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Umberto Viaro

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Chapter 1

Proof of the Routh criterion

Even if the Routh criterion is still an important tool in modern robust stability analysis and most control textbooks illustrate its implementation, the proof of this useful test is almost always omitted. The exclusion is probably due to the fact that the proofs available in the literature either require advanced mathematical concepts [1], [2] or provide little insight into the mechanism of the criterion [3], [4].

An alternative proof of the Routh test and the rules for determining the zero distribution of a real polynomial with respect to the imaginary axis was suggested in [5]. This proof uses only elementary geometric properties of polynomials that show pictorially how the numbers of sign permanences and variations along the first column of the Routh array are related to the number of the left half–plane (LHP) and right half–plane (RHP) roots of a polynomial.

As will be shown next, the adopted approach, which is reminiscent of the one in [6] and [10], proves useful also in the treatment of the critical cases that may arise in the construction of the Routh table.

1.1 Basic recursions

Consider the nth–degree polynomial:

$$P_n(s) = \sum_{k=0}^{n} a_{n,k}\, s^k, \quad a_{n,k} \in \mathbb{R},\ \forall k, \tag{1.1}$$

and decompose it into its even and odd parts as

$$P_n(s) = Q_n(s) + Q_{n-1}(s), \tag{1.2}$$

where

$$Q_n(s) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{n,n-2k}\, s^{n-2k} \tag{1.3}$$

and

$$Q_{n-1}(s) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,n-1-2k}\, s^{n-1-2k}. \tag{1.4}$$

The entries of every row of the Routh table coincide with the coefficients of the decreasing powers of s in the polynomials $Q_i(s)$, $0 \le i \le n$, formed, starting from (1.3) and (1.4), according to the Euclidean recursion:

$$Q_{i-2}(s) = Q_i(s) - q_{i-1}\, s\, Q_{i-1}(s), \tag{1.5}$$

where q_{i-1} is the ratio of the *leading* coefficients of $Q_i(s)$ and $Q_{i-1}(s)$, respectively. If all leading coefficients are different from 0, this procedure defines recursively a sequence of n numbers $\{q_i, 0 \le i \le n-1\}$.

Assume first that no leading coefficient is equal to zero. This implies, in particular, that $Q_n(s)$ and $Q_{n-1}(s)$, as well as the other polynomials $Q_i(s)$, are coprime. In fact, if $Q_n(s)$ and $Q_{n-1}(s)$ had a common factor $C_h(s)$ of degree h, the same factor would be present in $Q_i(s)$, $i \ge h$; hence, $Q_h(s)$ would turn out to be equal to $C_h(s)$, and $Q_{h-1}(s)$ identically equal to zero.

Consider now the polynomials:

$$P_i(s) = Q_i(s) + Q_{i-1}(s), \ \ 1 \le i \le n, \ \text{ and } \ P_0(s) = Q_0(s). \tag{1.6}$$

Combining (1.5) and (1.6) we obtain

$$P_{i-1}(s) = P_i(s) - q_{i-1}\, s\, Q_{i-1}(s). \tag{1.7}$$

In the following, it is shown first how the distribution of the roots of $P_i(s)$ is related to that of $P_{i-1}(s)$. Then, the root distribution of $P_n(s)$ is found by recursively applying this result starting from $P_0(s)$ and $P_1(s)$.

1.2 Root distribution

Relation (1.7) can be embedded in the locus:

$$\hat{P}_i(s;q) := P_i(s) - q\, s\, Q_{i-1}(s),\ \ q \in \mathcal{R},\ \ i = 1, 2, \cdots, n. \tag{1.8}$$

Clearly, $\hat{P}_i(s;0) = P_i(s)$ and $\hat{P}_i(s;q_{i-1}) = P_{i-1}(s)$. Moreover, q_{i-1} is the only value of parameter q for which $\hat{P}_i(s;q)$ has degree $i-1$.

Consider the i continuous arcs on the complex plane described by the i roots of $\hat{P}_i(s;q)$ as q varies from 0 to q_{i-1}. One and only one of these arcs tends to the point at infinity as q tends to q_{i-1} because, if no leading coefficient is equal to 0, $Q_{i-1}(s)$ and, thus, $P_{i-1}(s) = \hat{P}_i(s;q_{i-1})$ have $i-1$ finite roots, whereas for $q \neq q_{i-1}$ the polynomial $\hat{P}_i(s;q)$ has i finite roots. Concerning these arcs, the following lemma holds.

Lemma 1.2.1 *No arc crosses the imaginary axis at a point $s = \jmath\beta$ with β finite.*

Proof If a crossing occurred at $s = \jmath\beta$, then

$$\begin{aligned}\hat{P}_i(\jmath\beta;q_{i-1}) &= P_i(\jmath\beta) - q\,\jmath\beta\, Q_{i-1}(\jmath\beta) = \\ &[Q_i(\jmath\beta) - q\,\jmath\beta\, Q_{i-1}(\jmath\beta)] + Q_{i-1}(\jmath\beta) = 0\end{aligned} \tag{1.9}$$

for a suitable value of q. This would imply that both the real and the imaginary parts of $\hat{P}_i(\jmath\beta;q_{i-1})$, i.e., $Q_i(\jmath\beta) - q\,\jmath\beta\, Q_{i-1}(\jmath\beta)$ and $Q_{i-1}(\jmath\beta)$, vanish simultaneously, which contradicts the assumption that no leading element is zero and, consequently, $Q_i(s)$ and $Q_{i-1}(s)$ are coprime. □

The only arc that tends to infinity as $q \to q_{i-1}$ is confined to either the LHP or the RHP, as is the case for all of the other $i-1$ arcs. In fact, for $q \in [0, q_{i-1})$, this arc neither touches the imaginary axis nor passes to another half–plane through the point at infinity; consequently, it belongs to the half–plane on which its point corresponding to $q = 0$ lies. The following lemma indicates the location with respect to the imaginary axis of the unique arc that tends to infinity.

Lemma 1.2.2 *The arc that tends to infinity as $q \to q_{i-1}$ lies in the LHP if $q_{i-1} > 0$ and in the RHP if $q_{i-1} < 0$.*

Proof Denoting $P_i(s)$ as

$$P_i(s) = \sum_{j=0}^{i} a_{i,k}\, s^k, \tag{1.10}$$

so that

$$Q_{i-1}(s) = a_{i,i-1}s^{i-1} + a_{i,i-3}s^{i-3} + \cdots, \tag{1.11}$$

polynomial (1.8) may be written as

$$\hat{P}_i(s;q) = (a_{i,i} - q a_{i,i-1})s^i + a_{i,i-1}s^{i-1} + (a_{i,i-2} - q a_{i,i-3})s^{i-2} + \cdots \tag{1.12}$$

which, for $q = \bar{q} \in [0, q_{i-1})$ and $|q_{i-1} - \bar{q}| \ll |q_{i-1}|$, exhibits a root of very large magnitude close to the real root

$$\bar{z} = -\frac{a_{i,i-1}}{a_{i,i} - q a_{i,i-1}} = -\frac{1}{q_{i-1} - \bar{q}} \tag{1.13}$$

of the polynomial

$$(a_{i,i} - \bar{q} a_{i,i-1})s^i + a_{i,i-1}s^{i-1} \tag{1.14}$$

approximating $\hat{P}_i(s;\bar{q})$ for $|s|$ large. The sign of $\bar{z}$ is opposite to that of q_{i-1} because $|\bar{q}| < |q_{i-1}|$. It follows that $\bar{z}$ is in the LHP if $q_{i-1} > 0$, and in the RHP if $q_{i-1} < 0$. This is also true for the root of very large magnitude of $\hat{P}_i(s;\bar{q})$ close to $\bar{z}$ and for all of the points belonging to the corresponding arc. □

An immediate consequence of Lemmas 1.2.1 and 1.2.2 is the following corollary that relates the numbers $n_{l,i}$ and $n_{r,i}$ of LHP and RHP roots of $P_i(s)$ to the corresponding numbers $n_{l,i-1}$ and $n_{r,i-1}$ for $P_{i-1}(s)$.

Corollary 1.2.3 *If $q_{i-1} > 0$, then*

$$n_{l,i} = n_{l,i-1} + 1, \;\; n_{r,i} = n_{r,i-1}, \tag{1.15}$$

and, if $q_{i-1} < 0$, then

$$n_{l,i} = n_{l,i-1}, \;\; n_{r,i} = n_{r,i-1} + 1. \tag{1.16}$$

□

1.3 Routh's theorem

On the basis of the results of Section 1.2, the following theorem can be proved.

Theorem 1.3.1 *(Routh's Theorem) The numbers $n_{l,n}$ and $n_{r,n}$ of the LHP and RHP roots of $P_n(s)$ coincide, respectively, with the numbers of sign permanences and sign variations between consecutive elements along the first column of the Routh array for $P_n(s)$.*

Proof Observe that $q_{i-1} > 0$ if and only if $\mathrm{sgn}(a_{i,i}) = \mathrm{sgn}(a_{i-1,i-1})$, and $q_{i-1} < 0$ if and only if $\mathrm{sgn}(a_{i,i}) = -\mathrm{sgn}(a_{i-1,i-1})$. Therefore, a positive value of q_{i-1} corresponds to a sign permanence between the leading elements of the rows of order i and $i-1$, whereas a negative value of q_{i-1} corresponds to a sign variation between the same elements. By recursively applying Corollary 1.2.3 n times starting from $P_0(s)$ and $P_1(s)$, it turns out that $n_{r,n}$ coincides with the number of sign variations encountered along the first column, and $n_{l,n} = n - n_{r,n}$ coincides with the number of sign permanences, because each sign variation introduces an RHP root in the polynomial of immediately higher degree in the sequence (1.6) and each sign permanence an LHP root. □

An immediate consequence of Theorem 1.3.1 is the following.

Corollary 1.3.2 *(Routh's Stability Criterion) Polynomial $P_n(s)$ is Hurwitz (all roots in the open LHP) if and only if all of the leading coefficients of its Routh array have the same sign.* □

1.4 Critical cases

The critical cases arise when the leading coefficient of $Q_{n-1}(s)$ or of a polynomial generated by recursion (1.5) is equal to zero, that is, when a row of the Routh table begins with a zero. As is known, if all the entries of this row are equal to zero, then $Q_n(s)$ and $Q_{n-1}(s)$ have a common factor $C_h(s)$ of degree $h > 0$ whose coefficients coincide with the entries of the row of order h preceding the all–zero row of order $h-1$. The roots of $C_h(s)$ are symmetric with respect to the origin (quadrantal symmetry) and some, or all, of them may be on the imaginary axis; half of the remaining roots, if any, are in the LHP and half are in the

RHP. This situation is treated in the standard control literature: the procedure to evaluate the location of the roots is simple and unequivocal in this case. Therefore, here attention is limited to polynomials $P_n(s) = Q_n(s) + Q_{n-1}(s)$ with $Q_n(s)$ and $Q_{n-1}(s)$ coprime.

To overcome the difficulty due to the vanishing of a leading coefficient (but not of all coefficients of a row), various expedients that are not completely satisfactory have been suggested. A technique that draws on the considerations developed in the previous section is outlined next. For simplicity, it is assumed that the vanishing leading entry is followed, in the same row, by a nonzero entry. To this purpose, it is useful to adopt a new notation for the coefficients of $Q_i(s)$:

$$Q_i(s) = \sum_{k=0}^{\lfloor i/2 \rfloor} r_{i,i-2k} s^{i-2k} \tag{1.17}$$

so that (see (1.10))

$$r_{i,i-2k} = a_{i+1,i-2k} = a_{i,i-2k}. \tag{1.18}$$

Assume that the first row whose initial entry is equal to zero is the row of order i, so that $r_{i,i} = 0$ (but $r_{i,i-2} \neq 0$). Using this new notation, polynomial $P_{i+1}(s)$ can be written as

$$P_{i+1}(s) = r_{i+1,i+1}s^{i+1} + r_{i+1,i-1}s^{i-1} + r_{i,i-2}s^{i-2} + r_{i+1,i-3}s^{i-3} + \cdots . \tag{1.19}$$

Consider now the polynomial:

$$\tilde{P}_{i-2}(s) := (s+1)P_{i+1}(s) \tag{1.20}$$

whose roots are those of $P_{i+1}(s)$ with an additional LHP root at -1. Therefore, the first column of its Routh table exhibits the same number of sign variations as that for $P_{i+1}(s)$ and a number of sign permanences equal to that for $P_{i+1}(s)$ plus one. The upper three rows of the Routh table for (1.20) are:

$$\begin{array}{c|cccc} i+2 & r_{i+1,i+1} & r_{i+1,i-1} & r_{i+1,i-3} + r_{i,i-2} & \cdots \\ i+1 & r_{i+1,i+1} & r_{i+1,i-1} + r_{i,i-2} & r_{i+1,i-3} + r_{i,i-4} & \cdots \\ i & -r_{i,i-2} & r_{i,i-2} - r_{i,i-4} & r_{i,i-4} - r_{i,i-6} & \cdots \end{array} \tag{1.21}$$

Since the rows of order $i+2$ and $i+1$ in (1.21) begin with the same element, the sign permanence between their leading elements can be associated with the additional LHP root at -1. In this way, the numbers of sign permanences and variations from the row or order $i+1$ to that of order 0 account for the root distribution of $P_{i+1}(s)$ itself.

In practice, it is not necessary to actually form polynomial $\tilde{P}_{i-2}(s)$ according to (1.20). In fact, the row of order $i+2$ need not be computed, and the rows of orders $i+1$ and i in the above array can directly be obtained from those of the same orders in the array for $P_n(s)$. Specifically, the entries of the new row of order $i+1$ are given by the sum of the entries of the same column in the old rows of orders $i+1$ and i, and each entry of the new row of order i is given by the difference between the entry in the same position and that at its right (if any) in the old row of order i. In conclusion, to evaluate the root distribution of $P_n(s)$, it is enough: (i) to determine the number of sign permanencies and variations along the first column of the array from the row of order n to the row of order $i+1$ (like in the popular ϵ–method), (ii) to modify the rows of order $i+1$ and i as indicated above, and (iii) to evaluate the numbers of sign permanencies and variations from the new row of order $i+1$ to the row of order 0 (which avoids the problems associated with the use of cumbersome expressions involving ϵ–terms).

The extension of the procedure to the case in which more initial entries of a row are equal to zero is straightforward [5].

Example Consider the polynomial:

$$P_8(s) = s^8 + s^7 + 3s^6 + 2s^5 + 4s^4 + 3s^3 + 4s^2 + 2s + 1. \tag{1.22}$$

The upper five rows of its Routh table are

$$\begin{array}{c|ccccc} 8 & 1 & 3 & 4 & 4 & 1 \\ 7 & 1 & 2 & 3 & 2 & \\ 6 & 1 & 1 & 2 & 1 & \\ 5 & 1 & 1 & 1 & & \\ 4 & 0 & 1 & 1 & & \end{array} \tag{1.23}$$

With the suggested procedure, the rows of orders from 5 to 0 turn out to be

$$\begin{array}{c|ccc} 5 & 1(=1+0) & 2(=1+1) & 2(=1+1) \\ 4 & -1(=0-1) & 0(=1-1) & 1(1-0) \\ 3 & 2 & 3 & \\ 2 & 3/2 & 1 & \\ 1 & 5/3 & & \\ 0 & 1 & & \end{array} \tag{1.24}$$

By neglecting the row of order 4 in (1.23), there are three sign permanencies along the first column of (1.23). In the first column of (1.24) there are three sign permanencies and two sign variations. It follows that (1.22) has 6 LHP and 2 RHP roots. To obtain the same (correct) result with the ϵ–method, no approximation of the cumbersome expressions involving ϵ–terms is allowed.

1.5 Concluding remarks

The locations of the roots of two consecutive polynomials in the Routh sequence (1.6) have been related to each other by means of simple geometric properties. In particular, it has been shown that, on passing from $P_{i-1}(s)$ to $P_i(s)$, $i-1$ roots remain in the same half–plane, whereas the ith additional root is in the LHP if their leading coefficients have the same sign and in the RHP otherwise.

By applying this result iteratively starting from $P_0(s)$ and $P_1(s)$, a simple proof of the rule for counting the LHP and RHP roots of a given polynomial $P_n(s)$ has been provided. The adopted approach turns out to be particularly useful in the treatment of the critical cases.

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Often, short papers tend to be sharper than longer works because they focus on a single theme without lingering on unessential aspects, thus showing clearly the signicance of a contribution or an idea. The author of this book had the privilege of collaborating for over a quarter of a century with Antonio Lepschy (1931-2005), a recognized leader of the Italian control community.

Lepschy had a liking for the brief paper format, so that many results obtained by his research team were published in this way. The present compilation tells a few of these short stories, duly updated, trying to preserve their original flavour.

Umberto Viaro (http://umbertoviaro.blogspot.com/) has been professor of System and Control Theory at the University of Udine, Italy, since 1994. His 25-year-long collaboration with Antonio Lepschy resulted in more than 100 joint papers and two books. An essential role in this research activity was played by Wiesław Krajewski of the Systems Research Institute, Polish Academy of Sciences. The current research interests of Umberto Viaro concern optimal model reduction, robust control, switching and LPV control. He is the author or coauthor of 4 books and about 180 research papers.

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