however, clearly show how greatly FOURIER's method is facilitated by disencumbering it of these lengthy appendages.

It will be seen that this reduction of the work, taken in connexion with the very efficient means of subdividing the doubtful intervals, that will be unfolded in next chapter, confers upon the method of FOURIER a practical value, to which it could lay but comparatively little claim, in the form under which it has hitherto been presented.

CHAPTER XII.

SOLUTION OF EQUATIONS OF THE HIGHER ORDERS.

(162.) THE method of approximating to the real roots of numerical equations to be discussed in the present chapter is that which was first proposed by Mr. HORNER, and published by him in the *Philosophical Transactions*, in the year 1819. It is a process of remarkable simplicity; consisting merely of a series of easy transformations, conducted according to the directions given at (71), and uniformly adopted in the preceding chapters, and blending with each other in a continuous course of recurring operations, by which the figures of each root are evolved one by one.

The general principles of this method have already been explained, with very considerable detail, in the introductory volume on *Cubic and Biquadratic Equations*. It will therefore be sufficient here briefly to describe, in symbolical terms, the several steps of the process, whatever be the degree of the equation: and then, in connexion with the examples to be given in illustration of it, to examine into the practical difficulties that may sometimes retard its operations; and to ascertain what are the cases of peculiarity, in which a reference to other principles and other considerations may be of advantage, either in adding to the facility of its steps, or in increasing the certainty of its conclusions. In this inquiry we shall find the researches of FOURIER, as delivered in the preceding chapters, of considerable service: the methods of FOURIER and HORNER may indeed be made mutually subservient to one another: the analysis of FOURIER may be expedited by the method of approximation of HORNER; and the ambiguities and uncertainties, that would occasionally accompany the method of HORNER, may always be removed by the rules and tests of FOURIER; or by others that will be investigated presently.

(163.) When the first figure r of one of the roots of the equation

$$A_n x^n + \ldots\ldots A_3 x^3 + A_2 x^2 + Ax + N = 0$$

is determined, it is easy to obtain the transformed equation

$$A'_n x'^n + \ldots\ldots A'_3 x'^3 + A'_2 x'_2 + A'x' + N' = 0,$$

involving the remaining portion of the root; and, as this portion forms one of the entire roots of the transformed, if the first figure of it be found, we shall have the second figure of the original root, and, by a repetition of the process of transformation, we shall get a new equation, involving the following figures of the root. The evolution of any root would, therefore, be effected, by finding the first figure by trial, or by a previous analysis, and diminishing the roots by it; then finding the first figure of this reduced root from the transformed equation, diminishing the roots by it, and so on till the proposed root be entirely evolved, or determined to any required number of decimals.

It is evident that, after the determination of the first figure, and thence of the transformed equation, we shall not be left to conjecture the value of the following figure; for, as in the case of cubic and biquadratic equations, so fully developed in the introductory volume, we may regard N′, when transposed to the right, as a dividend; and, if the true first figure of the root x' be r', we shall have so to determine r' that, when the dividend is divided by

$$A'_n r'^{n-1} + \ldots\ldots A'_3 r'^2 + A'_2 r' + A',$$

the quotient may be r'; and we are evidently assisted in this determination of r' by A′, the known portion of the true divisor. The influence of this *trial divisor* will indeed be readily foreseen, after what has been done in the work referred to.

When, by help of the trial-divisor, the new figure r' of the root is ascertained, and the divisor completed, we may proceed to the next transformation by diminishing the roots of the last transformed equation by r'; we shall thus have an equation of the form

$$A''_n x''^n + \ldots\ldots A''_3 x''^2 + A''_2 x'' = N'';$$

the first figure r'', in the root of which, must be such that, when N$''$ is divided by

$$A''_n r''^{n-1} + \; . \; . \; . \; . \; A''_3 r'' + A''_2,$$

the quotient must be r'': and, for discovering r'', we have the trial-divisor A''_2, which is previously known.

It is plain, therefore, that the determination of the several root figures, r, r', r'', &c. in succession, is effected by a continuous and uniform arithmetical process; the several trial-divisors A', A''_2, &c., all presenting themselves as they are wanted, in passing from one transformation to another.

(164.) These trial-divisors, it must be observed, although always valuable aids towards suggesting the successive figures of the root, must not be regarded as unerring guides in this respect: the influence of the preceding coefficients, in the transformation which the anticipated figure is to complete, should always be estimated and allowed for, just as in the common operation for the square root an estimate of the influence of the suggested figure upon the divisor always operates in determining that figure.

It is of importance to keep this in remembrance; since the trial-divisor, if left unchecked by the consideration adverted to, may remain widely at fault through several leading steps of the development.

It is further worthy of observation, that if two roots do not commence with the same figure, r, then, in approximating to the root whose first figure is r, the first transformed equation cannot have more than one root, whose leading figure is below r in the numerical scale; that one root being that to which our approximation is at the outset directed. For if the transformed equation had two roots (k, k') below r in the numerical scale, then the original equation would have had two roots $x = r + k$, and $x = r + k'$, commencing with the same figure. Consequently, in passing through the successive transformations, in our pursuit of the single root fixed upon for development, we may be quite sure that no ambiguity can ever arise from *two* roots of any transformed existing in the numerical scale below that of the root-figure last reached. Similar reasoning shows that when two roots commence with the same figures, and not more than two, the transformed equations can have no other roots but these—diminished by the common figures already determined—occupying places in the numerical scale below the last of these figures. And so on, whatever number of roots commence with the same figures.

In approximating to these nearly equal roots, the theorem of BUDAN will always apprize us when any of them are inadvertently overstepped; and when not only nearly equal, but also imaginary roots are indicated between the same limits, the criterion of FOURIER, as also others, to be hereafter investigated, will always enable us to determine whether two real roots have been overstepped, or only an indicator of two imaginary roots. And thus in all cases may we carry on the approximation without the slightest ambiguity or embarrassment.

We shall commence with an example or two of no more than the ordinary degree of difficulty, merely for the purpose of showing the general method of arranging the operation. This arrangement admits of some little variety: we shall give the two forms most to be recommended for compactness and ease of execution. The plan, according to which the entrance of unnecessary decimals into the several columns of the work is provided against, is that already so fully explained in the introductory treatise: the intention of it is to secure the greatest possible accuracy with the least possible expenditure of figures. A different mode of contraction was adopted by Mr. HORNER; in which we believe he has been followed by every expositor of his method; but we have always considered this mode of abridgment as involving the chief practical defect in the operation; since in some parts of the process useless figures are retained, and in other parts really effective ones dismissed. The consequence is, that the final decimal of the root can never be depended upon, and not unfrequently error sensibly affects both the last and the last but one. In the method of contraction here to be employed, accuracy is in general secured to the nearest unit in the last figure of the root; and we therefore consider the change to be a real practical improvement in the mode of working. As an illustration of the correctness with which all the roots of an equation may be computed under the guidance of this principle, reference may be made to the introductory volume, pp. 221, 231.

(165.) 1. One root of the equation

$$x^5+4x^4-2x^3+10x^2-2x-962=0$$

is found to lie between 3 and 4, required the development of it to seven or eight places of decimals.

```
1  4      -2         10          -2          962 (3·35484874
   3      21         57         201          597
   —      —          —          ━━━          ——— 1
   7      19         67         199          365
   3      30        147         642       299·14833
   —      —         ———         ——— 1     ————————— 2
  10      49        214         841        65·85167
   3      39        264      156·1611      59·87805
   —      —         ——— 1    ━━━━━━━━     ————————— 3
  13      88        478      997·1611       5·97362
   3      48       42·537    169·4514       4·92583
   —     ——— 1    ———————   ————————— 2   ————————— 4
  16     136      520·537   1166·6125       1·04779
   3       5·79    44·301     30·9484        ·98760
 ———— 1  ——————   ———————   ━━━━━━━━━     ————————— 5
19·3     141·79   564·838   1197·560|9        6019
   3       5·88    46·092     31·353          4940
 ————    ——————   ——————— 2 ————————— 3       ————
19·6     147·67   610·930   1228·914          1079
   3       5·97     8·037      2·544           988
 ————    ——————   ———————   ━━━━━━━━━         ————
19·9     153·64   618·96|7  1231·45|8           91
   3       6·06     8·09       2·54             86
 ————    —————— 2 ———————   ————————— 4         —
20·2     159·70   627·0|6   1234·00              5
   3       1·03     8·2         ·50              5
 ———— 2  ——————   ——————— 3 ━━━━━━━━━            —
2|0|·5   160·7|3  635·3     1234·5|0
           1         ·6         ·5
        ———————  ———————   ————————— 5
         161|·7  6|3|5|·9  1|2|3|5|·0
           1
        ———————
         1|6|3
```

2. One root of the equation

$$x^5 + 6x^4 - 10x^3 - 112x^2 - 207x = 110$$

is found to lie between 4 and 5: required the development of it.

<table>
<tr><td>1</td><td>6</td><td>−10</td><td>−112</td><td>−207</td><td>110</td><td>(4·46410161</td></tr>
<tr><td></td><td>4</td><td>40</td><td>120</td><td>32</td><td>−700</td><td></td></tr>
<tr><td></td><td>—</td><td>—</td><td>—</td><td>—</td><td>— 1</td><td></td></tr>
<tr><td></td><td>10</td><td>30</td><td>8</td><td>−175</td><td>810</td><td></td></tr>
<tr><td></td><td>4</td><td>56</td><td>344</td><td>1408</td><td>667·05984</td><td></td></tr>
<tr><td></td><td>—</td><td>—</td><td>—</td><td>— 1</td><td>— 2</td><td></td></tr>
<tr><td></td><td>14</td><td>86</td><td>352</td><td>1233</td><td>142·94016</td><td></td></tr>
<tr><td></td><td>4</td><td>72</td><td>632</td><td>434·6496</td><td>133·46395</td><td></td></tr>
<tr><td></td><td>—</td><td>—</td><td>— 1</td><td>—</td><td>— 3</td><td></td></tr>
<tr><td></td><td>18</td><td>158</td><td>984</td><td>1667·6496</td><td>9·47621</td><td></td></tr>
<tr><td></td><td>4</td><td>88</td><td>102·624</td><td>477·4144</td><td>9·24089</td><td></td></tr>
<tr><td></td><td>—</td><td>— 1</td><td>—</td><td>— 2</td><td>— 4</td><td></td></tr>
<tr><td></td><td>22</td><td>246</td><td>1086·624</td><td>2145·0640</td><td>·23532</td><td></td></tr>
<tr><td></td><td>4</td><td>10·56</td><td>106·912</td><td>79·3352</td><td>·23158</td><td></td></tr>
<tr><td></td><td>— 1</td><td>—</td><td>—</td><td>—</td><td>—</td><td></td></tr>
<tr><td></td><td>26·4</td><td>256·56</td><td>1193·536</td><td>2224·399|2</td><td>374</td><td></td></tr>
<tr><td></td><td>·4</td><td>10·72</td><td>111·264</td><td>80·389</td><td>232</td><td></td></tr>
<tr><td></td><td>—</td><td>—</td><td>— 2</td><td>— 3</td><td>—</td><td></td></tr>
<tr><td></td><td>26·8</td><td>267·28</td><td>1304·800</td><td>2304·788</td><td>142</td><td></td></tr>
<tr><td></td><td>·4</td><td>10·88</td><td>17·453</td><td>5·434</td><td>139</td><td></td></tr>
<tr><td></td><td>—</td><td>—</td><td>—</td><td>—</td><td>—</td><td></td></tr>
<tr><td></td><td>27·2</td><td>278·16</td><td>1322·25|3</td><td>2310·22|2</td><td>3</td><td></td></tr>
<tr><td></td><td>·4</td><td>11·04</td><td>17·56</td><td>5·44</td><td>2</td><td></td></tr>
<tr><td></td><td>—</td><td>— 2</td><td>—</td><td>— 4</td><td>—</td><td></td></tr>
<tr><td></td><td>27·6</td><td>289·20</td><td>1339·8|1</td><td>2315·66</td><td>1</td><td></td></tr>
<tr><td></td><td>·4</td><td>1·68</td><td>17·6</td><td>14</td><td></td><td></td></tr>
<tr><td></td><td>— 2</td><td>—</td><td>— 3</td><td>—</td><td></td><td></td></tr>
<tr><td></td><td>28·|0</td><td>290·8|8</td><td>1357·4</td><td>2315·8|0</td><td></td><td></td></tr>
<tr><td></td><td></td><td>1·7</td><td>1·2</td><td>1</td><td></td><td></td></tr>
<tr><td></td><td></td><td>—</td><td>—</td><td>—</td><td></td><td></td></tr>
<tr><td></td><td></td><td>292·|6</td><td>1358·|6</td><td>2|3|1|5·9</td><td></td><td></td></tr>
<tr><td></td><td></td><td>1</td><td>1</td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td>—</td><td>—</td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td>|2|9|4</td><td>1|3|6|0</td><td></td><td></td><td></td></tr>
</table>

In the two preceding examples the steps of the general investigation have been rigidly conformed to, and the arrangement which the operation thus takes is that which is in general to be preferred on the ground of simplicity; it may however be varied so as to assume a form of greater compactness, and which may be the more convenient form to give to the work, when, from the magnitude of the coefficients, it would otherwise spread over too large a space. In the chapter on cubic equations in the introductory volume (Chap. III), we have shown how to reduce certain portions of the work when the leading coefficient is unity: a similar reduction may be applied to the more advanced equations. This reduction chiefly affects the first column of figures. But Mr. HORNER so arranged the process, that the subsequent columns up to that which supplies the divisors inclusive, were diminished in length: we shall exhibit the effect of both these principles of abbreviation in the solution of the following equation of the fourth degree, of which the roots have been developed, as in the foregoing example, at page 211 of the introductory treatise, and with which the operation that follows may be compared :—

3. Let the equation be

$$x^4 + 3x^3 + 2x^2 + 6x = 148{\cdot}6,$$

of which one root lies between 2 and 3.

```
                                                      3
                    2               6           148·6 (2·734400
      5 . . . .    10              24            60
                   —               ▬             ——— 1
                   12              30            88·6
                   14              28            82·9731
                   18              — 1           ————— 2
                   — 1             82             5·6269
                   44              36·533         4·8977
   11·7 . . . .     8·19           ━━━━━━         ———— 3
                   ————           118·533          ·7292
                   52·19            6·076          ·6626
                    8·68           ————— 2         ————
                    9·17          161·142            666
                   ———— 2           2·114            662
                   70·04           ━━━━━━            ——
   13|·8 . . .       ·41          163·25|6             4
                   ————                1
                   70·4|5          ———— 3
                     ·4           165·38
                   ——— 3              ·28
                   70·9            ━━━━━
                                1|6|5|·6|6
```

In this process the first column on the left is formed, like that in the operation for cubics ; the multiplier being the index of the degree, in this case 4. In the second column each addend is formed from the immediately preceding one, by adding to it the square of the last root figure. The addends in the third column are as usual formed by multiplying those of the preceding column by the last root figure ; and every *trial* divisor is the sum of the *three* numbers above it, as in cubic equations.

Precisely in this way may the first and second columns always be formed, whatever be the degree of the equation ; but when there are intermediate columns of work, between the second and that which supplies the trial-divisors, as must always happen when the equation is above the fourth degree, the addends in these must each be formed, in the abridged method, by multiplying the corresponding addend in the preceding column by the last root figure, and, *at the same time*, taking in the addend immediately above ; so that every addend in these intermediate columns helps to form the one immediately under it, by being incorporated with the product, which in the unabridged process, is carried from column to column. The following example, taken from Mr. HORNER's paper in the *Philosophical Transactions*, with no further alteration than that which concerns the decimal contractions, the abbreviation of the first column, and the preservation of the trial-divisors, will sufficiently illustrate our meaning :—

4. Required a root of the equation

$$x^5 + 12x^4 + 59x^3 + 150x^2 + 201x = 207.$$

```
                 59        150            201                12
       12·6...  7·56       39·936         113·9616        207(    ·638605803327
                ————       ——————         ————————        18·897696
                66·56      189·936        314·9616        ——————————— 1
                 7·92       44·688         26·8128         18·02304
                 8·28       49·656        ———————— 1       13·9304119743
                 8·64      ——————— 1      455·7360        ————————————— 2
                ————— 1    284·280          8·61106581      4·0926280257
                91·40        2·755527     ————————————      3·8031030022
    1|5|·0|3..   ·4509     ——————————     464·34706581    ————————————— 3
                ———————    287·035527         8307243        ·2895250235
                91·8509      2·769081     ———————————— 2     ·2867504754
                  ·4518      2·782662     473·04120405     ————————————— 4
                  ·4527    ——————————— 2    2·34667123         27745481
                  ·4536    292·587270     ————————————         23904795
                ——————— 2     ·746634     475·3878752|8      ————————— 5
                93·2090    ———————————       ·0059807           3840686
                  ·1202    293·33390|4    ———————————— 3        3824781
                ———————       ·7475|9     477·7405272          ————————
                93·329|2      ·7485|5        ·1769318             15905
                  ·120     ——————————— 3  ————————————            14343
                  ·120     294·830|0      477·917459|0            —————
                  ·120          56|4               34              1562
                ——————— 3  ——————————— 4  ———————————— 4           1434
                9|4        294·886|4      478·094425               ————
                                5|6             1475                128
                                5|6       ————————————               96
                            —————————— 5  478·0959|00                ——
                            2|9|5         ——————————— 5              32
                                          478·0974                   33
                                                 2                   —
                                          ————————————
                                          4|7|8·|0|9|7|6
```

Hence one root of the equation is ·638605803327.

The only objection that could be brought against this mode of arranging the numerical process, is, that in the third column of the work, such arrangement requires us to perform the operations of multiplication and addition simultaneously. But, in the case of a biquadratic equation, no such objection can apply, and, consequently, the foregoing arrangement of the work may perhaps be preferred on the score of practical facility.

(166.) We have observed above, that in the foregoing example, from Mr. Horner's paper, we have slightly modified the process, and it ought to be mentioned that, in so doing, we have, in fact, increased the length of the operation. This has arisen from our having *actually exhibited* the trial-divisor derivable at every step from the last root-figure.

By dispensing, however, with this, and merely writing under the true divisor the addend which is due to the formation of the next trial-divisor, we may, without actually performing the addition, readily foresee what the leading figures in that divisor would be, and thence discover the new figure of the root. In the foregoing example, no inconvenience can arise from this suppression of the trial-divisors, even from the commencement of the operation, on account of the smallness of the addends in the divisor column. But, where the addends are of considerable influence, we think it preferable always to exhibit the trial-divisors. The work of the last example stands in Mr. Horner's paper as below; and it is easy to see, from the circumstances just adverted to, that after the first step, every true divisor may be safely taken as a trial-divisor for the next figure of the root, for the addends recede sufficiently far to the right to allow the leading figures of the divisors to continue constant. After a step or two, such will indeed usually be the case when but a single root lies in the interval under examination; but not when two or more roots so lie, nor even when indicators of imaginary roots occupy the same interval, and occur in the divisor column, because this column will then tend to zero as well as the final column. In the case of nearly equal roots this is plain, since when roots accurately equal enter, diminishing by one of those roots reduces as many of the advanced columns accurately to zero as there are repetitions of the root, page 140. And in the case of imaginary roots the statement is authorized by (122). Moreover, in all cases of difficulty, where an appeal to the theorems of Budan and Fourier becomes necessary, the entire row of transformed coefficients should always be exhibited. Nevertheless, the compact form given to the work in the following arrangement, may recommend it to adoption in cases where the length of the operation is the only difficulty.

```
 12      59         150            201                 207·00000 (·638605803327
   6      75 6       3993 6        1139616             188·97696
 ----    -----      ---------      ---------           ---------
 126     665 6      18993 6        3149616              18·0230400000
  24      79 2       4468 8        1407744              13·9304119743
    3     82 8       4965 6          861106581          -------------
 ----     86 4        275 55 27    -----------           4·0926280257
 1503      4 509    -----------    4643470 6581          3·8031030021
|  12    -------    28703 55 27      86941382|4         -------------
|----    918 509      276 90 81      23466712|16         ·2895250236
|15        4 518      278 26|62    -------------         ·2867504754
|          4 527       74 66|32    475387875 2|66       -------------
           4|536    ---------|-      2352651|9 5 2        27745482
           1|20     29333 39|02        176931|8 4         23904795
        |-------       74 75|92    ----------|-|-       -----------
        |933 |29       74|85|50    477917459|0|6           3840687
        |  3 |60        5|64          1769|65|6|8          3824781
        |--- |      -----|--             1475|          -----------
        |94  |      2|9488|64      ----------|--            15906
        |    |       |   11|28     47809 59|00             14343
                     |----|                1|4|75         --------
                  |  2|95 |                   |23           1563
                                   -----------|--           1434
                                   4|7|8|0|9 7|6 1          ------
                                              |2 3           129
                                                              96
                                                             ---
                                                              33
                                                              33
                                                             ---
```

(167.) In each of the preceding examples the root developed is the only one lying between the same two consecutive numbers in the arithmetical scale, that is, no second root exists having the same leading figures. When such is the case we shall usually find, as here, that the trial-divisors become efficient in suggesting the successive figures of the root from a very early stage of the approximation. But, as already observed, this will not happen under different circumstances. For let a be an approximate value of a root of $f(x) = 0$, that is, a number consisting of one or more of the leading figures of that root; and let the remaining portion of the root be h. Then

$$f(a+h) = f(a) + f_1(a)h + \frac{f_2(a)}{2}h^2 + \frac{f_3(a)}{2\cdot 3}h^3 + \&\text{c.} = 0 \ldots .[1]$$

$$\therefore -\frac{f(a)}{f_1(a)} = h + \frac{f_2(a)}{2f_1(a)}h^2 + \frac{f_3(a)}{2\cdot 3f_1(a)}h^3 + \&\text{c.} = \frac{\text{N}'}{f_1(a)}$$

Now as, by the increase of a, h diminishes and becomes less than unit, the terms involving h^2, h^3, &c. diminish also; and if the coefficients of these also diminish with h, or even slightly increase, these terms must at length diminish more rapidly than h itself, and become with respect to it very small. Consequently the quotient $\frac{\text{N}'}{f_1(a)}$ will then give one or more of the leading figures of h: that is, the trial-divisor $f_1(a)$ of N′ will be fully efficient in determining the next figure of the root after a.

But if $f_1(a)$ also regularly and rapidly diminish with h, whilst $f_2(a)$ does not, then it is plain that the second term $\frac{f_2(a)}{2f_1(a)}h^2$, of the preceding development, as well as the first, must form an important part of the whole; so that $\frac{\text{N}'}{f_1(a)}$ will no longer give the leading figure of h but the leading figure of $h + \frac{f_2(a)}{2f_1(a)}h^2$. What we have now described must evidently happen if the equation $f_1(x) = 0$ has a root with several leading figures the same as those of the root $a + h$ of $f(x) = 0$; or if this latter equation have two nearly equal roots in the interval under examination (98).

If there be three such roots the trial-divisor will be still more at fault: for then $f_1(a)$ will diminish as h^2, and $f_2(a)$ will diminish as h: so that the third term as well as the second being thus of the same order of magnitude as the first, must be retained with it, as an approximation to the whole series. Thus, when nearly equal roots enter the equation, the difficulty of separating them will be very considerable if we depend entirely upon the trial-divisors for the discovery of the successive figures of the root. An example of the total inadequacy of these may be seen at page 227 of the introductory treatise.

(168.) In order to remedy this defect let us take the limiting equation of [1]: then we shall have

$$f_1(a) + f_2(a)h + \frac{f_3(a)}{2}h^2 + \&c. = 0 \;.\;.\;.\;.\; [2]$$

This would be accurately true in conjunction with [1] provided two roots, h, of [1] were accurately equal (97); and it must approach nearer to the truth as the roots approach nearer to equality. The leading figure of the root h of this latter equation is given by $-\frac{f_1(a)}{f_2(a)}$, agreeably to the general principle.

If three roots approach to equality, or continue the same for several decimals, then, continuing the derivation, we have for approximating to h the equation

$$f_2(a) + f_3(a)h + \&c. = 0 \;.\;.\;.\;.\; [3]$$

and consequently for the leading figure of the repeated root $-\frac{f_2(a)}{f_3(a)}$. And so on.

The transformed coefficients

$$-\,\mathrm{N}',\quad \mathrm{A}',\quad \mathrm{A}'_2,\quad \mathrm{A}'_3,\quad \mathrm{A}'_4,\;.\;.\;.\;.\;.\;.\;.\;\mathrm{A}'_n.$$

furnished by diminishing the roots by a, in HORNER's process, are (116)

$$f(a),\; f_1(a),\; \frac{f_2(a)}{2},\; \frac{f_3(a)}{2.3},\; \frac{f_4(a)}{2.3.4},\;.\;.\;.\;.\; \frac{f_n(a)}{2.3.4\ldots_n}$$

Hence, according as a single root, two roots, three roots, &c. lie in the same narrow interval should we employ, to obtain the leading figure of the root of the transformed equation, one or other of the expressions

$$\frac{N'}{A'}, -\frac{A'}{2A'_2}, -\frac{A'_2}{3A'_3}, -\frac{A'_3}{4A'_4}, \ldots -\frac{A'_{n-1}}{nA'_n}.$$

These conclusions supply a useful theorem for approximating to roots very nearly equal: cases of this kind are reduced by it to the same simplicity as the common case in which only a single root lies in the interval within which our search is directed.

(169.) If we write the equations [1], [2], [3], &c. in the usual notation, the preceding conclusions may perhaps present themselves with greater clearness: for we thus have

$$f(x) = A'_n x^n + \ldots A'_4 x^4 + A'_3 x^3 + A'_2 x^2 + A'x - N' = 0 \ldots [1]$$

$$f_1(x) = nA'_n x^{n-1} + \ldots 4A'_4 x^3 + 3A'_3 x^2 + 2A'_2 x + A' = 0 \ldots [2]$$

$$f_2(x) = n(n-1)A'_n x^{n-2} + \ldots 3.4A'_4 x^2 + 2.3A'_3 x + 2A'_2 = 0 \ldots [3]$$

$$f_3(x) = n(n-1)(n-2)A'_n x^{n-3} + \ldots 2.3.4A'_4 x + 2.3A'_3 = 0 \ldots [4]$$

&c. &c. &c.

where for two, three, four, &c. nearly equal roots the corresponding approximate equations are

$$f_1(x) = 0, \quad f_2(x) = 0, \quad f_3(x) = 0, \quad \&c.$$

and the leading figure, common to these roots, is by the general principle,

$$-\frac{A'}{2A'_2}, -\frac{A'_2}{3A'_3}, -\frac{A'_3}{4A'_4}, \&c.$$

It is plain that the process for computing the function $f(x)$ for any value of x supplies, in its progress, the computations of the subordinate functions $f_1(x)$, $f_2(x)$, $f_3(x)$, &c. for the same value; so that if there are m nearly equal roots, we approximate to the single root of the equation $f_{m-1}(x) = 0$ having the same leading figures in common with them, in the act of approximating to the proposed root in $f(x) = 0$.

We may consider this subject under a different aspect:—We know that when a single root only lies in a given interval, we speedily reach a transformed equation, which, by neglecting all the terms except the last two, $f_1(a)x - \mathrm{N}'$, furnishes a simple equation $f_1(a)x - \mathrm{N}' = 0$, from which a correct figure or two of the sought root may be determined. This simple equation becomes more and more effective as the approximation advances; till at length it, alone, is found sufficient to supply as many correct figures as are necessary to complete the stipulated amount of decimals in the root, and thus to close the operation. It is to this simple equation that the successive transformations tend in approximating to a single isolated root.

But when instead of a distinct individual root we approach simultaneously to a pair of roots, then of course the process in like manner tends towards a quadratic equation, the leading coefficient of which becomes less and less influenced by the addends to it accruing from the preceding columns of the work. Both roots of this quadratic, commencing, by hypothesis, with the same figure, this common figure will be found by dividing the second coefficient, with changed sign, by twice the first (see *Introductory Treatise*, p. 143). When three roots are thus simultaneously approximated to, the operation tends to merge into a cubic equation involving those roots diminished by the preceding root figures: and as by hypothesis these three roots commence with the same figure, it is plain that one third of their sum will commence with that figure. Hence, the common figure will be obtained by dividing the second coefficient of this approximate cubic equation, taken with changed sign, by three times the first, or simply minus the third by the second: and so on. And these are the conclusions established above.

We thus see that the last two coefficients of the equation of the $n-(m-1)$th degree which furnishes an approximation to a single one of the contiguous roots, are the first two in the equation of the mth degree which is the approximate equation involving all these m roots. It is interesting to notice that in the case of the approximate quadratic the leading figure common to both roots will otherwise be obtained by dividing twice its final term (N') by the preceding coefficient A', as is obvious. In the case of the approximate cubic the leading figure common to the three roots will be found by dividing three times N' by the preceding coefficient A': and so on.

(170.) We at once see, from these principles, how tedious it would be, when roots proceed together for several places of figures, to reach the place at which they separate by seeking each new figure through a set of trial-transformations commencing with $\frac{N'}{A'}$, or which is the same thing, with $\frac{f(a)}{f_1(a)}$, as in the method of FOURIER, and proceeding onwards from this, through all the intermediate transformations, till we arrive at twice, three times, four times, &c. this quantity, according to the number of roots in the interval. By preliminary transformations FOURIER finally renders the interval so narrow as to comprehend but two roots: if others are close to these their preliminary separation is very laborious: but even when the interval is made to comprehend only two roots proceeding together for five or six places of decimals, the separation of them, by commencing every new set of transformations with a trial-number only half the value it ought to be, thus rendering on the average three or four trial transformations necessary for every figure of the root, we say that even in this simplest case of contiguous roots the separation is so laborious as to justify the observations upon its impracticability which we ventured to make at (131).

(171.) From what has now been shown, it appears that when roots have leading figures in common, the proper expressions for suggesting the true figures up to the place at which the roots separate are as follow, the denominators of these expressions being the trial-divisors in reference to the numerators:

TWO ROOTS	THREE ROOTS	FOUR ROOTS	&c.
$-\frac{A'}{2A'_2}$,	$-\frac{A'_2}{3A'_3}$ or $-\frac{A'}{A'_2}$,	$-\frac{A'_3}{4A'_4}$,	&c. . . . . (c)

These, therefore, are to be employed in the cases under consideration, till a root becomes detached, when the ordinary trial-divisor may be brought into operation to carry forward the approximation of the single root thus separated. We shall generally be apprised of our arrival at this point by a discrepance between the figure given by the suitable expression above, and that given by the corresponding expression in the series

$$\frac{2N'}{A'}, \qquad \frac{3N'}{A'}, \qquad \frac{4N'}{A'}, \qquad \&c. \;.\;.\;.\;. \;(\text{D})$$

But by continuing to pursue the root of $f_{m-1}(x) = 0$, whose figures are given one after another by the proper expression in the former series, as long as these figures are the same as those of the roots to be separated, we shall of course always of necessity effect the separation desired, and be informed of it by the loss of variation in the series of transformed coefficients, agreeably to the theorem of Budan. It is true that this theorem, as well as the discrepance adverted to above, may indicate the separation of a group of roots: but as we should then treat each group by itself, we may here suppose a single root to detach itself—the least of the group.

Now when the separation is accomplished it will be advisable to pursue this smallest of the nearly equal roots, to at least one figure beyond that at which the separation takes place; and to employ the resulting transformation for the nearly equal roots that still remain, remembering that these will present themselves each diminished by the figure by which the preceding approximation was extended. For of the roots of this transformed equation, *one* will commence with *zero*, in consequence of the advanced step recommended, and the others will commence with a common figure, supposing the common figures of these to be not yet exhausted. Hence, as our approximate equation involving the roots in question, and of which the first coefficient is A_m, and the second A_{m-1}, has the sum of its m roots expressed by $\frac{A'_{m-1}}{A'_m}$; and since the leading figure of one of these roots is 0, and that of the others the same figure, it follows that $\frac{A'_{m-1}}{(m-1)A'_m}$, or this expression minus 1, will furnish that figure. And thus the common leading figure of the $m-1$ roots still remaining unseparated, may be easily discovered, and then the next root separated by the process above described, the trial-divisors, after the transformation by this leading figure, being advanced to the column next adjacent on the right, to that which supplied the divisors at first. But if after the separation of the first root, those that remain do not, as supposed, commence with the same figure, then the figure, determined conformably to that supposition, will either be the first of the root next in order, or will effect a new separation: in the former case, the figure increased by unit will effect the separation. And in this uniform manner are we to proceed till all the roots are separated and severally developed as far as necessary.

All that it is requisite to attend to in seeking the separation of roots one after another, which have several leading figures in common, may be summarily stated in the following precepts:—

(172.) 1. Find the leading figure common to all the nearly equal roots by a previous analysis of the equation, and effect with it the usual transformation.

2. Find the next figure common to the roots by the proper expression (C); the first being used if there be but two roots, the second if three, and so on. The figure thus determined will be the same as that suggested by the corresponding expression (D), as soon as (C) becomes effective for the root of $f_{m-1}(x) = 0$.

3. Continue to determine the successive common figures in this way, either till a change of sign in N′ informs us that a root has separated—which will of course be the least root, supposing all to have been rendered positive, or till a discrepance between the figures determined by (C) and (D) announces a separation. If several roots separate at once, we must deal in this way with each distinct group, till the least in that group detaches itself.

4. By aid of the ordinary trial-divisor, extend the approximation towards this least root one figure further, and employ the resulting transformed equation for the separation of another root.

5. To find the leading figure common to the remaining roots take the multiple of the divisor in (C), hitherto employed, a unit less; and employ the expression, thus modified, to furnish the first transformation; and then proceed anew as above, using, for future figures of the roots still unseparated, the expression in (C) immediately preceding the one before employed, till another root separates: this root increased by the figures of the former root, will exhibit a second root of the proposed equation. The supposed common leading figure will, *in all cases*, either be the true figure of the next root—or it will effect a separation—or it will exceed the first figure of the remotest root by 1.

Or, instead of being guided by the last two precepts, we may proceed as follows. Having separated, as above directed, the least root of the group in the interval within which our approximations are confined, we may take the transformed equation to which this separation is due; and remembering that our operations are now to be confined within the narrower interval comprising only the remaining roots of the group, after these have been diminished by the figures resulting from the former set of transformations, we are to direct our approximation towards the root of $f_{m-2}(x) = 0$, lying in the interval, just as at first it was directed towards the root of $f_{m-1}(x) = 0$. The proper expression suggestive of the root-figures is that one of (c) which precedes that before employed. This expression will continue to suggest the correct figures of the root of $f_{m-2}(x) = 0$, so long as these are the same as the figures in the roots still unseparated. A discrepance here will, as before, indicate that the figures or that figure place; have ceased to concur; and the true root-figure will effect a second separation. And so on till the analysis is completed.

(173.) We have only one more particular to notice in reference to the general theory of the trial-divisors:—It is, that in the earlier steps of our approximation to a root it will sometimes happen that the absolute number, which we seek to exhaust, will, instead of continually diminishing as true figures of the root become determined, increase and diminish alternately till a point is reached where all oscillations of this kind cease and the absolute number tends progressively to zero. A reference to the values assumed by N′ in the leading steps of the approximation at pages 181, 218 of the treatise on *Cubic and Biquadratic Equations* will exemplify these circumstances. The cause of this peculiarity may be easily discovered from an examination of the general expression for N′ at page 259: for from that expression we have

$$N' = f_1(a) \left\{ h + \frac{f_2(a)}{2f_1(a)} h^2 + \&c. \right\}$$

which will uniformly diminish with h provided neither $f_1(a)$ nor $f_2(a)$ tends to zero—disregarding for the present the functions included under the &c. The consequences of $f_1(a)$ tending to zero with $f(a)$ we have already adverted to: but if $f_2(a)$ alone tend to zero, then, as h diminishes, the coefficient of h^2 also diminishes; and if this coefficient pass through zero, it must afterwards increase, with changed sign.

It is obvious that, at an early stage of the approximation, before h has become very small, this change in the term referred to, from additive to subtractive, or vice versâ, may have sufficient influence upon N′ to produce the effect mentioned. If several such changes occur during the process—that is, if several roots of $f_2(x) = 0$ are passed over in our approximation to the single root of $f(x) = 0$, then several of these oscillations of N′ may take place. It is plain that similar reasoning will apply to the more advanced terms of the above general expression for N′; so that the peculiarities noticed, arise from derived functions vanishing, and reappearing with changed signs, during the development of a root of the primitive function. These fluctuations, however, become insensible when h has reached a certain small value.

We shall now proceed to illustrate what has been established in these latter articles by two examples of peculiar difficulty. We shall prepare for the development of the roots by an actual analysis of the equation by the method of STURM, which we shall at first present without any abbreviation.*

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* These equations were publicly proposed for solution by Mr. LOCKHART, a gentleman to whom we have been under obligations upon former occasions; and who has furnished many other examples admirably adapted to try the powers of the modern methods of solution. The laborious computations involved in the analysis and solution of the equations in the text were executed by two very promising young mathematicians—members of the senior mathematical class in Belfast College—Mr. WILLIAM FINLAY of Belfast, and Mr. SMYLIE ROBSON of Tobermore, in the county of Derry. The calculations of Mr. FINLAY are, for distinction, marked [F]; those of Mr. ROBSON with [R].

(174.) Required the analysis and solution of the equation

$$x^5 + 173x^4 + 2356x^3 + 10468x^2 - 14101x + 4183 = 0.$$

The function X_1 is $5x^4 + 692x^3 + 7068x^2 + 20936x - 14101$

$$\begin{array}{r|l}
5 & +\ 692 + 7068 + 20936 - 14101 \\
173 & -\ 4712 - 31404 + 56404 - 20915 \\
\hline
\end{array}$$

$$\begin{array}{r}
119716 + 1222764 + 3621928 - 2439473 \\
-23560 - 157020 + 282020 - 104575 \\
\hline
96156 + 1065744 + 3903948 - 2544048
\end{array} \qquad \text{[F.]}$$

or, by dividing all the terms by the common factor 4 × 9, we have

$$2671 + 29604 + 108443 - 70668$$

$$\begin{array}{c}
2671 - 29604 - 108443 + 70668 \\
\times \\
5 + 692 + 7068 + 20936 - 14101
\end{array}$$

$$\begin{array}{r|l}
1700312 & -\ 50336036448 - 184386934216 + 120157648416 \\
13355 & -\ 1448256265 + 943771140 \\
7134241 & \ \ 50424815388 + 149362469576 - 100599932341 \\
\hline
 & 1359477325 + 34080693500 - 19557716075
\end{array}$$

dividing all the terms by the common factor 5^2, we have

$$54379093 + 1363227740 - 782308643$$

$$\begin{array}{c}
54379093 - 1363227740 + 782308643 \\
\times \\
2671 + 29604 + 108443 - 70668
\end{array}$$

$$\begin{array}{r|l}
-2031342624368 & 27691826149828575 68320 - 1589136891937388812624 \\
145246557403 & 1136276372223625 34129 \\
2957085755502649 & 3206752505839737 65507 - 2089713361698611 99532 \\
\hline
 & -3203485502789193867956 + 1798108228107250012156
\end{array}$$

or, dividing by the common factor 1227089452, we have

$$-2610637307303 + 1465344050653$$

$$\begin{array}{c}
-2610637307303 - 1465344050653 \\
\times \\
54379093 + 1363227740 - 782308643
\end{array}$$

$$\begin{array}{r|r}
-3638577276801810382949 & 5331767565402726835634120783515697 \\
6815427150282258457133809 & -5331767565402670680575623788211187 \\
\hline
 & -5615505849699530 4510
\end{array}$$

Hence, STURM'S functions are,

$$X\ = x^5 + 173x^4 + 2356x^3 + 10468x^2 - 14101x + 4183$$

$$X_1 = 5x^4 + 692x^3 + 7068x^2 + 20936x - 14101$$

$$X_2 = 2671x^3 + 29604x^2 + 108443x - 70668$$

$$X_3 = 54379093x^2 + 1363227740x - 782308643$$

$$X_4 = -2610637307303x + 1465344050653$$

$$X_5 = -5615505849699530 4510$$

As the signs of the leading terms of these functions present one variation, we infer that the equation has a pair of imaginary,a nd consequently three real, roots: and as the signs of the final terms present two variations more than those of the leading terms it follows, page 216, that two of these roots are positive, and consequently one negative. The situations of these are found as follows:

	X	X_1	X_2	X_3	X_4	X_5	
For $x = -200$ . . .	$-$	$+$	$-$	$+$	$+$	$-$	*four variations.*
-100 . . .	$+$	$-$	$-$	$+$	$+$	$-$	*three* ,,
0 . . .	$+$	$-$	$-$	$-$	$+$	$-$	
1 . . .	$+$	$+$	$+$	$+$	$-$	$-$	*one* ,,

In developing the negative root we shall convert it into positive by changing the alternate signs of the equation.

Development of the root [150, 160.]

1	−173	2356	− 10468	−14101	4183	158·5612971
	100	−7300	−494400	−50486800	−5050090100	
	− 73	−4944	−504868	−50500901	5050094283 **1**	
	100	2700	−224400	−72926800	1120194950	
	27	−2244	−729268	−123427701 **1**	3929899333 **2**	
	100	12700	1045600	145831600	3611123672	[F.]
	127	10456	316332 **1**	22403899	318775661 **3**	
	100	22700	2600300	329221600	283510132	
	227	33156 **1**	2916632	351625499 **2**	35265529 **4**	
	100	18850	3667800	99764960	34518109	
	327 **1**	52006	6584432	451390459	747420 **5**	
	50	21350	4860300	108275680	576209	
	377	73356	11444732 **2**	559666139 **3**	171211 **6**	
	50	23850	1025888	7354124	115245	
	427	97206	12470620	56702026|3	55966	
	50	26350	1063840	738994	51861	
	477	123556 **2**	13534460	57441020 **4**	4105	
	50	4680	1102304	89162	4034	
	527	128236	14636764 **3**	5753018|2	71	
	50	4744	71484	8921	58	
	577 **2**	132980	1470824|8	5761939 **5**	13	
	8	4808	7164	149		
	585	137788	147798|9	576208|8		
	8	4872	718	15		
	593	142660 **3**	148517 **4**	576224 **6**		
	8	309	86	3		
	601	142969	14860|3	57|6|2|2|7		
	8	31	9			
	609	1432|8	1|4|8|6|9			
	8	3				
	6|1|7 **3**	1|4|3|6				

Development of the two roots [0,1].

A_5	A_4	A_3	A_2	A	N	
1	173	2356	10468	−14101	−4183	|·5612971
	·5	86·75	1221·375	5844·6875	−4128·15625 **1**	
	173·5	2442·75	11689·375	−8256·3125	−54·84375	
	·5	87·00	1264·875	6477·1250 **1**	−54·8189027424 **2**	
	174·0	2529·75	12954·250	−1779·1875	−248472576	
	·5	87·25	1308·500 **1**	865·53912096	−2354110341 **3**	
	174·5	2617·00	14262·750	−913·64837904	−130615419	
	·5	87·50 **1**	162·902016	875·35117584 **2**	−116596558 **4**	
	175·0	2704·50	14425·652016	−38·29720320	−14018861	
	·5 **1**	10·5336	163·534248	14·756099792	−13937178 **5**	
	175·5	2715·0336	14589·186264	−23·54110340|8	−81683	
	·06	10·5372	164·166696 **2**	14·75884680 **3**	−81544 **6**	
	175·56	2725·5708	14753·352960	−8·78225661	−139	
	6	10·5408	2·7468318	2·95242869	−117	
	175·62	2736·1116	14756·099791|8	−5·8298279|2	−22 **7**	
	6	10·5444 **2**	2·747008	2·9525386 **4**		
	175·68	2746·6560	14758·84680|0	−2·8772893		
	6	·17580	2·74718 **3**	1·3287140		
	175·74	2746·8318|0	14761·59398	−1·548575|3		[F.]
	6 **2**	·1758	·54948	1·328736 **5**		
	175·80	2747·007|6	14762·1434|6	−·219839		
	·001	·176	·5495	·103348		
	1|7|5|·8|0|1	2747·18|4	14762·693|0	−·11649|1		
		·18 **3**	·549 **4**	·10335 **6**		
		2747·36	14763·242	— 1314		
		4	·247	147		
		2|7|4|7|·4|0	14763·48|9	−116|7		
			·24	15 **7**		
			14763·7|3	101		
			·2			
			147|6|4|·0 **5**			
		14|7|6	−1314	−139	|·0000007	
			1033	−197		
			−28|1	58		
			103			
			75			

The trial-divisors whence the several figures of the root in the preceding development are obtained, are all supplied by the column A_2; the column A supplying the corresponding dividend. Thus :—The general form for the root figure is $-\frac{A'}{2A'_2}$, agreeably to the precept at p. 263 ; so that when the transformation **1** is obtained the fraction $\frac{177 \ldots}{2 \times 14 \ldots}$ supplies the corresponding root-figure 6. In like manner, from the transformation **2**, we get for the next figure $\frac{382 \ldots}{2 \times 14 \ldots} = 1$; and so on, the trial-divisor being constantly 29. The place at which the roots separate is determined as soon as there is a discrepance between $-\frac{A'}{2\,A'_2}$, and $\frac{2N'}{A}$ as stated at(171).

This discrepance in the above operation does not appear till the transformation **6** is reached; when we have

$$\frac{1314}{2 \times 147\ldots} = 4, \text{ and } \frac{2 \times 139}{131} = 2.$$

Hence the roots separate after the 7. Carrying on the approximation to the least root one figure beyond this, by aid of the ordinary trial-divisor — 1314, we find 1 for the seventh decimal. For 2 the absolute number in **6** becomes plus, and continues so for all numbers up to 7 inclusive; but for 8 it is again minus. Hence 7 is the seventh decimal of the other root. But if, agreeably to the precept 5 at p. 265, we had taken the transformation **7** and had divided — A' by A'_2, that is, 102 by 14|7, we should have got 6 ; which increased by the advanced figure 1 of last root gives 7 for the corresponding figure of the second root. Thus the roots are

·5612971 . . ., and ·5612977 . . .

(175.) In the following solution of the foregoing equation the analysis is conducted differently in the final step, and the subsequent developments are according to the more compact arrangement exhibited at (165).

$$x^5 + 173x^4 + 2356x^3 + 10468x^2 - 14101x + 4183 = 0$$

$$5x^4 + 692x^3 + 7068x^2 + 20936x \quad - 14101$$

5 | +692 +7068 +20936 −14101

173 | −4712 −31404 +56404 −20915

119716 + 1222764 + 3621928 − 2439473

−23560 −157020 +282020 −104575

96156 + 1065744 + 3903948 − 2544048

or ÷ 36

[R.]

2671 +29604 +108443 −70668

2671 − 29604 −108443 +70668

5 +692 +7068 + 20936 − 14101

1700312 | −50336036448 − 184386934216 + 120157648416

13355 | −1448256265 +943771140

7134241 | 50424815388 + 149362469576 − 100599932341

1359477325 +34080693500 −19557716075

or ÷ 25

54379093 +1363227740 −782308643

54379093 − 1363227740 +782308643

2671 +29604 +108443 − 70668

−2031342624368 | 2769182614982857568320 − 1589136891937388812624

145246557403 | 1136276372223625334129

2957085755502649 | 3206752505839737655507 −2089713361698611995532

−3203485502789193867956 + 1798108228107250012156

or ÷ 4

−800871375697298466989 +449527057026812503039

The value of x in this last function, when equated to zero, is ·56129744509275; which, substituted for x in the preceding function, gives ·0000029, &c. Hence the sign of X_5 is —. Consequently the functions are

$$X = x^5 + 173x^4 + 2356x^3 + 10468x^2 - 14101x + 4183$$

$$X_1 = 5x^4 + 692x^3 + 7068x^2 + 20936x - 14101$$

$$X_2 = 2671x^3 + 29604x^2 + 108443x - 70668$$

$$X_3 = 54379093x^2 + 1363227740x - 782308643$$

$$X_4 = -80087137569729846698 9x + 44952705702681250303 9$$

$$X_5 = -$$

From these functions we infer at once that there is one pair of imaginary roots; and that of the three real roots, two are positive.

	X	X_1	X_2	X_3	X_4	X_5	
For $x = -160$ . . .	$-$	$+$	$-$	$+$	$+$	$-$	*four variations*
$= -150$ . . .	$+$	$+$	$-$	$+$	$+$	$-$	*three* ,,
$= 0$ . . .	$+$	$-$	$-$	$-$	$+$	$-$	*three* ,,
$= 1$ . . .	$+$	$+$	$+$	$+$	$-$	$-$	*one* ,,

Thus both the positive roots lie between 0 and 1, and the following is the operation for finding them:—

Development of the two roots in the interval [0, 1].

```
          2356
173·5       86·75
        ----------
        2442·75  }
          87     }
          87·25  } 1
          87·5   }
175·56      10·5336
        ----------
        2715·0336 }
          10·5372 }
          10·5408 } 2
          10·5444 }
175·81        ·1758
        ----------
        2746·8318 }
            1758  }
            1758  } 3
            1758  }
             352
        ----------
        2747·3944 }
             352  }
             352  } 4
             352  }
             158
        ----------
       274|7·516
```

```
10468
 1221·375
-----------
11689·375
 1264·875  }
 1308·5    } 1
  162·902016
-------------
14425·652016 }
  163·534248 } 2
  164·166696 }
    2·746832
-------------
14756·099792 }
    2·747008 } 3
    2·747183 }
      549479
-------------
14762·143462 }
      549486 } 4
      549493 }
      247276
-------------
14763·489717 }
      24728  } 5
      24728  }
       1923
-------------
14764·00|35
```

```
 −14101
   5844·6875
  ------------
 −8256·3125  }
  6477·125   } 1
    865·53912096
  ---------------
 −913·64837904 }
  875·35117584 } 2
   14·756099792
  ---------------
 −23·541103408 }
   14·7588468  } 3
    2·952428692
  ---------------
 −5·829827916 }
  2·952538589 } 4
  1·328714075
  ---------------
 −1·54857525  }
  1·32873633  } 5
    10334802
  ---------------
  −116491|0
```

```
              173
−4183      (      0·561297
−4128·15625
-------------- 1
  −54·84375
  −54·8189027424
  --------------- 2
       −248472576
      −23541103408
      ------------- 3
       −1306154192
       −1165965583
       ------------ 4
        −140188609
        −139371773
        ----------- 5
           −816836
           −815437
           --------
             −1399
```

[R.]

If 1 be taken as the next figure of the root there will be no change of sign in the last column; but if 2 be taken there will be a change of sign. If 7 be taken as the next figure, there will be a change of sign, but if 8 be taken, there will be no change of sign. We infer, therefore, that the first seven places of figures of the two positive roots are ·5612971 and ·5612977.

It is plain that the results which complete each step in the third column when multiplied by 2 become the divisors, and the bracketed numbers in the next column the dividends for suggesting, as before, the successive figures of the root.

In the operation below, the negative root is, for the sake of convenience, converted into a positive one, by changing the alternate signs of the equation.

Development of the root in the interval [150, 160].

```
                                                   —173
        2356        —10468        —14101           4183(158·5612971
 —73   —7300       —494400     —50486800    —5050090100 1
       —4944       —504868     —50500901 1   5050094283
        2700       —224400 1   —72926800     1120194950 2
       12700 1     1045600     145831600     3929899333
       22700       2600300      22403899     3611123672 3
 377   18850       2916632     329221600 2    318775661
       52006       3667800 2    99764960      283510132 4
       21350       4860300     451390459 3     35265529
       23850 2     1025888     108275680       34518110 5   [R.]
       26350      12470620       7354124         747419
 585    4680       1063840 3   567020263         576209 6
      128236       1102304       7389944 4       171210
        4744         71484        891619         115245
        4808 3    14708248     575301826          55965
        4872         71639 4      892137 5        51860
 617     309         71794         14878           4105
      142969          8636     576208841 6         4034
         309      14860317         14878             71
         309 4        8638 5        2976
         309          8640     576226695
          37           144
      143933      14877739
          37           144 6
          37 5         144
          37            29
           1
      144045      14878056
```

(178.) Required the analysis and solution of the equation,

$$x^6 + 378x^5 + 38189x^4 + 492368x^3 - 572554x^2 + 213720x - 26352 = 0$$

The function X_1 is $3x^5 + 945x^4 + 76378x^3 + 738552x^2 - 572554x + 106860$

And if the process be carried on without any abbreviations Sturm's functions will be obtained as below. But the amount of labour employed in computing them with this strict accuracy is very great, and we are precluded from exhibiting the type of the several steps of the work for want of room. The step involving the computation of X_5 presents more than one hundred and fifty figures in a row. In the second solution of the present equation, to be hereafter given, the abbreviated form of working, suggested at (166) will be employed, and the steps of the process thus reduced within manageable space.

Sturm's functions are

$$X = x^6 + 378x^5 + 38189x^4 + 492368x^3 - 572554x^2 + 213720x - 26352 \qquad [F.]$$

$$X_1 = 3x^5 + 945x^4 + 76378x^3 + 738552x^2 - 572554x + 106860$$

$$X_2 = 10673x^4 + 2036631x^3 + 23836942x^2 - 18302601x + 3405618$$

$$X_3 = 160625580788x^3 + 10061221870437x^2 - 7442477781884x + 1368314891916$$

$$X_4 = 926557887071378530541x^2 - 679427162023871692444x + 124552732633275966924$$

$$X_5 = 750652461820512869711x - 275234602193506000862$$

$$X_6 = 4041325384229060478930527316980483171109556969694 00$$

From which we find the roots to be all real; and to lie in the following intervals: viz.

[0, 1]; [0, 1]; [0, 1]; [— 10, — 20]; [— 160, —170]; [— 190, — 200].

Development of the root [10, 20], alternate signs being changed.

1	−378	38189	—492368	—572554	—213720	26352	(16·36660027
	10	—3680	345090	—1472780	—20453340	—206670600	
	−368	34509	—147278	—2045334	—20667060	206696952 **1**	
	10	—3580	309290	1620120	—4252140	163107720	
	−358	30929	162012	—425214	—24919200 **1**	43589232 **2**	
	10	—3480	274490	4365020	52103820	35257454	
	−348	27449	436502	3939806 **1**	27184620	8331778 **3**	
	10	—3380	240690	4744164	84258300	7497738	
	−338	24069	677192 **1**	8683970	111442920 **2**	834040 **4**	
	10	—3280	113502	5359080	6081927	758111	
	−328	20789 **1**	790694	14043050	11752484|7	75929 **5**	
	10	—1872	102486	5909196	617849	75895	
	−318 **1**	18917	893180	19952246 **2**	12370334 **3**	34	
	6	—1836	91686	320844	125896	25	
	−312	17081	984866	2027309|0	1249623|0	9	
	6	—1800	81102	32189	12629	8	
	−306	15281	1065968 **2**	205949|8	1262252 **4**	1	
	6	—1764	3511	3229	1267		
	−300	13517	106947|9	209179 **3**	126351|9		
	6	—1728	349	648	127		
	−294	11789 **2**	10729|7	20982|7	126479 **5**		
	6	— 85	35	65	13		
	−288	1170|4	1076|5	2104|8	1|2|6|4|9|2		
	6	— 8	3	6			
	−2|8|2 **2**	1|1|6|2	1|0|8|0 **3**	2|1|1|1 **4**			

[F.]

The preceding development was placed first in order, that those which follow may each be presented entire at one opening of the book. The contractions of the several columns have commenced at a period of the work sufficiently remote, to secure accuracy in the root to about eight places of decimals.

In the following operation all curtailment of the decimals is postponed till the seventh decimal of the root is obtained: the reduction might however have been safely introduced after the completion of the second transformation, as is obvious from a glance at the final column of the work. A great deal of multiplication might thus have been spared and the work reduced in extent by about one third. But these reductions have been dispensed with, as in the case of the calculation of STURM'S functions, in order to secure greater perspicuity in a case generally considered to involve difficulties of a peculiar kind, and requiring more than ordinary caution in its management. The same example will be hereafter solved in a form in which economy of space and figures is more especially consulted.

As to the difficulty of executing the several steps of the following work, it involves in it nothing of what has hitherto been considered as peculiarly embarrassing when the separation of the roots is so long delayed. By the principles established at (171) the transformed coefficients in the third and fourth columns of the work supply with certainty and rapidity the successive figures of the root, till we reach the transformation 4. Extending the process to this, we find the suggested root-figure to be 4:* the employment of this, however, changes the sign of the absolute number; and one change only is lost by the transformation: hence the three roots cannot concur beyond ·3666. Resorting now to the ordinary trial-divisor we find 002 to be the true value of the next figure of the least root; and we may now extend the approximation to it as far as we please.

---

* The root-figure suggested by the other expression, viz. $\frac{3N'}{A'} = \frac{1599\ldots}{202\ldots}$ is 7, a discrepancy which, as remarked at (172), informs us that the hypothesis of the roots continuing together beyond the point reached is untenable.

Development of the roots

1 378	38189	492368	—572554
·3	113·49	11490·747	151157·6241
378·3	38302·49	503858·747	—421396·3759
·3	113·58	11524·821	154615·0704
378·6	38416·07	515383·568	—266781·3055
·3	113·67	11558·922	158082·7470 **1**
378·9	38529·74	526942·490	—108698·5585
·3	113·76	11593·050 **1**	32451·74090976
379·2	38643·50	538535·540	—76246·81759024
·3	113·85 **1**	2326·808496	32591·43148224
379·5	38757·35	540862·348496	—43655·38610800
·3 **1**	22·7916	2328·176208	32731·20413040 **2**
379·8	38780·1416	543190·524704	—10924·18197760
·06	22·7952	2329·544136	3288·505337219856
379·86	38802·9368	545520·068840	—7635·676640380144
6	22·7988	2330·912280 **2**	3289·904869836864
379·92	38825·7356	547850·981120	—4345·771770543280
6	22·8024	233·241749976	3291·304484572320 **3**
379·98	38848·5380	548084·222869976	—1054·467285970960
6	22·8060 **2**	233·255436168	329·2844160146368656
380·04	38871·3440	548317·478306144	—725·1828699563231344
6	2·280996	233·269122576	329·2984139686443264
380·10	38873·624996	548550·747428720	—395·8844559876788080
6 **2**	2·281032	233·282809200 **3**	329·3124120047745120 **4**
380·16	38875·906028	548784·030237920	—66·5720439829042960
·006	2·281068	23·329786474776	·1097754715963647843775968
380·166	38878·187096	548807·360024394776	—66·462268511307931215622403|2
6	2·281104	23·329923345768	
380·172	38880·468200	548830·689947740544	
6	2·281140 **3**	23·330060216976	
380·178	38882·749340	548854·020007957520	
6	·22811796	23·330197088400 **4**	
380·184	38882·97745796	548877·350205045920	
6	·22811832	·0077767780018879840	
380·190	38883·20557628	548877·357981823921887984|0	
6 **3**	·22811868		
380·196	38883·43369496		
·0006	·22811904		
380·1966	38883·66181400		
6	·22811940 **4**		
380·1972	38883·88993340		
6	·0000760399200		
380·1978	38883·8900094399201|0		
6			
380·1984			
6			
380·1990			
6 **4**			
380·1996			
·0000002			
380·199600|2			

in the interval [0, 1].

```
213720
—126418·91277
═════════════
 87301·08723
—80034·39165
─────────────  1
  7266·69558
—4574·8090554144
════════════════
  2691·8865245856
—2619·3231664800
────────────────  2
    72·5633581056
—45·814059842280864
═══════════════════
    26·749298263319136
—26·074630623259680
───────────────────  3
     ·674667640059456
—·43510972197379388064
═══════════════════════
     ·23955791808566211936
—·23753067359260728480
───────────────────────  4
     ·00202724449305483456
—·000013292453702261586243124480
════════════════════════════════
     ·00201395203935257297375687552|0
```

```
26352
26190·326169
────────────  1
  161·673831
  161·513191475136
  ────────────────  2
     ·160639524864
     ·160495789579914816
     ───────────────────  3
     ·000143735284085184
     ·00014373475085139727 1616
     ──────────────────────────  4
     ·000000000533233786728384
     ·000000000402790407870514594751375104
     ─────────────────────────────────────  5
     ·000000000130443378857869405248624896
```

(·3666002

[F.]

In the foregoing operation the expression which suggests the several figures of the root may be taken either

$$-\frac{A'_2}{3A'_3}, \text{ or } -\frac{A'}{A'_2}$$

as remarked at (171). Till we reach the transformation 4 all three of the expressions

$$-\frac{A'_2}{3A'_3}, \ -\frac{A'}{A'_2}, \ -\frac{3N'}{A'}$$

concur in furnishing the same root-figure: but at this point their determinations all differ. As remarked above, the first expression gives 4; as this produces the loss of a single variation, we conclude that the discrepancy adverted to arises from the passage of a single root—the least of the three, which separates from the next root after concurring with it as far as ·3666. The variations furnished by the above-mentioned 4, by which this root is separated, are two in number, the sign it gives to the absolute number, left untransposed, being *minus*: and it will of course continue minus till the next root is separated. But, as the following step will show, we need not, as heretofore, seek, by trial-transformations, whereabouts this separation takes place.

Development of the roots

380·1996	38883·88993340	548877·350205045920	—66·5720439829042960
·00006	·0228119796	2·333034764722776	32·93278099438863856656
380·19966	38883·9127453796	548879·683239810642776	—33·63926298851565743344
6	228119832	2·333036133441768	32·93292097655664507264
380·19972	38883·9355573628	548882·016275944084544	— ·70634201195901236080
6	228119868	2·333037502160976	32·93306095880677473120 5
380·19978	38883·9583693496	548884·349313446245520	32·22671894684776237040
6	228119904	2·333038870880400 5	
380·19984	38883·9811813400	548886·682352317125920	
6	228119940 5		
380·19990	38884·0039933340		
6 5			
380·19996			

In this operation the suggested root-figure, agreeably to what is stated at page 265 is $-\frac{A'_2}{2A'_3}$; instead of $-\frac{A'_2}{3A'_3}$ as before; since the transformation employed is that deduced in the former

380·19996	38884·0039933340	548886·682352317125920	32·22671894684776237040
·000001	·000380199961	·038884004373533961	·548886721236321499453961
380·199961	38884·004373533961	548886·721236321499453961	32·775605668084083869853961
1	380199962	38884004753733923	·548886760120326253187884
380·199962	38884·004753733923	548886·760120326253187884	33·324492428204410123041845
1	380199963	38884005133933886	·548886799004331387121770 5
380·199963	38884·005133933886	548886·799004331387121770	33·873379227208741510163615
1	380199964	38884005514133850 5	
380·199964	38884·005514133850	548886·837888336901255620	
1	380199965 5		
380·199965	38884·005894333815		
1 5			
380·199966			

In this final step we know that two roots, and only two, are comprehended in the interval [·0000600 . . . ., ·00000 . . . .] where the dots stand for unknown figures. The approximate quadratic will furnish the leading figures of these roots. But as the leading figure of the one sought is in the sixth place, and the *in the interval* [0, 1] *continued.*

·00202724449305483456		·000000000533233786728384		(·0000600
—·0020183557793109394460064		·000000000533322824633706839616	5	
·0000088887137438951139936		—·000000000000089037905322839616		
—· 423805207175407416480	5			
—·0000334918069736456276544		[F.]		

development from ·36660, the 0 being a figure beyond that at which the least root separates from the others. We know that this root-figure will either be the true first figure of the second root of the equation, when diminished by the former development, or it will separate, or else pass over, the remaining roots (172). As it leaves the variations given by overstepping the first root, unchanged, we conclude that no such separation or passage is effected; and that consequently the 6 is the true figure of the next root.

—·0000334918069736456276544		—·000000000000089037905322839616		(·0000010
·000032775605668084083869853961		—·000000000000716201305561543784546039	5	
—·000000716201305561543784546039		·000000000000627163400238704168546039		
·000033324492428204410123041845	5			
·000032608291122642866338495806				

sixth place of the other is zero, we infer, as above, that the leading figure sought is

$$\frac{A'_{m-1}}{(m-1)A'_m} = \frac{A'}{(2-1)A'_2} = \frac{A'}{A'_2} = \frac{334\ldots}{322\ldots} = 1$$

Thus the three roots are separated; and the first step, in each of the three distinct directions which they severally take at separating, is actually exhibited; so that we may now pursue them in these directions, to any extent we please, by aid of the ordinary trial-divisors.

Development of the root [160, 170].

<table>
<tr><td>1</td><td>—378</td><td></td><td>38189</td><td></td><td>—492368</td><td></td><td>—572554</td><td></td></tr>
<tr><td></td><td>100</td><td></td><td>—27800</td><td></td><td>1038900</td><td></td><td>54653200</td><td></td></tr>
<tr><td></td><td>—278</td><td></td><td>10389</td><td></td><td>546532</td><td></td><td>54080646</td><td></td></tr>
<tr><td></td><td>100</td><td></td><td>—17800</td><td></td><td>—741100</td><td></td><td>—19456800</td><td></td></tr>
<tr><td></td><td>—178</td><td></td><td>—7411</td><td></td><td>—194568</td><td></td><td>34623846</td><td></td></tr>
<tr><td></td><td>100</td><td></td><td>—7800</td><td></td><td>—1521100</td><td></td><td>—171566800</td><td></td></tr>
<tr><td></td><td>—78</td><td></td><td>—15211</td><td></td><td>—1715668</td><td></td><td>—136942954</td><td>1</td></tr>
<tr><td></td><td>100</td><td></td><td>2200</td><td></td><td>—1301100</td><td></td><td>—123013680</td><td></td></tr>
<tr><td></td><td>22</td><td></td><td>—13011</td><td></td><td>—3016768</td><td>1</td><td>—259956634</td><td></td></tr>
<tr><td></td><td>100</td><td></td><td>12200</td><td></td><td>966540</td><td></td><td>8850720</td><td></td></tr>
<tr><td></td><td>122</td><td></td><td>—811</td><td>1</td><td>—2050228</td><td></td><td>—251105914</td><td></td></tr>
<tr><td></td><td>100</td><td></td><td>16920</td><td></td><td>2197740</td><td></td><td>227547120</td><td></td></tr>
<tr><td></td><td>222</td><td>1</td><td>16109</td><td></td><td>147512</td><td></td><td>—23558794</td><td>2</td></tr>
<tr><td></td><td>60</td><td></td><td>20520</td><td></td><td>3644940</td><td></td><td>59042964</td><td></td></tr>
<tr><td></td><td>282</td><td></td><td>36629</td><td></td><td>3792452</td><td></td><td>35484170</td><td></td></tr>
<tr><td></td><td>60</td><td></td><td>24120</td><td></td><td>5308140</td><td></td><td>63610680</td><td></td></tr>
<tr><td></td><td>342</td><td></td><td>60749</td><td></td><td>9100592</td><td>2</td><td>99094850</td><td></td></tr>
<tr><td></td><td>60</td><td></td><td>27720</td><td></td><td>739902</td><td></td><td>68307996</td><td></td></tr>
<tr><td></td><td>402</td><td></td><td>88469</td><td></td><td>9840494</td><td></td><td>167402846</td><td>3</td></tr>
<tr><td></td><td>60</td><td></td><td>31320</td><td></td><td>761286</td><td></td><td>3669228</td><td></td></tr>
<tr><td></td><td>462</td><td></td><td>119789</td><td>2</td><td>10601780</td><td></td><td>17107207|4</td><td></td></tr>
<tr><td></td><td>60</td><td></td><td>3528</td><td></td><td>782886</td><td></td><td>368166</td><td></td></tr>
<tr><td></td><td>522</td><td></td><td>123317</td><td></td><td>11384666</td><td></td><td>1747537|3</td><td></td></tr>
<tr><td></td><td>60</td><td></td><td>3564</td><td></td><td>804702</td><td></td><td>36941</td><td></td></tr>
<tr><td></td><td>582</td><td>2</td><td>126881</td><td></td><td>12189368</td><td>3</td><td>1784478</td><td>4</td></tr>
<tr><td></td><td>6</td><td></td><td>3600</td><td></td><td>41392</td><td></td><td>7418</td><td></td></tr>
<tr><td></td><td>588</td><td></td><td>130481</td><td></td><td>1223076|0</td><td></td><td>179189|6</td><td></td></tr>
<tr><td></td><td>6</td><td></td><td>3636</td><td></td><td>4145</td><td></td><td>742</td><td></td></tr>
<tr><td></td><td>594</td><td></td><td>134117</td><td></td><td>122722|1</td><td></td><td>17993|2</td><td></td></tr>
<tr><td></td><td>6</td><td></td><td>3672</td><td></td><td>415</td><td></td><td>74</td><td></td></tr>
<tr><td></td><td>600</td><td></td><td>137789</td><td>3</td><td>12313|7</td><td></td><td>18067</td><td>5</td></tr>
<tr><td></td><td>6</td><td></td><td>185</td><td></td><td>42</td><td></td><td>7</td><td></td></tr>
<tr><td></td><td>606</td><td></td><td>13797|4</td><td></td><td>12356</td><td>4</td><td>1|8|0|7|4</td><td></td></tr>
<tr><td></td><td>6</td><td></td><td>19</td><td></td><td>8</td><td></td><td></td><td></td></tr>
<tr><td></td><td>612</td><td></td><td>1381|6</td><td></td><td>1|2|3|6|4</td><td></td><td></td><td></td></tr>
<tr><td></td><td>6</td><td></td><td>2</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td>6|1|8</td><td>3</td><td>1|3|8|4</td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

alternate signs being changed.

```
      —213720
   5408064600
   ==========
   5407850880
   3462384600
   ----------  1
   8870235480
 —15597398040
 ============
  —6727162560
 —15066354840
 ------------  2
 —21793517400
    212905020
 ============
 —21580612380
    594569100
 ------------  3
 —20986043280
     51321622
 ============
 —20934721658
      5242612
 ------------  4
 —2088229554
     1075138
 ===========
 —208715441|6
      107959
 ----------  5
 —208607483
      10845
 ==========
 —20859663|8
       1085
 ---------  6
 —20858579
       109
 =========
 —2085847|0
        11
 --------  7
 —2085836
        1
=========
—2|0|8|5|8|3|5
```

```
        26352      (166·3666600026
 540785088000
 ------------  1
—540785061648
—403629753600
 ------------  2
—137155308048
—129483674280
 ------------  3
  —7671633768
  —6280416497
   ----------  4
  —1391217271
  —1252292650
   ----------  5
   —138924621
   —125157983
    ---------  6
    —13766638
    —12515082
     --------  7
     —1251556
     —1251501
     --------
          —55
          —42
           --
          —13
          —12
           --
           —1
```

[F.]

Development of the root [190, 200],

1 −378	38189	— 492368	—572554
100	—27800	1038900	54653200
−278	10389	546532	54080646
100	—17800	— 741100	—19456800
−178	— 7411	— 194568	34623846
100	— 7800	—1521100	—171566800 **1**
− 78	—15211	—1715668	—136942954
100	2200	—1301100 **1**	—50630220
22	—13011	—3016768	—187573174
100	12200 **1**	2454210	463306680
122	— 811	−562558	275733506
100 **1**	28080	5710410	1335911580 **2**
222	27269	5147852	1611645086
90	36180	9695610	183626724
312	63449	14843462	1795271810
90	44280	14409810 **2**	191901000
402	107729	29253272	1987172810
90	52380	1351182	200343756 **3**
492	160109	30604454	2187516566
90	60480 **2**	1379046	10469795
582	220589	31983500	219798636|1
90	4608	1407126	1049180
672	225197	33390626	22084781|6
90 **2**	4644	1435422 **3**	105138 **4**
762	229841	34826048	22189920
6	4680	73268	21080
768	234521	3489931|6	2221100|0
6	4716	7334	2108
774	239237	349726|6	222320|8
6	4752 **3**	734	211 **5**
780	243989	35046|1	222532
6	239	73	21
786	24422|8	35119 **4**	22255|3
6	24	15	2
792	2444|7	3513|4	2|2|2|5|7
6 **3**	2	2	
7|9|8	2|4|4|7	3|5|1|4	

alternate signs being changed.

```
     —213720
  5408064600
  5407850880
  3462384600  1
  8870235480
—16881585660
— 8011350180
 24816015540  2
 16804665360
 10771630860
 27576296220
 11923036860  3
 39499333080
   659395908
 4015872898|8
   66254345   4
 4082127244
   13326600
 409545384|4
   1333925    5
 410879309
    133532
 41101284|1
    13354     6
 41114638
     1336
 4111597|4
     134      7
 4111731
      13
 411174|4
      1
4|1|1|1|7|5
```

```
        26352
 540785088000  1
—540785061648
—721021516200  2
 180236454552
 165457777320  3
  14778677232
  12047618696  4
   2731058536
   2457272306  5
    273786230
    246607705  6
     27178525
     24669584  7
      2508941
      2467047
        41894
        41118
          776
          411
          365
          329
           36
           37
```

(196·366661019

[F.]

(177.) The solution of the foregoing equation when the entire process is exhibited in its most compact form is as follows:—

$$X = x^6 + 378x^5 + 38189x^4 + 492368x^3 - 572554x^2 + 213720x - 26352$$

$$X_1 = 3x^5 + 945x^4 + 76378x^3 + 738552x^2 - 572554x + 106860$$

$$\begin{array}{r|rrrrr}
1 & +945 & +76378 & +738552 & -572554 & +106860 \\
126 & -76378 & -1477104 & +2290216 & -1068600 & +158112 \\ \hline
 & 119070 & +9623628 & +93057552 & -72141804 & +13464360 \\
 & -76378 & -1477104 & +2290216 & -1068600 & +158112 \\ \hline
 & 42692 & +8146524 & +95347768 & -73210404 & +13622472
\end{array}$$

$$\text{or} \div 4 \qquad\qquad\qquad \text{[R.]}$$

$$\begin{array}{rrrrr}
10673 & +2036631 & +23836942 & -18302601 & +3405618 \\ \hline\hline
10673 & -2036631 & -23836942 & +18302601 & -3405618 \\
3 & +945 & +76378 & +738552 & -572554 & +106860
\end{array}$$

$$\begin{array}{r|rrrr}
3976092 & -8097832226052 & -94777874390664 & +72772825415292 & -13541050484856 \\
32019 & -763235045898 & +586030981419 & -109044482742 & \\
113912929 & 8700441691162 & +84130621538808 & -65221303150666 & +12172735592940 \\ \hline
 & 160625580788 & +10061221870437 & -7442477781884 & +1368314891916
\end{array}$$

```
                   160625580788—10061221870437+7442477781884—1368314891916
                          10673+2036631+23836942—18302601+3405618
                        ____________________________________________________________________
1368098·5004045402275355|—13764742553182223029·25+10182042692689609229·31—1871989551711480143·71
                  10673|      79433565366047932        —14604024841419468
           160625580788|      3828822652959870296  —2939865915556029588      +547029369192066984
                        ____________________________________________________________________

                           9856486334856304801·25  —7227572752292160173·31+1324960182519413159·71
                       ____________________________________________________________________

     [R.]            9856486334856304801·25+7227572752292160173·31—1324960182519413159·71

                          160625580788    +10061221870437           —7442477781884+1368314891916
                                 ________________________________________________________________
    10179005534065·58421525|  73569503043443524276 * * * —13486777030281653088 * * *
               160625580788|  —212822498838155223 * * *
     9856486334856304801·25|—73356680554611308231 * * * +13486777033950435687 * * *
                            ______________________________________________________
                                     10005939178 * * *                —3668782598 * * *
                                    ______________________________________________
```

As the roots of $X_4 = 0$, found by the general method, are ·36662, &c., and ·36666051, and the root of $X_5 = 0$ is ·3666604, if the root of $X_5 = 0$ be substituted for x in X_4 the result will be negative. We infer, therefore, that the sign of the last function is *plus*. It will be seen that, in the last two steps of the process, there has been retained one, or perhaps two figures more than was sufficient to obtain this result.

Consequently, Sturm's functions are

$$X = x^6 + 378x^5 + 38189x^4 + 492368x^3 - 572554x^2 + 213720x - 26352$$

$$X_1 = 3x^5 + 945x^4 + 76378x^3 + 738552x^2 - 572554x + 106860$$

$$X_2 = 10673x^4 + 2036631x^3 + 23836942x^2 - 18302601x + 3405618$$

$$X_3 = 160625580788x^3 + 10061221870437x^2 - 7442477781884x + 1368314891916$$

$$X_4 = 9856486334856304801{\cdot}25x^2 - 7227572752292160173{\cdot}31x + 1324960182519413159{\cdot}71$$

$$X_5 = 10005939178x - 3668782598$$

$$X_6 = +$$

[R.]

	X	X_1	X_2	X_3	X_4	X_5	X_6	
For $x = -200$	+	−	+	−	+	−	+	gives 6 variations
$= -190$	−	−	+	−	+	−	+	,, 5 ,,
$= -170$	−	+	−	−	+	−	+	,, 5 ,,
$= -160$	+	+	−	−	+	−	+	,, 4 ,,
$= -20$	+	−	−	+	+	−	+	,, 4 ,,
$= -10$	−	+	+	+	+	−	+	,, 3 ,,
$=$ ·3	−	+	+	+	+	−	+	,, 3 ,,
$=$ ·4	+	+	+	+	+	+	+	,, no variation.

In the following operations, the negative roots are, for the sake of convenience, converted into positive ones by changing the alternate signs of the function.

Development of the root [190, 200.]

[R.]

```
                                                                     —378
            38189    —492368      —572554       —213720           26352 (196·3666610
    —278   —27800    1038900     54653200    5408064600    540785088000
            10389     546532     54080646    5407850880   —540785061648
           —17800    —741100    —19456800    3462384600   —721021516200
            —7800   —1521100   —171566800  —16881585660
                                                           180236454552
             2200   —1301100    —50630220
                                            —8011350180    165457777320
            12200    2454210
                               —187573174   24816015540
                                                            14778677232
     312    28080
                     —562558    463306680   10771630860
                                                            12047618697
            27269    5710410   1335911580
                                            27576296220
                                                             2731058535
            36180    9695610    183626724
                                            11923036860
                                                             2457272306
            44280   14409810
                               1795271810     659395908
            52380    1351182                                  273786229
                                191901000
                                            40158728988
            60480                                             246607706
                    30604454    200343756
                                              662543447
     768     4608
                     1379046     10469795                      27178523
                                              133266002
           225197    1407126                                   24669585
                               2197986361
                                            40954538437
             4644    1435422
                                 10491797                       2508938
                                                1333926
             4680      73268
                                 10513821                       2467047
                                                 133532
             4716
                    34899316      2108056
                                                                  41891
                                              411012843
             4752
                       73340
                               2221100035                         41118
                                                 133545
  79|8·3      239
                       73412
                                    21090
                                                  13356             773
           244228      73484
                                    21099
              239      14714                  411159744
                                     2111
              240                                   134
                    35134266
                                 22255301
              240                                    13
                         147
                                       21
              240
                                               41117|44
                         147
                                       21
    8|00       48
                         147
                                    2|226
          2|45235         15
                       3|518
```

Development of the root [160, 170].

−278
282
588
61|8·3
6|20

38189
−27800
10389
−17800
−7800
2200
12200
16920
16109
20520
24120
27720
31320
3528
123317
3564
3600
3636
3672
186
137975
186
186
186
186
37
1|38756

−492368
1038900
546532
—741100
−1521100
−1301100
966540
−2050228
2197740
3644940
5308140
739902
9840494
761286
782886
804702
41393
12230761
41449
41504
41560
8325
12363599
83
83
83
8
12|39

−572554
54653200
54080646
−19456800
−171566800
−123013680
—259956634
8850720
227547120
59042964
35484170
63610680
68307996
3669229
171072075
3681664
3694115
741816
179189670
74231
74281
7434
1807491
743
743
74
18|091

−213720
5408064600
5407850880
3462384600
−15597398040
−6727162560
—15066354840
212905020
—21580612380
594569100
51321623
—20934721657
52426122
10751380
—20871544155
10795919
1084495
—20859663741
1084941
108540
—208584703
1085
108
—208583|5

26352
540785088000
—540785061648
—403629753600
—137155308048
−129483674280
—7671633768
−6280416497
−1391217271
−1252292650
−138924621
−125157982
−13766639
−12515082
−1251557
−1251501
−56

—378
(166·3666600

[R.]

Development of the root [10, 20].

					—378	
	38189	—492368	—572554	—213720	26352	(16·3666002
—368	—3680	345090	—1472780	—20453340	—206670600	
	34509	—147278	—2045334	—20667060	206696952	
	—3580	309290	1620120	—4252140	163107720	
	—3480	274490	4365020	52103820	43589232	
	—3380	240690	4744164	27184620	35257454	
	—3280	113502	8683970	84258300	8331778	
—312	—1872	790694	5359080	6081927	7497738	[R.]
	18917	102486	5909196	117524847	834040	
	—1836	91686	320844	6178494	758111	
	—1800	81102	20273090	1258964	75929	
	—1764	3511	321890	124962305	75895	
	—1728	1069479	322928	1262857	34	
—28|2	—85	3486	64832	126714	25	
	11704	3461	20982740	126351876		
	—85	3435	64873	126753		
	—84	681	64914	12680		
	—84	1080542	6496	126491309		
	—84	680	21119023	12680		
—2|80	—17	679	6496	4		
	11350	678	6496	12|6503993		
	—17	68	650			
	—17	1082647	2|1132665			
	—17	68				
	—16	68				
	—1	67				
	11|282	108|285				

In explanation of the process on the next page it is necessary to remark that the three roots do not separate till after the fourth figure. The equation whose roots are (x — ·3666) is that below the first series of double lines, from inspecting which it appears that any significant figure in the fifth or sixth place, or 3 in the seventh place would produce a change of sign in the last column showing that a root had been passed over. The least root is, therefore, ·3666002. When 6 is put in the fifth place there is a change of sign in the last column, indicating the passage over the root already determined. Had 7 been put in the fifth place, *three* variations would have been lost, showing that the three roots had been passed over. Hence we infer that 6 is the fifth figure of both of the roots still sought, and the equation whose roots are (x— ·36666) is that below the second series of double lines. As any significant figure, either in the sixth or seventh place, would cause the loss of more variations, it appears that the middle root is ·3666600, and as 2 in the sixth place, or 1 in the sixth place and 1 in the seventh place, would make all the signs of the transformed positive, we have for the greatest root ·3666610.

Development of the three

	38189	492368	—572554
378·3	113·49	11490·747	151157·6241
	38302·49	503858·747	—421396·3759
	113·58	11524·821	154615·0704
	113·67	11558·922	158082·747
	113·76	11593·05	32451·74090976
	113·85	2326·808496	—76246·81759024
379·86	22·7916	540862·348496	32591·43148224
	38780·1416	2328·176208	32731·2041304
	22·7952	2329·544136	3288·505337219856
	22·7988	2330·91228	—7635·676640380144
	22·8024	233·241749976	3289·904869836864
	22·2806	548084·222869976	3291·30448457232
380·166	2·280996	233·255436168	329·284416014637
	38873·624996	233·269122576	—725·182869956323
	2·281032	233·2828092	329·298413968644
	2·281068	23·329786475	329·312412004775
	2·281104	548807·360024395	—66·572043982904
	2·28114	23·329923346	32·932780994389
380·1966	228118	23·330060217	
	38882·977458	23·330197088	—33·639262988515
	228118	548877·350205046	32·932920976557
	228119	2·333034765	32·933060958807
	228119		32·226718946848
	228119	548879·683239811	5488867212
	38883·889933	2·333036134	32·775605668
	22812	2·333037502	
	38883·912745	2·333038870	
	22812	548886·682352317	
	22812	38884	
	22812	548886·721236	
	22812		
	38884·003993		
	38		
	38884·00437		

roots in the interval [0, 1].

213720
—126418·91277

87301·08723
—80034·39165
—4574·8090554144

2691·8865245856
—2619·32316648
—45·814059842280864

26·749298263319136
—26·07463062325968
—435109721973794

239557918085662
—237530673592607

2027244493055
—2018355779311

8888713744
—42380520717

—33491806973
327756057

—7162013

26352
26190·326169

161·673831
161·513191475136

160639524864
160495789579914816

143735284085184
143734750851397

533233787
533322825

—89038
—716201

627163

—378

$$(0{\cdot}3666\begin{cases}002\\ 6\begin{cases}00\\ 10\end{cases}\end{cases}$$

[R.]

We need not subjoin to the preceding solution anything by way of comment or explanation; the remarks which accompany the former solution at page 281, equally apply to this. The same precepts guide us to the successive common figures of the roots, and to the places at which they separate and become distinct, whichever mode of arranging the work be adopted. In the method above, however, these precepts cannot be applied with the same facility as in the former mode of arrangement; but something, as regards facility of operation, must be expected to be sacrificed where brevity and compactness are the principal objects sought; and these are very completely attained in the process just exhibited.

(179.) The foregoing examples sufficiently confirm all that has been stated in the preceding parts of the present work as to the entire efficacy of the methods of Sturm and Horner in the analysis and solution of numerical equations, since they have been so framed as to put the competency of these methods to the severest test. The principal part of the labour involved in the solutions just given is that expended upon the preparatory analysis: this, on account of the large coefficients which enter the equations, is very considerable, more especially in the second example, even when the entrance of redundant figures is provided against, as at page 289. It is easy to see that this labour is entirely attributable to the magnitude of the coefficients, conjointly with the number of steps in the process, and not at all to the proximity of the roots to each other: and we can thus at once infer what are the circumstances which in general set bounds to the practicability of Sturm's method. In the subsequent development of the real roots, these circumstances affect the length and difficulty of the operation in but a comparatively small degree: the magnitude of the coefficients interfering but little with the progress of the work, after the first figure or two of the root has been correctly determined, and we are sure that our pursuit of the subsequent figures is in the proper direction.

(180.) The difficulty that has hitherto retarded the process—affecting every step of it with uncertainty till verified by trial-operations—is that which has always been found attendant upon the separation of roots having several leading figures in common. In such cases the work of the true development, as exhibited to the eye, could convey but a very inadequate notion of the time, and cautious deliberation, expended upon each step of the work, up to the figure at which the nearly equal roots are found to separate. By means of the principles established at (171) this difficulty is now entirely removed, and roots having a great number of leading figures in common may now be developed and separated with even more ease and expedition than so many roots already isolated. So long as the figures of the nearly equal roots actually keep together, the foregoing examples sufficiently illustrate the ease and rapidity with which their development may be carried on: when their point of separation is reached we shall find indications of the circumstance spontaneously to present themselves, either at the step itself, or at that immediately beyond: for if not at the place where the figures cease to be alike, yet at the place immediately beyond it, we shall find a contradiction between two expressions (C), (D), page 263, which on the principle of the non-separation of the roots must agree. The comparison of these tests, as we proceed from step to step, involves no trouble: it is made at a simple inspection. But we need not make the comparison at all, or may disregard it till it spontaneously offers itself to our notice. We may continue to develope the root of the function $f_{m-1}(x) = 0$, which lies between the nearly equal roots (see page 261) till the changes of sign in the transformed coefficients inform us that the separation is accomplished. It is true that the intervening root of $f_{m-1}(x) = 0$ spoken of, may itself concur with the figures of one of the sought roots, between which it lies, *beyond* the place at which these latter roots themselves separate; but it must detach itself from both at last; and thus make known, by the changes of sign referred to, that the roots sought cannot concur *beyond* the place thus reached, although they *may* separate before reaching that place: yet actually up to it will the figures of one of the sought roots be correctly exhibited. To find the exact place at which they separate—whether we direct our attention to these latter indications of the separation, or to the former—we must start anew with the transformed coefficients with which we are furnished by the step immediately beyond that at which the separation was discovered, as explained at (171), and proceed to develop the remaining roots in the interval as there directed: the first significant figure, with which this new course of operations commences, points out distinctly the place where the preceding root has separated from the others.

Generally speaking, the figure at which the root of $f_{m-1}(x) = 0$ separates the roots sought, will be that at which the roots themselves separate, as in the preceding examples; for it is improbable that a number, restricted only by the condition of lying between two others, should not become distinct from both at the point where they themselves separate from each other: but whether or not such be the case, the directions already given will infallibly make known whereabouts the separation must take place; and it is impossible that we can be carried more than a figure or two beyond it without the discrepancy adverted to above forcing itself upon our attention.

(181.) When, as in the improbable case just adverted to, the intervening root of the derived function continues to coincide with one of the two roots of the primitive between which it lies, after those roots themselves have separated, we should also become acquainted with the circumstance by increasing the last found figure of this intervening root by unity; for we should thus obtain a new transformation, making known, by its loss of variations, that a root or roots had been passed over in the interval between it and the former transformation. We say *roots*, because it is possible, where several lie close together, that the intervening root of which we speak may separate more than one on each side. Hence, if we continue our development till the discrepancy already mentioned presents itself, and then increase the last figure of the development by unity—should the roots remain unseparated—we shall assuredly separate them if real, or else discover that they are imaginary.

(182.) The more nearly the doubtful roots in any interval approach to equality; or to speak more accurately, the more minute the change in the constitution of an equation which suffices to make these roots equal, the longer, of course, will the determination of their true character be delayed. Such must be expected to be the case, whatever tests be employed for distinguishing imaginary roots from real. But, of all methods for effecting this object, that must be pronounced the best which, when the doubtful roots prove to be real, accomplishes the separation of them with most ease and expedition. For the tests in question become effective only when that point of the development is reached, the next step of which, to be correct, must involve an imaginary expression. Till this point is attained, the approximation is perfectly accurate, whether the roots approached to, eventually prove to be real, or imaginary: since that must be regarded as a true approximation to a value of x, in the equation $f(x) = 0$, which commences by furnishing for $f(x)$ a value $f(x) = a$, and thence a continuous series of values, in which a, by losing its leading figures one after another, tends continuously to zero. This is a sure criterion of a true approximation; and consequently, as far as it extends, we should seek in vain for any peculiarity among its steps that would enable us to predict whether this tendency of a towards zerocould continue indefinitely, by the continual accession of real increments to x; or whether, beyond a certain point, the leading figures of a can be destroyed only by making the increment imaginary (p. 163.) It is at this point only that peculiarity can be first introduced.

The operation by which this point is reached—supposing such to exist—is precisely that which separates the roots—supposing them to be real: so that, as stated above, the method which is competent to effect the separation of nearly equal roots with the greatest ease, is that which will also conduct us the most readily to the peculiarity alluded to when the roots prove to be imaginary.

(183.) This peculiarity may manifest itself in various ways:—by satisfying the criterion of FOURIER [A] at page 165; by fulfilling the conditions of BUDAN (144); by forbidding a tendency towards zero in the absolute number beyond a certain limit, as noticed above and explained at p. 163; or lastly, by causing the discrepancy, between the expressions (C), (D) at page 263, so much dwelt upon in the present chapter. All these may be regarded as concurrent—or, as very nearly concurrent—indications of the same thing. FOURIER's criterion keeps us in suspense during the steps of the approximation till a certain point is reached. This point is that at which the roots separate, if they are real:—it is that at which the approximation becomes defective, if they are imaginary. When they prove to be real, the operation, which has just discovered their character, has also supplied us with their approximate values; when they prove to be imaginary, the same operation has furnished a real value which more nearly satisfies the equation than any other. The same observations apply to the transformations of BUDAN. These equally furnish the figures of the real roots, up to the point of separation, and those of the imaginary roots up to the point where the imaginary part becomes essential to the further continuance of the approximation towards zero.

(184.) It would seem then to be of but comparatively little moment what test we employ to distinguish nearly equal roots from imaginary ones. What is of real importance is that we take the safest and shortest path towards the point which must be reached before such test can be effective; and that our arrival at this point be intimated to us by unequivocal marks or indications, so that we may be spared the labour of a continual appeal to the test at every step we make.

These two important objects have, we conceive, been accomplished in the present chapter: a rapid and certain means of separating the roots has been theoretically established, and practically illustrated in examples well calculated to try its efficiency: and means, as ready and certain, for determining the point where the *indicator* of imaginary roots lies, have also been added. The process of approximation is but an extension of Mr. HORNER's own method to the case of nearly equal roots:—a case but inadequately provided for by the general operation as applied to roots lying singly in distinct intervals. The additional precepts that have been furnished, for distinguishing between these nearly equal roots and those that are imaginary—as delivered at pages 299 and 290, are, we believe, altogether new: and from the ease with which they may be applied, as the approximation proceeds, without requiring any actual calculation, they appear to be well adapted to general use, and to possess obvious advantages over the methods hitherto employed.

(185.) In the researches of FOURIER, so fully discussed in Chapters VIII and IX, we have seen that the object first to be accomplished, in the analysis of a doubtful interval, is to reduce the last three of the indices to the values 0, 1, 2. Till we get three such consecutive indices, either for the three final functions, or for some preceding three, FOURIER's peculiar method of proceeding does not come into operation. In bringing about this arrangement of indices, the proposed interval must be narrowed, with a view to the detection of the situations of those roots of the derived equations which require to be passed over, or separated, before the desired arrangement can be presented. In the search after these extraneous roots, in the proposed interval, we are not supplied by FOURIER, with any guide; but are left principally to mere random conjecture, as to the situations where they are most likely to be found; so that this part of the analysis is purely tentative. When the desired arrangement of three consecutive indices is obtained, the trial-transformations, in the prosecution of the analysis, are limited within a certain definite range of values, fixed by the criterion [A] at page 165: and in our practical applications of FOURIER'S method, our substitutions have always been controlled by this formula; although FOURIER himself never makes any distinct reference to it for this purpose. Now in carrying out FOURIER'S plan—in connexion with HORNER'S method of effecting the several transformations, and of arranging the successive steps in one continuous process—we shall not be left without the necessary guidance to those roots of the derived equations, which occur in the same interval as those of the primitive, and which we wish to exclude from that interval. For the arrangement adverted to presents us with trial-divisors, suggestive of the figures which enter one after another into the development of the root to be excluded. These trial-divisors, it is true, may at first suggest figures wide of the truth: but the modifications which they undergo, in passing into the character of true divisors, are always to be more or less accurately anticipated from an inspection of the antecedent tributary columns of the work; which inspection the arrangement peculiar to the method greatly facilitates.

The correct figures of the root, to which our approximation is more immediately directed, may thus be rapidly, and accurately, determined; so that by increasing either of these figures by unity we can inclose the root spoken of within two consecutive numbers whenever we please, without expending any useless trial-transformations. And thus the desired arrangement in the advanced indices will be facilitated. When this arrangement is brought about, we have seen, by the foregoing precepts and examples, how we may pursue the analysis without any of that ambiguity attending our steps, which is unavoidable when those steps may range anywhere within the limits fixed by FOURIER'S criterion. The separation of the roots in the second of the preceding examples, with nothing but this criterion to guide us, would be nearly impracticable.

(186.) It is thus by combining the accurate and comprehensive theoretical views of FOURIER with HORNER'S practical development, extended, as above, to the hitherto perplexing case of nearly equal roots, that the problem of the solution of numerical equations of the more advanced degrees may be reduced to a practicable form.

The principles delivered in the present chapter, taken in conjunction with`what is established at (101) respecting the criteria of equal roots, and combined with the theorem of BUDAN (107), contain all the essential elements of this solution.

(187.) There is no absolute necessity for FOURIER'S formula [A], in order to complete the analysis which the theorem of BUDAN but partially accomplishes. The precepts (172) and the directions at pages 298-9 will, in general, effect the objects of that formula more readily, and with quite as much certainty. We should not in general, therefore, recommend any appeal to this test: unless indeed immediately before the initial step in the approximation for separating the roots suspected to be nearly equal—after the leading figure common to both has been found—or, which is the same thing, after the consecutive numbers between which they lie have been determined—by the theorem of BUDAN. But if in any case a further reference to FOURIER'S test be thought advisable, the foregoing directions will guide us to the exact spot where that reference should be made.

(188.) We shall now show how the analyses of the foregoing examples are to be effected by aid only of the principles and precepts here referred to.

The first of those examples is

$$x^5 + 173x^4 + 2356x^3 + 10468x^2 - 14101x + 4183 = 0.$$

And by (89) a superior limit to the positive roots is 3: hence, in seeking the situations of these, we have merely to contract the interval [0, 3]: it is plain, from the variations in the signs of the terms, that there can be but two such roots. It is further obvious, without any calculation, that the transformation (1) will have all its terms positive. Hence,

(0) . . $+1+173+2356+10468-14101+4183$ *two variations.*

0 1 2

(1) . . $+ \quad + \quad + \quad + \quad + \quad +$ *no variation.*

Consequently two roots are indicated in the interval [0, 1].

In order to examine the negative intervals it will be convenient to change the alternate signs, or, which is the same thing, to substitute $-x$ for x, and then change *all* the signs, writing the original polynomial thus

$$-f(-x) = x^5 - 173x^4 + 2356x^3 - 10468x^2 - 14101x - 4183.$$

As the superior limit here is so great as 174, it would be imprudent to advance in our transformations merely by *units*. Advancing at first by 10, we see, without any calculation, that the transformation (10) must render the third, and all the following coefficients, negative; so that two roots will be indicated between 0 and 10; or, in the proposed equation, between 0 and -10. Subdividing now this interval, in order to find between what consecutive numbers the roots are to be sought, we have the following results:

(0) . . . . $1-173+2356-10468-14101-4183$ *three var.*

(1) . . . . $1-168+1674-4428-28656-26568$

(2) . . . . $1-163+1012-404-33157-58145$

(3) . . . . $1-158+370+1664-31576-90856$

(4) . . . . $1-153-252+1836-27765-120555$

(5) . . . . $1-148-854+172-25456-146888$

0 1 2 2

(6) . . . . $1-143-1436-3268-28261-173173$ *one var.*

Hence, two roots are indicated between 5 and 6; or, in the original equation, between -5 and -6. As only one other root remains, the situation of it may be found from the proposed polynomial alone: we thus discover it to lie between -100 and -200.

Two doubtful intervals therefore remain to be examined; viz., the intervals [0, 1] and [-5, -6]; which latter, in reference to the transformation (5), is also the interval [0, 1]. If the roots of the equation $f_1(-x) = 0$; in this last interval, are real, they will either commence with the same leading decimal, or will separate at this decimal. They cannot commence with the same figure, because of the palpable discrepancy between $\frac{172}{3 \times 854}$ and $\frac{25456}{172}$: and, by actually performing the first step of the approximation to the root of $f_2(-x) = 0$, we find that the roots of $f_1(-x) = 0$ do not separate. We conclude therefore that they are imaginary. The same conclusion would have immediately resulted from the criterion of FOURIER; which gives

$$\frac{f_1(5)}{f_2(5)} = \frac{2546}{2 \times 172} > 1.$$

Examining now the the interval [0, 1], in the original equation, we find no inconsistency between the trial-expressions $\frac{14101}{2 \times 10468}$ and $\frac{2 \times 4183}{14101}$. It is true that ·6 is suggested by the former, and only ·5 by the latter; but, upon trial, we find ·6 to exceed the root of $f_1(x) = 0$, to which our approximation is directed: therefore ·5 is the true figure. Setting out with this, we proceed, step by step, guided by the expressions (C), (D) at page 263, as at page 272; where we see that the roots separate at the seventh decimal.

(189.) In the second example

$$x^6+378x^5+38189x^4+492368x^3-572554x^2+213720x-26352=0$$

3 is a superior limit to the positive roots (89). But it is easily

seen, from an inspection of the coefficients, that the transformation (1) will render them all positive; so that three variations will disappear in passing over the interval [0, 1]. We have, therefore,

(0)	$+1$	$+378$	$+38189$	$+492368$	-572554	$+213720$	-26352
				0	**1**	**2**	**3**
(1)	$+$	$+$	$+$	$+$	$+$	$+$	$+$

Now as there is a perfect consistency among the expressions

$$\frac{572554}{3\times 492368},\quad \frac{213720}{572554},\quad \frac{3\times 26352}{213720},$$

as far as the initial-figure in each quotient is concerned, we take this initial-figure, ·3, for the first figure of the indicated roots; and proceed, guided in like manner, to each successive figure, as at page 280; when we find the roots to separate as there shown.

As the remaining roots are wide apart, their situations will be made known by Budan's transformations, without the aid of any additional principle.

It may be proper to remark here, in reference to the doubtful interval [5, 6], in the former example, that if the roots of $f_1(-x)=0$ had been real, we should have continued to develop the root of $f_2(-x)=0$, as in this example, extending the work up to $f(-x)$, till the roots of $f_1(-x)=0$, or of $f(-x)=0$, had separated. And thus should we proceed in all cases where several final indices are alike, remembering to use for $-\mathrm{N}'$ the proper expression (169) due to the earliest of these indices.

(190.) In order to illustrate still further the practical efficacy of the theory established in the present chapter we shall apply it to the equation

$$12x^3 - 120x^2 + 326x - 127 = 0$$

which has already been analysed, by help of the criterion of Fourier, at page 244.

The doubtful interval is found, as at page 243, to be [4, 5]; that is, the first figure of the root of $f_1(x) = 0$, to which our approximation is to be directed, is 4; or taking the equation already obtained from transforming by this 4; agreeably to the partial analysis at page 243, the first figure will lie between 0 and 1. The true first figure is to be found, as usual, by aid of the trial-divisor, in this case 2×24, conformably to the precepts at (172): the dividend $-$ A′ being 58. Regard being paid, however, to the influence of the preceding coefficient 12, it is easily seen that the figure sought is ·7

$$\begin{array}{rrrrl}
12 & +24 & -58 & -25 & (\cdot 7 \\
 & 8\cdot 4 & 22\cdot 68 & -24\cdot 724 & \\
\cline{2-4}
 & 32\cdot 4 & -35\cdot 32 & -\cdot 276 & \\
 & 8\cdot 4 & 28\cdot 56 & & \\
\cline{2-3}
 & 40\cdot 8 & -6\cdot 76 & & \\
 & 8\cdot 4 & & & \\
\cline{2-2}
 & 49\cdot 2 & & &
\end{array}$$

It is obvious that from this step the trial-divisors become fully effective; that is, $-\frac{A'}{2A'_2}$ will henceforth always furnish the correct root-figure. If the doubtful roots of $f(x) = 0$, in the interval under examination, are real, they have ·7 for a common leading figure: they have also the number suggested by the second trial-divisor, that is, by $2 \times 49\cdot 2$, and which is the number 6, for the common second figure, provided this same 6 is furnished by $\frac{2N'}{A'}$; otherwise the roots, if real, must separate after the first figure (172). The latter alternative must have place, since the number given by the second expression is not 6 but 8. Transforming by ·06, the roots do not separate:—no changes are lost. Hence, if they are real they must of necessity separate for the transformation ·07 (181): but this causes *two* variations to disappear. Consequently the roots are imaginary.

Fourier's test applied to the transformation (·76) authorizes this conclusion immediately, as at page 244, and thus saves the additional transformation. The discrepancy, however, which it required this transformation to interpret, pointed out the exact place where Fourier's test ought to be introduced. But, in strictness, we should not regard the entire transformation, but only that portion of it which determines the absolute number, as expended in testing the character of the roots: we refer to the transformation (·07), which is deduced from the transformation (·06), by transforming by an additional unit. For where actual solution is the object, the process of development ought not to terminate as soon as the roots are ascertained to be imaginary. We ought, on the contrary, to continue this development till the absolute number becomes stationary in its leading figures, and converges towards zero only as respects its remote decimals: after which, it should be matter of deliberation whether we are to reject the root as imaginary, or to retain it as real, rejecting only the imaginary increment (p. 163). It is a very prevalent, but a very grave mistake, to admit only the approximate values of the real roots of an equation, and to reject indiscriminately all those that are imaginary. It is an important, though we believe hitherto an unnoticed fact, that in the case of imaginary roots the real development, carried on up to the point adverted to above, is of far more practical consequence than would be the complete determination of the imaginary roots themselves; since from these roots the real development could not be inferred. In other words, a more effective and available solution would be furnished to an equation by substituting the real development here mentioned for the imaginary roots, than by actually exhibiting those roots themselves in their complete form. This, doubtless, seems paradoxical; but the truth of the statement will appear presently.

(191.) We shall now return to our equation and pursue the development of the root of $f_1(x) = 0$ till N′ ceases to diminish in its leading decimals, and thus becomes convergent, not towards zero, but towards a finite constant.

```
12   +24          −58           −25             (·76705
       8·4         22·68        −24·724
     ———          ———           ——————— 1
      32·4        −35·32         −·276
       8·4         28·56         −·225888
     ———          ——— 1         ——————— 2
      40·8         −6·76         −·050112
       8·4          2·9952       −·002564044
     ——— 1        ———           ——————— 3
      49·2         −3·7648       −·047547956
        ·72         3·0384       −·000000151
     ———          ——— 2         ——————— 4
      49·92        −·7264        −·047547805
        ·72         ·360108
     ———          ———
      50·64        −·366292
        ·72         ·360696
     ——— 2        ——— 3
      51·36        −·005596
        84          ·002581
     ———          ———
      51·444       −·0030|15
        84           26
     ———          ——— 4
      51·528       −·0004
        84
     ——— 3
    5|1·6|12
```

It is not necessary to extend the operation further: the absolute number, as far as the first four or five figures of it, will obviously remain constant, however far the root of $f_1(x) = 0$ be carried. The necessity for an imaginary increment clearly discovers itself after the transformation **2**.

It may be remarked here, that after having obtained the transformation **2** above, we might, agreeably to the directions before given, increase the root-figure ·06, which led to that transformation, by unity, in order to decide whether the roots separate at the figure ·7, or are imaginary, and might perform the following bye-operation:

$$\begin{array}{lrr}
51{\cdot}36 & -{\cdot}7264 & -{\cdot}050112\ ({\cdot}01 \\
 & {\cdot}5136 & {\cdot}002128 \\
\cline{2-3}
 & -{\cdot}2128 & -{\cdot}047984
\end{array}$$

The final sign being minus shows that the roots do not separate, and that they are therefore imaginary.

(192.) The development just exhibited when the 4, by which the roots were at first diminished, is introduced, is 4·76705. It would obviously be a real root of the proposed equation if the remainder —·047 . . . above were actually zero; that is, if the original absolute number had been 127·047 . . . instead of 127. In other words the value 4·76705 is accurately a root of the equation

$$12x^3 - 120x^2 + 326x - 127{\cdot}047 \ldots = 0 \ldots [1]$$

as far as five places of decimals. And from the final result of the middle column of the work above, it is clearly also a root of the limiting equation

$$36x^2 - 240x + 326 = 0 \ldots [2]$$

to the same extent of decimals. Thus the equation [1] has a pair of roots equal to 4·76705. And this same pair must be regarded as approximate roots of the proposed equation

$$12x^3 - 120x^2 + 326x - 127 = 0,$$

provided the left-hand member of [1] be regarded as an approximation to the left-hand member of this last equation; or provided the nature of the inquiry be such that the 127 may be taken for 127·047 . . . without transgressing the limits of error which that inquiry may be permitted to involve, without appreciable injury to its conclusions.

We know that in almost every physical inquiry the numbers furnished by observation, and of which the coefficients of the equations employed in that inquiry are functions, are more or less affected with error: and thus, on the principle of disregarding imaginary roots, certain real solutions of importance might be rejected as imaginary, and on the other hand imaginary roots might be replaced by real. It would doubtless be the safer, as well as the more consistent plan to retain, as real approximate solutions, all those developments which, like that above, really indicate imaginary roots, but yet accurately solve equations which are close approximations to those under discussion.

If we were actually to determine the complete imaginary roots of the equation here treated, we should find, as stated above, that they would afford us no clue to those approximate real values which, for a proposed extent of decimals, satisfy the equation more nearly than any other real values of like extent: for these real values are not, in general, the real parts of the imaginary roots. In the example before us, the *real root* of the equation, computed to seven places of decimals, is ·4656774,[*] so that *twice*

---

[*] The actual development of this root is as follows:

```
12      — 120       + 326        127          (·46567742
           4·8      —46·08       111·968
        ———————     ———————      ———————
        —115·2      279·92       15·032
           4·8      —44·16       13·768032
        ———————     ———————      —————————
        —110·4      235·76        1·263968
           4·8       —6·2928      1·113504
        ———————     —————————    —————————
        —105·6      229·4672      ·150464
           ·72       —6·2496      ·133273
        ———————     —————————    —————————
        —104·88     223·2176       17191
           ·72       — ·5169       15544
        ———————     —————————    —————————
        —104·16     222·700|7       1647
           ·72        —·517         1554
        ———————     —————————    —————————
        —103·44     222·184           93
             6         —62            89
        ———————     —————————         ——
        —103·3|8     222·12|2          4
              6           6
        ————————    —————————
       —1|0|3|·3    2|2|2|·0|6
```

the real part of the imaginary pair is $\frac{120}{12}$ — ·4656774 : that is, the imaginary roots are 4·76716 . . . $\pm \beta \sqrt{-1}$. And thus the imaginary increment $\alpha \pm \beta \sqrt{-1}$ (foot-note, page 163), required to complete the approximate root 4·76705, is ·00011 . . . $\pm \beta \sqrt{-1}$; but we have nothing to guide us to the real part of this imaginary increment, whether the imaginary part be determined or not: so that it is really true that the development 4·76705 furnishes a more accurate and effective real solution to the equation proposed than the complete imaginary roots themselves could supply.

(193.) It might not be amiss to call such developments *imperfect roots* of the equation. As remarked at page 162 (foot-note), these developments are something more than mere *indicators* of imaginary roots: they would be perfect real roots of the equation were a slight correction introduced into the absolute term: and whether in reference to this correction, or in reference to the imaginary increment omitted, the designation of *imperfect roots* seems sufficiently expressive of the peculiarities by which they are distinguished: and thus, real numerical results, available in actual practice, become redeemed from the neglected mass of imaginary quantities.

With respect to the second class of imaginary roots, commented upon at page 162, and which are *merely indicated* by the developments of roots of certain limiting equations inferior in degree to $f_1(x) = 0$, they are not to be replaced by real imperfect roots, like those of the first class: the imperfect roots which supply these latter indications replace imaginary roots in the inferior limiting equations, and not those in the primitive equation. We shall merely remark further, that after it was ascertained, as above, that the roots under examination were imaginary, we might have developed the corresponding pair of imperfect roots to any extent of decimals without regarding the final column of the work, which is useless for that purpose. But, by completing this column, simultaneously with the others, we ascertain the amount of correction for N that would render the imperfect root perfect, which correction it is necessary to know. We might thus express the general solution of the preceding equation as follows:

One root, ·4656774 . . . . : One pair of imperfect roots, 4·76705 : Correction of N = ·047.

(194.) From what has now been delivered it appears that we have at least three efficient and easily applied methods for determining the true character of doubtful roots. Each is individually fully competent to remove the doubt: but they are so related to one another that, as already observed at (183), each becomes conclusive at nearly the same stage of the analysis; at which point either may be indifferently applied; or, if need be, they may be concurrently employed as mutually confirmatory of one another. The chief matter of importance is not which criterion shall be used, but how we may approach, with most certainty and ease, the point where either becomes effective. And for this we have furnished ample directions in the present chapter.

The first of the methods here adverted to is that of FOURIER, taken in connexion with the improved method of subdividing the doubtful interval, explained and illustrated in the preceding rules and examples.

The second method is that developed at (168-9), embodied in the precepts (172), and commented upon at pages 297-302; and which dispenses with the criterion of FOURIER.

The third method is founded upon the principle delivered in the foot-note at page 163 and adverted to further, at (192-3). Agreeably to this principle we are to develop the intervening single root of the derived equation immediately inferior in degree to that in which the doubtful pair occurs in the same interval, just as in the other methods. This development we are to continue, as in the example above, being guided to the successive figures by the expressions [c] at page 263 either till it becomes obvious that the absolute number N′ is converging, not to zero, but to a finite constant, and consequently can never change its sign however far that development be continued—a clear proof of imaginary roots (182), or till a change of sign actually presents itself in N′, thus announcing that the roots have really separated.

It is probable that this last method, being altogether independent of external tests or bye-operations, may be found, in practice, as useful and convenient as either of the others. For if the roots under examination turn out to be real, we shall thus have found the development of one of them at the same time as we have determined their character; and if they prove to be imaginary, we shall, in like manner, with the detection of their nature, have accomplished the development of the pair of imperfect roots which supply their place.

(195.) We are thus furnished with means, more than sufficient, for overcoming all the difficulties so long attendant upon the problem of the general solution of numerical equations:—there is no conceivable case of this problem which the methods now developed can be found inadequate to cope with.

The practical facility with which these methods may be brought into operation depends mainly, as we have abundantly shown, upon the ease and rapidity with which we can develop a single isolated root, situated in a known interval, or having a known leading figure. In further attempts to expedite the numerical process, attention should be chiefly directed therefore to this—the simplest of the various cases that can occur. The trial-divisors, which are to direct us to the successive figures of this isolated root, will become gradually more and more efficient as the preceding tributary coefficients in the successive transformations become more and more unimportant in numerical magnitude, in relation to those divisors. When therefore the proposed equation is such that the coefficients diminish considerably in magnitude from the first to the last, the case will, in general, be unfavorable to the early efficiency of the trial-divisors. But it may be converted into a favorable case by simply reversing the order of the coefficients; thus changing the equation into another whose roots are the reciprocals of those in the original. This is the transformation which we adverted to at page 90, as occasionally useful in reference to HORNER's method of developing the roots of equations.

(196.) We might here terminate these researches: but the beautiful theorem of STURM, which they in a great measure supersede as respects equations beyond the fourth degree, demands a few additional observations, more especially in reference to the abbreviated form of computing the functions exhibited at pages 273 and 288-9.

It is obvious, from inspecting these computations, that a very considerable saving, both of space and labour, is effected by means of the reductions there employed. The calculation of the complete coefficients that enter the advanced functions in STURM's series is, as we have sufficiently seen in the preceding examples, a work of great arithmetical labour in high equations with rather large coefficients: and the principal part of this labour is expended upon what invariably turns out to be mere useless redundancies. It is superfluous, in the computation of STURM's functions, to aim at a greater degree of accuracy than is sufficient to secure correctness in the leading figure of the final constant, or in fact in the sign merely of this figure. We are not, however, in possession of any means by which all superfluous figures may be excluded; all that we can do is to provide against the entrance, into the final result, of more than a predetermined extent of figures, by some such method of setting bounds to their increase, as that recommended at (156).

We have acknowledged at p. 235 that this method is open to the objection of presenting the final result, in certain extreme cases, abridged of all its significant figures, leaving only a row of zeros, from which of course the character of one pair of roots would remain dubious.

In the case of a pair of roots, accurately equal, the final result would be accurately a row of zeros; and the repeated root would be accurately obtained by equating the preceding function, of the first degree, to zero: this function being the common measure of X and X_1, and hence the root of it a common root of $X = 0$ and $X_1 = 0$. But when significant figures occur in the final result, then there is only an approach to these circumstances; which approach must evidently become more and more close, as the first significant figure recedes more and more to the right; leaving a larger number of consecutive zeros, or unoccupied places, before it. It has been mentioned in the introductory treatise so often referred to, page 224, and will be proved hereafter, that the pair of *nearly* equal roots thus indicated must concur in their leading figures to about half the number which expresses these unoccupied places; that is, supposing the roots indicated to be real and not imaginary. But in either case it is clear, that a slight variation in the final term of the original equation would convert the roots indicated into a real equal pair. Hence, although the character of the roots indicated by STURM's final remainder, when preceded by several unoccupied places may be altogether unknown, from our curtailments having caused all but the zeros to disappear, yet we may confidently infer that an equation, differing from that proposed only by a minute variation in its final term, will have two equal roots, each given by equating the function of the first degree to zero, its other roots being the same as those of the proposed. If six or seven zeros, or blank places, supply leading figures in the remainder, then, by the principle adverted to above, we may conclude that the two equal roots mentioned will agree with the two roots of the proposed equation to about three places of figures; if eight or nine zeros occur, to about four places, and so on, when the latter roots are real. When they are imaginary, the same equal roots may nevertheless be taken, to the same extent of places, as approximate roots of the proposed equation, or as what we have called real *imperfect roots;* and which, as we have already seen, deserve to be taken into account in the practical solution of numerical equations.

Suppose then that, in employing STURM's method for the analysis of equations, we so regulate our abbreviations, where large numbers are involved, as to secure accuracy to the extent of ten or twelve places in the final remainder. Should these places all turn out to be blank, we may safely take the root of the simple equation, furnished by the preceding remainder, for one of a pair of equal roots of the proposed.*

---

* It should be borne in mind that in all practical inquiries it is a waste of effort to attempt an accuracy in which even the data of our calculations is deficient: such an attempt is moreover as likely to lead us wrong as right.

The following observations, by Professor PEACOCK, are so completely in harmony with some of the views set forth in the present chapter, that the author feels it obligatory upon him to quote them:

"If the root of an equation be determined approximately, the equation may

If, however, for any other purpose, apart from practical utility, it be desired to determine the exact character of the doubtful roots then, without caring to secure half the above-mentioned number of true places in the final remainder, we may proceed to develop the root of $X_1 = 0$, commencing our approximation with the leading figure furnished by the simple equation adverted to, and extending the process up to X, as at page 271, till, by aid of **Fourier's** criterion, or the other tests proposed in the present chapter, the nature of the roots is ascertained.

If the function immediately preceding the last—the function of the first degree—have both its coefficients in like manner preceded by blank places or zeros, we may infer the approach of three roots of the proposed towards equality, or else of two distinct pairs of roots : for it is plain that the antecedent function of the second degree would, under these circumstances, require no change to be made in the *leading* figures of its coefficients to

be depressed, and the general processes of solution, or of approximation, may be applied to find the roots of the quotient of the division. Thus in the equation

$$x^3 - 3x + 2{\cdot}0000001 = 0$$

one of the roots is very nearly equal to 1 : if we divide the equation by $x - 1$, and neglect the small remainder which results from the division, we shall get the quotient

$$x^2 - x - 2 = (x - 1)(x + 2) = 0$$

whose roots are 1 and -2; or we may suppose one of the roots to be 1·0001, the second ·9999, and the third -2; or we may suppose two of the roots to be imaginary, namely, $1 \pm {\cdot}001\sqrt{-1}$. All these roots are approximate values of the roots of the equation, which different processes, whether tentative or direct, may determine: and it is obvious that when two roots are equal, or nearly so, an inaccuracy of the approximation to those roots which are employed in the depression of the primitive equation, may convert real roots into imaginary, or conversely. Such consequences will never follow when the limits and nature of the roots are previously ascertained, and every root is determined independently of the rest: but it is not very easy to prevent their occurrence when methods of approximation are applied without any previous inquiries into the nature and limits of the roots, though the resulting conversion of imaginary roots into real, and of real roots into imaginary, may not deprive them of the character of true approximations to the values of the roots which are required to be determined."—*Report of the Third Meeting of the British Association*, p. 349.

render it an exact divisor of the cubic that precedes it, or, which is the same thing, to render the coefficients of the functions of the first degree, referred to, accurately zero, and, consequently, the quadratic function accurately a common divisor of X, and X_1. Hence, if the roots of the quadratic are equal, three such roots must enter the equation $X = 0$; if they are unequal, each must enter twice into $X = 0$. An examination of the leading figures of the roots of this quadratic must determine to which of these circumstances the case before us approximates. These leading figures will supply, as before, the first steps in the approximations towards the doubtful roots.

In this manner may all ambiguity that might otherwise attend the more advanced functions of STURM, when extensively curtailed, be satisfactorily cleared up. And instead of thus computing all the functions up to the last, we may, if we please, stop at the quadratic, as recommended in the treatise on *Cubic and Biquadratic Equations;* and instead of examining the intervals thus left in doubt, by the method there taught, we may proceed to discuss them agreeably to the directions given in the present chapter.

We thus see that by combining the methods of FOURIER and HORNER with that of STURM, the calculations which would otherwise enter into this last method may be very considerably reduced.

(197.) But before finally dismissing this method of STURM it may be proper to show that it is in itself fully adequate, not only to determine the character of the roots of an equation under all circumstances, but likewise to remove every ambiguity that might present itself in the course of their subsequent development, without the aid of any other theorem whatever—even the theorem of DESCARTES, called the *rule of signs.*

The following example, taken from STURM's *Mémoire,* is sufficient to show this; and to point out the mode of proceeding whenever any such ambiguity occurs.

Let the equation

$$x^3 + 11x^2 - 102x + 181 = 0$$

be proposed for analysis and solution.

First, in order to ascertain the number and situation of the roots, we form the functions

$$\begin{aligned}
X &= x^3 + 11x^2 - 102x + 181 \\
X_1 &= 3x^2 + 22x - 102 \\
X_2 &= 854x - 2751 \\
X_3 &= + ;
\end{aligned}$$

from which, as all the leading signs are +, we infer that all the roots are real (page 214.)

To determine the intervals of the positive roots, we make the substitutions

$x = 0$, which gives	+	−	−	+	*two variations*
$x = 1$ . . .	+	−	−	+	
$x = 2$ . . .	+	−	−	+	
$x = 3$ . . .	+	−	−	+	*two variations*
$x = 4$ . . .	+	+	+	+	*no variation.*

Hence the equation has two positive roots, both comprised between 3 and 4; so that the first figure, common to both, is 3. Therefore, by our method of approximation, the first step of the process will be as follows:

$$\begin{array}{lrr}
 & & 11 \\
 & -102 & -181\ (3 \\
14\ .\ .\ . & 42 & -180 \\
\hline
 & -60 & -1 \\
 & 9 & \\
\hline
 & -9 & \\
20\ .\ .\ . & &
\end{array}$$

and the resulting transformed equation, whose roots are those of the original, diminished by 3, is

$$x'^3 + 20x'^2 - 9x' + 1 = 0 \quad . \; . \; . \; . \; [2].$$

The first figure of the root of this, or the second figure in the quotient above, appears to be ·2, because, of all numbers occupying the place of the second figure, we find this to be the one which produces a result nearest to — 1. Still we cannot always affirm, independently of all reference to any other principle, that the number which produces a result nearest to the absolute number, or which, when the terms are all arranged on one side, produces a result the nearest to zero, is necessarily the first figure of the root, unless the next figure in the scale produces a change of sign in the absolute number, which is not the case here. To test the figure ·2, therefore, we transform all the other functions, as well as the first X, by diminishing the value of x in each, by 3, as above; and we find these results, viz.

$$\begin{aligned} X' &= x'^3 + 20x'^2 - 9x' + 1 \\ X'_1 &= 3x'^2 + 40x' - 9 \\ X'_2 &= 854x' - 189 \\ X'_3 &= + ; \end{aligned}$$

which, for $x' = \cdot 2$, gives the series $+ \; - \; - \; +$ *two variations*

and for $x' = \cdot 3$ . . . . $+ \; + \; + \; +$ *no variation ;*

so that two roots of [2] are comprised between ·2 and ·3, and thus ·2 is the correct second figure of both roots of [1].

If the substitutions, ·2, ·3, had not given series of signs, differing by two variations, we should have concluded that the root figure, ·2, was incorrect ; and should have continued to substitute, in the transformed functions, the successive values, 0, ·1, ·2, ·3, ·4, ... 1, till such a difference of variations had been obtained, and should have taken the less of the two numbers, to which the change was due, for the true second figure.

We may now, therefore, complete the second step of our solution, and shall thus have

		11	
	— 102	— 181	(3·2
14 . . .	42	— 180	
	— 60	— 1	
	9	— ·992	
	— 9	— ·008	
20·2 . . .	4·04		
	— 4·96		
	4		
	— ·88		
20·6 . . .			

and may thence proceed, without hesitation, to any extent, with our approximation; for the trial-divisor indicates ·01 as the third figure of the root, and, trying ·02, we find a change of sign in the absolute number, a sure indication that the two roots separate at the second figure, and become distinct. The roots, as far as three decimals, are 3·213 and 3·229.*

To determine the remaining two roots of an equation after the others have been computed.

(198.) Let the second term of the proposed equation be removed by (79) before any of the roots are developed; or, if this term be retained, let the developed roots be reduced to those belonging to the equation after the removal of the second term. We are to suppose that all the roots but two have thus been found,

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* For a further discussion of the theorem of STURM, with examples of its application to equations of the seventh degree, the reader is referred to the *Mathematical Dissertations*, by the author of the present work.

and our object is to obtain a formula for the determination of these two. It will be sufficient, to illustrate the method proposed, to confine our investigation to equations of the sixth and inferior degrees. Let then the equation of the sixth degree, deprived of its second term as proposed, be generally represented by

$$x^6 + px^4 + qx^3 + \ldots\ldots = 0.$$

and let its six roots be denoted by

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,$$

the sum of which must be zero on account of the absence of the second term of the equation; so that

$$x_1 + x_2 + x_3 + x_4 = -(x_5 + x_6) \ldots\ldots [1].$$

Now p being the sum of the products of all the roots taken two and two, we have

$$p=(x_1+x_2+x_3+x_4)\ (x_5+x_6)+x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4+x_5x_6$$

$$=-(x_1+x_2+x_3+x_4)^2+x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4+x_5x_6$$

$$\therefore p+(x_1+x_2+x_3+x_4)^2-(x_1x_2+x_1x_3\ldots\ldots x_3x_4)=x_5x_6\ldots\ldots[2].$$

Equations [1] and [2] furnish expressions for the sum and product of the two roots x_5, x_6; and consequently for the coefficients of the quadratic involving them. Hence, for the roots of this quadratic, that is, for the values of x_5 and x_6, we have the formula

$$-\frac{x_1+x_2+x_3+x_4}{2}\pm\surd\left\{-3\left(\frac{x_1+x_2+x_3+x_4}{2}\right)^2-p+x_1x_2+x_1x_3+\ldots x_3x_4\right\}\ ..[3]$$

which will give the remaining two roots of an equation of the sixth degree after four are determined.

If the equation be only of the fifth degree then $x_4=0$; and the formula becomes

$$-\frac{x_1+x_2+x_3}{2}\pm\surd\left\{-3\left(\frac{x_1+x_2+x_3}{2}\right)^2-p+x_1(x_2+x_3)+x_2x_3\right\}\ ..[4].$$

If it be of the fourth degree, then $x_3=0$, and for the two remaining roots, after two are determined, we have

$$-\frac{x_1+x_2}{2}\pm\surd\left\{-3\left(\frac{x_1+x_2}{2}\right)^2-p+x_1x_2\right\}\ldots[5]$$

And finally when it is of the third degree, or $x_2=0$, we have for the required form the expression

$$-\frac{x_1}{2}\pm\surd\left\{-3\left(\frac{x_1}{2}\right)^2-p\right\}\ldots[6].$$

This last expression is, we find, also given by GARNIER, in his *Analyse Algébrique*, page 216. The others are, we believe, new. It would be easy to vary their forms: indeed different forms for the third and fourth degrees have already been given in the introductory treatise, pp. 236-241. But the above are the most convenient on account of their simplicity, involving no division operations, nor any of the advanced coefficients of the original equation: so that if all the roots but two are computed, without depriving the equation of its second term, then, in making the requisite change in it, for the application of the preceding formulas, we may stop the process of transformation when p is obtained. But the most convenient way of obtaining p is from the formula

$$p=\frac{A_{n-2}}{A_n}-\frac{n(n-1)}{2}\left\{\frac{A_{n-1}}{nA_n}\right\}^2$$

which is readily deduced from the expressions at page 86.

For introducing the coefficient A_n of the first term into those expressions, in order to give them the greater generality, we have for the third term of the transformed equation

$$\frac{n(n-1)}{2}r^2+(n-1)\frac{A_{n-1}}{A_n}r+\frac{A_{n-2}}{A_n}$$

in which, that the second term of the transformed may vanish, we must have

$$r=-\frac{A_{n-1}}{nA_n}.$$

Putting therefore this in the third term above, that term becomes

$$\frac{n\,(n-1)}{2}\left\{\frac{\mathrm{A}_{n-1}}{n\mathrm{A}_n}\right\}^2-(n-1)\,\frac{\mathrm{A}^2{}_{n-1}}{n\mathrm{A}^2{}_n}+\frac{\mathrm{A}_{n-2}}{\mathrm{A}_n}$$

$$=\frac{\mathrm{A}_{n-2}}{\mathrm{A}_n}-\frac{n\,(n-1)}{2}\left\{\frac{\mathrm{A}_{n-1}}{n\mathrm{A}_n}\right\}^2$$

As an example of the application of these formulas we may take the equation

$$x^5-41x^3-12x^2+292x-240=0$$

whose roots are 1, 2, 6, -4, -5; and any three of these substituted in [4] will determine the remaining two.

But the utility of these expressions is not confined to this object: a reference to them will often save many steps of calculation in the analysis of equations. Thus, in treating the equation of the fifth degree, at (188,) by the method of FOURIER a good deal of calculation was found necessary in order to determine that two of the doubtful roots were imaginary: if these had been allowed to remain doubtful till the others had been developed, the character of the former might have been discovered, with but very little trouble, from the formula [4], after the determination of p, which is soon effected: thus,

$$p=2356-10\,(34{\cdot}6)^2=2356-11971{\cdot}6=-9615{\cdot}6$$

and it is obvious, from the formula, when the roots already developed are each increased by 34·6, in order to reduce them to those belonging to the equation after the removal of the second term, that the quantity under the radical will be

$$-3\left(\frac{53{\cdot}64}{2}\right)^2+9615{\cdot}6-123{\cdot}96\times70{\cdot}326+(35{\cdot}16)^2=-23.$$

so that the two roots x_4, x_5 are imaginary.

In equations of the fourth degree it will sometimes be found more convenient to substitute the following for the formula [5] above, viz.

$$-\frac{x_1+x_2}{2}\pm\sqrt{\left\{-2\left(\frac{x_1+x_2}{2}\right)^2-\left(\frac{x_1-x_2}{2}\right)^2-p\right\}}$$

As an example let the equation

$$x^4 + 312x^3 + 23337x^2 - 14874x + 2360 = 0$$

be proposed.

Two roots of this equation are found, in the introductory treatise, page 231, to be −126·3166644731 and −186·3166651784. The determination of the remaining roots, in the treatise referred to, was attended with considerable trouble, on account of their character remaining long doubtful from their close proximity to each other. If we change the proposed equation into another of the form

$$x^4 + 0x^3 + px^2 + \&c. = 0$$

by increasing the roots by $\frac{312}{4} = 78$, the two roots just exhibited will be changed into −48·3166644731 and −108·3166651784: and for p we shall have

$$p = 23337 - 6(78)^2 = -13167$$

so that the preceding formula will become

$$78{\cdot}3166648257 \pm \sqrt{\{-2(78{\cdot}3166648257)^2 - (30{\cdot}0000003526)^2 + 13167\}}$$

the two values given by which are 78·3166651 . . . . and 78·3166644 . . . . Consequently, subtracting the 78, by which the original roots were increased, we have for the two remaining roots of the proposed equation

$$x_3 = {\cdot}3166651 \ldots, \quad x_4 = {\cdot}3166644 \ldots$$

On the Determination of the Integral Roots by the Method of Divisors.

(199.) It was demonstrated at (62) that no equation in which the coefficient of the first term is unity, and those of the other terms integers, can have a fractional root; so that the roots of every such equation can comprise only whole numbers, and interminable decimals. These latter we have shown above how to approximate to as closely as we please; and, although the same method will furnish us, figure by figure, with every integral root also, yet it is worth while to explain here a distinct process for the discovery and determination of every such root. The method we advert to was proposed by NEWTON, and is called the *method of divisors.* We may apply it to detect fractional roots by (82).

Let

$$N + Ax + A_2x^2 + A_3x^3 + A_4x^4 + \dots . x^n = 0$$

be an equation of the nth degree, in which the coefficients are all whole numbers; and let a be an integral root of it, then we must have

$$N + Aa + A_2a^2 + A_3a^3 + A_4a^4 + \dots . a^n = 0$$

$$\therefore \frac{N}{a} = - A - A_2a - A_3a^2 - A_4a^3 - \dots . - a^{n-1}.$$

from which we infer that $\frac{N}{a}$ must be a whole number; hence every integral root must always be a divisor of the last term N. Call the quotient of this division Q, then, by transposing — A, and dividing by a, the last equation will become

$$\frac{Q + A}{a} = - A_2 - A_3a - A_4a^2 - \dots . - a^{n-2};$$

consequently, $\frac{Q + A}{a}$ is also a whole number, which, calling Q_2, and transposing $- A_2$, we have, after division by a,

$$\frac{Q_2 + A_2}{a} = - A_3 - A_4a - \dots . - a^{n-3};$$

hence $\frac{Q_2 + A_2}{a}$, or Q_3, is also a whole number; and, continuing this process, we shall obviously have the quotients

$$Q, \; Q_2, \; Q_3, \; Q_4, \dots . Q_n$$

all whole numbers, and the last Q_n, will be -1.

(200.) We infer, therefore, that for a to be an integral root of an equation, the last term must be divisible by it, and so must the sum of the quotient and next coefficient; and throughout, the sum of each coefficient and preceding quotient must be divisible by a, the final quotient being always -1; which are conclusions analogous to those at page 185.

Hence, after having determined all the divisors of the absolute term in an equation, we must submit all those of them which are between the limits $-L$ and $+L'$ of the roots, found by the rules in Chapter VI, to the foregoing tests, and retain only those divisors which satisfy them all.

(201.) When, however, one divisor is found to succeed, we need not, in order to test the others, return to the original coefficients, since, as it is easy to show, the quotients Q, Q_2, Q_3, &c., are no other than the coefficients of the depressed equation with their signs changed, or, which is the same thing, the coefficients in the quotient of $N + Ax + A_2x^2 \ldots\ldots x^n$ by $a - x$; for, by actually performing the division, and recollecting that

$$N = Qa,\ Q + A = Q_2a,\ Q_2 + A_2 = Q_3a,\ \&c. \ldots\ldots [1],$$

we have

$$\begin{array}{l} a - x)\ N + Ax + A_2x^2 + A_3x^3 \ldots\ldots (Q + Q_2x + Q_3x^2 \ldots\ldots \\ \qquad N - Qx \\ \qquad \overline{\qquad\qquad\qquad} \\ \qquad\qquad Q_2ax + A_2x^2 \\ \qquad\qquad Q_2ax - Q_2x^2 \\ \qquad\qquad \overline{\qquad\qquad\qquad\qquad} \\ \qquad\qquad\qquad Q_3ax^2 + A_3x^3 \\ \qquad\qquad\qquad Q_3ax^2 - Q_3x^3 \\ \qquad\qquad\qquad \overline{\qquad\qquad\qquad\qquad} \\ \qquad\qquad\qquad\qquad Q_4ax^3 + A_4x^4 \\ \qquad\qquad\qquad\qquad\qquad \&c. \end{array}$$

It follows, therefore, that a being a root of the proposed equation, the equation

$$Q + Q_2x + Q_3x^2 \;.\;.\;.\;. - x^{n-1} = 0 \;.\;.\;.\;. \; [2]$$

will be the depressed equation involving the remaining roots, for changing the signs of all the terms does not change the roots. Hence the other integral roots of the original equation will also be roots of this; so that, for the discovery of them, we may employ this depressed equation instead of the proposed. If we multiply every term of the depressed equation by a, keeping in mind the conditions [1] above, it will become

$$N + (Q + A)\,x + (Q_2 + A_2)\,x^2 \ldots . - ax^{n-1} = 0 \ldots . [3],$$

the roots of which are, of course, the same as those of [2]; so that, for the discovery of another integral root, we may, if we please, use the form [3] instead of [2], in which case the final quotient must be $-a$.

As an example, let us take the equation

$$x^5 + 5x^4 + x^3 - 16x^2 - 20x - 16 = 0.$$

The divisors of 16 are

$$\pm 16, \quad \pm 8, \quad \pm 4, \quad \pm 2, \quad \pm 1.$$

A superior limit to the positive roots is, by (87),

$$1 + \sqrt[3]{16} \text{ or } 4;$$

and, by substituting $-x$ for x in the proposed, or, which is the same thing, by changing the signs of the alternate terms, the equation will be

$$x^5 - 5x^4 + x^3 + 16x^2 - 20x + 16 = 0,$$

and a superior limit to its positive roots is, by (89), 6; but it is easy to see at a glance that 5 must also exceed the greatest positive root, therefore -5 is a limit to the negative roots of the proposed. Hence the divisors not within the limits -5, 4, that is, the divisors

$$\pm 16, \quad \pm 8, \quad +4,$$

must be rejected; we have, therefore, to try only the divisors ± 2 and -4:

$$\begin{array}{lrrrrrr}
+2)\,-16 & & -20 & -16 & +1 & +5 & +1 \\
& & -8 & -14 & -15 & -7 & -1\ (a) \\
\cline{3-7}
-2) & & -28 & -30 & -14 & -2 & 0 \\
& & 8 & 10 & 10 & 2\ (b) & \\
\cline{3-6}
-4) & & -20 & -20 & -4 & 0 & \\
& & 4 & 4 & 4\ (c) & & \\
\cline{3-5}
& & -16 & -16 & 0 & &
\end{array}$$

Hence $+2$, -2, and -4, are integral roots: the coefficients (a), (b), (c), are those of the successive depressed equations, reversed: the final depressed equation is

$$4x^2 + 4x + 4 = 0,$$

or rather

$$x^2 + x + 1 = 0,$$

the roots of which are imaginary. (*Algebra*, art. 107.)

We have not applied the method to the divisors $+1$ and -1, because it is easy to ascertain whether or not these are roots of the equation, and to depress the equation accordingly by (51). In fact the method of art. (51) will equally serve for the discovery of all the suitable divisors, and is perhaps on the whole but little inferior in facility to that above. We should indeed, by the method of (51), have in all cases to arrive at the final term of the transformation, before we could affirm that the number under trial was a root or not; whereas, in the method here explained, there is a chance of detecting the unsuitable divisors at every division, as the quotient may be fractional. It is scarcely necessary to observe that, when such quotients occur, the work is to be erased, and a new divisor tried; thus: suppose it were required to find whether the equation

$$x^3 - 37x + 72 = 0$$

has any integral positive roots. We readily see that 5 is a superior limit to the positive roots; so that the only divisors of 72 to be tried are 2, 3, and 4. Trying 2, we have

$$\begin{array}{rrrrr} 2) & 72 & -37 & 0 & +1 \\ & & 36 & & \\ \cline{3-3} & & -1 & & \end{array}$$

the divisor 2 must be rejected, as the next quotient would be fractional. Trying 3, we have

$$\begin{array}{rrrrr} 3) & 72 & -37 & 0 & +1 \\ & & 24 & & \\ \cline{3-3} & & -13 & & \end{array}$$

the divisor 3 is also unsuitable, as this gives Q_2 fractional. Lastly, trying 4, we have

$$\begin{array}{rrrrr} 4) & 72 & -37 & 0 & +1 \\ & & 18 & & \\ \cline{3-3} & & -19 & & \end{array}$$

which must be rejected for a like reason, so that there are no positive integral roots.

When the divisors of the last term between the limits $-L$ and $+L'$ are very numerous, the trials may become tiresome; but it is easy to devise a contrivance for diminishing the number of superfluous divisors thus:

(202.) We have seen (201) that

$$\frac{N + Ax + A_2x^2 + A_3x^3 + \ldots\ldots}{a - x} = Q + Q_2x + Q_3x^2 + \ldots\ldots$$

the second member being an integer for every integral value of x, because the coefficients are all integral; the simplest integral values of x are $+1$ and -1; hence the first member shows that when $+1$ is put for x, in the original polynomial $f(x)$, no divisor a can be admissible which does not render $\frac{f(1)}{a-1}$ an integer; and, putting -1 for x, we see that no divisor can be admissible which does not render $\frac{f(-1)}{a+1}$ an integer. The divisors between the limits may, therefore, be advantageously submitted to these tests before those at (200) are applied to them. We know from (53) that $f(1)$ will be the last term of the transformed equation in $(x-1)$, and $f(-1)$ will be the last term of the transformed equation in $(x+1)$; hence the best mode of proceeding will be, to effect one step of each transformation by (71), and to divide the final term in the first by each divisor *minus* 1, and the final term in the second by the same, *plus* 1; and then to employ only those divisors which furnish integral quotients. Should the final term in either transformation be 0, it will be a proof that the divisor unity is a root, and then we must employ the depressed equation for the other roots; the coefficients of this depressed equation will have been written down in proceeding to the final term, as at (51).

(203.) Let the equation

$$x^3 - 5x^2 - 18x + 72 = 0$$

be proposed.

The limit of the positive roots is $18 + 1 = 19$; and, changing the alternate signs, we have (88)

$$-\left(\frac{72}{5} + 1\right) = -15\tfrac{2}{5}$$

for the limit of the negative roots; hence the only divisors of 72 which can comprise the integral roots, are

$$2, \quad 3, \quad 4, \quad 6, \quad 8, \quad 9, \quad 12, \quad 18,$$

$$-2, \quad -3, \quad -4, \quad -6, \quad -8, \quad -9, \quad -12.$$

Let us, therefore, now proceed to determine $f(1)$ and $f(-1)$,

$$\begin{array}{rrrl} 1 & -5 & -18 & +72\ (1, -1 \\ & -4 & -22 & 50 = f(1) \\ & -6 & -12 & 84 = f(-1). \end{array}$$

Now those among the foregoing divisors, which, diminished by 1, divide 50, and which, increased by 1, divide 84, are

$$2, \quad 3, \quad -4, \quad 6;$$

and, by trying these in succession, we find 2 to fail; but for 3, -4, and 6, we have

$$\begin{array}{rrrr} 3)\ 72 & -18 & -5 & +1 \\ & 24 & 2 & -1 \\ \cline{2-4} -4) & 6 & -3 & 0 \\ & -18 & 3 & \\ \cline{2-3} 6) & -12 & 0 & \\ & 12 & & \\ \cline{2-2} & 0 & & \end{array}$$

hence the roots are all integral, and are 3, -4, and 6.

Newton's Method of approximating to the Incommensurable Roots of an Equation.

(204.) The method proposed by Newton for approximating to the incommensurable roots which may still exist in an equation, after the integral roots have been removed by the method of divisors, requires, like all other approximative methods, that we know the intervals in which the roots are situated. It requires, moreover, that before commencing the approximation to any root, we render the interval so narrow, that the extreme limits of it may not differ by more than $\frac{1}{10}$; in which case, either limit must be within the fraction $\frac{1}{10}$ of the value of the root. Call the initial value, thus obtained, x', and its difference from the true root δ: then, in the proposed equation $f(x) = 0$, we have

$$x = x' + \delta;$$

and, consequently,

$$f(x) = f(x' + \delta) = f(x') + f_1(x')\delta + \frac{f_2(x')}{2}\delta^2 + \&c. = 0;$$

and, since δ is less than $\frac{1}{10}$, δ^2 must be less than $\frac{1}{100}$, δ^3 less than $\frac{1}{1000}$, &c.; hence, rejecting the terms into which these diminishing factors enter, we have, for a first approximation to the value of the correction δ, the expression

$$\delta = -\frac{f(x')}{f_1(x')};$$

which will give the value true to two places of decimals: adding, therefore, this approximate correction to x', we obtain a nearer value, x'', to the root, the error δ' being below $\frac{1}{100}$.

For a second approximation, put

$$x = x'' + \delta',$$

then, proceeding as before, we have

$$\delta' = -\frac{f(x'')}{f_1(x'')},$$

which will usually give the value of the correction, as far as four places of decimals, and this correction applied to x'' will give the more correct value x''' for x, being the true value, as far as about four decimals; and, by repeating the operation, we shall get a new value, true to about eight decimals, and so on.

The following is the example chosen by **Newton** to illustrate his method, viz.

$$f(x) = x^3 - 2x - 5 = 0, \ \therefore f_1(x) = 3x^2 - 2.$$

The root of this equation lies between 2 and 3; to narrow these limits, diminish the roots of the transformed in $x - 3$, by ·5, and we shall find no change of sign in the final term; hence the root is between 2 and 2·5. Diminish the roots of this transformed by ·4, and still the final sign is preserved; hence the root is between 2 and 2·1, so that the first two figures of it must be 2·0, that is,

$$x = 2{\cdot}0 + \delta;$$

also,

$$\delta = -\frac{f(2{\cdot}0)}{f_1(2{\cdot}0)} = -\frac{-1}{10} = {\cdot}1$$

$$\therefore x = 2{\cdot}1 + \delta'$$

$$\delta' = -\frac{f(2{\cdot}1)}{f_1(2{\cdot}1)} = -\frac{{\cdot}061}{11{\cdot}23} = -{\cdot}0054$$

$$\therefore x = 2{\cdot}0946$$

$$\delta'' = -\frac{f(2{\cdot}0946)}{f_1(2{\cdot}0946)} = -\frac{{\cdot}0005416}{11{\cdot}16205} = -{\cdot}00004852.$$

$$\therefore x = 2{\cdot}09455148.$$

In this particular example the approximation is very rapid; this arises from the circumstance that, in the expressions for δ', δ'', &c. the numerators are very small when compared with the denominators; such, however, will not be the case, when the root, to which we are approaching, differs but little from another root; because, as the roots approach to equality, the expression $f_1(x)$, when the value of one of these roots is put for x, approaches to zero (98); and hence the denominators of the foregoing fractions will be very small, as well as the numerators. In such a case, too, the terms rejected in the values of δ', δ'', &c. might exceed in magnitude those preserved, and thus no approximation to the true corrections would be obtained. These imperfections in Newton's process render its application unsafe, when the root sought differs by only a small decimal from any of the other real roots, unless, indeed, at each approximation, we test the value obtained, by actually substituting it in the proposed equation.

As an illustration, let the equation,

$$x^3 - 7x + 7 = 0,$$

be proposed.

After a few trials, a root is found to lie between 1·3 and 1·4, and to be nearer to 1·4 than to 1·3. Let us assume then

$$x = 1{\cdot}4 + \delta,$$

then we have

$$\delta = -\frac{f(1{\cdot}4)}{f_1(1{\cdot}4)} = -\frac{{\cdot}056}{1{\cdot}12} = -{\cdot}05$$

$$\therefore x = 1{\cdot}35 + \delta'.$$

To verify this approximation, let 1·35 and 1·36 be separately put for x in the proposed equation, the results are

$$\text{for } x = 1{\cdot}35, \quad f(x) = +{\cdot}010375$$

$$\text{for } x = 1{\cdot}36, \quad f(x) = -{\cdot}004544$$

which, being of contrary signs, shows that our approximation is correct.

For a second approximation, we have

$$\delta'' = -\frac{f(1{\cdot}35)}{f_1(1{\cdot}35)} = \frac{{\cdot}010375}{1{\cdot}5325} = {\cdot}0068$$

$$\therefore x = 1{\cdot}3568.$$

To verify this approximation, let 1·3568 and 1·3569 be substituted for x, in the proposed, the results will be

$$\text{for } x = 1{\cdot}3568, \quad f(x) = + {\cdot}000141586432$$

$$\text{for } x = 1{\cdot}3569, \quad f(x) = - {\cdot}000006100991\,;$$

which, being of contrary signs, proves the correctness of our approximation: hence the root is between 1·3568 and 1·3569, the former number is, therefore, the true value, as far as four places of decimals.

It will not escape the observation of the student, that the process for the determination of the successive values of $f(x')$, $f(x'')$, $f_1(x')$, $f_1(x'')$, &c. as also the operations for verifying the several approximations, may all be conducted with great advantage, agreeably to the method of arranging the transformations uniformly employed throughout this volume.

(205.) Had this been the arrangement adopted by the original cultivators of the Newtonian method, that method would, no doubt, soon have been perfected into the more general and compact process of HORNER; and NEWTON'S divisor—which we see is nothing more than that which in the preceding pages has been employed as a trial-divisor—instead of being used to *determine* the figures of the root, would, in the early steps of the development, merely have been referred to to suggest those figures, and thence have been perfected into a true divisor before the operation with it came to be actually performed.

If we take NEWTON'S example at page 333, and arrange the steps of the work there given agreeably to the method of transformation referred to, the operation will stand thus:—

1	0	−2	5 (2·1,054485184577
	2	4	4
	—	—	— 1
	2	2	1
	2	8	1·061
	—	— 1	— 2
	4	10	−·061
	2	·61	−55992625
	— 1	—	— 3
	6	10·61	−5007375
	·1	·62	−4465824464
	—	— 2	— 4
	6·1	11·23	−541550536
	·1	−31475	−446471845
	—	—	— 5
	6·2	11·198525	−95078691
	·1	−31450	−89291957
	— 2	— 3	— 6
	6·3	11·167075	−5786734
	−05	−251384	−5580721
	—	—	—
	6·295	11·16456116	−206013
	−05	−251368	−111614
	—	— 4	—
	6·290	11·16204748	−94399
	−05	−25135	−89291
	— 3	—	—
	6·285	11·1617961|3	−5108
	−4	−2513	−4464
	—	— 5	—
	6·2846	11·1615448	−644
	−4	−502	−558
	—	—	—
	6·2842	11·161494|6	−86
	−4	−50	−78
	— 4	— 6	—
	|6|·2|8|3|8	11·161444	−8
		−3	−8
		—	—
		1|1|·1|6|1|4|4|1	

The root-figures after the comma are negative, since in the second transformation the root is overstepped; and in this manner may we always proceed whenever, either by mistake or otherwise, too great a root-figure is employed in any step. By actually subtracting the negative portion of the root from the positive we find $x = 2{\cdot}0945514815423$.

On account of the smallness of the correction by which each trial-divisor is converted into a true divisor the preceding example is peculiarly favorable to the method of NEWTON: yet the operation conducted as above, and by which a single figure only is determined at each step, is even in this case far more brief and simple than the corresponding process of NEWTON, though several figures of the root are furnished at each step. This will be more clearly seen by comparing the foregoing work with the details of the calculation as given by FOURIER, in pages 212-216 of the *Analyse des Equations,* according to what he considers an improved mode of conducting the Newtonian operation.

Before closing this long chapter on the general solution of equations we have briefly to notice a new method of solving equations just published by Mr. WEDDLE of Newcastle-upon-Tyne. It is an ingenious and useful addition to the means we previously possessed for overcoming the practical difficulties of this important problem; more especially in reference to equations of advanced degrees in which several terms are absent. For it has the peculiarity of conducting its steps by aid of transformations, through all of which the zero coefficients in the original equation are transmitted, and are never, as in other methods, replaced by significant numbers; so that every zero coefficient in the proposed equation enables us to dispense with an entire column of the work, and thus to compress the solutions of certain equations of a very high degree into a comparatively small space. The extent to which the researches in the present chapter have been carried precludes our doing more than to make this brief mention of Mr. WEDDLE's performance, and to recommend it to the perusal of those interested in the progress of improvement in the general solution of Numerical Equations.*

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* The work is published in a quarto tract under the title of "A new simple and general method of solving Numerical Equations of all orders. By THOMAS WEDDLE;" and is sold in London by Hamilton, Adams, and Co. price *5s.*

CHAPTER XIII.

SOLUTION OF RECURRING AND BINOMIAL EQUATIONS.

Recurring Equations.

(206.) It has been shown at (74) that every equation of an even degree, of the form

$$x^{2n}+Ax^{2n-1}+A_2x^{2n-2}+A_3x^{2n-3}+\ldots.+A_3x^3+A_2x^2+Ax+1=0,$$

in which the coefficients of any two terms, equally distant from the extremes, are alike both in magnitude and sign, has one half of the entire system of roots, the reciprocals of the other half; that is, if n of the roots be

$$a_1,\ a_2,\ a_3,\ .\ .\ .\ .\ a_n,$$

then the other n roots will be

$$\frac{1}{a_1},\ \frac{1}{a_2},\ \frac{1}{a_3},\ .\ .\ .\ .\ \frac{1}{a_n};$$

and, moreover, that even when the equidistant coefficients are like only in magnitude, and unlike in sign, the same relations will exist, provided only the middle term of the equation be absent.

It has also been shown, that if the equation is of an odd degree, then, whether the equal and equidistant coefficients have like signs or not, the same relations among the roots will have place, and that one root will always be $+ 1$ or $- 1$, according as the sign of the last term is $-$ or $+$; so that a recurring equation of an odd degree may always be depressed to a recurring equation of a degree lower.

On account of these peculiar properties of recurring equations, they may always be reduced to others of inferior degrees; in fact, every such equation of an odd degree may, as we have just remarked, be at once reduced to the next inferior even degree; and this, as we shall now prove, may be further reduced to an equation of half the dimensions.

(207.) Suppose the exponent $2n$, in the general equation above, to be successively 2, 4, 6, &c. then dividing every term by x^n, we shall have the several equations

$$x + \frac{1}{x} + \mathrm{A} = 0, \text{ which may be written } z + \mathrm{A} = 0$$

$$(x^2 + \frac{1}{x^2}) + \mathrm{A}\,(x + \frac{1}{x}) + \mathrm{A}_2 = 0 \ \therefore\ z^2 - 2 + \mathrm{A}z + \mathrm{A}_2 = 0$$

$$(x^3 + \frac{1}{x^3}) + \mathrm{A}\,(x^2 + \frac{1}{x^2}) + \mathrm{A}_2\,(x + \frac{1}{x}) + \mathrm{A}_3 = 0$$

$$\therefore\ (z^3 - 3z) + \mathrm{A}\,(z^2 - 2) + \mathrm{A}_2 z + \mathrm{A}_3 = 0$$

&c. &c.

These several equations in z are of a lower degree, by one half, than those from which they have been deduced; and, if in either of these the value of z be found, x will be obtained by the solution of a quadratic, from the condition

$$x + \frac{1}{x} = z.$$

It is worthy of remark, that the depressed equations in z are formed according to a certain law, easily discovered from the general relation,

$$(x^{n} + \frac{1}{x^{n}})\,(x + \frac{1}{x}) = x^{n+1} + \frac{1}{x^{n+1}} + x^{n-1} + \frac{1}{x^{n-1}};$$

which, by replacing $x + \frac{1}{x}$ by z, gives

$$x^{n+1} + \frac{1}{x^{n+1}} = (x^{n} + \frac{1}{x^{n}})\,z - (x^{n-1} + \frac{1}{x^{n-1}});$$

a formula from which the expression $x^{n+1} + \frac{1}{x^{n+1}}$ is obtained in terms of the two preceding expressions; hence we have

$$x^{2} + \frac{1}{x^{2}} = (x + \frac{1}{x})\,z - (x^{0} + \frac{1}{x^{0}}) = z^{2} - 2$$

$$x^{3} + \frac{1}{x^{3}} = (x^{2} + \frac{1}{x^{2}})\,z - (x + \frac{1}{x}) = z^{3} - 3z$$

$$x^{4} + \frac{1}{x^{4}} = (x^{3} + \frac{1}{x^{3}})\,z - (x^{2} + \frac{1}{x^{2}}) = z^{4} - 4z^{2} + 2$$

$$x^{5} + \frac{1}{x^{5}} = (x^{4} + \frac{1}{x^{4}})\,z - (x^{3} + \frac{1}{x^{3}}) = z^{5} - 5z^{3} + 5z$$

&c. &c. &c.

the expressions in z, obviously forming a recurring series, of which the scale of relation is $(-1, z)$, (*Algebra*, art. 172).

(208.) Let now the recurring equation,

$$4x^{6} - 24x^{5} + 57x^{4} - 73x^{3} + 57x^{2} - 24x + 4 = 0,$$

be proposed for solution; or, which is the same thing, the equation

$$4\,(x^{3} + \frac{1}{x^{3}}) - 24\,(x^{2} + \frac{1}{x^{2}}) + 57\,(x + \frac{1}{x}) - 73 = 0,$$

which, by putting

$$x + \frac{1}{x} = z, \text{ or } x^2 - zx = -1,$$

and, taking account of the foregoing expressions, becomes

$$4z^3 - 24z^2 + 45z - 25 = 0;$$

an equation of a degree, lower by one half than the proposed.

One root of this equation we find to be 1; thus

$$\begin{array}{rrrrl} 4 & -24 & 45 & -25 & (1 \\ & 4 & -20 & 25 & \\ & — & — & — & \\ & -20 & 25 & 0 & \end{array}$$

and, for the depressed equation, containing the other roots, we have

$$4z^2 - 20z + 25 = 0;$$

of which the first member is a perfect square, because the square of half the middle term is equal to the product of the extremes; its root is evidently $2z - 5$; hence z has two values equal to $\frac{5}{2}$; and, therefore, the six values of x are given by the three quadratic equations,

$$x^2 - x = -1, \; x^2 - \frac{5}{2}x = -1, \; x^2 - \frac{5}{2}x = -1;$$

the roots of the proposed equation are, therefore,

$$\frac{1 \pm \sqrt{-3}}{2}, \; \frac{1}{2}, \; 2, \; \frac{1}{2}, \; 2.$$

That the first pair of roots, viz.

$$\frac{1 + \sqrt{-3}}{2}, \text{ and } \frac{1 - \sqrt{-3}}{2}$$

are the reciprocals of each other, will be readily seen by multiplying the terms of the latter by

$$1 + \sqrt{-3}.$$

Again, let the equation

$$x^5 - 11x^4 + 17x^3 + 17x^2 - 11x + 1 = 0,$$

be proposed for solution.

Then, as this equation has necessarily the root $x = -1$, we immediately get the depressed biquadratic,

$$x^4 - 12x^3 + 29x^2 - 12x + 1 = 0,$$

or, dividing by x^2, and bringing the equidistant terms together,

$$(x^2 + \frac{1}{x^2}) - 12\,(x + \frac{1}{x}) + 29 = 0,$$

which, by means of the assumed relation,

$$x + \frac{1}{x} = z, \text{ or } x^2 - zx = -1,$$

becomes

$$z^2 - 12z + 27 = 0.$$

By solving this quadratic, we have, for z, the values 9 and 3; and consequently, the values of x in the preceding biquadratic equation are involved in the two quadratics following, viz.

$$x^2 - 9x = -1, \text{ and } x^2 - 3x = -1;$$

these values are, consequently,

$$\frac{9}{2} \pm \frac{1}{2}\sqrt{77}, \ \frac{3}{2} \pm \frac{1}{2}\sqrt{5};$$

hence the five roots of the proposed equation are

$$-1, \frac{9 + \sqrt{77}}{2}, \ \frac{9 - \sqrt{77}}{2}, \ \frac{3 + \sqrt{5}}{2}, \ \frac{3 - \sqrt{5}}{2};$$

or, if the terms of the second of these fractions be multiplied by $9 - \sqrt{77}$, and those of the last fraction by $3 + \sqrt{5}$, the four last roots will assume the following forms, viz.

$$\frac{9 + \sqrt{77}}{2}, \ \frac{2}{9 + \sqrt{77}}; \ \frac{3 + \sqrt{5}}{2}, \ \frac{2}{3 + \sqrt{5}};$$

each being accompanied by its reciprocal.

(209.) It has been observed above that an equation of an even degree is recurring only when the equidistant coefficients are like in sign as well as magnitude; if, however, the signs are unlike, the equation may be reduced to a recurring one, by dividing its first member by $x - 1$; for it is plain that a root of the equation

$$x^{2n} + Ax^{2n-1} + A_2 x^{2n-2} + \cdots\cdots - A_2 x^2 - Ax - 1 = 0$$

is 1, since the substitution of this for x renders the first member zero; this first member is, therefore, divisible by $x - 1$; and the resulting quotient must evidently be the same as that which we should get by dividing

$$1 + Ax + A_2 x^2 + \cdots\cdots - A_2 x^{2n-2} - A_3 x^{2n-1} - x^{2n}$$

by $1 - x$, because this dividend and divisor are no other than the former with changed signs; the terms, however, of the latter quotient would be those of the former, reversed.

The *coefficients* of the first quotient would, it is plain, be all obtained by dividing

$$1 + A + A_2 + \cdots\cdots - A_2 - A - 1$$

by 1 — 1; and the coefficients of the second quotient would be obtained by dividing

$$1 + A + A_2 + \cdots\cdots - A_2 - A - 1$$

by 1 — 1; the same series of coefficients are, therefore, produced in both cases; but this latter series is no other than the former, taken in reverse order, therefore the coefficients in the quotient, arising from dividing the proposed polynomial by $x - 1$, furnish the same series, whether taken in the direct or in the reverse order. The depressed equation, therefore, resulting from the elimination of the root 1, is a recurring equation of an odd degree, whose equidistant terms are equal in magnitude and sign. This depressed equation has, therefore, the root -1, and, consequently, equations of the kind, here considered, have always two roots equal to $+1$, and -1, which may be eliminated, and the resulting equation lowered to one of half its degree.

Binomial Equations.

(210.) Binomial Equations are those which consist of merely two terms; the one being some power of the unknown quantity, and the other the absolute number. The general form of such equations is, therefore,

$$y^n \pm a^n = 0,$$

in which a^n represents a known quantity. By substituting ax for y, the form becomes

$$a^n x^n \pm a^n = 0;$$

or, more simply,

$$x^n \pm 1 = 0,$$

to which form we shall always suppose the equation to be reduced.

(211.) The following particulars respecting these equations, result from the simplest considerations.

1. If n be even, the equation $x^n - 1 = 0$, or $x^n = 1$, has two real roots, viz. $+1$ and -1, and no more. That it has these two roots is plain, for an even power of ± 1 is always $+1$; and that it has no other real root is equally obvious, because no other number can, by its involution, produce 1. Hence the binomial $x^n - 1$ is divisible by $(x + 1)(x - 1) = x^2 - 1$. By actually

performing the division, we have the equation

$$x^{n-2} + x^{n-4} + x^{n-6} + \cdots + x^4 + x^2 + 1 = 0,$$

a recurring equation in which all the $n - 2$ roots must be imaginary. For example, the equation

$$x^6 - 1 = 0,$$

divided by $x^2 - 1$, gives

$$x^4 + x^2 + 1 = 0,$$

whence

$$x = \pm \sqrt{\frac{-1 \pm \sqrt{-3}}{2}};$$

so that the six roots of the proposed equation are

$$+1, \qquad -1$$

$$+\sqrt{\frac{-1+\sqrt{-3}}{2}}, \quad -\sqrt{\frac{-1+\sqrt{-3}}{2}}$$

$$+\sqrt{\frac{-1-\sqrt{-3}}{2}}, \quad -\sqrt{\frac{-1-\sqrt{-3}}{2}}$$

2. If n be odd, the equation $x^n - 1 = 0$ has only one real root, viz. $+ 1$; for $+ 1$ is the only real number of which the odd powers are $+ 1$; hence $x - 1$ is the only real simple factor of $x^n - 1$: dividing by this, we have the recurring equation

$$x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x^2 + x + 1 = 0,$$

of which all the $n - 1$ roots are imaginary.

For example, the equation

$$x^3 - 1 = 0,$$

divided by $x - 1$, gives

$$x^2 + x + 1 = 0,$$

whence

$$x = \frac{-1 \pm \sqrt{-3}}{2};$$

so that the three roots of the proposed equation are

$$1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}.$$

3. If n be even, the equation $x^n + 1 = 0$, or $x^n = -1$, has no real root, for $\sqrt[n]{-1}$ is then impossible; so that all the roots of this equation are imaginary. For example, the four roots of the equation

$$x^4 + 1 = 0,$$

as determined by the rules for recurring equations are

$$\frac{-1+\sqrt{-1}}{\sqrt{2}}, \frac{-1-\sqrt{-1}}{\sqrt{2}}, \frac{1+\sqrt{-1}}{\sqrt{2}}, \frac{1-\sqrt{-1}}{\sqrt{2}}.$$

4. If n be odd, the equation $x^n + 1 = 0$, or $x^n = -1$, has one real root, viz. -1, and no more, because this is the only real number of which an odd power is -1; hence, if the equation $x^3 + 1 = 0$ be proposed, the first member being divisible by $x + 1$, we have the equation

$$x^2 - x + 1 = 0;$$

so that the three roots of the proposed equation are

$$-1, \frac{1+\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}.$$

5. The roots of the equation

$$x^n \pm 1 = 0$$

are all unequal; for the limiting polynomial nx^{n-1} having no divisor in common with $x^n \pm 1$, the proposed cannot have equal roots (98).

PROPOSITION I.

(212.) If α be one of the imaginary roots of the equation $x^n - 1 = 0$, then any power of α will be also a root.

For, since α is one root of the equation $x^n - 1 = 0$, therefore $\alpha^n = 1$, and consequently,

$$\alpha^{2n}=1,\ \alpha^{3n}=1,\ \alpha^{4n}=1,\ \text{\&c., also } \alpha^{-n}=1,\ \alpha^{-2n}=1,\ \alpha^{-3n}=1,\ \text{\&c.}$$

the values

$$\alpha,\ \alpha^2,\ \alpha^3 \ldots\ldots,\ \alpha^{-1},\ \alpha^{-2},\ \alpha^{-3},\ldots\ldots,$$

thus satisfying, the conditions of the equation are roots of it.

Cor. 1. It hence appears that the roots of the equation $x^n - 1 = 0$ may be represented under an infinite variety of forms, each term in the following series being a root, viz.

$$\ldots\ldots\alpha^{-3},\alpha^{-2},\alpha^{-1},1,\alpha,\alpha^2,\alpha^3,\ldots\ldots\alpha^{n-1},\alpha^n,\alpha^{n+1},\ldots\ldots\alpha^{2n},\alpha^{2n+1},\ldots\ldots$$

in which series, however, there cannot be more than n quantities essentially different, otherwise the equation would have more than n roots.

PROPOSITION II.

(213.) If α be one of the imaginary roots of the equation $x^n + 1 = 0$, then any odd power of α will be also a root.

For, since α is one root of the equation $x^n = -1$, therefore, $\alpha^n = -1$; and, since every odd power, whether positive or negative, of -1 is also -1, therefore

$$\alpha^{3n} = -1,\ \alpha^{5n} = -1,\ \alpha^{7n} = -1,\ \text{\&c., also}$$

$$\alpha^{-3n} = -1,\ \alpha^{-5n} = -1,\ \alpha^{-7n} = -1,\ \text{\&c.};$$

so that the quantities

$$\alpha,\ \alpha^3,\ \alpha^5 \ldots\ldots,\ \alpha^{-1},\ \alpha^{-3},\ \alpha^{-5},\ldots\ldots,$$

are roots of the equation. These roots, therefore, assume an infinite variety of forms, although there cannot be more than n essentially different.

PROPOSITION III.

(214.) When n is a prime number, the roots of the equation $x^n - 1 = 0$ are all exhibited in the series

$1, \quad \alpha, \quad \alpha^2, \quad \alpha^3, \ldots\ldots \alpha^{n-1},$

or $\alpha^n, \quad \alpha^{n+1}, \quad \alpha^{n+2}, \quad \alpha^{n+3}, \ldots\ldots \alpha^{2n-1},$

or $\alpha^{2n}, \quad \alpha^{2n+1}, \quad \alpha^{2n+2}, \quad \alpha^{2n+3}, \ldots\ldots \alpha^{3n-1}$

. . . .

or $1, \quad \alpha^{-1}, \quad \alpha^{-2}, \quad \alpha^{-3}, \ldots\ldots \alpha^{-(n-1)}$

or $\alpha^{-n}, \quad \alpha^{-n-1}, \quad \alpha^{-n-2}, \quad \alpha^{-n-3}, \ldots\ldots \alpha^{-n-(n-1)}$

. . . .

It has been shown in prop. I., that every one of the foregoing quantities is a root of the equation if α is a root; if, therefore, no two of the n quantities in each series are the same under a different form, each series will exhibit all the n roots of the equation.

Now, if we suppose any two of the roots in either series to be equal, as for instance α^p and α^t, in which $t > p$, then by dividing the equation $\alpha^t = \alpha^p$ by α^p, we have $\alpha^{t-p} = 1$; and that this equation is impossible may be proved as follows:

Because n is a prime number, and $t - p$ necessarily less than n, therefore the numbers n and $t - p$ are prime to each other; and consequently two whole numbers, x' and y', may always be found such that

$$(t - p)\, x' = ny' + 1\,;^*$$

and, as $\alpha^{t-p} = 1$, therefore $\alpha^{(t-p)x'} = 1$, and consequently $\alpha^{ny'+1} = 1$, that is, $\alpha^{ny'} . \alpha = 1$; but $\alpha_{y'}$ is a root (prop. I.); hence $\alpha^{ny'} = 1$, therefore, from the last equation, $\alpha = 1$, which is impossible, because, by hypothesis α is imaginary. Hence, each of the series announced above, comprises the n roots of the equation under different forms.

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[*] See the *Algebra*, third edition, page 271.

PROPOSITION IV.

(215.) When p and q have no common measure, then the equations $x^p - 1 = 0$ and $x^q - 1 = 0$ have no common root except unity.

If possible, let α be a root common to both equations, and different from unity, then we have $\alpha^p = 1$ and $\alpha^q = 1$; and, since p and q are prime to each other, two whole numbers, x' and y', may always be found such that $px' = qy' + 1$ (*Algebra*, p. 271.) Consequently $\alpha^{px'} = \alpha^{qy'+1} = \alpha^{qy'} . \alpha$. But $\alpha^{y'}$ is a root of each equation: hence $\alpha^{qy'} = 1$, therefore $\alpha = 1$, which is contrary to the hypothesis. Hence the root common to the two proposed equations can be no other than unity.

Cor. When the equations $x^n - 1 = 0$, $x^m - 1 = 0$, have an imaginary root in common, the exponents m, n, must have a common measure.

PROPOSITION V.

(216.) When n is a composite number, formed of the factors p, q, r, &c., then the roots of the equations $x^p - 1 = 0$, $x^q - 1 = 0$, $x^r - 1 = 0$, &c., must all of them be roots of the equation $x^n - 1 = 0$.

This is obvious; for the two quantities x^n, and 1, may be regarded as the result of the two quantities x^p, and 1, raised to the qr &c. power, or as the result of x^q and 1 raised to the pr &c. power, &c.; and the difference of two powers is always accurately divisible by the difference of their roots. (*Algebra*, p. 201.)

PROPOSITION VI.

(217.) When n is the product of two prime numbers, p and q, the roots of the equation $x^n - 1 = 0$ will be expressed by the n products arising from multiplying every root of the equation $x^p - 1 = 0$, by every root of the equation $x^q - 1 = 0$.

Let the roots of the equation $x^p - 1 = 0$ be

$$1 \quad \alpha, \quad \alpha^2, \quad \alpha^3, \cdots\cdot \alpha^{p-1},$$

and those of the equation $x^q - 1 = 0$,

$$1 \quad \beta, \quad \beta^2, \quad \beta^3, \cdots\cdot \beta^{q-1}.$$

These two series of roots are all different, with the exception of the common root unity (prop. IV.), and are, therefore, so many different roots of the equation $x^n - 1 = 0$, (prop. V.). The pq products also, arising from multiplying the one series by the other, will be so many roots of the proposed equation. For, let $\alpha^h \beta^k$ represent any one of these products, then, since α^h and β^k are roots of $x^n - 1 = 0$, we have $\alpha^{hn} = 1$ and $\beta^{kn} = 1$; and consequently, $(\alpha^h \beta^k)^n = 1$, or $(\alpha^h \beta^k)^n - 1 = 0$; hence $\alpha^h \beta^k$ must be a root of $x^n - 1 = 0$. The products are all different: for, if possible, let

$$\alpha^h \beta^k = \alpha^g \beta^m$$

$$\therefore \alpha^{h-g} = \beta^{m-k}.$$

Now, whether $h - g$ and $m - k$ be positive or negative numbers, the expression α^{h-g} is, necessarily, a root of $x^p - 1 = 0$, and the expression β^{m-k}, a root of $x^q - 1 = 0$; and as these roots are, by prop. IV., essentially different, except when they are both unity, it follows that the equation deduced from our hypothesis cannot exist; that hypothesis, therefore, is not true, so that no two products can be equal to each other. As, therefore, the products are pq in number, all different, and all satisfy the equation $x^n - 1 = 0$, they must express the pq roots of that equation.

In the foregoing reasoning, it is, of course, presumed that the component factors, p, q, are unequal. If they are equal, then the roots of the equation, $x^n - 1 = 0$, will not all be comprised in the aforesaid products.

As an example of the application of this proposition, let it be required to determine the six roots of the equation, $x^6 - 1 = 0$.

As 6 is composed of the two prime numbers, 2 and 3, we have first to find the roots of

$$x^2 - 1 = 0, \quad \text{and} \quad x^3 - 1 = 0.$$

The roots of $x^2 - 1 = 0$ are 1 and -1. The roots of $x^3 - 1 = 0$ are, p. 347,

$$1, \frac{-1 + \sqrt{-3}}{2}, \frac{-1 - \sqrt{-3}}{2}.$$

Consequently, the six roots sought are the six products, arising from multiplying these three roots by 1, -1, and are, therefore,

$$1, -1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}, \frac{1+\sqrt{-3}}{2}.$$

PROPOSITION VII.

(218.) To determine the roots of the equation $x^n - 1 = 0$, when n is the square of a prime number p.

Put $x^p = z$, then $x^p - z = 0$, and $z^p - 1 = 0$, and let the roots of this last equation be 1, β, β^2, β^3, ... β^{p-1}, then, by substitution,

$$x^p - z = \begin{cases} x^p - 1 = 0, \\ x^p - \beta = 0, \\ x^p - \beta^2 = 0, \\ x^p - \beta^3 = 0, \end{cases}$$

&c. &c.

hence the pp values of x, in these p equations, will evidently be all different, and will be the roots of the equation $x^{pp} - 1 = 0$.

To determine these roots, it will be sufficient to advert to art. (75), which proves that the roots of $x^p - \beta = 0$ are equal to the roots of $x^p - 1 = 0$ multiplied by $\sqrt[p]{\beta}$; and, in a similar manner, the roots of $x^p - \beta^2 = 0$ are equal to the roots of

$x^p - 1 = 0$, multiplied by $\sqrt[p]{\beta^2}$, &c.; therefore we immediately conclude that the roots of

$$\left.\begin{array}{l} x^p - 1 = 0 \text{ are } 1, \beta, \beta^2, \beta^3 \ldots\ldots\ldots\ldots \beta^{p-1} \\ x^p - \beta = 0 \ldots \sqrt[p]{\beta}, \beta\sqrt[p]{\beta}, \beta^2\sqrt[p]{\beta} \ldots\ldots \beta^{p-1}\sqrt[p]{\beta} \\ x^p - \beta^2 = 0 \ldots \sqrt[p]{\beta^2}, \beta\sqrt[p]{\beta^2}, \beta^2\sqrt[p]{\beta^2} \ldots \beta^{p-1}\sqrt[p]{\beta^2} \end{array}\right\} \begin{array}{l} = \text{the } n \text{ roots of} \\ \quad x^n - 1 = 0, \end{array}$$

$$\&c. \qquad \&c. \qquad \&c.$$

For example, let it be required to find the roots of $x^9 - 1 = 0$. The roots of $x^3 - 1 = 0$ are

$$1, \quad \frac{-1 + \sqrt{-3}}{2}, \quad \frac{-1 - \sqrt{-3}}{2};$$

hence the roots of $x^9 - 1 = 0$ are

$$1, \quad \frac{-1 + \sqrt{-3}}{2}, \quad \frac{-1 - \sqrt{-3}}{2},$$

$$\sqrt[3]{\frac{-1 + \sqrt{-3}}{2}}, \quad \frac{-1 + \sqrt{-3}}{2}\sqrt[3]{\frac{-1 + \sqrt{-3}}{2}},$$

$$\frac{-1 - \sqrt{-3}}{2}\sqrt[3]{\frac{-1 + \sqrt{-3}}{2}}, \quad \sqrt[3]{\frac{-1 - \sqrt{-3}}{2}},$$

$$\frac{-1 + \sqrt{-3}}{2}\sqrt[3]{\frac{-1 - \sqrt{-3}}{2}}, \quad \frac{-1 - \sqrt{-3}}{2}\sqrt[3]{\frac{-1 - \sqrt{-3}}{2}}.$$

From the preceding propositions we may infer, as at (42), that every root has as many values as there are units in its index; for, as there are n different quantities which satisfy the equation $x^n = 1$, it follows that $\sqrt[n]{1}$ has n different values; and it is plain that if each of these values be multiplied by the common arithmetical value of $\sqrt[n]{a}$, the n products will all be different, and such that, if each be raised to the nth power, the result will always be a; hence the products of which we speak are so many different values of $\sqrt[n]{a}$. The determination, therefore, of the n roots of $\sqrt[n]{a}$ requires that we are able to extract the nth arithmetical root of a, and to exhibit all the imaginary roots of $\sqrt[n]{1}$. The foregoing propositions have been devoted chiefly to an examination of the properties and relations of these roots, and not to the actual exhibition of their values, although, as in the proposition above, one or two examples of the solution have been given to illustrate the reasoning. To obtain the imaginary roots, however, in their simplest form, that is, in the form $a + b\sqrt{-1}$, and for all values of the exponent, requires the aid of a theorem, borrowed from the science of Trigonometry.

(219.) The theorem to which we refer, is the well-known formula of De Moivre given in most books on Analytical Trigonometry, viz. (see *Trigonometry*, second edition, page 59,)

$$(\cos a \pm \sin a \,.\, \sqrt{-1})^{n} = \cos na \pm \sin na \,.\, \sqrt{-1}\,;$$

which, if the arc $2k\pi$, (π being a semicircumference, and k any integer,) be substituted for na, becomes

$$(\cos \frac{2k\pi}{n} \pm \sin \frac{2k\pi}{n} \,.\, \sqrt{-1})^{n} = \cos 2k\pi \pm \sin 2k\pi \,.\, \sqrt{-1}\,;$$

that is, since

$$\cos 2k\pi = 1, \text{ and } \sin 2k\pi = 0,$$

$$(\cos \frac{2k\pi}{n} \pm \sin \frac{2k\pi}{n} \cdot \sqrt{-1})^{n} = 1;$$

so that the expression

$$\cos \frac{2k\pi}{n} \pm \sin \frac{2k\pi}{n} \,.\, \sqrt{-1},$$

comprehends in it all the n roots of unity, or all the particular values of x, which satisfy the equation $x^{n} - 1 = 0$.

Although, in this general expression, the value of k is quite arbitrary, yet, assume it what we will, the expression can never furnish more than n different values. These different values will arise from the several substitutions of

$$0, \quad 1, \quad 2, \quad 3 \; \ldots \; .$$

up to the number $\frac{n-1}{2}$, inclusively, if n is odd, and up to $\frac{n}{2}$, if n

is even; and for substitutions beyond these limits the preceding results will recur. To prove this, let us actually substitute as proposed: we shall thus have the following series of results, viz.

$$\text{for } k = 0 \;.\;.\;.\;.\; x = \cos 0 \pm \sin 0 \,.\, \sqrt{-1} = 1$$

$$k = 1 \;.\;.\;.\;.\; x = \cos \frac{2\pi}{n} \pm \sin \frac{2\pi}{n} \cdot \sqrt{-1}$$

$$k = 2 \;.\;.\;.\;.\; x = \cos \frac{4\pi}{n} \pm \sin \frac{4\pi}{n} \cdot \sqrt{-1}$$

$$k = 3 \;.\;.\;.\;.\; x = \cos \frac{6\pi}{n} \pm \sin \frac{6\pi}{n} \cdot \sqrt{-1}$$

$$\vdots \qquad\qquad \vdots$$

$$k = \frac{n-1}{2} \;.\;.\;.\; x = \cos \frac{(n-1)\pi}{n} \pm \sin \frac{(n-1)\pi}{n} \cdot \sqrt{-1}$$

Each of these expressions, except the first, involves two distinct values; so that, omitting the value given by $k = 0$, there are $n - 1$ values, and, consequently, altogether, there are n values; and that they are all different, is plain, because the arcs

$$0, \frac{2\pi}{n}, \;\frac{4\pi}{n}, \;\frac{6\pi}{n}, \;.\;.\;.\;.\; \frac{(n-1)\pi}{n},$$

being all different, and less than π, have all different cosines. The arcs which would arise from continuing the substitutions, are

$$\frac{(n+1)\pi}{n}, \;\frac{(n+3)\pi}{n}, \;\frac{(n+5)\pi}{n}, \text{ \&c.};$$

or, which are the same,

$$2\pi - \frac{(n-1)\pi}{n}, \;2\pi - \frac{(n-3)\pi}{n}, \;2\pi - \frac{(n-5)\pi}{n}, \text{ \&c.},$$

and the sines and cosines of these are respectively the same as the sines and cosines of the arcs

$$\frac{(n-1)\pi}{n}, \frac{(n-3)\pi}{n}, \frac{(n-5)\pi}{n}, \text{ \&c.,}$$

which have already occurred.*

If n is an even number, the final substitution for k must be $\frac{n}{2}$ instead of $\frac{n-1}{2}$, as above; and therefore the final pair of conjugate values for x will be

$$x = \cos\pi \pm \sin\pi . \sqrt{-1} = -1,$$

which values of x differ from all the other values, because in them no arc occurs so great as π.

The arcs which would arise from continuing the substitutions beyond $k = \frac{n}{2}$ are

$$\frac{(n+2)\pi}{n}, \frac{(n+4)\pi}{n}, \frac{(n+6)\pi}{n}, \text{ \&c.;}$$

or, which are the same,

$$2\pi - \frac{(n-2)\pi}{n}, 2\pi - \frac{(n-4)\pi}{n}, 2\pi - \frac{(n-6)\pi}{n}, \text{ \&c.,}$$

and the sines and cosines of these are respectively the same as the sines and cosines of the arcs

$$\frac{(n-2)\pi}{n}, \frac{(n-4)\pi}{n}, \frac{(n-6)\pi}{n}, \text{ \&c ,}$$

which have already occurred.*

It is easy to see that in every pair of conjugate roots, each is the reciprocal of the other. In fact whatever be k,

$$(\cos\frac{2k\pi}{n} + \sin\frac{2k\pi}{n} \cdot \sqrt{-1})\ (\cos\frac{2k\pi}{n} - \sin\frac{2k\pi}{n} . \sqrt{-1}) =$$

$$\cos^2\frac{2k\pi}{n} + \sin^2\frac{2k\pi}{n} = 1,$$

---

* The *signs* of the sines will, however, be different; but the only effect of this difference is evidently to furnish each pair of conjugate roots in an inverse order.

which shows that the two factors in the first member are of the form $\alpha, \frac{1}{\alpha}$.

We have proved (212) that every power of an imaginary root of the binomial equation is also a root; but, unless n be a prime number, we could not infer that all the roots would ever be produced by involving any one of them. Such would not indeed be the case. There is always, however, one among the imaginary roots of which the involution will furnish all the others; it is the first imaginary root, or that due to the substitution $k = 1$, in the foregoing series of values; for, by DE MOIVRE's formula, the powers of this produce all the others, thus:

$$(\cos\frac{2\pi}{n} + \sin\frac{2\pi}{n} \cdot \sqrt{-1})^2 = \cos\frac{4\pi}{n} + \sin\frac{4\pi}{n} \cdot \sqrt{-1}$$

$$(\cos\frac{2\pi}{n} + \sin\frac{2\pi}{n} \cdot \sqrt{-1})^3 = \cos\frac{6\pi}{n} + \sin\frac{6\pi}{n} \cdot \sqrt{-1}$$

$$\vdots \qquad\qquad \vdots$$

$$(\cos\frac{2\pi}{n} + \sin\frac{2\pi}{n} \cdot \sqrt{-1})^{\frac{n-1}{2}} = \cos\frac{n-1}{n}\pi + \sin\frac{n-1}{n}\pi \cdot \sqrt{-1}.$$

These powers of the first imaginary root, which we may call α, thus furnish one half of the entire number of imaginary roots, and the reciprocals of these being the other half, all of them are determined from the first; the imaginary roots are, therefore,

$$\alpha, \quad \alpha^2, \quad \alpha^3, \; . \; . \; . \; . \quad \alpha^{\frac{n-1}{2}}$$

$$\frac{1}{\alpha}, \quad \frac{1}{\alpha^2}, \quad \frac{1}{\alpha^3}, \; . \; . \; . \; . \quad \frac{1}{\alpha^{\frac{n-1}{2}}}.$$

When n is even, the last power will be

$$(\cos\frac{2\pi}{n} + \sin\frac{2\pi}{n} \cdot \sqrt{-1})^{\frac{n}{2}} = \cos\pi + \sin\pi \, . \, \sqrt{-1};$$

and the imaginary roots are, therefore,

$$\alpha,\quad \alpha^2,\quad \alpha^3, \ldots \quad \alpha^{\frac{n}{2}}$$

$$\frac{1}{\alpha},\ \frac{1}{\alpha^2},\ \frac{1}{\alpha^3},\ \ldots\ \frac{1}{\alpha^{\frac{n}{2}}}.$$

(220.) By the same general formula (page 354), we are enabled to determine all the roots of the equation

$$x^n + 1 = 0,$$

for, since

$$\cos (2k + 1)\pi = -1, \text{ and } \sin (2k + 1)\pi = 0,$$

that formula gives

$$(\cos \frac{2k+1}{n}\pi \pm \sin \frac{2k+1}{n}\pi . \sqrt{-1})^n =$$

$$\cos (2k + 1)\pi \pm \sin (2k + 1)\pi . \sqrt{-1} = -1;$$

hence the n values of x are all comprised in the general expression

$$x = \cos \frac{2k+1}{n}\pi \pm \sin \frac{2k+1}{n}\pi . \sqrt{-1};$$

which, by putting for k the values 0, 1, 2, 3, &c. in succession, furnishes the following series of separate values, viz.

$$\text{for } k = 0 \ \ldots\ldots \ x = \cos\frac{\pi}{n} \pm \sin\frac{\pi}{n} . \sqrt{-1}$$

$$k = 1 \ \ldots\ldots \ x = \cos\frac{3\pi}{n} \pm \sin\frac{3\pi}{n} . \sqrt{-1}$$

$$k = 2 \ \ldots\ldots \ x = \cos\frac{5\pi}{n} \pm \sin\frac{5\pi}{n} . \sqrt{-1}$$

$$\vdots$$

$$k = \frac{n-1}{2} \ldots x = \cos\pi \pm \sin\pi . \sqrt{-1} = -1;$$

or, when n is even,

$$k = \frac{n-2}{2} \ \ldots\ x = \cos\left(\pi - \frac{\pi}{n}\right) \pm \sin\left(\pi - \frac{\pi}{n} \,.\, \sqrt{-1}\right).$$

Now that the foregoing system of n roots are all different is obvious, since

$$\frac{\pi}{n},\ \frac{3\pi}{n},\ \frac{5\pi}{n} \ \ldots\ldots\ \frac{n\pi}{n}, \text{ or } \pi - \frac{\pi}{n},$$

are all different arcs, of which the greatest does not exceed a semicircumference. If the preceding series be extended, it will be easy to prove, after what has been done at page 355, that the values formerly obtained will recur.

As in the former case of the general problem, so here, each root may be derived from the first pair of the series: thus, denoting the first root, $\cos\frac{\pi}{n} \pm \sin\frac{\pi}{n} \,.\, \sqrt{-1}$, by α or $\frac{1}{\alpha}$, according as the upper or lower sign is taken, we evidently have, for the preceding series, the following equivalent expressions, viz.

$$\left.\begin{array}{l} \alpha,\ \ \alpha^3,\ \ \alpha^5, \ldots\ldots\ \alpha^n \\ \frac{1}{\alpha},\ \frac{1}{\alpha^3},\ \frac{1}{\alpha^5}, \ldots\ldots\ \frac{1}{\alpha^n} \end{array}\right\} \text{when } n \text{ is odd.}$$

and

$$\left.\begin{array}{l} \alpha,\ \ \alpha^3,\ \ \alpha^5, \cdots\cdots\ \alpha^{n-1} \\ \frac{1}{\alpha},\ \frac{1}{\alpha^3},\ \frac{1}{\alpha^5}, \cdots\cdots\ \frac{1}{\alpha^{n-1}} \end{array}\right\} \text{when } n \text{ is even.}$$

For further researches on the theory of binomial equations, the student may consult LAGRANGE's Traité de la Résolution des Equations Numériques, Note 14; LEGENDRE's Théorie des Nombres, Part v.; the Disquisitiones Arithmeticæ of GAUSS; BARLOW's Theory of Numbers; and IVORY's article on Equations, in the Encyclopædia Britannica.

CHAPTER XIV.

ON CONTINUED FRACTIONS.

(221.) LET α represent either a fractional or an incommensurable number; and let a be the greatest integer below the value of α, and which, if α be less than 1, will of course be 0. Then, since $\alpha - a$ is less than 1, it follows that $\frac{1}{\alpha - a}$ must be greater than 1. Put

$$\frac{1}{\alpha - a} = \beta \therefore \alpha = a + \frac{1}{\beta},$$

and let b be the integer which in value is immediately below β; then $\beta - b$ is less than 1, and consequently $\frac{1}{\beta - b}$ must be greater than 1. Put

$$\frac{1}{\beta - b} = \gamma \therefore \beta = b + \frac{1}{\gamma},$$

and let c be the greatest integer below the value of γ; then will $\gamma - c$ be less than 1, and therefore $\frac{1}{\gamma - c}$ greater than 1. Put

$$\frac{1}{\gamma - c} = \delta \therefore \gamma = c + \frac{1}{\delta}.$$

Continuing this process, we obviously have, by substituting in the foregoing expression for α the values of β, γ, &c. in succession, the following development of the value α, viz.

$$\alpha = a + \frac{1}{\beta}$$

$$= a + \frac{1}{b + \frac{1}{\gamma}}$$

$$= a + \frac{1}{b + \frac{1}{c + \frac{1}{\delta}}}$$

&c.

which development is called a *continued fraction.*

If either of the quantities β, γ, δ, &c. is a whole number, the development must of course terminate at that number; and this will necessarily be the case if α be rational, or a finite fraction; but if α be irrational, then the fraction representing its development must be interminable. This is readily admissible; it is, however, an unavoidable conclusion from what follows.

(222.) If α be a rational fraction $\frac{A}{B}$, we may very easily arrive at its equivalent continued fraction. For the first term a will be the quotient of A by B; and, calling the remainder C, we shall have

$$\frac{A}{B} - a = \frac{C}{B} \therefore \beta = \frac{B}{C}.$$

In like manner, the division of B by C gives b; and putting D for the remainder, we have

$$\frac{B}{C} - b = \frac{D}{C} \therefore \gamma = \frac{C}{D}.$$

Similarly the division of C by D gives c, and so on.

Hence a, b, c, &c. are no other than the quotients which successively arise in the process for finding the common measure of the terms of the proposed fraction $\frac{\mathrm{A}}{\mathrm{B}}$; thus:

$$\begin{array}{l}
\mathrm{B})\ \mathrm{A}\ (a \\
\quad\ \underline{\mathrm{B}a} \\
\quad\ \mathrm{C})\ \mathrm{B}\ (b \\
\qquad\ \underline{\mathrm{C}b} \\
\qquad\ \mathrm{D})\ \mathrm{C}\ (c \\
\qquad\quad\ \underline{\mathrm{D}c} \\
\qquad\qquad \mathrm{E} \\
\qquad\quad\ \ \&c.
\end{array}$$

It is easy to see that when α is a rational fraction, the expression deduced for it, in the preceding article, is readily derivable from this operation of the common measure; indeed the form of the continued fraction, as deduced from this process, will have greater generality than that given in last article. For without restricting the foregoing quotients to be integral and positive, we shall evidently have, in every case,

$$\alpha = \frac{\mathrm{A}}{\mathrm{B}} = a + \frac{\mathrm{C}}{\mathrm{B}} = a + \frac{1}{\beta}$$

$$\beta = \frac{\mathrm{B}}{\mathrm{C}} = b + \frac{\mathrm{D}}{\mathrm{C}} = b + \frac{1}{\gamma}$$

$$\gamma = \frac{\mathrm{C}}{\mathrm{D}} = c + \frac{\mathrm{E}}{\mathrm{D}} = c + \frac{1}{\delta}$$

$$\&c. \qquad \&c. \qquad \&c.$$

so that

$$\frac{\mathrm{A}}{\mathrm{B}} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \&c.}}}$$

in which a, b, c, &c. are quotients, positive or negative, integral or fractional, derived by the foregoing operation. In most applications of continued fractions, integral and positive quotients only are employed; but it is useful to show that these restrictions are not essential to the form of the development; which is preserved, whatever be the character of the quotients. This is a truth that we shall have occasion to avail ourselves of at the close of the Chapter; at present, however, we shall require only positive and integral quotients.

(223.) As a particular application, let the proposed fraction be $\frac{1103}{887}$; then, by applying the process for the common measure, the several quotients furnish the following development, viz.

$$\frac{1103}{887} = 1 + \cfrac{1}{4 + \cfrac{1}{9 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4}}}}}},$$

and if the fraction be $\frac{1171}{9743}$, we have the following equivalent development, viz.

$$\frac{1171}{9743} = \cfrac{1}{8 + \cfrac{1}{3 + \cfrac{1}{8 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}}}}}}.$$

Since the process of seeking the greatest common divisor of two numbers always terminates, it follows that every rational fraction may be expressed in a finite continued fraction.

(224.) We might obviously, by reduction, collect into one the successive portions

$$\frac{1}{a}, \quad \cfrac{1}{a + \cfrac{1}{b}}, \quad \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c}}}, \text{ \&c.}$$

of a continued fraction, by putting for a, in the first, $a + \frac{1}{b}$;

then $b + \frac{1}{c}$ for b, and so on; we should thus have the results

$$\frac{1}{a}, \quad \frac{b}{ab + 1}, \quad \frac{bc + 1}{a\,(ab + 1) + c}, \text{ \&c.}$$

so that every finite continued fraction may be reduced to an ordinary finite fraction; but an incommensurable quantity cannot be expressed by a terminate continued fraction.

The partial sums which we have just obtained are called *converging fractions;* for, as we shall presently demonstrate, they approach nearer and nearer to the whole value of the continued fraction.

For the sake of simplicity, let us represent the series of converging fractions by

$$\frac{\mathrm{A}}{\mathrm{A}'}, \quad \frac{\mathrm{B}}{\mathrm{B}'}, \quad \frac{\mathrm{C}}{\mathrm{C}'}, \text{ \&c.}$$

then we shall always be able recognize the particular fraction represented, by observing that the capitals A, B, C, &c. correspond to the quotients a, b, c, &c. last introduced; so that $\frac{\mathrm{B}}{\mathrm{B}'}$ will represent

$$\cfrac{1}{a + \cfrac{1}{b}},$$

$\frac{\mathrm{C}}{\mathrm{C}'}$ will represent

$$\cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c}}},$$

and so on. This notation being agreed upon, we may readily demonstrate the following proposition: viz.

In any three consecutive converging fractions

$$\frac{\mathrm{P}}{\mathrm{P}'}, \quad \frac{\mathrm{Q}}{\mathrm{Q}'}, \quad \frac{\mathrm{R}}{\mathrm{R}'},$$

we shall always have the property

$$\frac{R}{R'} = \frac{Qr + P}{Q'r + P'},$$

r being, as observed above, the quotient last introduced into the value of $\frac{R}{R'}$.

As to the first three converging fractions, viz.

$$\frac{1}{a}, \quad \frac{b}{ab + 1}, \quad \frac{bc + 1}{a(ab + 1) + c},$$

or

$$\frac{A}{A'}, \quad \frac{B}{B'}, \quad \frac{C}{C'},$$

it is plain that the property announced has place; for we immediately recognize the relation

$$\frac{C}{C'} = \frac{Bc + A}{B'c + A'}.$$

If then we can show from this, that the succeeding fraction must have the same property, similar reasoning would apply to the next following fraction, and so on throughout the whole. We have only then, in order to establish the proposition, to prove that from the condition

$$\frac{C}{C'} = \frac{Bc + A}{B'c + A'} \text{ we must have } \frac{D}{D'} = \frac{Cd + B}{C'd + B'}.$$

The expression for $\frac{D}{D'}$ differs from the expression for $\frac{C}{C'}$ only by having $c + \frac{1}{d}$ in place of c; so that, by changing in $\frac{C}{C'}$, c into $c + \frac{1}{d}$ we must have $\frac{D}{D'}$; therefore

$$\frac{\mathrm{D}}{\mathrm{D}'} = \frac{\mathrm{B}\,(c + \frac{1}{d}) + \mathrm{A}}{\mathrm{B}'(c + \frac{1}{d}) + \mathrm{A}'} = \frac{(\mathrm{B}c\ + \mathrm{A})\,d + \mathrm{B}}{(\mathrm{B}'c + \mathrm{A}')\,d + \mathrm{B}'}$$

$$= \frac{\mathrm{C}d + \mathrm{B}}{\mathrm{C}'d + \mathrm{B}'};$$

hence, generally,

$$\frac{\mathrm{R}}{\mathrm{R}'} = \frac{\mathrm{Q}r + \mathrm{P}}{\mathrm{Q}'r + \mathrm{P}'},$$

which shows that both numerators and denominators go on continually increasing. By means of this property we may form the series of converging fractions with great facility, when only the first two are given; and we may thence arrive at the entire sum of the series when it terminates, and thus obtain the value of the original fraction.

For example, let it be required to determine the fraction of which the development is

$$1 + \cfrac{1}{4 + \cfrac{1}{9 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4}}}}}}.$$

Here the first two converging fractions are $\frac{1}{1}$, $\frac{5}{4}$, from which we deduce the third by multiplying the two terms of the second each by 9, and adding in the corresponding terms of the first fraction; from the third we get the fourth, using the next quotient 2 as the multiplier, and adding in the corresponding terms of the second fraction, and so on, as follows:

$$\begin{array}{ccccccc} & & 9 & 2 & 1 & 1 & 4 \\ \frac{1}{1}, & \frac{5}{4}, & \frac{46}{37}, & \frac{97}{78}, & \frac{143}{115}, & \frac{240}{193}, & \frac{1103}{887}. \end{array}$$

(225.) We can now show the propriety of calling these results *converging fractions,* by proving that they continually approach nearer and nearer to the true value of the continued fraction.

That these fractions are alternately less and greater than the developed form may be readily seen, without the aid of the above property; for, calling the entire value x, we have the first, $\frac{a}{1}$, less than x, because the positive quantity $\frac{1}{b + \&c.}$ is neglected. The second $a + \frac{1}{b}$ is greater than x, for the denominator is less than it ought to be, by the positive quantity $\frac{1}{c + \&c.}$, yet, if we take in $\frac{1}{c}$, that denominator will be increased too much, because $\frac{1}{c}$ is greater than $\frac{1}{c + \&c.}$; so that $a + \cfrac{1}{b + \cfrac{1}{c}}$ is less than x, and so on. But to prove the proposition announced in a general manner, we shall employ the equation

$$\frac{R}{R'} = \frac{Qr + P}{Q'r + P'}$$

before established, either member of which will necessarily express the value of the entire fraction x, if we substitute in it $r + \frac{1}{s + \&c.}$ for r. The quantity $r + \frac{1}{s + \&c.}$ is always greater than unity, because r is not less than unity. Calling it y we have

$$x = \frac{Qy + P}{Q'y + P'};$$

and, consequently, by subtracting first $\frac{P}{P'}$ and then $\frac{Q}{Q'}$ from each side, we have the equations

$$x - \frac{P}{P'} = \frac{(QP' - Q'P)y}{(Q'y + P')P'}, \quad x - \frac{Q}{Q'} = \frac{Q'P - QP'}{(Q'y + P')Q'},$$

which show that the differences $x - \frac{P}{P'}$, $x - \frac{Q}{Q'}$, have contrary signs; so that if x be greater than $\frac{P}{P'}$, it will be less than $\frac{Q}{Q'}$, and vice versâ; and, as x is greater than the first converging fraction $\frac{a}{1}$, (or $\frac{0}{1}$ if a is 0), it follows that, throughout the series of converging fractions, the 1st, 3d, 5th, 7th, &c. of them are each below the true value; and the 2d, 4th, 6th, 8th, &c. above the true value.

As to the relative values of the differences $x - \frac{P}{P'}$, $x - \frac{Q}{Q'}$, it is plain that the latter is less than the former, because y is greater than 1, and Q' greater than P', since the denominators increase as the fractions advance (224). It follows, therefore, that the converging fractions approach continually nearer and nearer to the true value of the continued fraction; and, therefore, this value may be approximated to as closely as we please when the first two converging fractions are found. It follows, moreover, that the odd terms of the series of converging fractions form an increasing series of values, approximating to the truth, and that the even terms form a decreasing series of approximating values.

(226.) Let us now inquire what is the limit to the error we commit, in taking any one of these converging fractions for the complete value.

In the first place, it is clear that this error cannot be so great as the difference between the fraction taken and that which immediately follows it, because the true value lies between these two. Now the *numerator* of the difference between two consecutive fractions is obtained by multiplying the terms crosswise, and subtracting; the denominator is obtained by multiplying together those of the given fractions. Let, then, $\frac{P}{P'}$, $\frac{Q}{Q'}$, be any two consecutive fractions, and we shall have, for the numerator of their difference, the expression

$$PQ' - P'Q;$$

and, for the numerator of the difference between $\frac{Q}{Q'}$, $\frac{R}{R'}$, or, which is the same thing, between $\frac{Q}{Q'}$, $\frac{Qr + P}{Q'r + P'}$, we shall have the expression

$$QQ'r + P'Q - QQ'r - PQ' = P'Q - PQ';$$

the very same as the former difference, only with contrary sign. Hence, throughout the series, if the difference between each fraction and the next following be taken, the numerators of the results will always be the same in magnitude, but will have alternate signs. To determine the actual value of the numerators, we have, therefore, only to ascertain it in one instance. Let us then take the two leading fractions, which are $\frac{1}{a}$, $\frac{b}{ab+1}$, and we have

$$(ab + 1) - ab = 1;$$

hence the numerators in question are always unity, so that the error we commit in taking the converging fraction, $\frac{Q}{Q'}$, for the true value, is always less than $\frac{1}{Q'R'}$. This leads to a valuable property of these fractions: which is, that between any two consecutive terms, it will be impossible to insert a fraction of intermediate value, whose denominator shall not exceed that of each of the proposed fractions, for it is obvious that no fraction can be smaller than $\frac{1}{Q'R'}$, unless its denominator be greater. Hence, the converging fractions not only approximate continually to the value of x, but they present themselves in the most simple forms possible; so that it would be impracticable to substitute for any one of them another, more approximative, that would not be more complex. That the converging fractions always present themselves in the lowest terms is plain from the condition just referred to, viz.

$$P'Q - PQ' = 1$$

the first member of which would admit of an integral divisor if $\frac{P}{P'}$, $\frac{Q}{Q'}$, were not in their lowest terms, which integral divisor is forbidden by the second member.

These converging fractions are, therefore, highly useful for the purpose of enabling us to express, in small numbers, a near value of a ratio of which the terms may be too large to be easily managed in computation. For instance, the ratio of the diameter of a circle to its circumference is known to be very nearly as 100000 to 314159; and to get a series of other ratios, more simply expressed, and continually approximating to this, we proceed as follows:

```
100000) 314159 (3
        300000
        ------
         14159) 100000 (7
                 99113
                 -----
                   887) 14159 (15
                          887
                         ----
                         5289
                         4435
                         ----
                          854) 887 (1
                               854
                               ---
                                33) 854 (25
                                     66
                                    ---
                                    194
                                    165
                                    ---
                                     29) 33 (1
                                         29
                                         --
                                          4) 29 (7
                                             28
                                             --
                                              1) 4 (4
                                                 4
                                                 -
```

∴ the quots. are 3 7 15 1 25 1 7 4

and conv. frac. $\frac{1}{3}, \frac{7}{22}, \frac{106}{333}, \frac{113}{355}, \frac{2931}{9208}, \frac{3044}{9563}, \frac{24239}{76149}, \frac{100000}{314159}$.

The second of these ratios, viz. 7 to 22, is that which was first given by ARCHIMEDES, and is sufficiently near the truth for many practical purposes; the ratio, 113 to 355, is that of METIUS, and is a still nearer approximation. The ratio of ARCHIMEDES differs from the truth, by a quantity less than $\frac{1}{22 \times 333}$, and the ratio of METIUS differs from the truth, by a quantity less than $\frac{1}{355 \times 9208}$, as appears from the foregoing expression for the limit of the error.

(227.) We may easily obtain a limit to the error, that shall be independent of the denominator of the fraction which follows that at which we stop; although such a limit will not be so small as that just deduced. For, since the denominators increase, we must have

$$R' > Q' \therefore Q'R' > Q'^2 \therefore \frac{1}{Q'R'} < \frac{1}{Q'^2};$$

hence the error committed by taking the converging fraction, $\frac{Q}{Q'}$, for the value of x, must be less than $\frac{1}{Q'^2}$.

From this expression for the limit of error, we can always determine a converging fraction, which shall approach as near to the true value as we please, or which shall differ from that value by less than any assigned quantity Δ; for, in order that $\frac{Q}{Q'}$ may be the fraction, it will be sufficient that $\frac{1}{Q'^2}$ do not exceed Δ, or, that Q' be not less than $\sqrt{\frac{1}{\Delta}}$.

The property established in (226), that $P'Q - PQ' = 1$, will also furnish an expression for the inferior limit of the error, as well as for the superior limit; for, in consequence of this property, we have (225)

$$x - \frac{P}{P'} = \frac{y}{(Q'y + P')\,P'};$$

and, therefore, dividing the numerator and only *part* of the denominator by y, we have

$$x - \frac{P}{P'} > \frac{1}{(Q' + P')\,P'}$$

Hence, taking either of the converging fractions, $\frac{P}{P'}$, for the true value of x, we have the following limits to the error, viz.

$$x - \frac{P}{P'} < \frac{1}{P'Q'} < \frac{1}{P'^2}$$

$$x - \frac{P}{P'} > \frac{1}{(Q' + P')\,P'} > \frac{1}{2P'Q'}.$$

(228.) In the examples hitherto given of the development of α, in the form of a continued fraction, α has been considered to be a rational fraction, and the several quantities a, b, c, &c. have been obtained by means of the operation to find the greatest common measure of the terms of the proposed fraction. But, when α is an irrational quantity, it is obvious that we must determine a, b, c, &c. by some other means. Let us here recall the principles with which we set out at the commencement of the Chapter, and, from which, without any restriction as to the rationality of α, we arrived at the expressions

$$\alpha = a + \frac{1}{\beta} = a + \frac{1}{b} + \frac{1}{\gamma} \quad = a + \frac{1}{b} + \frac{1}{c} + \frac{1}{\delta};$$

and so on; and let us follow the successive steps there pointed out, in order to effect the reduction of $\sqrt{19}$ into a continued fraction; that is, let $\alpha = \sqrt{19}$.

Now the greatest integer in $\sqrt{19}$ is 4, $\therefore a = 4$, and consequently, $\frac{1}{\sqrt{19} - 4} = \beta$. In order to perceive more readily the greatest integer in this, multiply both numerator and denominator by $\sqrt{19}+4$, then $\dfrac{\sqrt{19}+4}{3}=\beta$, in which the greatest integer is obviously 2; hence $\dfrac{\sqrt{19}+4}{3}-2=\dfrac{\sqrt{19}-2}{3}\therefore\dfrac{3}{\sqrt{19}-2}=\gamma$, and, by proceeding in this way, we have

$$\alpha=\sqrt{19}=4+\frac{\sqrt{19}-4}{1}\;\ldots\ldots\ldots\;\therefore a=4$$

$$\beta=\frac{1}{\sqrt{19}-4}=\frac{\sqrt{19}+4}{3}=2+\frac{\sqrt{19}-2}{3}\;\therefore b=2$$

$$\gamma=\frac{3}{\sqrt{19}-2}=\frac{\sqrt{19}+2}{5}=1+\frac{\sqrt{19}-3}{5}\;\therefore c=1$$

$$\delta=\frac{5}{\sqrt{19}-3}=\frac{\sqrt{19}+3}{2}=3+\frac{\sqrt{19}-3}{2}\;\therefore d=3$$

$$\epsilon=\frac{2}{\sqrt{19}-3}=\frac{\sqrt{19}+3}{5}=1+\frac{\sqrt{19}-2}{5}\;\therefore e=1$$

$$\zeta=\frac{5}{\sqrt{19}-2}=\frac{\sqrt{19}+2}{3}=2+\frac{\sqrt{19}-4}{3}\;\therefore f=2$$

$$\eta=\frac{3}{\sqrt{19}-4}=\frac{\sqrt{19}+4}{1}=8+\sqrt{19}-4\;\therefore g=8$$

$$\theta=\frac{1}{\sqrt{19}-4}.$$

As we have now arrived at the same expression as that which we have already had for β, it is plain that the series b, c, &c.

must recur; and that the continued fraction, as far as one period, will be

$$\sqrt{19} = 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{8 + \&c.}}}}}}$$

and the series of converging fractions, which may be carried to any extent, now that we have got a, b, c, for a complete period, will be

$$\begin{array}{cccccccc} 2 & 1 & 3 & 1 & 2 & 8 & 2 & \&c. \\ \frac{1}{2} & \frac{1}{3} & \frac{4}{11} & \frac{5}{14} & \frac{14}{39} & \frac{117}{326} & \frac{248}{691} & \&c. \end{array}$$

$$\therefore \sqrt{19} = 4 + \frac{248}{691} \text{ nearly};$$

which does not differ from the truth by so much as $\frac{1}{(691)^2}$.

It is not only in the particular example which we have here chosen, that the continued fraction is periodical, for it is the property of all quadratic surds to give rise to these recurring fractions; but, for the proof of it, we must refer the student to Barlow's Theory of Numbers, or to the Théorie des Nombres of Legendre, page 43.

Application of Continued Fractions to the Summation of Infinite Series.

(229.) In our treatise on *Algebra*, page 248, we promised to furnish, in the present volume, a direct and easy method of summing every infinite series of which the generating function is rational. The method to which we alluded, is one of the many deductions from the doctrine of continued fractions, and may, therefore, without impropriety, be given in this place.

1. Let the sum of the infinite series

$$1 - 3x + 5x^2 - 7x^3 + 9x^4 - 11x^5 + 13x^6 - \&c.$$

be required.

Regarding this series as the numerator of a fraction, whose denominator is unity, and, dividing the denominator by the numerator, we obtain for quotient 1 and for remainder

$$3x - 5x^2 + 7x^3 - 9x^4 + 11x^5 - \&c.$$

dividing the former divisor, that is, the original series, by this remainder, we have, for quotient $\frac{1}{3x}$, and for remainder,

$$-\frac{4}{3}x + \frac{8}{3}x^2 - \frac{12}{3}x^3 + \frac{16}{3}x^4 - \frac{20}{3}x^5 + \&c.$$

dividing the last divisor by this, we obtain for quotient $-\frac{9}{4}$, and for remainder,

$$x^2 - 2x^3 + 3x^4 - 4x^5 + 5x^6 - 6x^7 + \&c.$$

and, lastly, dividing the preceding divisor by this, we get, for quotient $-\frac{4}{3x}$, and for remainder, zero. Hence the proposed series may be replaced by the continued fraction,

$$\cfrac{1}{1 + \cfrac{1}{\frac{1}{3x} + \cfrac{1}{-\frac{9}{4} + \cfrac{1}{-\frac{4}{3x}}}}}$$

We may get rid of the fractional denominators, one by one, in the usual way, thus: omitting the leading term, multiply numerator and denominator of the remaining fraction by $3x$, then omitting the second term, multiply numerator and denominator of the remaining fraction by 4, and, finally, omitting the preceding terms, multiply numerator and denominator of the remaining fraction by $3x$, and the continued fraction will then be

$$\frac{1}{1} + \frac{3x}{1} + \frac{12x}{-9} + \frac{12x}{-4};$$

or rather

$$\frac{1}{1} + \frac{3x}{1} - \frac{4x}{3} + \frac{x}{1}.$$

This may be easily reduced to an ordinary fraction, by collecting the several terms, commencing at the last; and we thus find, for the sum of the proposed series, the expression

$$\frac{1-x}{(1+x)^2}.$$

2. As a second example, let the series

$$1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \&c.$$

be proposed.

By proceeding as in the former example, we find the following series of quotients and remainders, viz.

QUOTIENTS.	REMAINDERS.
1	$2x - 3x^2 + 4x^3 - 5x^4 + 6x^5 - \&c.$
$\frac{1}{2x}$	$-\frac{x}{2} + \frac{2x^2}{2} - \frac{3x^3}{2} + \frac{4x^4}{2} - \frac{5x^5}{2} + \&c.$
-4	$x^2 - 2x^3 + 3x^4 - 4x^5 + \&c.$
$-\frac{1}{2x}$	0

Hence the equivalent continued fraction is

$$\frac{1}{1} + \cfrac{1}{\cfrac{1}{2x} + \cfrac{1}{-4 + \cfrac{1}{-\cfrac{1}{2x};}}}$$

or rather,

$$\frac{1}{1} + \cfrac{2x}{1 - \cfrac{x}{2 + \cfrac{x}{1};}}$$

which, reduced to an ordinary fraction, is

$$\frac{1}{(1+x)^2};$$

of which the proposed series is the development.

3. Let the series be

$$1 + 5x + 9x^2 + 13x^3 + 17x^4 + 21x^5 + \text{\&c.}$$

Then proceeding as in the last example, we have the following table of quotients and remainders, viz.

QUOTIENTS.	REMAINDERS.
1	$-\ 5x\ -\ 9x^2\ -13x^3 - 17x^4 - 21x^5 - \text{\&c.}$
$-\frac{1}{5x}$	$\frac{16x}{5} + \frac{32x^2}{5} + \frac{48x^3}{5} + \frac{64x^4}{5} + \frac{80x^5}{5} + \text{\&c.}$
$-\frac{25}{16}$	$x^2 + 2x^3 + 3x^4 + 4x^5 + \text{\&c.}$
$\frac{16}{5x}$	0

hence the equivalent continued fraction is

$$\frac{1}{1} + \cfrac{1}{-\frac{1}{5x} + \cfrac{1}{-\frac{25}{16} + \cfrac{1}{\frac{16}{5x}}}},$$

or rather

$$\frac{1}{1} - \cfrac{5x}{1 + \cfrac{16x}{5 - \cfrac{x}{1}}};$$

which, reduced to a common fraction, is

$$\frac{1 + 3x}{(1 - x)^2};$$

the development of which is the proposed series.

By treating, in a similar way, the series

$$4x + 15x^2 + 40x^3 + 85x^4 + 156x^5 + \&c.$$

we find its generating rational fraction to be

$$\frac{x(1 + x^2)(4 - x)}{(1 - x)^4}.$$

These examples are sufficient to show that the foregoing process, founded on the determination of the greatest common divisor, between unity and the proposed series, furnishes a direct and simple method of summing every infinite series of which the generating function is rational. We are indebted for it to a paper by M. Le Barbier, published in the *Annales de Mathématiques*, for March, 1831.*

Application of Continued Fractions to the Solution of Equations.

The method of approximating to the incommensurable roots

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* It is proper to mention, that the preceding method is also embodied in a paper by Mr. Horner, " On the use of continued Fractions in the Summation of Series," published in the *Annals of Philosophy* for June, 1826.

of an equation, by continued fractions, is due to LAGRANGE. An example or two will suffice to illustrate it.

1. Let the equation

$$x^3 - 2x - 5 = 0,$$

be proposed. It is soon seen that 2 is the first figure of the real root, the other two are imaginary, because $4(-2)^3 + 27 \times 5^2 > 0$ (Introductory Treatise, page 106). Substitute, then, $2 + \frac{1}{x'}$ for x, and we have the following transformed equation, in which the root x' must necessarily exceed unity:

$$x'^3 - 10x'^2 - 6x' - 1 = 0.$$

Of course we effect this transformation, not by the actual substitution of $2 + \frac{1}{x'}$ for x, in the proposed equation, as LAGRANGE did, but by operating as in (73), thus:

$$\begin{array}{cccc}
1 & 0 & -2 & -5\ (2 \\
 & \underline{2} & \underline{4} & \underline{4} \\
 & 2 & 2 & -1 \\
 & \underline{2} & \underline{8} & \\
 & 4 & 10 & \\
 & \underline{2} & & \\
 & 6 & &
\end{array}$$

and, consequently (73), the transformed equation is

$$x'^3 - 10x'^2 - 6x' - 1 = 0.$$

The first figure of the root of this equation, found by trial, is 10; putting, therefore,

$$x' = 10 + \frac{1}{x''},$$

we have, for a new transformed, the equation

$$61x''^3 - 94x''^2 - 20x'' - 1 = 0:$$

the first figure in the root of which is 1. Put, therefore,

$$x'' = 1 + \frac{1}{x'''},$$

and effect a third transformation, which will be

$$54x'''^3 + 25x'''^2 - 89x''' - 61 = 0;$$

in which the first figure of the root is 1. Continue this process, and we shall have, for the leading figures of the root of the original equation, the expression

$$x = 2 + \cfrac{1}{10 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \&c.}}}}$$

which furnishes the converging fractions following,

$$\begin{array}{cccccccccc} & & 1 & 1 & 2 & 1 & 3 & 1 & 1 & 12 \\ 2, & \frac{21}{10}, & \frac{23}{11}, & \frac{44}{21}, & \frac{111}{53}, & \frac{155}{74}, & \frac{576}{275}, & \frac{731}{349}, & \frac{1307}{624}, & \frac{16415}{7837}, \&c. \end{array}$$

and these are alternately below and above the true value of the root. The fraction $\frac{16415}{7837}$ is greater than the true value; but the error being less than $\frac{1}{(7837)^2}$, by (227), that is, less than 0000000163, it follows that the approximation, $\frac{16415}{7837}$, will be true as far as the seventh decimal. The root is, therefore, 2·0945514, true to seven places.

In each of the transformed equations, which occur in the foregoing operation, the root is necessarily greater than unity, and but one real root exists in each; so that, in searching for the first figure, we are to limit our trials to the numbers 0, 1, 2, 3, ... 10, 11, &c.

When the equation has several real roots, they may all be separately evolved, as above, provided we know their number and situation. This knowledge the application of Sturm's theorem will always supply, and the method of Lagrange may thus be perfected. The same thing may also be effected by the transformations of Budan, or by the methods expounded in Chap. XII. But, as remarked at (146), the method of Budan is that which best unites with Lagrange's process for developing the roots. We shall exemplify it in its application to the equation

$$x^3 - 7x + 7 = 0.$$

By the first series of transformations exhibited at page 198, we find the interval [1, 2] to be doubtful: hence if the equation have any real positive roots, they must lie between 1 and 2, so that we must have

$$x = 1 + \frac{1}{x'},$$

where x' must be determined from the transformed equation

$$x'^3 - 4x'^2 + 3x' + 1 = 0.$$

Applying Budan's method to this, we are led to the second series of transformations at page 198, which resolves the doubt, and at the same time supplies the leading figure of each of the two positive values of x'; that of one of these values being found to be 1, and that of the other 2. It is thus that Budan's transformations, for determining the character of a doubtful interval, facilitates the actual development of the roots under examination whenever they eventually prove to be real: for, as here, each series of transformations supplies an additional term in the continued fraction by which the developments are expressed, till the roots separate and proceed singly, lying in distinct intervals.

In order to approximate to the first of the above roots, which have now separated, put as before

$$x' = 1 + \frac{1}{x''},$$

and we shall have the transformed equation

$$x''^3 - 2x''^2 - x'' + 1 = 0;$$

to which there will be no necessity to apply the transformations of Budan, because we know that it has one root, and only one, greater than unity, so that two consecutive numbers in the series 0, 1, 2, ... , must, when substituted for x, give results with contrary signs. These numbers are 2 and 3.

To approximate to the other positive root, we must put

$$x' = 2 + \frac{1}{x_{''}},$$

which will furnish the transformed equation

$$x_{''}^{3} + x_{''}^{2} - 2x_{''} - 1 = 0;$$

which has one, and only one, root greater than unity; and, therefore, its situation may be easily found by trial to be between 1 and 2. We have, therefore, now to make the substitutions

$$x'' = 2 + \frac{1}{x'''}$$

$$x_{''} = 1 + \frac{1}{x_{'''}},$$

and we thus have the new equations

$$x'''^{3} - 3x'''^{2} - 4x''' - 1 = 0$$

$$x_{'''}^{3} - 3x_{'''}^{2} - 4x_{'''} - 1 = 0,$$

each of which has one, and only one, root greater than unity; and the first figure of each is found by trial to be 4. The next transformation is consequently

$$x''''^{3} - 20x''''^{2} - 9x'''' - 1 = 0,$$

the first figure of the root of which is 20; and therefore the next transformation is

$$181x'''''^{3} - 391x'''''^{2} - 40x''''' - 1 = 0,$$

the first figure of the root of which is 2; the next transformed equation is

$$197x''''''^3 - 568x''''''^2 - 695x'''''' - 181 = 0,$$

the first figure of the root of which is 3; and, by continuing these transformations, we have, for the values of x sought, the following developments, viz.

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{20 + \cfrac{1}{2 + \cfrac{1}{3 +, \&c.}}}}}}$$

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{20 + \cfrac{1}{2 + \cfrac{1}{3 +, \&c.}}}}}}$$

The converging fractions deduced from these are

$$\begin{array}{cccccccc} & & 2 & 4 & 20 & 2 & 3 & \\ \frac{1}{1}, & \frac{2}{1}, & \frac{5}{3}, & \frac{22}{13}, & \frac{445}{263}, & \frac{912}{539}, & \frac{3181}{1880}, & \&c. \end{array}$$

$$\begin{array}{cccccccc} & & 1 & 4 & 20 & 2 & 3 & \\ \frac{1}{1}, & \frac{3}{2}, & \frac{4}{3}, & \frac{19}{14}, & \frac{384}{283}, & \frac{787}{580}, & \frac{2745}{2023}, & \&c. \end{array}$$

Hence, for near values of x, we have

$$x = \frac{3181}{1880}, \quad x = \frac{2745}{2023};$$

or, in decimals,

$$x = 1{\cdot}6920213, \quad x = 1{\cdot}3568957,$$

which are true as far as six decimals, and are but a unit below the truth in the seventh place.

(230.) The two equations just solved are both given by Lagrange in illustration of the method, proposed by Waring and himself, for discovering the character of roots by means of an auxiliary equation, technically called *the equation of the squares of the differences*, and also for the purpose of exhibiting, as above, the actual development of those that are thus ascertained to be real, by his peculiar process of continued fractions. The formation of the auxiliary equation, which will be explained hereafter, is in general impracticable beyond very narrow limits, on account of the long computations required: and it is therefore advantageously replaced by the transformations of Budan, as in the example just considered. As before remarked, these transformations actually supply the leading terms of Lagrange's development, up to the point where the roots are found to separate, whenever those roots turn out to be real; so that of all the methods of analysis that might be proposed, that of Budan would seem to claim a preference in connexion with Lagrange's process of development. But in those cases of difficulty, so fully discussed in the preceding chapters of the present work, where roots approach very near to equality, Budan's trial-transformations become in general too numerous to be available in actual practice. On the average four or five of these transformations must be calculated upon for each effective step in the analysis; so that the separation of the two contiguous roots, in the equation of the fifth degree proposed at page 268, might be expected to involve about four times the labour which was expended upon the same object at page 271. (See articles 145, 146.)

CHAPTER XV.

THE SOLUTION OF TWO EQUATIONS, CONTAINING TWO UNKNOWN QUANTITIES: THEORY OF ELIMINATION.

(231.) Two equations, each containing two unknown quantities, x, y, together with known numbers, may be thus expressed, viz.

$$F(x, y) = 0, \quad f(x, y) = 0 \quad . \ . \ . \ . \ [1];$$

and their solution consists in determining the system of values for x and y, which simultaneously satisfy both equations.

In order that y may have a value β, which will equally belong to both equations, it is obviously necessary, and it is sufficient, that there exist a value for x, competent to satisfy the two equations

$$F(x, \beta) = 0, \quad f(x, \beta) = 0;$$

that is, these two equations in x must have a common root, and therefore the polynomials $F(x, \beta)$, $f(x, \beta)$, must have a common factor, or admit of a common measure in x. In order, therefore, to ascertain whether any proposed value, β, for y is consistent with the conditions [1], we should have to perform the operation for the common measure upon the functions $F(x, \beta)$, $f(x, \beta)$. If a common measure, which is a function of x, be found to exist, the proposed value for y is admissible, and the common measure, equated to zero, will be an equation whose roots will be the corresponding values of x; but if no such common measure exist, we must then reject the assumed value of y, as being incompatible with the conditions [1].

To assume different values for one of the unknowns, and, in this way, to try their eligibility, would, in many cases, require an endless series of operations. The most direct and obvious mode of proceeding, in order to obtain values for y, which must necessarily cause the functions in x to have a common measure, would seem to be this, viz. to arrange the terms of each polynomial according to the powers of x, and to operate upon them, for the common measure, till we arrive at a remainder independent of x, and then to equate this remainder in y to zero. For the values of y, which satisfy this equation, are all such as to cause the remainder to vanish.

It must be remembered, however, that, in the operation of finding the greatest common measure of two algebraical expressions, we have frequent occasion to suppress certain factors, and to introduce others, and, before we could affirm with confidence that the values of y, which cause the remainder to vanish, necessarily fulfil the proposed conditions, we must examine whether or not this remainder is affected by the factors, which may have been rejected or introduced. If, however, the process for the common measure, in any particular case, requires neither the suppression nor the introduction of a factor, we may then safely infer that the final remainder, or that which is independent of x, will, when equated to zero, furnish all the values of y consistent with the proposed conditions; because, if each value thus determined were to be put for y, in the original polynomials, and the common measure in each case found, we should, obviously, arrive at the very same series of collateral expressions for x.

For example, suppose the equations

$$x^3 + 3yx^2 + (3y^2 - y + 1)\,x + y^3 - y^2 + 2y = 0$$

$$x^2 + 2yx + y^2 - y = 0$$

were proposed for solution.

As the polynomials are already arranged according to the decreasing powers of x, we may at once commence the operation for the common measure, which is as follows:

$$
\begin{array}{l}
x^2+2yx+y^2-y]x^3+3yx^2+(3y^2-y+1)x+y^3-y^2+2y[x+y \\
\qquad\qquad\qquad x^3+2yx^2+(\ y^2-y)x \\
\hline
\qquad\qquad\qquad\qquad yx^2+(2y^2+1)x+y^3-y^2+2y \\
\qquad\qquad\qquad\qquad yx^2+2y^2x+y^3-y^2 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad x+2y]x^2+2yx+y^2-y[x \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x^2+2yx \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y^2-y.
\end{array}
$$

Having now got a remainder, independent of x, we have for the determination of all those values of y, which cause the proposed polynomials to have a common measure, the equation

$$y^2-y=0 \therefore y=0, \quad y=1;$$

and the values of x, corresponding to these, are, of course, those furnished by equating the common measure to zero; they are, therefore,

$$x=0, \quad x=-2.$$

It is plain that $y=0$, and $y=1$, are the only values which, when substituted in the proposed expression, will cause the preceding operation to terminate; if other values for y existed, the final remainder above would necessarily contain them.

If a like process be performed with the two equations

$$x^4+2yx^3+(2y^2+1)x^2+(y^3+9y^2+y-81)x+y^2=0$$

$$x^3+2yx^2+2y^2x+y^2+9y^3-81=0,$$

we should find, without suppressing or introducing any factor, the expression $9y^2 - 81$, for the remainder in y, and the expression $x^2 + yx + y^2$, for the corresponding divisor; hence the final equations for determining x and y, are

$$y^2 - 9 = 0, \quad x^2 + yx + y^2 = 0,$$

the solution of which will furnish the values which satisfy the proposed equations.

But let us examine the consequences of introducing or suppressing factors in the course of the process for finding the common measure, or of arriving at a remainder Y independent of x.

There are three distinct cases to consider, viz.

1. The value attributed to y may reduce to zero neither of the factors which have been introduced or suppressed.

2. It may reduce to zero one of the factors which have been introduced.

3. The value may be such as to reduce to zero one of the factors which have been suppressed.

(232.) 1. Suppose a value to be attributed to y that does not render any of the factors introduced or suppressed zero. If we substitute this value in the two polynomials, and perform the operation with the resulting functions of x, we shall obtain for remainder the same value that would be furnished by the substitution of y in Y, or else a value equal to the result of this substitution multiplied or divided by a numerical factor. For every algebraic factor introduced or suppressed is, by the substitution of the proposed value for y, reduced to a number, because, by hypothesis, none of them are rendered zero; these factors, therefore, affect only the numerical factors of the several remainders which arise in the course of the operation. Hence, in order that the value of y may satisfy the proposed equations, it is necessary and sufficient that it satisfies the equation $Y = 0$.

(233.) 2. Let the value attributed to y destroy one of the factors introduced into a dividend to render the division possible;

the dividend thus modified will, for that particular value of y, become zero; so that, in order to carry on the division, we have introduced a factor that causes a dividend to vanish, which is of course not allowable; for, with such a dividend, the process would always terminate, whether there was a common measure or not; we cannot, therefore, affirm that the value of y, which causes one of the factors that have been introduced to vanish, satisfies the proposed equations, although it may fulfil the condition $Y = 0$.

(234.) 3. Lastly, let the value attributed to y destroy one of the factors which have been suppressed, and yet not satisfy the condition $Y = 0$; then, such a value of y causes the process to terminate at that remainder in which the factor has been suppressed, because, when the assumed value is put for y in the polynomials, this remainder becomes zero; hence the preceding divisor is a common measure of those polynomials, and thus a common measure may exist for values of y which do not satisfy the condition $Y = 0$. It must be remarked, however, that if in any part of the operation which precedes the suppression of the vanishing factor, a factor has been introduced which also vanishes for the same value of y, the above conclusion would not necessarily follow.

(235.) From the foregoing considerations we see, that to obtain the values of y which belong to the proposed equations, we must equate to zero the remainder, which is independent of x, as also each of the factors in y which have been suppressed in the course of the operation, and resolve each equation separately; secondly, that among the values thus obtained, there may be found some which are extraneous, and which must therefore be rejected as not being consistent with the proposed conditions. If no factor has been suppressed in the course of the operation, the equation $Y = 0$ alone will furnish all the suitable values of y, and may also contain values not admissible, provided factors have been introduced; but when no factor has been either introduced or suppressed, then the values of y in the equation $Y = 0$ all belong to the proposed equations, to the exclusion of all other values.

Having thus examined the influence of the factors introduced or suppressed in the course of the operation upon the final remainder in y, let us now return to the original polynomials $F(x, y)$, $f(x, y)$, and analyse the process by which we must arrive at this remainder.

(236.) The proposed functions being arranged according to the descending powers of x, will each be of the form

$$ax^m + bx^{m-1} + cx^{m-2} \ldots = 0,$$

where the coefficients a, b, c, &c. are all independent of x.

As these coefficients may have a function of y for a common divisor, let us suppose that the greatest common divisor of the coefficients in $F(x, y)$ is $F_{\prime}(y)$, and that the greatest common divisor of the coefficients in $f(x, y)$ is $f_{\prime}(y)$; also of these two divisors let the greatest common divisor be $\phi(y)$, which will therefore be the greatest divisor common to all the coefficients of both equations. If now we represent by A the quotient of $F(x, y)$ by $F_{\prime}(y)$; by B the quotient of $f(x, y)$ by $f_{\prime}(y)$; by $F'(y)$ the quotient of $F_{\prime}(y)$ by $\phi(y)$; and by $f'(y)$ the quotient of $f_{\prime}(y)$ by $\phi(y)$, we shall obviously have

$$F(x, y) = \phi(y) \times F'(y) \times A = 0$$

$$f(x, y) = \phi(y) \times f'(y) \times B = 0,$$

both of which equations will be satisfied by the condition

$$\phi(y) = 0,$$

as also by either of the following pairs of conditions, viz.

1. $F'(y) = 0, \quad B = 0.$
2. $f'(y) = 0, \quad A = 0.$
3. $A = 0, \quad B = 0.$

The conditions

$$F'(y) = 0, \quad f'(y) = 0,$$

it is evident, cannot exist, because they involve but one unknown quantity, and their first members have no common factor.

As to the equation $\phi(y) = 0$, it furnishes certain values of y for which x is indeterminate; for the proposed equations will evidently be satisfied for any value of x in conjunction with these values of y.

To find the solutions of the system $F'(y) = 0$, $B = 0$, we must resolve the first equation, which contains only y, and substitute the resulting values separately in B, and we shall thus have so many equations in x to determine the corresponding values. The system $f'(y) = 0$, $A = 0$, requires similar treatment.

This preliminary examination being disposed of, the equations will be thus reduced to the simplest form for the application of the general method, viz., to the system $A = 0$, $B = 0$, in which neither A nor B has any factor in y. To determine the solutions which satisfy this system, we must apply the process for finding the common measure.

(237.) 1. Suppose that the first step of this process conducts to a remainder, R, of a lower degree in x than the divisor, without our being obliged to use any preparation to render the division possible, or to avoid the occurrence of y as a denominator in the quotient Q; then, if A is the polynomial taken for the dividend, we shall have the identity

$$A = BQ + R,$$

which shows that whatever values of x and y satisfy the equations $A = 0$, $B = 0$, the same must also satisfy the equation $R = 0$; and that whatever values satisfy the equations $B = 0$, $R = 0$, satisfy also the equation $A = 0$; so that the solutions of the proposed equations

$$A = 0, \quad B = 0,$$

are exactly the same as those of the equations

$$B = 0, \quad R = 0,$$

which are more simple than the former system, inasmuch as one is of an inferior degree in x. The same conclusions evidently follow when the dividend A is multiplied at the outset by any numerical factor.

It is easy to prove that the consequences just deduced could not have place if the quotient Q contained y in a denominator. For suppose the form of the quotient to be $Q = \frac{H}{K}$, K being a quantity containing y; the identity above would then be

$$A = \frac{BH}{K} + R.$$

If we gave to x and y all the values which fulfil the conditions

$$A = 0, \quad B = 0,$$

among these values there might be some for y which, for aught we know to the contrary, might render K zero, in which case $\frac{BH}{K}$ would become $\frac{0}{0}$, which is not necessarily zero; so that $A = 0$, $B = 0$, would not necessarily imply $R = 0$; and we could not therefore assert that all the solutions of the system $A = 0$, $B = 0$, were equally given by the system $B = 0$, $R = 0$.

(238.) 2. Let us now suppose that, to avoid fractions in the quotient, it be necessary to introduce an algebraical factor into the dividend A: call it C, and let Q, R be the corresponding quotient and remainder, as before. We shall thus have the identity

$$CA = BQ + R.$$

which shows that the solutions of the equations

$$B = 0, \quad R = 0,$$

are the same as those of the equations

$$CA = 0, \quad B = 0.$$

Now this last system divides itself into two others, viz.

$$A = 0, \quad B = 0, \text{ and } C = 0, \quad B = 0.$$

Consequently the equation $B = 0$, $R = 0$, will give all the solutions of the proposed system $A = 0$, $B = 0$, but they will give in addition, solutions to the system $C = 0$, $B = 0$.

These latter solutions we can separate from the others; for $C = 0$, containing only y, will furnish all the values of y which are doubtful, and the values of x, corresponding to these, are given by the solutions to $B = 0$, $R = 0$. Those pairs of these values which, substituted in the equation $A = 0$, satisfy its conditions are admissible, the others are to be rejected.

(239.) From the preceding discussion it appears that the solution of the two equations proposed is reducible to the solution of the two equations

$$B = 0, \quad R = 0.$$

As the polynomial B contains no factors depending only on y, if R contain any such factors, we may of course suppress them; but then we must take account of the solutions which reduce to zero these factors, connecting each value of y with that value of x which satisfies $B = 0$, when the said value of y is substituted in B.

We shall now give an example or two of the application of this theory.

EXAMPLES.

1. Let the system of equations be

$$x^2 + (8y - 13)\,x + y^2 - 7y + 12 = 0$$
$$x^2 - (4y + 1)\,x + y^2 + 5y = 0.$$

Here the coefficients having no common measure, these equations may be regarded as the equations $A = 0$, $B = 0$, treated above; and from these we are to determine, agreeably to the general theory, the system $B=0$, $R=0$, which will contain all the solutions required. Dividing A by B, we have

$$x^2-(4y+1)\,x+y^2+5y]\;x^2+(8y-13)x+y^2-7y+12\;[1$$

$$\underline{x^2-(4y+\;1)x+y^2+5y}$$

$$R=(12y-12)x-12y+12=12(y-1)(x-1),$$

the equations which furnish the solutions are, therefore,

$$\begin{array}{c|c} y-1=0 & x-1=0 \\ x^2-(4y+1)x+y^2+5y=0 & x^2-(4y+1)\,x+y^2+5y=0, \end{array}$$

and each of these systems may be solved without repeating the divisions; the solutions are

$$\begin{array}{c|c|c|c} y=1 & y=1 & y=0 & y=-1 \\ x=3 & x=2 & x=1 & x=1. \end{array}$$

2. Let the equations

$$x^3+2yx^2+2y\,(y-2)\,x+y^2-4=0$$

$$x^2+2yx\;+2y^2-5y+2=0$$

be proposed.

The coefficients having no common measure, we have, by dividing the first polynomial by the second, the following remainder, viz.

$$R=(y-2)\,x+y^2-4=(y-2)\,(x+y+2);$$

hence the solutions to the proposed equations are those of the systems

$$\begin{array}{c|c} y-2=0 & x+y+2=0 \\ x^2+2yx+2y^2-5y+2=0 & x^2+2yx+2y^2-5y+2=0. \end{array}$$

The first system furnishes the solutions

$$\begin{array}{c|c} y=2 & y=2 \\ x=0 & x=-4. \end{array}$$

To solve the other system, proceed as at first; that is, divide the second polynomial by the first, and there will result the remainder

$$y^2 - 5y + 6;$$

hence the system is replaced by the new system

$$y^2 - 5y + 6 = 0$$

$$x + \ \ y + 2 = 0,$$

which gives for solutions

$$\begin{array}{l|l} y = 2 & y = 3 \\ x = -4 & x = -5; \end{array}$$

so that there are, in all, four solutions to the proposed equations.

3. Let the equations

$$(y - 1)\, x^2 + 2x - \ \ 5y + 3 = 0$$

$$yx^2 + 9x - 10y = 0$$

be proposed.

Multiplying the first polynomial by y, to render it divisible by the second, and then performing the division, we have

$$yx^2 + 9x - 10y]\ (y - 1)\, yx^2 + 2yx - 5y^2 + 3y\ [y - 1$$

$$(y - 1)\, yx^2 + (9y - 9)\, x - 10y^2 + 10y$$

$$\overline{\qquad (-7y + 9)\, x + 5y^2 - \ \ 7y. \qquad}$$

As we have multiplied the dividend by the factor y, the equation $y = 0$ may be a solution to test. Substitute 0 for y in the proposed equations; one, viz. the divisor, furnishes the value $x = 0$, which value does not satisfy the other; hence the factor introduced supplies no solution. We must now proceed with the polynomials B and R; and, in order to this, must multiply the dividend B by $(-7y+9)$, and we shall have, for the remainder arising from the division, the polynomial

$$25y^5 - 70y^4 - 126y^3 + 414y^2 - 243y.$$

The final equations are, therefore,

$$(-7y+9)\,x + 5y^2 - 7y = 0$$

$$25y^5 - 70y^4 - 126y^3 + 414y^2 - 243y = 0;$$

the roots of the second are

$$y = 0,\ \ y = 1,\ \ y = 3,\ \ y = \frac{-3 \pm 3\sqrt{10}}{5}$$

and to these correspond the following values of x, deduced from the first, viz.

$$x = 0,\ \ x = 1,\ \ x = 2,\ \ x = -5 \mp \sqrt{10}.$$

No extraneous solution has been introduced by means of the factor $-7y+9$, by which we have multiplied the second dividend, because none of the above values of y cause it to vanish; but an inadmissible solution has been introduced by the factor y, which multiplies the first dividend, viz. the solution $y = 0$, $x = 0$; rejecting this, therefore, we have, for the entire number of true solutions, the four systems following, viz.

$$\begin{array}{l|l|l|l} y = 1 & y = 3 & y = \dfrac{-3 + 3\sqrt{10}}{5} & y = \dfrac{-3 - 3\sqrt{10}}{5} \\ x = 1 & x = 2 & x = -5 - \sqrt{10} & x = -5 + \sqrt{10} \end{array}$$

4. Let the equations proposed for solution be

$$(y^2 - 1)\,x^2 + (2y^3 - 2y)\,x + y^4 - 2y^2 + 1 = 0$$

$$(y^2 - 3y + 2)\,x^2 - y^4 - 3y^3 + 7y^2 + 15y - 18 = 0.$$

The coefficients of the first polynomial admit of the common divisor $y^2 - 1$; and those of the second admit of the common divisor $y^2 - 3y + 2$; these two factors have themselves a common divisor, which is $y - 1$; so that the proposed equations may be written thus:

$$(y-1)(y+1)(x^2+2yx+y^2-1)=0$$

$$(y-1)(y-2)(x^2-y^2-6y-9)=0.$$

These are satisfied by the values $y = 1$, combined with any value of x whatever, as observed at page 391.

They are also satisfied by the values which satisfy

$$y+1=0, \quad x^2-y^2-6y-9=0;$$

which values are

$$\begin{array}{l|l} y=-1 & y=-1 \\ x=2 & x=-2 \end{array}$$

They are also satisfied by the values which satisfy

$$y-2=0, \quad x^2+2yx+y^2-1=0;$$

which values are

$$\begin{array}{l|l} y=2 & y=2 \\ x=-1 & =-2. \end{array}$$

The remaining solutions are involved in the equations A $= 0$, B $= 0$, (page 391), that is, in the equations

$$x^2+2yx+y^2-1=0$$
$$x^2-y^2-6y-9=0;$$

to which we may apply the method of the common divisor; but, as it is easy to see that the second equation gives

$$x=\pm(y+3),$$

we may substitute these values in the first equation. The first value, $y + 3$, will reduce it to

$$y^2+3y+2=0;$$

which furnishes the values $y = -1$, $y = -2$; and, from the relation $x = y + 3$, we have, for the corresponding values of x, $x = 2$, $x = 1$. If we substitute, in the first equation, the other value, $-(y + 3)$, for x, it will be reduced to $8 = 0$; this value, therefore, furnishes no solution.

5. As a last example, let the equations

$$(y - 2)x^2 - 2x + 5y - 2 = 0$$

$$yx^2 - 5x + 4y = 0,$$

be taken.

The coefficients having no common divisor, we at once commence the operation for finding R; but, to avoid fractions in the quotient, we must prepare the dividend by multiplying it by y.

$$\begin{array}{rl} yx^2 - 5x + 4y] & (y-2)yx^2 - 2yx + 5y^2 - 2y\,[y - 2 \\ & \underline{(y-2)yx^2 - (5y - 10)x + 4y^2 - 8y} \\ & \mathrm{R} = (3y - 10)x + y^2 + 6y. \end{array}$$

It is necessary now to repeat the operation with B and R; and, for this purpose, we must multiply the dividend B by $3y - 10$; the resulting remainder will be found to be

$$y^5 + 12y^4 + 87y^3 - 200y^2 + 100y;$$

so that the final equations are

$$(3y - 10)x + y^2 + 6y = 0$$

$$y^5 + 12y^4 + 87y^3 - 200y^2 + 100y = 0.$$

The second of which is satisfied, for $y = 0$, to which corresponds $x = 0$, in the first; but this is an inadmissible solution, as it does not satisfy the proposed equations. It is due to the factor y introduced in the first division. Suppressing this factor, the final equation in y becomes

$$y^4 + 12y^3 + 87y^2 - 200y + 100 = 0;$$

which cannot involve any inadmissible values of y, because the only circumstance which could cause their introduction is the introduction of the factor, $3y - 10$, in the second division, and this is reduced to zero, by the value $y = \frac{10}{3}$. But, as this value is fractional, it cannot be a root of the equation above (62). We also see, from other causes, that the factor, $3y - 10$, can introduce no solution; the conditions

$$3y - 10 = 0, \quad (3y - 10) x + y^2 + 6y = 0,$$

are incompatible.

The final equation in y has the root $y = 1$, to which corresponds $x = 1$, and the depressed equation in y is

$$y^3 + 13y^2 + 100y - 100 = 0;$$

the roots of which involve interminable decimals. Hence, the remaining solutions can be obtained only by approximation.

In the Chapter next following, a method will be found of obtaining the final equation in y, which shall comprise all the solutions to the proposed equations, and be unembarrassed with inadmissible values.

We shall now proceed to one or two applications of the theory just delivered.

On Irrational Equations.

(240.) All the *direct* methods employed for the solution of equations suppose that the unknown quantities in them are not affected with any radical sign; when therefore, the unknown is found under a radical sign, it will be necessary, before applying the process of solution, to employ some preparatory method of rendering the equation rational. Such a method is at once suggested by the theory of elimination. For, if we equate each of the irrational terms with an unknown quantity, and remove the radical from each of these new equations by involution, we shall have a series of equations (including the original one, with its irrational terms replaced by the new symbols,) without radicals, from which the quantities, temporarily introduced, may be eliminated, and thence a rational equation obtained, involving only the original unknown quantities.

The following examples will fully illustrate the mode of proceeding:

1. Let the equation be

$$x - \sqrt{x-1} + \sqrt[3]{x+1} = 0.$$

Put

$$y = \sqrt{x-1}, \quad z = \sqrt[3]{x+1};$$

and we then have the three following rational equations, from which we may eliminate y and z, viz.

$$y^2 = x - 1, \; z^3 = x + 1, \; x - y + z = 0.$$

From the last equation we get $y = x + z$, and, by substituting this value in the first, y becomes eliminated, and we have these two equations in x and z, viz.

$$z^3 - x + 1 = 0$$

$$z^2 + 2xz + x^2 - x + 1 = 0;$$

and, to eliminate z from these, we apply the process explained in the preceding articles, and thus get the final equation

$$x^6 - 3x^5 + 8x^4 + x^3 + 7x^2 - 7x + 2 = 0.$$

2. Let the equation be

$$\sqrt[3]{4x+7} + 2\sqrt{x-4} = 1.$$

Putting

$$y = \sqrt[3]{4x+7}, \quad z = \sqrt{x-4},$$

we have the system of equations

$$y^3 = 4x + 7, \quad z^2 = x - 4,$$

$$y + 2z = 1.$$

From the last we find $y = 1 - 2z$, and this value of y, substituted in the first, reduces the system to the two equations

$$8z^3 - 12z^2 + 2z + 6 = 0$$

$$z^2 - x + 4 = 0,$$

from which, by the process already explained, we obtain the final equation,

$$16x^3 - 184x^2 + 801x - 1405 = 0.$$

It will be remembered, in conformity with the remarks at (46) that the operations above, by means of which irrational equations are rendered rational, introduce foreign roots into the final result whenever the signs prefixed to the original irrational quantities are intended to indicate the character of the roots. The rational equation will always have a greater number of roots, or be satisfied for a greater number of values of x, than the irrational equation which it is intended to replace, unless the signs of the irrational terms be perfectly unrestricted.

Method of TSCHIRNHAUSEN *for Solving Equations.*

(241.) As another application of the theory of elimination we shall briefly illustrate the principle upon which TSCHIRNHAUSEN proposed to accomplish the general solution of equations, but which, as observed at (81), was soon found to be of but very limited application, not extending beyond equations of the fourth degree; and even within this extent too laborious for general use. The principle consists in connecting with the proposed an auxiliary equation of inferior degree with undetermined coefficients, and of as simple a form as possible consistently with the office it is to perform, but involving, besides the unknown quantity x, a second unknown y. The unknown, common to both equations, is then eliminated according to the preceding theory, and a final equation in y thus obtained, of which the coefficients are functions of the undetermined coefficients in the auxiliary equation. The arbitrary quantities, thus entering the coefficients of the final equation in y, are then determined so as to cause certain of these coefficients to vanish; by which means the equation is ultimately reduced to a prescribed form, supposed to be solvable by known methods.

(242.) As an example, let it be required to reduce the cubic equation

$$x^3 + ax^2 + bx + c = 0 \quad . \ . \ . \ . \ [1]$$

to the binomial form

$$y^3 + k = 0.$$

Assume an auxiliary equation

$$x^2 + a'x + b' + y = 0 \quad . \ . \ . \ . \ [2]$$

and eliminate x from [1] and [2] in the usual way. The remainder arising from dividing the first member of [1] by the first member of [2] is

$$(a'^2 - aa' + b - b' - y)\, x + (a' - a)\,(b' + y) + c$$

which equated to zero gives

$$x = \frac{(a - a')\,(b' + y) - c}{a'^2 - aa' + b - b' - y}$$

and this value of x, substituted in the proposed equation, transforms it, after reduction, into the form

$$y^3 + hy^2 + iy + k = 0 \quad . \ . \ . \ . \ [3]$$

where

$$h = 3b' - aa' + a^2 - 2b$$

$$i = 3b'^2 - 2b'\,(aa' - a^2 + 2b) + a'^2 b$$

$$+ (3c - ab)\, a' + b^2 - 2ac$$

$$k = b'^3 - ab'^2 a' + bb'a'^2 - ca'^3 + (a^2 - 2b)\, b'^2 +$$

$$(3c - ab)\, a'b' + aca'^2 + (b^2 - 2ac)\, b' - bca' + c^2$$

Hence, in order to reduce [3] to the prescribed form, we must determine the arbitrary quantities a', b' conformably to the conditions $h = 0$, $i = 0$; that is, these quantities must satisfy the equations

$$3b' - aa' + a^2 - 2b = 0$$

$$3b'^2 - 2b'(aa' - a^2 + 2b) + a'^2b +$$

$$(3c - ab)a' + b^2 - 2ac = 0$$

of which the first is of the first degree with respect to a' and b', and the other of the second degree; so that their values may be determined by a quadratic equation. And these values, or rather the expression for them in terms of the given coefficients, being substituted in the preceding expression for k, render that symbol known; and thus the required form

$$y^3 + k = 0$$

is obtained.

(243.) In a similar manner may the general equation of the fourth degree

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

be transformed into one of the form

$$y^4 + hy^2 + k = 0$$

which is virtually a quadratic, by eliminating x from the pair of equations

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

$$x^2 + a'x + b' + y = 0$$

which elimination will conduct to a final equation in y of the form

$$y^4 + gy^3 + hy^2 + iy + k = 0$$

from which the second and fourth terms will vanish by the equations of condition

$$g = 0, \quad i = 0$$

the first of which will be of the first degree as regards the arbitrary quantities a', b', and the second of the third: both quantities are therefore determinable by means of an equation of the third degree, and thence the quantities h, k, which are known functions of them.

All this is very laborious; but it really does effect the object proposed thus far: that is, it reduces the solution of equations of the third and fourth degrees to those of inferior degrees: but beyond this point the method fails, as the conditional equations resolve themselves ultimately into a final equation that exceeds in degree that which they are intended to simplify.

As already remarked (81) Mr. Jerrard has greatly extended the principle of Tschirnhausen, and has succeeded in reducing the general equation of the fifth degree

$$x^5 + A_4 x^4 + A_3 x^3 + A_2 x^2 + Ax + N = 0$$

to the remarkably simple forms

$$x^5 + ax^4 + b = 0$$

$$x^5 + ax^3 + b = 0$$

$$x^5 + ax^2 + b = 0$$

$$x^5 + ax \ + b = 0$$

so that the solution of the general equation of the fifth degree might be considered as accomplished if either of the above forms could be solved in general terms.

For a very masterly analysis of Mr. Jerrard's researches, the reader is referred to the paper of Sir W. R. Hamilton in the Report of the sixth meeting of the British Association.

On the Equation of the Squares of the Differences.

(244.) We have already remarked that the equation of the squares of the differences is an auxiliary equation, employed by LAGRANGE for the purpose of separating the real roots of any algebraical equation proposed for numerical solution.

This auxiliary equation is such as to furnish, for its roots, the squares of the differences between every two roots of the proposed equation; so that when we have ascertained the inferior and superior limits of the positive roots of an equation, if we substitute, successively, for x, in it, a series of numbers, increasing from the inferior limit, up to the superior, by differences, Δ, not exceeding the least difference found to exist between the sought roots, by means of the auxiliary equation, no two roots, however close together, can exist in any interval between two consecutive substitutions; and, therefore, in thus proceeding from limit to limit, there will necessarily be presented as many successive changes of sign, in the final term, as there are positive roots between the limits, so that the situation of each root will become known. By determining the limits of the negative roots of the proposed equation, they also may be separated in a similar manner.

When the auxiliary equation, from which the value of Δ is to be deduced, is found, we shall not be required actually to solve it for this purpose; it will, obviously, be sufficient to determine the inferior limit of its positive roots, which limit, being less than the square of the least difference which exists among the roots of the proposed equation, the square root of it may be taken for Δ.

(245.) Except in the simplest cases, the inferior limit spoken of will be a fraction $\frac{1}{l}$ less than unit, and $\sqrt{l}$ will in general be incommensurable; so that it will be convenient to replace $\sqrt{l}$ by the whole number k which is immediately superior to it, taking $\frac{1}{k}$ for Δ. Thus, having found L for the superior limit of the positive roots of the proposed equation, and $-$ L′ for that of the negative roots, it will only remain, in order to detect the number and situations of the positive roots, to substitute for x the following numbers in succession, viz.

$$\frac{1}{k}, \ \frac{2}{k}, \ \frac{3}{k}, \ \frac{4}{k}, \ \ldots . \ L$$

and to detect the negative roots to substitute in like manner the numbers

$$-\frac{1}{k}, -\frac{2}{k}, -\frac{3}{k}, -\frac{4}{k}, \ \ldots . \ -L'$$

the former series will furnish as many changes of sign in the absolute number as there are positive roots, and the latter series as many as there are negative roots.

But fractional substitutions may be altogether avoided by transforming the proposed equation into another whose roots are k times as great (75); that is, by substituting $\frac{y}{k}$ for x. The transformed equation in y will thus have the differences of its roots k times the differences of the roots of the equation in x; and as $\frac{1}{k}$ is less than any of the latter differences, 1 must be less than any of the former differences, that is, the differences of the roots of the equation in y are all greater than unit; so that the roots will all be separated by means of the two series

$$1, \quad 2, \quad 3, \quad 4, \ \ldots . \quad kL$$

$$-1, -2, -3, -4, \ \ldots . \ -kL'$$

As the squares of the differences of all the real roots are positive, it follows that if negative roots occur in the auxiliary equation they must arise from the imaginary roots in the proposed. And moreover, if any of the roots of the auxiliary equation are zero the proposed equation must have equal roots.

(246.) It is easy to see that the foregoing method of separating the real roots, and, consequently, of discovering the number of imaginary roots in an equation is infallible; but, as we have before observed, the great length of the calculations which are necessary to the formation of the equation of the squares of the differences, when the proposed equation is above the third or fourth degree, renders the method nearly impracticable. This is now no longer a matter of regret, as the solution of the important problem of the separation of the roots has been rendered, by the researches to which the preceding chapters have been devoted, altogether independent of the equation of the squares of the differences; this latter problem, therefore, will henceforth be regarded with interest only on account of its connexion with the name of LAGRANGE, and with the history of algebraical research.

We advert to the problem here merely to explain its meaning and object to the student, and to furnish an additional example in elimination.

(247.) Let the proposed equation be

$$f(x) = 0 \quad . \ . \ . \ . \quad [1],$$

and let a be any one indifferently of its n roots, a_1, a_2, $a_3 \ldots a_n$; then, in order to obtain an equation whose roots may be the differences between those of the proposed, it will be sufficient to establish the relation

$$y = x - a, \text{ or } x = a + y;$$

which transforms the proposed into

$$f(a + y) = 0;$$

of which the development is

$$f(a) + f_1(a)\,y + f_2(a)\,\frac{y^2}{1.2} + f_3(a)\,\frac{y^3}{1.2.3} + \&c. = 0;$$

but since, by hypothesis, a is a root of the proposed equation,

$$f(a) = 0;$$

hence, suppressing this term, and dividing by y, we have

$$f_1(a) + f_2(a)\,\frac{y}{1.2} + f_3(a)\,\frac{y^2}{1.2.3} + \ldots\ldots = 0 \ldots [2]$$

the roots, or values of y, in this equation, are, by the condition above, the differences between any assumed root, a, and the $n - 1$ other roots of the proposed equation. By putting, in succession, all the values for a—that is, in fact, all the values of x, deduced from [1]—in [2], the corresponding values of y, will, together, furnish all the possible differences between the roots of [1]. In other words, all the possible differences will be obtained by substituting the values of x, deduced from the equation

$$f(x) = 0,$$

in the equation

$$f_1(x) + f_2(x)\frac{y}{1.2} + f_3(x)\frac{y^2}{1.2.3} + \ldots\ldots = 0\,;$$

and this is tantamount to saying that the differences sought arise from the solution of this system of equations.

It is easy to foresee the degree of the final equation in y arising from the elimination of x from these two equations; for, as its roots are equal to the remainders obtained, by subtracting from each of the n roots of the proposed, all the other $n - 1$ roots in succession, there are, obviously, in the whole, $n(n - 1)$ roots or remainders, hence the final equation, furnishing these roots, is of the $n(n - 1)$th degree.

(248.) We shall apply this process to the following equation of the third degree

$$x^3 + px + q = 0,$$

where

$$f(x) = x^3 + px + q, \quad f_1(x) = 3x^2 + p$$

$$f_2(x) = 6x, \quad f_3(x) = 6.$$

Hence the two equations, from which x is to be eliminated, are

$$x^3 + px + q = 0$$

$$3x^2 + 3yx + y^2 + p = 0,$$

and the operation is as follows:

$$\begin{array}{rl}
3x^2 + 3yx + y^2 + p] & 3x^3 + 3px + 3q\,[x - y \\
& 3x^3 + 3yx^2 + (y^2 + p)x \\
\hline
& \quad - 3yx^2 - (y^2 - 2p)x + 3q \\
& \quad - 3yx^2 - 3y^2x - y^3 - py \\
\hline
& \qquad 2(y^2 + p)x + y^3 + py + 3q
\end{array}$$

This remainder is now to be taken for a new divisor, and the former divisor for a dividend, and the operation continued; but as the foregoing divisor, B, does not admit of division by the remainder, R, in its present form, we multiply it by the factor $2(y^2 + p)$, in order to render the division possible, and proceed with the operation as follows:

$$\begin{array}{rl}
2(y^2+p)x+y^3+py+3q] & 6(y^2+p)x^2+6(y^2+p)yx+2(y^2+p)^2\,[3x+3(y^3+py-3q) \\
& 6(y^2+p)x^2+3(y^3+py+3q)x \\
\hline
& 3(y^3+py-3q)x+2(y^2+p)^2 \\
\text{or, mult. by } 2\,(y^2+p), & 6(y^2+p)(y^3+py-3q)x+4(y^2+p)^3 \\
& 6(y^2+p)(y^3+py-3q)x+3(y^3+py+3q)(y^3+py-3q) \\
\hline
& \qquad 4(y^2+p)^3-3(y^3+py+3q)(y^3+py-3q)
\end{array}$$

In the second division we have multiplied twice by $y^2 + p$, in order to render the division possible; but this factor introduces no extraneous value of y, for the value which reduces it to zero, being given by $y^2 + p = 0$, reduces the divisor, R, last employed, to q, and not to zero, as it ought, to be a solution. The final equation in y, involving the values sought, and those only, is, therefore,

$$y^6 + 6py^4 + 9p^2y^2 + 4p^3 + 27q^2 = 0,$$

which is *the equation of the differences;* and, by putting z for y^2, we have

$$z^3 + 6pz^2 + 9p^2z + 4p^3 + 27q^2 = 0$$

for *the equation of the squares of the differences.*

In the particular case of the equation

$$x^3 - 7x + 7 = 0 \quad . \; . \; . \; . \; [1]$$

we have

$$p = -7, \quad q = 7,$$

and therefore the equation in z is

$$z^3 - 42z^2 + 441z - 49 = 0 \quad . \; . \; . \; . \; [2]$$

In order to find an inferior limit to the positive roots of this equation, we change it into another, of which the roots are the reciprocals of those of [2], agreeably to the directions at p. 101. This reciprocal equation is

$$z^3 - 9z^2 + \frac{42}{49}z - \frac{1}{49} = 0$$

to the positive roots of which 10 is a superior limit (86): we may ascertain whether this is the closest superior limit by transforming the equation successively by 9, 8, &c. We thus find that 9 is a limit still closer. Hence, in the present case $k = 3$; and consequently $\triangle = \frac{1}{3}$. Transform now the proposed equation into another whose roots are three times as great: that is, put $\frac{y}{3}$ for x in [1], and we thus have

$$y^3 - 63y + 189 = 0 \quad . \; . \; . \; . \; [3]$$

in which we have only to substitute in succession the numbers 1, 2, 3, &c. for y, and we thus find the only intervals within which the absolute number changes sign to be [4, 5] and [5, 6]. Hence the equation [3] has two positive roots, one in each of these intervals, and consequently the proposed equation has two positive roots, one between $\frac{4}{3}$ and $\frac{5}{3}$ and the other between $\frac{5}{3}$ and $\frac{6}{3}$. The situations of these roots have been otherwise determined at pp. 198, 381.

It is obvious that every equation of the differences, as well as that just deduced, will be of an even degree, and will contain only even powers of y; because every root, as $a_1 - a_2$, is accompanied by another $a_2 - a_1$, the roots being equal to the differences $a_1 - a_2, a_1 - a_3, \ldots a_1 - a_n$; $a_2 - a_1, a_2 - a_3, \ldots a_2 - a_n$, &c. of the roots of the proposed equation; so that the polynomial in y is of the form

$$(y - \alpha)\,(y + \alpha)\,(y - \beta)\,(y + \beta)\,\ldots.,$$

or of the form

$$(y^2 - \alpha^2)\,(y^2 - \beta^2)\,\ldots.,$$

and therefore involves only even powers of y.

(249.) If in this expression we put y equal to zero, and change the sign of the result, we shall obtain an expression for the last of STURM's functions, or for that function multiplied by a positive numerical factor. This will appear evident from comparing the investigations from which the functions of STURM, and the equation of the squares of the differences, respectively result. Hence, calling the numerical factor adverted to, P, the final function of STURM will be expressed by

$$\mathrm{P}(\alpha\beta\gamma\,.\,.\,.\,.)^2$$

where α, β, γ, &c. represent the differences of the roots of the proposed equation.

If two of the roots a_1, a_2, a_3, &c. have leading figures in common, then one of the preceding differences will have so many blank places, where otherwise there would have been significant leading figures; and, consequently, the above-written square will have twice that number, or twice the number plus or minus 1, of leading places blank that would otherwise have been occupied by significant leading figures.

We may infer therefore that when two roots have leading figures in common, STURM's final remainder will, in general, be preceded by twice that number of blank places, or twice that number plus or minus 1.

If three roots have any number of leading figures in common, then three of the differences, α, β, γ, will each have that number of places vacant, which would otherwise be occupied with leading significant figures: their product will therefore have in general about three times that number of unoccupied leading places, and the square of the product may, consequently, be expected to have six times that number—at least within about three or four places.

Hence, in the case of two nearly equal roots, half the number of blank places in Sturm's final remainder—or that half plus or minus 1—will, in general, denote the number of places in which the two contiguous roots concur.

In the case of three nearly equal roots one sixth the number of blank places—or that sixth plus or minus 1—will, in general, denote the number of places in which the three contiguous roots concur. And, generally, in the case of m nearly equal roots, if the number of blank places, that would have been occupied by leading figures in Sturm's final remainder, had these roots differed in their leading figures, be divided by $m(m-1)$, the quotient—or the quotient plus or minus 1—will, in general, denote the number of places in which the m nearly equal roots concur. When m exceeds two, blank places will occur in one or more of the preceding remainders, according to the excess of m above 2 (p. 316): these blanks are, of course, to be added to those additional blanks by which the final remainder is preceded.

(250.) In each of these cases it is plain that the function of the first degree, preceding the final remainder, will be an approximate common measure of the original polynomial X and its derived function X_1; and this approximate common measure must evidently have the character of a true common measure as far as the final blank places or zeros extend, since if a true common measure could exist, this approximate one would thus far coincide with it.

In the case of two nearly equal roots, therefore, the function of the first degree, equated to zero, will give for x a value lying between the two roots of $X = 0$, and to as many places as there are blanks in the final remainder, actually coinciding with the root of $X_1 = 0$ lying in the same interval; seeing that the simple equation actually divides this latter to the extent specified.

As the function of the first degree is thus an approximate common measure of X, X_1, it is of course equally so of the subsequent functions of STURM.

This is the principle that we have applied to the analysis and solution of equations in the introductory treatise, page 224.

In the case of three nearly equal roots, the root of the function of the first degree, being still the nearest to the common measure, within the extent mentioned, will not coincide throughout that extent with a root of either of the preceding functions, but must interpose itself between each of the two contiguous roots that enter into all these, up to X_1, inclusive; just as in the case above, it interposed itself between the two contiguous roots of $X = 0$.

Similar remarks apply when the nearly equal roots exceed three in number.*

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* As observed in the introductory treatise the connexion between the blank places in STURM's final remainder, and the concurring figures in two nearly equal roots, was first noticed by Mr. RUTHERFORD. We know not whether any proof of the principle in this particular case, or any generalization of it, has been given.

CHAPTER XVI.

ON THE SYMMETRICAL FUNCTIONS OF THE ROOTS OF AN EQUATION.

(251.) A *symmetrical function* of the roots of an equation is any expression in which all the roots are similarly involved, so that any of them may be interchanged without affecting the form or composition of the function. The coefficients, for example, of every equation are each of them symmetrical functions of its roots; for it has been shown (60) that if the roots of any equation be a_1, a_2, a_3 . . . . a_n, the successive coefficients will be the following functions of them, viz.

$$-(a_1 + a_2 + a_3 + \;.\;.\;.\;.\; a_n),$$

$$a_1 a_2 + a_1 a_3 + a_2 a_3 + \;.\;.\;.\;.\; a_{n-1}\, a_n,$$

$$-(a_1 a_2 a_3 + a_1 a_2 a_4 + \;.\;.\;.\;.\; a_{n-2}\, a_{n-1}\, a_n),$$

. . . .

. . . .

$$a_1 a_2 a_3 a_4 \;.\;.\;.\;.\; a_n(-1)^n;$$

and each of these is a symmetrical function, because, however we interchange the roots, the function itself will remain unchanged.

The preceding forms are, we see, immediately given by the coefficients of the proposed equation; and it is the object of the present Chapter to show that not only these, but every other rational and symmetrical function of the roots, may always be expressed in terms of the coefficients, without the aid of the roots themselves.

Determination of the Sums of the Powers of the Roots of an Equation.

(252.) As usual, let us represent the general equation of the nth degree by

$$f(x) = x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} \dots Ax + N = 0 \dots [1],$$

and its roots by

$$a_1, \quad a_2, \quad a_3, \quad a_4, \dots \dots a_n.$$

Then $f_1(x)$, being the first derived function from the polynomial [1], we know (97) that

$$\frac{f_1(x)}{f(x)} = \frac{1}{x-a_1} + \frac{1}{x-a_2} + \frac{1}{x-a_3} + \dots \frac{1}{x-a_n};$$

and, consequently,

$$f_1(x) = \frac{f(x)}{x-a_1} + \frac{f(x)}{x-a_2} + \frac{f(x)}{x-a_3} + \dots \frac{f(x)}{x-a_n} \dots [2],$$

Performing now the actual division for any one of these fractions, as, for instance, for the fraction $\frac{f(x)}{x-a_m}$, or, which is the same thing, depressing the original equation [1], by any one of its roots, a_m, we shall get the polynomial which follows, the coefficients being formed by the rule at (51),

$$\begin{array}{r|r|r|r}
x^{n-1}+a_m & x^{n-2}+a_m{}^2 & x^{n-3}+a_m{}^3 & x^{n-4}+\ldots a_m{}^{n-1} \\
A_{n-1} & A_{n-1}a_m & A_{n-1}a_m{}^2 & A_{n-1}a_m{}^{n-2} \\
 & A_{n-2} & A_{n-2}a_m & A_{n-2}a_m{}^{n-3} \\
 & & A_{n-3} & A_{n-3}a_m{}^{n-4} \\
 & & & \vdots \\
 & & & N\,;
\end{array}$$

and this will equally represent the development of either of the other fractions, by putting the corresponding value for m. Conceiving, therefore, m to be successively 1, 2, 3, &c. to n, and, putting for abridgment,

$$\Sigma_m = a_1{}^m + a_2{}^m + a_3{}^m + \ldots . a_n{}^m,$$

we shall, obviously, have for the sum of all the developments, that is for $f_1(x)$, the polynomial

$$\begin{array}{r|r|r|r}
nx^{n-1}+\Sigma_1 & x^{n-2}+\Sigma_2 & x^{n-3}+\Sigma_3 & x^{n-4}\ldots\Sigma_{n-1} \\
nA_{n-1} & A_{n-1}\Sigma_1 & A_{n-1}\Sigma_2 & A_{n-1}\Sigma_{n-2} \\
 & nA_{n-2} & A_{n-2}\Sigma_1 & A_{n-2}\Sigma_{n-3} \\
 & & A_{n-3} & A_{n-3}\Sigma_{n-4} \\
 & & & \vdots \\
 & & & nN.
\end{array}$$

But the development of $f_1(x)$ is also, p. 110,

$$nx^{n-1}+(n-1)\,A_{n-1}\,x^{n-2}+(n-2)\,A_{n-2}\,x^{n-3}+\ldots 2A_2\,x+A\,;$$

hence, by the method of indeterminate coefficients,

$$\begin{aligned}
&\Sigma_1 + nA_{n-1} = (n-1)\,A_{n-1} \\
&\Sigma_2 + A_{n-1}\,\Sigma_1 + nA_{n-2} = (n-2)\,A_{n-2} \\
&\Sigma_3 + A_{n-1}\,\Sigma_2 + A_{n-2}\,\Sigma_1 + A_{n-3} = (n-3)\,A_{n-3} \\
&\qquad\vdots \\
&\Sigma_{n-1} + A_{n-1}\,\Sigma_{n-2} + A_{n-2}\,\Sigma_{n-3} + A_{n-3}\,\Sigma_{n-4} + \ldots . nN = A\,;
\end{aligned}$$

that is,

$$\begin{aligned}
&\Sigma_1 + A_{n-1} = 0 \\
&\Sigma_2 + A_{n-1}\Sigma_1 + 2A_{n-2} = 0 \\
&\Sigma_3 + A_{n-1}\Sigma_2 + A_{n-2}\Sigma_1 + 3A_{n-3} = 0 \\
&\qquad\qquad \vdots \\
&\Sigma_{n-1} + A_{n-1}\Sigma_{n-2} + A_{n-2}\Sigma_{n-3} + \ldots . (n-1)A = 0.
\end{aligned}$$

By means of these equations the functions Σ_1, Σ_2, Σ_3, &c. may be easily calculated in succession up to the function Σ_{n-1}.

(253.) The foregoing equations may be extended so as to include the functions Σ_n, Σ_{n+1}, Σ_{n+2}, ... Σ_{n+p}; for, from the original equation, we have

$$\begin{aligned}
&a_1^n + A_{n-1}a_1^{n-1} + A_{n-2}a_1^{n-2} + \ldots . Aa_1 + N = 0 \\
&a_2^n + A_{n-1}a_2^{n-1} + A_{n-2}a_2^{n-2} + \ldots . Aa_2 + N = 0 \\
&\qquad\qquad \vdots \\
&a_n^n + A_{n-1}a_n^{n-1} + A_{n-2}a_n^{n-2} + \ldots . Aa_n + N = 0;
\end{aligned}$$

and, by multiplying these equations respectively by a_1^p, a_2^p .... a_n^p, and adding them together, there results the equation

$$\Sigma_{n+p} + A_{n-1}\Sigma_{n+p-1} + A_{n-2}\Sigma_{n+p-2} + \ldots . A\Sigma_{p+1} + N\Sigma_p = 0,$$

which, by putting 0, 1, 2, &c. for p, furnishes the following continuation of the foregoing relations, viz.

$$\begin{aligned}
&\Sigma_n + A_{n-1}\Sigma_{n-1} + A_{n-2}\Sigma_{n-2} + \ldots . A\Sigma_1 + nN^* = 0 \\
&\Sigma_{n+1} + A_{n-1}\Sigma_n + A_{n-2}\Sigma_{n-1} + \ldots . A\Sigma_2 + N\Sigma_1 = 0 \\
&\Sigma_{n+2} + A_{n-1}\Sigma_{n+1} + A_{n-2}\Sigma_n + \ldots . A\Sigma_3 + N\Sigma_2 = 0 \\
&\qquad\qquad \vdots \\
&\Sigma_{n+p} + A_{n-1}\Sigma_{n+p-1} + A_{n-2}\Sigma_{n+p-2} + \ldots . A\Sigma_{p+1} + N\Sigma_p = 0.
\end{aligned}$$

---

* It is plain that $\Sigma_0 = a_1^0 + a_2^0 + a_3^0 + \ldots . a_n^0 = n$.

Hence, by means of the coefficients merely, we may calculate the sums of the powers of the roots of an equation, in succession, to any extent; and it is plain, from the foregoing expressions, that the several sums will all be integral functions of the coefficients.

As a particular application of the preceding general formulas, let it be required to find the sum of the sixth powers of the roots of the equation

$$x^4 + x^3 - 7x^2 - x + 6 = 0.$$

$$\Sigma_1 = -A_{n-1} = -1$$

$$\Sigma_2 = -A_{n-1}\,\Sigma_1 - 2A_{n-2} = 1 + 14 = 15$$

$$\Sigma_3 = -A_{n-1}\,\Sigma_2 - A_{n-2}\,\Sigma_1 - 3A_{n-3} = -15 - 7 + 3 = -19$$

$$\Sigma_4 = -A_{n-1}\,\Sigma_3 - A_{n-2}\,\Sigma_2 - A_{n-3}\,\Sigma_1 - 4A = 19 + 105 - 1 - 24 = 99$$

$$\Sigma_5 = -A_{n-1}\,\Sigma_4 - A_{n-2}\,\Sigma_3 - A_{n-3}\,\Sigma_2 - A\Sigma_1 = -99 - 133 + 15 + 6 = -211$$

$$\Sigma_6 = -A_{n-1}\,\Sigma_5 - A_{n-2}\,\Sigma_4 - A_{n-3}\,\Sigma_3 - A\Sigma_2 = 211 + 693 - 19 - 90 = 795.$$

If the sums of the negative powers of the roots of an equation be required, we might derive suitable formulas from the general table above, by considering p to be negative; but it will be preferable in this case to transform the equation to another in $\frac{1}{x}$ by (73), and then to employ the formulas in their present state.*

(254.) By means of the general expressions in last article, we may find the values of the coefficients A_{n-1}, A_{n-2}, &c. in terms of the sums of the powers of the roots; thus:

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* For another mode of investigating the expressions for the sums of the powers of the roots of an equation, and for the use which NEWTON and LAGRANGE made of these sums for approximating to the greatest root, see the Author's *Essay on the Computation of Logarithms*, page 97, second edition.

$$A_{n-1} = -\Sigma_1$$

$$A_{n-2} = -\frac{\Sigma_2 + A_{n-1}\Sigma_1}{2}$$

$$A_{n-3} = -\frac{\Sigma_3 + A_{n-1}\Sigma_2 + A_{n-2}\Sigma_1}{3}$$

$$A_{n-4} = -\frac{\Sigma_4 + A_{n-1}\Sigma_3 + A_{n-2}\Sigma_2 + A_{n-3}\Sigma_1}{4}.$$

&c. &c.

Determination of any Combination of the Powers of the Roots of an Equation.

(255.) By multiplying together the two expressions

$$\Sigma_m = a_1^m + a_2^m + a_3^m + a_4^m + \dots$$

$$\Sigma_p = a_1^p + a_2^p + a_3^p + a_4^p + \dots,$$

we have the two following series of partial products, viz.

$$\Sigma_m \times \Sigma_p = a_1^{m+p} + a_2^{m+p} + a_3^{m+p} + a_4^{m+p} + \dots$$
$$+ a_1^m a_2^p + a_1^m a_3^p + a_1^m a_4^p + a_2^m a_1^p + \dots$$

Each of these series is a symmetrical function of the roots; the first being the sum of their $m+p$ powers, and the second being the sum of the products of every two roots raised, the one to the power m, and the other to the power p. This latter function may be represented briefly by $S(a_1^m a_2^p)$; so that we shall have

$$\Sigma_m \times \Sigma_n = \Sigma_{m+p} + S(a_1^m a_2^p)$$

$$\therefore S(a_1^m a_2^p) = \Sigma_m \times \Sigma_p - \Sigma_{m+p}.$$

Hence the function $S(a^m a_2^p)$ is determinable in terms of the coefficients of the equation.

Again, if we multiply together the expressions

$$S(a_1^{m} a_2^{p}) = a_1^{m} a_2^{p} + a_1^{m} a_3^{p} + a_1^{m} a_4^{p} + a_2^{m} a_1^{p} + \ldots .$$

and $$\Sigma_{q} = a_1^{q} + a_2^{q} + a_3^{q} + a_4^{q} + \ldots .$$

we shall have a result consisting of three series of partial products, the terms of each distinct series involving like combinations of the roots; viz. The first series will consist of the products of every two roots raised, the one to the power $m + q$, and the other to the power p, and which series may be denoted by $S\ (a_1^{m+q} a_2^{p})$. The second series will be formed of the products of every two roots raised, the one to the power $p + q$, and the other to the power m, which series may be expressed by $S\ (a_1^{p+q} a_2^{m})$. The third series will be the products of every three roots raised, one to the power m, one to the power p, and one to the power q; and which will be represented by $S\ (a_1^{m} a_2^{p} a_3^{q})$. That is, we shall have

$$S\ (a_1^{m} a_2^{p}) \times \Sigma_{q} = S\ (a_1^{m+q} a_2^{p}) + S\ (a_1^{p+q} a_2^{m}) + S\ (a_1^{m} a_2^{p} a_3^{q}).$$

and, therefore, by transposing and replacing the functions,

$$S\ (a_1^{m} a_2^{p}), \quad S\ (a_1^{m+q} a_2^{p}), \quad S\ (a_1^{p+q} a_2^{m}),$$

by their values in last page, we have

$$S(a_1^{m} a_2^{p} a_3^{q}) = \Sigma_{m}\Sigma_{p}\Sigma_{q} - \Sigma_{m+p}\Sigma_{q} - \Sigma_{m+q}\Sigma_{p} - \Sigma_{p+q}\Sigma_{m} + 2\Sigma_{m+p+q}.$$

by which equation the triple function $S\ (a_1^{m} a_2^{p} a_3^{q})$ may be obtained in terms of the coefficients.

By continuing this process of deduction, we may obtain expressions for the succeeding combinations. The functions thus determined are called the elementary symmetrical functions, and it is from the union of these that every complex, rational and integral, symmetrical function is formed. We shall give a few examples of these combinations in the following article.

Before proceeding to these, however, it may be proper to show how the above general functions become modified, when the exponents m, p, q, &c. are not unequal.

The expression $\mathrm{S}\,(a_1{}^{\mathrm{m}}\,a_2{}^{\mathrm{p}})$ is truly the representation of

$$a_1{}^{\mathrm{m}}a_2{}^{\mathrm{p}} + a_1{}^{\mathrm{m}}\,a_3{}^{\mathrm{p}} + a_1{}^{\mathrm{m}}a_4{}^{\mathrm{p}} + \ldots\ldots + a_2{}^{\mathrm{m}}a_1{}^{\mathrm{p}} + a_3{}^{\mathrm{m}}a_1{}^{\mathrm{p}} + \ldots\ldots$$

only when m and p are unequal; for, when $m = p$, this series consists of terms which are equal two and two; so that, in that case, only half the entire sum will be expressed by $\mathrm{S}\,(a_1{}^{\mathrm{m}}a_2{}^{\mathrm{m}})$. Hence

$$\mathrm{S}\,(a_1{}^{\mathrm{m}}\,a_2{}^{\mathrm{m}}) = \frac{(\Sigma_{\mathrm{m}})^2 - \Sigma_{2\mathrm{m}}}{2}.$$

For similar reasons,

$$\mathrm{S}\,(a_1{}^{\mathrm{m}}a_2{}^{\mathrm{p}}a_3{}^{\mathrm{p}}) = \frac{\Sigma_{\mathrm{m}}(\Sigma_{\mathrm{p}})^2 - 2\Sigma_{\mathrm{m+p}}\,\Sigma_{\mathrm{p}} - \Sigma_{\mathrm{m}}\,\Sigma_{2\mathrm{p}} + 2\Sigma_{\mathrm{m+2p}}}{2}.$$

Lastly, when the exponents in this latter function are all three equal, the terms represented will be equal six and six; so that

$$\mathrm{S}\,(a_1{}^{\mathrm{m}}a_2{}^{\mathrm{m}}a_3{}^{\mathrm{m}}) = \frac{(\Sigma_{\mathrm{m}})^3 - 3\Sigma_{2\mathrm{m}}\Sigma_{\mathrm{m}} + 2\Sigma_{3\mathrm{m}}}{6}.$$

Transformation of an Equation into another whose Roots shall be given Functions of those of the original Equation.

(256.) Let it be required to form the equation whose roots are the sums of the roots of the equation $f(x) = 0$, taken two and two.

If, as usual, we represent the roots of the proposed equation by

$$a_1, \quad a_2, \quad a_3, \quad a_4 \;\ldots\ldots\; a_n,$$

those of the transformed equation, $\mathrm{F}(y) = 0$, will be

$$a_1 + a_2, \quad a_1 + a_3, \quad a_1 + a_4, \quad a_2 + a_3, \quad \&c.,$$

and will amount in number to the number of different combinations which can be formed with the roots of the proposed, taken two and two. If each were to be combined with every one of the other roots, the whole number of combinations would obviously be $n\,(n-1)$; but it is plain that every combination would then occur twice; so that the correct number of combinations must be $\frac{n(n-1)}{2}$.* Hence the number $\frac{n(n-1)}{2}$ denotes the degree of the transformed equation. Let us proceed to the composition of its coefficients.

The sums of the powers of the roots of the transformed equation will be expressed by the formulas

$$\Sigma'_1=(a_1+a_2)+(a_1+a_3)+(a_1+a_4)+(a_2+a_3)+\&\text{c.}=$$

$$2(a_1+a_2+a_3+a_4+\&\text{c.})=2\text{S}(a_1)$$

$$\Sigma'_2=(a_1+a_2)^2+(a_1+a_3)^2+(a_1+a_4)^2+(a_2+a_3)^2+\&\text{c.}=$$

$$2(a_1^2+a_2^2+a_3^2+a_4^2+\&\text{c.})+2(a_1a_2+a_1a_3+a_1a_4+a_2a_3+\&\text{c.})$$

$$=2\text{S}(a_1^2)+2\text{S}(a_1a_2)$$

$$\Sigma'_3=(a_1+a_2)^3+(a_1+a_3)^3+(a_1+a_4)^3+(a_2+a_3)^3+\&\text{c.}=$$

$$2(a_1^3+a_2^3+a_3^3+a_4^3)+3(a_1^2a_2+a_1^2a_3+a_1^2a_4+a_2^2a_3)+\&\text{c.}$$

$$=2\text{S}(a_1^3)+3\text{S}(a_1^2a_2)$$

&c. &c.

Hence the sums of the powers of the roots of the transformed equation may be obtained in terms of the sums of the powers of the original roots and their elementary combinations; the sums of the powers being thus known, for the transformed equations, the coefficients of this equation are found by the formulas, at (254), as in the following example:

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* The doctrine of combinations and permutations is given in almost every English treatise on Arithmetic.

Let it be required to transform the equation,

$$x^3 + A_2x^2 + Ax + N = 0,$$

into another,

$$y^3 + A'_2y^2 + A'y + N' = 0,$$

whose roots shall be the sums of the roots of the former equation, taken two and two.

Let us first calculate the values of Σ_1, Σ_2, Σ_3, as at page 418 ;

$$\Sigma_1 = - A_2$$

$$\Sigma_2 = - A_2\Sigma_1 - 2A = A_2^2 - 2A$$

$$\Sigma_3 = - A_2\Sigma_2 - A\Sigma_1 - 3N = - A_2^3 + 3AA_2 - 3N.$$

The value of $S(a_1a_2)$ is, by p. 421, A ; and for $S(a_1^2a_2)$, we have (255)

$$S(a_1^2a_2) = \Sigma_2 \times \Sigma_1 - \Sigma_3 = - AA_2 + 3N.$$

Consequently, for the expressions on last page we have

$$\Sigma'_1 = 2\Sigma_1 = - 2A_2$$

$$\Sigma'_2 = 2\Sigma_2 + 2S\,(a_1a_2) = 2A_2^2 - 2A$$

$$\Sigma'_3 = - 2\Sigma_3 + 3S\,(a_1^2\,a_2) = - 2A_2^3 + 3AA_2 + 3N.$$

Finally, by the formulas (254),

$$A'_2 = - \Sigma'_1 = 2A_2$$

$$A' = - \frac{\Sigma'_2 + A'_2\,\Sigma'_1}{2} = A^2_2 + A$$

$$N' = - \frac{\Sigma'_3 + A'_2\,\Sigma'_2 + A'\,\Sigma'_1}{3} = A_2\,A - N.$$

Hence the transformed equation is

$$y^3 + 2A_2 y^2 + (A_2^2 + A) y + A_2 A - N = 0.$$

(257.) By proceeding in a similar manner, we may form an equation of which the roots are combinations of those of the original equation, of the form $a_1 + a_2 + ka_1a_2$, $a_1 + a_3 + ka_1a_3$, &c. k being any given number. The degree of this equation will, of course, be the same as the degree of that formed from the sums of the roots; it will, therefore, be denoted by the number $\frac{n(n-1)}{2}$. The expressions for the sums of the powers of the roots of the transformed equation are

$$\Sigma'_1 = (a_1 + a_2 + ka_1a_2) + (a_1 + a_3 + ka_1a_3) + \&c.$$

$$= (n-1)\,S\,(a_1) + kS\,(a_1a_2)$$

$$\Sigma'_2 = (a_1 + a_2 + ka_1a_2)^2 + (a_1 + a_3 + ka_1a_3)^2 + \&c.$$

$$= (n-1)\,S\,(a_1^2) + 2S\,(a_1a_2) + 2kS\,(a_1^2a_2) + k^2S\,(a_1^2a_2^2)$$

$$\Sigma'_3 = (a_1 + a_2 + ka_1a_2)^3 + (a_1 + a_3 + ka_1a_3)^3 + \&c.$$

$$= (n-1)\,S\,(a_1^3) + 3S\,(a_1^2a_2) + 3kS\,(a_1^3a_2) + 6kS\,(a_1^2a_2^2) +$$

$$3k^2\,S\,(a_1^3a_2^2) + k^3\,S\,(a_1^3a_2^3)$$

&c. &c.

from which it is evident that the coefficients of the transformed equation may be expressed in functions of the coefficients of the given equation.

(258.) As a second application, let it be required to form the equation of the squares of the differences of the roots of a given equation.

The proposed equation being $f(x) = 0$, and its n roots as before, the roots of the transformed equation will be

$$(a_1 - a_2)^2,\ (a_1 - a_3)^2,\ (a_1 - a_4)^2 \ \ldots\ (a_2 - a_3)^2,\ (a_2 - a_4)^2 \ \ldots$$

the number of which, obviously, amounts to the number of combinations, two and two, that can be formed with the n quantities, a_1, a_2, a_3, &c. Hence the degree of the required equation is $\frac{n(n-1)}{2}$; and, to find its coefficients, we must, as before, first determine the values of Σ'_1, Σ'_2, Σ'_3, &c. by the following formulas:

$$\Sigma'_1 = (a_1 - a_2)^2 + (a_1 - a_3)^2 + (a_1 - a_4)^2 + \&c.$$

$$= (n-1)\,S\,(a_1^2) - 2S\,(a_1 a_2)$$

$$\Sigma'_2 = (a_1 - a_2)^4 + (a_1 - a_3)^4 + (a_1 - a_4)^4 + \&c.$$

$$= (n-1)\,S\,(a_1^4) - 4S\,(a_1^3 a_2) + 6S\,(a_1^2 a_2^2)$$

$$\Sigma'_3 = (a_1 - a_2)^6 + (a_1 - a_3)^6 + (a_1 - a_4)^6 + \&c.$$

$$= (n-1)\,S\,(a_1^6) - 6S\,(a_1^5 a_2) + 15S\,(a_1^4 a_2^2) - 20S\,(a_1^3 a_2^3)$$

$$\&c. \qquad\qquad \&c.$$

As a particular example, let the equation, already considered at page 410, viz.

$$x^3 - 7x + 7 = 0,$$

be proposed, in which

$$A_2 = 0, \ \ A = -7, \ \ N = 7.$$

By the formulas at p. 418 we have

$$\Sigma_1 = 0, \ \Sigma_2 = 14, \ \Sigma_3 = -21, \ \Sigma_4 = 98, \ \Sigma_5 = -245, \ \Sigma_6 = 833.$$

Consequently,

$$S\,(a_1^2) = \Sigma_2 = 14$$

$$S\,(a_1^4) = \Sigma_4 = 98$$

$$S\,(a_1^6) = \Sigma_6 = 833.$$

Also, from (255),

$$S(a_1a_2) \;= A = -7$$

$$S(a_1^3a_2) \;= -\Sigma_4 = -98$$

$$S(a_1^5a_2) \;= \Sigma_5\,\Sigma_1 - \Sigma_6 = -833$$

$$S(a_1^2a_2^2) = \frac{(\Sigma_2)^2 - \Sigma_4}{2} = \frac{196 - 98}{2} = 49$$

$$S(a_1^4a_2^2) = \Sigma_4\,\Sigma_2 - \Sigma_6 = 539$$

$$S(a_1^3a_2^3) = \frac{(\Sigma_3)^2 - \Sigma_6}{2} = -196.$$

Therefore, making these substitutions in the formulas on last page,

$$\Sigma'_1 = 2S(a_1^2) - 2S(a_1a_2) = 42$$

$$\Sigma'_2 = 2S(a_1^4) - 4S(a_1^3a_2) + 6S(a_1^2a_2^2) = 882$$

$$\Sigma'_3 = 2S(a_1^6) - 6S(a_1^5a_2) + 15S(a_1a_2^2) - 20S(a_1^3a_2^3) = 18669;$$

and hence, finally,

$$A'_2 = -\Sigma'_1 = -42$$

$$A' \;= -\frac{\Sigma'_2 + A'_2\,\Sigma'_1}{2} = 441$$

$$N' \;= -\frac{\Sigma'_3 + A'_2\,\Sigma'_2 + A'\Sigma'_1}{3} = -49,$$

so that the transformed equation is

$$y^3 - 42y^2 + 441y - 49 = 0.$$

On the Degree of the Final Equation, resulting from the Elimination of one of the Unknown Quantities from two Equations, containing two Unknowns.

(259.) Let the two equations be

$$f(x)=x^n+A_{n-1}x^{n-1}+A_{n-2}x^{n-2}+\ldots.A_2x^2+Ax+N=0\ldots.[1]$$

$$F(x)=x^m+B_{m-1}x^{m-1}+B_{m-2}x^{m-2}+\ldots.B_2x^2+Bx+M=0\ldots.[2];$$

in which the coefficients, $A, A_2, A_3, \ldots$; $B, B_2, B_3, \ldots$ are functions of y.

If we could resolve the first of these equations, we should obtain for x, n values, a, b, c, &c. which would be functions of y, and which, when substituted in the second, would furnish the n equations,

$$F(a)=a^m+B_{m-1}a^{m-1}+B_{m-2}a^{m-2}+\ \ldots\ .\ B_2a^2+Ba+M=0$$

$$F(b)=b^m+B_{m-1}b^{m-1}+B_{m-2}b^{m-2}+\ \ldots\ .\ B_2b^2+Bb+M=0$$

$$F(c)=c^m+B_{m-1}c^{m-1}+B_{m-2}c^{m-2}+\ \ldots\ .\ B_2c^2+Bc+M=0$$

. . . .

. . . .

and these equations being solved for y, would make known the corresponding values of this quantity.

It is, however, in but few cases that we can actually solve the equation [1] for x; if we could, the determination of the corresponding values of x would not require the solution of the n separate equations, just obtained, because they may all be combined in a single equation, viz. the equation

$$F(a)\,F(b)\,F(c)\ \ldots\ .=0\ \ldots\ .\ [3];$$

and it is plain that the product, which forms its first member, undergoes no alteration, however we interchange a, b, c, &c. in the factors; that is, this product will contain none but rational and symmetrical functions of the roots of the equation [1]. Hence, the first member of the equation [3] may be determined, by means of the coefficients of the equation [1], and the final equation in y, thus obtained.

As an example, let us take the equations

$$(y - 2)\, x^2 - 2x + 5y - 2 = 0$$

$$yx^2 - 5x + 4y = 0.$$

Let us represent the values of x in terms of y, which satisfy the second equation by a and b. These, substituted in the first, furnish the two equations,

$$(y - 2)\, a^2 - 2a + 5y - 2 = 0$$

$$(y - 2)\, b^2 - 2b + 5y - 2 = 0\,;$$

of which the product is

$$(y - 2)^2\text{S}(a^2b^2) - 2\,(y - 2)\text{S}(a^2b) +$$

$$(y - 2)(5y - 2)\Sigma_2 - 2(5y - 2)\Sigma_1 +$$

$$4\text{S}(ab) + (5y - 2)^2 = 0.$$

The coefficients A_2, A_1, N of the terms in the second equation, are

$$\text{A}_2 = -\frac{5}{y}, \quad \text{A} = 4, \quad \text{N} = 0\,;$$

consequently, we have (253, 255)

$$\Sigma_1 = \frac{5}{y},\ \Sigma_2 = \frac{25 - 8y^2}{y^2},\ \Sigma_3 = \frac{125 - 6y^2}{y^3}$$

$$\text{S}(ab) = 4, \quad \text{S}(a^2b) = \frac{20}{y}, \quad \text{S}(a^2b^2) = 16\,;$$

and, substituting these values in the preceding equation, there results

$$y^5 + 12y^3 + 87y^2 - 200y + 100 = 0,$$

which is the final equation sought.

It must be confessed, however, that the foregoing method of deducing the final equation is usually very tedious; yet it has the advantage of presenting that equation unencumbered with extraneous roots. But the principal value of the foregoing investigation consists in its readily leading to the establishment of this theorem, first demonstrated by BEZOUT, namely:

The degree of the final equation, which results from the elimination of one of the unknowns, from two equations, of any degree whatever, involving two unknown quantities, can never surpass the product of the degrees of the two equations; and it is exactly equal to that product when the proposed equations are in their most general form.

In order to determine the degree in y, of the equation [3], we must consider that each term of the product [3], is formed by the multiplication of one term of the first factor, one of the second, one of the third, &c. Let, then, Ka^{h}, $K'b^{h'}$, $K''c^{h''}$ . . . be terms, taken at random, in each of the n factors [3]; the corresponding term of the product, will be

$$KK'K'' \ldots \times a^{h}b^{h'}c^{h''} \ldots;$$

moreover, the entire product is symmetrical in a, b, c, &c. so that this term forms part of one of the symmetrical functions which enter into the composition of [3], which partial function may be represented by

$$KK'K'' \ldots \times S(a^{h}b^{h'}c^{h''} \ldots) \ldots [4].$$

It will, therefore, be sufficient to determine the highest degree in y of this function.

Now, as by supposition, Ka^{h} is one of the terms in the polynomial $F(a)$ of the mth degree, it follows that the degree of y, in K, cannot exceed the $m - h$ degree. In like manner, the degree of y in K' cannot exceed $m - h'$; the degree of y in K'' cannot exceed $m - h''$, &c. Consequently, the product of the n polynomials $KK'K'' \ldots$ cannot exceed the degree $mn - h - h' - h'' \ldots$

Let us now ascertain the degree which the polynomial $S(a^h b^{h'} c^{h''} \ldots)$ cannot surpass.

Referring to the general expressions involving Σ_1, Σ_2, Σ_3, &c. at page 418, and recollecting that in our equation [1], page 427, the coefficient, A_{n-1}, cannot exceed the first degree in y, the coefficient, A_{n-2}, cannot exceed the second degree, and so on, we shall immediately see that the expressions for Σ_1, Σ_2, Σ_3, &c., deduced from our equation [1], cannot exceed the degree in y, denoted by the index suffixed to the symbol Σ. Referring, in like manner, to the general expressions in (255), which exhibit the double, triple, &c. functions, we there also recognise that, in $S(a^h b^{h'} c^{h''} \ldots)$, the degree in y cannot exceed $h + h' + h'' \ldots$ Hence, in the expression [4], the degree in y cannot exceed mn.

If the coefficients, A_{n-1}, A_{n-2}, A_{n-3}, &c. in the equation [1], and those in the equation [2], are in their most general form, that is, if they exhibit a series of functions of y, regularly ascending, in degree, the expressions Σ_1, Σ_2, Σ_3, &c. will have the degree denoted by their suffixed indices, and hence the degree of $S(a^h b^{h'} c^{h''} \ldots)$, will be $h + h' + h'' \ldots$ It is plain, too, that in this case, K, K', K'', &c. being, in their most general form, their degrees in y will be exactly $m - h$, $m - h'$, $m - h''$, &c. Consequently, the degree of the final equation in y, will, under these circumstances, be exactly mn.

CHAPTER XVII.

THE DETERMINATION OF THE IMAGINARY ROOTS OF EQUATIONS.

(260.) It has already been proved that imaginary roots always enter into equations in conjugate pairs of the form $\alpha \pm \beta \sqrt{-1}$. And this previous knowledge of the form which every imaginary root must take, suggests a method for the actual determination of the proper numerical values for α and β in any proposed case. The method is as follows:

Let

$$x^n + A_{n-1} x^{n-1} + \ldots . Ax + N = 0$$

be an equation containing imaginary roots; then, by substituting $\alpha + \beta \sqrt{-1}$ for x, we have

$$(\alpha+\beta\sqrt{-1})^n + A_{n-1}(\alpha+\beta\sqrt{-1})^{n-1} + ..A(\alpha+\beta\sqrt{-1}) + N = 0;$$

or, by developing the terms by the binomial theorem, and collecting the real and imaginary quantities separately, we have the form

$$M + N\sqrt{-1} = 0,$$

an equation which cannot exist except under the conditions

$$M = 0, \quad N = 0 \ldots . [1].$$

From these two equations, therefore, in which M, N contain only the quantities α, β, combined with the given coefficients, all the systems of values of α and β may be determined; and these, substituted in the expression $\alpha + \beta \sqrt{-1}$, will make known all the imaginary roots of the proposed equation; those that are real corresponding to $\beta = 0$.

It is obvious from the theory of elimination as developed in Chapter XV, and from the method of numerical solution explained in Chapter XII, that the labour of deducing from this pair of equations the final equation involving only one of the unknowns α, β, and of afterwards solving the equation for that unknown, will in general be impracticable for equations above the third degree. LAGRANGE, by combining with the principle of this solution the method of the squares of the differences explained at (247), avoids both the elimination and subsequent solution here spoken of. It is easy to see how this may be brought about if we have any independent means of determining one of the unknowns β: for the adoption of these means would enable us to dispense with the elimination; and as the substitution of the value of β in both of the equations [1] would convert those equations into two simultaneous equations involving but one unknown quantity, their first members would necessarily have a common factor of the first degree in α, which, equated to zero, would furnish for α the proper value to accompany β; and thus, instead of solving the final equation referred to, we should only have to find the common measure between the two polynomials M, N containing the unknown quantity α.

Now corresponding to every pair of imaginary roots $\alpha + \beta\sqrt{-1}$, $\alpha - \beta \sqrt{-1}$, there necessarily exists, in the equation of the squares of the differences, a real negative root, $-4\beta^2$; so that if all the negative roots of the latter equation be found, the quantity $-4\beta^2$ must appear among them; from which the value of β would be immediately obtained, and thence, by aid of the common measure as just explained, the corresponding value of α.

But the equation of the squares of the differences may have a greater number of negative roots than there are pairs of imaginary roots in the proposed; which however cannot happen except two non-conjugate imaginary roots have equal real parts, or except a real root be equal to the real part of an imaginary root. LAGRANGE discusses these peculiarities, and establishes the exactness and generality of the principle in question, as follows:

When the real parts α, γ, &c. of the imaginaries

$$\alpha + \beta\sqrt{-1},\quad \alpha - \beta\sqrt{-1}$$
$$\gamma + \delta\sqrt{-1},\quad \gamma - \delta\sqrt{-1}$$
$$\text{\&c.}\qquad\qquad \text{\&c.}$$

are unequal, as well when compared with one another as when compared with the real roots a, b, c, &c. it is evident that the equation of the squares of the differences cannot have any other negative roots than those furnished by the several pairs of conjugate imaginary roots, and which are

$$-4\beta^2,\quad -4\delta^2,\quad \text{\&c.}$$

All the other roots, not arising from the differences furnished by the real roots a, b, c, &c. will evidently be imaginary; those between the real and imaginary roots supplying the forms

$$(\alpha - a + \beta\sqrt{-1})^2,\quad (\alpha - a - \beta\sqrt{-1})^2$$
$$(\alpha - b + \beta\sqrt{-1})^2,\quad (\alpha - b - \beta\sqrt{-1})^2$$
$$\text{\&c.}\qquad\qquad \text{\&c.}$$

and those between the non-conjugate roots the forms

$$\{(\alpha - \gamma) + (\beta - \delta)\sqrt{-1}\}^2,\quad \{(\alpha - \gamma) - (\beta - \delta)\sqrt{-1}\}^2$$
$$\{(\alpha - \gamma) + (\beta + \delta)\sqrt{-1}\}^2,\quad \{(\alpha - \gamma) - (\beta + \delta)\sqrt{-1}\}^2$$

so that in this case every negative root in the auxiliary equation will indicate a pair of imaginary roots in the proposed, and will moreover supply the value of the imaginary part. But if it happen that among the quantities α, γ, &c. there be found any equal among themselves or equal to any of the quantities a, b, c, &c. then the auxiliary equation will necessarily have negative roots, corresponding to which there can be no imaginary pair in the proposed equation.

For let $a=\alpha$; then the two imaginary roots $(\alpha-a+\beta\sqrt{-1})^2$, $(\alpha-a-\beta\sqrt{-1})^2$ will become $-\beta^2$ and $-\beta^2$, and consequently real and negative; so that if the proposed equation contain only two imaginary roots, $\alpha+\beta\sqrt{-1}$ and $\alpha-\beta\sqrt{-1}$, then, in the case of $a=\alpha$, the equation of the squares of the differences will contain, besides the real negative root $-4\beta^2$, the two $-\beta^2$, $-\beta^2$, both negative and equal.

We thus see that when the equation of the squares of the differences has three negative roots, of which two are equal to one another, the proposed may have either three pairs of imaginary roots, or but a single pair.

If the proposed contains four imaginary roots, $\alpha+\beta\sqrt{-1}$, $\alpha-\beta\sqrt{-1}$, $\gamma+\delta\sqrt{-1}$, $\gamma-\delta\sqrt{-1}$, then the equation of the squares of the differences must contain the two negative roots $-4\beta^2$, and $-4\delta^2$; if $\alpha=a$, it must also contain the two equal negative roots $-\beta^2$, $-\beta^2$; and if moreover $\gamma=b$ it must contain, in addition to these, the negative pair $-\delta^2$, $-\delta^2$: and lastly, if $\alpha=\gamma$ the four imaginary roots

$$\{(\alpha-\gamma)+(\beta-\delta)\sqrt{-1}\}^2,\ \{(\alpha-\gamma)-(\beta-\delta)\sqrt{-1}\}^2$$

$$\{(\alpha-\gamma)+(\beta+\delta)\sqrt{-1}\}^2,\ \{(\alpha-\gamma)-(\beta+\delta)\sqrt{-1}\}^2$$

will be converted into the two negative pairs

$$-(\beta-\delta)^2,\ -(\beta-\delta)^2;\ -(\beta+\delta)^2,\ -(\beta+\delta)^2$$

Hence we may deduce the following conclusions, viz.

1. When all the real negative roots of the equation of the squares of the differences are unequal, then the proposed will necessarily have so many pairs of imaginary roots.

If in this case we call any one of these negative roots $-w$, we shall have $\beta=\frac{\sqrt{w}}{2}$; and if this value be substituted for β in the two equations [1], and the operation for the common measure of their first members be carried on till we arrive at a remainder of the first degree in α, the proper value of α will be obtained by equating this remainder to zero. Thus each negative root $-w$ will furnish two conjugate imaginary roots $\alpha+\beta\sqrt{-1}$, and $\alpha-\beta\sqrt{-1}$.

2. If among the negative roots of the equation of the squares of the differences equal roots are found, then each unequal root, if any such occur, will, as in the preceding case, always furnish a pair of imaginary roots. Each pair of equal roots may, however, give either two pair of imaginary roots or no imaginary roots, so that two equal roots will give either four imaginary roots or none; three equal roots will give either six imaginary roots or two; four equal roots will give either eight imaginary roots, or four, or none: and so on.

Suppose two of the negative roots $-w$, $-w$, are equal: then putting, as above, $\beta=\frac{\sqrt{w}}{2}$, we shall substitute this value of β in the two polynomials [1], and shall carry on the process for the common measure between these polynomials till we arrive at a remainder of the second degree in α; since the polynomials must have a common divisor of the second degree in α, seeing that the equations [1] must have two roots in common, on account of the double value of β.

Equating then this quadratic remainder to zero, we shall be furnished with two values for α: these may be either both real or both imaginary. In the former case call the two values α' and α''; we shall then have the four imaginary roots

$$\alpha'+\beta\sqrt{-1},\ \alpha'-\beta\sqrt{-1},\ \alpha''+\beta\sqrt{-1},\ \alpha''-\beta\sqrt{-1}$$

In the second case, the values of α being imaginary—contrary to the conditions by which the fundamental equations [1] are governed— we infer, that to the equal negative roots $-w$, $-w$, there cannot correspond any imaginary roots in the proposed equation.

If the equation of the squares of the differences have three equal negative roots, $-w$, $-w$, $-w$, then, putting as before $\beta=\frac{\sqrt{w}}{2}$, we should operate on the polynomials [1], for the common measure, till we reach a remainder of the third degree in α : this remainder equated to zero will furnish three values of α, which will either be all real, or one real and two imaginary. In the first case six imaginary roots will be implied: in the second only two; the imaginary values of α being always rejected, as not coming within the conditions implied in [1].

The foregoing principles are theoretically correct: but the practical application of them, beyond equations of the third and fourth degrees, is too laborious for them to become available in actual computation. We give the following illustration of them from LAGRANGE.

To determine the imaginary roots of the equation

$$x^3 - 2x - 5 = 0.$$

Computing the equation of the squares of the differences from the general formula for the third degree at (248), viz.

$$z^3 + 6pz^2 + 9p^2z + 4p^3 + 27q^2 = 0$$

in which $p = -2$ and $q = -5$, we have

$$z^3 - 12z^2 + 36z + 643 = 0$$

In order to determine the negative roots of this equation, change the alternate signs, or put $z = -w$, and then change all the signs, converting the equation into

$$w^3 + 12w^2 + 36w - 643 = 0$$

and seek the positive root, which is found by trial to lie between 5 and 6. Adopting LAGRANGE's development, p. 379, this root proves to be

$$w = 5 + \cfrac{1}{6 + \cfrac{1}{5 + \cfrac{1}{6 + \&c.}}}$$

from which we get the converging fractions

$$5, \quad \frac{31}{6}, \quad \frac{160}{31}, \quad \frac{991}{192}, \ \&c.$$

Knowing thus an approximate value of w, we know $\beta = \frac{\sqrt{w}}{2}$.

In order now to get the equations [1], page 231, substitute $\alpha + \beta\sqrt{-1}$ for x in the proposed equation; and form two equations, one with the real terms of the result, the other with the imaginary terms: we shall thus have the equations [1] referred to, viz.

$$\alpha^3 - (3\beta^2 + 2)\alpha - 5 = 0$$

$$3\alpha^2 - \beta^2 - 2 = 0$$

in which β is known.

Seeking now the greatest common measure of the first members of these equations, stopping the operation at the remainder of the first degree in α, and equating that remainder to zero, we have

$$\alpha = -\frac{15}{8\beta^2 + 4}$$

and thus both α and β are determined in approximate numbers.

(261.) There is another method of proceeding for the determination of imaginary roots, somewhat different from the preceding, being independent of the equation of the squares of the differences. It is suggested from the following considerations:

Since the quadratic involving a pair of imaginary conjugate roots is always of the form

$$x^2 - 2\alpha x + \alpha^2 + \beta^2 = 0,$$

every equation into which such roots enter must always be accurately divisible by a quadratic divisor of this form: that is, the proper values of α and β are such that the remainder of the first degree in x, resulting from the division must be zero. This furnishes a condition from which those proper values of α and β may be determined; the condition, namely, that the remainder spoken of, $Ax - B$, must be equal to zero, independent of particular values of x; and this implies the twofold condition

$$A = 0, \quad B = 0,$$

from which α and β, of which A and B are functions, may be determined.

As an example let the equation proposed be

$$x^4 + 4x^3 + 6x^2 + 4x + 5 = 0.$$

Dividing the first member by

$$x^2 - 2\alpha x + \alpha^2 + \beta^2$$

we have for quotient

$$x^2 + (4 + 2\alpha)\, x + 6 + 8\alpha + 3\alpha^2 - \beta^2$$

and for the remainder of the first degree in x

$$(4 + 12\alpha + 12\alpha^2 + 4\alpha^3 - 4\alpha\beta^2 - 4\beta^2)x -$$

$$(\alpha^2 + \beta^2)\,(6 + 8\alpha + 3\alpha^2 - \beta^2) + 5$$

which, being equal to zero whatever be the value of x, furnishes the two equations

$$4 + 12\alpha + 12\alpha^2 + 4\alpha^3 - 4\alpha\beta^2 - 4\beta^2 = 0$$

$$(\alpha^2 + \beta^2)\,(6 + 8\alpha + 3\alpha^2 - \beta^2) + 5 = 0$$

From the first of these we get

$$\beta^2 = (1 + \alpha)^2$$

and this, substituted in the second, gives

$$4\alpha^4 + 16\alpha^3 + 24\alpha^2 + 16\alpha = 0$$

two roots of which are 0 and -2 : the other two are imaginary, and must consequently be rejected as contrary to the hypothesis as to the form of the indeterminate quadratic divisor.

The two real values of α, substituted in the expression above for β^2, give

$$\text{for } \alpha = \quad 0,\ \beta^2 = 1^2 \qquad \therefore \beta = +1$$

$$\alpha = -2,\ \beta^2 = (-1)^2 \therefore \beta = -1$$

and consequently the component factors of the original quadratic divisor, viz. the factors

$$x - \alpha - \beta\sqrt{-1}, \quad x - \alpha + \beta\sqrt{-1},$$

furnish these two pairs of imaginary roots, viz.

$$x = \sqrt{-1}, \quad x = -\sqrt{-1}$$

$$\text{and } x = -2 - \sqrt{-1}, \quad x = -2 + \sqrt{-1}$$

This method, like that before given, is impracticable beyond very narrow limits, because of the high degree to which the final equation in α usually rises. And it is further to be observed of both, and indeed of all methods for determining imaginary roots by aid of the real roots of certain numerical equations, that whenever, as is usual, these real roots are obtained only approximately, our results may, under peculiar circumstances, be erroneous. For instance, in the two methods just explained we have two equations $f(\alpha) = 0$, $\mathrm{F}(\beta) = 0$, where the coefficients of α in the first are functions of β, and the coefficients of β in the second functions of α; hence, whichever of these symbols be computed approximately, in order to furnish determinate values for the coefficients of the other, these coefficients must vary slightly from the true coefficients; and, consequently, under this slight variation of the coefficients, real roots may become converted into imaginary and imaginary into real,—see page 315.

CHAPTER XVIII.

ON THE SOLUTION OF CUBIC AND BIQUADRATIC EQUATIONS BY GENERAL FORMULÆ.

(262.) In the former chapters of the present treatise ample instructions have been given for the complete solution of every algebraical equation whose coefficients are expressed in known numbers.

It still remains for us to give a concise account of the labours of mathematicians, as far as they have been successful, in the solution of equations with literal coefficients. The problem we now propose to consider is therefore this, viz. to determine finite expressions for the roots of an equation in functions of the coefficients; a problem long regarded as the most important in Algebra, because of its involving the complete solution of numerical equations. But the recent discoveries, unfolded in the former part of the present work, have reduced this celebrated problem to one of comparative insignificance; and have removed that regret, which was so long and so universally felt, on account of the failure of every attempt to extend the solution of literal equations beyond the first four degrees. We shall, therefore, content ourselves with briefly explaining the principal formulas which have been proposed for the solution of cubic and biquadratic equations.

Solution of a Cubic Equation by the Method of CARDAN.

(263.) Let the proposed equation be first deprived of its second term by the rule at (79), it will then have the form

$$x^3 + px + q = 0 \quad . \quad . \quad . \quad . \quad [1]$$

Assume x equal to the sum of two other unknown quantities; that is, put

$$x = y + z,$$

we shall then have

$$x^3 = y^3 + z^3 + 3yz\,(y + z)\,;$$

that is, replacing $y + z$ by x, and transposing,

$$x^3 - 3yzx - y^3 - z^3 = 0,$$

and, in order that this may be identical with the proposed equation, we must determine y and z so as to satisfy these conditions, viz.

$$yz = -\frac{p}{3}, \quad y^3 + z^3 = -q.$$

The problem is therefore reduced to the determination of y and z from these two equations.

From the first we have

$$y^3z^3 = -\frac{p^3}{27};$$

hence, combining this with the second, we have the sum of two quantities, $y^3 + z^3$, and their product, y^3z^3, given to determine the quantities: a problem which we know may be solved by help of a quadratic equation (*Algebra*, p. 156), viz. the equation

$$v^2 + qv - \frac{p^3}{27} = 0 \quad . \quad . \quad . \quad . \quad [2]$$

of which the two roots, or values of v, will be the expressions for y^3 and z^3 sought. Hence, solving the equation, and separating the two roots, we have

$$y^3 = -\frac{q}{2} + \sqrt{\left\{\frac{q^2}{4} + \frac{p^3}{27}\right\}} \; . \; . \; . \; . \; [3]$$

$$z^3 = -\frac{q}{2} - \sqrt{\left\{\frac{q^2}{4} + \frac{p^3}{27}\right\}} \; . \; . \; . \; . \; [4]$$

and consequently, since $x = y + z$, there results the following general expression for the roots of the proposed equation, viz.

$$x = \left\{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right\}^{\frac{1}{3}} + \left\{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right\}^{\frac{1}{3}},$$

which is the formula of CARDAN.

Since the cube root of y^3 may be represented indifferently by either of the three expressions, page 352,

$$y, \quad \frac{-1+\sqrt{-3}}{2}y, \quad \frac{-1-\sqrt{-3}}{2}y,$$

and the cube root of z^3 by either of the expressions

$$z, \quad \frac{-1+\sqrt{-3}}{2}z, \quad \frac{-1-\sqrt{-3}}{2}z,$$

it would seem that $x = y + z$ admits of nine values, or that the proposed equation has nine roots. It must be remembered, however, that in all cases when we assign the root of any expression, a tacit reference is made to the generation of the proposed power, the root being in fact assumed to be the expression from which the given power has been actually produced. When we speak of any proposed power having a multiplicity of roots, we merely refer to the various expressions from either of which that power *might* be generated; and as many of these as prove inconsistent with the conditions involved in the production of the power, are of course to be rejected. Now one of the conditions in virtue of which y^3 and z^3 have been produced, is

$$yz = -\frac{p}{3};$$

that is to say, the products of the roots y, z, must be possible; but of the nine products which the preceding expressions for y and z are competent to furnish, six will be found to be imaginary: such a combination of values must therefore be rejected, as inconsistent with the conditions to be fulfilled; the other three products are possible.

Hence the only admissible solutions are the three following, where the product of y and z fulfils the condition above

$$y + z,$$

$$\frac{-1+\sqrt{-3}}{2}y + \frac{-1-\sqrt{-3}}{2}z,$$

$$\frac{-1-\sqrt{-3}}{2}y + \frac{-1+\sqrt{-3}}{2}z.$$

But a form will be given to CARDAN'S formula hereafter, that shall be unencumbered with superfluous values.

The equation [2], upon the solution of which that of the proposed cubic is made to depend, being of the next inferior degree, is called the *reducing equation*. The roots of this reducing equation are exhibited in [3] and [4]. If these are real, the formula for x will consist of the cube roots of two real quantities: but if the roots of the reducing equation are imaginary, then the expression for x will be the cube roots of two imaginary quantities; and, consequently, each root must itself be imaginary: that is to say, if the relation between p and q be such that

$$\frac{q}{4} + \frac{p^3}{27} < 0,$$

the expression for x will consist of two parts, each of which is imaginary.

But the sum of these parts, that is, the complete expression for x, must of necessity be real; because the preceding relation between p and q is that which necessitates the reality of all the roots of the equation. (See *Introductory Treatise*, page 106.) Still, with the exception of a few particular cases, it is impossible to deliver the compound expression for the roots from its imaginary symbols, without developing each part separately, and then by incorporating the two, representing the aggregate by an infinite series, from which the imaginary symbol shall have disappeared.

The cases of exception referred to are those in which [3] and [4] are each complete cubes. For then the roots involved in the expression for x will be expressible in finite terms: and as they must be of the forms, page 54,

$$P + Q\sqrt{-1} \text{ and } P - Q\sqrt{-1} \quad \text{--- - - [5]}$$

it follows that x will be expressed, without series, by the finite quantity 2P.* Moreover, in what may be regarded as the extreme case of the above condition respecting p and q, viz. the case in which

$$\frac{q^2}{4} + \frac{p^3}{27} = 0$$

which implies two equal roots in the reducing equation, and also two equal roots in the cubic, the formula for x becomes real and finite, because the quantity under the quadratic radical vanishes.

With these exceptions we may lay it down as a general rule, that CARDAN'S formula is incompetent to furnish either of the roots of a cubic equation, in a finite form, when all three are real. The formula can present a real root in a finite form only when the remaining two roots are imaginary, or the expression under the quadratic radical *positive*.

The case in which the three roots are real has hence been

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* In the above observations we refer merely to the fact that, when the expressions in question are complete cubes, x must admit of an expression in finite terms: whether the roots of these complete cubes can be actually determined in all cases, and thence the finite value of x deduced, is a distinct inquiry. It is considered in the foot-note at the bottom of next page.

called the *irreducible case* of cubic equations, and many attempts have been made to convert the irreducible expressions in this case to a finite real form.

It is obvious that these attempts can succeed only to the extent mentioned in the instances of exception adverted to above. The two parts of the formula for x can never be replaced by finite expressions such as [5], unless [3] and [4] be perfect cubes. It has been regarded as anomalous and paradoxical that the value of x should not appear under a real finite form whenever the roots represented by x are known to be real; and yet to expect such to be the case is to expect that every expression of the form $a \pm \sqrt{-b}$ must be a perfect cube.*

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* In order to deduce a general expression for the cube root of $a + \sqrt{b}$, whether b be positive or negative, put

$$\sqrt[3]{(a + \sqrt{b})} = A + \sqrt{B} \ldots . [a]$$

then, cubing each member, we have

$$a + \sqrt{b} = A^3 + 3A^2 \sqrt{B} + 3\,AB + B \sqrt{B};$$

and equating the rational terms with the rational, and the irrational with the irrational, we have the following equations of condition for determining A and B, viz.

$$a = A^3 + 3\,AB$$

$$\sqrt{b} = (3A^2 + B) \sqrt{B}, \text{ or } b = (3\,A^2 + B)^2\,B.$$

If from the square of the first of these we subtract the second, we shall have

$$a^2 - b = (A^2 - B)^3.$$

It appears therefore that, in order that in any case the proposed extraction may be possible, in finite terms, $a^2 - b$ must be a perfect cube. This condition, it will be observed, is fulfilled by each of the expressions [3], [4], in the text; otherwise the cube roots of these could never be obtained in finite terms.

Putting for brevity $a^2 - b = c^3$, we have

$$A^2 - B = c \therefore B = A^2 - c \ldots . [b];$$

substituting this in the first of the above equations of condition, we have the final equation

$$4\,A^3 - 3c\,A - a = 0;$$

or multiplying by 2, and putting x for 2 A,

$$x^3 - 3\,cx - 2\,a = 0 \ldots . [c].$$

Thus the proposed cube root necessarily involves in its expression the roots of a

Each root of such a cube would take the form $P \pm Q\sqrt{-1}$; and by multiplying any one of them by the respective cube roots of unity, all the values implied in the original cubic expression would be obtained. Now P must be a *rational* function of a and b: for if radicals entered P, the assumed form would involve more values than the original. Hence, when a and b are commensurable numbers, as they must be when the coefficients of the equation are, P, and consequently 2 P, one of the roots, must be commensurable. But such is not necessarily the case; the roots being, in general, incommensurable. Consequently, the supposed finite form for the cube root is impossible, except in particular cases.

(264.) When one root x, of a cubic equation, is determined, we have seen (198) that the formula for the other two roots is

$$-\frac{x_1}{2} \pm \sqrt{\left\{-3\left(\frac{x_1}{2}\right)^2 - p\right\}}$$

Consequently putting, agreeably to the foregoing hypothesis, $y + z$ for x_1, and $3yz$ for $-p$, we have for the other two roots

$$-\frac{y+z}{2} \pm \sqrt{\left\{-\frac{3(y+z)^2}{4} + 3yz\right\}}$$

or

$$-\tfrac{1}{2}\left\{(y+z) \mp (y-z)\sqrt{-3}\right\}$$

As remarked above, the expression $y + z$, when each term of it is developed in a series, by the binomial theorem, becomes in the irreducible case $(P + Q\sqrt{-1}) + (P - Q\sqrt{-1})$, P and Q re-

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cubic equation; and when this cube root is that of either of the expressions [3], [4], these being in the irreducible case—the cubic just deduced will be also in the irreducible case: for b being then negative, c^3 is positive, and greater than a^2; hence, in the final cubic above, the coefficient of x is negative, and the twenty-seventh part of the cube of it greater than the square of half the absolute number. Thus in attempting to convert CARDAN's irreducible forms into finite expressions, we are invariably conducted back to the same forms; so that actually to exhibit the first member of [a] in the form given to the second member, whether b be positive or negative, we must determine x or 2A from [c], either by trial or by numerical solution, and thence deduce B from [b.] It is plain that in general both A and B will be incommensurable numbers.

presenting real series. Hence, p. 54, $y - z$ will be expressed by $(P + Q\sqrt{-1}) - (P - Q\sqrt{-1}.)$ Consequently, the three real values are

$$2\,P, \text{ and } - P \pm Q\sqrt{3}.$$

It is not worth while actually to exhibit here the general series for P and Q, as they are of little practical importance in the computation of the roots, on account of the trouble of calculating the successive terms, and their general slow convergency.

(265.) The irreducible case may be otherwise solved by help of a table of sines and cosines.

Thus, by Trigonometry, (*Trig.* p. 59,)

$$\cos 2\,\theta = 2\cos^2\theta - 1$$

$$\cos 3\,\theta = 2\cos 2\,\theta\cos\theta - \cos\theta$$

Substituting the first expression in the second,

$$\cos 3\,\theta = 4\cos^3\theta - 3\cos\theta$$

whence

$$\cos^3\theta - \tfrac{3}{4}\cos\theta - \tfrac{1}{4}\cos 3\theta = 0\;.\;.\;.\;.\;[1].$$

In the proposed cubic equation

$$x^3 - px^2 + q = 0\;.\;.\;.\;.\;[2],$$

put the unknown $r\cos\theta$ for x; or, which is the same thing, put $\frac{x}{r}$ for $\cos\theta$; and [1] becomes

$$x^3 - \tfrac{3}{4}r^2x - \tfrac{1}{4}r^3\cos 3\,\theta = 0.$$

Comparing this with [2] we have

$$\tfrac{1}{4}r^3\cos 3\,\theta = -q,$$

and

$$\tfrac{3}{4}r^2 = p \;\therefore\; r = 2\sqrt{\frac{p}{3}}$$

$$\therefore \cos 3\,\theta = -\frac{3q}{pr} = -\frac{q}{2}\sqrt{\frac{27}{p^3}}$$

Consequently, the trigonometrical solution of the proposed cubic, that is, the determination of θ, and thence of $r\cos\theta$, depends upon the *trisection of an arc*, or the determination of $\cos\theta$ from $\cos 3\theta$, which latter cosine is, we see, given in terms of the known coefficients. The mode of proceeding by aid of trigonometrical tables is obvious: we are to seek in the table of cosines for the angle whose cosine is $-\frac{q}{2}\sqrt{\frac{27}{p^3}}$, this will be the angle 3θ, and consequently one third of it will be θ; and the cosine of this multiplied by r, or $2\sqrt{\frac{p}{3}}$, will give $r\cos\theta = x$, for one of the real roots of the equation [2]. As the given cosine, $-\frac{q}{2}\sqrt{\frac{27}{p^3}}$, belongs equally to *three* arcs, viz. 3θ, $2\pi + 3\theta$, and $2\pi - 3\theta$; by taking the cosine of one third of each of the latter two, we shall have the values of the remaining roots. Thus all the three roots will be expressed as follows, viz.

$$2\sqrt{\frac{p}{3}}\,.\cos\theta,\ 2\sqrt{\frac{p}{3}}\,.\cos\tfrac{1}{3}(2\pi + 3\theta),\ 2\sqrt{\frac{p}{3}}\,.\cos\tfrac{1}{3}(2\pi - 3\theta)$$

or using the supplements of the two latter arcs instead of the arcs themselves, and remembering that the cosine of an arc is equal to minus the cosine of its supplement, we have, somewhat more simply, the three values of x in the following form:

$$2\sqrt{\frac{p}{3}}\,.\cos\theta,\ -2\sqrt{\frac{p}{3}}\,.\cos(60^\circ - \theta),\ -2\sqrt{\frac{p}{3}}\,.\cos(60^\circ + \theta).$$

It is worthy of notice that this trigonometrical method of solving a cubic equation applies, with a single exception, exclusively to the irreducible case: for as the trigonometrical cosine of an arc is always less than unit, except when that arc is a multiple of 180°, we must have

$$\frac{q}{2}\sqrt{\frac{27}{p^3}} < 1 \therefore \frac{q^2}{4} < \frac{p^3}{27}$$

$$\text{or } \frac{q^2}{4} - \frac{p^3}{27} < 0$$

When 3θ is a multiple of 180°, two roots must be equal, p. 444.

EULER'S *Method of Solving a Biquadratic Equation.*

(266.) Let the proposed biquadratic, when deprived of its second term, be

$$x^4 + px^2 + qx + r = 0.$$

Assume x equal to the sum of three other unknown quantities; that is, put

$$x = u + v + w \dots . [1],$$

then

$$x^2 = u^2 + v^2 + w^2 + 2\,(uv + uw + vw).$$

Put P for $u^2 + v^2 + w^2$, and we shall have

$$\begin{aligned}(x^2 - \mathrm{P})^2 &= 4\,(uv + uw + vw)^2 \\ &= 4\,(u^2 v^2 + u^2 w^2 + v^2 w^2) + \\ &\quad 8\,uvw\,(u + v + w)\,;\end{aligned}$$

that is, putting Q for $u^2 v^2 + u^2 w^2 + v^2 w^2$, and replacing $u + v + w$ by x,

$$x^4 - 2\mathrm{P}x^2 + \mathrm{P}^2 = 4\mathrm{Q} + 8uvwx$$

$$\therefore\ x^4 - 2\mathrm{P}x^2 - 8uvwx + \mathrm{P}^2 - 4\mathrm{Q} = 0,$$

and, in order that this may be identical with the proposed equation, we must have these conditions, viz.

$$\mathrm{P} = u^2 + v^2 + w^2 = -\frac{p}{2}$$

$$\mathrm{Q} = u^2v^2 + u^2w^2 + v^2w^2 = \frac{\mathrm{P}^2 - r}{4} = \frac{p^2 - 4r}{16}$$

$$uvw = -\frac{q}{8}, \text{ or } u^2v^2w^2 = \frac{q^2}{64}.$$

The conditions show that the quantities u^2, v^2, w^2, must be the roots of the cubic equation

$$y^3 + \frac{p}{2} y^2 + \frac{p^2 - 4r}{16} y - \frac{q^2}{64} = 0 \; . \; . \; . \; . \; [2];$$

or, putting

$$y = \frac{z}{4},$$

the roots of the equation

$$z^3 + 2pz^2 + (p^2 - 4r) z - q^2 = 0 \; . \; . \; . \; . \; [3].$$

Call these roots z', z'', z''', then the roots of [2] will be

$$\frac{z'}{4}, \; \frac{z''}{4}, \; \frac{z'''}{4},$$

and hence the expression [1] for x takes the following forms, viz.

$$\frac{\sqrt{z'}}{2} + \frac{\sqrt{z''}}{2} + \frac{\sqrt{z'''}}{2}, \quad -\frac{\sqrt{z'}}{2} - \frac{\sqrt{z''}}{2} - \frac{\sqrt{z'''}}{2},$$

$$\frac{\sqrt{z'}}{2} + \frac{\sqrt{z''}}{2} - \frac{\sqrt{z'''}}{2}, \quad \frac{\sqrt{z'}}{2} - \frac{\sqrt{z''}}{2} + \frac{\sqrt{z'''}}{2},$$

$$-\frac{\sqrt{z'}}{2} + \frac{\sqrt{z''}}{2} + \frac{\sqrt{z'''}}{2}, \quad -\frac{\sqrt{z'}}{2} - \frac{\sqrt{z''}}{2} + \frac{\sqrt{z'''}}{2},$$

$$-\frac{\sqrt{z'}}{2} + \frac{\sqrt{z''}}{2} - \frac{\sqrt{z'''}}{2}, \quad \frac{\sqrt{z'}}{2} - \frac{\sqrt{z''}}{2} - \frac{\sqrt{z'''}}{2}.$$

But some of these values are inadmissible, since a necessary condition is, that $uvw = -\frac{q}{8}$; hence, we must preserve only those of the foregoing trinomial expressions of which the product of the radicals gives always a sign contrary to that of q, and these are, when q is positive,

$$x = \frac{-\sqrt{z'} - \sqrt{z''} - \sqrt{z'''}}{2}$$

$$x = \frac{-\sqrt{z'} + \sqrt{z''} + \sqrt{z'''}}{2}$$

$$x = \frac{\sqrt{z'} + \sqrt{z''} - \sqrt{z'''}}{2}$$

$$x = \frac{\sqrt{z'} - \sqrt{z''} + \sqrt{z'''}}{2};$$

and when q is negative

$$x = \frac{\sqrt{z'} + \sqrt{z''} + \sqrt{z'''}}{2}$$

$$x = \frac{-\sqrt{z'} + \sqrt{z''} - \sqrt{z'''}}{2}$$

$$x = \frac{-\sqrt{z'} - \sqrt{z''} + \sqrt{z'''}}{2}$$

$$x = \frac{\sqrt{z'} - \sqrt{z''} - \sqrt{z'''}}{2}.$$

And these formulas exhibit the four roots of the proposed equation: they will be given in a better form hereafter.

Solution of an Equation of the Fourth Degree, by the Method of LOUIS FERRARI.

(267.) Taking the same general form as before, viz.

$$x^4 + px^2 + qx + r = 0;$$

we have, by transposition,

$$x^4 = -px^2 - qx - r.$$

Add the quantity

$$2kx^2 + k^2,$$

to both sides, and we shall then have

$$(x^2 + k)^2 = (2k - p)\,x^2 - qx + (k^2 - r);$$

and it remains to determine k, so that the second member of this equation may be a complete square. In order to this, k must fulfil the condition

$$(2k - p)\,(k^2 - r) = \frac{q^2}{4},$$

since, in every perfect square, four times the product of the extreme terms is equal to the square of the middle one.

Actually multiplying the two factors, and dividing by the coefficient, 2, of k^3, we have, finally, the cubic equation,

$$k^3 - \frac{p}{2}\,k^2 - rk + \frac{pr}{2} - \frac{q^2}{8};$$

and a root of this being determined by the rule of CARDAN, or otherwise, the solution of the proposed biquadratic is reduced to that of the two quadratics following, viz.

$$x^2 + k = \sqrt{2k - p}\,.\,x - \sqrt{k^2 - r}$$

$$x^2 + k = -\sqrt{2k - p}\,.\,x + \sqrt{k^2 - r}$$

For a better form, see the Scholium at the end.

Solution of the Biquadratic by the Method of DESCARTES.

(268.) Taking the same general form as before, viz.

$$x^4 + px^2 + qx + r = 0 \quad . \; . \; . \; . \; [1]$$

let the first member be decomposed into two quadratic factors, by assuming it equal to the product

$$(x^2 + kx + f)\,(x^2 - kx + g)$$

and then determining the unknown coefficients k, f, g, so as to bring about the desired identity between the two expressions. For this purpose we have by actually executing the multiplication

$$x^4 + (f + g - k^2)\,x^2 + (gk - fk)\,x + fg \quad . \; . \; . \; . \; [2]$$

and equating the coefficients of the like terms in [1] and [2]

$$f + g - k^2 = p, \ gk - fk = q, \ fg = r \ . \ . \ . \ . \ [3]$$

Multiplying the first of these by k, and then adding and subtracting the second, we have

$$2gk = k^3 + pk + q$$

$$2fk = k^3 + pk - q$$

Lastly, multiplying these together and introducing the third condition, viz. $4fg = 4r$, we get finally

$$k^6 + 2pk^4 + (p^2 - 4r)\,k^2 - q^2 = 0$$

or, putting z for k^2, the reducing cubic

$$z^3 + 2pz^2 + (p^2 - 4r)\,z - q^2 = 0$$

which is the same as that in Euler's solution.

When z is determined from this equation, k becomes known, being the square root of z: and from the first two of the equations of condition [3] we have

$$f + g = k^2 + p, \ g - f = \frac{q}{k}$$

$$\therefore \ g = \tfrac{1}{2}(k^2 + p + \frac{q}{k})$$

$$f = \tfrac{1}{2}(k^2 + p - \frac{q}{k})$$

and, consequently, the two quadratics for determining the four values of x are

$$x^2 + kx + \tfrac{1}{2}(k^2 + p - \frac{q}{k}) = 0$$

$$x^2 - kx + \tfrac{1}{2}(k^2 + p + \frac{q}{k}) = 0$$

All the preceding methods of investigation involve particular analytical artifices. That which follows, and which we believe to be new, has the peculiarity of being altogether independent of these artifices. It furnishes at the same time the results both of

Descartes and Euler, which are shown to be immediate inferences from the known composition of the coefficients of the biquadratic, without the aid of any additional principle.

New Method for Equations of the Fourth Degree.

(269.) As in the preceding cases let the proposed equation be

$$x^4 + px^2 + qx + r = 0$$

and let its four roots be represented by x_1, x_2, x_3, x_4: then, since the sum of any two of these is equal to the sum of the remaining two, the quadratic equation involving the roots x_3, x_4, will be

$$x^2 + (x_1 + x_2)\,x + x_3x_4 = 0$$

and, consequently, the roots themselves will be expressed by

$$-\frac{x_1 + x_2}{2} \pm \sqrt{\left\{\left(\frac{x_1 + x_2}{2}\right)^2 - x_3\,x_4\right\}}$$

This expression, as remarked at (198) may be converted into a variety of different forms from attending to the known composition of the coefficients p and q. Two of these forms are exhibited at (198); a third is obtained as follows:

Since

$$p = (x_1 + x_2)(x_3 + x_4) + x_1x_2 + x_3x_4$$
$$= -(x_1 + x_2)^2 + x_1x_2 + x_3x_4$$

it follows that the expression under the radical is

$$\left(\frac{x_1+x_2}{2}\right)^2 - x_3x_4 = -\frac{p}{2} - \left(\frac{x_1+x_2}{2}\right)^2 + \frac{x_1x_2 + x_3x_4}{2} - x_3x_4$$

But
$$q = -(x_1 + x_2)\,x_3x_4 - (x_3 + x_4)\,x_1x_2$$
$$= (x_1 + x_2)\,(x_1x_2 - x_3x_4)$$
$$\therefore \frac{q}{2\,(x_1 + x_2)} = \frac{x_1x_2 - x_3x_4}{2} = \frac{x_1x_2 + x_3x_4}{2} - x_3x_4,$$

so that, substituting this in the expression referred to, we have

$$-\frac{x_1+x_2}{2} \pm \sqrt{\left\{-\frac{p}{2} - \left(\frac{x_1+x_2}{2}\right)^2 + \frac{q}{2(x_1+x_2)}\right\}}$$

If the other two roots, x_3, x_4 be substituted for these, then since $x_3 + x_4 = -(x_1 + x_2)$ we shall have for all four of the roots the formula

$$\mp\frac{x_1+x_2}{2} \pm \sqrt{\left\{-\frac{p}{2} - \left(\frac{x_1+x_2}{2}\right)^2 \pm \frac{q}{2(x_1+x_2)}\right\}}$$

or putting m for $x_1 + x_2$,

$$\mp\frac{m}{2} \pm \frac{1}{2}\sqrt{\left\{-2p - m^2 \pm 2\frac{q}{m}\right\}} \quad . \ . \ . \ . \ [1]$$

where, it is to be observed, that the double sign of the radical applies whether the upper or the lower sign of m be taken. Now if we multiply all these four values together the product must evidently be r. This multiplication is very easily performed in consequence of the double sign adverted to furnishing for each sign of m a pair of expressions, forming the one the sum and the other the difference of two quantities, so that, taking the upper sign of m, we have for the product of the corresponding pair of values

$$\frac{m^2}{4} - \frac{1}{4}\left\{-2p - m^2 + 2\frac{q}{m}\right\} = \frac{m^2}{2} + \frac{p}{2} - \frac{q}{2m}$$

In like manner, for the product of the other pair of values, we have

$$\frac{m^2}{4} - \frac{1}{4}\left\{-2p - m^2 - 2\frac{q}{m}\right\} = \frac{m^2}{2} + \frac{p}{2} + \frac{q}{2m}$$

These two products form also the sum and difference of two quantities: hence the final product is

$$\left(\frac{m^2}{2} + \frac{p}{2}\right)^2 - \left(\frac{q}{2m}\right)^2 = r$$

that is

$$\frac{m^4}{4} + \frac{pm^2}{2} + \frac{p^2}{4} - \frac{q^2}{4m^2} = r$$

or finally

$$m^6 + 2pm^4 + (p^2 - 4r)\,m^2 - q^2 = 0$$

And this, when z is put for m^2, becomes

$$z^3 + 2pz^2 + (p^2 - 4r)\,z - q^2 = 0 \;.\;.\;.\;.\; [2]$$

the reducing cubic of **Euler** and **Descartes**. If a root of this cubic be found, and the square root of it be put for m in the general formula [1], the four roots of the biquadratic will be exhibited.

(270.) These four roots are evidently those given by the pair of quadratic equations in the solution of **Descartes**.

If all three of the roots of the reducing cubic be found, then representing them by z', z'', z'''; and observing, from the coefficient of the second term of that cubic, that

$$-\,2p = z' + z'' + z'''$$

and from the final term that $q^2 = z'z''z'''$, the general expression [1] for the four roots of the biquadratic may be written thus :

$$\mp\frac{\sqrt{z'}}{2} \pm \tfrac{1}{2}\sqrt{\{\,z' + z'' + z''' - z' \pm 2\sqrt{z''z'''}\,\}}$$

$$= \mp\frac{\sqrt{z'}}{2} \pm \tfrac{1}{2}\sqrt{\{\,z'' \pm 2\sqrt{z''z'''} + z'''\,\}}$$

$$= \mp\frac{\sqrt{z'}}{2} \pm \tfrac{1}{2}\,(\sqrt{z''} \pm \sqrt{z'''})$$

which expression resolves itself into the four expressions of **Euler** for the roots of the biquadratic.

It should be remarked that, as in [1], the second of the ambiguous signs in this result always preserves its double signification, giving a pair of values corresponding to each single value of the preceding term. But the third ambiguous sign, that of $\sqrt{z'''}$, is not equally arbitrary. It must always be fixed by the condition that the product of the three radicals in each of the four results may furnish a sign opposite to that of q, as in **Euler**'s investigation.

Thus, taking any three of the combinations which those radicals represent: as, for instance, the three

$$(x_1 + x_2),\ (x_1 + x_3),\ (x_1 + x_4)$$

we have for their product

$$(x_1 + x_2)\left\{ x_1^2 + x_1(x_3 + x_4) + x_3x_4 \right\}$$
$$= (x_1 + x_2)\left\{ x_1^2 + x_1(x_1 + x_2) + x_3x_4 \right\}$$
$$= (x_1 + x_2)(x_3x_4 - x_1x_2) = -q \qquad \text{(page 454.)}$$

But for the purposes of actual computation, the formula [1] at first deduced, where any one of the three roots of [2] may be put for m^2, is the more convenient. In the Scholium at the end of next Chapter, we shall offer some remarks upon certain imperfections with which the formulas of CARDAN, EULER, and FERRARI are chargeable.

The methods just exhibited for solving by general formulas equations of the third and fourth degrees are isolated, having no observable bond of connexion with one another. Other investigations have been prosecuted with more general views, and by more uniform and connected processes, in the hope of extending the powers of the analysis beyond equations of the fourth degree. But the great difficulties with which the inquiry is beset, when carried on beyond equations of the fourth degree, and the fruitlessness of the attempts hitherto made to vanquish them, have at length led to the entire abandonment of the research. We shall however give, in conclusion, a short chapter exhibiting the general character of the investigation by which LAGRANGE sought to extend the formulas for the solution of equations beyond the fourth degree; referring the student, for another mode of considering the same comprehensive problem, to a paper, by Sir JOHN W. LUBBOCK, in the *Philosophical Magazine,* 1839: as also to the NOTES appended to the *Traité de la Résolution,* &c. of LAGRANGE.

CHAPTER XIX.

SOLUTION OF EQUATIONS OF THE THIRD AND FOURTH DEGREES, BY MEANS OF SYMMETRICAL FUNCTIONS.

(271.) We shall now explain the methods which Lagrange has employed for the general solution of equations of the third and fourth degrees by means of equations of inferior degrees. These methods, which are founded upon the theory of symmetrical functions, were first developed by Lagrange, in the *Berlin Memoirs* for 1770 and 1771, and are also given with some modifications in the *Traité de la Résolution des Equations Numériques*, Note XIII.

Equation of the Third Degree.

Let the proposed equation be

$$x^3 + px + q = 0,$$

in which the second term is absent. Call the roots a, a_2, a_3, then we immediately have the relation

$$a + a_2 + a_3 = 0;$$

and, if we could discover two other equations of the first degree in a, a_2, a_3, the values of these quantities might be easily determined by elimination.

Let us assume the relation

$$la + ma_2 + na_3 = z;$$

then, as there is nothing to distinguish one root from either of the others, the relation which we have just assumed may be indifferently one or the other of the six following, viz.

$$la + ma_2 + na_3 = z$$
$$la + ma_3 + na_2 = z$$
$$la_2 + ma + na_3 = z$$
$$la_2 + ma_3 + na = z$$
$$la_3 + ma + na_2 = z$$
$$la_3 + ma_2 + na = z;$$

and these could all be given by the solution of an equation of the sixth degree, in z. But, in order that such an equation might be solved as a quadratic, it must be of the form

$$z^6 + Az^3 + B = 0 \quad . \; . \; . \; . \; [1];$$

which, if we put u for z^3, becomes

$$u^2 + Au + B = 0$$

$$\therefore u = -\frac{A}{2} \pm \sqrt{\frac{A^2}{4} - B};$$

hence, putting

$$z'^3 = -\frac{A}{2} + \sqrt{\frac{A^2}{4} - B}$$

$$z''^3 = -\frac{A}{2} - \sqrt{\frac{A^2}{4} - B};$$

and recollecting that the three cube roots of unity are 1, α, α^2, we have, for the six values of z, the following expressions, viz.

$$z', \quad \alpha z', \quad \alpha^2 z', \quad z'', \quad \alpha z'', \quad \alpha^2 z'';$$

taking, then, any two of the six expressions above, for z' and z'', as, for instance,

$$la + ma_2 + na_3 = z', \quad la + ma_3 + na_2 = z'';$$

the four others must fulfil the following conditions, viz.

$$\begin{aligned} la_2 + ma + na_3 &= \alpha\,(la + ma_3 + na_2) \\ la_2 + ma_3 + na &= \alpha^2(la + ma_2 + na_3) \\ la_3 + ma + na_2 &= \alpha\,(la + ma_2 + na_3) \\ la_3 + ma_2 + na &= \alpha^2(la + ma_3 + na_2) \end{aligned}$$

which must be formed so that the coefficients of a, a_2, a_3, in one member of each, shall be different from those in other members, in order to avoid contradictory conditions.

The four equations, just deduced, are transformable into the following, viz.

$$\begin{aligned} (l - \alpha n)a_2 + (m - \alpha l)a + (n - \alpha m)a_3 &= 0 \\ (l - \alpha^2 m)a_2 + (m - \alpha^2 n)a_3 + (n - \alpha^2 l)a &= 0 \\ (l - \alpha n)a_3 + (m - \alpha l)a + (n - \alpha m)a_2 &= 0 \\ (l - \alpha^2 m)a_3 + (m - \alpha^2 n)a_2 + (n - \alpha^2 l)a &= 0; \end{aligned}$$

which will evidently be satisfied if we can fulfil the conditions

$$l = \alpha n, \quad m = \alpha l, \quad n = \alpha m$$
$$l = \alpha^2 m, \quad m = \alpha^2 n, \quad n = \alpha^2 l,$$

which are reducible to the two following, viz.

$$m = \alpha l, \quad \text{and} \quad n = \alpha^2 l;$$

for, from $\alpha^3 = 1$, we have $\alpha = \frac{1}{\alpha^2}$, and $\alpha^2 = \frac{1}{\alpha}$; so that $m = \alpha l$ is the same as $m = \frac{1}{\alpha^2} l$, whence $l = \alpha^2 m$. In like manner,

$n = \alpha^2 l$ is the same as $n = \frac{1}{\alpha} l$; whence $l = \alpha n$. Lastly, the relations, $m = \alpha l$, $n = \alpha^2 l$, divided the one by the other, give $\frac{m}{n} = \frac{1}{\alpha}$ $\therefore m = \frac{1}{\alpha} n = \alpha^2 n$, and $n = \alpha m$; hence it will be sufficient to consider the two relations,

$$m = \alpha l, \quad n = \alpha^2 l;$$

from which, as we have just seen, all the others are deducible. We thus have m and n expressed in terms of l, which, being arbitrary, put it for simplicity equal to unity; then we shall have

$$m = \alpha, \quad n = \alpha^2,$$

and thus the three values, l, m, n, are no other than the three cube roots of unity.

Substituting these values in the expressions

$$la + ma_2 + na_3 = z', \quad la + ma_3 + na_2 = z'';$$

they become

$$a + \alpha a_2 + \alpha^2 a_3 = z', \quad a + \alpha a_3 + \alpha^2 a_2 = z''.$$

We may, in like manner, substitute the same values in the four remaining equations, and afterwards form, by multiplication, the equation in z; as, however, we know that this equation is to be of the form [1], its six roots must be comprised in the two equations,

$$z^3 = z'^3 = (a + \alpha a_2 + \alpha^2 a_3)^3$$

$$z^3 = z''^3 = (a + \alpha a_3 + \alpha^2 a_2)^3$$

or, which is the same thing, in the single equation,

$$\{ z^3 - (a + \alpha a_2 + \alpha^2 a_3)^3 \} \{ z^3 - (a + \alpha a_3 + \alpha^2 a_2)^3 \} = 0.$$

By actually performing the multiplication here indicated, and comparing the coefficients of the resulting terms with those of the corresponding terms in [1], we have these conditions, viz.

$$\mathrm{A} = - \{ (a + \alpha a_2 + \alpha^2 a_3)^3 + (a + \alpha a_3 + \alpha^2 a_2)^3 \}$$

$$\mathrm{B} = \quad (a + \alpha a_2 + \alpha^2 a_3)^3 . (a + \alpha a_3 + \alpha^2 a_2)^3.$$

Hence the coefficients of the equation,

$$z^6 + Az^3 + B = 0,$$

are symmetrical functions of the roots proposed.

If we develop these values of A and B, and keep in mind that

$$\alpha^3 = 1, \quad \alpha^4 = \alpha, \quad \alpha^5 = \alpha^2, \quad \alpha^6 = 1, \text{ \&c.}$$

and, because the coefficient of the second term in $y^3 - 1 = 0$ is zero, that

$$1 + \alpha + \alpha^2 = 0, \text{ and, therefore, } \alpha + \alpha^2 = -1,$$

we shall have these values, viz. (see page 422),

$$\begin{aligned} A &= -\{ 2S(a^3) + 3(\alpha + \alpha^2) S(a^2a_2) + 12aa_2a_3 \\ &= -\{ 2S(a^3) - 3S(a^2a_2) + 12aa_2a_3 \\ B &= \{ S(a^2) + (\alpha + \alpha^2) S(aa_2) \}^3 \\ &= \{ S(a^2) - S(aa_2) \}^3. \end{aligned}$$

Now, from the original equation,

$$x^3 + px + q = 0,$$

we obtain, in terms of the coefficients, p, q, the following values for these symmetrical functions (255), viz.

$$S(a^2) = \Sigma_2 = -2p, \quad S(a^3) = \Sigma_3 = -3q, \quad S(aa_2) = p,$$

$$S(a^2a_2) = \Sigma_2 \Sigma_1 - \Sigma_3 = -\Sigma_3 = 3q.$$

Moreover,

$$aa_2a_3 = -q;$$

hence, substituting these values in the preceding expressions for A and B, we have

$$\begin{aligned} A &= -\{ -6q - 9q - 12q \} = 27q \\ B &= \quad \{ -2p - p \}^3 = (-3p)^3 = -27p^3; \end{aligned}$$

consequently, the equation in z is now fully determined; it is

$$z^6 + 27qz^3 - 27p^3 = 0;$$

from which we get

$$z^3 = -\frac{27q}{2} \pm 27 \sqrt{\frac{q^2}{4} + \frac{p^3}{27}};$$

and thence

$$z' = 3\left\{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right\}^{\frac{1}{3}}$$

$$z'' = 3\left\{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right\}^{\frac{1}{3}}.$$

These values being now known, we have, for the determination of the roots, a, a_2, a_3, the three simple equations,

$$a + a_2 + a_3 = 0$$

$$a + \alpha a_2 + \alpha_2 a_3 = z'$$

$$a + \alpha^2 a_2 + \alpha\, a_3 = z''.$$

By adding these equations together, taking account of the property,

$$1 + \alpha + \alpha^2 = 0,$$

we have

$$3a = z' + z'';$$

which gives, for the root a, the value

$$a = \frac{z' + z''}{3};$$

that is, substituting for z', z'', their values in terms of p and q, as expressed above,

$$a = \left\{-\frac{q}{2} + \sqrt{\frac{q}{4} + \frac{p^3}{27}}\right\}^{\frac{1}{3}} + \left\{-\frac{q}{2} - \sqrt{\frac{q}{4} + \frac{p^3}{27}}\right\}^{\frac{1}{3}};$$

which agrees with the formula, before found, by the method of CARDAN, (p. 442).

To obtain the other two roots, multiply the second of the three equations above by α, the third by α^2, and add the results to the first, we shall thus have

$$3a_3 = \alpha z' + \alpha^2 z'' \therefore a_3 = \frac{\alpha z' + \alpha^2 z''}{3}.$$

Lastly, multiply the second by α^2, the third by α, add as before, and we shall have

$$3a_2 = \alpha^2 z' + \alpha z'' \therefore a_2 = \frac{\alpha^2 z' + \alpha z''}{3};$$

and thus all three of the roots are determined.

Equation of the Fourth Degree.

(272.) Let the equation,

$$x^4 + px^2 + qx + r = 0,$$

be proposed for solution.

As the second term is absent, one relation among the roots is

$$a + a_2 + a_3 + a_4 = 0.$$

Let us endeavour to obtain three other relations of the first degree, in a, a_2, a_3, a_4. For this purpose, assume

$$ka + la_2 + ma_3 + na_4 = z;$$

then, as there is no distinction between the roots expressed in this relation, it may represent indifferently any one of the 24 equations which arise from permuting the letters a, a_2, a_3, a_4, in all the ways possible. Hence the equation in z, which would be satisfied for any one of these 24 values, indifferently, must be of the 24th degree; and, in order that it may be resolvable by the formula for equations of the third degree, it must take the form

$$z^{24} + Az^{16} + Bz^{8} + C = 0.$$

It is possible to reduce the degree of this equation; for, since *k*, *l*, *m*, *n*, are indeterminate, we may suppose $k = l$, and thus reduce the number of distinct equations to twelve. By supposing, moreover, $m = n$, the equations are further reduced to six, which are as follows:

$$\begin{aligned}
l(a\ \ + a_2) + m(a_3 + a_4) &= z\\
l(a\ \ + a_3) + m(a_2 + a_4) &= z\\
l(a\ \ + a_4) + m(a_2 + a_3) &= z\\
l(a_3 + a_4) + m(a\ \ + a_2) &= z\\
l(a_2 + a_4) + m(a\ \ + a_3) &= z\\
l(a_2 + a_3) + m(a\ \ + a_4) &= z.
\end{aligned}$$

The equation in *z* will, therefore, under these restrictions, be only of the sixth degree, and, in order to solve it, it must be of the form

$$z^6 + Az^4 + Bz^2 + C = 0,$$

$$\text{or } (z^2)^3 + A\,(z^2)^2 + Bz^2 + C = 0,$$

and whatever value of *z* satisfies this equation, the same value, with contrary sign, will also satisfy it. The roots are therefore equal in magnitude two and two, but of contrary signs; and it is plain that the six values of *z* exhibited above will represent these relations by putting $l = -m = 1$; in fact, we shall then have for the values of *z* the expressions,

$$\left\{\begin{aligned}
a\ \ + a_2 - (a_3 + a_4) &= z\\
a\ \ + a_3 - (a_2 + a_4) &= z\\
a\ \ + a_4 - (a_2 + a_3) &= z
\end{aligned}\right.$$

$$\left\{\begin{aligned}
a_3 + a_4 - (a\ \ + a_2) &= z\\
a_2 + a_4 - (a\ \ + a_3) &= z\\
a_2 + a_3 - (a\ \ + a_4) &= z
\end{aligned}\right.$$

where the last three values of z are, in magnitude, the same as the first three, but with opposite signs. Hence, by transposing, and multiplying the several pairs of factors together, we have the following single equation in z, involving all the six values above, viz.,

$$\{ z^2 - (a + a_2 - a_3 - a_4)^2 \} \times$$
$$\{ z^2 - (a + a_3 - a_2 - a_4)^2 \} \times$$
$$\{ z^2 - (a + a_4 - a_2 - a_3)^2 \} = 0;$$

and as this involves none but symmetrical functions of a, a_2, a_3, a_4, its coefficients may be expressed by means of the coefficients of the proposed equation; but the following considerations will facilitate their determination. By actually squaring the quantities within the brackets, we have

$$(a + a_2 - a_3 - a_4)^2 =$$
$$(a + a_2 + a_3 + a_4)^2 - 4(aa_3 + aa_4 + a_2 a_3 + a_2 a_4);$$

but

$$a + a_2 + a_3 + a_4 = 0$$
$$aa_2 + aa_3 + aa_4 + a_2 a_3 + a_2 a_4 + a_3 a_4 = p;$$

therefore,

$$- (a + a_2 - a_3 - a_4)^2 = 4p - 4 (aa_2 + a_3 a_4).$$

In like manner,

$$- (a + a_3 - a_2 - a_4)^2 = 4p - 4 (aa_3 + a_2 a_4)$$
$$- (a + a_4 - a_2 - a_3)^2 = 4p - 4 (aa_4 + a_2 a_3);$$

putting, therefore, for abridgment,

$$z^2 + 4p = 4u,$$

the equation in z will be transformed into the following, viz.

$$\{ u-(aa_2+a_3a_4) \} \{ u-(aa_3+a_2a_4) \} \{ u-(aa_4+a_2a_3) \} =0:$$

which is of the form,

$$u^3 + A'u^2 + B'u + C' = 0;$$

its coefficients being

$$A' = -(aa_2 + aa_3 + aa_4 + a_2 a_3 + a_2 a_4 + a_3 a_4) = -p$$

$$B' = (aa_2 + a_3 a_4)(aa_3 + a_2 a_4) +$$

$$(aa_2 + a_3 a_4)(aa_4 + a_2 a_3) +$$

$$(aa_3 + a_2 a_4)(aa_4 + a_2 a_3) = S(a^2 a_2 a_3)$$

$$C' = -(aa_2 + a_3 a_4)(aa_3 + a_2 a_4)(aa_4 + a_2 a_3).$$

Now, from the formula at page 421, we have,

$$S(a^2 a_2 a_3) = \frac{\Sigma_2 (\Sigma_1)^2 - 2\Sigma_3 \Sigma_1 - (\Sigma_2)^2 + 2\Sigma}{2},$$

$$S(a^2 a_2^2 a_3^2) = \frac{(\Sigma_2)^3 - 3\Sigma_4 \Sigma_2 + 2\Sigma_6}{6};$$

in which

$$\Sigma_1 = 0,\ \Sigma_2 = -2p,\ \Sigma_3 = -3q$$

$$\Sigma_4 = -p\Sigma_2 - 4r = 2p^2 - 4r,$$

$$\Sigma_6 = -2p^3 + 4pr + 3q^2 + 2pr = -2p^3 + 6pr + 3q^2.$$

Hence, substituting these values in the expressions for B′ and C′, we have

$$B' = \frac{-4p^2 + 4p^2 - 8r}{2} = -4r,$$

$$C' = -q^2 + 2pr + 2pr = 4pr - q^2;$$

and thus the equation in u is

$$u^3 - pu^2 - 4ru + 4pr - q^2 = 0;$$

and replacing u by its value $\frac{z^2 + 4p}{4} = \frac{z^2}{4} + p$, or, for simplicity, by $z_1 + p$, we have, for the final cubic, the equation

$$z_1^3 + 2pz_1^2 + (p^2 - 4r) z_1 - q^2 = 0.$$

Calling the roots of this equation z', z'', z''', we shall have $4z'$, $4z''$, $4z'''$ for the squares of the expressions $a + a_2 - a_3 - a_4$,

$a + a_3 - a_2 - a_4$, $a + a_4 - a_2 - a_3$; that is, these expressions are

$$a + a_2 - a_3 - a_4 = \pm 2 \sqrt{z'}$$

$$a + a_3 - a_2 - a_4 = \pm 2 \sqrt{z''}$$

$$a + a_4 - a_2 - a_4 = \pm 2 \sqrt{z'''}$$

also, $$a + a_2 + a_3 + a_4 = 0.$$

By adding these, we find

$$4a = \pm 2 \sqrt{z'} \pm 2 \sqrt{z''} \pm 2 \sqrt{z'''};$$

$$\therefore a = \pm \tfrac{1}{2} \sqrt{z'} \pm \tfrac{1}{2} \sqrt{z''} \pm \tfrac{1}{2} \sqrt{z'''}.$$

Again, adding the first and fourth, and subtracting the sum of the other two from the result, we have

$$4a_2 = \pm 2 \sqrt{z'} \mp 2 \sqrt{z''} \mp 2 \sqrt{z'''}$$

$$\therefore a_2 = \pm \tfrac{1}{2} \sqrt{z'} \mp \tfrac{1}{2} \sqrt{z''} \mp \tfrac{1}{2} \sqrt{z'''}.$$

Similarly,

$$a_3 = \mp \tfrac{1}{2} \sqrt{z'} \pm \tfrac{1}{2} \sqrt{z''} \mp \tfrac{1}{2} \sqrt{z'''}$$

$$a_4 = \mp \tfrac{1}{2} \sqrt{z'} \pm \tfrac{1}{2} \sqrt{z''} \mp \tfrac{1}{2} \sqrt{z'''}.$$

As to the proper signs of the radicals $\sqrt{z'}$, $\sqrt{z''}$, $\sqrt{z'''}$, we must observe that since

$$(a + a_2 - a_3 - a_4)(a + a_3 - a_2 - a_4)(a + a_4 - a_2 - a_3) = -8q,$$

these signs ought to be such as to render their product positive if q is negative, and negative if q is positive. The values thus deduced are the same as those otherwise determined in the preceding chapter, and commented upon in the Scholium following.

(273.) The above method of investigating analytical expressions for the roots of equations by means of symmetrical functions, may be extended to equations of higher degrees than the fourth; but the auxiliary equation in z, to which the investigation leads, is, after the fourth degree, of a higher order than the proposed. In equations of the fifth degree the auxiliary one rises to the 120th degree, which, by means of certain artifices, is, however, capable of depression. But no method has yet been devised, whereby an equation of the fifth degree can be solved by help of an auxiliary equation below the sixth. It has indeed been the object of analysts of late to demonstrate the utter impossibility of such a solution by any combination of algebraic symbols; on which subject the student is referred to the profound paper of Sir WILLIAM HAMILTON in the *Transactions of the Royal Irish Academy*, vol. xvi.

It is of course to be understood that the impossibility referred to does not exclude particular classes of equations from becoming solvible in general symbols. It would, indeed, be easy to assume particular irrational forms for the roots, and, by eliminating the radicals, thence to deduce an equation of any degree we please, the coefficients of which should be literal quantities, ready to receive any numerical values we may choose to give them. But these would only be equations of a particular class; the coefficients of which would be related to one another by a fixed law, virtually implied in the form assumed for the roots; and could, therefore, be only of very rare application in actual practice. Such particular forms have, however, been investigated by DE MOIVRE, WARING, and EULER: an account of the mode of deducing them may be seen in the second volume of the *Algebra* of MEYER HIRSCH.* The formula of DE MOIVRE, for a certain class of equations of the nth degree, we shall give in the scholium below; because of the direct and simple manner in which it may be investigated without assuming the irrational form of the roots, and also because of the celebrity which it has acquired.

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* This work of HIRSCH contains a very comprehensive discussion of the theory of the symmetrical functions of the roots of equations: a subject upon which we have been precluded from dwelling at any great length in the present volume on account of the space occupied by more useful inquiries. In addition to this work of HIRSCH, we would also direct the attention of the student to the researches of Mr. MURPHY, as contained in his *Theory of Equations*, and in his papers in the *Cambridge Transactions* for 1831, and in those of the *Royal Society* for 1837. A view of the method of approximating to the roots of equations by the method of Recurring Series may be seen in EULER's *Algebra*; and in a more extended form in MURPHY's *Equations* adverted to above.

Scholium.

(274.) The formula of Cardan for the roots of a cubic equation, and the expressions of Euler for the roots of a biquadratic, furnish exemplifications of the doctrine at pp. 26, 40 respecting the limited sense in which irrational results must be taken when the signs of the radicals are brought under any control by stipulated conditions. One of the irrational terms in Cardan's formula takes a fixed and determinate sign as soon as the sign of the other irrational term is determined upon, on account of the controlling condition at page 443. And in like manner any one of the three irrational terms in Euler's forms becomes fixed in sign as soon as the other two radicals, which are entirely independent and uncontrolled as to sign, become fixed, because of the overruling condition $8uvw = -q$, page 449.

In the expressions for the four roots of the biquadratic at p. 451 the signs prefixed to the radicals thus point out the character of the roots, and limit the generality that would otherwise belong to them. But it is easy to avoid all inadmissible generality in the formulas referred to, and to deduce them at once free from extraneous values: and it is undoubtedly better to do this than to encumber our results with what must afterwards be rejected upon adjusting the formula to the case in hand. The necessity for any such supplementary adjustment does in fact imply some imperfection in our method, or some mistake in our reasoning: unless indeed the original restrictions of the problem are such as unavoidably to limit the generality of the ordinary symbols of operation from the outset. This is not the case in either of the problems referred to: the limitation to which the result ought to conform has been introduced in the course of the investigation, and has then been improperly disregarded, leaving the final result uncontrolled by it.

The condition $z = -\frac{p}{3y}$, introduced into the investigation of Cardan's formula, ought to have had its influence upon every subsequent step: z^3 should have been replaced by $\left(-\frac{p}{3y}\right)^3$; and the final result should accordingly have been written thus:

$$x = \left\{ -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right\}^{\frac{1}{3}} - \frac{\frac{1}{3}p}{\left\{ -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right\}^{\frac{1}{3}}},$$

from which all superfluous values are excluded.

(275.) The formulas of **Euler** are open to the same objection. The condition

$$\sqrt{z'}\sqrt{z''}\sqrt{z'''} = -q, \text{ or } \sqrt{z'''} = \frac{-q}{\sqrt{z'}\sqrt{z''}}$$

introduced into the investigation, has been disregarded; and thus the result is uncontrolled by its restrictions, and involves a superfluous radical, like the formula of **Cardan.** When freed from this imperfection, the result of **Euler**'s investigation will take the form

$$x = \frac{\sqrt{z'}}{2} + \frac{\sqrt{z''}}{2} + \frac{-q}{2\sqrt{z'}\sqrt{z''}},$$

which comprehends all that is implied in the four distinct expressions at page 451, and involves no superfluous values, the radicals being unrestricted in generality.

It will have been observed that in the formulas of **Euler** at page 451 the signs prefixed to the radicals to designate the character of the roots are sufficiently explicit for that purpose: but in the formula of **Cardan**, at pages 442 and 463, similar precision cannot be impressed upon the expressions by means of the prefixed signs, so that in applying the formula it is essential that the overruling condition $3yz = -p$ be always borne in remembrance.

(276.) Like observations apply to the formulas of **Ferrari** at page 452: they are not only faulty inasmuch as they preserve no trace of the restrictions that have been introduced into the investigation, but, as in the case of **Cardan**'s expressions, they are also incapable of the necessary adjustment by putting any restrictions upon the signs of the radicals; so that as these signs are entirely inoperative, it is matter of indifference whether they be plus or minus. The requisite precision is only to be attained by conforming the values to the condition

$$\sqrt{(2k-p)}\sqrt{(k^2-r)} = -\frac{q}{2}$$

$$\text{or } \sqrt{(k^2-r)} = \frac{-\frac{1}{2}q}{\sqrt{(2k-p)}}$$

But if, as in the cases discussed above, we actually introduce this condition into the results themselves, the expressions become freed from all ambiguity, and take the explicit form

$$x^2 + k = \sqrt{(2k-p)}x + \frac{-\frac{1}{2}q}{\sqrt{(2k-p)}}$$

where the sign of the radical is entirely unrestricted. And it is in this form that the expression should always be written.

(277.) It may be remarked here that LAGRANGE, at the end of the last edition of his work on *Equations*, incidentally notices the above mode of writing EULER's form as that in which it *may* be expressed. But he does not insist upon this as the form in which it always ought to be written, and which is the only form strictly in accordance with the conditions impressed upon the symbols in the course of the investigation. POINSOT, in a note appended to the above-mentioned work, animadverts somewhat upon LAGRANGE's remark, and proposes what he regards as a preferable mode of interpretation as respects EULER's ambiguous forms. We have not space to transcribe this note of POINSOT, but must express our entire dissent from the doctrine there propounded.

(278.) It may be observed, in conclusion, that the formula of CARDAN is only a particular case of a much more general expression, an expression which in fact exhibits under a similar form the roots of an equation of the nth degree. But the practical utility of this general form ceases with the equation of the third degree, because for the higher degrees it is in general impossible to transform the proposed equation into another belonging to the particular class of equations comprehended under the general form. This class of equations was first solved by DE MOIVRE, and every equation belonging that class is said to come under DE MOIVRE's *solvible form.*

The general investigation may be conducted in a manner analogous to that which has been employed in the particular case;

the problem to be solved being, indeed, simply this, viz., To find two quantities, y and z, such that the sum of their nth powers may be a and their product b.

The value y determined by these conditions (*Algebra*, third ed. p. 156), is

$$y = \{ \tfrac{1}{2} a + \sqrt{(a^2 - 4b^n)} \}^{\frac{1}{n}}$$

and consequently, since $z = \frac{b}{y}$ we must have for z the expression

$$z = \frac{b}{\{ \frac{1}{2} a + \sqrt{(a^2 - 4b^n)} \}^{\frac{1}{n}}}$$

And (*Algebra*, p. 157), $y + z$ is the value of x in the equation

$$x^n - nbx^{n-2} + \frac{n(n-3)}{2} b^2 x^{n-4} - \frac{n(n-3)(n-4)}{2.3} b^3 x^{n-6} + \ldots . = a.$$

When $n = 3$ the equation is

$$x^3 - 3bx - a = 0,$$

a form to which every cubic may be reduced, as a and b may be any values whatever. Making $a = -q$, and $3b = -p$, we have the general form at (263), and for $x = y + \frac{b}{y}$ the expression already deduced.

When $n = 5$ the general form becomes

$$x^5 - 5bx^3 + 5b^2x - a = 0,$$

of which the roots are represented by the formula

$$x = \{ \tfrac{1}{2} a + \sqrt{(a^2 - 4b^5)} \}^{\frac{1}{5}}$$
$$+ \frac{b}{\{ \frac{1}{2} a + \sqrt{(a^2 - 4b^5)} \}^{\frac{1}{5}}}$$

Many attempts have been made to reduce the general equation of the fifth degree to the above solvible form, but they have all been fruitless. We shall merely add, that by substituting in the preceding equation c for $5b$, we may write the form thus,

$$x^5 - cx^3 + \tfrac{1}{5} c^2 x - a = 0,$$

where a and c are unrestricted.

NOTE.

THE principle established at pp. 260-3 and there affirmed to remove an imperfection in HORNER'S method of appoximation was, we find, known to Mr. HORNER himself. The author of the present work did not discover this till long after the sheet referred to was printed. Mr. HORNER announces the principle as follows :

"When an equation has α extensive or incommensurable equal roots, the following *indications* will be observed as the work proceeds : simultaneous and regular diminution in the last α terms ; the preceding terms constant in regard of signs, and, to an increasing extent, in their first digits also ; the last $\alpha + 1$ signs alternately + and –, or – and + ; that mutual relation among the last $\alpha + 1$ terms which the expansion of $h(x - r)^{\alpha}$ indicates.

"From that law of expansion, also, the *value* of the approximation r may be readily found, and in various ways. *E. g.* (A) If the last term but one be made the search divisor, the dividend must be α times the last term. (B) The αth term, reckoning backward, may be the dividend, and the search divisor be α times the preceding. (C) The last term being the dividend, and the $\alpha + 1$th term, reckoning backward, the search divisor, the αth root of the quotient may be tried for r.

"The ordinary rules, which are only adapted to the case $\alpha = 1$, of course fail when applied to other cases ; but by attending to the indications here made out, the new method is divested of perplexity, and the transformations may proceed under every circumstance without interruption." (*Leybourn's Repository*, No. 19, page 63.)

The preceding observations apply to the case of equal roots : when the roots are nearly equal Mr. HORNER proceeds as follows :

"When two or more roots of an equation agree in their first digit or digits, and are at the same time smaller or greater than any other root, abstracting the signs, the evolution of the portion in which they agree will necessarily be attended with indications very similar to those which we have traced to the influence of equal roots. Consequently they present the same resistance to the Newtonian and similar methods; a fact which LAGRANGE has not failed to dilate upon. But they also share in the facilities pointed out in our remarks above; to such a degree at least, that an approximation may thence be generally obtained which shall at once place the remaining solution within the reach of the ordinary methods." (*Ibid.* page 67.)

Mr. HORNER prosecutes the subject no further than this: he illustrates the preceding observation by means of the equation of LAGRANGE, viz. $x^3 - 7x + 7 = 0$; from which it would appear that his view of the principle is, after all, very different from ours; as he employs it to suggest the second figure, 6, of one of the roots, which figure is widely different from the second figure of the other root. We employ the principle, only so long as the figures in the contiguous roots *concur*, to suggest those figures: and we have shown its competency to detect the place at which this concurrence ceases. The suggestion of the true root-figure, in LAGRANGE's example, is mere matter of chance; and proves nothing as to the value of the principle in question. But had such examples as those developed, by aid of this principle, in the text, occurred to Mr. HORNER, there is no doubt that he would have prosecuted the developments in the same manner. The difficulty, however, of ascertaining the exact place at which a root separates, and the method of determining the leading figure of those that remain—particulars of much importance, would still have remained unprovided for.*

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* It may be proper to add to these remarks, that Mr. LOCKHART, by examining a great variety of equations of the fifth degree, each having a pair of roots very nearly equal—or concurring in several leading figures—found from experience, that $\frac{2N'}{A'}$ always separated those roots; so that the successive values of this expression, as the approximation advanced, always furnished the concurring figures. See *Resolution of Equations by means of Inferior and Superior Limits.* 1842.

The precept 5 at page 265 accomplishes the latter of these particulars in a very direct and easy manner: the other mode of proceeding, explained in the directions which follow this precept, is purely tentative; and of course far less eligible.

We may here remark that in comparing FOURIER'S method of analysis with that exemplified at p. 306, it is implied at p. 307, that the former method dispenses with a transformation necessarily involved in the latter: it must not be overlooked, however, that the transformation (4.8), employed in FOURIER'S analysis at p. 245, is equivalent to the additional transformation adverted to; and, therefore, that the desired conclusion is reached in both methods by the same number of steps.

THE END.

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