POLSKA AKADEMIA NAUK
Instytut Badań Systemowych

# ZASTOSOWANIA INFORMATYKI W NAUCE, TECHNICE I ZARZĄDZANIU

Redakcja:
Jan Studziński
Ludosław Drelichowski
Olgierd Hryniewicz

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**Seria: BADANIA SYSTEMOWE**
**Tom 41**

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Redaktor naukowy:

**Prof. dr hab. Jakub Gutenbaum**

Warszawa 2005

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Książka wydana dzięki dotacji KOMITETU BADAŃ NAUKOWYCH

Książka zawiera wybór artykułów poświęconych omówieniu aktualnego stanu badań w kraju, w zakresie rozwoju modeli, technik i systemów informatycznych oraz ich zastosowań w różnych dziedzinach gospodarki. Kilka artykułów omawia aplikacyjne wyniki projektów badawczych i celowych Ministerstwa Nauki i Informatyzacji.

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# THE INFLUENCE OF A CHANGE OF THE NUMBER OF ALTERNATIVES ON A GROUP JUDGEMENT IN POSITIONAL METHODS

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*In the paper the method worked out by D.G. Saari making it possible to determine group judgement for the whole set of alternatives on the basis of results obtained for the subsets of this set is analyzed in detail. Considerations are illustrated with numerical example in order to show the practical usefulness of the method considered.*

**Keywords:** Group opinion, expert judgement, positional methods.

## 1. Introduction

It is well known that group judgement determined with the use of well defined methods in some situations may exhibit not desirable properties (so called paradoxes can occur) or the results obtained by means of different methods may differ (even may be opposite). In such a case two approaches are possible. First one consists in developing new methods that are free of disadvantages of the former ones; the second one provides explanation of the reasons of possible inconsistencies. This approach is represented by D.G. Saari (Saari, 1995).

Saari proposed to decompose the set of experts' judgements into components responsible for particular voting paradoxes. It is assumed that expert judgements have the form of preference orders and no equivalent alternatives occur. If the number of alternatives is n, then they can be ordered in n! different ways.

Assume $t_j$ denotes the number of experts whose judgement has the form of preference order $P^j$ (j=1,..., n!). If the number of experts is K then

$$\sum_{j=1}^{K} t_j = K \qquad (1)$$

A vector $T = (t_1, ...., t_{n!})$ is called the profile.
The profile comprises – subject to the assumptions given – the complete description of results of an expertise.

The basic idea of Saari's approach is to decompose a profile into components that influence the group judgement in a specific way.

The number of components of a profile to be analysed depends on the number of alternatives considered. For the case of three alternatives, a profile consists of four components: basic, reversal, Condorcet and neutral. The number of components increases with the number of alternatives and new components are to be introduced.

In 2000 in the Journal of Economic Theory appeared a comprehensive two-part paper of D.G. Saari containing a mathematical approach to the problem of voting paradoxes (Saari, 2000a; 2000b).

In the papers mentioned above Saari analysed the case of more than three alternatives. He proposed to modify some components of a profile derived for the case of three alternatives and introduced new ones. For the case of four alternatives the profile consists of the following components: basic, double reversal, departure, modified departure, Condorcet and neutral.

The first part of the paper mentioned concerns pairwise comparisons, the second one positional methods. One of the problems examined in the latter is the influence of a change of the number of alternatives to be ordered by experts on the group judgement.

In his former book Saari showed (Saari, 1995) that the changes mentioned do not influence so called basic profiles.

Most of voting paradoxes is related to profiles orthogonal to the space of basic profiles (Ratliff 2002), (Saari, 1995; 2000b). They are called departure profiles to emphasize their effect upon situations that do not occur in the case of basic profiles.

Saari proved (Saari, 2000b) that the measure of this departure is the difference among the properties of a given method and the method of Borda (Nurmi, 1987), (Saari, 1995). In Saari's opinion (Saari, 1995; 2000b) the Borda method may be considered as the most interesting one, because for this method the number of possible paradoxes is not significant.

When determining group judgement the problem of taking into account the full set of alternatives or restricting considerations to its subsets is of some importance. The situation when subsets are analyzed only, may lead to the loss of assumption of the rationality of experts' (Condorcet paradox).

The method proposed by Saari (Saari, 1995; 2000b) makes it possible to determine group judgement for the whole set of alternatives on the basis of results obtained for subsets of this set. It is derived from the outranking matrix approach (Nurmi, 1987; 1992; 2002). This method - being very useful in applications - will be described in detail. Considerations are exemplified with an example of four alternatives.

## 2. Positional methods

Information on the position taken by an alternative in experts' preference orders is of crucial importance for positional methods (Nurmi, 1987; 1992), (Saari, 1995; 2000b). The notion of voting vector (Saari 1995, 2000b) plays the fundamental role in these methods.

Assume that the number of experts is equal K and the number of alternatives is equal n.

**Definition 1** (Saari, 1995; 2000b)

A voting vector $w^n$ for n alternatives is a convex combination of (n-1) normalised vectors $\{v_j^n\}$ [1), j=1, ..., n.

$$\text{where } v_j^n = \{ \underbrace{1}_{1^{st}\text{ component}}, ..., \underbrace{1}_{j^{th}\text{component}}, 0, ... \underbrace{0}_{n^{th}\text{component}} \}. \tag{2}$$

$v_j^n$ vector may be interpreted as follows: each alternative placed in the position from 1 to j receives one vote, remaining alternatives obtain no votes.

The simplex $P^n$ is defined as follows

$$P^n: w^n = \sum_{j=1}^{n-1} \lambda_j v_j^n, \qquad \lambda_j \geq 0, \quad \sum_{j=1}^{n-1} \lambda_j = 1. \tag{3}$$

In the analysis of positional methods, the Borda method is considered as the reference one. The voting vector for this method, denoted as $b^n$, is as follows

$$b^n = \frac{1}{n-1} \sum_{j=1}^{n-1} v_j^n . \tag{4}$$

From (3) and (4) it can be noticed that $\lambda_j = \dfrac{1}{n-1}$, j=1, ..., n-1.

The $b^n$ vector is the barycenter of the simplex $P^n$ (Sikorski, 1969). This statement makes it possible to introduce the orthogonal coordinate system with the origin in the barycenter (Saari, 2000b)

$$w^n = b^n + \sum_{j=2}^{n-1} \alpha_j E_j^n \tag{5}$$

where only one component of the vector $E_j^n$ is not equal 0, namely the j-th one is equal 1.

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[1) Components of the normalised voting vector take values from the [0,1] interval.

## 3. The analysis of four alternatives case

As it was mentioned earlier, it is assumed that expert judgements have the form of preference orders and there are no equivalent alternatives.

To show the practical usefulness of approach proposed the example of four alternatives $O = \{O_1, O_2, O_3, O_4\}$ will be considered in detail.

Given a preference order of four alternatives

$$P_4^i : O_{i_1} \succ O_{i_2} \succ O_{i_3} \succ O_{i_4} \tag{6}$$

The subscript denotes the number of alternatives taken into account, the superscript points a particular preference order considered.

The preference order $P_4^i$ can be decomposed into $\binom{4}{3} = 4$ preference orders of three alternatives:

$$\begin{aligned} &P_3^{1i} : O_{i_1} \succ O_{i_2} \succ O_{i_3} \\ &P_3^{2i} : O_{i_1} \succ O_{i_2} \succ O_{i_4} \\ &P_3^{3i} : O_{i_1} \succ O_{i_3} \succ O_{i_4} \\ &P_3^{4i} : O_{i_2} \succ O_{i_3} \succ O_{i_4} \end{aligned} \tag{7}$$

Let us assume now, according to (2) that the voting vector $v_1^3$ is of the form $v_1^3 = (1, 0, 0)$. It follows from Definition 1 that the number of votes a particular alternative receives is as given in Table 1.

**Table 1**

| Preference order \ Alternative | $O_{i_1}$ | $O_{i_2}$ | $O_{i_3}$ | $O_{i_4}$ |
|---|---|---|---|---|
| $P_3^{1i}: O_{i_1} \succ O_{i_2} \succ O_{i_3}$ | 1 | 0 | 0 | 0 |
| $P_3^{2i}: O_{i_1} \succ O_{i_2} \succ O_{i_4}$ | 1 | 0 | 0 | 0 |
| $P_3^{3i}: O_{i_1} \succ O_{i_3} \succ O_{i_4}$ | 1 | 0 | 0 | 0 |
| $P_3^{4i}: O_{i_2} \succ O_{i_3} \succ O_{i_4}$ | 0 | 1 | 0 | 0 |
| Total | **3** | **1** | **0** | **0** |

If we assume that the voting vector $v_2^3$ is of the form $v_2^3 = (1, 1, 0)$, the number of votes a particular alternative receives is given in Table 2.

**Table 2**

| Preference order \ Alternative | $O_{i_1}$ | $O_{i_2}$ | $O_{i_3}$ | $O_{i_4}$ |
|---|---|---|---|---|
| $P_3^{1i}: O_{i_1} \succ O_{i_2} \succ O_{i_3}$ | 1 | 1 | 0 | 0 |
| $P_3^{2i}: O_{i_1} \succ O_{i_2} \succ O_{i_4}$ | 1 | 1 | 0 | 0 |
| $P_3^{3i}: O_{i_1} \succ O_{i_3} \succ O_{i_4}$ | 1 | 0 | 1 | 0 |
| $P_3^{4i}: O_{i_2} \succ O_{i_3} \succ O_{i_4}$ | 0 | 1 | 1 | 0 |
| Total | **3** | **3** | **2** | **0** |

Under the assumptions given, four alternatives may be ordered in 4!=24 ways. Generally, these preference orders are partitioned into four groups with respect to an alternative placed in the first position (Bury, Wagner, 2004).

Making use of data from Tables 1 and 2 one can determined for each alternative under consideration the number of votes it receives for voting vectors $v_1^3$ and $v_2^3$ and each of 24 preference orders. They are given in Table 3. It should be noticed that Saari (Saari, 2000a, b) numbered the preference orders analyzed in another way, according to geometric interpretation of the problem. In the first left column of Table 3 Saari's enumeration is given.

**Table 3**

| Preference order | | | | | | | | | Number of votes | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | $v_1^3 = (1,0,0)$ | | | | $v_2^3 = (1,1,0)$ | | | |
| | | | | | | | | | $O_1$ | $O_2$ | $O_3$ | $O_4$ | $O_1$ | $O_2$ | $O_3$ | $O_4$ |
| $[P^1]$ | $P^1$: | $O_1$ | > | $O_2$ | > | $O_3$ | > | $O_4$ | 3 | 1 | 0 | 0 | 3 | 3 | 2 | 0 |
| $[P^{24}]$ | $P^2$: | $O_1$ | > | $O_2$ | > | $O_4$ | > | $O_3$ | 3 | 1 | 0 | 0 | 3 | 3 | 0 | 2 |
| $[P^{23}]$ | $P^3$: | $O_1$ | > | $O_4$ | > | $O_2$ | > | $O_3$ | 3 | 0 | 0 | 1 | 3 | 2 | 0 | 3 |
| $[P^8]$ | $P^4$: | $O_1$ | > | $O_4$ | > | $O_3$ | > | $O_2$ | 3 | 0 | 0 | 1 | 3 | 0 | 2 | 3 |
| $[P^7]$ | $P^5$: | $O_1$ | > | $O_3$ | > | $O_4$ | > | $O_2$ | 3 | 0 | 1 | 0 | 3 | 0 | 3 | 2 |
| $[P^2]$ | $P^6$: | $O_1$ | > | $O_3$ | > | $O_2$ | > | $O_4$ | 3 | 0 | 1 | 0 | 3 | 2 | 3 | 0 |
| $[P^6]$ | $P^7$: | $O_2$ | > | $O_1$ | > | $O_3$ | > | $O_4$ | 1 | 3 | 0 | 0 | 3 | 3 | 2 | 0 |
| $[P^{19}]$ | $P^8$: | $O_2$ | > | $O_1$ | > | $O_4$ | > | $O_3$ | 1 | 3 | 0 | 0 | 3 | 3 | 0 | 2 |
| $[P^{20}]$ | $P^9$: | $O_2$ | > | $O_4$ | > | $O_1$ | > | $O_3$ | 0 | 3 | 0 | 1 | 2 | 3 | 0 | 3 |
| $[P^{17}]$ | $P^{10}$: | $O_2$ | > | $O_4$ | > | $O_3$ | > | $O_1$ | 0 | 3 | 0 | 1 | 0 | 3 | 2 | 3 |
| $[P^{18}]$ | $P^{11}$: | $O_2$ | > | $O_3$ | > | $O_4$ | > | $O_1$ | 0 | 3 | 1 | 0 | 0 | 3 | 3 | 2 |
| $[P^5]$ | $P^{12}$: | $O_2$ | > | $O_3$ | > | $O_1$ | > | $O_4$ | 0 | 3 | 1 | 0 | 2 | 3 | 3 | 0 |
| $[P^3]$ | $P^{13}$: | $O_3$ | > | $O_1$ | > | $O_2$ | > | $O_4$ | 1 | 0 | 3 | 0 | 3 | 2 | 3 | 0 |
| $[P^{12}]$ | $P^{14}$: | $O_3$ | > | $O_1$ | > | $O_4$ | > | $O_2$ | 1 | 0 | 3 | 0 | 3 | 0 | 3 | 2 |
| $[P^{11}]$ | $P^{15}$: | $O_3$ | > | $O_4$ | > | $O_1$ | > | $O_2$ | 0 | 0 | 3 | 1 | 2 | 0 | 3 | 3 |
| $[P^{14}]$ | $P^{16}$: | $O_3$ | > | $O_4$ | > | $O_2$ | > | $O_1$ | 0 | 0 | 3 | 1 | 0 | 2 | 3 | 3 |
| $[P^{13}]$ | $P^{17}$: | $O_3$ | > | $O_2$ | > | $O_4$ | > | $O_1$ | 0 | 1 | 3 | 0 | 0 | 3 | 3 | 2 |
| $[P^4]$ | $P^{18}$: | $O_3$ | > | $O_2$ | > | $O_1$ | > | $O_4$ | 0 | 1 | 3 | 0 | 2 | 3 | 3 | 0 |
| $[P^{22}]$ | $P^{19}$: | $O_4$ | > | $O_1$ | > | $O_2$ | > | $O_3$ | 1 | 0 | 0 | 3 | 3 | 2 | 0 | 3 |
| $[P^9]$ | $P^{20}$: | $O_4$ | > | $O_1$ | > | $O_3$ | > | $O_2$ | 1 | 0 | 0 | 3 | 3 | 0 | 2 | 3 |
| $[P^{10}]$ | $P^{21}$: | $O_4$ | > | $O_3$ | > | $O_1$ | > | $O_2$ | 0 | 0 | 1 | 3 | 2 | 0 | 3 | 3 |
| $[P^{15}]$ | $P^{22}$: | $O_4$ | > | $O_3$ | > | $O_2$ | > | $O_1$ | 0 | 0 | 1 | 3 | 0 | 2 | 3 | 3 |
| $[P^{16}]$ | $P^{23}$: | $O_4$ | > | $O_2$ | > | $O_3$ | > | $O_1$ | 0 | 1 | 0 | 3 | 0 | 3 | 2 | 3 |
| $[P^{21}]$ | $P^{24}$: | $O_4$ | > | $O_2$ | > | $O_1$ | > | $O_3$ | 0 | 1 | 0 | 3 | 2 | 3 | 0 | 3 |

This table makes it possible to determine directly the number of votes a particular alternative receives for experts' judgements given.

**Example 1** (Saari, 2000b)
Ten experts are asked to order the set of four alternatives. Their preference orders are given in Table 4.

**Table 4**

| Number of experts | Preference order |
|---|---|
| 2 | $O_1 \succ O_2 \succ O_3 \succ O_4$ |
| 1 | $O_1 \succ O_3 \succ O_4 \succ O_2$ |
| 2 | $O_1 \succ O_4 \succ O_3 \succ O_2$ |
| 2 | $O_3 \succ O_2 \succ O_4 \succ O_1$ |
| 3 | $O_4 \succ O_2 \succ O_3 \succ O_1$ |

For the preference orders given in Table 4 one can use results of Table 3. They are as follows.

**Table 5**

| Preference order | | | | | | | | Number of votes | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | $v_1^3 = (1, 0, 0)$ | | | | $v_2^3 = (1, 1, 0)$ | | | |
| | | | | | | | | $O_1$ | $O_2$ | $O_3$ | $O_4$ | $O_1$ | $O_2$ | $O_3$ | $O_4$ |
| $P^1$: | $O_1$ | > | $O_2$ | > | $O_3$ | > | $O_4$ | 3 | 1 | 0 | 0 | 3 | 3 | 2 | 0 |
| $P^4$: | $O_1$ | > | $O_4$ | > | $O_3$ | > | $O_2$ | 3 | 0 | 0 | 1 | 3 | 0 | 2 | 3 |
| $P^5$: | $O_1$ | > | $O_3$ | > | $O_4$ | > | $O_2$ | 3 | 0 | 1 | 0 | 3 | 0 | 3 | 2 |
| $P^{17}$: | $O_3$ | > | $O_2$ | > | $O_4$ | > | $O_1$ | 0 | 1 | 3 | 0 | 0 | 3 | 3 | 2 |
| $P^{23}$: | $O_4$ | > | $O_2$ | > | $O_3$ | > | $O_1$ | 0 | 1 | 0 | 3 | 0 | 3 | 2 | 3 |

Making use of data given in Tables 3 and 4, it is possible to determine the number of votes assigned to each of the alternatives investigated.

**Table 6**[1)]

| No. of experts | Preference order | | | | | | | | Number of votes | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | $v_1^3 = (1, 0, 0)$ | | | | $v_2^3 = (1, 1, 0)$ | | | |
| | | | | | | | | | $O_1$ | $O_2$ | $O_3$ | $O_4$ | $O_1$ | $O_2$ | $O_3$ | $O_4$ |
| 2 | $P^1$: | $O_1$ | > | $O_2$ | > | $O_3$ | > | $O_4$ | 6 | 2 | 0 | 0 | 6 | 6 | 4 | 0 |
| 2 | $P^4$: | $O_1$ | > | $O_4$ | > | $O_3$ | > | $O_2$ | 6 | 0 | 0 | 2 | 6 | 0 | 4 | 6 |
| 1 | $P^5$: | $O_1$ | > | $O_3$ | > | $O_4$ | > | $O_2$ | 3 | 0 | 1 | 0 | 3 | 0 | 3 | 2 |
| 2 | $P^{17}$: | $O_3$ | > | $O_2$ | > | $O_4$ | > | $O_1$ | 0 | 2 | 6 | 0 | 0 | 6 | 6 | 4 |
| 3 | $P^{23}$: | $O_4$ | > | $O_2$ | > | $O_3$ | > | $O_1$ | 0 | 3 | 0 | 9 | 0 | 9 | 6 | 9 |
| Total | | | | | | | | | **15** | **7** | **7** | **11** | **15** | **21** | **23** | **21** |

[1)] The numbers of votes given in this table are those of Table 5 multiplied by the number of xperts.

It should be noted that the voting vector $v_1^3$ defines the plurality winner and the voting vector $v_2^3$ determines the antiplurality winner. For both cases the winning alternative is $O_1$.

## 4. The analysis of n alternatives case

The approach presented above for the case of 4 alternatives can be easily generalized.
Given a preference order of n alternatives, n≥3

$$P_n^i : O_{i_1} \succ O_{i_2} \succ \ldots \succ O_{i_{n-1}} \succ O_{i_n} \tag{8}$$

This preference order can be written as n preference orders of (n-1) alternatives:

$$\begin{array}{lcccccccc} P_{n-1}^{1i} & : O_{i_1} & \succ & O_{i_2} & \succ & \cdots & \succ & O_{i_{n-1}} \\ P_{n-1}^{2i} & : O_{i_1} & \succ & O_{i_2} & \succ & \cdots & \succ & O_{i_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \succ & O_{i_n} \\ P_{n-1}^{(n-1)i} & : O_{i_1} & \succ & O_{i_3} & \succ & \cdots & \ddots & O_{i_n} \\ P_{n-1}^{ni} & : O_{i_2} & \succ & O_{i_3} & \succ & \cdots & \succ & O_{i_n} \end{array} \tag{9}$$

For the case of (n-1) alternatives the voting vectors $v_j^{n-1}$ are of the form

$$v_j^{n-1} = \{ \underbrace{1}_{1^{st}\ position}, \ldots, \underbrace{1}_{j^{th}\ position}, 0, \ldots, \underbrace{0}_{(n-1)^{th}\ position} \}. \tag{10}$$

The analysis of the preference orders (9) makes it possible to determine the number of votes a particular alternative receives in the case of considering the whole set of n alternatives. The numbers obtained are given in Table 7.

**Table 7**

| Voting vector \ Alternatives | $O_{i_1}$ | $O_{i_2}$ | $O_{i_3}$ | $O_{i_4}$ | … | $O_{i_{n-2}}$ | $O_{i_{n-1}}$ | $O_{i_n}$ |
|---|---|---|---|---|---|---|---|---|
| $v_1^{n-1} = \{1,0,\ldots,0\}$ | (n-1) | 1 | 0 | 0 | … | 0 | 0 | 0 |
| $v_2^{n-1} = \{1,1,0,\ldots,0\}$ | (n-1) | (n-1) | 2 | 0 | … | 0 | 0 | 0 |
| $v_3^{n-1} = \{1,1,1,0,\ldots,0\}$ | (n-1) | (n-1) | (n-1) | 3 | … | 0 | 0 | 0 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ | ⋮ | ⋮ | ⋮ |
| $v_{n-2}^{n-1} = \{1,1,\ldots,1,0\}$ | (n-1) | (n-1) | (n-1) | (n-1) | | (n-1) | (n-2) | 0 |

It is worth to note that for the voting vector $v_j^{n-1}$, j=1,...,n-2, the number of votes assigned to alternatives $O_{i_1}, O_{i_2}, \ldots, O_{i_j}$ is equal (n-1), alternative $O_{i_{j+1}}$ receives j votes and alternatives $O_{i_{j+2}}, \ldots, O_{i_n}$ obtain no votes.

The components $\ell_{i_s}^j$ (s=1, ..., n) of the voting vector $u_j^n$ for n alternatives are as follows

$$\frac{1}{(n-1)}\ell_{i_s}^j = \begin{cases} 1 & s = 1,\ldots,j \\ \dfrac{j}{(n-1)} & s = j+1 \\ 0 & s = j+2,\ldots,n \end{cases} \tag{11}$$

where $\ell_{i_s}^j$ is the number of votes the alternative $O_{i_s}$ receives in the case when the voting vector is of the form $v_j^{n-1}$.

This formula without proof and interpretation is given in the paper (Saari 2000b). For the case considered in the previous Section this formula takes the form

$$\frac{1}{3}\ell_{i_s}^1 = \begin{cases} 1 & s = 1 \\ \dfrac{1}{3} & s = 2 \\ 0 & s = 3,4 \end{cases} \qquad \frac{1}{3}\ell_{i_s}^2 = \begin{cases} 1 & s = 1,2 \\ \dfrac{2}{3} & s = 3 \\ 0 & s = 4 \end{cases} \tag{12}$$

For Example 1 the number of votes assigned to a particular alternative - determined with the use of (12) - is given in Table 8.

**Table 8**

| No. of experts | Preference order | | | | | | | | Number of votes: $u_1^4$ $O_1$ | $u_1^4$ $O_2$ | $u_1^4$ $O_3$ | $u_1^4$ $O_4$ | $u_2^4$ $O_1$ | $u_2^4$ $O_2$ | $u_2^4$ $O_3$ | $u_2^4$ $O_4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | $P^1$ | $O_1$ | > | $O_2$ | > | $O_3$ | > | $O_4$ | 6 | 2 | 0 | 0 | 6 | 6 | 4 | 0 |
| 2 | $P^4$ | $O_1$ | > | $O_4$ | > | $O_3$ | > | $O_2$ | 6 | 0 | 0 | 2 | 6 | 0 | 4 | 6 |
| 1 | $P^5$: | $O_1$ | > | $O_3$ | > | $O_4$ | > | $O_2$ | 3 | 0 | 1 | 0 | 3 | 0 | 3 | 2 |
| 2 | $P^{17}$ | $O_3$ | > | $O_2$ | > | $O_4$ | > | $O_1$ | 0 | 2 | 6 | 0 | 0 | 6 | 6 | 4 |
| 3 | $P^{23}$ | $O_4$ | > | $O_2$ | > | $O_3$ | > | $O_1$ | 0 | 3 | 0 | 9 | 0 | 9 | 6 | 9 |
| | | | | | | | | $\frac{1}{3}\ell_{i_s}^j$ | **5** | $\mathbf{\frac{7}{3}}$ | $\mathbf{\frac{7}{3}}$ | $\mathbf{\frac{11}{3}}$ | **5** | **7** | $\mathbf{\frac{23}{3}}$ | **7** |
| | | | | | | | Total | $\ell_{i_s}^j$ | **15** | **7** | **7** | **11** | **15** | **21** | **23** | **21** |

Results obtained are the same as in Table 6.

## 5. The departure vector

**Definition 2** (Saari 2000b)
The derived set $D^k$ of voting vectors related to the simplex $P^k$ (3) consists of all voting vectors that can be expressed in the form of (3).
The derived set $D^k$ is spanned by vectors $u_j^k$, j=1, ..., k-2. The dimension of $D^k$ is (k-3). The $b^k$ vector, given by (4), belongs to that space. Saari (Saari, 2000b) formulated the following theorem.

**Theorem** (Saari, 2000b).
A vector $d^k$, called the departure vector, that is normal to the space $D^k$ is as follows

$$d^k = \left[0, \binom{k-1}{1}, -\binom{k-1}{2}, \dots, (-1)^{k-1}\binom{k-1}{k-2}, 0\right] \tag{13}$$

The proof of the above, given in (Saari, 2000b), may be simplified due to a known combinatorial theorem (Gradsztejn, Ryżik, 1963)

$$\sum_{z=0}^{a}(-1)^z\binom{m}{z} = (-1)^a\binom{m-1}{a}, \quad a \le m-1 \tag{14}$$

To show, that the given vector $d^k$ (13) is normal to the space $D^k$, it sufficient to prove that it is orthogonal to the vector $(u_j^k - b^k)$, i.e. that the scalar product of these vectors is equal zero (Borsuk, 1976)

$$(u_j^k - b^k)\cdot d^k = 0 \tag{15}$$

From properties of the scalar product it follows that

$$(u_j^k - b^k)d^k = u_j^k d^k - b^k d^k \tag{16}$$

Applying formula (13) that defines the $d^k$ vector, one can write its components, except for the first and the last ones, as follows

$$d_s^k = (-1)^s\binom{k-1}{s-1}, \text{ s=2, ..., k-1.} \tag{17}$$

Taking into account the form of $u_j^k$ (11), the scalar product $u_j^k \cdot d^k$ can be written in the following way

$$u_j^k \cdot d^k = \sum_{s=2}^{j}(-1)^s\binom{k-1}{s-1} + \frac{j}{(k-1)}(-1)^{j+1}\binom{k-1}{j} \tag{18}$$

However

$$\frac{j}{(k-1)}(-1)^{j+1}\binom{k-1}{j}=\frac{j}{(k-1)}(-1)^{j+1}\frac{(k-1)!}{j!(k-1-j)!}=$$
$$(-1)^{j+1}\frac{(k-2)!}{(j-1)![k-2-(j-1)]!}=(-1)^{j+1}\binom{k-2}{j-1} \tag{19}$$

Making use of theorem (14) and assuming (s-1) = z, one has

$$\sum_{s=2}^{j}(-1)^{s}\binom{k-1}{s-1}=\sum_{z=1}^{j-1}(-1)^{z+1}\binom{k-1}{z}=-\sum_{z=1}^{j-1}(-1)^{z}\binom{k-1}{z}=$$
$$(-1)^{0}\binom{k-1}{0}-\sum_{z=0}^{j-1}(-1)^{z}\binom{k-1}{z}=-(-1)^{(j-1)}\binom{k-2}{j-1}+1 \tag{20}$$

Summing expressions (19) and (20) one obtains

$$u_j^k\cdot d^k=1-(-1)^{j-1}\binom{k-2}{j-1}+(-1)^{j}\binom{k-2}{j-1}=1+\binom{k-2}{j-1}\left[(-1)^{j-1}-(-1)^{j+1}\right]=1 \tag{21}$$

Vector $b^k$ is as follows

$$\left(1,\frac{k-2}{k-1},\ldots,\frac{1}{k-1},0\right) \tag{22}$$

so its components can be written as

$$b_s^k=\frac{k-s}{k-1}\quad s=2,\ldots,k \tag{23}$$

Taking into account (17) and (23) the scalar product $b^k d^k$ may be written as follows

$$b^k d^k=\sum_{s=2}^{k-1}\left(\frac{k-s}{k-1}\right)(-1)^{s}\binom{k-1}{s-1}=\sum_{s=2}^{k-1}(-1)^{s}\binom{k-2}{s-1}=$$
$$\sum_{s=2}^{k-2}(-1)^{s}\binom{k-2}{s-1}+(-1)^{k-1}\binom{k-2}{k-2}=\sum_{s=2}^{k-2}(-1)^{s}\binom{k-2}{s-1}-(-1)^{k} \tag{24}$$

Using (14) one can rewrite (24) in the form

$$b^k\cdot d^k=1-(-1)^{k}+(-1)^{k-3}\binom{k-3}{j-3}=1+(-1)^{k}\left[-1-(-1)^{j-3}\right]=1 \tag{25}$$

Finally

$$(u_j^k - b^k)\cdot d^k = u_j^k \cdot d^k - b^k \cdot d^k = 0 \qquad (26)$$

*q.e.d.*

Vector $d^k$ (13) is used to construct so called departure profile.

**Definition 3**

The departure profile defined for the alternative $O_i$, $D_i^k$ assigns $(-1)^s \binom{n-1}{s-1}$ votes to preference orders in which the alternative $O_i$ takes the s-th position.
It can be shown (Saari, 2000b) that

$$\sum_{i=1}^{k} D_i^k = 0 \qquad (27)$$

The profile $D_i^k$ and its modification plays an important role in the analysis of the results of an expertise when the method of decomposition of the profiles is applied.

## 6. Final remarks

The paper discusses a method of deriving the voting vector for n alternatives in the case when the voting vector is known for the set of (n-1) alternatives. This approach may be generalized, i.e. one can apply the result obtained for $n_1$ alternatives to get the result for $(n_1+1)$ alternatives and again to apply the latter to obtain result for $(n_2+1)$ alternatives and so on.

Such approach is effort consuming. However, it makes possible to carefully analyze the process of constructing voting vectors for sets with larger and larger number of alternatives. Moreover, it enables one to disclose reasons of paradox occurrence.

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**ZASTOSOWANIA INFORMATYKI**
**W NAUCE, TECHNICE I ZARZĄDZANIU**

Monografia zawiera wybór artykułów dotyczących informatyzacji procesów zarządzania, prezentując bieżący stan rozwoju informatyki stosowanej w Polsce i na świecie. Zamieszczone artykuły opisują metody, algorytmy i techniki obliczeniowe stosowane do rozwiązywania złożonych problemów zarządzania, a także omawiają konkretne zastosowania informatyki w różnych sektorach gospodarki. Kilka prac przedstawia wyniki projektów badawczych Ministerstwa Nauki i Informatyzacji, dotyczących rozwoju metod informatycznych i ich zastosowań.

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