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# Stability Analysis for Parametric Vector Optimization Problems

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# Preface

We study stability of minimal points and solutions to parametric (or perturbed) vector optimization problems in the framework of real topological vector spaces and, if necessary, normed spaces. Because of particular importance of finite-dimensional problems, called multicriteria optimization problems, which model various real-life phenomena, a special attention is paid to the finite-dimensional case. Since one can hardly expect the sets of minimal points and solutions to be singletons, set-valued mappings are natural tools for our studies.

Vector optimization problems can be stated as follows. Let $X$ be a topological space and let $Y$ be a topological vector space ordered by a closed convex pointed cone $\mathcal{K} \subset Y$. Vector optimization problem

$$\begin{array}{ll} \mathcal{K} - \min \ f_0(x) & \\ \text{subject to } x \in A_0\,, & \qquad (P_0) \end{array}$$

where $f : X \to Y$ is a mapping, and $A_0 \subset X$ is a subset of $X$, relies on finding the set $\mathrm{Min}(f_0, A_0, \mathcal{K}) = \{y \in f_0(A_0) \mid f_0(A_0) \cap (y - \mathcal{K}) = \{y\}\}$ called the **Pareto** or **minimal point** set of $(P_0)$, and the **solution set** $S(f_0, A_0, \mathcal{K}) = \{x \in A_0 \mid f_0(x) \in \mathrm{Min}(f_0, A_0, \mathcal{K})\}$. We often refer to problem $(P_0)$ as the **original problem** or **unperturbed one**. The space $X$ is the **argument** space and $Y$ is the **outcome** space.

Let $U$ be a topological space. We embed the problem $(P_0)$ into a family $(P_u)$ of vector optimization problems parametrised by a parameter $u \in U$,

$$\begin{array}{ll} \mathcal{K} - \min \ f(u,x) & \\ \text{subject to } x \in A(u)\,, & \qquad (P_u) \end{array}$$

where $f : U \times X \to Y$ is the parametrised objective function and $A : U \rightrightarrows Y$, is the feasible set multifunction, $(P_0)$ corresponds to a parameter value $u_0$. The performance multifunction $\mathcal{M} : U \rightrightarrows Y$, is defined as $\mathcal{M}(u) = \text{Min}(f(u,\cdot), A(u), \mathcal{K})$, and the solution multifunction $\mathcal{S} : U \rightrightarrows Y$, is given as $\mathcal{S}(u) = S(f(u,\cdot), A(u), \mathcal{K})$, and $f : U \times X \to Y$, $A(u) \subset X$.

Our aim is to study continuity properties of $\mathcal{M}$ and $\mathcal{S}$ as functions of the parameter $u$. Continuous behaviour of solutions as functions of parameters is of crucial importance in many aspects of the theory of vector optimization as well as in applications(correct formulation of the model and/or approximation) and numerical solution of the problem in question.
We investigate continuity in the sense of Hausdorff and Hőlder of the multivalued mappings of minimal points $\mathcal{M}(u)$ and solutions $\mathcal{S}(u)$ as functions of the parameter $u$ under possibly weak assumptions. We attempt to avoid as much as possible compactness assumptions which are frequently over-used (see eg [83]).
It is a specific feature of vector optimization that the outcome space is equipped with a partial order generated by a cone the properties of which are important for stability analysis. In many spaces cones of nonnegative elements have empty interiors and because of this we derive stability results for cones with possibly empty interior. This kind of results are specific for vector optimization and do not have their counterpart in scalar optimization.
We introduce two new concepts: the notion of containment(with some variants for cones with empty interiors),[16], and the notion of strict minimality, [12].

The containment property $(CP)$, defined in topological vector spaces, is introduced to study upper semicontinuities (in the sense of Hausdorff) of minimal points, [11, 16]. It is a variant of the domination property $(DP)$, which appears frequently in the context of stability of solutions to parametric vector optimization problems. Although it is not a commonly adopted view point, the domination property may be accepted as a solution concept which generalizes the standard concept of a solution to scalar optimization problem. In consequence, the containment property $(CP)$ may also be seen as a solution concept in vector optimization. To investigate more deeply this aspect we interpret the containment property as a generalization of the concept of the set of $\phi-$local solutions appearing in the

context of Lipschitz continuity of solutions to scalar optimization problems. Under mild assumptions the containment property imply that the set weakly minimal points equals the set of minimal points. This equality, in turn, is a typical ingradient of standard finite-dimensional sufficient conditions for upper semicontinuity of minimal points.
To study Hőlder upper continuity of minimal points we define the rate of containment of a set with respect to a cone, which is a real-valued function of a scalar argument, see [14, 15]. The rate of growth of this function influence decisively the rate of Hőlder continuity of minimal points, [15].

Strictly minimal points are introduced to study lower semicontinuities (lower Hausdorff, lower Hőlder) of minimal points [20, 13]. The definition of a strictly minimal point is given in topological vector spaces and it is a generalization of the notion of a super efficient point in the sense of Borwein and Zhuang defined in normed spaces. We discuss strict minimality in vector optimization by proving that it is a vector counterpart of the concept of $\phi-$ local solution to scalar optimization problem.

Theory of vector optimization may be considered as an abstract study of optimization problems with mappings taking values in the outcome space equipped with a partial order structure. As such, it contains many concepts and results which generalize and/or have their counterparts in scalar optimization. The very definition of the set of minimal points of vector optimization problem in the outcome space may serve as an example here. This is a counterpart of the optimal value of scalar optimization problem. Another example is the concept of well-posed optimization problem. In subsequent developments we often compare our results and considerations with the corresponding approaches in scalar optimization. For instance, we define several classes of well-posed vector optimization problems by generalizing the concept of scalar minimizing sequence and in these classes we investigate continuity of solutions. For scalar optimization problems, the existing approaches and results on well-posedness are extensively discussed in the monograph by Dontchev and Zolezzi [33].

Convergence and rates of convergence of solutions to perturbed optimization problems is one of crucial topics of stability analysis in optimization both from theoretical and numerical points of view. For scalar optimization it was investigated by many authors see eg., [72], [32], [47], [78], [55], [81], [59], [60], [82], [2], and many others. An exhaustive survey of current state of research is given in the recent monograph by Bonnans and Shapiro [26]. In vector optimization the results on Lipschitz continuity of solutions are not so numerous, and concern some classes of problems, for linear case see eg.,[28], [29], [30], for convex case see eg., [25], [31].

The organization of the material is as follows. In Chapter 2 we investigate upper Hausdorff continuity of the multivalued mapping $M$, $M(u) = \mathrm{Min}(\Gamma(u)|\mathcal{K})$ assigning to a given parameter value $u$ from a topological space $U$ the set of minimal points of the set $\Gamma(u) \subset Y$ with respect to cone $\mathcal{K} \subset Y$, where for any subset $A$ of a topological vector space $Y$ the set of minimal points is defined as $\mathrm{Min}(A|\mathcal{K}) = \{y \in A \mid A \cap (y - \mathcal{K}) = \{y\}\}$, and $\Gamma : U \rightrightarrows Y$, is a given multivalued mapping. The main tool which allows us to obtain the general result is the containment property $(CP)$. Some infinite-dimensional examples are discussed. A special attention is paid to the containment property $(CP)$ in finite-dimensional case, when $Y = R^m$.

In Chapter 3 we discuss upper Hőlder continuity of the minimal point multivalued mapping $M$. To this aim we introduce the rate of containment $\delta$ which is a one-variable nondecreasing function, defined for a given set $A$ and the order generating cone $\mathcal{K}$. The assumption of sufficiently fast growth rate of this function appears to be the crucial assumption for all upper Hőlder stability results of Chapter 3.

In Chapter 4 we apply the results obtained in Chapters 2 and 3 to derive conditions for upper Hausdorff and upper Hőlder stability of minimal points to parametric vector optimization problems by taking $\Gamma(u) = f(u, A(u))$. Moreover, we introduce the concept of $\Phi-$ strong solutions to vector optimization problem $(P_0)$, which is a generalization of the concept of a $\phi-$local minimizer to scalar optimization problem, the latter being introduced by Attouch and

Wets [6].

In Chapter 5 we investigate the lower continuity and lower Hőlder continuity of the minimal point multivalued mapping $M$. To this aim we introduce the notion of strict minimality mentioned above and the rate of strict minimality. In Section 5.5 we apply the results obtained in Chapter 5 to parametric vector optimization problems and we derive sufficient conditions for lower and lower Hőlder continuity of Pareto point multivalued mapping $\mathcal{M}$. An important tool here is the notion of $\Phi-$ strict solution to vector optimization problem introduced in Section 6.1. This notion can be interpreted as another possible generalization of the concept of $\phi-$local minimizer.

In Chapter 6 we propose several definitions of a well-posed vector optimization problem. All these definitions are based on properties of $\varepsilon-$solutions to vector optimization problems. For well-posed vector optimization problems we prove upper Hausdorff continuity of solution multivalued mapping $S$, $S(u) = S(f(u,\cdot), A(u), \mathcal{K})$.

### 6.4 Continuity of solutions to vector optimization problems

## 7 Stability results

Now we prove the following stability results.

**Theorem 7.0.1** *Let $X$, $Y$, $U$ , $Y$ , and $\mathcal{K}$ be as in Lemma 6.1, . Assume that $u_0 \in U$ and*

*(i) $\mathcal{A}$ is u.s.c. at $u_0$ and $\mathcal{F}\mathcal{A}$ is sup-lower continuous at $(x, u_0)$ for $x \in M(f, A)$ uniformly on $M(f, A)$, $f$ is $\mathcal{K}$-lower continuous on $X$,*

*(ii) $(P)$ is well-posed in the sense of Definition 6.2.2,*

*(iii) the set $f(A)$ has the containment property, and the domination property,*

*(iv) $M(f, A) = WM(f, A)$.*

*Then $\mathcal{S}$ is u.s.c. at $u_0$.*

**Proof.** Suppose that $\mathcal{S}$ is not u.s.c. at $u_0$. This means that there exist a closed subset $F \subset X$, $F \cap \mathcal{S}(u_0) = \emptyset$, and a sequence $(u_k)$, $lim_k u_k = u_0$ such that $F \cap \mathcal{S}(u_k) \neq \emptyset$. Hence, there exists a sequence $(x_k)$, $x_k \in F \cap \mathcal{S}(u_k)$, $k = 1, 2,$ .

Two situations are possible now:

(1) there exists a subsequence $(x_{k_m})$ of the sequence $(x_k)$ such that $x_{k_m} \in \mathcal{A}(u_{k_m}) \setminus \mathcal{A}(u_0)$, $m = 1, 2, ....$

(2) for all $k$ sufficiently large $x_k \in \mathcal{A}(u_k) \cap \mathcal{R}(u_0)$.

Consider the case (1) and denote $z_m = x_{k_m}$, $m = 1, 2, ....$ If $(z_m)$ does not contain any convergent subsequence or contains a subsequence with a limit point $x_0$ not belonging to $\mathcal{A}(u_0)$, then $\mathcal{A}$ is not u.s.c. at $u_0$ which contradicts $(i)$. Hence, suppose that $(z_m)$ contains a convergent subsequence with a limit point $z_0 \in \mathcal{A}(u_0)$. Without loss of generality we can assume that $lim_m z_m = z_0$. By Lemma 6.1, $z_0$ must be in $\mathcal{S}(u_0)$, which contradicts the assumption that $F \cap \mathcal{S}(u_0) = \emptyset$.

Consider now the case (2), ie. for all $k$ sufficiently large $x_k \in \mathcal{A}(u_k) \cap \mathcal{A}(u_0)$. Hence, we have

$$f(x_k) \in \mathcal{F}\mathcal{A}(u_k) \cap \mathcal{F}\mathcal{A}(u_0).$$

for all $k$ sufficiently large. By Lemma 6.2, the sequence $(x_k)$ must be a minimizing sequence. By $(ii)$, and Proposition 6.2.1, $(x_k)$ contains a convergent subsequence with a limit point belonging to $S(f, A)$. This, however, contradicts the assumption that $S(f, A) \cap F = \emptyset$.

□

Since sup-lower continuity is implied by the lower semicontinuity and the uniformity is achieved on compact subsets we obtain the following variant of Theorem 7.0.1

**Theorem 7.0.2** *Let $X$, $Y$, $U$ and $\mathcal{K}$ be as in Lemma 6.1. Assume that*

*$(i)$ $\mathcal{A}$ is u.s.c. and l.s.c. at $u_0$, and $f$ is continuous on $X$,*

*$(ii)$ $(P)$ is well-posed in the sense of Definition 6.2.2,*

*$(iii)$ the set $f(A)$ has the containment property, and the domination property,*

*$(iv)$ $M(f, A) = WM(f, A)$,*

*$(v)$ the set $M(f, A)$ is compact.*

*Then the Pareto solution multifunction $\mathcal{S}$ is u.s.c. at $u_0$.*

**Theorem 7.0.3** *Let $X$, $Y$, $U$ and $\mathcal{K}$ be as in Lemma 6.1.*

*Assume that*

*$(i)$ $\mathcal{FA}$ is u.H.c. at $u_0$ and sup-lower continuous at $(x, u_0)$, $x \in M(f, A)$ uniformly on $M(f, A)$, $f$ is $\mathcal{K}$-lower continuous on $X$, uniformly on $A$,*

*$(ii)$ $(P)$ is well-posed in the sense of Definition 6.2.4,*

*$(iii)$ the set $f(A)$ has the containment property, and the domination property,*

*$(iv)$ $M(f, A) = WM(f, A)$.*

*Then $\mathcal{S}$ is u.H.c. at $u_0$.*

**Proof.** Suppose that $\mathcal{S}$ is not u.H.c. at $u_0$. This means that there exist a neighbourhood $\overline{W}$ and a sequence $(u_k)$, $lim_k u_k = u_0$, and a sequence $(x_k)$, $x_k \in \mathcal{S}(u_k)$ such that

$$x_k \notin \mathcal{S}(u_0) + \overline{W}, \ k = 1, 2, .... \quad (*)$$

Suppose that there exists an infinite subset of integers $M_1$ such that

$$x_k \in \mathcal{A}(u_k) \setminus \mathcal{A}(u_0) \ for \ k \in M_1. \qquad (**)$$

Let $(z_m)$ denote the sequence $(x_k)_{k\in M_1} = (z_m)$.

If $cl((z_m)) \cap \mathcal{A}(u_0) \neq \emptyset$, then there exists a point $z_0$ which is a limit point of the sequence from $(z_m)$ and belongs to $\mathcal{A}(u_0)$. By Lemma 6.1, $z_0$ must be in $\mathcal{S}(u_0)$, and we obtain a contradiction with the assumption $(*)$.

Hence, it must be $cl((z_m)) \cap \mathcal{A}(u_0) = \emptyset$. If there exists a neighbourhood $\overline{W1}$ of zero such that $cl((z_m)) \cap [\mathcal{A}(u_0) + \overline{W1}] = \emptyset$, then $\mathcal{A}$ is not u.H.c. at $u_0$. Otherwise, for each neighbourhood $W$ of zero

$$cl((z_m)) \cap [\mathcal{A}(u_0) + W] \neq \emptyset.$$

Hence, for each $W$ there exists $M$ such that for a certain $(x_m) \in \mathcal{A}(u_0)$ we have

$$z_m \in x_m + W \quad for \quad m \geq M.$$

By the uniform lower-continuity of $f$ on $A$, for each neighbourhood $U$ of zero in $Y$ there exists an index $M1$ such that

$$f(z_m) \in f(x_m) + U + \mathcal{K} \quad for \quad m \geq M1.$$

There exists a neighbourhood $\overline{U1}$ such that (for an infinite number of m, say $m \in M3$) we have

$$f(x_m) \notin [M(f, A) + \overline{U1}], \ m \in M3. \qquad (***)$$

Indeed, if for each neighbourhood $U$ and all $m$ sufficiently large, say $m \geq \overline{M}$ we would have $f(x_m) \in [M(f, A) + U]$, ie. $f(x_m) \in \eta_m + U$, $m \geq \overline{M}$ then $(x_m)$ would contain a minimizing sequence. Without loss of generality $(x_m)$ itself could be assumed to be minimizing. And, by assumptions, it would be contained in the set $S(f, A) + W$ for any $W$ and all $m$ sufficiently large. In consequence, also $(z_m)$ for $m$ sufficiently large would be contained in any set of the form $S(f, A) + W$. This, however would contradict the assumption $(*)$.

Thus, $(***)$ must hold. Since all $x_m$, for $m \geq \overline{M}$, belong to $\mathcal{A}(u_0)$, for each $m$ there exists an element $y_m \in M(f, A)$ such that $p_m = f(x_m) - y_m \in \mathcal{K}$. Moreover, because of $(***)$, $p_m \notin \overline{U1}$. By the containement property, there exists a neighbourhood $\overline{O}$ such that

$$p_m + \overline{O} \in \mathcal{K} \quad for\ m \geq \overline{M}.$$

On the other hand, be the uniform sup-lower continuity of $\mathcal{FA}$ we have

$$(y_m + 1/2\overline{O} - \mathcal{K}) \cap \mathcal{FA}(u_m) \neq \emptyset$$

for all $m$ sufficiently large, ie.there exists a sequence $(w_m)$ such that

$$f(w_m) \in \mathcal{FA}(u_m) \ and\ f(w_m) \in y_m + 1/2\overline{0} - \mathcal{K}$$

for all $m$ sufficiently large and $m \geq \overline{M}$. This gives

$$\begin{aligned} f(z_m) - f(w_m) &= \\ &[f(z_m) - f(x_m] + [f(x_m) - y_m] + [y_m - f(w_m)] \\ &\in 1/2\overline{O} + p_m + 1/2\overline{O} + \mathcal{K} \\ &\in \mathcal{K} + \mathcal{K} \subset \mathcal{K}. \end{aligned}$$

which contradicts the fact that $z_m \in \mathcal{S}(u_m)$.

Consider now the opposite situation and assume that for all $k$ sufficiently large

$$x_k \in \mathcal{A}(u_k) \cap \mathcal{A}(u_0). \quad (****)$$

Consequently, we have also that

$$f(x_k) \in \mathcal{FA}(u_k) \cap \mathcal{FA}(u_0)$$

for all $k$ sufficiently large. Now, by Lemma 6.2, the sequence $(x_k)$ must be a minimizing sequence or $x_k \notin \mathcal{S}(u_k)$. Since $x_k \in \mathcal{S}(u_k)$ for all $k$ $(x_k)$ must be a minimizing sequence. By well-posedness, for each $W$ there exists an index $K$ such that

$$x_k \in \mathcal{S}(u_0) + W \quad for\ \ k \geq K.$$

This, however, contradicts $(*)$ which finishes the proof.

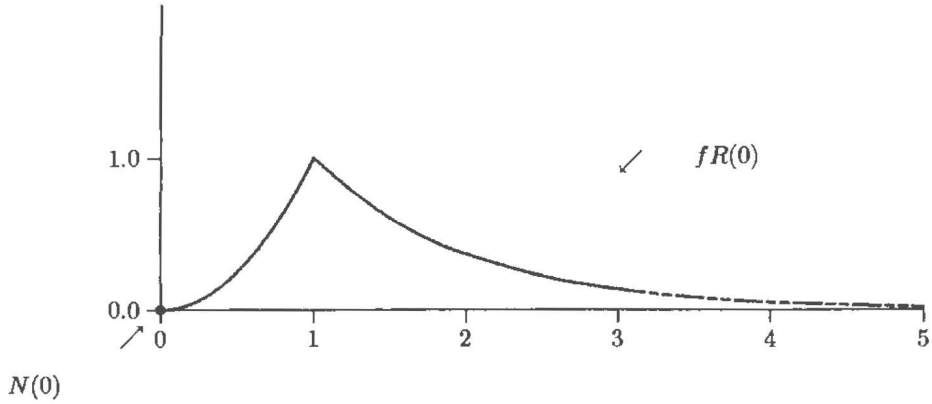


Figure 1: The feasible and optimal value sets in the space $Y$ for examples 7.0.1 and 7.0.2

□

The following examples show that well–posedness does not imply the containment property of the set $f(A)$.

**Example 7.0.1** *Let us consider the vector optimization problem of the function*

$$f(x) = \begin{cases} (x, e^{1-x}) & \text{if } x \geq 1 \\ (x, x^2) & \text{if } 0 \leq x \leq 1 \end{cases}$$

*under the constraints*

$$0.0 \leq x$$

The problem in Example 7.0.1 is well–posed but the set $f(R)$ does not have the containment property. In a simple modification presented below the set $f(A)$ has the containment property.

**Example 7.0.2** *Let us consider the vector optimization problem for the function*

$$f(x) = \begin{cases} (x, e^{1-x}) & \text{if } x \geq 1 \\ (x, x^2) & \text{if } 0 \leq x \leq 1 \end{cases}$$

*under the constraints*

$$0.0 \leq x$$

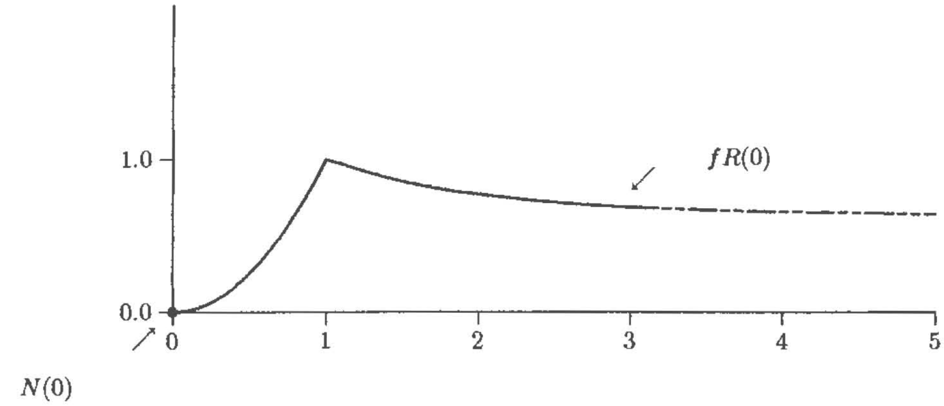


Figure 2: The feasible and optimal sets in the objective space for Examples 7.0.1 and 7.0.2

### 7.0.1 Concluding remarks.

Two variants of Theorem 7.0.1 and Theorem 7.0.3 can be obtained by applying Propositions **??** and **??**.

For instance, according to Proposition **??** the assumptions *(iii)* and *(iv)* of Theorems 7.0.1 and 7.0.3 can be replaced by the assumption

$(iv)'$ – *the set* $f(A)$ *has the strong domination property.*

It should be remembered, however, that this assumption is stronger than the previous ones. For instance, in Example 7.0.2 the strong domination property is not satisfied but it is easy to introduce perturbations to the problem in such a way that the assumptions of Theorems 7.0.1 and 7.0.3 are satisfied and the solution multifunction has the respective continuity properties.

If we consider the definition of well–posed scalar optimization problems as introduced eg. in [19], then one of the consequences of well–posedness in that sense is that the problem in question has nonempty solution set and, in consequence $Min(f(A)|\mathcal{R}^+)$ is also nonempty. According to the Remark **??** the assumptions *(iii)* and *(iv)* of Theorems 7.0.1 and 7.0.3 are then automatically satisfied and we obtain the scalar counterparts of the above theorems (see eg. [19]).

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